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Two models for yeast cell communication

Vincent Calvez ^{*}, Thomas Lepoutre [†] Nicolas Meunier Nicolas Muller [‡]

March 30, 2017

Abstract

We study two separate models for yeast cell-cell communication. Each model consists of a coupled system of two non-linear, non-local equations in one dimension of space. The first model describes the two cells in the transversal direction, orthogonal to the membrane, whereas the second model describes the two cells in the tangential direction, along the membrane. We study long time dynamics for each model, ranging from convergence to stable steady states (bistability in model 1), to possible finite time blow-up (formation of singularity in model 2). The biological interpretation of the mathematical results is discussed.

1 Introduction

In this work we propose and analyze two models describing some aspects of yeast cell-cell communication. Each of the two models is based on a system of coupled non-linear and non-local convection-diffusion equations. In both cases the convection-diffusion equation, without coupling, was introduced and studied in previous works to describe protein dynamics in a single yeast cell [12, 4, 2, 16, 20]. Here, from the mathematical viewpoint, in both models, the novelty is the coupling between two non-linear and non-local convection-diffusion equations.

1.1 The transversal model

Let us start with the first model which writes:

$$\partial_t n_i = \partial_{xx} n_i + \chi \mu_1 \mu_2 \partial_x n_i, \quad (1)$$

for $i = 1, 2$, where $n_i = n_i(t, x)$ is defined on $t \geq 0, x \geq 0$, with an attachment and detachment kinetic at the boundaries located at $x = 0$:

$$\frac{d}{dt} \mu_i(t) = n_i(t, 0) - \mu_i(t), \quad t > 0, \quad (2)$$

and flux boundary conditions:

$$\frac{d}{dt} \mu_i(t) = \partial_x n_i(t, 0) + \chi \mu_1(t) \mu_2(t) n_i(t, 0), \quad t > 0, \quad (3)$$

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which ensure the conservation, in each cell, of the following quantity

$$M_i := \mu_i(t) + \int_0^\infty n_i(t, x) dx. \quad (4)$$

We refer to Section 2 for a detailed presentation of the model with biological motivations.

Since the advection is bounded, $\mu_i(t) \leq M_i$, global existence of solutions to the Cauchy problem (1)–(2)–(3) holds true. Here, our aim is to precise the long time behaviour. Since there is no comparison principle on equation (1), our method is based on a concentration-comparison principle that is obtained when equation (1) is integrated in space, see [16]. This principle allows constructing some remarkable sub/supersolutions and performing a non-linear stability analysis.

Before stating the results, we give some notations. First for simplicity, throughout this work we will assume that

$$M_1 = M_2 = 1. \quad (5)$$

Moreover, for $i = 1, 2$ and for $t \geq 0, x \geq 0$, let the function $N_i(t, x)$ be defined by

$$N_i(t, x) = \mu_i(t) + \int_0^x n_i(t, y) dy, \quad (6)$$

where $(n_i, \mu_i)_{1 \leq i \leq 2}$ is the solution to (1)–(2)–(3) with (5). For $\mu \in (0, 1)$ let the function N_μ be defined by

$$N_\mu(x) = \mu + (1 - \mu)(1 - e^{-\chi\mu^2 x}). \quad (7)$$

Furthermore, let P be the polynomial

$$P(m) = \chi m^2 - \chi m + 1, \quad (8)$$

whose roots are real numbers $\mu_- < \mu_+$ if $\chi > 4$. Finally, let n_+ be the function defined by

$$n_+(x) = \chi\mu_+^2 e^{-\chi\mu_+^2 x}. \quad (9)$$

We start with a linear stability result. In this work we assume that $\chi \geq 4$, which is the interesting case for which there exists non trivial stationary states. It is to be noticed that χ is linked to intra-cellular distance as it is discussed below after the second model.

Proposition 1.1 (Linear stability) *If $\chi \geq 4$, then the system (1)–(2)–(3) and (5) admits a steady state, denoted by $(\bar{n}_i, \bar{\mu}_i)_{i=1,2}$. Any steady state satisfies $\bar{n}_1 = \bar{n}_2$ and $\bar{\mu}_1 = \bar{\mu}_2$. If $\chi > 4$, then, there are two steady states (\bar{n}_-, μ_-) and (\bar{n}_+, μ_+) with $\mu_- < \mu_+$. Furthermore, (\bar{n}_+, μ_+) is linearly stable while (\bar{n}_-, μ_-) is linearly unstable.*

In addition we perform a non-linear stability analysis. For large enough initial conditions, we prove the convergence of the solution to (1)–(2)–(3) with (5) towards the steady state (\bar{n}_+, μ_+) .

Proposition 1.2 (Non-linear stability of the largest equilibrium) *Let*

$(n_i, \mu_i)_{1 \leq i \leq 2}$ be the solution to (1)–(2)–(3) with (5) and with initial data $(n_i^0, \mu_i^0)_{1 \leq i \leq 2}$ with finite entropy $\int_0^\infty n_i^0(x + \log n_i^0) dx < +\infty$. Assume that $\chi > 4$ and that there exists two real numbers $(\underline{\mu}, \bar{\mu}) \in (0, 1)^2$ such that

$$\begin{cases} \mu_- < \underline{\mu} < \mu_+ < \bar{\mu}, \\ N_{\underline{\mu}}(x) \leq N_i(0, x) \leq N_{\bar{\mu}}(x) \quad \text{for all } x \geq 0 \text{ and } i = 1, 2, \end{cases}$$

and assume in addition that

$$\mu_i^0 \in (\underline{\mu}, \bar{\mu}) \quad i = 1, 2.$$

Then, for $i = 1, 2$, the convergence of (n_i, μ_i) towards the steady state (\bar{n}_+, μ_+) holds true in the following sense:

$$\begin{cases} \lim_{t \rightarrow \infty} \mu_i(t) = \mu_+, \\ \lim_{t \rightarrow \infty} \|n_i(t, \cdot) - (1 - \mu_+)n_+(\cdot)\|_{L^1(\mathbb{R}_+)} = 0, \end{cases}$$

where n_+ is defined by (9).

On the other hand, for small initial conditions we prove the self-similar behaviour of the solution. Let G be the normalized Gaussian distribution on the half line

$$G(x) = \sqrt{\frac{2}{\pi}} e^{-\frac{x^2}{2}}. \quad (10)$$

Proposition 1.3 (Non-linear stability of the zero steady state) Assume that $\chi > 4$ and that there exists $\mu_0 \in (0, \mu_-)$ such that for $i = 1, 2$

$$\mu_i(0) < \mu_0 \quad \text{and} \quad \forall x \geq 0, \quad N_i(0, x) \leq N_{\mu_0}(x),$$

and assume in addition that $\int_0^\infty x^2 n_i(0, x) dx < +\infty$. Then, the following convergences hold true:

$$\text{for } i = 1, 2 \quad \begin{cases} \lim_{t \rightarrow \infty} \mu_i(t) = 0, \\ \lim_{t \rightarrow \infty} \left\| n_i(t, \cdot) - \frac{1}{\sqrt{1+2t}} G\left(\frac{\cdot}{\sqrt{1+2t}}\right) \right\|_{L^1(\mathbb{R}_+)} = 0, \end{cases}$$

with an exponential rate for the second one.

In the context of yeast cell communication, the main interest of this first model is to link the output of cell communication to protein aggregation on both cell membranes. The next Table summarizes the long time dynamics results contained in Proposition 1.1 and Proposition 1.2 in a informal way. Notice the bistability of the communicating state (for large μ) vs. the silent state (for small μ). Alternatively speaking, when the cells do not invest enough in the communication, they do not respond to each other, and no dialog can take place between them.

μ	$0 < \mu < \mu_-$	μ_-	$\mu_- < \mu < \mu_+$	μ_+	$\mu_+ < \mu$
N_μ	supersolution	lin. instable steady state	subsolution	lin. stable steady state	supersol.
Cv	If $N_i(0, x) < N_{\mu_-}(x)$, $n_i \rightarrow 0$ and $\mu_i \rightarrow 1$ no communication		If $N_{\mu_-}(x) < N_i(0, x)$, $n_i \rightarrow (1 - \mu_+)n_+$ cell communication		

Table 1: Summary of the dynamics when $\chi > 4$

1.2 The tangential model

Let us now present the second model that will be designed as the tangential model since it focuses on the membrane dynamics. Let $\Omega_1 = (x \in \mathbb{R}, y \leq 0)$, $\Omega_2 = (x \in \mathbb{R}, y \geq h)$, $\Gamma_1 = (x \in \mathbb{R}, y = 0)$, $\Gamma_2 = (x \in \mathbb{R}, y = h)$, and Ψ be the intercellular space, see Fig. 1. For $i = 1, 2$, $\mu_i(t, x)$ is now a function of both time and space and it satisfies

$$\partial_t \mu_i = \partial_{xx} \mu_i + \partial_x (\mu_i \mathcal{H}(S_j \mu_i)), \quad \text{on } \Gamma_i, \quad (11)$$

where \mathcal{H} is the Hilbert Transform, whose definition is

$$\mathcal{H}(f)(x) := \frac{1}{\pi} \text{p.v.} \int_{\mathbb{R}} \frac{f(y)}{x-y} dy = \frac{1}{\pi} \lim_{\varepsilon \rightarrow 0} \int_{|x-y| > \varepsilon} \frac{f(y)}{x-y} dy, \quad (12)$$

and where S_j satisfies the following elliptic equation

$$\begin{cases} -\Delta S_j + \lambda S_j = 0, & \text{on } \Psi, \\ \nabla S_j \cdot \mathbf{e}_j = \mu_j, & \text{on } \Gamma_j, \end{cases} \quad (13)$$

where \mathbf{e}_j denotes the unit outward normal vector to Γ_j .

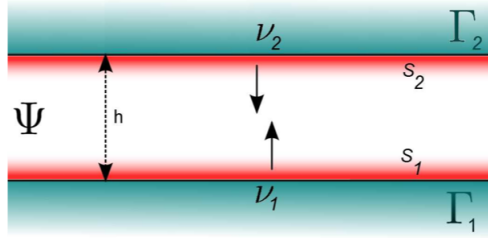


Figure 1: Tangential model for the communication between two cells

In the two dimensional case the solution to (13) can be explicitly computed. For all $t > 0$ and $x \in \mathbb{R}$, one has on Γ_i :

$$S_j(t, x) = \frac{1}{\pi} \int_{\mathbb{R}} F(\sqrt{(x-x')^2 + h^2}) \mu_j(t, x') dx', \quad \text{for } j \in \{1, 2\}, \quad (14)$$

where F is a decreasing nonnegative function on \mathbb{R}^+ , vanishing at $+\infty$. Indeed, we know that $F(r) = C \int_0^\infty e^{-t - \frac{r^2}{4t}} dt$, see [10], where C is a positive constant whose value is given in the appendix 8.

Let F_h and f_h be the functions defined by

$$F_h(x) = f_h(x^2) = F\left(\sqrt{x^2 + h^2}\right), \quad (15)$$

then, $y \rightarrow f_h(y)$ is a convex function on \mathbb{R}^+ . Indeed one has $f_h(u) = \int_0^\infty e^{-t - \frac{h^2}{4t}} e^{-\frac{u}{4t}} dt$ and differentiating twice we obtain

$$f_h''(u) = \int_0^\infty \frac{1}{16t^2} e^{-t - \frac{h^2}{4t}} e^{-\frac{u}{4t}} dt \geq 0.$$

Since F_h is smooth, let $K(h)$ be defined by $K(h) := \left\| \frac{d}{dx} F_h \right\|_\infty < +\infty$.

For this tangential model, (11)–(13), we can expect blow-up of the solution. This can be seen by performing a formal computation on the time derivative of the joint second moment defined by

$$\mathcal{E}(t) := \iint_{\mathbb{R}^2} \frac{(x-y)^2}{4M^2} \mu_1(t, x) \mu_2(t, y) dx dy > 0, \quad (16)$$

and by proving that \mathcal{E} cannot remain positive for all time, which is an obstruction to global existence. Such a technique was first used by Nagai [21], then by many authors in various contexts (see [22] and references therein).

Proposition 1.4 *Assume that $f_h(0) < \frac{2\pi}{M^2}$, where f_h is defined by (15). Assume in addition that $\mathcal{E} = \mathcal{E}(t=0)$ satisfies*

$$\frac{M^4 K(h)}{\pi} \sqrt{\mathcal{E}} - \frac{M^4}{2\pi} f_h(\mathcal{E}) \leq \frac{M^2 f_h(0)}{2\pi} - 1.$$

Then, there is an obstruction to global existence of the solution (μ_1, μ_2) to (11)–(13).

Such a criterium is in agreement with biological experiments in which it is observed that above a threshold-distance, beyond some distance, two cells cannot polarize simultaneously.

We describe the plan of the paper. First in Section 2 we justify the two models from a modelling point of view. Section 3 deals with notations and mathematical useful inequalities. In Section 4 we state a comparison principle. Section 5 is devoted to the proof of the non-linear stability of the larger steady state, Proposition 1.2. In Section 6, in the case of small initial conditions, we prove extinction, and convergence towards a self-similar profile in the rescaled coordinates, Proposition 1.3. Finally we study in Section 7 the tangential model and we prove Proposition 1.4. Since this work is focused on the non-linear stability analysis, we postpone in the appendix the linear stability analysis and the proof of Proposition 1.1.

2 Motivation

How do cells communicate with each other? This question, which seems simple, has not been answered so far. Cell communication plays a fundamental role in many cellular processes including cell division and differentiation, directional movement as well as morphogenesis. Defects in cell-cell communication are also implied in the development of cancer.

