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COLLISIONS OF SEVERAL WALKERS IN RECURRENT RANDOM ENVIRONMENTS

ALEXIS DEVULDER, NINA GANTERT, AND FRANÇOISE PÈNE

ABSTRACT. We consider d independent walkers on \mathbb{Z} , m of them performing simple symmetric random walk and $r = d - m$ of them performing recurrent RWRE, in I independent non-deterministic environments. We show that the product is recurrent, almost surely, if and only if $m \leq 1$ or $m = d = 2$. In the transient case with $r \geq 1$, we prove that the walkers meet infinitely often, almost surely, if and only if $m = 2$ and $r \geq I = 1$. In particular, while I does not have an influence for the recurrence or transience, it does play a role for the probability to have infinitely many meetings.

1. INTRODUCTION AND STATEMENT OF THE MAIN RESULTS

Recurrence and transience of products of simple symmetric random walks on \mathbb{Z}^d is well-known since the works of Pólya [P21]. If the product of several walks is transient, one may ask if they meet infinitely often. It is also well-known and goes back to Dvoretzky and Erdős, see ([DE51], p. 367) that 3 independent simple symmetric random walks (SRW) in dimension 1 meet infinitely often almost surely while 4 walks meet only finitely often, almost surely. In fact, Pólya's original interest in recurrence/transience of simple random walk came from a question about collisions of two independent walkers on the same grid, see [P84], "Two incidents".

The classical topic of meetings/collisions of two or more walkers walking on the same graph has found recent interest, see [KP04], [BSP12], where the grid is replaced by more general graphs. It is well-known that if a graph is recurrent for simple random walk, two independent walkers do not necessarily meet infinitely often, see [KP04]. Since on a *transitive* recurrent graph, two independent walkers do meet infinitely often, almost surely, see [KP04], the "infinite collision property" describes how far the graph is from being transitive. For motivation from physics, see [CC12].

We investigate this question for products of recurrent random walks in random environment (RWRE) and of simple symmetric random walks on \mathbb{Z} . It is known already that, for any n , a product of n independent recurrent RWRE in n i.i.d. environments is recurrent, see [Z01], and that n independent walkers in the same recurrent environment meet infinitely often in the origin, see [GKP14]. Here, we consider several walkers in one-dimensional random/non-random environment, with the additional twist that not all walkers are necessarily using the same environment.

Let d, m, r be nonnegative integers such that $m + r = d \geq 1$. We consider d walkers, m of them performing SRW $S^{(1)}, \dots, S^{(m)}$ and the r others performing random walks $Z^{(1)}, \dots, Z^{(r)}$ in I independent random environments, with $I \leq r$. More precisely, we

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consider r collections of i.i.d. random variables $\omega^{(1)} := (\omega_x^{(1)})_{x \in \mathbb{Z}}, \dots, \omega^{(r)} := (\omega_x^{(r)})_{x \in \mathbb{Z}}$, taking values in $(0, 1)$ and defined on the same probability space $(\Omega, \mathcal{F}, \mathbf{P})$, such that $\omega^{(1)}, \dots, \omega^{(I)}$ are independent and such that the others are exact copies of some of these I collections, i.e., for every $j \in \{I + 1, \dots, r\}$, there exists an index $J_j \in \{1, \dots, I\}$ such that $\omega^{(j)} \equiv \omega^{(J_j)}$. A realization of $\omega := (\omega^{(1)}, \dots, \omega^{(r)})$ will be called an *environment*. We set

$$Y_n := (S_n^{(1)}, \dots, S_n^{(m)}, Z_n^{(1)}, \dots, Z_n^{(r)}), \quad n \in \mathbb{N},$$

and make the following assumptions. Given $\omega = (\omega^{(1)}, \dots, \omega^{(r)})$ and $x \in \mathbb{Z}^d$, under P_ω^x , $S^{(1)}, \dots, S^{(m)}, Z^{(1)}, \dots, Z^{(r)}$ are independent Markov chains such that $P_\omega^x(Y_0 = x) = 1$ and for all $y \in \mathbb{Z}$ and $n \in \mathbb{N}$,

$$P_\omega^x[S_{n+1}^{(i)} = y + 1 | S_n^{(i)} = y] = \frac{1}{2} = P_\omega^x[S_{n+1}^{(i)} = y - 1 | S_n^{(i)} = y], \quad i \in \{1, \dots, m\}, \quad (1)$$

$$P_\omega^x[Z_{n+1}^{(j)} = y + 1 | Z_n^{(j)} = y] = \omega_y^{(j)} = 1 - P_\omega^x[Z_{n+1}^{(j)} = y - 1 | Z_n^{(j)} = y], \quad j \in \{1, \dots, r\}. \quad (2)$$

We set $S^{(i)} := (S_n^{(i)})_{n \in \mathbb{N}}$ and $Z^{(j)} := (Z_n^{(j)})_{n \in \mathbb{N}}$ for every $i \in \{1, \dots, m\}$ and every $j \in \{1, \dots, r\}$. Note that, for every j , $Z^{(j)} = (Z_n^{(j)})_{n \in \mathbb{N}}$ is a *random walk on \mathbb{Z} in the environment $\omega^{(j)}$* , and that the $S^{(i)}$'s are independent SRW, independent of the $Z^{(j)}$'s and of their environments. We call $P_\omega := P_\omega^0$ the *quenched law*. Here and in the sequel we write 0 for the origin in \mathbb{Z}^d . We also define the *annealed law* as follows:

$$\mathbb{P}[\cdot] := \int P_\omega[\cdot] \mathbf{P}(d\omega).$$

Setting $\rho_k^{(j)} := \frac{1 - \omega_k^{(j)}}{\omega_k^{(j)}}$ for $j \in \{1, \dots, r\}$ and $k \in \mathbb{Z}$, we assume moreover that there exists $\varepsilon_0 \in (0, 1/2)$ such that for every $j \in \{1, \dots, r\}$,

$$\mathbf{P}[\omega_0^{(j)} \in [\varepsilon_0, 1 - \varepsilon_0]] = 1, \quad \mathbf{E}[\log \rho_0^{(j)}] = 0, \quad \sigma_j^2 := \mathbf{E}[(\log \rho_0^{(j)})^2] > 0, \quad (3)$$

where \mathbf{E} is the expectation with respect to \mathbf{P} . The $Z^{(j)}$ are RWRE, often called *Sinai's walks* due to the famous result of [S82]. Solomon [S75] proved the recurrence of $Z^{(j)}$ for \mathbf{P} -almost every environment.

Our first result concerns the recurrence/transience of $Y := (Y_n)_n$. Recurrence of Y means that $S^{(1)}, \dots, S^{(m)}, Z^{(1)}, \dots, Z^{(r)}$ meet simultaneously at 0 infinitely often. As explained previously, this result is known for SRW (i.e. if $m = d$) since [P21] and more recently for RWRE (i.e. if $r = d$, that is, if $m = 0$) in the case where the environments $\omega^{(j)}$ are independent (i.e. $I = r = d$, see [Z01, GKP14]) and in the case where the environment $\omega^{(j)}$ is the same for all the RWRE (i.e. $r = d, I = 1$, see [GKP14]). See also [Ga13] for related results.

Theorem 1.1. *If $m \leq 1$, or if $m = d = 2$, then, for \mathbf{P} -almost every ω , the random walk Y is recurrent with respect to P_ω^0 . Otherwise, for \mathbf{P} -almost every ω , the random walk Y is transient with respect to P_ω^0 .*

In particular, a product of two recurrent RWRE and one SRW is recurrent, while a product of two SRW and one recurrent RWRE is transient.

When Y is transient, a natural question is the study of the simultaneous meetings (i.e., collisions) of $S^{(1)}, \dots, S^{(m)}, Z^{(1)}, \dots, Z^{(r)}$. That is, we would like to extend the results of [P21, DE51] to the case in which some of the random walks are in random environments (when $r \geq 1$). We recall that when $r = 0$, the number of collisions is, by [P21, DE51], almost surely infinite if $m \leq 3$ and almost surely finite when $m \geq 4$. Interestingly, compared to Theorem 1.1, the behaviour depends on whether $I = 1$ (when the RWRE are all in the same environment) or $I \geq 2$ (at least two RWRE are in independent environments).

Theorem 1.2. *We distinguish the 3 following different cases.*

(i) *If $m \geq 3$ and $r \geq 1$, then, for \mathbf{P} -almost every environment ω ,*

$$P_\omega^0[S_n^{(1)} = S_n^{(2)} = S_n^{(3)} = Z_n^{(1)} \text{ infinitely often}] = 0,$$

i.e. almost surely, the walks $S^{(1)}, S^{(2)}, S^{(3)}, Z^{(1)}$ meet simultaneously only a finite number of times. A fortiori, $S^{(1)}, \dots, S^{(m)}, Z^{(1)}, \dots, Z^{(r)}$ also meet simultaneously only a finite number of times.

(ii) *If $m = 2$ and $r \geq I = 1$, then for \mathbf{P} -almost every environment ω ,*

$$P_\omega^0[S_n^{(1)} = S_n^{(2)} = Z_n^{(1)} = \dots = Z_n^{(r)} \text{ infinitely often}] = 1,$$

i.e. almost surely, the walks $S^{(1)}, S^{(2)}, Z^{(1)}, \dots, Z^{(r)}$ meet simultaneously infinitely often.

(iii) *If $m = 2$ and $r \geq I \geq 2$, then for \mathbf{P} -almost every environment ω ,*

$$P_\omega^0[S_n^{(1)} = S_n^{(2)} = Z_n^{(1)} = Z_n^{(2)} \text{ infinitely often}] = 0,$$

i.e. almost surely, the walks $S^{(1)}, S^{(2)}, Z^{(1)}, Z^{(2)}$, and a fortiori the walks $S^{(1)}, S^{(2)}, Z^{(1)}, \dots, Z^{(r)}$, meet simultaneously only a finite number of times.

This last result can be summarized in the following manner. Assume that $r \geq 1$ and that Y is transient (i.e. $m \geq 2$ and $r \geq 1$), then $S^{(1)}, \dots, S^{(m)}, Z^{(1)}, \dots, Z^{(r)}$ meet simultaneously infinitely often if and only if $m = 2$ and $I = 1$. Hence our results cover collisions of an arbitrary number of random walks in equal or independent random (or degenerate) environments.

The article is organized as follows. In Section 2, we give an estimate on the return probability of recurrent RWRE, see Proposition 2.1, which is of independent interest. Our main results for direct products of walks are proved in Section 3. The proofs concerning the simultaneous meetings of random walks are based on two key lemmas on recurrent RWRE proved in Sections 4 and 5. A key result of Section 4 is a localization result with probability $1 - (\log n)^{-2+\varepsilon}$ of a recurrent RWRE in a low potential area defined with several valleys (see Proposition 4.4). In Section 5, we prove that, with high probability on ω (for \mathbf{P}), the quenched probability $P_\omega[|Z_n - b(N)| \leq 1]$ is larger than a positive constant, for any $n \in [N^{1-\varepsilon}, N]$ for some $\varepsilon > 0$, where $b(N)$ is the bottom of some valley of the potential V of the recurrent RWRE Z (see Proposition 5.2).

2. A RETURN PROBABILITY ESTIMATE FOR THE RWRE

We consider a recurrent one dimensional RWRE $Z = (Z_n)_{n \in \mathbb{N}}$ in the random environment $\omega = (\omega_x)_{x \in \mathbb{Z}}$, where the $\omega_x \in (0, 1)$, $x \in \mathbb{Z}$, are i.i.d. (that is, $Z_0 = 0$ and (2) is satisfied with Z and ω instead of $Z^{(1)}$ and $\omega^{(1)}$). We assume the existence of $\varepsilon_0 \in (0, 1/2)$ such that

$$\mathbb{P}[\omega_0 \in [\varepsilon_0, 1 - \varepsilon_0]] = 1, \quad \mathbb{E}[\log \rho_0] = 0, \quad \mathbb{E}[(\log \rho_0)^2] > 0, \quad (4)$$

where $\rho_k := \frac{1-\omega_k}{\omega_k}$, $k \in \mathbb{Z}$. The following result completes [GKP14, Theorem 1.1] which says that, for every $0 \leq \vartheta < 1$, we have for \mathbb{P} -almost every environment ω ,

$$\sum_{n \geq 1} \frac{P_\omega^0[Z_n = 0]}{n^\vartheta} = \infty.$$

Proposition 2.1. *For \mathbb{P} -almost every environment ω ,*

$$\sum_{n \geq 1} \frac{P_\omega^0[Z_n = 0]}{n} < \infty.$$

Before proving this result, we introduce some more notations. First, let

$$\tau(x) := \inf\{n \geq 1 : Z_n = x\}, \quad x \in \mathbb{Z}.$$

In words, $\tau(x)$ is the hitting time of the site x by the RWRE Z . As usual, we consider the *potential* V , which is a function of the environment ω and is defined on \mathbb{Z} as follows:

$$V(x) := \begin{cases} \sum_{i=1}^x \log \frac{1-\omega_i}{\omega_i} & \text{if } x > 0, \\ 0 & \text{if } x = 0, \\ -\sum_{i=x+1}^0 \log \frac{1-\omega_i}{\omega_i} & \text{if } x < 0. \end{cases} \quad (5)$$

The potential is useful since it relates to the description of the RWRE as an electric network. It can be used to estimate ruin probabilities for the RWRE. In particular, we have (see e.g. [Z01, (2.1.4)] and [D14, Lemma 2.2] coming from [Z01, p. 250]),

$$P_\omega^b[\tau(c) < \tau(a)] = \left(\sum_{j=a}^{b-1} e^{V(j)} \right) \left(\sum_{j=a}^{c-1} e^{V(j)} \right)^{-1}, \quad a < b < c \quad (6)$$

and

$$E_\omega^b[\tau(a) \wedge \tau(c)] \leq \varepsilon_0^{-1} (c-a)^2 \exp \left[\max_{a \leq \ell \leq k \leq c-1; k \geq b} (V(k) - V(\ell)) \right], \quad a < b < c, \quad (7)$$

where E_ω^b denotes the expectation with respect to P_ω^b and $u \wedge v := \min(u, v)$, $(u, v) \in \mathbb{R}^2$. For symmetry reasons, we also have

$$E_\omega^b[\tau(a) \wedge \tau(c)] \leq \varepsilon_0^{-1} (c-a)^2 \exp \left[\max_{a \leq \ell \leq k \leq c-1, \ell \leq b-1} (V(\ell) - V(k)) \right], \quad a < b < c. \quad (8)$$

Moreover, we have (see Golosov [G84], Lemma 7)

$$P_\omega^b[\tau(c) < k] \leq k \exp \left(\min_{\ell \in [b, c-1]} V(\ell) - V(c-1) \right), \quad b < c. \quad (9)$$

and by symmetry, we get (similarly as in Shi and Zindy [SZ07], eq. (2.5) but with some slight differences for the values of ℓ)

$$P_\omega^b[\tau(a) < k] \leq k \exp\left(\min_{\ell \in [a, b-1]} V(\ell) - V(a)\right), \quad a < b. \quad (10)$$

Lemma 2.2. *Let $\gamma > 0$. For \mathbb{P} -almost every ω , there exists $N(\omega)$ such that for every $n \geq N(\omega)$,*

$$n^{\frac{1}{2}-\gamma} \leq \max_{k \in \{0, \dots, n\}} V(k) \leq n^{\frac{1}{2}+\gamma}, \quad -n^{\frac{1}{2}+\gamma} \leq \min_{k \in \{0, \dots, n\}} V(k) \leq -n^{\frac{1}{2}-\gamma},$$

and such that the same inequalities hold with $\{-n, \dots, 0\}$ instead of $\{0, \dots, n\}$.

Proof. Observe that it is enough to prove that \mathbb{P} -almost surely,

$$n^{\frac{1}{2}-\gamma} \leq \max_{1 \leq k \leq n} V(k) \leq n^{\frac{1}{2}+\gamma} \quad (11)$$

if n is large enough (up to a change of $\log \rho_i$ in $-\log \rho_i$, in $\log \rho_{1-i}$ or in $-\log \rho_{1-i}$). The first inequality of (11) is given by [H65, Theorem 2]. The second inequality of (11) is a consequence of [C48, Equation (10)]. \blacksquare

Proof of Proposition 2.1. Let $\eta \in (0, 1)$ and $n \geq 3$. We define

$$z_+ := \inf\{y \geq 1 : V(y) \leq -(\log n)^{1-\eta}\}, \quad z_- := \sup\{y \leq -1 : V(y) \leq -(\log n)^{1-\eta}\}.$$

Due to the previous lemma, \mathbb{P} -almost surely, if n is large enough, the following inequalities hold:

$$|z_\pm| \leq \frac{(\log n)^{2-\eta}}{2} \quad \text{and} \quad \max_{z_- \leq i, j \leq z_+} (V(i) - V(j)) \leq (\log n)^{1-\eta/10}. \quad (12)$$

We have by the strong Markov property,

$$\begin{aligned} P_\omega^0[Z_n = 0] &\leq P_\omega^0[\tau(z_+) > n, \tau(z_-) > n] + \sum_{k=0}^n P_\omega^0[\tau(z_+) = k] P_\omega^{z_+}[Z_{n-k} = 0] \\ &\quad + \sum_{k=0}^n P_\omega^0[\tau(z_-) = k] P_\omega^{z_-}[Z_{n-k} = 0]. \end{aligned} \quad (13)$$

Recall that, given ω , the Markov chain Z is an electrical network where, for every $x \in \mathbb{Z}$, the conductance of the bond $(x, x+1)$ is $C_{(x, x+1)} = e^{-V(x)}$ (in the sense of Doyle and Snell [DS84]). In particular, the reversible measure μ_ω (unique up to a multiplication by a constant) is given by

$$\mu_\omega(x) := e^{-V(x)} + e^{-V(x-1)}, \quad z \in \mathbb{Z}. \quad (14)$$

So we have

$$\begin{aligned} P_\omega^{z_\pm}[Z_{n-k} = 0] &= P_\omega^0[Z_{n-k} = z_\pm] \frac{\mu_\omega(0)}{\mu_\omega(z_\pm)} \leq \frac{\mu_\omega(0)}{\mu_\omega(z_\pm)} = \frac{e^{-V(0)} + e^{-V(-1)}}{e^{-V(z_\pm)} + e^{-V(z_\pm-1)}} \\ &\leq \frac{e^{-V(0)} + e^{-V(-1)}}{e^{-V(z_\pm)}} \leq (e^{-V(0)} + e^{-V(-1)}) \exp[-(\log n)^{1-\eta}]. \end{aligned}$$

Hence,

$$\sum_{k=0}^n P_\omega^0[\tau(z_\pm) = k] P_\omega^{z_\pm}[Z_{n-k} = 0] \leq (e^{-V(0)} + e^{-V(-1)}) \exp[-(\log n)^{1-\eta}]. \quad (15)$$

Moreover we have due to (7) and to Markov's inequality,

$$\begin{aligned} P_\omega^0[\tau(z_+) > n, \tau(z_-) > n] &\leq n^{-1} E_\omega^0[\tau(z_+) \wedge \tau(z_-)] \\ &\leq n^{-1} \varepsilon_0^{-1} (z_+ - z_-)^2 \exp\left[\max_{z_- \leq \ell \leq k \leq z_+ - 1} [V(k) - V(\ell)]\right]. \end{aligned}$$

Now using (12), P-almost surely, we have

$$P_\omega^0[\tau(z_+) > n, \tau(z_-) > n] \leq \varepsilon_0^{-1} n^{-1} (\log n)^{4-2\eta} \exp[(\log n)^{1-\eta/10}]$$

for every n large enough. This combined with (13), (15) and $e^{-V(-1)} \leq \varepsilon_0^{-1}$ gives P-almost surely for large n

$$P_\omega^0[Z_n = 0] \leq 5\varepsilon_0^{-1} \exp[-(\log n)^{1-\eta}].$$

Consequently, $\sum_{n \geq 1} \frac{P_\omega^0[Z_n = 0]}{n} < \infty$ P-almost surely, which ends the proof of Proposition 2.1. \blacksquare

3. DIRECT PRODUCT OF WALKS

We start with a proof of Theorem 1.1.