From the biological point of view, a prototypical model for cell communication is given by yeast cell mating. Yeast cell communication involves some intra-cellular proteins (Cdc42), the cell cytoskeleton and extra-cellular pheromone molecules, Fig. 2.

Several studies have proposed mathematical models that incorporate many aspects of the molecular mechanisms involved in pheromone-induced protein aggregation. Although some of these models have been tested for their ability to fit quantitative data [9, 24, 11, 19, 7, 18, 6, 15], they have not been quantitatively assessed for their ability to make accurate predictions with no additional free parameter. Here we will use a model which was first introduced in [12], then studied in [4, 2] and finally tested for its ability to predict experimental data in [20]. This model relies on a coarse-grained description of the cytoskeleton and it is expressed by a non-linear and non-local partial differential equation. In the present work we enrich this model in order to study cell-cell communication.

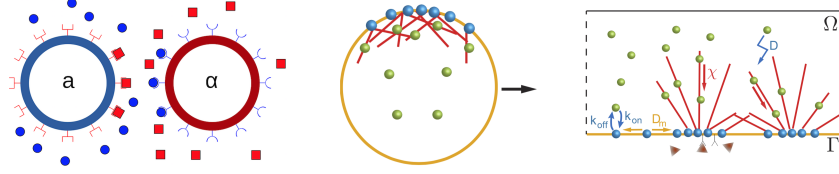


Figure 2: Model for yeast cell communication. On the left, yeast cells of both types secrete some pheromone (\mathbf{a} or α) and bear a pheromone receptor to detect the pheromone produced by the cells of the opposite type. On the middle and on the right a two-dimensional model of protein dynamics inside each cell. The middle panel shows a cell, the right a more detailed view. Actin is polymerized into short filaments, that interact with each other and these are bundled together to form actin cables (which form the cytoskeleton) that cross the cell. The nucleation of filaments is proportional to both the local density of Cdc42 (the proteins that are transported by the cell cytoskeleton in each cell) and to the concentration of pheromone.

Denoting respectively by n and c the concentrations of Cdc42 and actin filaments in the cytoplasm of the cell, which is described by a bounded domain $\Omega \subset \mathbb{R}^2$, and denoting by μ the Cdc42 concentration on the boundary of the cell, denoted by Γ , the model is:

$$\begin{cases} \partial_t n = \Delta n - \chi \nabla \cdot (n \nabla c), & \text{on } \Omega, \\ \partial_t \mu = \partial_{ss} n + n - \mu, & \text{on } \Gamma, \end{cases} \quad (17)$$

where s is a parametrisation of the boundary Γ .

In this model, active transport of proteins is modeled as the gradient of the concentration of actin filaments, $\chi \nabla c$.

Nucleation of new filaments is assumed to occur at the plasma membrane, under the combined action of Cdc42 and pheromone molecules. After a dimensional analysis, the model that describes the cytoskeletal density is:

$$\begin{cases} -\Delta c = 0, & \text{on } \Omega, \\ -\nabla c \cdot \mathbf{e} = S\mu, & \text{on } \Gamma, \end{cases} \quad (18)$$

where \mathbf{e} is the unit outward normal vector and S is the pheromone-generated signal trace on the cell membrane.

Equations (17) and (18) are complemented by initial conditions and by an additional boundary condition on the cell membrane which guarantees the conservation of the total Cdc42 pool on each cell:

$$(\nabla n - \chi n \nabla c) \cdot \mathbf{e} = -\partial_t \mu \quad \text{on } \Gamma. \quad (19)$$

In the one dimensional case where the cytoplasm of the cell is modelled by the half line $x > 0$ and the membrane is located at $x = 0$, the model (17)–(18) simply rewrites as

$$\partial_t n = \partial_{xx} n + S\mu \partial_x n, \quad t > 0, \quad x > 0, \quad (20)$$

with an additional flux boundary condition that assures mass conservation. This latter equation has been mathematically studied in [4, 2, 16], its dynamics is well understood and is reminiscent of the Keller-Segel model in two dimensions. The principal result of [2] was to identify regimes in which non homogeneous stationary states, that were interpreted as polarised states, emerge.

In the more realistic two dimensional case, numerical simulations, [3], show that for large enough values of the total cell protein pool, the majority of the Cdc42 molecules are located in the neighborhood of the cell membrane. Hence postulating that $\int n(t, \mathbf{x}) d\mathbf{x}_\perp = \mu(t, s)$, the dynamics of μ can be formally written by integrating equation (17) with respect to the normal coordinate \mathbf{x}_\perp to the membrane Γ . This yields a one dimensional non-linear and non-local convection diffusion equation in which the advective field is given by the Hilbert transform \mathcal{H} , a non-local operator:

$$\partial_t \mu = \partial_{ss} \mu + \chi \partial_s (\mu \mathcal{H}(S\mu)) , \quad (21)$$

where, s is a parametrisation of the boundary Γ . This latter equation is known to have a solution which aggregates in a finite time (blow-up) if $\chi \int_\Gamma \mu$ is large enough as compared to the value of S , see [5]. In our setting, this analysis means that for small values of the total mass, $\int_\Gamma \mu$, the cell remains unpolarised while for large values it gets polarized.

In nature, the budding yeast, *Saccharomyces cerevisiae*, exists as haploid cells in two types (**a** and α). Cells of both types secrete some pheromone (**a** or α), Fig. 2, and bear a pheromone receptor to detect the pheromone produced by the cells of the opposite type, [13]. In the present work we propose two models where the production of extra-cellular pheromone (**a** or α) is function of the concentration of the protein Cdc42 at the membrane. Furthermore according to biological literature, [17, 1, 25, 26], we assume that the pheromone contributes to the nucleation of new filaments at the plasma membrane of the cell of the opposite type, see Fig 2. To describe the protein dynamics on each cell membrane we use and enrich either the model (17)–(18) in the one-dimensional case or the tangential model (21). This yields to two systems of two coupled one-dimensional non-linear and non-local convection-diffusion equations. Throughout this work, for simplicity, we will denote by the subscript $i = 1, 2$ the cell type and we will assume that the total protein pool of each cell is equal to 1.

2.1 The transversal model

To simplify, the cytoplasm of each cell is modelled by the half line, the membranes are localized at the boundaries.

In the one-dimensional version of (17)–(18) the advection field is simply $-\chi\mu(t)S(t)$. According to the previously described biological scenario, in the cell of type i , we consider that the advection field is $-\chi\mu_i(t)S_j(t)$ where S_j is the concentration of pheromone produced by the cell of the opposite type j . Moreover, in this work we assume that $S_j = \mu_j$, which means that the pheromone produced by cell of type i is equal to the concentration of proteins Cdc42 attached at its membrane multiplied by a damping factor, χ , which depends on the inter-cellular distance. Hence, in both cells the advection field is $-\chi\mu_1(t)\mu_2(t)$. Hence the first system of coupled equations is (1)–(2)–(3). From the modelling point of view this first model describes whether, in each cell, proteins Cdc42 are located in the cytoplasm, the bulk of the cell, or aggregates on the boundary, the cell membrane, this latter case will be referred as a polarized state. The case where aggregation on the membrane occurs in both cell will be referred as stable dialog between the two cells.

2.2 The tangential model

We suppose that the two cells occupy respectively the two half-planes $\Omega_1 = (x \in \mathbb{R}, y \leq 0)$, $\Omega_2 = (x \in \mathbb{R}, y \geq h)$, see Fig. 1, that the two membranes are denoted by $\Gamma_1 = (x \in \mathbb{R}, y = 0)$ and $\Gamma_2 = (x \in \mathbb{R}, y = h)$, and that the intercellular space is Ψ .

In the one-cell case the tangential model writes as (21), the advection field is $\chi\mathcal{H}(S\mu)$. According to the previously described biological scenario, in the cell of type i , we consider that the advection

field is $\chi\mu_i\mathcal{H}(S_j\mu_i)$ where S_j is the concentration of pheromone produced by the cell of the opposite type, j . Hence the second model is (11) – (13). From the modelling point of view the second model describes whether, in each cell, proteins Cdc42 have an homogeneous distribution on the membrane or aggregates on given part of the boundary, this latter case will be referred as a polarized state. Since both membranes are lines, this model can provide information about both polarisomes (places where proteins aggregates) alignment. The situation where both polarisomes are at the same location will be referred as co-polarization or stable dialog between the two cells.

3 Notations and inequalities

Let us start by introducing some classical notations.

Definition 3.1 *Given two probability measures p, q on \mathbb{R}_+ , we define the relative entropy of p with respect to q by*

$$\mathcal{H}(p|q) = \int_0^\infty p(x) \log \frac{p(x)}{q(x)} dx = \int_0^\infty q(x) \left(\frac{p(x)}{q(x)} \log \frac{p(x)}{q(x)} - \frac{p(x)}{q(x)} + 1 \right) dx \geq 0.$$

The Fisher information of p with respect to q is defined as the quantity

$$\mathcal{I}(p|q) = \int_0^\infty p(x) \left(\partial_x \log \frac{p(x)}{q(x)} \right)^2 dx.$$

Moreover if p, q have finite second moment, [28], the quadratic Wasserstein distance $W(p, q)$ is defined by

$$W(p, q) = \inf_{\pi \in \Pi(p, q)} \sqrt{\iint_{\mathbb{R}_+ \times \mathbb{R}_+} |x - y|^2 d\pi(x, y)},$$

where $\Pi(p, q)$ denotes the set of probability measures on $\mathbb{R}_+ \times \mathbb{R}_+$ with marginals p and q .

There are several results concerning various representations for the quadratic Wasserstein distance when it is specialized to the real line. In such a case it may considerably be simplified in terms of the distribution functions $F(x) = \int_0^x p(y) dy$, $x \in (0, \infty)$, associated to probability measures p , [28].

Theorem 3.2 (Representation for W) *Let p and q be probability measures on \mathbb{R}^+ with respective distribution functions F and G . Then*

$$W(p, q) = \int_0^1 |F^{-1}(t) - G^{-1}(t)| dt,$$

where F^{-1} is the pseudo-inverse function:

$$F^{-1}(t) = \inf\{x \in \mathbb{R} : F(x) \geq t, 0 < t < 1\}.$$

In the sequel we will use two important inequalities linking the relative entropy and the Fisher information, see [28].

Lemma 3.3 (Log-Sob inequality) *Assume q has a Gaussian concentration i.e. $q(x) = e^{-V(x)}$ with $V''(x) \geq c > 0$, then the logarithmic Sobolev inequality holds true*

$$\mathcal{I}(p|q) \geq 2c\mathcal{H}(p|q).$$

Lemma 3.4 (HWI inequality for exponential measure) Assume that $q(x) = \lambda e^{-\lambda x}$ then the following inequality holds true

$$\mathcal{I}(p|q) \geq \left(\frac{\mathcal{H}(p|q)}{W(p, q)} \right)^2.$$

In order to obtain a rate of convergence for the L^1 norm we will use the Csiszár-Kullback inequality, [8, 14].

Proposition 3.5 (Csiszár-Kullback inequality) For any non-negative functions $f, g \in L^1(\mathbb{R}_+)$ such that $\int_{\mathbb{R}_+} f(x) dx = \int_{\mathbb{R}_+} g(x) dx = M$, we have that

$$\|f - g\|_1^2 \leq 2M \int_0^\infty f(x) \log \left(\frac{f(x)}{g(x)} \right) dx. \quad (22)$$

4 Comparison principle and consequences

We start noticing that there is no comparison principle on (1)–(2)–(3). In this section, we first establish a concentration comparison principle reminiscent of [16] on the quantities N_i whose definition we recall now

$$N_i(t, x) = \mu_i(t) + \int_0^x n_i(t, y) dy, \quad i = 1, 2, \quad (23)$$

where $(n_i, \mu_i)_{i=1,2}$ is the solution to (1)–(2)–(3) and (5). Then, in a second step we construct some remarkable sub/supersolutions and in the following two sections we use this principle to perform a non-linear stability analysis.

4.1 Concentration comparison principle

For $i = 1, 2$, the integrated equations associated with (1)–(2)–(3) and (5) are

$$\begin{cases} \partial_t N_i(t, x) - \partial_{xx} N_i(t, x) - \chi \mu_1(t) \mu_2(t) \partial_x N_i(t, x) = 0, \\ N_i(t, 0) = \mu_i(t), \quad \lim_{x \rightarrow \infty} N_i(t, x) = 1, \\ \frac{d}{dt} \mu_i(t) = \partial_x N_i(t, 0) - \mu_i(t). \end{cases} \quad (24)$$

We now define supersolution and subsolution to (24).

Definition 4.1 A supersolution (resp. subsolution) to (24) is a couple of nondecreasing functions (\bar{N}_1, \bar{N}_2) (resp. $(\underline{N}_1, \underline{N}_2)$) satisfying

$$\begin{cases} \partial_t \bar{N}_i(t, x) - \partial_{xx} \bar{N}_i(t, x) - \chi \bar{\mu}_1(t) \bar{\mu}_2(t) \partial_x \bar{N}_i(t, x) \geq 0, \\ \bar{N}_i(t, 0) = \bar{\mu}_i(t), \quad \lim_{x \rightarrow \infty} \bar{N}_i(t, x) = 1, \\ \frac{d}{dt} \bar{\mu}_i(t) \geq \partial_x \bar{N}_i(t, 0) - \bar{\mu}_i(t), \end{cases} \quad (25)$$

with similar definition for a subsolution by changing \geq into \leq .

We now state the concentration comparison principle.

Lemma 4.2 (Comparison principle) Let (\bar{N}_1, \bar{N}_2) and $(\underline{N}_1, \underline{N}_2)$ be respectively smooth super and subsolution to (24) defined on $[0, T] \times \mathbb{R}_+$. Assume that for $i = 1, 2$

$$\bar{N}_i(0, x) \geq \underline{N}_i(0, x), \quad \forall x \geq 0, \quad \text{and} \quad \bar{\mu}_i(0) > \underline{\mu}_i(0).$$

Then, for all $t \in (0, T)$, for all $x \geq 0$, one has

$$\bar{N}_1(t, x) \geq \underline{N}_1(t, x) \quad \text{and} \quad \bar{N}_2(t, x) \geq \underline{N}_2(t, x).$$

Proof For $i = 1, 2$, denoting $F_i = \bar{N}_i - \underline{N}_i$, one has

$$\begin{cases} \partial_t F_i - \partial_{xx} F_i - \chi \bar{\mu}_1(t) \bar{\mu}_2(t) \partial_x F_i \geq \chi \left(\bar{\mu}_1(t) \bar{\mu}_2(t) - \underline{\mu}_1(t) \underline{\mu}_2(t) \right) \partial_x \underline{N}_i, \\ F_i(t, 0) = \bar{\mu}_i(t) - \underline{\mu}_i(t), \quad \lim_{x \rightarrow \infty} F_i(t, x) = 0, \\ \frac{d}{dt} \left(\bar{\mu}_i(t) - \underline{\mu}_i(t) \right) \geq \partial_x F_i(t, 0) - (\bar{\mu}_i(t) - \underline{\mu}_i(t)), \\ F_i(0, x) = \bar{N}_i(0, x) - \underline{N}_i(0, x) \geq 0. \end{cases}$$

The bootstrap acts as follows. Since we are dealing with nondecreasing functions N_i , the quantity $\chi \left(\bar{\mu}_1(t) \bar{\mu}_2(t) - \underline{\mu}_1(t) \underline{\mu}_2(t) \right) \partial_x \underline{N}_i$ is nonnegative as long as

$$\bar{\mu}_1(t) \bar{\mu}_2(t) - \underline{\mu}_1(t) \underline{\mu}_2(t) \geq 0,$$

holds true. This is in particular the case if

$$\bar{\mu}_1(t) \geq \underline{\mu}_1(t) \quad \text{and} \quad \bar{\mu}_2(t) \geq \underline{\mu}_2(t). \quad (26)$$

Recalling the assumption $\bar{\mu}_i(0) > \underline{\mu}_i(0)$, we denote by $T > 0$ the first time for which an equality in (26) occurs. Let us say that $\bar{\mu}_1(T) = \underline{\mu}_1(T)$. Let us define the function $z(t)$ by

$$z(t) := \bar{\mu}_1(t) - \underline{\mu}_1(t).$$

Then, for all $(t, x) \in (0, T) \times (0, \infty)$, one has

$$\begin{cases} \partial_t F_1(t, x) - \partial_{xx} F_1(t, x) - \chi \bar{\mu}_1(t) \bar{\mu}_2(t) \partial_x F_1(t, x) \geq 0, \\ F_1(t, 0) = \bar{\mu}_1(t) - \underline{\mu}_1(t) = z(t) \geq 0, \quad \lim_{x \rightarrow \infty} F_1(t, x) = 0, \\ \frac{d}{dt} z(t) \geq \partial_x F_1(t, 0) - z(t). \end{cases}$$

Recalling in addition the assumptions $F_i(0, x) \geq 0$ for all $x \geq 0$ and that the functions are smooth, we can assume that there exists a nonnegative compactly (in $(0, +\infty)$) supported function f such that $f(x) \leq F_1(0, x)$ and $f(0) = 0$. Next, we consider the solution to the parabolic equation

$$\begin{cases} \partial_t g(t, x) - \partial_{xx} g(t, x) - \chi \bar{\mu}_1(t) \bar{\mu}_2(t) \partial_x g(t, x) = 0 & x \in (0, \infty), \\ g(t, 0) = 0, \\ g(0, x) = f(x). \end{cases}$$

On the first hand, by standard maximum principle, see [10], one deduces that $F_1 \geq g$ on $[0, T] \times [0, \infty)$. On the second hand by maximum principle again one can see that $g(t, x) > 0$ for all $(t, x) \in (0, T) \times (0, \infty)$. Hence, applying the Hopf Lemma, [10], it follows that $\partial_x g(T, 0) > 0$. Next, from the equality $F_1(T, 0) = g(T, 0) = 0$, we deduce that $\partial_x F_1(T, 0) \geq \partial_x g(T, 0) > 0$. Consequently one has $\frac{d}{dt} z(T) > 0$ which contradicts $z > 0$ on $[0, T)$ and $z(T) = 0$.

4.2 Remarkable sub/supersolutions

For $\mu \in (0, 1)$ and $m \geq 0$, the function $N_{\mu, m}$ is defined by

$$N_{\mu, m}(x) = \mu + (1 - \mu)(1 - e^{-mx}). \quad (27)$$

We first establish

Lemma 4.3 (remarkable static sub/supersolutions) *Assume that*

$$\frac{\mu_i}{1 - \mu_i} \geq m_i \geq \chi\mu_1\mu_2, \quad i = 1, 2. \quad (28)$$

Then, $(N_{\mu_1, m_1}, N_{\mu_2, m_2})$ is a supersolution to (24). Moreover similar result holds true for a subsolution by changing \geq into \leq .

Proof Differentiating (27) one obtains

$$\begin{cases} -N''_{\mu_i, m_i}(x) - \chi\mu_1\mu_2 N'_{\mu_i, m_i}(x) = (1 - \mu_i)e^{-m_i x} (m_i^2 - \chi\mu_1\mu_2 m_i), \\ N'_{\mu_i, m_i}(0) - \mu_i = (1 - \mu_i)m_i - \mu_i, \end{cases}$$

and the result follows from the definition 4.1.

A practical example of supersolution (resp. a subsolution) is the following. For $\mu \in (0, 1)$ let us recall the definition of N_μ :

$$N_\mu(x) = \mu + (1 - \mu)(1 - e^{-\chi\mu^2 x}). \quad (29)$$

Lemma 4.4 *The couple (N_μ, N_μ) is a supersolution (resp. a subsolution) to (24) if*

$$\chi\mu^2 - \chi\mu + 1 \geq 0 \quad (\text{resp. } \leq 0).$$

If $\chi > 4$, then there are 2 steady states (\bar{n}_-, μ_-) and (\bar{n}_+, μ_+) to (1)–(2)–(3) with $\mu_- < \mu_+$, whose linear stability is described by the Table 1.

4.3 Comparison to specific symmetric solutions

Firstly, we notice that any solution $(n_i, \mu_i)_{i=1,2}$ to (1)–(2)–(3) with identical initial values on both cells, i.e. such that $n_1^0 = n_2^0 = n^0$ and $\mu_1(0) = \mu_2(0)$, will have identical values for both cells at all time, i.e. $n_1(t, x) = n_2(t, x) = n(t, x)$ and $\mu_1(t) = \mu_2(t) = \mu(t)$. Moreover, n, μ is the solution to the following system

$$\begin{cases} \partial_t n(t, x) - \partial_{xx} n(t, x) - \chi\mu(t)^2 \partial_x n(t, x) = 0, & (t, x) \in (0, \infty)^2, \\ \partial_x n(t, 0) + \chi\mu(t)^2 n(t, 0) = \frac{d}{dt} \mu(t) = n(t, 0) - \mu(t). \end{cases} \quad (30)$$

In the sequel we will say that a solution $(n_i, \mu_i)_{i=1,2}$ with identical values on both cells, i.e. solution to (30), is a symmetric solution to (1)–(2)–(3).

In the next sections we will derive quantitative properties for solutions to system (30) and we will use the comparison principle with specific symmetric solutions that we describe hereafter.

Given initial data $(n_i^0, \mu_i(0))_{i=1,2}$, we define $N_i(0, x) = \mu_i + \int_0^x n_i^0(y) dy$ as before, the real numbers $\underline{\mu}(0), \bar{\mu}(0)$ and the functions $\underline{N}(0, x), \bar{N}(0, x)$ by

$$\begin{cases} \underline{\mu}(0) = \min_i(\mu_i(0)), & \underline{N}(0, x) = \min_i(N_i(0, x)), \\ \bar{\mu}(0) = \max_i(\mu_i(0)), & \bar{N}(0, x) = \max_i(N_i(0, x)), \\ \underline{n}^0(x) = \frac{d}{dx}\underline{N}(0, x), & \bar{n}^0(x) = \frac{d}{dx}\bar{N}(0, x). \end{cases} \quad (31)$$

Let us denote by $(\underline{n}(t, x), \underline{\mu}(t))$ and $(\bar{n}(t, x), \bar{\mu}(t))$ the solutions to the symmetric system (30) starting respectively from the initial data $(\underline{n}(0, x), \underline{\mu}(0))$ and $(\bar{n}(0, x), \bar{\mu}(0))$. We first note that the initial relation between $\underline{\mu}, \underline{N}, \bar{\mu}, \bar{N}$ and μ_i, N_i stated in (31) do not remain true for $t \neq 0$: for $t > 0$, one might have $\bar{N}(t, x) \neq \max_i(N_i(t, x))$ for instance. However, the comparison principle 4.2 ensures that

$$\forall i = 1, 2, \quad \underline{N}(t, x) \leq N_i(t, x) \leq \bar{N}(t, x) \quad \text{for } (t, x) \in (0, \infty)^2.$$

In particular, we have $\underline{\mu}(t) \leq \mu_i(t) \leq \bar{\mu}(t)$.

Remark 1 *In the sequel, we will have to estimate decays of Lyapunov type functional F of the following form*

$$F(n, \mu) = \int_0^\infty f(x, n(x)) dx + g(\mu).$$

We first note that $\{\mu_1(0), \mu_2(0)\} = \{\underline{\mu}(0), \bar{\mu}(0)\}$ and $\{n_1^0(x), n_2^0(x)\} = \{\underline{n}^0(x), \bar{n}^0(x)\}$ for almost every $x > 0$, by definition, hence

$$\begin{aligned} \sum_{i=1}^2 \int_0^\infty f(x, n_i^0(x)) dx + g(\mu_i) &= \int_0^\infty f(x, \underline{n}^0(x)) dx + g(\underline{\mu}(0)) \\ &\quad + \int_0^\infty f(x, \bar{n}^0(x)) dx + g(\bar{\mu}(0)). \end{aligned}$$

In particular if f and g take only nonnegative values, then

$$F(m^0, \nu(0)) \leq F(n_1^0, \mu_1(0)) + F(n_2^0, \mu_2(0)),$$

for $(m^0, \nu(0)) \in \{(\underline{n}^0, \underline{\mu}(0)), (\bar{n}^0, \bar{\mu}(0))\}$. We will apply the previous inequality to establish that initial data with finite entropy can be compared to symmetric solutions with initially finite entropy.

5 Non-linear stability of (\bar{n}_+, μ_+) . Proof of Proposition 1.2

We break the proof in several steps. We first prove the result for symmetric solutions, Proposition 5.1, and then we extend it to general solutions by using the comparison principle.

5.1 Symmetric solutions

In this section we will prove the following result.

Proposition 5.1 *Let (n, μ) be the solution to (30) with $M = 1$ and with initial data (n^0, μ^0) with finite entropy. Assume that $\chi > 4$, that there exists two real numbers $(\underline{\mu}, \bar{\mu}) \in (0, 1)^2$ such that*

$$\begin{cases} \mu_- < \underline{\mu} < \mu_+ < \bar{\mu}, \\ N_{\underline{\mu}}(x) \leq N(0, x) \leq N_{\bar{\mu}}(x) \quad \text{for all } x \geq 0, \end{cases}$$

and assume in addition that

$$\mu^0 \in (\underline{\mu}, \bar{\mu}).$$

Then, (n, μ) converges towards the steady state (\bar{n}_+, μ_+) of (30) in the following sense:

$$\begin{cases} \lim_{t \rightarrow \infty} \mu(t) = \mu_+, \\ \lim_{t \rightarrow \infty} \|n(t, \cdot) - (1 - \mu_+)n_+(\cdot)\|_{L^1(\mathbb{R}_+)} = 0, \end{cases}$$

where n_+ is defined by (9).

We split the proof of this result into several Lemmas. In a first step, in Lemma 5.2, we prove entropy dissipation. Then, in Lemma 5.3, we provide lower bounds on the terms involved in entropy dissipation allowing to prove that the Lyapunov functional tends to zero.

For symmetric solutions close to the linearly stable steady state (\bar{n}_+, μ_+) solution to (69), in order to prove entropy dissipation, we define the Lyapunov functional L :

$$L(t) := (1 - \mu)\mathcal{H}(\tilde{n}|n_+) = \int_0^\infty n(t, x) \log \frac{\tilde{n}(t, x)}{n_+(x)} dx \geq 0, \quad (32)$$

where n_+ is defined by (9) and

$$\tilde{n}(t, x) = \frac{n(t, x)}{\int_0^\infty n(t, x) dx} = \frac{n(t, x)}{1 - \mu(t)}, \quad (33)$$

since $\int_0^\infty n dx = 1 - \mu$. We additionally define the function f by

$$f(\mu) = \mu \log \frac{\mu}{\mu_+} + (1 - \mu) \log \frac{1 - \mu}{1 - \mu_+} + \frac{\chi\mu^3}{3} - \chi\mu_+^2\mu + 2\frac{\chi\mu_+^3}{3}, \quad (34)$$

and the polynomial P by

$$P(x) = \chi x^2 - \chi x + 1. \quad (35)$$

We notice that the quantity $(\mu^2 - \mu_+^2)P(\mu)$ is nonnegative for all $\mu \in [\mu_-, 1]$.