Proof. 1. If $m \geq 1$ and $r = 0$, then $(Y_n)_n$ is a simple random walk on \mathbb{Z}^m . So it is recurrent if $m \in \{1, 2\}$, and transient if $m \geq 3$.

2. If $m \geq 3$ and $r \geq 1$, then the 3-tuple of the three first coordinates of $(Y_n)_n$ is $(S_n^{(1)}, S_n^{(2)}, S_n^{(3)})_n$ which is a random walk on \mathbb{Z}^3 , hence is transient. So $(Y_n)_n$ is transient for P-almost every ω .

3. If $m = 2$ and $r \geq 1$, then applying the local limit theorem (see e.g. Lawler and Limic [LL10] Prop. 2.5.3) for $S^{(1)}$ and $S^{(2)}$ for $n \in \mathbb{N}^*$,

$$P_\omega^0[Y_n = 0] = \prod_{i=1}^2 P[S_n^{(i)} = 0] \prod_{j=1}^r P_{\omega^{(j)}}^0[Z_n^{(j)} = 0] \leq \frac{c}{n} P_{\omega^{(1)}}^0[Z_n^{(1)} = 0],$$

where $c > 0$ is a constant. This and Proposition 2.1 yield $\sum_{n=0}^\infty P_\omega^0[Y_n = 0] < \infty$ for P-almost ω . Hence, $(Y_n)_n$ is P-almost surely transient.

4. We now assume $m \leq 1$. We choose some $\delta \in (0, 1/5)$ such that $3\delta r < \frac{1-2\delta}{2}$. We denote by $\lfloor x \rfloor$ the integer part of x for $x \in \mathbb{R}$. For $L \in \mathbb{N}$, we have

$$\begin{aligned} \sum_{n \geq 0} P_\omega^0[Y_n = 0] &\geq \sum_{n = \lfloor \frac{e^{(1-2\delta)L}}{2} \rfloor + 1}^{\lfloor e^{(1-2\delta)L} \rfloor} P_\omega^0[Y_{2n} = 0] \\ &= \sum_{n = \lfloor \frac{e^{(1-2\delta)L}}{2} \rfloor + 1}^{\lfloor e^{(1-2\delta)L} \rfloor} \prod_{i=1}^m P[S_{2n}^{(i)} = 0] \prod_{j=1}^r P_{\omega^{(j)}}^0[Z_{2n}^{(j)} = 0]. \end{aligned}$$

Due to [GKP14] (Propositions 3.2, 3.4 and (3.22)), since $\delta \in (0, 1/5)$, there exist $C(\delta) > 0$ and a sequence $(\Gamma(L, \delta))_{L \in \mathbb{N}}$ of elements of \mathcal{F} (that is, depending only on ω) such that

$$\mathbb{P} \left[\bigcap_{N \geq 0} \bigcup_{L \geq N} \Gamma(L, \delta) \right] = 1 \quad (16)$$

and such that, for every $L \in \mathbb{N}$, on $\Gamma(L, \delta)$, we have

$$\forall i \in \{1, \dots, r\}, \forall k_i \in \{\lfloor e^{3\delta L} \rfloor + 1, \dots, \lfloor e^{(1-2\delta)L} \rfloor\}, \quad P_{\omega^{(i)}}^0[Z_{2k_i}^{(i)} = 0] \geq C(\delta) e^{-3\delta L}. \quad (17)$$

Due to the local limit theorem, this gives on $\Gamma(L, \delta)$, for large L so that $\frac{e^{(1-2\delta)L}}{2} \geq e^{3\delta L}$,

$$\sum_{n \geq 0} P_{\omega}^0[Y_n = 0] \geq \frac{e^{(1-2\delta)L}}{3} \left(\frac{c}{e^{(1-2\delta)L/2}} \right)^m \left(\frac{C(\delta)}{e^{3\delta L}} \right)^r \geq c_1(\delta) e^{[(1-2\delta)/2 - 3\delta r]L},$$

which goes to infinity as L goes to infinity due to our choice of δ , $c_1(\delta)$ being a positive constant. Thanks to (16), this gives $\sum_{n \geq 0} P_{\omega}^0[Y_n = 0] = +\infty$ for \mathbb{P} -almost all ω . Consequently, $(Y_n)_n$ is recurrent for \mathbb{P} -almost every environment ω . \blacksquare

Remark 3.1. Recall that Sinai [S82] (see also Golosov [G84]) proved the convergence in distribution of $(Z_n^{(i)}/(\log n)^2)_n$. Recall also that, due to de Moivre's theorem, $(S_n^{(i)}/\sqrt{n})_n$ converges in distribution. Due to Theorem 1.1, Y is recurrent iff $\sum_n 1/(n^{\frac{m}{2}}((\log n)^2)^r) = \infty$, where $n^{\frac{m}{2}}((\log n)^2)^r$ is the product of the normalizations of the coordinates of Y under the (non Markovian) annealed law \mathbb{P} .

Note also that Theorem 1.2 and the previous paragraph lead to the following statement (only for $r \geq 1$): if $\sum_{n \geq 1} \frac{1}{n^{m/2}(\log n)^{2I-2}} < \infty$, then almost surely, $S_n^{(1)}, \dots, S_n^{(m)}, Z_n^{(1)}, \dots, Z_n^{(r)}$ meet simultaneously only a finite number of times; otherwise, they almost surely meet simultaneously infinitely often.

Now we will start to prove Theorem 1.2. Note that the case $m \leq 1$ is already treated in Theorem 1.1 which says that in this case the random walks meet infinitely often at 0. Moreover, the case $r = 0$ is already known, as said before.

Proof of Theorem 1.2. Let $A_n := \{S_n^{(1)} = \dots = S_n^{(m)} = Z_n^{(1)} = \dots = Z_n^{(r)}\}$ for $n \geq 0$.

Proof of (i). Assume $m = 3$ and $r = 1$. Observe that for large n ,

$$\begin{aligned} \mathbb{P}_{\omega}^0[A_n] &= \sum_{k \in \mathbb{Z}} \mathbb{P}_{\omega}^0[Z_n^{(1)} = k] (\mathbb{P}[S_n^{(1)} = k])^3 \\ &\leq C \sum_{k \in \mathbb{Z}} \frac{\mathbb{P}_{\omega}^0[Z_n^{(1)} = k]}{n\sqrt{n}} = \frac{C}{n\sqrt{n}} \end{aligned}$$

for some $C > 0$ since for every $k \in \mathbb{Z}$ and $n \in \mathbb{N}$, $\mathbb{P}[S_{2n}^{(1)} = k] \leq \mathbb{P}[S_{2n}^{(1)} = 0] \sim_{n \rightarrow +\infty} (\pi n)^{-1/2}$ due to the local limit theorem. Hence $\sum_n \mathbb{P}_{\omega}^0[S_n^{(1)} = S_n^{(2)} = S_n^{(3)} = Z_n^{(1)}] < \infty$ and (i) follows by the Borel-Cantelli lemma in this case and a fortiori when $m \geq 3$ and $r \geq 1$.

Proof of (ii). Assume $m = 2$ and $r \geq I = 1$. We use the generalization of the second Borel Cantelli lemma due to Kochen and Stone [KS64] combined with a result by Doob. To simplify notations, we also write ω for $\omega^{(1)}$, so $\omega^{(i)} = \omega$ for every $1 \leq i \leq r$.

We first prove that $\sum_n \mathbb{P}_\omega(A_n) = \infty$ a.s. More precisely, we fix an initial condition $x = (x_1, x_2, y_1, \dots, y_r) \in (2\mathbb{Z})^{2+r} \cup (2\mathbb{Z} + 1)^{2+r}$. We have for all n and ω ,

$$P_\omega^x[A_n] = \sum_{k \in \mathbb{Z}} \mathbb{P}[x_1 + S_n^{(1)} = k] \mathbb{P}[x_2 + S_n^{(2)} = k] \prod_{j=1}^r P_\omega^{y_j}[Z_n^{(j)} = k].$$

Notice that, for every $i \in \{1, 2\}$, due to the de Moivre-Laplace theorem (see e.g. [LL10, Prop. 2.5.3 and Corollary 2.5.4] or [F68, VII-3]),

$$\sup_{k \in x_i + n + 2\mathbb{Z}, |k| \leq (\log n)^3} \left| \mathbb{P}[x_i + S_n^{(i)} = k] - \frac{\sqrt{2}}{\sqrt{\pi n}} e^{-(k-x_i)^2/(2n)} \right| = o(n^{-1/2}).$$

Consequently for large even n , for every ω ,

$$P_\omega^x[A_n] \geq \sum_{|k| \leq (\log n)^3, k - (x_1 + n) \in (2\mathbb{Z})} \frac{1}{\pi n} \prod_{j=1}^r P_\omega^{y_j}[Z_n^{(j)} = k] = \frac{1}{\pi n} \sum_{|k| \leq (\log n)^3} \prod_{j=1}^r P_\omega^{y_j}[Z_n^{(j)} = k].$$

This remains true for large odd n . Hence for large n ,

$$P_\omega^x[A_n] \geq \frac{1}{\pi n} P_\omega^{(y_1, \dots, y_r)}[Z_n^{(1)} = \dots = Z_n^{(r)}] - \frac{1}{\pi n} P_\omega^{y_1}[|Z_n^{(1)}| > (\log n)^3]. \quad (18)$$

Recall that $(Z_n^{(1)}/(\log n)^3)_n$ converges almost surely to 0 with respect to the annealed law (see [DR86] Theorem 4, or more recently [HS98] Theorem 3). This holds also true for $P_\omega^{y_1}$ for \mathbb{P} -almost every ω , so the last probability in (18) goes to 0 as $n \rightarrow +\infty$, which yields $\lim_{N \rightarrow +\infty} \frac{1}{\log N} \sum_{n=1}^N \frac{1}{n} P_\omega^{y_1}[|Z_n^{(1)}| > (\log n)^3] = 0$. Hence for \mathbb{P} -almost every ω ,

$$\limsup_{N \rightarrow +\infty} \frac{1}{\log N} \sum_{n=1}^N P_\omega^x[A_n] \geq \frac{c(\omega)}{\pi}, \quad (19)$$

with $c(\omega) := \inf_{(y_1, \dots, y_r) \in ((2\mathbb{Z})^r \cup (2\mathbb{Z}+1)^r)} \limsup_{N \rightarrow +\infty} \frac{1}{\log N} \sum_{n=1}^N \frac{1}{n} P_\omega^{(y_1, \dots, y_r)}[Z_n^{(1)} = \dots = Z_n^{(r)}]$. If $r = 1$, then $c(\omega) = 1$. If $r > 1$, due to Lemma 5.1, $c(\omega) > 0$ for \mathbb{P} -almost every environment ω . This implies that

$$\sum_{n \geq 1} P_\omega^x[A_n] = +\infty. \quad (20)$$

Moreover, let $C > 0$ be such that for all $n \geq 1$ and $k \in \mathbb{Z}$, $\mathbb{P}[S_n^{(1)} = k] \leq Cn^{-1/2}$, which exists e.g. since $\mathbb{P}[S_{2n}^{(1)} = k] \leq \mathbb{P}[S_{2n}^{(1)} = 0] \sim_{n \rightarrow +\infty} (\pi n)^{-1/2}$ by the local limit theorem. So for $1 \leq n < m$, we have by Markov property,

$$\begin{aligned} P_\omega^x[A_n \cap A_m] &= \sum_{k, \ell \in \mathbb{Z}} P_\omega^{(y_1, \dots, y_r)}[Z_n^{(1)} = \dots = Z_n^{(r)} = k, Z_m^{(1)} = \dots = Z_m^{(r)} = \ell] \\ &\quad \times \mathbb{P}[x_1 + S_n^{(1)} = k] \mathbb{P}[x_2 + S_n^{(2)} = k] (\mathbb{P}[S_{m-n}^{(1)} = \ell - k])^2 \\ &\leq \sum_{k \in \mathbb{Z}} P_\omega^{(y_1, \dots, y_r)}[Z_n^{(1)} = \dots = Z_n^{(r)} = k] P_\omega^{(k, \dots, k)}[Z_{m-n}^{(1)} = \dots = Z_{m-n}^{(r)}] \frac{C^4}{n(m-n)} \\ &\leq \frac{C^4}{n(m-n)}. \end{aligned}$$

Consequently, for large N ,

$$\sum_{1 \leq n, m \leq N, m \neq n} P_\omega^x[A_n \cap A_m] \leq 2 \sum_{n=1}^N \frac{C^4}{n} \sum_{\ell=1}^{N-n} \frac{1}{\ell} \leq 3C^4(\log N)^2.$$

Applying this and (19) we get for \mathbf{P} -almost every ω , for every initial condition x ,

$$\limsup_{N \rightarrow +\infty} \frac{\left(\sum_{n=1}^N P_\omega^x[A_n]\right)^2}{\sum_{1 \leq n, m \leq N} P_\omega^x[A_n \cap A_m]} \geq \frac{(c(\omega))^2}{3\pi^2 C^4}. \quad (21)$$

Due to the Kochen and Stone extension of the second Borel-Cantelli lemma (see Item (iii) of the main theorem of [KS64] applied with $X_n = \sum_{i=1}^n \mathbf{1}_{A_i}$, or [S76, p. 317]), (21) and (20) imply that $P_\omega^x[A_n \text{ i.o.}] > (c(\omega))^2/(3\pi^2 C^4) > 0$, where i.o. means infinitely often. Now for \mathbf{P} -almost every ω , due to a result by Doob (see for example Proposition V-2.4 in [N64]), since $E := \{A_n \text{ i.o.}\}$ is invariant (with respect to the shifts of the sequence (Y_0, Y_1, Y_2, \dots)), for every $x \in (2\mathbb{Z})^{2+r} \cup (2\mathbb{Z} + 1)^{2+r}$, $(P_\omega^{(S_n^{(1)}, S_n^{(2)}, Z_n^{(1)}, \dots, Z_n^{(r)})}[E])_n$ converges P_ω^x -almost surely to $\mathbf{1}_E$. But $\inf_{x \in (2\mathbb{Z})^{2+r} \cup (2\mathbb{Z} + 1)^{2+r}} P_\omega^x[E] \geq (c(\omega))^2/(3\pi^2 C^4) > 0$, so we conclude that $\mathbf{1}_E = 1$ P_ω^x -almost surely, thus $P_\omega^x(E) = 1$, for \mathbf{P} -almost every environment ω .

Proof of (iii). Assume $m = 2$ and $r = I = 2$. We have

$$\begin{aligned} \mathbb{P}^0[A_n] &= \sum_{k \in \mathbb{Z}} \mathbb{P}[Z_n^{(1)} = Z_n^{(2)} = k] (\mathbb{P}[S_n^{(1)} = k])^2 \\ &\leq \frac{C^2}{n} \mathbb{P}[Z_n^{(1)} = Z_n^{(2)}] = O(n^{-1}(\log n)^{-3/2}), \end{aligned}$$

due to Proposition 4.1 and the local limit theorem. Hence $\sum_n \mathbb{P}^0[A_n] < \infty$ and (iii) follows due to the Borel-Cantelli lemma. \blacksquare

So there only remains to prove Proposition 4.1 and Lemma 5.1.

4. PROBABILITY OF MEETING FOR TWO INDEPENDENT RECURRENT RWRE IN INDEPENDENT ENVIRONMENTS

Let $Z^{(1)}$ and $Z^{(2)}$ be two independent recurrent RWRE in independent environments $\omega^{(1)}$ and $\omega^{(2)}$ satisfying (4). The following lemma is a key result in the proof of case (iii) of Theorem 1.2.