Lemma 5.2 *Let (n, μ) be the solution to (30). The following inequality holds true*

$$\frac{d}{dt} \left(L(t) + f(\mu(t)) \right) \leq -D^2 - \chi(\mu^2 - \mu_+^2)\mu P(\mu),$$

where f is defined by (34) and

$$D^2 = \int_0^\infty n (\partial_x \log n + \chi\mu^2)^2 dx = (1 - \mu)I \left(\tilde{n} | \chi\mu^2 e^{-\chi\mu^2 x} \right).$$

Proof of Lemma 5.2 We first notice that $f(\mu_+) = 0$, and that

$$f'(\mu) = \log \frac{\mu(1 - \mu_+)}{\mu_+(1 - \mu)} + \chi(\mu^2 - \mu_+^2).$$

Recalling that $P(\mu_+) = \chi\mu_+^2 - \chi\mu_+ + 1 = 0$, we see that $\chi\mu_+^2 = \frac{\mu_+}{1 - \mu_+}$ and thereby

$$f'(\mu) = \log \frac{\mu}{\chi\mu_+^2(1 - \mu)} + \chi(\mu^2 - \mu_+^2). \quad (36)$$

In particular, the function f is nonincreasing on $(0, \mu_+)$ and nondecreasing on $(\mu_+, 1)$, hence it is nonnegative on $(0, 1)$.

Next, we see that

$$\begin{aligned} L(t) &= \int_0^\infty (n \log n - n \log n_+ - n \log(1 - \mu)) \, dx \\ &= \int_0^\infty (n \log n - n \log n_+) \, dx - (1 - \mu) \log(1 - \mu). \end{aligned}$$

Hence, differentiating L

$$\frac{d}{dt}L(t) = \int_0^\infty \partial_t n (1 + \log n - \log n_+) \, dx + (1 + \log(1 - \mu(t))) \frac{d}{dt}\mu(t),$$

and using that $\int_0^\infty n \, dx + \mu = 1$, we obtain

$$\frac{d}{dt}L(t) = \int_0^\infty \partial_t n \log \frac{n}{n_+} \, dx + \log(1 - \mu(t)) \frac{d}{dt}\mu(t).$$

Next, recalling the definition (9) of $n_+(x) = \chi\mu_+^2 e^{-\chi\mu_+^2 x}$ and that (n, μ) is solution to (30), an integration by parts yields that

$$\begin{aligned} \frac{d}{dt}L(t) &= \int_0^\infty \log \frac{n}{n_+} (\partial_{xx}n + \chi\mu^2 \partial_x n) \, dx + \log(1 - \mu) \frac{d}{dt}\mu \\ &= -\log \frac{n(t, 0)}{n_+(0)} \frac{d}{dt}\mu - \int_0^\infty n (\partial_x \log n + \chi\mu_+^2) (\partial_x \log n + \chi\mu^2) \, dx \\ &\quad + \log(1 - \mu) \frac{d}{dt}\mu \\ &= -\log \frac{n(t, 0)}{(1 - \mu)n_+(0)} \frac{d}{dt}\mu - \underbrace{\int_0^\infty n (\partial_x \log n + \chi\mu^2)^2 \, dx}_{D^2} \\ &\quad + \chi(\mu^2 - \mu_+^2) \int_0^\infty (\partial_x n + \chi\mu^2 n) \, dx \\ &= -D^2 - \log \frac{n(t, 0)}{(1 - \mu)n_+(0)} \frac{d}{dt}\mu + \chi(\mu^2 - \mu_+^2)(-n(t, 0) + \chi\mu^2(1 - \mu)). \end{aligned}$$

Using now that $\frac{d}{dt}\mu = n(t, 0) - \mu$, we deduce that

$$\begin{aligned} \frac{d}{dt}L(t) &= -D^2 + \left(\log \frac{\mu}{n(t, 0)} + \log \frac{(1 - \mu)n_+(0)}{\mu} \right) \frac{d}{dt}\mu \\ &\quad + \chi(\mu^2 - \mu_+^2) \left(-\frac{d}{dt}\mu - \mu + \chi\mu^2(1 - \mu) \right) \\ &= -D^2 + \underbrace{(n(t, 0) - \mu) \log \frac{\mu}{n(t, 0)}}_{\leq 0} + \chi(\mu^2 - \mu_+^2)\mu (-\chi\mu^2 + \chi\mu - 1) \\ &\quad + \left(\chi\mu_+^2 - \chi\mu^2 + \log \frac{(1 - \mu)\chi\mu_+^2}{\mu} \right) \frac{d}{dt}\mu, \end{aligned}$$

and this achieves the proof of Lemma 5.2 by using (36) together with the definition of P .

Under some assumptions on the initial condition it is possible to find lower bounds on $D^2 + \chi(\mu^2 - \mu_+^2)\mu P(\mu)$ allowing to prove that $L + f$ tends to zero. The keystone is to prove that μ stays away from μ_- in order the term $(\mu^2 - \mu_+^2)P(\mu)$ to stay nonnegative.

Lemma 5.3 *Under the same hypothesis as in Proposition 5.1, one has*

$$\frac{d}{dt} \left(L(t) + f(\mu(t)) \right) \leq -C \left(L(t) + f(\mu(t)) \right)^2.$$

Proof of Lemma 5.3 We first note that the assumptions made here correspond to the ones made in Lemma 4.2, hence the comparison principle applies

$$\forall (t, x) \in (0, \infty) \times \mathbb{R}_+, \quad N_{\underline{\mu}}(x) \leq N(t, x) \leq N_{\bar{\mu}}(x), \quad (37)$$

and

$$\forall t > 0, \quad \underline{\mu} \leq \mu(t) \leq \bar{\mu}. \quad (38)$$

A consequence of the previous inequality (37) is that the first momentum is bounded, as one has formally

$$\int_0^\infty xn \, dx = [x(N-1)]_0^\infty + \int_0^\infty (1-N) \, dx = \int_0^\infty (1-N) \, dx.$$

Using (37), we see that the previous computation is in fact rigorous since N has an exponential profile at infinity and one obtains

$$\int_0^\infty xn \, dx \leq \frac{(1-\mu_-)}{\chi\mu_-^2}. \quad (39)$$

Next, let us define the function \tilde{N} by

$$\tilde{N}(t, x) = \frac{N(t, x) - \mu(t)}{1 - \mu(t)} \quad (t, x) \in (0, \infty) \times \mathbb{R}_+. \quad (40)$$

From the inequality (37), it follows that

$$\frac{N_{\underline{\mu}} - \mu}{1 - \mu} \leq \tilde{N} \leq \frac{N_{\bar{\mu}} - \mu}{1 - \mu}.$$

Hence, recalling the definition (29) of N_μ :

$$\frac{\underline{\mu} - \mu + (1 - \underline{\mu})(1 - e^{-\chi\underline{\mu}^2 x})}{1 - \mu} \leq \tilde{N} \leq \frac{\bar{\mu} - \mu + (1 - \bar{\mu})(1 - e^{-\chi\bar{\mu}^2 x})}{1 - \mu},$$

which becomes

$$1 - \frac{(1 - \underline{\mu})e^{-\chi\underline{\mu}^2 x}}{1 - \mu} \leq \tilde{N} \leq 1 - \frac{(1 - \bar{\mu})e^{-\chi\bar{\mu}^2 x}}{1 - \mu}.$$

Consequently, looking at the pseudo-inverse, we obtain

$$\begin{aligned} \tilde{N}^{-1}(u) &= \inf \{ x, \tilde{N}(t, x) \geq u \} \\ &\leq \inf \left\{ x, 1 - \frac{(1 - \underline{\mu})e^{-\chi\underline{\mu}^2 x}}{1 - \mu} \geq u \right\} = -\frac{1}{\chi\underline{\mu}^2} \log \frac{(1-u)(1-\mu)}{1-\underline{\mu}} \\ &\leq -\frac{1}{\chi\underline{\mu}^2} \log \frac{(1-u)(1-\bar{\mu})}{1-\underline{\mu}}. \end{aligned}$$

Therefore, using the representation result for W given in theorem 3.2 together with the previous pseudo-inverse estimation, we deduce that there exists a positive constant C_W such that

$$W\left(\tilde{n}, \chi\mu^2 e^{-\chi\mu^2 x}\right) \leq C_W.$$

Applying now the HWI inequality, recalled in lemma 3.4, to the exponential measure $\chi\mu^2 e^{-\chi\mu^2 x}$ leads to

$$\begin{aligned} D^2 &= (1-\mu)\mathcal{I}\left(\tilde{n}|\chi\mu^2 e^{-\chi\mu^2 x}\right) \geq (1-\mu)\left(\frac{\mathcal{H}\left(\tilde{n}|\chi\mu^2 e^{-\chi\mu^2 x}\right)}{W\left(\tilde{n}, \chi\mu^2 e^{-\chi\mu^2 x}\right)}\right)^2 \\ &\geq (1-\mu)\left(\frac{\mathcal{H}\left(\tilde{n}|\chi\mu^2 e^{-\chi\mu^2 x}\right)}{C_W}\right)^2, \end{aligned}$$

hence for all $\varepsilon \leq 1/C_W^2$, the following estimate holds true

$$D^2 \geq (1-\mu)\varepsilon \mathcal{H}\left(\tilde{n}|\chi\mu^2 e^{-\chi\mu^2 x}\right)^2. \quad (41)$$

Using now the definition of the relative entropy, $\mathcal{H}(n|p) = \int_0^\infty n \log n/p \, dx$, we see that

$$\mathcal{H}\left(\tilde{n}|\chi\mu^2 e^{-\chi\mu^2 x}\right) = \mathcal{H}(\tilde{n}|n_+) + \int_0^\infty \tilde{n} \, dx \log \frac{\mu_+^2}{\mu^2} + \chi(\mu^2 - \mu_+^2) \int_0^\infty x \tilde{n} \, dx.$$

Moreover, from the bounds (38) and (39) on μ and on the first momentum, it follows that

$$\mathcal{H}\left(\tilde{n}|\chi\mu^2 e^{-\chi\mu^2 x}\right) \geq \mathcal{H}(\tilde{n}|n_+) - K|\mu - \mu_+|,$$

hence, for all $\varepsilon \leq 1/C_W^2$, the contribution of the entropy dissipation satisfies the following inequality

$$D^2 \geq (1-\mu)\varepsilon \left(\mathcal{H}(\tilde{n}|n_+)^2 - K^2|\mu - \mu_+|^2\right). \quad (42)$$

Furthermore, there exists a positive constant C_2 (depending on $\underline{\mu}, \bar{\mu}$), such that

$$\forall \mu \in [\underline{\mu}, \bar{\mu}], \quad \chi(\mu^2 - \mu_+^2)\mu P(\mu) \geq C_2(\mu - \mu_+)^2. \quad (43)$$

Consequently, for all $\varepsilon \in (0, 1/C_W^2)$

$$D^2 + \chi(\mu^2 - \mu_+^2)\mu P(\mu) \geq (1-\mu)\varepsilon \mathcal{H}(\tilde{n}|n_+)^2 + \left(C_2 - (1-\mu)\varepsilon K^2\right)(\mu - \mu_+)^2,$$

and adjusting the value of ε , we see that there exists a constant $C > 0$ such that

$$D^2 + \chi(\mu^2 - \mu_+^2)\mu P(\mu) \geq C\left(\mathcal{H}(\tilde{n}|n_+)^2 + (\mu - \mu_+)^2\right).$$

Recalling the definition of f and of its derivative (34) and (36), we deduce that there exists a positive constant C_3 such that $(\mu - \mu_+)^2 \geq C_3 f(\mu)^2$. This achieves the proof of Lemma 5.3 with a not explicit constant.

Corollary 5.4 *Under the same hypothesis as in Proposition 5.1, the following estimates hold*

$$\forall t > 0, \quad 0 \leq \int_0^\infty n \log \frac{\tilde{n}}{n_+} dx + (\mu - \mu_+)^2 \leq \frac{C}{1+t},$$

and

$$\forall t > 0, \quad \|\tilde{n} - n_+\|_{L^1(\mathbb{R}_+)} \leq \frac{C}{\sqrt{1+t}},$$

where n_+ is defined by (9).

Proof The first inequality is a consequence of lemma 5.3. The second one is obtained by the triangle inequality:

$$\|n - (1 - \mu_+)n_+\|_{L^1(\mathbb{R}_+)} \leq (1 - \mu_+)\|\tilde{n} - n_+\|_{L^1(\mathbb{R}_+)} + |\mu - \mu_+|,$$

where we have used the definition (33) of \tilde{n} . We apply Csiszar-Kullback inequality

$$\|\tilde{n} - n_+\|_{L^1(\mathbb{R}_+)} \leq \sqrt{\frac{2L(t)}{1 - \mu}} \leq \frac{C}{\sqrt{1+t}},$$

and we conclude by using the inequality $|\mu - \mu_+|^2 \leq C/\sqrt{1+t}$.

5.2 Nonsymmetric data: application of the comparison principle.

In this section, in Lemma 5.5, we give some of the convergence and estimate of Proposition 1.2. The proof relies on symmetric results stated in the previous section. We postpone in the Appendix 9 the much technical step in which entropy dissipation is computed in order to obtain a convergence in entropy.

Lemma 5.5 *Let $(n_i, \mu_i)_{i=1,2}$ be the solution to (1)–(2)–(3) with (5), with initial data $(n_i^0, \mu_i^0)_{1 \leq i \leq 2}$ with finite entropy $\int_0^\infty n_i^0(x + \log n_i^0) dx < +\infty$. Assume that $\chi > 4$ and that there exists two real numbers $(\underline{\mu}, \bar{\mu}) \in (0, 1)^2$ such that*

$$\begin{cases} \mu_- < \underline{\mu} < \mu_+ < \bar{\mu}, \\ N_{\underline{\mu}}(x) \leq N_i(0, x) \leq N_{\bar{\mu}}(x) \quad \text{for all } x \geq 0 \text{ and } i = 1, 2, \end{cases}$$

and assume in addition that

$$\mu_i^0 \in (\underline{\mu}, \bar{\mu}) \quad i = 1, 2.$$

Then, for $i = 1, 2$, the following convergence and estimate hold true:

$$\begin{cases} \forall x \geq 0, \quad N_i(t, x) \rightarrow N_+(x) \quad \text{as } t \rightarrow \infty, \\ |\mu_i - \mu_+| \leq \frac{C}{\sqrt{1+t}}, \end{cases}$$

for some constant C , where $N_+(x) = \mu_+ + \int_0^x n_+(y) dy$.

Proof Given initial data, $(n_i^0, \mu_i^0)_{i=1,2}$ satisfying the condition of finite entropy and such that $\mu_i(0) \in (\underline{\mu}, \bar{\mu})$, we can define the functions N_i^0 as before, and the functions and real numbers $\bar{N}^0, \underline{N}^0, \bar{n}^0, \underline{n}^0, \bar{\mu}^0$ and $\underline{\mu}^0$ by (31). Next, we can build solutions $(\bar{\mu}, \bar{n})$ and $(\underline{\mu}, \underline{n})$ to the symmetric system (30) with respective initial data $(\bar{\mu}^0, \bar{n}^0)$ and $(\underline{\mu}^0, \underline{n}^0)$. Corollary 5.4 then applies and it provides the result.

6 Self-similar behaviour for small data. Proof of Proposition 1.3

In this section we establish attraction towards self-similar profile for small data for symmetric solution and the non-symmetric case is postponed in Appendix 10. In the non-symmetric case the density is expected to decay in a self-similar way. To catch this asymptotic behaviour we rescale the density with the classical parabolic rescaling:

$$n(t, x) = \frac{1}{\sqrt{1+2t}} u\left(\frac{1}{2} \log(1+2t), \frac{x}{\sqrt{1+2t}}\right) = \frac{1}{\sqrt{1+2t}} u(\tau, y),$$

which is mass-preserving

$$\int_0^\infty u(\tau, y) dy = \int_0^\infty n\left(\frac{e^{2\tau}-1}{2}, x\right) dx = 1 - \mu\left(\frac{e^{2\tau}-1}{2}\right) = 1 - \tilde{\mu}(\tau),$$

where we have set $\mu(t) = \mu\left(\frac{e^{2\tau}-1}{2}\right) = \tilde{\mu}(\tau)$. Since (n, μ) is solution to (30), $(u, \tilde{\mu})$ satisfies the boundary value problem:

$$\begin{cases} \partial_\tau u(\tau, y) - \partial_{yy} u(\tau, y) - \partial_y(yu(\tau, y)) - \chi \tilde{\mu}^2 e^\tau \partial_y u(\tau, y) = 0, & (\tau, y) \in (0, \infty)^2, \\ \frac{d}{d\tau} \tilde{\mu}(\tau) = \partial_y u(\tau, 0) + \chi \tilde{\mu}^2 e^\tau u(\tau, 0) = e^\tau u(\tau, 0) - e^{2\tau} \tilde{\mu}(\tau), \end{cases} \quad (44)$$

the additional left-sided drift $\partial_y(yu(\tau, y))$ contributes to confine the mass in the new frame (τ, y) .

Our goal is now to prove the two following convergences when $\tau \rightarrow \infty$:

$$\begin{cases} u(\tau, \cdot) \rightarrow G(\cdot) & \text{strongly in } L^1(0, \infty), \\ \tilde{\mu} \rightarrow 0. \end{cases}$$

We will in fact be more quantitative and will establish in particular that $\tilde{\mu}$ behaves like $O(e^{-\tau})$. As a first step, for several quantities involving $\tilde{\mu}, \tilde{\mu}e^\tau$ we will study time integrated quantities instead of directly $\tilde{\mu}, \tilde{\mu}e^\tau$.

We break the proof in several steps. We start with the symmetric case and we prove the following result.

Lemma 6.1 *Let (n, μ) be a solution to (30) with $\chi > 4$ and let $\mu_- < \mu_+$ be the roots of P . Assume that there exists $\mu_0 \in (0, \mu_-)$ such that $\mu(0) < \mu_0$ and $N(0, x) \leq N_{\mu_0}(x)$, for $x \geq 0$, and that $\int_0^\infty x^2 n(0, x) dx < +\infty$. Then, the following convergences hold true: $\lim_{t \rightarrow \infty} \mu(t) = 0$ and*

$$\lim_{t \rightarrow \infty} \left\| n(t, \cdot) - \frac{1}{\sqrt{1+2t}} G\left(\frac{\cdot}{\sqrt{1+2t}}\right) \right\|_{L^1(\mathbb{R}_+)} = 0.$$

Proof of Lemma 6.1 The assumptions made here correspond to the ones made in lemma 4.2, hence the comparison principle applies: $N(t, x) \leq N_{\mu_0}(x)$ for all $t > 0$ and for all $x \geq 0$. In particular, we have $\mu(t) \leq \mu_0 < \mu_-$ hence $P(\mu(t)) > 0$ for all times, by definition of μ_- .

As a first step we establish that $\int_0^\tau \tilde{\mu}(\tau') e^{\tau'} d\tau' = O(\tau)$. We start with a useful result.

Lemma 6.2 *Let $0 \leq f \in L^1_{loc}([0, +\infty[)$. Assume that there exists $C > 0$ such that for all $\tau > 0$,*

$$\int_0^\tau f(\tau') d\tau' \leq C(1 + \tau).$$

Then, the following upper bound holds true:

$$\forall \lambda > 0, \quad \int_0^\infty e^{-\lambda \tau'} f(\tau') d\tau' < +\infty.$$

The proof is a consequence of the following computations

$$\begin{aligned} \int_0^n e^{-\lambda\tau'} f(\tau') d\tau' &\leq \sum_{k=0}^{n-1} e^{-\lambda k} \int_k^{k+1} f(\tau') d\tau' \leq C \sum_{k=0}^{n-1} e^{-\lambda k} (1+k) \\ &\leq C \sum_{k=0}^{\infty} e^{-\lambda k} (1+k) < +\infty. \end{aligned}$$

Lemma 6.3 *There exist two constants M_1 and M_2 depending only on $\int_0^\infty y^2 u(0, y) dy$ such that*

$$\forall \tau > 0, \quad \int_0^\infty y u(\tau, y) dy \leq M_1 \quad \text{and} \quad \int_0^\infty y^2 u(\tau, y) dy \leq M_2.$$

Moreover, one has

$$\int_0^\tau (\tilde{\mu}(\tau') + \tilde{\mu}^2(\tau')) e^{\tau'} d\tau' \leq C(1 + \tau),$$

and

$$\forall \alpha > 0, \quad \int_0^\infty \tilde{\mu}(\tau) e^{(1-\alpha)\tau} d\tau < +\infty, \quad (45)$$

and

$$\forall \tau > 0, \quad \int_0^\tau \tilde{\mu}(\tau')^2 e^{\tau'} \int_0^\infty y u(\tau', y) dy d\tau' \leq C(1 + \tau), \quad (46)$$

for some $C > 0$.

Proof of Lemma 6.3 We first see that

$$\frac{d}{d\tau} \int_0^\infty y^2 u(\tau, y) dy = 2(1 - \tilde{\mu}(\tau)) - 2\chi \tilde{\mu}(\tau)^2 e^\tau \int_0^\infty y u(\tau, y) dy - 2 \int_0^\infty y^2 u(\tau, y) dy, \quad (47)$$

from which it follows that

$$\int_0^\infty y^2 u(\tau, y) dy \leq 1 + e^{-2\tau} \left(\int_0^\infty y^2 u(0, y) dy - 1 \right),$$

hence we can choose $M_2 = \max(1, \int_0^\infty y^2 u(0, y) dy)$. Using the Cauchy-Schwarz inequality, we deduce that

$$\int_0^\infty y u(\tau, y) dy \leq \sqrt{1 - \tilde{\mu}(\tau)} \sqrt{M_2} \leq \sqrt{M_2} =: M_1.$$

Consequently $\int_0^\infty (e^{-\tau} + y) u(\tau, y) dy$ is uniformly bounded and

$$\frac{d}{d\tau} \int_0^\infty (e^{-\tau} + y) u(\tau, y) dy = \tilde{\mu}(\tau) e^\tau P(\tilde{\mu}) - \int_0^\infty (e^{-\tau} + y) u(\tau, y) dy,$$

therefore

$$\begin{aligned} &\int_0^\infty (e^{-\tau} + y) u(\tau, y) dy + \int_0^\tau \int_0^\infty (e^{-\tau'} + y) u(\tau', y) dy d\tau' \\ &= \int_0^\infty (1 + y) u(0, y) dy + \int_0^\tau \tilde{\mu}(\tau') e^{\tau'} P(\tilde{\mu}(\tau')) d\tau'. \end{aligned}$$

Since $\int_0^\infty (e^{-\tau} + y)u(\tau, y) dy$ is uniformly bounded and recalling that $P(\tilde{\mu}) \geq P(\mu_0) > 0$, we have

$$P(\mu_0) \int_0^\tau \tilde{\mu}(\tau') e^{\tau'} d\tau' \leq \int_0^\tau \tilde{\mu}(\tau') e^{\tau'} P(\tilde{\mu}(\tau')) d\tau' \leq C(1 + \tau),$$

hence the first bound (45) by lemma 6.2.

Estimate (46) then follows from the uniform bounds $\tilde{\mu}(\tau) \leq \mu_0$ and $\int_0^\infty yu(\tau, y) d\tau \leq M_1$.

Let us now prove entropy dissipation. To do so, we compare the solution u to the normalized gaussian G on the half line. We consider the following Lyapunov functional \mathcal{L} :

$$\begin{aligned} \mathcal{L}(\tau) = & \underbrace{(1 - \tilde{\mu}(\tau)) \int_0^\infty G(y) H\left(\frac{u(\tau, y)}{(1 - \tilde{\mu}(\tau))G(y)}\right) dy}_{\mathcal{L}_1} \\ & + \underbrace{H(1 - \tilde{\mu}(\tau))}_{\mathcal{L}_2} + \underbrace{G(0)e^{-\tau} H\left(\frac{\tilde{\mu}(\tau)}{G(0)e^{-\tau}}\right)}_{\mathcal{L}_3}, \end{aligned} \quad (48)$$

where $H(x) = x \log x - x + 1$ and where we have separated the various contributions \mathcal{L}_1 , \mathcal{L}_2 and \mathcal{L}_3 that we will study separately. Since H is non-negative, \mathcal{L} is also non-negative. Moreover, \mathcal{L} can be rewritten as

$$\begin{aligned} \mathcal{L}(\tau) = & \int_0^\infty \left(u(\tau, y) \log\left(\frac{u(\tau, y)}{G(y)}\right) - u(\tau, y) \right) dy \\ & + 1 + \tilde{\mu}(\tau) \log\frac{\tilde{\mu}(\tau)}{G(0)e^{-\tau}} - \tilde{\mu}(\tau) + G(0)e^{-\tau}. \end{aligned} \quad (49)$$

Recalling the expression of the Fisher information:

$$I(u|G) = \int_0^\infty u(\tau, y) (\partial_y \log u(\tau, y) + y)^2 dy \geq 0, \quad (50)$$

$$I(u|G_\mu) = \int_0^\infty u(\tau, y) (\partial_y \log u(\tau, y) + y + \chi \tilde{\mu}(\tau, y)^2 e^\tau)^2 dy \geq 0, \quad (51)$$

we can derive upper bounds on $\frac{d}{d\tau} \mathcal{L}(\tau)$ in two different ways.

Lemma 6.4 *The following upper bounds on $\frac{d}{d\tau} \mathcal{L}(\tau)$ hold true:*

$$\begin{aligned} \frac{d}{d\tau} \mathcal{L}(\tau) \leq & -I(u|G) + \tilde{\mu}(\tau) - G(0)e^{-\tau} + \chi \tilde{\mu}(\tau)^3 e^{2\tau} \\ & + \frac{d}{d\tau} \frac{\chi \tilde{\mu}(\tau)^3}{3} - \chi \tilde{\mu}(\tau)^2 e^\tau \int_0^\infty yu(\tau, y) dy, \end{aligned} \quad (52)$$

and

$$\begin{aligned} \frac{d}{d\tau} \mathcal{L}(\tau) \leq & -I(u|G_\mu) + \tilde{\mu}(\tau) - G(0)e^{-\tau} - \frac{d}{d\tau} \frac{\chi \tilde{\mu}(\tau)^3}{3} \\ & + \chi \tilde{\mu}(\tau)^2 e^\tau \int_0^\infty yu(\tau, y) dy - \chi \tilde{\mu}(\tau)^3 e^{2\tau} P(\tilde{\mu}(\tau)), \end{aligned} \quad (53)$$

where P is defined by (35).