Proposition 4.1. *For every $\varepsilon > 0$, $\mathbb{P}[Z_n^{(1)} = Z_n^{(2)}] = O((\log n)^{-2+\varepsilon})$.*

The main idea of the proof of this lemma is that $Z_n^{(1)}$ and $Z_n^{(2)}$ are localized with high (annealed) probability in two areas (depending on the environments, see Proposition 4.4) which have no common point with high probability (see Lemma 4.5). Due to [S82], we know that, with high probability, $Z_n^{(i)}$ is close to the bottom $B_n^{(i)}$ of some valley (containing 0 and of height larger than $\log n$) for the potential $V^{(i)}$. Here and in the following, $V^{(i)}$ denotes the potential corresponding to $\omega^{(i)}$, defined as in (5) with ω replaced by $\omega^{(i)}$. An intuitive idea to prove Proposition 4.1 should then be that $p_n := \mathbb{P}\left[\max_{i=1,2} |Z_n^{(i)} - B_n^{(i)}| \geq |B_n^{(1)} - B_n^{(2)}|/2\right]$ is very small. More precisely we would

like to prove that $p_n = O((\log n)^{-1-\varepsilon})$. (In view of the proof of (iii) above, it would suffice to show that $\sum_n \frac{p_n}{n} < \infty$). However, this seems difficult to prove and we are not even sure that it is true. Indeed, in view of Lemma 4.3 below (proved for $W^{(i)} \approx V^{(i)}$), we think that with probability $> 1/\log n$, 0 belongs to a valley of height between $\log n - 2 \log \log n$ and $\log n$ and that the annealed probability that $Z_n^{(i)}$ is close to the bottom of this valley (which is not $B_n^{(i)}$) should be greater than $1/\log n$. Hence, to prove Proposition 4.1, we will work with several valleys instead of a single one.

4.1. proof of Proposition 4.1. We first recall the definition of h -extrema introduced in [NP89]. If $w : \mathbb{R} \rightarrow \mathbb{R}$ is a continuous function and $h > 0$, we say that $y_0 \in \mathbb{R}$ is an h -*minimum* for w if there exist real numbers a and c such that $a < y_0 < c$, $w(y_0) = \inf_{[a,c]} w$, $w(a) \geq w(y_0) + h$ and $w(c) \geq w(y_0) + h$. We say that y_0 is an h -*maximum* for w if y_0 is an h -minimum for $-w$. In any of these two cases, we say that y_0 is an h -*extremum* for w .

As in [C05], we denote by \mathcal{W} the set of functions $w : \mathbb{R} \rightarrow \mathbb{R}$ such that the three following conditions are satisfied: **(a)** w is continuous on \mathbb{R} ; **(b)** for every $h > 0$, the set of h -extrema of w can be written $\{x_k(w, h), k \in \mathbb{Z}\}$, with $(x_k(w, h))_{k \in \mathbb{Z}}$ strictly increasing, unbounded from below and above, and with $x_0(w, h) \leq 0 < x_1(w, h)$, notation that we use in the rest of the paper on \mathcal{W} ; **(c)** for all $k \in \mathbb{Z}$ and $h > 0$, $x_k(w, h)$ is an h -minimum if and only if $x_{k+1}(w, h)$ is an h -maximum. We now introduce, for $w \in \mathcal{W}$, $i \in \mathbb{Z}$ and $h > 0$,

$$b_i(w, h) := \begin{cases} x_{2i}(w, h) & \text{if } x_0(w, h) \text{ is an } h\text{-minimum,} \\ x_{2i+1}(w, h) & \text{otherwise.} \end{cases}$$

As a consequence, the $b_i(w, h)$ are the h -minima of w . We denote by $M_i(w, h)$ the unique h -maximum of w between $b_i(w, h)$ and $b_{i+1}(w, h)$. That is, $M_i(w, h) = x_{j+1}(w, h)$ if $b_i(w, h) = x_j(w, h)$.

For $w \in \mathcal{W}$, $h > 0$ and $i \in \mathbb{Z}$, the restriction of $w - w(x_i(w, h))$ to $[x_i(w, h), x_{i+1}(w, h)]$ is denoted by $T_i(w, h)$ and is called an h -*slope*, as in [C05]. If $x_i(w, h)$ is an h -minimum (resp. h -maximum), then $T_i(w, h)$ is a nonnegative (resp. nonpositive) function, and its maximum (resp. minimum) is attained at $x_{i+1}(w, h)$. We also introduce, for each slope $T_i(w, h)$, its *height* $H(T_i(w, h)) := |w(x_{i+1}(w, h)) - w(x_i(w, h))| \geq h$, and its *excess height* $e(T_i(w, h)) := H(T_i(w, h)) - h \geq 0$.

These h -extrema are useful to localize RWRE and diffusions in a random potential. Indeed, a diffusion in a two-sided Brownian potential W (resp. in a $(-\kappa/2)$ -drifted Brownian potential W_κ with $0 < \kappa < 1$) is localized at large time t with high probability in a small neighborhood of $b_0(W, \log t)$ (resp. some of the $b_i(W_\kappa, \log t - \sqrt{\log t})$, $i \geq 0$) see e.g. [C05] and [C08] (resp. [AD15]). For some applications to recurrent RWRE, see e.g. [BF08] and [D14].

Let $C_1 > 2$ and $\alpha > 2$. Define $\log_2 x = \log \log x$ for $x > 1$. As in [D14], we use the Komlós-Major-Tusnády almost sure invariance principle [KMT75], which ensures that:

Lemma 4.2. *Up to an enlargement of $(\Omega, \mathcal{F}, \mathbb{P})$, there exist two independent two-sided Brownian motions $(W^{(i)}(s), s \in \mathbb{R})$ ($i \in \{1, 2\}$) with $\mathbb{E}[(W^{(i)}(1))^2] = \mathbb{E}[(V^{(i)}(1))^2] = \sigma_i^2$*

and a real number $\tilde{C}_1 > 0$ such that for all n large enough,

$$\mathbb{P} \left[\sup_{|t| \leq (\log n)^\alpha} \left| V^{(i)}(\lfloor t \rfloor) - W^{(i)}(t) \right| > \tilde{C}_1 \log_2 n \right] \leq (\log n)^{-C_1}, \quad i \in \{1, 2\}.$$

Proof. Notice that $V^{(1)}$ and $V^{(2)}$ are independent, since $\omega^{(1)}$ and $\omega^{(2)}$ are independent. Due to ([KMT75], Thm. 1), there exist positive constants a , b and c such that for $N \in \mathbb{N}$, up to an enlargement of $(\Omega, \mathcal{F}, \mathbb{P})$, there exist two independent two-sided Brownian motions $(W^{(i)}(s), s \in \mathbb{R})$ ($i \in \{1, 2\}$) on $(\Omega, \mathcal{F}, \mathbb{P})$ with $\mathbb{E}[(W^{(i)}(1))^2] = \mathbb{E}[(V^{(i)}(1))^2] = \sigma_i^2$ such that

$$\forall x \in \mathbb{R}, \forall i \in \{1, 2\}, \quad \mathbb{P} \left[\sup_{|k| \leq N} \left| V^{(i)}(k) - W^{(i)}(k) \right| > a \log N + x \right] \leq b e^{-cx}. \quad (22)$$

Applying this result to $N := \lfloor (\log n)^\alpha \rfloor + 1$ and $x := (\log(2b) + C_1 \log_2 n)/c$ and taking $\tilde{C}_1 > 2(a\alpha + \frac{C_1}{c})$, we obtain that

$$\mathbb{P} \left[\sup_{|k| \leq \lfloor (\log n)^\alpha \rfloor + 1} \left| V^{(i)}(k) - W^{(i)}(k) \right| > \frac{\tilde{C}_1}{2} \log_2 n \right] \leq \frac{1}{2} (\log n)^{-C_1}, \quad (23)$$

for all n large enough. Moreover, for every n large enough,

$$\begin{aligned} & \mathbb{P} \left[\sup_{|t| \leq (\log n)^\alpha} \left| W^{(i)}(\lfloor t \rfloor) - W^{(i)}(t) \right| > \frac{\tilde{C}_1}{2} \log_2 n \right] \\ & \leq 3(\log n)^\alpha \mathbb{P} \left[\sup_{0 \leq t < 1} \left| W^{(i)}(t) \right| > \frac{\tilde{C}_1}{2} \log_2 n \right] \leq 6(\log n)^\alpha \mathbb{P} \left[\left| W^{(i)}(1) \right| > \frac{\tilde{C}_1}{2} \log_2 n \right] \\ & \leq 6(\log n)^\alpha \frac{2}{\sqrt{2\pi}} e^{-\frac{(\tilde{C}_1)^2}{8\sigma_i^2} (\log_2 n)^2} = \frac{12}{\sqrt{2\pi}} (\log n)^{\alpha - \frac{(\tilde{C}_1)^2}{8\sigma_i^2} \log_2 n} \leq \frac{1}{2} (\log n)^{-C_1}, \end{aligned}$$

since $\sup_{[0,1]} W^{(i)} =_{law} |W^{(i)}(1)|$. This combined with (23) proves the lemma. \blacksquare

In the remaining of the paper, we use the $W^{(i)}$ introduced in Lemma 4.2. We will use the valleys for the $W^{(i)}$. Fix some $C_2 \geq 2\alpha + 2 + 10\tilde{C}_1$. Let $h_n := \log n - 5C_2 \log_2 n$. We know from ([C05], Lemma 8) that $W^{(i)} \in \mathcal{W}$ almost surely. Moreover, using [HS98, Th 2.1] with $0 < a = b$, we have $\mathbb{P}[\sup_{0 \leq s \leq t} |W^{(i)}(s) - \underline{W}^{(i)}(s)| < b] \leq (4/\pi) \exp[-\pi^2 \sigma_i^2 t / (8b^2)]$, where $\underline{W}^{(i)}(s) := \inf_{[0,s]} W^{(i)}$. Applying this several times to $W^{(i)}$ and $-W^{(i)}$ with $t = (\log n)^\alpha / 10$ and $b = h_n$, the following holds with a probability $1 - o((\log n)^{-2})$ (since $\alpha > 2$),

$$\forall i \in \{1, 2\}, \quad -(\log n)^\alpha \leq b_{-4}(W^{(i)}, h_n) \leq M_3(W^{(i)}, h_n) \leq (\log n)^\alpha. \quad (24)$$

The following lemma shows that Proposition 4.1 is more subtle than it may seem at first sight.

Lemma 4.3. *Let W be a two-sided standard Brownian motion and $\sigma > 0$. Then, for every n large enough,*

$$\mathbb{P}[H(T_0(\sigma W, h_n)) \leq \log n] \geq C_2 (\log_2 n) (\log n)^{-1}, \quad (25)$$

$$\mathbb{P}[\#\{j \in \{-5, \dots, 5\}, H(T_j(\sigma W, h_n - 2C_2 \log_2 n)) \leq \log n + C_2 \log_2 n\} \geq 2] \quad (26)$$

$$\begin{aligned}
&= O((\log_2 n)^2(\log n)^{-2}), \\
&\quad \mathbb{P}[\exists j \in \{-5, \dots, 5\}, H(T_j(\sigma W, h_n - 2C_2 \log_2 n)) \leq \log n + C_2 \log_2 n] \\
&= O((\log_2 n)(\log n)^{-1}).
\end{aligned} \tag{27}$$

In particular, the probability that the height of the central valley for $W^{(i)}$ is less than $\log n$ is not negligible. However, with large enough probability, all the valleys close to 0 except maybe one are large, with height greater than $\log n + C_2 \log_2 n$.

Proof of Lemma 4.3. Let $\tilde{h}_n := h_n - 2C_2 \log_2 n$. First, due to ([NP89], Prop. 1, see also [C05] eq. (8)), $e(T_i(\sigma W, \tilde{h}_n))/\tilde{h}_n$ is for $i \neq 0$ an exponential random variable with mean 1. Consequently, for $i \neq 0$ and large n ,

$$\mathbb{P}[H(T_i(\sigma W, \tilde{h}_n)) \leq \log n + C_2 \log_2 n] = \mathbb{P}[e(T_i(\sigma W, \tilde{h}_n)) \leq 8C_2 \log_2 n] \leq \frac{9C_2 \log_2 n}{\log n}.$$

Observe that $e(T_0(\sigma W, \tilde{h}_n))/\tilde{h}_n$ is by scaling equal in law to $e(T_0(W, 1))$, which has a density equal to $(2x + 1)e^{-x}\mathbf{1}_{(0, \infty)}(x)/3$ due to ([C05], formula (11)). Hence for large n ,

$$\begin{aligned}
C_2(\log_2 n)(\log n)^{-1} &\leq \mathbb{P}[e(T_0(\sigma W, \tilde{h}_n)) \leq 5C_2 \log_2 n] \\
&\leq \mathbb{P}[e(T_0(\sigma W, \tilde{h}_n)) \leq 8C_2 \log_2 n] \leq 9C_2(\log_2 n)(\log n)^{-1}.
\end{aligned}$$

This remains true if \tilde{h}_n is replaced by h_n . These inequalities already prove (25) and (27). Moreover, due to ([NP89], prop. 1), the slopes $T_i(\sigma W, \tilde{h}_n)$, $i \in \mathbb{Z}$ are independent, so the random variables $H(T_i(\sigma W, \tilde{h}_n))$, $i \in \mathbb{Z}$ are independent. This and the previous inequalities lead to (26). \blacksquare

Because of (25), it seems reasonable to consider strictly more than one valley of height at least h_n if we want to localize a recurrent RWRE with probability $\geq 1 - (\log n)^{-(1+\varepsilon)}$.

We first introduce some notation. Let, for $i \in \{1, 2\}$, $j \in \mathbb{Z}$ and $n \geq 3$,

$$\begin{aligned}
&\Xi_{n,j}(W^{(i)}) \\
&:= \{x \in [M_{j-1}(W^{(i)}, h_n), M_j(W^{(i)}, h_n)], W^{(i)}(x) \leq W^{(i)}(b_j(W^{(i)}, h_n)) + C_2 \log_2 n\}.
\end{aligned}$$

Loosely speaking, $\Xi_{n,j}(W^{(i)})$ is the set of points with low potential in the j -th valley for $W^{(i)}$. We also define

$$\Xi_n(W^{(i)}) := \bigcup_{j=-2}^2 \Xi_{n,j}(W^{(i)}).$$

Proposition 4.4. *Let $\varepsilon > 0$ and $i \in \{1, 2\}$. For all n large enough, we have*

$$\mathbb{P}[Z_n^{(i)} \notin \Xi_n(W^{(i)})] \leq q_n := (\log n)^{-2+\varepsilon}.$$

In Proposition 4.4, we localize the RWRE $Z^{(i)}$ in a set of points which are close to the $b_j(\cdot)$ "vertically", instead of "horizontally" as in Sinai's theorem (see [S82]).

Proposition 4.1 is then an easy consequence of Proposition 4.4 and of the following estimate on the environments.

Lemma 4.5. *Let $\varepsilon > 0$. For large n ,*

$$\mathbb{P}[\Xi_n(W^{(1)}) \cap \Xi_n(W^{(2)}) \neq \emptyset] \leq (\log n)^{-2+\varepsilon}.$$

Proof of Lemma 4.5. First, let $k \in \Xi_n(W^{(i)})$ for some $i \in \{1, 2\}$ and $n \geq 3$. Hence $k \in [M_{j-1}(W^{(i)}, h_n), M_j(W^{(i)}, h_n)]$ and $W^{(i)}(k) \leq W^{(i)}(b_j(W^{(i)}, h_n)) + C_2 \log_2 n$ for some $j \in \{-2, -1, 0, 1, 2\}$. By definition of h_n -minima, we notice that the two Brownian motions $(W^{(i)}(x+k) - W^{(i)}(k), x \geq 0)$ and $(W^{(i)}(-x+k) - W^{(i)}(k), x \geq 0)$ hit $h_n - C_2 \log_2 n$ before $-2C_2 \log_2 n$. By independence, it follows that, for n large enough, for every $k \in \mathbb{Z}$ and $i \in \{1, 2\}$,

$$\begin{aligned} \mathbb{P}[k \in \Xi_n(W^{(i)})] &\leq \mathbb{P}[T_{(i)}^+(h_n - C_2 \log_2 n) < T_{(i)}^+(-2C_2 \log_2 n)] \\ &\quad \times \mathbb{P}[|T_{(i)}^-(h_n - C_2 \log_2 n)| < |T_{(i)}^-(-2C_2 \log_2 n)|] \\ &\leq O((\log_2 n) / \log n)^2, \end{aligned}$$

where $T_{(i)}^+(z) := \inf\{x > 0 : W^{(i)}(x) = z\}$ and $T_{(i)}^-(z) := \sup\{x < 0 : W^{(i)}(x) = z\}$. Consequently, since $W^{(1)}$ and $W^{(2)}$ are independent, we have uniformly on $k \in \mathbb{Z}$,

$$\begin{aligned} \mathbb{P}[k \in \Xi_n(W^{(1)}) \cap \Xi_n(W^{(2)})] &= \mathbb{P}[k \in \Xi_n(W^{(1)})] \mathbb{P}[k \in \Xi_n(W^{(2)})] \\ &\leq O((\log_2 n)^4 / (\log n)^4). \end{aligned} \quad (28)$$

Finally, (24) applied with $2 + \varepsilon > 2$ instead of α and (28) lead to

$$\begin{aligned} &\mathbb{P}[\Xi_n(W^{(1)}) \cap \Xi_n(W^{(2)}) \neq \emptyset] \\ &\leq o((\log n)^{-2}) + \sum_{k=-\lfloor (\log n)^{2+\varepsilon} \rfloor}^{\lfloor (\log n)^{2+\varepsilon} \rfloor} \mathbb{P}[k \in \Xi_n(W^{(1)}) \cap \Xi_n(W^{(2)})] \\ &\leq O((\log_2 n)^4 (\log n)^{-2+\varepsilon}) \leq (\log n)^{-2+2\varepsilon}, \end{aligned}$$

for every n large enough. Since this is true for every $\varepsilon > 0$, this proves the lemma. \blacksquare

Proof of Proposition 4.1. We have for large n , due to Proposition 4.4,

$$\begin{aligned} \mathbb{P}[Z_n^{(1)} = Z_n^{(2)}] &\leq \mathbb{P}[Z_n^{(1)} = Z_n^{(2)}, Z_n^{(1)} \in \Xi_n(W^{(1)}), Z_n^{(2)} \in \Xi_n(W^{(2)})] \\ &\quad + \mathbb{P}[Z_n^{(1)} \notin \Xi_n(W^{(1)})] + \mathbb{P}[Z_n^{(2)} \notin \Xi_n(W^{(2)})] \\ &\leq \mathbb{P}[\Xi_n(W^{(1)}) \cap \Xi_n(W^{(2)}) \neq \emptyset] + 2q_n. \end{aligned}$$