Proof of lemma 6.4 Differentiating (49), we obtain

$$\begin{aligned}
\frac{d}{d\tau}\mathcal{L}(\tau) &= \int_0^\infty \log \frac{u(\tau, y)}{G(y)} \partial_\tau u(\tau, y) dy + \log \frac{\tilde{\mu}(\tau)}{G(0)e^{-\tau}} \frac{d}{d\tau}\tilde{\mu}(\tau) + \tilde{\mu}(\tau) - G(0)e^{-\tau} \\
&= -\log \frac{u(\tau, 0)}{G(0)} \frac{d}{d\tau}\tilde{\mu}(\tau) + \log \frac{\tilde{\mu}(\tau)}{G(0)e^{-\tau}} \frac{d}{d\tau}\tilde{\mu}(\tau) + \tilde{\mu}(\tau) - G(0)e^{-\tau} \\
&\quad - \int_0^\infty u(\tau, y) (\partial_y \log u(\tau, y) + y) (\partial_y \log u(\tau, y) + y + \chi\tilde{\mu}(\tau)^2 e^\tau) dy \\
&= -\log \frac{u(\tau, 0)}{\tilde{\mu}(\tau)e^\tau} \frac{d}{d\tau}\tilde{\mu}(\tau) + \tilde{\mu}(\tau) - G(0)e^{-\tau} \\
&\quad - \int_0^\infty u(\tau, y) (\partial_y \log u(\tau, y) + y) (\partial_y \log u(\tau, y) + y + \chi\tilde{\mu}(\tau)^2 e^\tau) dy.
\end{aligned} \tag{54}$$

On the first hand, recalling (44), it follows that $-\frac{d}{d\tau}\tilde{\mu}(\tau) \log \frac{u(\tau, 0)}{\tilde{\mu}(\tau)e^\tau} \leq 0$, hence, using the definition (50) of $I(u|G)$, it yields

$$\begin{aligned}
\frac{d}{d\tau}\mathcal{L}(\tau) &\leq -I(u|G) + \tilde{\mu}(\tau) - G(0)e^{-\tau} \\
&\quad - \chi\tilde{\mu}(\tau)^2 e^\tau \int_0^\infty u(\tau, y) (\partial_y \log u(\tau, y) + y) dy \\
&\leq -I(u|G) + \tilde{\mu}(\tau) - G(0)e^{-\tau} + \chi\tilde{\mu}(\tau)^2 e^\tau u(\tau, 0) \\
&\quad - \chi\tilde{\mu}(\tau)^2 e^\tau \int_0^\infty yu(\tau, y) dy \\
&\leq -I(u|G) + \tilde{\mu}(\tau) - G(0)e^{-\tau} + \chi\tilde{\mu}(\tau)^3 e^{2\tau} + \frac{d}{d\tau} \frac{\chi\tilde{\mu}(\tau)^3}{3} \\
&\quad - \chi\tilde{\mu}^2 e^\tau \int_0^\infty yu(\tau, y) dy,
\end{aligned}$$

which is exactly (52).

On the other hand, starting again from (54) and using the definition (51) of $I(u|G_\mu)$, we have as well

$$\begin{aligned}
\frac{d}{d\tau}\mathcal{L}(\tau) &\leq -I(u|G_\mu) + \tilde{\mu}(\tau) - G(0)e^{-\tau} \\
&\quad + \chi\tilde{\mu}(\tau)^2 e^\tau \int_0^\infty u(\tau, y) (\partial_y \log u(\tau, y) + y + \chi\tilde{\mu}(\tau)^2 e^\tau) dy \\
&\leq -I(u|G_\mu) + \tilde{\mu}(\tau) - G(0)e^{-\tau} - \chi\tilde{\mu}(\tau)^2 e^\tau u(\tau, 0) \\
&\quad + \chi\tilde{\mu}(\tau)^2 e^\tau \int_0^\infty yu(\tau, y) dy + (\chi\tilde{\mu}(\tau)^2 e^\tau)^2 (1 - \tilde{\mu}(\tau)) \\
&\leq -I(u|G_\mu) + \tilde{\mu}(\tau) - G(0)e^{-\tau} - \chi\tilde{\mu}(\tau)^3 e^{2\tau} - \frac{d}{d\tau} \frac{\chi\tilde{\mu}(\tau)^3}{3} \\
&\quad + \chi\tilde{\mu}(\tau)^2 e^\tau \int_0^\infty yu(\tau, y) dy + (\chi\tilde{\mu}(\tau)^2 e^\tau)^2 (1 - \tilde{\mu}(\tau)) \\
&\leq -I(u|G_\mu) + \tilde{\mu}(\tau) - G(0)e^{-\tau} - \frac{d}{d\tau} \frac{\chi\tilde{\mu}(\tau)^3}{3} \\
&\quad + \chi\tilde{\mu}(\tau)^2 e^\tau \int_0^\infty yu(\tau, y) dy - \chi\tilde{\mu}(\tau)^3 e^{2\tau} P(\tilde{\mu}(\tau)),
\end{aligned}$$

which is the second estimate (53).

We establish some bounds that will be used later on.

Lemma 6.5 *The following quantities are finite:*

$$\int_0^\infty \tilde{\mu}(\tau')^3 e^{2\tau'} d\tau' < +\infty, \quad (55)$$

$$\int_0^\infty \tilde{\mu}(\tau')^2 e^{\tau'} d\tau' < +\infty. \quad (56)$$

Proof of Lemma 6.5 Integrating (53) and using the nonnegativity of $\mathcal{L}(\tau) + \chi \frac{\tilde{\mu}(\tau)^3}{3}$, we obtain

$$\begin{aligned} -\left(\mathcal{L} + \chi \frac{\tilde{\mu}^3}{3}\right)(0) &\leq \int_0^\tau \left(\tilde{\mu}(\tau') - G(0)e^{-\tau'}\right) d\tau' \\ &+ \chi \int_0^\tau \tilde{\mu}(\tau')^2 e^{\tau'} \int_0^\infty yu(\tau', y) dy d\tau' - \chi \int_0^\tau \tilde{\mu}(\tau')^3 e^{2\tau'} P(\tilde{\mu}(\tau')) d\tau'. \end{aligned}$$

Moreover, recalling the estimate (45) in lemma 6.3, we know that $\tilde{\mu} - G(0)e^{-\tau}$ is integrable, hence there exists a constant $C > 0$ such that

$$\chi \int_0^\tau \tilde{\mu}(\tau')^3 e^{2\tau'} P(\tilde{\mu}(\tau')) d\tau' \leq C + \chi \int_0^\tau \tilde{\mu}(\tau')^2 e^{\tau'} \int_0^\infty yu(\tau', y) dy d\tau'.$$

Using the lower bound $P(\tilde{\mu}) \geq P(\mu_0)$ and the upper bound $\int_0^\infty yu \leq M_1$, this leads to

$$\int_0^\tau \tilde{\mu}(\tau')^3 e^{2\tau'} d\tau' \leq C \left(1 + \int_0^\tau \tilde{\mu}(\tau')^2 e^{\tau'} d\tau'\right).$$

Moreover, by Cauchy-Schwarz inequality, we have

$$\begin{aligned} \int_0^\tau \tilde{\mu}(\tau')^2 e^{\tau'} d\tau' &\leq \left(\int_0^\tau \tilde{\mu} d\tau'\right)^{1/2} \left(\int_0^\tau \tilde{\mu}(\tau')^3 e^{2\tau'} d\tau'\right)^{1/2} \\ &\leq \left(\int_0^\infty \tilde{\mu} d\tau'\right)^{1/2} \left(\int_0^\tau \tilde{\mu}(\tau')^3 e^{2\tau'} d\tau'\right)^{1/2}, \end{aligned}$$

hence

$$\int_0^\tau \tilde{\mu}(\tau')^3 e^{2\tau'} d\tau' \leq C \left(1 + \sqrt{\int_0^\tau \tilde{\mu}(\tau')^3 e^{2\tau'} d\tau'}\right),$$

which implies

$$\forall \tau > 0, \quad \int_0^\tau \tilde{\mu}(\tau')^3 e^{2\tau'} d\tau' \leq C,$$

and consequently

$$\int_0^\tau \tilde{\mu}(\tau')^2 e^{\tau'} d\tau' \leq \left(\int_0^\infty \tilde{\mu}(\tau) d\tau\right)^{1/2} \sqrt{C},$$

which achieves the proof of Lemma 6.5.

We can now state the first major intermediate result.

Lemma 6.6 *The functional \mathcal{L} is bounded and tends to 0 as τ goes to ∞ .*

Proof of Lemma 6.6 We break the proof in two steps. First, we prove that \mathcal{L} has a limit and then that this limit is 0.

Summing (52) and (53), and using lemma 6.5, we obtain that

$$\frac{d}{d\tau}\mathcal{L}(\tau) \leq g(\tau),$$

where g is nonnegative and belongs to $L^1(\mathbb{R}_+)$. Since \mathcal{L} is nonnegative, the previous inequality implies that \mathcal{L} is bounded by $\mathcal{L}(0) + \int_0^\infty g(\tau) d\tau$. To summarize, we have proved that

$$\forall \tau > 0, \quad \mathcal{L}(\tau) \geq 0, \quad \text{and} \quad \left(\frac{d}{d\tau}\mathcal{L} \right)_+ \in L^1(\mathbb{R}_+).$$

Using now Lemma 6.7 p171 of [23] we deduce that \mathcal{L} belongs to $BV(\mathbb{R}_+)$ and that it admits a limit L_∞ .

Let us now prove that the limit L_∞ is 0. To do so let us treat the different contributions $\mathcal{L}_1, \mathcal{L}_2$ and \mathcal{L}_3 of \mathcal{L} , defined by (48), in a suitable way.

On the first hand, we consider the contribution \mathcal{L}_2 and we see that

$$\forall \tilde{\mu} \in [0, 1], \quad 0 \leq \mathcal{L}_2(\tau) = H(1 - \tilde{\mu}) \leq \tilde{\mu}, \quad (57)$$

hence $\mathcal{L}_2 \in L^1(\mathbb{R}_+)$.

On the second hand, we see that

$$\begin{aligned} \mathcal{L}_3(\tau) &= G(0)e^{-\tau} H \left(\frac{\tilde{\mu}(\tau)}{G(0)e^{-\tau}} \right) = \tilde{\mu}(\tau) \log \tilde{\mu}(\tau) - \tilde{\mu}(\tau) \log G(0) + \tau \tilde{\mu}(\tau) - \tilde{\mu}(\tau) \\ &\quad + G(0)e^{-\tau} \\ &\leq -\tilde{\mu}(\tau) \log G(0) + \tau \tilde{\mu}(\tau) + G(0)e^{-\tau}. \end{aligned} \quad (58)$$

Since the right hand side of the previous inequality belongs to $L^1(\mathbb{R}_+)$, so does \mathcal{L}_3 .

Let us now consider the term \mathcal{L}_1 . Recalling the definition (50) of $I(u|G)$, the Fisher information \mathcal{I} and the logarithmic Sobolev inequality (since G is the standard Gaussian function), we obtain

$$\begin{aligned} I(u|G) &= (1 - \tilde{\mu}) \mathcal{I} \left(\frac{u}{1 - \tilde{\mu}} \middle| G \right) \\ &\geq 2(1 - \tilde{\mu}) \mathcal{H} \left(\frac{u}{1 - \tilde{\mu}} \middle| G \right) = 2\mathcal{L}_1(\tau). \end{aligned}$$

Using inequality (52), the nonnegativity of \mathcal{L} and the integrability of the other contributions involved in (52), we deduce that

$$\int_0^\infty I(u|G) d\tau < +\infty.$$

Consequently, \mathcal{L} , written as (48), is the sum of three nonnegative integrable functions. Since \mathcal{L} has a limit when τ tends to ∞ , we have

$$\lim_{\tau \rightarrow \infty} \mathcal{L}(\tau) = 0 \quad \text{and} \quad \forall i \in \{1, 2, 3\}, \quad \lim_{\tau \rightarrow \infty} \mathcal{L}_i(\tau) = 0.$$

This ends the proof of lemma 6.6. In particular, it implies that $\lim_{\tau \rightarrow \infty} \tilde{\mu}(\tau) = 0$.

Let us now prove that convergence of \mathcal{L} towards zero is exponential. The crucial step is to prove that $\mathcal{L}(\tau)e^{\alpha\tau}$ is bounded.

Lemma 6.7 *For all $\alpha < 1$, $\mathcal{L}(\tau)e^{\alpha\tau}$ is uniformly bounded.*

Proof of Lemma 6.7 In order to cancel out the terms $\tilde{\mu}(\tau)^3 e^{2\tau}$ in $\frac{d}{d\tau}\mathcal{L}(\tau)$, we make a convex combination with coefficients $P(\tilde{\mu})/(1 + P(\tilde{\mu}))$ and $1/(1 + P(\tilde{\mu}))$ of the two inequalities (52) and (53) stated in Lemma 6.4 and we obtain

$$\begin{aligned} \frac{d}{d\tau}\mathcal{L}(\tau) &\leq -\frac{P(\tilde{\mu})}{1 + P(\tilde{\mu})}I(u|G) - \frac{1}{1 + P(\tilde{\mu})}I(u|G_\mu) + \tilde{\mu}(\tau) - G(0)e^{-\tau} \\ &\quad + \frac{P(\tilde{\mu}) - 1}{P(\tilde{\mu}) + 1} \frac{d}{d\tau} \frac{\chi\tilde{\mu}(\tau)^3}{3} + \frac{1 - P(\tilde{\mu})}{P(\tilde{\mu}) + 1} \chi\tilde{\mu}(\tau)^2 e^\tau \int_0^\infty yu(\tau, y) dy. \end{aligned}$$

Let g be a function such that its derivative satisfies

$$g'(\tilde{\mu}) = \frac{\chi\tilde{\mu}^2(1 - P(\tilde{\mu}))}{1 + P(\tilde{\mu})}. \quad (59)$$

Since $\tilde{\mu} \in (0, \mu_-)$, it follows that $P(\tilde{\mu}) \in (0, 1)$, hence

$$g'(\tilde{\mu}) \geq 0.$$

Moreover $g(0) = 0$ so that g is a nonnegative increasing function on the domain of interest.