This and Lemma 4.5 prove Proposition 4.1. \blacksquare

4.2. Proof of Proposition 4.4. We fix $\varepsilon > 0$ and $i \in \{1, 2\}$. To simplify notations we write V for $V^{(i)}$, Z_n for $Z_n^{(i)}$, W for $W^{(i)}$. We also introduce for $j \in \mathbb{Z}$,

$$x_j := \lfloor x_j(W, h_n) \rfloor, \quad b_j := \lfloor b_j(W, h_n) \rfloor, \quad M_j := \lfloor M_j(W, h_n) \rfloor.$$

We denote by \mathcal{G}_n the set of **good environments** ω satisfying (24) together with the following properties (see Figure 1):

$$\sup_{|t| \leq (\log n)^\alpha} |V(\lfloor t \rfloor) - W(t)| \leq \tilde{C}_1 \log_2 n, \quad (29)$$

$$\#\{j \in \{-5, \dots, 5\}, H(T_j(W, h_n - 2C_2 \log_2 n)) \leq \log n + C_2 \log_2 n\} \leq 1. \quad (30)$$

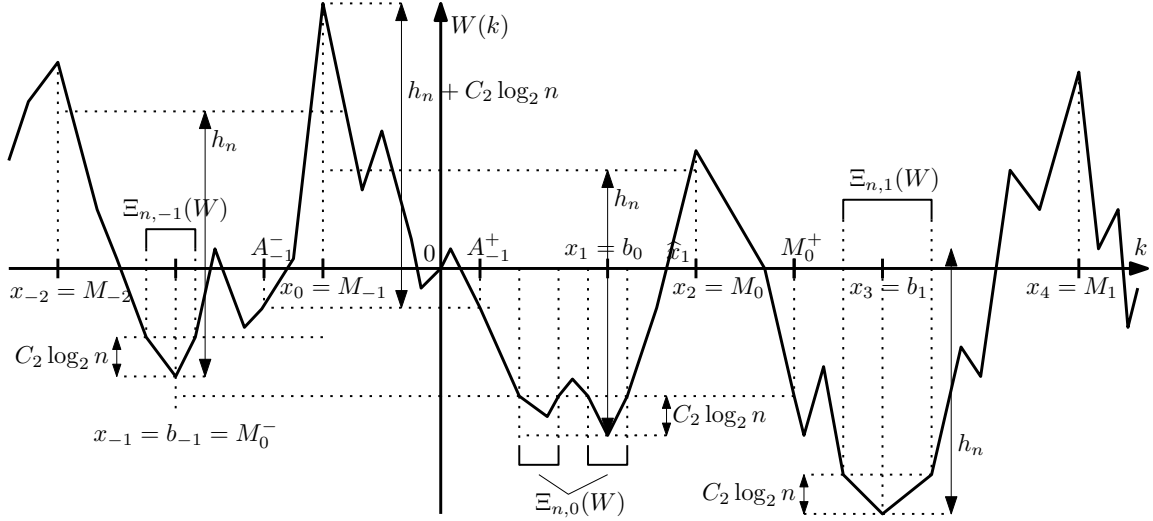


Figure 1. Pattern of W for a good environment $\omega \in \mathcal{G}_n$ and representation of different quantities.

For every n large enough, we have

$$\mathbb{P}[(\mathcal{G}_n)^c] \leq (\log n)^{-2+\varepsilon}, \quad (31)$$

due to (24) and to Lemmas 4.2 and 4.3, since $C_1 > 2$.

We now consider $\omega \in \mathcal{G}_n^+$ where $\mathcal{G}_n^+ := \mathcal{G}_n \cap \{x_1(W, h_n) \text{ is an } h_n\text{-minimum}\}$, that is,

$$b_{-1}(W, h_n) = x_{-1}(W, h_n) < x_0(W, h_n) = M_{-1}(W, h_n) \leq 0 < b_0(W, h_n) = x_1(W, h_n).$$

Indeed, the other case, that is, $x_0(W, h_n)$ is an h_n -minimum, or equivalently $\omega \in \mathcal{G}_n^-$ with $\mathcal{G}_n^- := \mathcal{G}_n \setminus \mathcal{G}_n^+$, is similar by symmetry.

Proof of Proposition 4.4. Let us see how we can derive Proposition 4.4 from (31) and from Lemmas 4.6, 4.8 and 4.10 below. Applying Lemma 4.6 with $y = 0$ and $j = -1$ on \mathcal{G}_n^+ , the random walk Z goes quickly to b_{-1} or b_0 with high probability. More precisely, setting $E_1 := \{\tau(b_{-1}) \wedge \tau(b_0) \leq n(\log n)^{-3C_2}\}$, there exists $\tilde{n}_0 \in \mathbb{N}$ such that, for every $n \geq \tilde{n}_0$,

$$\forall \omega \in \mathcal{G}_n^+, \quad P_\omega(E_1) \geq 1 - (\log n)^{-2}. \quad (32)$$

Due to Lemmas 4.8 and 4.10, there exists $\tilde{n}_1 \in \mathbb{N}$ such that, for every $n \geq \tilde{n}_1$,

$$\forall \omega \in \mathcal{G}_n^+, \quad P_\omega[E_1, Z_n \notin \Xi_n(W)] \leq 11(\log n)^{-2} \quad (33)$$

and so

$$\forall n \geq \max(\tilde{n}_0, \tilde{n}_1), \quad \forall \omega \in \mathcal{G}_n^+, \quad P_\omega[Z_n \notin \Xi_n(W)] \leq 12(\log n)^{-2}.$$

By symmetry, this remains true with \mathcal{G}_n^+ replaced by \mathcal{G}_n^- . Therefore, due to (31), for every n large enough,

$$\mathbb{P}[Z_n \notin \Xi_n(W)] \leq \int_{\mathcal{G}_n} P_\omega[Z_n \notin \Xi_n(W)] \mathbb{P}(d\omega) + \mathbb{P}[(\mathcal{G}_n)^c] \leq 2(\log n)^{-2+\varepsilon}.$$

Since this is true for every $\varepsilon > 0$, this proves Proposition 4.4. ■

We will use the following property. For $j \in \mathbb{Z}$, let

$$\begin{aligned}\widehat{\mu}_j(x) &:= \exp[-(V(x) - V(b_j))] + \exp[-(V(x-1) - V(b_j))] \\ &= \exp[V(b_j)]\mu_\omega(x), \quad x \in \mathbb{Z},\end{aligned}$$

with reversible measure μ_ω defined in (14). It follows from reversibility that

$$\forall k \in \mathbb{N}, \forall x \in \mathbb{Z}, \forall b \in \mathbb{Z}, \quad P_\omega^b[Z_k = x] = \frac{\mu_\omega(x)}{\mu_\omega(b)} P_\omega^x[Z_k = b] \leq \frac{\mu_\omega(x)}{\mu_\omega(b)} \leq \exp[V(b)]\mu_\omega(x). \quad (34)$$

In particular,

$$\forall j \in \mathbb{Z}, \forall k \in \mathbb{N}, \forall x \in \mathbb{Z}, \quad P_\omega^{b_j}[Z_k = x] \leq \widehat{\mu}_j(x). \quad (35)$$

Lemma 4.6. *There exists $n_0 \in \mathbb{N}$ such that, for every $n \geq n_0$, every $\omega \in \mathcal{G}_n$, every $j \in \{-2, \dots, 1\}$ and every integer $y \in]b_j, b_{j+1}[$,*

$$P_\omega^y[\tau(b_j) \wedge \tau(b_{j+1}) > n(\log n)^{-3C_2}] \leq (\log n)^{-2}. \quad (36)$$

Proof. Let $j \in \{-2, \dots, 1\}$ and $\omega \in \mathcal{G}_n$. Assume for example that $y \in [M_j, b_{j+1}[$, the proof being symmetric in the case when $y \in]b_j, M_j]$. We set (see Figure 1 for $j = -1$)

$$\begin{aligned}A_j^+ &:= \min(b_{j+1}, \inf\{k \geq M_j : W(k) \leq W(M_j(W, h_n)) - h_n - C_2 \log_2 n\}), \\ A_j^- &:= \max(b_j, \sup\{k \leq M_j : W(k) \leq W(M_j(W, h_n)) - h_n - C_2 \log_2 n\}).\end{aligned}$$

1. If $y \in [M_j, A_j^+[$, due to (7), (24) and (29), applying Markov's inequality, we get

$$\begin{aligned}P_\omega^y \left[\tau(A_j^-) \wedge \tau(A_j^+) > \frac{e^{h_n + 2C_2 \log_2 n}}{2} \right] &\leq \frac{2\varepsilon_0^{-1} [b_{j+1} - b_j]^2}{e^{h_n + 2C_2 \log_2 n}} \exp \left[\max_{[b_j, b_{j+1}]} V - \min_{[A_j^-, A_j^+]} V \right] \\ &\leq \frac{2\varepsilon_0^{-1} [M_2 - b_{-2}]^2}{e^{h_n + 2C_2 \log_2 n}} e^{(W(M_j(W, h_n)) + \tilde{C}_1 \log_2 n) - (W(M_j(W, h_n)) - h_n - C_2 \log_2 n - \tilde{C}_1 \log_2 n - \log \varepsilon_0^{-1})} \\ &\leq \frac{8\varepsilon_0^{-1} (\log n)^{2\alpha} e^{h_n + (C_2 + 2\tilde{C}_1) \log_2 n + \log \varepsilon_0^{-1}}}{e^{h_n + 2C_2 \log_2 n}} \leq 8\varepsilon_0^{-2} (\log n)^{2\alpha + 2\tilde{C}_1 - C_2} \leq \frac{1}{3} (\log n)^{-2}\end{aligned}$$

for every n large enough, where we used $\sup_{[b_j(W, h_n), b_{j+1}(W, h_n)]} W = W(M_j(W, h_n))$ and $V(A_j^\pm) \geq V(A_j^\pm \mp 1) - \log \frac{1-\varepsilon_0}{\varepsilon_0}$ in the second line and $C_2 > 2\alpha + 2\tilde{C}_1 + 2$ in the last one. Hence by the strong Markov property, for n large enough, for every $y \in [M_j, A_j^+[$,

$$\begin{aligned}P_\omega^y [\tau(b_j) \wedge \tau(b_{j+1}) > e^{h_n + 2C_2 \log_2 n}] \\ \leq \frac{1}{3} (\log n)^{-2} + P_\omega^{A_j^-} [\tau(b_j) > e^{h_n + 2C_2 \log_2 n} / 2] + P_\omega^{A_j^+} [\tau(b_{j+1}) > e^{h_n + 2C_2 \log_2 n} / 2]. \quad (37)\end{aligned}$$

2. Assume now that $y \in [A_j^+, b_{j+1}[$ (and so $A_j^+ < b_{j+1}$). Observe that W admits no h_n -maximum in the interval $]M_j(W, h_n), b_{j+1}(W, h_n)]$ by definition of $M_j(\cdot)$, so

$$\max_{M_j(W, h_n) \leq u \leq v \leq b_{j+1}(W, h_n)} (W(v) - W(u)) < h_n.$$

Hence due to (7), (24), (29), and to Markov's inequality, we have

$$P_\omega^y [\tau(M_j) \wedge \tau(b_{j+1}) > e^{h_n + 2C_2 \log_2 n} / 2] \leq \frac{2[M_2 - b_{-2}]^2}{\varepsilon_0 e^{h_n + 2C_2 \log_2 n}} \exp \left[\max_{M_j \leq \ell \leq k \leq b_{j+1}} (V(k) - V(\ell)) \right]$$

$$\leq \frac{8\varepsilon_0^{-1}(\log n)^{2\alpha} e^{h_n+2\tilde{C}_1 \log_2 n}}{e^{h_n+2C_2 \log_2 n}} \leq \frac{1}{6}(\log n)^{-2} \quad (38)$$

for every n large enough, since $2C_2 > 2\alpha + 2\tilde{C}_1 + 2$. Moreover, due to (6), (24) and (29), and since there is no h_n -maximum in $[A_j^+, b_{j+1}]$ and so $\sup_{[A_j^+, b_{j+1}]} W < W(A_j^+) + h_n$,

$$\begin{aligned} P_\omega^y [\tau(M_j) < \tau(b_{j+1})] &\leq \left(\sum_{\ell=A_j^+}^{b_{j+1}-1} e^{V(\ell)} \right) \left(\sum_{\ell=M_j}^{b_{j+1}-1} e^{V(\ell)} \right)^{-1} \\ &\leq [b_{j+1} - A_j^+] \exp \left(\max_{\ell \in \{A_j^+, \dots, b_{j+1}\}} V(\ell) \right) \exp(-V(M_j)) \\ &\leq 2(\log n)^\alpha \exp[W(A_j^+) + h_n - W(M_j(W, h_n)) + 2\tilde{C}_1 \log_2 n] \\ &\leq 2(\log n)^{\alpha-C_2+2\tilde{C}_1} \leq \frac{1}{6}(\log n)^{-2} \end{aligned} \quad (39)$$

for every n large enough, since $C_2 > \alpha + 2\tilde{C}_1 + 2$. Gathering (38) and (39), we get, for every n large enough, for every $y \in [A_j^+, b_{j+1}[$, uniformly on \mathcal{G}_n as the previous inequalities,

$$P_\omega^y [\tau(b_{j+1}) > n(\log n)^{-3C_2}/2] = P_\omega^y [\tau(b_{j+1}) > e^{h_n+2C_2 \log_2 n}/2] \leq \frac{1}{3}(\log n)^{-2}. \quad (40)$$

This already proves (36) for $y \in [A_j^+, b_{j+1}[$.

3. For symmetry reasons, we also get that, for every n large enough, for every $y \in]b_j, A_j^-]$,

$$P_\omega^y [\tau(b_j) > e^{h_n+2C_2 \log_2 n}/2] \leq \frac{1}{3}(\log n)^{-2}. \quad (41)$$

Finally, combining (37) with (40) and (41) proves (36) for $y \in [M_j, A_j^+[$. \blacksquare

We consider $\mathcal{I}_1 \in \{-1, 0\}$ such that $\tau(b_{\mathcal{I}_1}) = \tau(b_{-1}) \wedge \tau(b_0)$. Applying Lemma 4.6 with $y = 0$ and $j = -1$, on an event $E_1 = E_1(n)$ such that

$$\forall \omega \in \mathcal{G}_n^+, \quad P_\omega(E_1) \geq 1 - (\log n)^{-2}, \quad (42)$$

we have $\tau(b_{\mathcal{I}_1}) = \tau(b_{-1}) \wedge \tau(b_0) \leq \alpha_n := n(\log n)^{-3C_2}$. We consider the event $E_2 = E_2(n)$ on which Z first goes to the bottom of a "large" valley:

$$\begin{aligned} E_2^+(j) &:= \{W[M_j(W, h_n)] - W[b_j(W, h_n)] > \log n + C_2 \log_2 n\}, \\ E_2^-(j) &:= \{W[M_{j-1}(W, h_n)] - W[b_j(W, h_n)] > \log n + C_2 \log_2 n\}, \\ E_2 &:= E_2^+(\mathcal{I}_1) \cap E_2^-(\mathcal{I}_1). \end{aligned}$$

Notice that this event depends on ω but also on the first steps of Z up to time $\tau(b_{\mathcal{I}_1})$. Similarly as in (27), this event happens with probability $1 - O((\log n)^{-1} \log_2 n)$, so we cannot neglect E_2^c . We will treat separately the two events E_2 and E_2^c (the study of E_2^c being more complicated). Before considering these two events, we state the following useful result. We introduce for $j \in \mathbb{Z}$,

$$M_j^+ := b_{j+1} \wedge \inf\{k \geq M_j, W(k) \leq W(b_j(W, h_n)) + C_2 \log_2 n\}, \quad (43)$$

$$M_j^- := b_{j-1} \vee \sup\{k \leq M_{j-1}, W(k) \leq W(b_j(W, h_n)) + C_2 \log_2 n\}, \quad (44)$$

where $u \vee v := \max(u, v)$, $(u, v) \in \mathbb{R}^2$, so that

$$\forall k \in]M_j^-, M_{j-1}^-] \cup [M_j^+, M_j^+[, \quad W(k) > W(b_j(W, h_n)) + C_2 \log_2 n. \quad (45)$$

Lemma 4.7. *For every n large enough,*

$$\forall \omega \in \mathcal{G}_n, \forall j \in \{-2, \dots, 2\}, \quad \sup_{k \geq 0} P_\omega^{b_j}(Z_k \in [M_j^-, M_j^+] \setminus \Xi_n(W)) \leq (\log n)^{-2}. \quad (46)$$

Proof. Notice that thanks to (29) and (24), $V(x) \geq W(b_j(W, h_n)) + C_2 \log_2 n - \tilde{C}_1 \log_2 n - \log \varepsilon_0^{-1}$ for every integer $x \in ([M_j^-, M_j^+] \setminus \Xi_{n,j}(W))$ for $j \in \{-2, \dots, 2\}$, thanks to the definition of $\Xi_{n,j}(W)$ if $x \in [M_{j-1}, M_j]$ and (45) and $|V(y) - V(y-1)| \leq \log \frac{1-\varepsilon_0}{\varepsilon_0}$ otherwise. So, due to (35), (24) and (29), for large n , for all $\omega \in \mathcal{G}_n$ and $j \in \{-2, \dots, 2\}$,

$$\begin{aligned} & \sup_{k \geq 0} P_\omega^{b_j}(Z_k \in [M_j^-, M_j^+] \setminus \Xi_n(W)) \\ & \leq \sum_{x=M_j^-}^{M_j^+} \mathbf{1}_{\Xi_n(W)^c}(x) \hat{\mu}_j(x) \leq \sum_{x=M_j^-}^{M_j^+} \mathbf{1}_{\Xi_{n,j}(W)^c}(x) e^{V(b_j)} [e^{-V(x)} + e^{-V(x) + \log \frac{1-\varepsilon_0}{\varepsilon_0}}] \\ & \leq 2(\log n)^\alpha \varepsilon_0^{-1} e^{(W(b_j(W, h_n)) + \tilde{C}_1 \log_2 n) - (W(b_j(W, h_n)) + (C_2 - \tilde{C}_1) \log_2 n - \log \varepsilon_0^{-1})} \\ & = 2\varepsilon_0^{-2} (\log n)^{\alpha + 2\tilde{C}_1 - C_2} \leq (\log n)^{-2}, \end{aligned}$$

since $C_2 \geq 2\alpha + 2 + 10\tilde{C}_1$. ■

Lemma 4.8 (Simplest case). *There exists $n_1 \in \mathbb{N}$ such that for all $n \geq n_1$,*

$$\forall \omega \in \mathcal{G}_n^+, \quad P_\omega(E_1, E_2, Z_n \notin \Xi_n(\omega)) \leq 3(\log n)^{-2}.$$

Proof. Due to (9), (24) and (29), we have for large n , for all $\omega \in \mathcal{G}_n^+$ and all $j \in \{-2, \dots, 2\}$ uniformly on $E_2^+(j)$,

$$\begin{aligned} P_\omega^{b_j}[\tau(M_j) < n] & \leq n e^{\min[b_j, M_{j-1}]V - V(M_{j-1})} \leq n e^{V(b_j) - V(M_{j-1})} \\ & \leq n \exp[(W(b_j(W, h_n)) + \tilde{C}_1 \log_2 n) - (W(M_j(W, h_n)) - \tilde{C}_1 \log_2 n - \log \varepsilon_0^{-1})] \\ & \leq n e^{-(\log n + C_2 \log_2 n) + 2\tilde{C}_1 \log_2 n + \log \varepsilon_0^{-1}} = \varepsilon_0^{-1} (\log n)^{2\tilde{C}_1 - C_2} \leq (\log n)^{-2}, \quad (47) \end{aligned}$$

since $C_2 \geq 2\alpha + 2 + 10\tilde{C}_1$. Similarly, using (10) instead of (9), we have for large n , for all $\omega \in \mathcal{G}_n^+$ and all $j \in \{-2, \dots, 2\}$, uniformly on $E_2^-(j)$,

$$P_\omega^{b_j}[\tau(M_{j-1}) < n] \leq (\log n)^{-2}. \quad (48)$$

Let

$$\tau(x, y) := \inf\{k \geq 0, Z_{\tau(x)+k} = y\}, \quad x \in \mathbb{Z}, y \in \mathbb{Z}.$$

In particular, on $E_1 \cap E_2 \cap \{\tau(b_{\mathcal{I}_1}, M_{\mathcal{I}_1-1}) \geq n\} \cap \{\tau(b_{\mathcal{I}_1}, M_{\mathcal{I}_1}) \geq n\}$,

$$\tau(b_{\mathcal{I}_1}) \leq \alpha_n \leq n \leq \tau(b_{\mathcal{I}_1}) + \tau(b_{\mathcal{I}_1}, M_{\mathcal{I}_1-1}) \wedge \tau(b_{\mathcal{I}_1}, M_{\mathcal{I}_1}),$$

and so $Z_n \in [M_{\mathcal{I}_1-1}, M_{\mathcal{I}_1}] \subset [M_{\mathcal{I}_1}^-, M_{\mathcal{I}_1}^+]$. Applying (47) and (48) combined with the strong Markov property at time $\tau(b_{\mathcal{I}_1})$, and then (46), we get for large n , for every $\omega \in \mathcal{G}_n^+$,

$$\begin{aligned} & P_\omega[E_1, E_2, Z_n \notin \Xi_n(W)] \\ & \leq P_\omega[E_1, E_2, Z_n \notin \Xi_n(W), \tau(b_{\mathcal{I}_1}, M_{\mathcal{I}_1-1}) \geq n, \tau(b_{\mathcal{I}_1}, M_{\mathcal{I}_1}) \geq n] \\ & \quad + P_\omega[E_2, \tau(b_{\mathcal{I}_1}, M_{\mathcal{I}_1-1}) < n] + P_\omega[E_2, \tau(b_{\mathcal{I}_1}, M_{\mathcal{I}_1}) < n] \end{aligned}$$

$$\begin{aligned}
&\leq E_\omega [1_{E_1} P_\omega^{b_{\mathcal{I}_1}} (Z_{n-k} \in [M_{\mathcal{I}_1-1}, M_{\mathcal{I}_1}] \setminus \Xi_n(W))_{|k=\tau(b_{\mathcal{I}_1})}] + 2(\log n)^{-2} \\
&\leq 3(\log n)^{-2}.
\end{aligned} \tag{49}$$

This proves the lemma. \blacksquare

For the event E_2^c , we will use the following lemma, which is actually true for any Markov chain.

Lemma 4.9. *Let $a \neq b$. We have,*

$$\forall k \in \mathbb{N}, \quad P_\omega^b[\tau(a) = k] \leq P_\omega^b[\tau(a) < \tau(b)].$$

Proof. Let $k \in \mathbb{N}^*$. We have, by the Markov property,

$$\begin{aligned}
P_\omega^b[\tau(a) = k] &= \sum_{n=0}^k P_\omega^b[\tau(a) = k, Z_n = b, \forall n < \ell \leq k, Z_\ell \neq b] \\
&\leq \sum_{n=0}^k P_\omega^b[Z_n = b] P_\omega^b[\tau(a) = k - n, \tau(a) < \tau(b)] \\
&\leq P_\omega^b[\tau(a) \in [0, k], \tau(a) < \tau(b)] \\
&\leq P_\omega^b[\tau(a) < \tau(b)],
\end{aligned}$$

where we used $P_\omega^b[Z_n = b] \leq 1$ in the second inequality. \blacksquare

Lemma 4.10 (Most difficult case). *There exists $n'_1 \in \mathbb{N}$ such that for all $n \geq n'_1$,*

$$\forall \omega \in \mathcal{G}_n^+, \quad P_\omega(E_1, E_2^c, Z_n \notin \Xi_n(\omega)) \leq 8(\log n)^{-2}.$$

Proof. An essential remark is that if we are on E_2^c with $\omega \in \mathcal{G}_n^+$, then, due to (30), either we are on $E_2^-(\mathcal{I}_1) \setminus E_2^+(\mathcal{I}_1)$ or on $E_2^+(\mathcal{I}_1) \setminus E_2^-(\mathcal{I}_1)$. In the first case we set

$$\mathcal{I}_2 := \mathcal{I}_1 + 1, \quad A := M_{\mathcal{I}_1}^+, \quad B := M_{\mathcal{I}_1}(W, h_n) \text{ and } D := M_{\mathcal{I}_1}^-.$$

whereas in the second case we set

$$\mathcal{I}_2 := \mathcal{I}_1 - 1, \quad A := M_{\mathcal{I}_1}^-, \quad B := M_{\mathcal{I}_1-1}(W, h_n) \text{ and } D := M_{\mathcal{I}_1}^+.$$

Loosely speaking, with large probability, $b_{\mathcal{I}_2}$ is the bottom of the second valley reached by Z , and Z can reach it before time n or not, so we have to consider both cases.

We introduce $\tau'(A, b_{\mathcal{I}_2}) := \inf\{k \geq 0, Z_{\tau(b_{\mathcal{I}_1})+\tau(b_{\mathcal{I}_1}, A)+k} = b_{\mathcal{I}_2}\}$ and

$$\begin{aligned}
E_3 &:= \{\tau(b_{\mathcal{I}_1}) + \tau(b_{\mathcal{I}_1}, A) < n - 2n(\log n)^{-6C_2}\} \cap \{\tau'(A, b_{\mathcal{I}_2}) \leq n(\log n)^{-6C_2}\}, \\
E_4 &:= \{\tau(b_{\mathcal{I}_1}) + \tau(b_{\mathcal{I}_1}, A) \in [n - 2n(\log n)^{-6C_2}, n]\}, \\
E_5 &:= \{\tau(b_{\mathcal{I}_1}) + \tau(b_{\mathcal{I}_1}, A) > n\} \cap \{\tau(b_{\mathcal{I}_1}, D) > n\}, \\
E_6 &:= \{\tau(b_{\mathcal{I}_1}, D) \leq n\}, \\
E_7 &:= \{\tau'(A, b_{\mathcal{I}_2}) \geq n(\log n)^{-6C_2}\}.
\end{aligned}$$

Notice that

$$E_2^c \subset E_3 \cup E_4 \cup E_5 \cup E_6 \cup E_7. \tag{50}$$

- Control on E_6 . First, $E_2^c \cap \{\mathcal{I}_2 = \mathcal{I}_1 + 1\} \subset E_2^-(\mathcal{I}_1)$, so by (48) and since $D = M_{\mathcal{I}_1}^- < M_{\mathcal{I}_1-1} < b_{\mathcal{I}_1}$ when $\mathcal{I}_2 = \mathcal{I}_1 + 1$, we have for large n for every $\omega \in \mathcal{G}_n^+$,

$$P_\omega(E_2^c \cap \{\mathcal{I}_2 = \mathcal{I}_1 + 1\} \cap E_6) \leq E_\omega(1_{E_2^-(\mathcal{I}_1)} P_\omega^{b_{\mathcal{I}_1}}(\tau(M_{\mathcal{I}_1-1}) < \tau(D) \leq n)) \leq (\log n)^{-2}. \quad (51)$$

The case $\mathcal{I}_1 = \mathcal{I}_2 - 1$ follows similarly from (47), and so

$$P_\omega(E_2^c \cap E_6) \leq 2(\log n)^{-2}$$

for large n for every $\omega \in \mathcal{G}_n^+$.

- Control on E_4 . We start by proving that for every n large enough, for every $\omega \in \mathcal{G}_n^+$, uniformly on E_2^c ,

$$\forall x \in \mathbb{N}, \quad P_\omega^{b_{\mathcal{I}_1}}[\tau(A) \in [n - 2n(\log n)^{-6C_2} - x, n - x]] \leq (\log n)^{-2}. \quad (52)$$

Using Lemma 4.9 and then (6), we obtain on $E_2^-(\mathcal{I}_1) \setminus E_2^+(\mathcal{I}_1)$, since $b_{\mathcal{I}_1} < M_{\mathcal{I}_1} < A$,

$$\begin{aligned} P_\omega^{b_{\mathcal{I}_1}}[\tau(A) = \ell] &\leq P_\omega^{b_{\mathcal{I}_1}}[\tau(A) < \tau(b_{\mathcal{I}_1})] = \omega_{b_{\mathcal{I}_1}} P_\omega^{b_{\mathcal{I}_1}+1}[\tau(A) < \tau(b_{\mathcal{I}_1})] \\ &\leq e^{V(b_{\mathcal{I}_1})-V(M_{\mathcal{I}_1})} \leq e^{W[b_{\mathcal{I}_1}(W, h_n)]-W[M_{\mathcal{I}_1}(W, h_n)]+2\tilde{C}_1 \log_2 n} \\ &\leq e^{-h_n+2\tilde{C}_1 \log_2 n} = (\log n)^{5C_2+2\tilde{C}_1}/n \leq (\log n)^{-2}/(3n(\log n)^{-6C_2}), \end{aligned}$$

for every $\ell \in \mathbb{N}$ and $\omega \in \mathcal{G}_n^+$ for every n large enough, since $C_2 \geq 2\alpha + 2 + 10\tilde{C}_1$. Summing over ℓ proves (52) in this case, the other case $E_2^+(\mathcal{I}_1) \setminus E_2^-(\mathcal{I}_1)$ being very similar.

Due to (52), for large n , for every $\omega \in \mathcal{G}_n^+$, by the strong Markov property,

$$\begin{aligned} P_\omega(E_2^c \cap E_4) &= E_\omega[1_{E_2^c} P_\omega^{b_{\mathcal{I}_1}}(\tau(A) \in [n - 2n(\log n)^{-6C_2} - x, n - x])|_{x=\tau(b_{\mathcal{I}_1})}] \\ &\leq (\log n)^{-2}. \end{aligned} \quad (53)$$

- Control on E_7 . Let us prove that for n large enough,

$$\forall \omega \in \mathcal{G}_n^+, \quad P_\omega(E_2^c \cap E_7) \leq (\log n)^{-2}. \quad (54)$$

Due to the strong Markov property, it is enough to prove that for large n , for every $\omega \in \mathcal{G}_n^+$, uniformly on E_2^c ,

$$P_\omega^A[\tau(b_{\mathcal{I}_2}) \geq n(\log n)^{-6C_2}] \leq (\log n)^{-2}. \quad (55)$$

Recall that h_n -extrema are a fortiori $(h_n - 2C_2 \log_2 n)$ -extrema. Let us observe that due to (30) and since $W(B) - W(b_{\mathcal{I}_1}(W, h_n)) \leq \log n + C_2 \log_2 n$, the only possible slope $T_j[W, h_n - 2C_2 \log_2 n]$, $-5 \leq j \leq 5$ with height $\leq \log n + C_2 \log_2 n$ is $[B, b_{\mathcal{I}_1}(W, h_n)]$ (or $[b_{\mathcal{I}_1}(W, h_n), B]$) so $W(B) - W(b_{\mathcal{I}_2}(W, h_n)) > \log n + C_2 \log_2 n$. For the same reason, there is no $(h_n - 2C_2 \log_2 n)$ -extrema between B and $b_{\mathcal{I}_2}(W, h_n)$, and so $\sup_{B \leq u \leq v \leq b_{\mathcal{I}_2}(W, h_n)} (W(v) - W(u)) < h_n - 2C_2 \log_2 n$ in the case $\mathcal{I}_2 = \mathcal{I}_1 + 1$. Hence in this case, due to (7), (24), (29) and to Markov's inequality, and since $[B] = M_{\mathcal{I}_1} < A < b_{\mathcal{I}_2}$,

$$\begin{aligned} P_\omega^A[\tau([B]) \wedge \tau(b_{\mathcal{I}_2}) \geq n(\log n)^{-6C_2}] &\leq \frac{\varepsilon_0^{-1} 4(\log n)^{2\alpha} e^{h_n - 2C_2 \log_2 n + 2\tilde{C}_1 \log_2 n}}{n(\log n)^{-6C_2}} \\ &\leq \varepsilon_0^{-1} 4(\log n)^{2\alpha - C_2 + 2\tilde{C}_1} \leq \frac{1}{2}(\log n)^{-2}, \end{aligned} \quad (56)$$

for every n large enough since $C_2 \geq 2\alpha + 2 + 10\tilde{C}_1$. This is also true in the case $\mathcal{I}_2 = \mathcal{I}_1 - 1$ by (8). Moreover in the case $\mathcal{I}_2 = \mathcal{I}_1 + 1$, we have $\max_{[A, b_{\mathcal{I}_2}]} V \leq \sup_{[M_{\mathcal{I}_1}^+, b_{\mathcal{I}_2}(W, h_n)]} W + \tilde{C}_1 \log_2 n \leq W(M_{\mathcal{I}_1}^+) + (h_n - 2C_2 \log_2 n) + \tilde{C}_1 \log_2 n$ due to the previous remark, (24) and (29). Also, $W(M_{\mathcal{I}_1}^+) \leq W(b_{\mathcal{I}_1}(W, h_n)) + C_2 \log_2 n$ by (43), otherwise we would have $M_{\mathcal{I}_1}^+ = b_{\mathcal{I}_1+1}$ and $W(b_{\mathcal{I}_1+1}) \geq W(b_{\mathcal{I}_1}(W, h_n)) + C_2 \log_2 n \geq W(M_{\mathcal{I}_1}(W, h_n)) - \log n$ due to our hypothesis in this case $\mathcal{I}_2 = \mathcal{I}_1 + 1$, which in turn would give $W(M_{\mathcal{I}_1}(W, h_n)) - W(b_{\mathcal{I}_1+1}(W, h_n)) \leq \log n + 2\tilde{C}_1 \log_2 n$, which contradicts (30) since $2\tilde{C}_1 < C_2$. So by (6), (24) and (29), recalling that $W(b_{\mathcal{I}_1}(W, h_n)) + h_n \leq W(M_{\mathcal{I}_1}(W, h_n))$ and $B = M_{\mathcal{I}_1}$, we get

$$\begin{aligned} P_\omega^A [\tau(\lfloor B \rfloor) < \tau(b_{\mathcal{I}_2})] &\leq (b_{\mathcal{I}_2} - A) \exp \left[\max_{[A, b_{\mathcal{I}_2}]} V - V(B) \right] \\ &\leq 2(\log n)^\alpha e^{(W(b_{\mathcal{I}_1}(W, h_n)) + h_n + (\tilde{C}_1 - C_2) \log_2 n) - (W(M_{\mathcal{I}_1}(W, h_n)) - \tilde{C}_1 \log_2 n)} \\ &\leq 2(\log n)^{\alpha + 2\tilde{C}_1 - C_2} \leq (\log n)^{-2}/2 \end{aligned} \quad (57)$$

for every n large enough since $C_2 \geq 2\alpha + 2 + 10\tilde{C}_1$. We prove similarly (57) in the case $\mathcal{I}_2 = \mathcal{I}_1 - 1$. Then, (56) and (57) prove (55). Finally, (55) combined with the strong Markov property lead to (54).