Recalling the definition of P together with the upper bound of the first momentum we first deduce that

$$\frac{1 - P(\tilde{\mu})}{P(\tilde{\mu}) + 1} \chi\tilde{\mu}(\tau)^2 e^\tau \int_0^\infty yu(\tau, y) dy \leq \chi^2 \tilde{\mu}^3 e^\tau M_1,$$

hence, using the logarithmic Sobolev inequality we have

$$\frac{d}{d\tau} (\mathcal{L}(\tau) + g(\tilde{\mu}(\tau))) \leq -\frac{2P(\tilde{\mu})}{1 + P(\tilde{\mu})} \mathcal{L}_1(\tau) + \tilde{\mu}(\tau) - G(0)e^{-\tau} + \chi^2 \tilde{\mu}^3 e^\tau M_1.$$

Consequently, from the definition of \mathcal{L}_i , we deduce that

$$\begin{aligned} \frac{d}{d\tau} (e^{\alpha\tau} \mathcal{L}(\tau) + e^{\alpha\tau} g(\tilde{\mu}(\tau))) &\leq \left(\alpha - \frac{2P(\tilde{\mu})}{1 + P(\tilde{\mu})} \right) e^{\alpha\tau} \mathcal{L}_1(\tau) \\ &\quad + \tilde{\mu}(\tau)e^{\alpha\tau} - G(0)e^{(\alpha-1)\tau} + \chi^2 \tilde{\mu}^3 e^{1+\alpha\tau} M_1 \\ &\quad + \alpha e^{\alpha\tau} (g(\tilde{\mu}(\tau)) + \mathcal{L}_2(\tau) + \mathcal{L}_3(\tau)). \end{aligned} \quad (60)$$

Since we have already established that $\lim_{\tau \rightarrow \infty} \tilde{\mu}(\tau) = 0$, it follows that

$$\forall \alpha < 1, \quad \lim_{\tau \rightarrow +\infty} \left(\alpha - \frac{2P(\tilde{\mu}(\tau))}{1 + P(\tilde{\mu}(\tau))} \right) < 0.$$

Therefore the first contribution in the right-hand side of inequality (60) is controlled. It remains to prove that the other contributions of (60) belong to L^1 . For the second line of the right hand side it is an immediate consequence of (45) and (55) together with the fact that $\alpha < 1$. For the third line, recalling the definition (59) of g' together with $g(0) = 0$, we deduce that $g(\tilde{\mu}) = O(\tilde{\mu}^4)$ and therefore using (55) we have

$$\forall \alpha < 2, \quad \int_0^\infty e^{\alpha\tau} g(\tilde{\mu}(\tau)) d\tau < +\infty.$$

Finally the two following inequalities

$$\begin{aligned}\mathcal{L}_2(\tau)e^{\alpha\tau} &\leq \tilde{\mu}(\tau)e^{\alpha\tau}, \\ \mathcal{L}_2(\tau)e^{\alpha\tau} &\leq -\tilde{\mu}(\tau)e^{\alpha\tau} \log G(0) + \tau\tilde{\mu}(\tau)e^{\alpha\tau} + G(0)e^{-(1-\alpha)\tau},\end{aligned}$$

ensure that for $\alpha < 1$

$$\int_0^\infty e^{\alpha\tau}(\mathcal{L}_2(\tau) + \mathcal{L}_3(\tau)) d\tau < +\infty.$$

Summing up altogether, for $\tau \geq \tau_\alpha$, where τ_α is such that $\alpha - \frac{2P(\tilde{\mu})}{1+P(\tilde{\mu})} < 0$, we obtain that there exists $h \in L^1$ such that

$$\frac{d}{d\tau} (e^{\alpha\tau} \mathcal{L}(\tau) + e^{\alpha\tau} g(\tilde{\mu}(\tau))) \leq h.$$

Therefore, $e^{\alpha\tau} \mathcal{L}(\tau) + e^{\alpha\tau} g(\tilde{\mu}(\tau))$ is bounded and since $g(\tilde{\mu}) \geq 0$, so is $e^{\alpha\tau} \mathcal{L}(\tau)$, this achieves the proof of Lemma 6.7.

7 A tangential model for two cells mating

In this section we first study the tangential model (21) and then we enrich it to study cell-cell communication.

In this part for simplicity we will denote $\int \mu dx$ for $\int_\Gamma \mu(t, x) dx$. We recall that $M = \int \mu dx$ is a preserved quantity.

7.1 Global existence for small data for model (21)

In this paragraph we study global existence for small data for model (21) when S is a bounded function in time and space. It is worth noticing that this model does not present any generic nor immediate blow up. We perform log type estimate with subcritical mass. For simplicity we assume here that $\chi = 1$.

Lemma 7.1 *Assume that $\int \mu_0 |\log \mu_0| dx + \int x^2 \mu_0 dx < +\infty$, $C_2 M \|S\|_\infty < 1$ and that S is globally uniformly Lipschitz, then there exists a global solution to (21).*

Proof We differentiate $\int \mu \log \mu dx$ and prove that it decreases. Recalling that \mathcal{H} is an isometry on L^2 , see [27], we have

$$\begin{aligned}\frac{d}{dt} \int \mu \log \mu dx &= - \int \frac{|\partial_x \mu|^2}{\mu} dx - \int \mathcal{H}(S\mu) \partial_x \mu dx \\ &\leq - \int \frac{|\partial_x \mu|^2}{\mu} dx + \|\sqrt{\mu}\|_\infty \left\| \frac{\partial_x \mu}{\sqrt{\mu}} \right\|_2 \|\mathcal{H}(S\mu)\|_2 \\ &\leq - \int \frac{|\partial_x \mu|^2}{\mu} dx + \|\mu\|_\infty^{1/2} \left\| \frac{\partial_x \mu}{\sqrt{\mu}} \right\|_2 \|S\mu\|_2 \\ &\leq - \int \frac{|\partial_x \mu|^2}{\mu} dx + \|\mu\|_\infty^{1/2} \left\| \frac{\partial_x \mu}{\sqrt{\mu}} \right\|_2 \|S\|_\infty \|\mu\|_1^{1/2} \|\mu\|_\infty^{1/2}.\end{aligned}$$

Using next that $\|\mu\|_\infty \leq \|\partial_x \mu\|_1$ and that $\|\mu\|_1 = M$, we obtain

$$\begin{aligned} \frac{d}{dt} \int \mu \log \mu \, dx &\leq - \int \frac{|\partial_x \mu|^2}{\mu} \, dx + C_2 \left\| \frac{\partial_x \mu}{\sqrt{\mu}} \right\|_2 \|S\|_\infty M^{1/2} \|\partial_x \mu\|_1 \\ &\leq - \int \frac{|\partial_x \mu|^2}{\mu} \, dx + C_2 \left\| \frac{\partial_x \mu}{\sqrt{\mu}} \right\|_2^2 \|S\|_\infty M \\ &= (C_2 \|S\|_\infty M - 1) \int \frac{|\partial_x \mu|^2}{\mu} \, dx. \end{aligned} \quad (61)$$

In order to obtain equiintegrability, we need a bound on the second moment. Since all the computations can be performed by imposing firstly that $|x - x'| \geq \varepsilon$ and $|y - y'| \geq \varepsilon$ and then by letting ε go to 0, for simplicity we will omit to write the principal value in the definition of the Hilbert transform, (12). Hence, we obtain

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \int x^2 \mu \, dx &= M - \int x \mu \mathcal{H}(S\mu) \, dx \\ &= M - \frac{1}{2\pi} \iint \frac{xS(y) - yS(x)}{x - y} \mu(t, x) \mu(t, y) \, dx \, dy \\ &= M - \frac{1}{2\pi} \iint \left(S(y) + y \frac{S(y) - S(x)}{x - y} \right) \mu(t, x) \mu(t, y) \, dx \, dy \\ &\leq M + \frac{M^2 \|S\|_\infty}{2\pi} - \frac{1}{2\pi} \iint y \frac{S(y) - S(x)}{x - y} \mu(t, x) \mu(t, y) \, dx \, dy \\ &\leq M + \frac{M^2 \|S\|_\infty}{2\pi} + \frac{\|S'\|_\infty M^{3/2}}{2\pi} \sqrt{\int x^2 \mu \, dx}, \end{aligned} \quad (62)$$

which provides an upper bound on $\int x^2 \mu$. Combining the two estimates (61) and (62), we obtain equiintegrability of the solution.

7.2 Blow up of the joint second moment. Proof of Proposition 1.4

Here we give general condition under which blow up can be established for the system (11)–(13). Let us define I by

$$I(t) := \frac{1}{2} \iint (x - y) \left(\mathcal{H}(S_2 \mu_1)(t, x) - \mathcal{H}(S_1 \mu_2)(t, y) \right) \frac{\mu_1(t, x) \mu_2(t, y)}{M^2} \, dx \, dy. \quad (63)$$

Recalling the definition (16) of \mathcal{E} , we see that

$$\frac{d}{dt} \mathcal{E}(t) = \iint \frac{(x - y)^2}{4M^2} (\mu_1(t, x) \partial_t \mu_2(t, y) + \partial_t \mu_1(t, x) \mu_2(t, y)) \, dx \, dy,$$

hence

$$\frac{d}{dt} \mathcal{E}(t) = 1 - I(t).$$

Before stating a useful Lemma, let us give some notation. For simplicity, we omit the principal value in the definition (12) of the Hilbert transform \mathcal{H} but all the computations can be performed by imposing firstly that $|x - x'| \geq \varepsilon$ and $|y - y'| \geq \varepsilon$ and then by letting ε go to 0. We also omit to write the t variable. Moreover for brevity we will sometimes denote

$$\iiint \iiint_{x, y, x', y'} f(x, y, x', y') \, dy' \, dx' \, dy \, dx = \iiint \iiint f.$$

Lemma 7.2 Assume that $S_1 = S_2 = 1$. As long as the solution (μ_1, μ_2) to (11)–(13) is smooth enough to perform the computations, the following equality holds true:

$$I(t) = \frac{M}{2\pi}.$$

Proof of Lemma 7.2 With the previous notations, a first computation yields that

$$\begin{aligned} I &= \frac{1}{2\pi} \iint (x - y) \left(\int_{x'} \frac{\mu_1(x')}{x - x'} dx' - \int_{y'} \frac{\mu_2(y')}{y - y'} dy' \right) \frac{\mu_1(x)\mu_2(y)}{M^2} dx dy \\ &= \frac{1}{2\pi M^3} \iiint \int (x - y) \left(\frac{1}{x - x'} - \frac{1}{y - y'} \right) \mu_1 \mu_1' \mu_2 \mu_2' \\ &= \frac{1}{4\pi M^3} \iiint \int (x - y - x' + y') \left(\frac{1}{x - x'} - \frac{1}{y - y'} \right) \mu_1 \mu_1' \mu_2 \mu_2' \\ &= \frac{M^4}{2\pi M^3} - \frac{1}{4\pi M^3} \iiint \int \left(\frac{y - y'}{x - x'} + \frac{x - x'}{y - y'} \right) \mu_1 \mu_1' \mu_2 \mu_2', \end{aligned}$$

where we have used that $\iiint \int \mu_1 \mu_1' \mu_2 \mu_2' = M^4$. Using next the symmetry of y, y' and x, x' , it follows that

$$\iiint \int \frac{y - y'}{x - x'} \mu_1 \mu_1' \mu_2 \mu_2' = \iiint \int \frac{x - x'}{y - y'} \mu_1 \mu_1' \mu_2 \mu_2' = 0.$$

The result then follows.