- Control on E_5 . On $E_1 \cap E_2^c \cap E_5$, we have $\tau(b_{\mathcal{I}_1}) \leq n \leq \tau(b_{\mathcal{I}_1}) + \tau(b_{\mathcal{I}_1}, A) \wedge \tau(b_{\mathcal{I}_1}, D)$ and in particular $Z_n \in [A \wedge D, A \vee D] = [M_{\mathcal{I}_1}^-, M_{\mathcal{I}_1}^+]$. Applying (46) as in the simplest case, we get for large n , for all $\omega \in \mathcal{G}_n$,

$$\begin{aligned} &P_\omega(E_1 \cap E_2^c \cap E_5, Z_n \notin \Xi_n(W)) \\ &\leq E_\omega [1_{\{\tau(b_{\mathcal{I}_1}) \leq \alpha_n\}} P_\omega^{b_{\mathcal{I}_1}} (Z_{n-k} \in [M_{\mathcal{I}_1}^-, M_{\mathcal{I}_1}^+] \setminus \Xi_n(W))_{|k=\tau(b_{\mathcal{I}_1})}] \leq (\log n)^{-2}. \end{aligned} \quad (58)$$

- Control on E_3 . On $E_1 \cap E_2^c \cap E_3$, we have $\tau(b_{\mathcal{I}_2}) \leq \tau(b_{\mathcal{I}_1}) + \tau(b_{\mathcal{I}_1}, A) + \tau'(A, b_{\mathcal{I}_2}) < n$. Moreover, the height of the valley $[M_{\mathcal{I}_2-1}(W, h_n), M_{\mathcal{I}_2}(W, h_n)]$ is at least $\log n + C_2 \log_2 n$ on E_2^c due to (30), that is, we are on $E_2^-(I_2) \cap E_2^+(I_2)$. Also we get $P_\omega^{b_{\mathcal{I}_2}} [\tau(M_{\mathcal{I}_2-1}) \wedge \tau(M_{\mathcal{I}_2}) < n] \leq 2(\log n)^{-2}$ by (47) and (48) uniformly on $E_2^c \cap \mathcal{G}_n^+$ for large n . Using (46) and $[M_{\mathcal{I}_2-1}, M_{\mathcal{I}_2}] \subset [M_{\mathcal{I}_2}^-, M_{\mathcal{I}_2}^+]$, this gives for large n for every $\omega \in \mathcal{G}_n^+$,

$$\begin{aligned} &P_\omega(E_1 \cap E_2^c \cap E_3, Z_n \notin \Xi_n(W)) \\ &\leq E_\omega [1_{\{\tau(b_{\mathcal{I}_2}) < n\}} P_\omega^{b_{\mathcal{I}_2}} (Z_{n-k} \in [M_{\mathcal{I}_2}^-, M_{\mathcal{I}_2}^+] - \Xi_n(W))_{|k=\tau(b_{\mathcal{I}_2})}] \\ &\quad + E_\omega [1_{E_2^c} P_\omega^{b_{\mathcal{I}_2}} [\tau(M_{\mathcal{I}_2-1}) \wedge \tau(M_{\mathcal{I}_2}) < n]] \\ &\leq 3(\log n)^{-2}. \end{aligned} \quad (59)$$

Finally, (50) and the controls on E_i , $3 \leq i \leq 7$ prove Lemma 4.10, which ends the proof of Proposition 4.4. \blacksquare

5. PROBABILITY OF SIMULTANEOUS MEETING OF INDEPENDENT RECURRENT RWRE
IN THE SAME ENVIRONMENT

The following lemma plays a crucial role in the proof of the case (ii) of Theorem 1.2 when $r > 1$ and $I = 1$. Let $r > 1$ and let $Z^{(1)}, \dots, Z^{(r)}$ be r independent recurrent RWRE in the same environment ω satisfying (4).

Lemma 5.1. *For \mathbb{P} -almost every ω , there exists $c(\omega) > 0$ such that, for every $(y_1, \dots, y_r) \in [(2\mathbb{Z})^r \cup (2\mathbb{Z} + 1)^r]$, we have*

$$\limsup_{N \rightarrow +\infty} \frac{1}{\log N} \sum_{n=1}^N \frac{1}{n} \sum_{k \in \mathbb{Z}} \prod_{j=1}^r P_\omega^{y_j} [Z_n^{(j)} = k] \geq c(\omega).$$

Lemma 5.1 comes from the next intermediate result.

Proposition 5.2. *Let $\delta \in (0, 1)$. There exist events $\Delta_N(\delta)$, $N \geq 1$ and $\widehat{b}(N) \in 2\mathbb{Z}$ depending only on the environment ω , and constants $c(\delta) > 0$, $\varepsilon(\delta) \in (0, 1)$, with*

$$\liminf_{N \rightarrow +\infty} \mathbb{P}[\Delta_N(\delta)] \geq 1 - \delta, \quad (60)$$

such that

$$\begin{aligned} \forall (y_1, \dots, y_r) \in (2\mathbb{Z})^r, \exists N_1 \in \mathbb{N}, \forall N \geq N_1, \forall \omega \in \Delta_N(\delta), \forall j \in \{1, \dots, r\}, \\ \forall n \in [N^{1-\varepsilon(\delta)}, N] \cap (2\mathbb{N}), \quad P_\omega^{y_j} [Z_n^{(j)} = \widehat{b}(N)] \geq c(\delta). \end{aligned} \quad (61)$$

This remains true if $(2\mathbb{Z})^r$ and $2\mathbb{N}$ are replaced respectively by $(2\mathbb{Z} + 1)^r$ and $2\mathbb{N} + 1$.

Proof of Lemma 5.1. Assume that we have proved Proposition 5.2. Let $\delta \in (0, 1)$. First, notice that by (60),

$$\mathbb{P} \left[\limsup_{N \rightarrow +\infty} \Delta_N(\delta) \right] = \mathbb{P} \left[\bigcap_{N \in \mathbb{N}} \bigcup_{n \geq N} \Delta_n(\delta) \right] = \lim_{N \rightarrow +\infty} \mathbb{P} \left[\bigcup_{n \geq N} \Delta_n(\delta) \right] \geq \liminf_{N \rightarrow +\infty} \mathbb{P}[\Delta_N(\delta)] \geq 1 - \delta. \quad (62)$$

Now, let $(y_1, \dots, y_r) \in (2\mathbb{Z})^r$. There exists $N_1 \in \mathbb{N}$ such that for every $N \geq N_1$, on $\Delta_N(\delta)$,

$$\sum_{n=1}^N \frac{1}{n} \sum_{k \in \mathbb{Z}} \prod_{j=1}^r P_\omega^{y_j} [Z_n^{(j)} = k] \geq \sum_{n=N^{1-\varepsilon(\delta)}}^N \frac{\mathbf{1}_{2\mathbb{N}}(n)}{n} [c(\delta)]^r \geq [c(\delta)]^r \frac{\varepsilon(\delta)}{4} \log N$$

if N is large enough. Consequently, we have on $\limsup_{N \rightarrow +\infty} \Delta_N(\delta) = \{\omega \in \Delta_N(\delta) \text{ i.o.}\}$,

$$\limsup_{N \rightarrow +\infty} \frac{1}{\log N} \sum_{n=1}^N \frac{1}{n} \sum_{k \in \mathbb{Z}} \prod_{j=1}^r P_\omega^{y_j} [Z_n^{(j)} = k] \geq [c(\delta)]^r \frac{\varepsilon(\delta)}{4} > 0.$$

This and (62) prove Lemma 5.1 in the case $(y_1, \dots, y_r) \in (2\mathbb{Z})^r$. The proof in the case $(y_1, \dots, y_r) \in (2\mathbb{Z} + 1)^r$ is similar. \blacksquare

Now, it remains to prove Proposition 5.2.

5.1. Construction of $\Delta_N(\delta)$. Let $\delta \in (0, 1)$. The aim of this section is the construction of the set of environments

$$\Delta_N(\delta) := \bigcap_{i=0}^6 \Delta_N^{(i)},$$

where the set $\Delta_N^{(i)}$ also depends on δ . In what follows, ε_i is for $i > 0$ a positive constant depending on δ and used to define the set $\Delta_N^{(i)}$. As in the previous section, we will approximate the potential V by a two-sided Brownian motion W such that $\text{Var}(W(1)) = \text{Var}(V(1))$ (see Figure 2 for patterns of the potential V and of W in $\Delta_N(\delta)$). We start with $\Delta_N^{(1)}, \dots, \Delta_N^{(5)}$ which are W -measurable. Using the same notation as before for h -extrema, for a two-sided Brownian motion W , we define

$$\Delta_N^{(1)} := \{W \in \mathcal{W}\} \cap \bigcap_{i=-1}^1 \{H[T_i(W, (1 - 2\varepsilon_1) \log N)] \geq (1 + 2\varepsilon_1) \log N\},$$

$$\Delta_N^{(R)} := \{x_1(W, (1 - 2\varepsilon_1) \log N) \text{ is a } ((1 - 2\varepsilon_1) \log N)\text{-minimum for } W\},$$

and $\Delta_N^{(L)} := [\Delta_N^{(R)}]^c$, where R stands for right and L for left, $\Delta_N^{(2)} := \Delta_N^{(2,R)} \cup \Delta_N^{(2,L)}$ with

$$\Delta_N^{(2,R)} := \left\{ \max_{[0, x_1(W, (1-2\varepsilon_1) \log N)]} W < W(x_0(W, (1 - 2\varepsilon_1) \log N)) - \varepsilon_2 \log N \right\} \cap \Delta_N^{(R)},$$

$$\Delta_N^{(2,L)} := \left\{ \max_{[x_0(W, (1-2\varepsilon_1) \log N), 0]} W < W(x_1(W, (1 - 2\varepsilon_1) \log N)) - \varepsilon_2 \log N \right\} \cap \Delta_N^{(L)};$$

$$\Delta_N^{(3)} := \left\{ -\varepsilon_3^{-1}(\log N)^2 \leq x_{-1}[W, (1 - 2\varepsilon_1) \log N] \leq x_2[W, (1 - 2\varepsilon_1) \log N] \leq \varepsilon_3^{-1}(\log N)^2 \right\};$$

$$\Delta_N^{(4)} := \bigcap_{i=0}^1 \{|W(x_i(W, (1 - 2\varepsilon_1) \log N))| > \varepsilon_4 \log N\}$$

and $\Delta_N^{(5)} := \Delta_N^{(5,R)} \cup \Delta_N^{(5,L)}$, where

$$\Delta_N^{(5,L)} := \left\{ \min_{[0, x_1(W, (1-2\varepsilon_1) \log N)]} W > W(x_0(W, (1 - 2\varepsilon_1) \log N)) + \varepsilon_5 \log N \right\} \cap \Delta_N^{(L)},$$

$$\Delta_N^{(5,R)} := \left\{ \min_{[x_0(W, (1-2\varepsilon_1) \log N), 0]} W > W(x_1(W, (1 - 2\varepsilon_1) \log N)) + \varepsilon_5 \log N \right\} \cap \Delta_N^{(R)}.$$

Lemma 5.3. *Let W be a two-sided Brownian motion such that $\text{Var}(W(1)) = \text{Var}(V(1))$. There exist $(\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4) \in (0, 1/10)^4$ (and $\varepsilon_5 = \varepsilon_2$) such that, for every $i \in \{1, \dots, 5\}$, $\mathbb{P}[\Delta_N^{(i)}] > 1 - \delta/10$.*

Proof. First, by the same arguments as in the proof of Lemma 4.3, there exists $\varepsilon_1 \in (0, 1/10)$ such that $\mathbb{P}[\Delta_N^{(1)}] \geq 1 - \delta/10$.

We now introduce $\widetilde{W}_N(x) := W[x(\log N)^2]/\log N$, which has the same law as W by scaling. We notice that $x_0(\widetilde{W}_N, 1 - 2\varepsilon_1)$ is a local extremum for \widetilde{W}_N , so $\mathbb{P}[x_0(\widetilde{W}_N, 1 - 2\varepsilon_1) = 0] = 0$. So $x_0(\widetilde{W}_N, 1 - 2\varepsilon_1) < 0 < x_1(\widetilde{W}_N, 1 - 2\varepsilon_1)$ a.s. We start with the case $x_1(\widetilde{W}_N, 1 - 2\varepsilon_1)$ is a $(1 - 2\varepsilon_1)$ -minimum for \widetilde{W}_N , that is, the bottom $b_0(W, (1 - 2\varepsilon_1) \log N)$ of the central

$2\varepsilon_1)) = \min_{[x_0(\widetilde{W}_N, 1-2\varepsilon_1), 0]} \widetilde{W}_N < 0$ a.s. on $\Delta_N^{(L)} \cap \mathcal{W}$, so $|\widetilde{W}_N(x_0(\widetilde{W}_N, 1-2\varepsilon_1))| > 0$ a.s. Similarly, $|\widetilde{W}_N(x_1(\widetilde{W}_N, 1-2\varepsilon_1))| > 0$ a.s.

Replacing W by $-W$ in $\Delta_N^{(2)}$ proves that with $\varepsilon_5 := \varepsilon_2 > 0$, the event $\Delta_N^{(5)}$ satisfies $\mathbf{P}[\Delta_N^{(5)}] = \mathbf{P}[\Delta_N^{(2)}] \geq 1 - \delta/10$. \blacksquare

From now on, $\varepsilon_1, \dots, \varepsilon_5$ are the ones given by Lemma 5.3. Set $\varepsilon := \min(\varepsilon_1, \dots, \varepsilon_5)/9$.

Lemma 5.4. *Up to an enlargement of $(\Omega, \mathcal{F}, \mathbf{P})$, there exist a two-sided Brownian motion $(W(s), s \in \mathbb{R})$ defined on Ω such that $\mathbf{Var}(W(1)) = \mathbf{Var}(V(1))$ and a real number $\xi > 0$ such that*

$$\mathbf{P} \left[\sup_{|t| \leq 2\varepsilon_3^{-1}(\log N)^2} |V(\lfloor t \rfloor) - W(t)| > \varepsilon \log N \right] = O(N^{-\xi}).$$

Proof. Due to (22) (with its notations), for N large enough, possibly on an enlarged probability space, there exists a Brownian motion $(W(s), s \in \mathbb{R})$ such that

$$\mathbf{P} \left[\sup_{|k| \leq 2\varepsilon_3^{-1}(\log N)^2} |V(k) - W(k)| > \frac{\varepsilon}{2} \log N \right] \leq N^{-c \frac{\varepsilon}{10}}$$

and such that $\mathbf{Var}(W(1)) = \mathbf{Var}(V(1))$. Moreover,

$$\begin{aligned} \mathbf{P} \left[\sup_{|t| \leq 2\varepsilon_3^{-1}(\log N)^2} |W(t) - W(\lfloor t \rfloor)| > \frac{\varepsilon}{2} \log N \right] &\leq 5\varepsilon_3^{-1}(\log N)^2 \mathbf{P} \left[\sup_{|t| \leq 1} |W(t)| > \frac{\varepsilon}{2} \log N \right] \\ &= O((\log N)^2 \exp[-\varepsilon^2(\log N)^2/(8\sigma^2)]). \end{aligned}$$

Combining these two inequalities proves the lemma. \blacksquare

From now on, W is the Brownian motion W coming from Lemma 5.4 and $\Delta_N^{(1)}, \dots, \Delta_N^{(5)}$ are the corresponding events. We set

$$\Delta_N^{(0)} := \left\{ \sup_{|t| \leq 2\varepsilon_3^{-1}(\log N)^2} |V(\lfloor t \rfloor) - W(t)| \leq \varepsilon \log N \right\}.$$

For N large enough, $\mathbf{P}[\Delta_N^{(0)}] > 1 - \delta/10$. In particular on $\Delta_N^{(3)}$, we can apply the inequalities of $\Delta_N^{(0)}$ to any $t \in [x_{-1}(W, (1-2\varepsilon_1)\log N), x_2(W, (1-2\varepsilon_1)\log N)]$. We now introduce (here this is for V directly, not for W)

$$\begin{aligned} \theta_N^{(R)} &:= \inf \left\{ i \in \mathbb{N}, V(i) - \min_{0 \leq j \leq i} V(j) \geq (1 + \varepsilon_1) \log N \right\}, \\ \beta_N^{(R)} &:= \sup \left\{ i < \theta_N^{(R)}, V(i) = \min_{0 \leq j \leq \theta_N^{(R)}} V(j) \right\}, \\ \theta_N^{(L)} &:= \sup \left\{ i \in (-\mathbb{N}), V(i) - \min_{i \leq j \leq 0} V(j) \geq (1 + \varepsilon_1) \log N \right\}, \\ \beta_N^{(L)} &:= \inf \left\{ i > \theta_N^{(L)}, V(i) = \min_{\theta_N^{(L)} \leq j \leq 0} V(j) \right\}. \end{aligned}$$

By ([DGPS07], eq. (4.33)), there exists $\varepsilon_6 > 0$ such that if N is large enough, $\mathbf{P}[\Delta_N^{(6,R)}] \geq 1 - \delta/10$, where

$$\Delta_N^{(6,R)} := \left\{ \sum_{i=0}^{\theta_N^{(R)}-1} e^{-[V(i)-V(\beta_N^{(R)})]} \leq \varepsilon_6^{-1} \right\}, \quad \Delta_N^{(6,L)} := \left\{ \sum_{i=\theta_N^{(L)}}^{-1} e^{-[V(i)-V(\beta_N^{(L)})]} \leq \varepsilon_6^{-1} \right\}. \quad (63)$$

Replacing $V(\cdot)$ by $V(-\cdot)$ gives $\mathbf{P}[\Delta_N^{(6,L)}] \geq 1 - \delta/10$. Consequently, $\mathbf{P}[\Delta_N^{(6)}] \geq 1 - 2\delta/10$, where $\Delta_N^{(6)} := \Delta_N^{(6,R)} \cap \Delta_N^{(6,L)}$. Combining the previous estimates leads to:

Lemma 5.5. *For large N , $\mathbf{P}[\Delta_N(\delta)] \geq 1 - \delta$. So (60) is true.*

5.2. Random walk in an environment $\omega \in \Delta_N(\delta)$. The aim of this subsection is to prove Proposition 5.2 with the $\Delta_N(\delta)$ constructed in the previous section. Let $\delta \in (0, 1)$. We write Δ_N for $\Delta_N(\delta)$. We also fix $(y_1, \dots, y_r) \in (2\mathbb{Z})^r$. There exists $N_2 \in \mathbb{N}$ such that for $N \geq N_2$, $\mathbf{P}[\Delta_N] \geq 1 - \delta$ (due to Lemma 5.5), $A \leq \varepsilon \log N$, and $\max_{1 \leq j \leq r} |y_j| < \min(\varepsilon_2, \varepsilon_4)(\log N)/(4A)$, where $A := \log((1 - \varepsilon_0)/\varepsilon_0)$.

We introduce

$$\widehat{b}(N) := 2 \lfloor \beta_N^{(R)}/2 \rfloor \mathbf{1}_{\Delta_N^{(R)}} + 2 \lfloor \beta_N^{(L)}/2 \rfloor \mathbf{1}_{\Delta_N^{(L)}}. \quad (64)$$

We do the proof in the case $\omega \in \Delta_N \cap \Delta_N^{(R)}$. Indeed the case $\omega \in \Delta_N \cap \Delta_N^{(L)}$ is similar by symmetry. We define $\widehat{x}_i := \lfloor x_i(W, (1 - 2\varepsilon_1) \log N) \rfloor$, and

$$D_N^{(1)} := \{ \tau(\widehat{b}(N)) < \tau(\widehat{x}_0) \}, \quad D_N^{(2)} := \{ \tau(\widehat{x}_0) \wedge \tau(\widehat{b}(N)) \leq N^{1-\varepsilon_1} \}.$$

We sometimes write x_i instead of $x_i(W, (1 - 2\varepsilon_1) \log N)$ in the following.

In the following lemma, we prove that Z goes quickly to $\widehat{b}(N)$, which is nearly the bottom of the potential V in the central valley $[\widehat{x}_0, \widehat{x}_2]$, with large probability under $P_\omega^{y_j}$, uniformly on $\Delta_N \cap \Delta_N^{(R)}$ and j .

Lemma 5.6. *There exists $N_3 \in \mathbb{N}$ such that for all $N \geq N_3$,*

$$\forall \omega \in \Delta_N \cap \Delta_N^{(R)}, \forall j \in \{1, \dots, r\}, \quad P_\omega^{y_j} [D_N^{(1)}] \geq 1 - N^{-(\varepsilon_1 \wedge \varepsilon_2)/4}, \quad P_\omega^{y_j} [D_N^{(2)}] \geq 1 - N^{-\varepsilon_1/4}.$$

Proof. Let $N \geq N_2$, $\omega \in \Delta_N \cap \Delta_N^{(R)}$ and $j \in \{1, \dots, r\}$. First, notice that $W(x_2) - W(x_1) = H[T_1(W, (1 - 2\varepsilon_1) \log N)] \geq (1 + 2\varepsilon_1) \log N$ because $\omega \in \Delta_N^{(1)}$. This gives

$$V(\widehat{x}_2) - V(\widehat{x}_1) \geq W(x_2) - W(x_1) - 2\varepsilon \log N \geq (1 + \varepsilon_1) \log N \quad (65)$$

since $\omega \in \Delta_N^{(3)} \cap \Delta_N^{(0)}$. Hence $0 \leq \widehat{b}(N) \leq \beta_N^{(R)} \leq \theta_N^{(R)} \leq \widehat{x}_2 \leq \varepsilon_3^{-1}(\log N)^2$.

Now, assume that $\theta_N^{(R)} < x_1$. Since $V(\theta_N^{(R)}) - V(\beta_N^{(R)}) \geq (1 + \varepsilon_1) \log N$, the previous inequalities would give, on $\Delta_N^{(0)} \cap \Delta_N^{(3)}$, $W(\theta_N^{(R)}) - W(\beta_N^{(R)}) \geq (1 + \varepsilon_1 - 2\varepsilon) \log N \geq (1 - 2\varepsilon_1) \log N$. So, recalling that $W(x_1) = \min_{[0, x_1]} W$, there would exist a $((1 - 2\varepsilon_1) \log N)$ -maximum for W in $]0, x_1[$, which is not possible. Hence $x_1 \leq \theta_N^{(R)}$.

So, $V(\beta_N^{(R)}) \leq V(\widehat{x}_1) \leq W(x_1) + \varepsilon \log N < -8\varepsilon_4(\log N)/9$ because $\omega \in \Delta_N^{(0)} \cap \Delta_N^{(3)} \cap \Delta_N^{(4)}$. If $y_j > 0$, then $\min_{[0, y_j]} V \geq -|y_j|A \geq -\varepsilon_4(\log N)/4 > V(\beta_N^{(R)}) + 2A$, because $N \geq N_2$. Since similarly, $\max_{[0, y_j]} V \leq \varepsilon_4(\log N)/4$ and $\varepsilon_4 < 1$, we get successively $y_j \leq \theta_N^{(R)}$ and

$y_j \leq \beta_N^{(R)} - 2 \leq \widehat{b}(N) - 1$. If $y_j < 0$, we prove similarly that $\widehat{x}_0 < y_j$ since $V(\widehat{x}_0) \geq 8\varepsilon_4(\log N)/9$. Hence in every case, $\widehat{x}_0 < y_j < \widehat{b}(N)$.

We now prove that

$$\max_{[y_j, \widehat{b}(N)]} V - V(\widehat{x}_0) \leq -[(\varepsilon_1 \wedge \varepsilon_2)/2] \log N. \quad (66)$$

To this aim, notice that $\max_{[0, \widehat{x}_1]} V - V(\widehat{x}_0) \leq -\varepsilon_2(\log N)/2$ since $\omega \in \Delta_N^{(2,R)} \cap \Delta_N^{(0)} \cap \Delta_N^{(3)}$, and that if $y_j < 0$, we have $\max_{[y_j, 0]} V - V(\widehat{x}_0) \leq |y_j|A - (8/9)\varepsilon_2 \log N \leq -\varepsilon_2(\log N)/2$ since $W(x_0) \geq \varepsilon_2 \log N$ on $\Delta_N^{(2,R)}$ and so $V(\widehat{x}_0) \geq (8/9)\varepsilon_2 \log N$. This gives (66) when $\widehat{b}(N) \leq \widehat{x}_1$.

Assume now $\widehat{x}_1 < \widehat{b}(N)$. We have seen after (65) that $0 \leq \widehat{b}(N) \leq \theta_N^{(R)} \leq \widehat{x}_2$, moreover, $V(\widehat{b}(N)) \leq V(\beta_N^{(R)}) + A$ and we have proved that $V(\beta_N^{(R)}) \leq V(\widehat{x}_1)$, so we obtain

$$V(\widehat{b}(N)) - \varepsilon \log N - A \leq V(\widehat{x}_1) - \varepsilon \log N \leq W(x_1) \leq W(\widehat{b}(N)) \leq V(\widehat{b}(N)) + \varepsilon \log N$$

since $W(x_1) = \min_{[x_0, x_2]} W$, so that

$$|W(x_1) - V(\widehat{b}(N))| \leq \varepsilon \log N + A \leq 2 \min(\varepsilon_1, \varepsilon_2)(\log N)/9. \quad (67)$$

Moreover there is no $((1 - 2\varepsilon_1) \log N)$ -maximum for W in (x_0, x_2) , therefore,

$$\max_{[x_1, \widehat{b}(N)]} W < W(\widehat{b}(N)) + (1 - 2\varepsilon_1) \log N \leq W(x_1) + (1 - 2\varepsilon_1 + 3\varepsilon_1/9) \log N, \quad (68)$$

by $\Delta_N^{(0)}$ applied to $\widehat{b}(N)$ followed by (67). Since $V(\widehat{x}_0) \geq V(\widehat{x}_1) + (1 + \varepsilon_1) \log N$ on $\Delta_N^{(1)} \cap \Delta_N^{(0)} \cap \Delta_N^{(3)}$, this gives $\max_{[\widehat{x}_1, \widehat{b}(N)]} V - V(\widehat{x}_0) \leq -\varepsilon_1 \log N$ (since $\omega \in \Delta_N^{(0)} \cap \Delta_N^{(3)}$). Recapitulating all this gives (66) also when $\widehat{x}_1 < \widehat{b}(N)$.

So by (6) and (66), we get uniformly on $\Delta_N \cap \Delta_N^{(R)}$ and j for large N ,

$$P_\omega^{y_j} [(D_N^{(1)})^c] \leq [\widehat{b}(N) - y_j] \exp \left[\max_{[y_j, \widehat{b}(N)]} V - V(\widehat{x}_0) \right] \leq \frac{2\varepsilon_3^{-1}(\log N)^2}{N^{(\varepsilon_1 \wedge \varepsilon_2)/2}} \leq N^{-(\varepsilon_1 \wedge \varepsilon_2)/4},$$

where we used $\omega \in \Delta_N^{(3)}$ and $\widehat{x}_0 < y_j < \widehat{b}(N) < \widehat{x}_2$. This proves the first inequality of the lemma.

We now turn to $D_N^{(2)}$. Notice that $|\widehat{b}(N) - \widehat{x}_0| \leq |\widehat{x}_2 - \widehat{x}_0| \leq 3\varepsilon_3^{-1}(\log N)^2$ on Δ_N since $0 \leq \widehat{b}(N) \leq \widehat{x}_2$ as proved after (65). Moreover, there is no $((1 - 2\varepsilon_1) \log N)$ -maximum for W in (x_0, x_1) , so $\max_{x_0 \leq u \leq v \leq x_1} (W(v) - W(u)) < (1 - 2\varepsilon_1) \log N$. Also if $x_1 < \widehat{b}(N)$, $\min_{[x_0, \widehat{b}(N)]} W = W(x_1)$ and (68) lead to $\max_{x_1 \leq u \leq v \leq \widehat{b}(N)} (W(v) - W(u)) \leq (1 - 2\varepsilon_1 + 3\varepsilon_1/9) \log N$. Since $\omega \in \Delta_N^{(0)} \cap \Delta_N^{(3)}$, this gives

$$\max_{\widehat{x}_0 \leq \ell \leq k \leq \widehat{b}(N)} (V(k) - V(\ell)) \leq (1 - 13\varepsilon_1/9) \log N. \quad (69)$$

Hence, we have by (7),

$$E_\omega^{y_j} [\tau(\widehat{x}_0) \wedge \tau(\widehat{b}(N))] \leq \frac{[\widehat{b}(N) - \widehat{x}_0]^2}{\varepsilon_0} \exp \left[\max_{\widehat{x}_0 \leq \ell \leq k \leq \widehat{b}(N)} (V(k) - V(\ell)) \right] \leq \frac{9(\log N)^4 N^{1 - \frac{13\varepsilon_1}{9}}}{\varepsilon_0 \varepsilon_3^2}.$$

So due to Markov's inequality, $P_\omega^{y_j} [(D_N^{(2)})^c] \leq N^{-\varepsilon_1/4}$, uniformly in $\omega \in \Delta_N \cap \Delta_N^{(R)}$ and j , for large N . \blacksquare

In the following lemma, we prove that with large quenched probability, uniformly on $\Delta_N \cap \Delta_N^{(R)}$, after first hitting $\widehat{b}(N)$, the random walk Z stays in the central valley $[\widehat{x}_0, \widehat{x}_2]$ at least up to time N . To this aim, we now define

$$D_N^{(3)} := \{\forall k \in [\tau(\widehat{b}(N)), \tau(\widehat{b}(N)) + N - 1], \widehat{x}_0 < Z_k < \widehat{x}_2\}.$$

Lemma 5.7. *We have for large N ,*

$$\forall \omega \in \Delta_N \cap \Delta_N^{(R)}, \forall j \in \{1, \dots, r\}, \quad P_\omega^{y_j} [D_N^{(3)}] = P_\omega^{\widehat{b}(N)} [\tau(\widehat{x}_0) \wedge \tau(\widehat{x}_2) \geq N] \geq 1 - 2e^{2A} N^{-\varepsilon_1}.$$

Proof. Let $\omega \in \Delta_N \cap \Delta_N^{(R)}$. We recall that $|V(k) - V(k-1)| \leq A$ for every $k \in \mathbb{Z}$. We have, since $x_1 \leq \theta_N^{(R)}$ and so $V(\beta_N^{(R)}) \leq V(\widehat{x}_1)$, and by (65),

$$V(\widehat{b}(N)) - V(\widehat{x}_2) \leq V(\widehat{x}_1) + A - V(\widehat{x}_2) \leq A - (1 + \varepsilon_1) \log N. \quad (70)$$

Similarly,

$$V(\widehat{b}(N)) - V(\widehat{x}_0) \leq A - (1 + \varepsilon_1) \log N. \quad (71)$$

Hence (9) and (10) lead respectively to

$$P_\omega^{\widehat{b}(N)} (\tau(\widehat{x}_2) < N) \leq N \exp \left(\min_{[\widehat{b}(N), \widehat{x}_2-1]} V - V(\widehat{x}_2 - 1) \right) \leq N e^{2A - (1 + \varepsilon_1) \log N} \leq e^{2A} N^{-\varepsilon_1},$$

$$P_\omega^{\widehat{b}(N)} (\tau(\widehat{x}_0) < N) \leq N \exp \left(\min_{[\widehat{x}_0, \widehat{b}(N)-1]} V - V(\widehat{x}_0) \right) \leq N e^{2A - (1 + \varepsilon_1) \log N} \leq e^{2A} N^{-\varepsilon_1}.$$

These two inequalities yield $P_\omega^{\widehat{b}(N)} [\tau(\widehat{x}_0) \wedge \tau(\widehat{x}_2) < N] \leq 2e^{2A} N^{-\varepsilon_1}$, uniformly on $\Delta_N \cap \Delta_N^{(R)}$, which proves the lemma. \blacksquare

Now, similarly as in Brox [B86] for diffusions in random potentials (see also [AD15, p. 45]), we introduce a coupling between Z (under $P_\omega^{\widehat{b}(N)}$) and a reflected random walk \widehat{Z} defined below. More precisely, we define, for fixed N , $\widehat{\omega}_{\widehat{x}_0} := 1$, $\widehat{\omega}_x := \omega_x$ if $\widehat{x}_0 < x < \widehat{x}_2$, and $\widehat{\omega}_{x_2} := 0$. We consider a random walk $(\widehat{Z}_n)_{n \in \mathbb{N}}$ in the environment $\widehat{\omega}$, starting from $x \in [\widehat{x}_0, \widehat{x}_2]$, and denote its law by P_ω^x . That is, \widehat{Z} satisfies (2) with $\widehat{\omega}$ instead of ω and $\omega^{(j)}$ and \widehat{Z} instead of $Z^{(j)}$. In words, \widehat{Z} is a random walk in the environment ω , starting from $x \in [\widehat{x}_0, \widehat{x}_2]$, and reflected at \widehat{x}_0 and \widehat{x}_2 . Also, let

$$\widehat{\mu}(\widehat{x}_0) := e^{-V(\widehat{x}_0)}, \quad \widehat{\mu}(\widehat{x}_2) := e^{-V(\widehat{x}_2-1)}, \quad \widehat{\mu}(x) := e^{-V(x)} + e^{-V(x-1)}, \quad \widehat{x}_0 < x < \widehat{x}_2,$$

and $\widehat{\mu}(x) = 0$ if $x \notin [\widehat{x}_0, \widehat{x}_2]$. Notice that $\widehat{\mu}(\cdot)/\widehat{\mu}(\mathbb{Z})$ is an invariant probability measure for \widehat{Z} . As a consequence,

$$\widehat{\nu}(x) := \widehat{\mu}(x) \mathbf{1}_{2\mathbb{Z}}(x) / \widehat{\mu}(2\mathbb{Z}), \quad x \in \mathbb{Z}, \quad (72)$$

is an invariant probability measure for $(\widehat{Z}_{2n})_{n \in \mathbb{N}}$ for fixed $\widehat{\omega}$. That is, $P_\omega^{\widehat{\nu}}(\widehat{Z}_{2k} = x) = \widehat{\nu}(x)$ for every $x \in \mathbb{Z}$ and $k \in \mathbb{N}$, where $P_\omega^{\widehat{\nu}}(\cdot) := \sum_{x \in \mathbb{Z}} \widehat{\nu}(x) P_\omega^x(\cdot)$. Notice that $\widehat{\nu}$ and $\widehat{\mu}$ depend on N and ω .

We can now, again for fixed N and ω , build a coupling Q_ω of Z and \widehat{Z} , such that

$$Q_\omega(\widehat{Z} \in \cdot) = P_\omega^{\widehat{\nu}}(\widehat{Z} \in \cdot), \quad Q_\omega(Z \in \cdot) = P_\omega^{\widehat{b}(N)}(Z \in \cdot), \quad (73)$$

such that under Q_ω , these two Markov chains move independently until

$$\tau_{\widehat{Z}=Z} := \inf \{k \geq 0, \widehat{Z}_k = Z_k\},$$

which is their first meeting time, then $\widehat{Z}_k = Z_k$ for every $\tau_{\widehat{Z}=Z} \leq k < \tau_{exit}$, where τ_{exit} is the next exit time of Z from the central valley $[\widehat{x}_0, \widehat{x}_2]$, that is,

$$\tau_{exit} := \inf \{k > \tau_{\widehat{Z}=Z}, Z_k \notin [\widehat{x}_0, \widehat{x}_2]\},$$

and then \widehat{Z} and Z move independently again after τ_{exit} .

Now, we would like to prove that under Q_ω , Z and \widehat{Z} collide quickly, that is, $\tau_{\widehat{Z}=Z}$ is very small compared to N . To this aim, we introduce

$$\begin{aligned} \widehat{L}^- &:= \sup \{k \leq \widehat{b}(N), V(k) - V(\widehat{b}(N)) \geq (1 - \varepsilon_1) \log N\}, \\ \widehat{L}^+ &:= \inf \{k \geq \widehat{b}(N), V(k) - V(\widehat{b}(N)) \geq (1 - \varepsilon_1) \log N\}. \end{aligned}$$

Let $u \vee v := \max(u, v)$. We have the following:

Lemma 5.8. *We have for large N , $\tau(\cdot)$ denoting hitting times by Z as before,*

$$\forall \omega \in \Delta_N \cap \Delta_N^{(R)}, \quad Q_\omega[\tau(\widehat{L}^-) \vee \tau(\widehat{L}^+) > N^{1-\varepsilon_1/2}] \leq 4N^{-\varepsilon_1/4}.$$

Proof. Let $N \geq N_2$ and $\omega \in \Delta_N \cap \Delta_N^{(R)}$. Notice that $\widehat{x}_0 \leq \widehat{L}^- < \widehat{b}(N) < \widehat{L}^+ \leq \theta_N^{(R)} \leq \widehat{x}_2$ similarly as after (65). Because $\omega \in \Delta_N^{(5,R)} \cap \Delta_N^{(0)} \cap \Delta_N^{(3)}$ and due to (67), we have since $\varepsilon_5 = \varepsilon_2$,

$$\forall k \in [\widehat{x}_0, -1], \quad V(k) - V(\widehat{b}(N)) \geq W(x_1) + (\varepsilon_5 - \varepsilon) \log N - V(\widehat{b}(N)) \geq (\varepsilon_5/2) \log N. \quad (74)$$

Moreover $\min_{[0, \theta_N^{(R)}]} V = V(\beta_N^{(R)}) \geq V(\widehat{b}(N)) - A$, so $\min_{[\widehat{x}_0, \widehat{L}^+]} V \geq \min_{[\widehat{x}_0, \theta_N^{(R)}]} V \geq V(\widehat{b}(N)) - A$. Notice also for further use that, for every $k \in [\theta_N^{(R)}, \widehat{x}_2]$, we have $V(\theta_N^{(R)}) - V(k) \leq W(\theta_N^{(R)}) - W(k) + 2\varepsilon \log N < (1 - 2\varepsilon_1 + 2\varepsilon) \log N$ since $\omega \in \Delta_N^{(0)} \cap \Delta_N^{(3)}$ and because there is no $(1 - 2\varepsilon_1) \log N$ maximum for W in $(\widehat{x}_1, \widehat{x}_2)$ and $\widehat{x}_1 \leq \theta_N^{(R)} \leq k \leq \widehat{x}_2$, as proved after (65). Since $V(\theta_N^{(R)}) - V(\widehat{b}(N)) \geq (1 + \varepsilon_1) \log N - A$, this gives

$$\begin{aligned} \forall k \in [\theta_N^{(R)}, \widehat{x}_2], \quad V(k) - V(\widehat{b}(N)) &= V(k) - V(\theta_N^{(R)}) + V(\theta_N^{(R)}) - V(\widehat{b}(N)) \\ &\geq 2\varepsilon_1 \log N. \end{aligned} \quad (75)$$

Putting together these inequalities gives in particular $\min_{[\widehat{x}_0, \widehat{x}_2]} V \geq V(\widehat{b}(N)) - A$. Furthermore,

$$\max_{[\widehat{b}(N), \widehat{L}^+]} V \leq V(\widehat{b}(N)) + (1 - \varepsilon_1) \log N + A. \quad (76)$$

Hence,

$$\max_{\widehat{x}_0 \leq \ell \leq k \leq \widehat{L}^+ - 1, k \geq \widehat{b}(N)} [V(k) - V(\ell)] \leq \max_{[\widehat{b}(N), \widehat{L}^+]} V - \min_{[\widehat{x}_0, \widehat{L}^+]} V \leq (1 - \varepsilon_1) \log N + 2A.$$

This, (7), Markov's inequality and $\omega \in \Delta_N^{(3)}$ give

$$P_\omega^{\widehat{b}(N)}[\tau(\widehat{x}_0) \wedge \tau(\widehat{L}^+) > N^{1-\varepsilon_1/2}] \leq N^{-(1-\varepsilon_1/2)} \varepsilon_0^{-1} 4\varepsilon_3^{-2} (\log N)^4 N^{1-\varepsilon_1} e^{2A} \leq N^{-\varepsilon_1/4}$$

uniformly for large N . Moreover by (6), (71), (76) and $\omega \in \Delta_N^{(3)}$,

$$P_\omega^{\widehat{b}(N)}[\tau(\widehat{x}_0) < \tau(\widehat{L}^+)] \leq (\widehat{L}^+ - \widehat{b}(N)) \exp \left[\max_{[\widehat{b}(N), \widehat{L}^+]} V - V(\widehat{x}_0) \right] \leq \frac{(\log N)^2 e^{2A}}{\varepsilon_3 N^{2\varepsilon_1}} \leq \frac{1}{N^{\varepsilon_1/4}}$$

uniformly for large N . Consequently,

$$\begin{aligned} Q_\omega[\tau(\widehat{L}^+) > N^{1-\varepsilon_1/2}] &= P_\omega^{\widehat{b}(N)}[\tau(\widehat{L}^+) > N^{1-\varepsilon_1/2}] \\ &\leq P_\omega^{\widehat{b}(N)}[\tau(\widehat{x}_0) \wedge \tau(\widehat{L}^+) > N^{1-\varepsilon_1/2}] + P_\omega^{\widehat{b}(N)}[\tau(\widehat{x}_0) < \tau(\widehat{L}^+)] \\ &\leq 2N^{-\varepsilon_1/4}. \end{aligned}$$

We prove similarly that $Q_\omega[\tau(\widehat{L}^-) > N^{1-\varepsilon_1/2}] \leq 2N^{-\varepsilon_1/4}$ uniformly for large N , using (8) and (70) instead of (7) and (71) respectively, and because $\min_{[\widehat{x}_0, \widehat{x}_2]} V \geq V(\widehat{b}(N)) - A$ which we proved after (75). This proves Lemma 5.8. \blacksquare

Lemma 5.9. *For large N ,*

$$\forall \omega \in \Delta_N \cap \Delta_N^{(R)}, \quad \widehat{\nu}([\widehat{x}_0, \widehat{L}^-]) + \widehat{\nu}([\widehat{L}^+, \widehat{x}_2]) \leq N^{-\varepsilon_1/4}.$$

Proof. Let $N \geq N_2$ and $\omega \in \Delta_N \cap \Delta_N^{(R)}$. Recall that $\widehat{x}_0 \leq \widehat{L}^- < \widehat{b}(N) < \widehat{L}^+ \leq \widehat{x}_2$, which is proved before (74). Notice that $\widehat{L}^- \leq x_1 \leq \widehat{L}^+$, which is proved similarly as $x_1 \leq \theta_N^{(R)}$ after (65). Using the same method as for (75) with \widehat{L}^+ instead of $\theta_N^{(R)}$, we get $V \geq V(\widehat{b}(N)) + (\varepsilon_1/3) \log N$ on $[\widehat{L}^+, \widehat{x}_2]$. Also, $V(\widehat{L}^+ - 1) \geq V(\widehat{b}(N)) + (\varepsilon_1/3) \log N$. Since $\widehat{\mu}(2\mathbb{Z}) \geq e^{-V(\widehat{b}(N))}$, this leads to

$$\widehat{\nu}([\widehat{L}^+, \widehat{x}_2]) \leq [\widehat{x}_2 - \widehat{L}^+ + 2] e^{-V(\widehat{b}(N))} N^{-\varepsilon_1/3} / \widehat{\mu}(2\mathbb{Z}) \leq 3\varepsilon_3^{-1} (\log N)^2 N^{-\varepsilon_1/3} \leq N^{-\varepsilon_1/4} / 2$$

uniformly for large N , where we used $\omega \in \Delta_N^{(3)}$. We prove similarly that $\widehat{\nu}([\widehat{x}_0, \widehat{L}^-]) \leq N^{-\varepsilon_1/4} / 2$ uniformly for large N , which ends the proof of the lemma. \blacksquare

Lemma 5.10. *There exists $N_4 \in \mathbb{N}$ such that for $N \geq N_4$ for every $\omega \in \Delta_N \cap \Delta_N^{(R)}$,*

$$Q_\omega[\tau_{\widehat{Z}=Z} > N^{1-\varepsilon_1/2}] \leq 5N^{-\varepsilon_1/4} \quad (77)$$

and

$$Q_\omega[\tau_{\text{exit}} \leq N] \leq Q_\omega[\tau(\widehat{x}_0) \wedge \tau(\widehat{x}_2) < N] = P_\omega^{\widehat{b}(N)}[\tau(\widehat{x}_0) \wedge \tau(\widehat{x}_2) < N] \leq 2e^{2A} N^{-\varepsilon_1}. \quad (78)$$

Proof. Due to Lemma 5.8, we have for large N for all $\omega \in \Delta_N \cap \Delta_N^{(R)}$,

$$\begin{aligned} &Q_\omega[\tau_{\widehat{Z}=Z} > N^{1-\varepsilon_1/2}] \\ &\leq Q_\omega[\tau(\widehat{L}^-) \vee \tau(\widehat{L}^+) < \tau_{\widehat{Z}=Z}] + Q_\omega[\tau(\widehat{L}^-) \vee \tau(\widehat{L}^+) > N^{1-\varepsilon_1/2}] \\ &\leq Q_\omega[\tau(\widehat{L}^-) < \tau_{\widehat{Z}=Z}, \widehat{Z}_0 < \widehat{b}(N)] + Q_\omega[\tau(\widehat{L}^+) < \tau_{\widehat{Z}=Z}, \widehat{Z}_0 \geq \widehat{b}(N)] + 4N^{-\varepsilon_1/4}. \end{aligned}$$

Notice that under Q_ω , $Z_0 = \widehat{b}(N) \in (2\mathbb{Z})$ by (73) and (64), and $\widehat{Z}_0 \in (2\mathbb{Z})$ by (73) and (72). So the process $(\widehat{Z}_k - Z_k)_{k \in \mathbb{N}}$ starts at $(\widehat{Z}_0 - \widehat{b}(N)) \in (2\mathbb{Z})$ and only makes jumps belonging to $\{-2, 0, 2\}$, and thus up to time $\tau_{\widehat{Z}=Z} - 1$ it is < 0 (resp. > 0) on $\{\widehat{Z}_0 < \widehat{b}(N)\}$ (resp. $\{\widehat{Z}_0 \geq \widehat{b}(N)\}$), and in particular at time $\tau(\widehat{L}^-)$ on $\{\tau(\widehat{L}^-) < \tau_{\widehat{Z}=Z}, \widehat{Z}_0 < \widehat{b}(N)\}$

(resp. at time $\tau(\widehat{L}^+)$ on $\{\tau(\widehat{L}^+) < \tau_{\widehat{Z}=Z}, \widehat{Z}_0 \geq \widehat{b}(N)\}$). This gives for large N for all $\omega \in \Delta_N \cap \Delta_N^{(R)}$,

$$\begin{aligned}
& Q_\omega[\tau_{\widehat{Z}=Z} > N^{1-\varepsilon_1/2}] \\
& \leq Q_\omega[\tau(\widehat{L}^-) < \tau_{\widehat{Z}=Z}, \widehat{Z}_{\tau(\widehat{L}^-)} < \widehat{L}^-] + Q_\omega[\tau(\widehat{L}^+) < \tau_{\widehat{Z}=Z}, \widehat{Z}_{\tau(\widehat{L}^+)} > \widehat{L}^+] + 4N^{-\varepsilon_1/4} \\
& \leq Q_\omega[\tau(\widehat{L}^-) < \tau_{\widehat{Z}=Z}, \widehat{Z}_{2\lceil\tau(\widehat{L}^-)/2\rceil} \leq \widehat{L}^-] + Q_\omega[\tau(\widehat{L}^+) < \tau_{\widehat{Z}=Z}, \widehat{Z}_{2\lceil\tau(\widehat{L}^+)/2\rceil} \geq \widehat{L}^+] \\
& \quad + 4N^{-\varepsilon_1/4} \\
& \leq \widehat{\nu}([\widehat{x}_0, \widehat{L}^-]) + \widehat{\nu}([\widehat{L}^+, \widehat{x}_2]) + 4N^{-\varepsilon_1/4},
\end{aligned}$$

where the last inequality comes from the fact that $Q_\omega(\widehat{Z}_{2k} = x) = P_\omega^\widehat{Z}(\widehat{Z}_{2k} = x) = \widehat{\nu}(x)$ for every $x \in \mathbb{Z}$ and every (deterministic) $k \in \mathbb{N}$ as explained after (72), and from the independence of \widehat{Z} with Z and then $\tau(\cdot)$ up to $\tau_{\widehat{Z}=Z}$. Now, applying Lemma 5.9, this gives (77) for large N for every $\omega \in \Delta_N \cap \Delta_N^{(R)}$.

Due to (73) and Lemma 5.7, for large N for every $\omega \in \Delta_N \cap \Delta_N^{(R)}$, (78) holds. \blacksquare

Proof of Proposition 5.2. Recall that we have fixed $\delta \in (0, 1)$ and that (60) comes from Lemma 5.5. Let us prove (61). To this aim, we fix $(y_1, \dots, y_r) \in (2\mathbb{Z})^r$. Let $N_1 \in \mathbb{N}$ be such that $N_1 \geq \max(N_2, N_3, N_4)$ and such that for every $N \geq N_1$, $\varepsilon_3^{-1}(\log N)^2[N^{-\varepsilon_5/2} + 2N^{-2\varepsilon_1}] \leq \varepsilon_6^{-1}$, $N^{-(\varepsilon_1 \wedge \varepsilon_2 \wedge \varepsilon_4)/4} \leq 1/8$, $N^{1-\varepsilon_1/3} \geq N^{1-\varepsilon_1} + N^{1-\varepsilon_1/2}$ and $5N^{-\varepsilon_1/4} + 2e^{2A}N^{-\varepsilon_1} \leq \varepsilon_6 e^{-A}/6$. Now, we would like to give a lower bound for $P_\omega^{y_j}[Z_n = \widehat{b}(N)]$ for n even. Let $N \geq N_1$, $\omega \in \Delta_N \cap \Delta_N^{(R)}$, $j \in \{1, \dots, r\}$, and $n \in (2\mathbb{N})$, with $n \in [N^{1-\varepsilon_1} + N^{1-\varepsilon_1/2}, N]$. We have by the strong Markov property,

$$\begin{aligned}
P_\omega^{y_j}[Z_n = \widehat{b}(N)] & \geq P_\omega^{y_j}[Z_n = \widehat{b}(N), \tau(\widehat{b}(N)) \leq N^{1-\varepsilon_1}] \\
& = E_\omega^{y_j}[\mathbf{1}_{\{\tau(\widehat{b}(N)) \leq N^{1-\varepsilon_1}\}} P_\omega^{\widehat{b}(N)}(Z_k = \widehat{b}(N))_{|k=n-\tau(\widehat{b}(N))}] \\
& \geq P_\omega^{y_j}[\tau(\widehat{b}(N)) \leq N^{1-\varepsilon_1}] \inf_{k \in [N^{1-\varepsilon_1/2}, N] \cap (2\mathbb{N})} P_\omega^{\widehat{b}(N)}(Z_k = \widehat{b}(N)) \\
& \geq (1 - N^{-(\varepsilon_1 \wedge \varepsilon_2)/4} - N^{-\varepsilon_1/4}) \inf_{k \in [N^{1-\varepsilon_1/2}, N] \cap (2\mathbb{N})} P_\omega^{\widehat{b}(N)}(Z_k = \widehat{b}(N)) \quad (79)
\end{aligned}$$

because $\widehat{b}(N)$ and y_j are even (see (64)) and then $\tau(\widehat{b}(N))$ is also even under $P_\omega^{y_j}$, and where we used Lemma 5.6 in the last line. Moreover, for $k \in [N^{1-\varepsilon_1/2}, N] \cap (2\mathbb{N})$,

$$\begin{aligned}
P_\omega^{\widehat{b}(N)}(Z_k = \widehat{b}(N)) & = Q_\omega(Z_k = \widehat{b}(N)) \\
& \geq Q_\omega(Z_k = \widehat{b}(N), \tau_{\widehat{Z}=Z} \leq N^{1-\varepsilon_1/2}, \tau_{exit} > N) \\
& = Q_\omega(\widehat{Z}_k = \widehat{b}(N), \tau_{\widehat{Z}=Z} \leq N^{1-\varepsilon_1/2}, \tau_{exit} > N) \\
& \geq Q_\omega(\widehat{Z}_k = \widehat{b}(N)) - Q_\omega(\tau_{\widehat{Z}=Z} > N^{1-\varepsilon_1/2}) - Q_\omega(\tau_{exit} \leq N) \\
& \geq \widehat{\nu}(\widehat{b}(N)) - 5N^{-\varepsilon_1/4} - 2e^{2A}N^{-\varepsilon_1}, \quad (80)
\end{aligned}$$

where we used (73) in the first and last line, $Z_k = \widehat{Z}_k$ for $k \in [\tau_{\widehat{Z}=Z}, \tau_{exit})$ under Q_ω in the third line, and $Q_\omega(\widehat{Z}_k = x) = P_\omega^\widehat{Z}(\widehat{Z}_k = x) = \widehat{\nu}(x)$ since k is even, (77) and (78) in the last line since $N \geq N_4$.

Notice that $\widehat{\mu}(2\mathbb{Z}) = e^{-V(\widehat{b}(N))} \sum_{i=\widehat{x}_0}^{\widehat{x}_2-1} e^{-[V(i)-V(\widehat{b}(N))]}$, with

$$\sum_{i=\widehat{x}_0}^{-1} e^{-[V(i)-V(\widehat{b}(N))]} \leq |\widehat{x}_0| N^{-\varepsilon_5/2} \leq \varepsilon_3^{-1} (\log N)^2 N^{-\varepsilon_5/2} \leq \varepsilon_6^{-1}$$

since $N \geq N_1$, $\omega \in \Delta_N^{(3)}$ and thanks to (74).

Moreover, by (75), $\sum_{i=\theta_N^{(R)}}^{\widehat{x}_2-1} e^{-[V(i)-V(\widehat{b}(N))]} \leq 2\varepsilon_3^{-1} (\log N)^2 N^{-2\varepsilon_1} \leq \varepsilon_6^{-1}$ because $N \geq N_1$.

Finally, $\sum_{i=0}^{\theta_N^{(R)}-1} e^{-[V(i)-V(\beta_N^{(R)})]} \leq \varepsilon_6^{-1}$ since $\omega \in \Delta_N^{(6,R)}$ (see (63)). Moreover, $|V(\widehat{b}(N)) - V(\beta_N^{(R)})| \leq A$. Hence, $\widehat{\mu}(2\mathbb{Z}) \leq 3\varepsilon_6^{-1} e^A e^{-V(\widehat{b}(N))}$. Moreover, $\widehat{\mu}(\widehat{b}(N)) \geq e^{-V(\widehat{b}(N))}$ since $\widehat{x}_0 < \widehat{b}(N) < \widehat{x}_2$, and $\widehat{b}(N)$ is even by (64), so by (72), $\widehat{\nu}(\widehat{b}(N)) = \widehat{\mu}(\widehat{b}(N))/\widehat{\mu}(2\mathbb{Z}) \geq \varepsilon_6 e^{-A}/3$. This, (79) and (80) give for $N \geq N_1$,

$$\forall \omega \in \Delta_N \cap \Delta_N^{(R)}, \forall n \in [N^{1-\varepsilon_1/3}, N] \cap (2\mathbb{N}), \forall j \in \{1, \dots, r\}, \quad P_\omega^{y_j} [Z_n = \widehat{b}(N)] \geq \varepsilon_6 e^{-A}/8.$$

The proof is similar for $\omega \in \Delta_N \cap \Delta_N^{(L)}$ by symmetry. This, combined with Lemma 5.5, ends (61) with $c(\delta) = \varepsilon_6 e^{-A}/8 > 0$ and $\varepsilon(\delta) = \varepsilon_1/3$. To prove that this remains true if $(2\mathbb{Z})^r$ and $2\mathbb{N}$ are replaced respectively by $(2\mathbb{Z}+1)^r$ and $2\mathbb{N}+1$, we just condition $P_\omega^{y_j} [Z_n = \widehat{b}(N)]$ by Z_1 , and apply the Markov property and (61) to $(y_1 \pm 1, \dots, y_r \pm 1)$. ■

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