Using the previous Lemma 7.2, for any constant \bar{S} , to be chosen later, for any solution (μ_1, μ_2) to (11) and (14) we have

$$\begin{aligned} \frac{d}{dt} \mathcal{E}(t) &= 1 - \frac{M\bar{S}}{2\pi} \\ &- \underbrace{\frac{1}{2} \iint (x - y) \left(\mathcal{H}((S_2 - \bar{S})\mu_1)(x) - \mathcal{H}((S_1 - \bar{S})\mu_2)(y) \right) \frac{\mu_1(x)\mu_2(y)}{M^2} dx dy}_{=:R}. \end{aligned} \tag{64}$$

Let us choose $\bar{S} = MF_h(0)$, where we recall that $F_h(x) = f_h(x^2)$ are defined by (16) and $f_h(\cdot)$ is a convex function. Then, we have

$$S_i(x) - \bar{S} = \int_{\mathbb{R}} (F_h(x - x') - F_h(0)) \mu_i(x') dx'.$$

Recalling the expression of $\mathcal{H}((S_2 - \bar{S})\mu_1)$ together with the fact that the function F_h is even and omitting the principal value, we deduce that

$$\begin{aligned} \mathcal{H}((S_2 - \bar{S})\mu_1)(x) &= \frac{1}{\pi} \int_{\mathbb{R}} \frac{S_2(x') - \bar{S}}{x - x'} \mu_1(x') dx' \\ &= \frac{1}{\pi} \iint_{\mathbb{R}^2} \frac{F_h(x' - y') - F_h(0)}{x - x'} \mu_1(x') \mu_2(y') dy' dx'. \end{aligned}$$

Consequently R , defined by (64), can be rewritten as

$$R = \frac{1}{2\pi M^2} \iiint \int (x - y) \left(\frac{F_h(x' - y')}{x - x'} - \frac{F_h(y' - x')}{y - y'} \right) \mu_1 \mu_1' \mu_2 \mu_2'. \tag{65}$$

Using the symmetry of the variables, we obtain

$$\begin{aligned}
& \iiint \int (x-y) \frac{F_h(x'-y')}{x-x'} \mu_1 \mu'_1 \mu_2 \mu'_2 \\
&= \frac{1}{2} \iiint \int \frac{(x-y)F_h(x'-y') - (x'-y)F_h(x-y')}{x-x'} \mu_1 \mu'_1 \mu_2 \mu'_2 \\
&= \frac{1}{2} \iiint \int \left((x-y) \frac{F_h(x'-y') - F_h(x-y')}{x-x'} + F_h(x-y') \right) \mu_1 \mu'_1 \mu_2 \mu'_2,
\end{aligned}$$

and similarly,

$$\begin{aligned}
& - \iiint \int (x-y) \frac{F_h(x'-y')}{y-x'} \mu_1 \mu'_1 \mu_2 \mu'_2 \\
&= \frac{1}{2} \iiint \int \left((x-y) \frac{F_h(x'-y) - F_h(x'-y')}{y-y'} + F_h(x'-y') \right) \mu_1 \mu'_1 \mu_2 \mu'_2.
\end{aligned}$$

Hence, R defined by (64), writes as

$$\begin{aligned}
R &= \frac{1}{2\pi M^2} \iiint \int F_h(x-y) \mu_1 \mu'_1 \mu_2 \mu'_2 \\
&+ \frac{1}{4\pi M^2} \iiint \int (x-y) \left(\frac{F_h(x'-y) - F_h(x'-y')}{y-y'} + \frac{F_h(x'-y') - F_h(x-y')}{x-x'} \right) \mu_1 \mu'_1 \mu_2 \mu'_2.
\end{aligned} \tag{66}$$

It remains to prove that $R = \mathcal{O}(\sqrt{\mathcal{E}})$. Recalling that the potential F_h is smooth, we have in particular

$$\left\| \frac{d}{dx} F_h \right\|_{\infty} = K(h) < +\infty.$$

Using the mean value theorem, this allows controlling the second term in the expression (66) of R :

$$\begin{aligned}
& \frac{1}{4\pi} \left| \iiint \int (x-y) \left(\frac{F_h(x'-y) - F_h(x'-y')}{y-y'} + \frac{F_h(x'-y') - F_h(x-y')}{x-x'} \right) \mu_1 \mu'_1 \mu_2 \mu'_2 \right| \\
&\leq \frac{K(h)}{2\pi} \iiint \int |x-y| \mu_1 \mu'_1 \mu_2 \mu'_2.
\end{aligned}$$

Moreover using the Cauchy-Schwarz inequality, it follows that

$$\iiint \int |x-y| \mu_1 \mu'_1 \mu_2 \mu'_2 = M^2 \iint |x-y| \mu_1 \mu_2 \leq 2M^4 \sqrt{\mathcal{E}}.$$

Let us now consider the first term in (66). Recalling that $F_h(x) = f_h(x^2)$ with f_h convex and using Jensen's inequality, we obtain

$$\begin{aligned}
\frac{1}{2\pi} \iiint \int F_h(x'-y') \mu_1 \mu'_1 \mu_2 \mu'_2 &= \frac{M^4}{2\pi} \iint f_h((x-y)^2) \frac{\mu_1 \mu_2}{M^2} \\
&\geq \frac{M^4}{2\pi} f_h \left(\iint (x-y)^2 \frac{\mu_1 \mu_2}{M^2} \right) \\
&= \frac{M^4}{2\pi} f_h(\mathcal{E}).
\end{aligned}$$

Finally, this leads to

$$-\frac{M^4 K(h)}{\pi} \sqrt{\mathcal{E}} \leq -R \leq \frac{M^4 K(h)}{\pi} \sqrt{\mathcal{E}} - \frac{M^4}{2\pi} f_h(\mathcal{E})$$

This gives the blow up result for instance under the (not very explicit) condition: $M^2 F_h(0) < 2\pi$ and $\mathcal{E}(t=0)$ satisfies $\frac{M^4 K(h)}{\pi} \sqrt{\mathcal{E}} - \frac{M^4}{2\pi} f_h(\mathcal{E}) \leq \frac{M^2 F_h(0)}{2\pi} - 1$. This achieves the proof of Proposition 1.4.

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Appendix

8 Computation of the normalisation constant

In order to compute the normalisation constant C in the expression of the function F of equation (14) and to prove that $C = -4\pi$, let us introduce the auxiliary function H defined by

$$H(x, y) = \int_0^\infty \int_{\mathbb{R}} \frac{e^{-t - \frac{y^2 + (x-x')^2}{4t}}}{t} \mu_j(x') dx' dt,$$

then,

$$\frac{\partial}{\partial y} H|_{y=0} = \lim_{y \rightarrow 0} - \int_0^\infty \int_{\mathbb{R}} \frac{y}{2t^2} e^{-t - \frac{y^2 + (x-x')^2}{4t}} \mu_j(x') dx' dt.$$

In order to fix the multiplicative constant C let us compute the value of the following expression:

$$I(y) = - \int_0^\infty \int_{\mathbb{R}} \frac{y}{2t^2} e^{-t - \frac{y^2 + (x-x')^2}{4t}} dx' dt,$$

which rewrites as

$$\begin{aligned} I(y) &= - \int_0^\infty \frac{y}{2t^{3/2}} e^{-t - \frac{y^2}{4t}} \int_{\mathbb{R}} \frac{e^{-\frac{(x-x')^2}{4t}}}{\sqrt{t}} dx' dt \\ &= -2\sqrt{\pi} \int_0^\infty e^{-\frac{y^2}{u^2} - \frac{u^2}{4}} du \\ &\xrightarrow{y \rightarrow 0} -4\pi. \end{aligned}$$

where we have performed the change of variable $u = y/\sqrt{t}$ for $y > 0$. Consequently one obtains that

$$\begin{aligned} \frac{\partial}{\partial y} H|_{y=0} + 4\pi\mu_j(x) &= \lim_{y \rightarrow 0} - \int_0^\infty \int_{\mathbb{R}} \frac{y}{2t^2} e^{-t - \frac{y^2 + (x-x')^2}{4t}} (\mu_j(x') - \mu_j(x)) dx' dt \\ &= \lim_{y \rightarrow 0} - \int_0^\infty \int_{\mathbb{R}} \frac{y}{2t^{3/2}} e^{-t - \frac{y^2 + z^2}{4t}} (\mu_j(x+z) - \mu_j(x)) \frac{dz}{\sqrt{t}} dt \\ &= \lim_{y \rightarrow 0} - \int_0^\infty \int_{\mathbb{R}} \frac{y}{2t^{3/2}} e^{-t - \frac{y^2}{4t} - z^2} (\mu_j(x + \sqrt{t}z) - \mu_j(x)) dz dt \\ &= \lim_{y \rightarrow 0} - \int_0^\infty \int_{\mathbb{R}} e^{-\frac{u^2}{y^2} - \frac{u^2}{4} - z^2} \left(\mu_j\left(x + \frac{yz}{u}\right) - \mu_j(x) \right) dz du \\ &= 0. \end{aligned}$$

9 From symmetric solution to nonsymmetric solutions

In this section we briefly describe the steps to prove quantitative convergence for nonsymmetric initial data as stated in Proposition 1.2.

We first introduce the Lyapunov functional

$$L(t) = \sum_{i=1,2} (1 - \mu_i) \mathcal{H}(\tilde{n}_i | n_+).$$

For $i = 1, 2$, we denote by $D_i^2 = (1 - \mu_i) I(\tilde{n}_i | \chi_{\mu_1 \mu_2} e^{-\chi_{\mu_1 \mu_2} x})$ the dissipations and, performing similar computations as in the symmetric case, we obtain that

$$\begin{aligned} \frac{d}{dt} L(t) &= - \sum_{i=1,2} \left(D_i^2 + \underbrace{\log \frac{\mu_i}{n(t,0)} \frac{d}{dt} \mu_i}_{\leq 0} + \log \frac{(1 - \mu_i)n_+(0)}{\mu_i} \frac{d}{dt} \mu_i \right. \\ &\quad \left. + \chi(\mu_1 \mu_2 - \mu_+^2) (-n_i(t,0) + \chi_{\mu_1 \mu_2} (1 - \mu_i)) \right). \end{aligned} \tag{67}$$

Let the function f_{log} be defined by

$$f_{log}(\mu_+) = 0, \quad f'_{log}(\mu) = \log \frac{\mu}{(1-\mu)n_+(0)} = \log \frac{\mu(1-\mu_+)}{(1-\mu)\mu_+}.$$

Since $f'_{log}(\mu_+) = 0$ and the second derivative satisfies $f''_{log}(\mu) > 0$ for $\mu \in (0, 1)$, locally in the neighborhood of μ_+ , the function f_{log} behaves as $(\mu - \mu_+)^2$. Next, we see that

$$\begin{aligned} \chi(\mu_1\mu_2 - \mu_+^2)(n_1(t, 0) + n_2(t, 0)) &= 2 \frac{d}{dt} f\left(\frac{\mu_1 + \mu_2}{2}\right) \\ &+ \chi\left(\left(\frac{\mu_1 + \mu_2}{2}\right)^2 - \mu_+^2\right)(\mu_1 + \mu_2) - \chi \frac{(\mu_1^2 + \mu_2^2)}{4}(n_1(t, 0) + n_2(t, 0)), \end{aligned}$$

where the function f is defined by

$$f(\mu) = \chi\left(\frac{\mu^3}{3} - \mu_+^2\mu + 2\frac{\mu_+^3}{3}\right) = \chi\left(\frac{(\mu - \mu_+)^2(\mu + 2\mu_+)}{3}\right).$$

Let us define the functional \tilde{L} by

$$\tilde{L}(t) = L(t) + f_{log}(\mu_1) + f_{log}(\mu_2) + 2f\left(\frac{\mu_1 + \mu_2}{2}\right).$$

From the comparison principle and the HWI inequality, it follows that $D_1^2 + D_2^2 \geq cL^2(t)$, hence (67) reads

$$\frac{d}{dt} \tilde{L}(t) \leq -cL(t)^2 + g(\mu_1, \mu_2) + \chi \frac{(\mu_1^2 + \mu_2^2)}{4}(n_1(t, 0) + n_2(t, 0)),$$

where g is defined by

$$g(\mu_1, \mu_2) = \chi^2 \mu_1 \mu_2 (\mu_1 \mu_2 - \mu_+^2)(2 - \mu_1 - \mu_2) - \chi \left(\left(\frac{\mu_1 + \mu_2}{2} \right)^2 - \mu_+^2 \right) (\mu_1 + \mu_2).$$

Since

$$g(\mu_+, \mu_+) = 0, \quad \nabla g|_{(\mu_+, \mu_+)} = 0,$$

and the matrix $\nabla^2 g|_{(\mu_+, \mu_+)}$ is symmetric definite negative, then, locally in the neighborhood of (μ_+, μ_+) , there exists a positive constant still denoted by c such that

$$g(\mu_1, \mu_2) \leq -c((\mu_1 - \mu_+)^2 + (\mu_2 - \mu_+)^2).$$

Therefore, for $t \geq t_0$ large enough, so that the μ_i are close enough to μ_+ , up to a change of the value of the constant $c > 0$, we have

$$\frac{d}{dt} \tilde{L}(t) \leq -c\tilde{L}(t)^2 + \chi \frac{(\mu_1^2 + \mu_2^2)}{4}(n_1(t, 0) + n_2(t, 0)).$$

Note that in the symmetric case, we were able to conclude here. In the non-symmetric case, by using the comparison principle, we deduce that

$$\frac{d}{dt} \tilde{L}(t) \leq -c\tilde{L}(t)^2 + \frac{C}{1+t}(n_1(t, 0) + n_2(t, 0)), \quad (68)$$

from which it follows that \tilde{L} is bounded. Indeed, let us denote $F(t) = (1+t)\tilde{L}(t)$, then the previous inequality reads

$$F'(t) \leq \underbrace{\tilde{L}(t) - c(1+t)\tilde{L}(t)^2}_{\leq 1/(4c)} + C(n_1(t,0) + n_2(t,0)),$$

which becomes after integration and recalling that $\frac{d}{dt}\mu_i = n_i(t,0) - \mu_i$:

$$F(t) \leq F(0) + \frac{t}{4c} + C(\mu_1 + \mu_2) + C \int_0^t (\mu_1 + \mu_2) \leq F(0) + 2C + t \left(\frac{1}{4c} + 2C \right),$$

hence dividing back by $1+t$

$$\tilde{L}(t) \leq \tilde{L}(0) + 2C + \left(\frac{1}{4c} + 2C \right) = L_\infty.$$

Furthermore, for $t \geq t' \geq t_0$, after an integration by parts we have

$$\begin{aligned} \int_{t'}^t \frac{n_i(s,0)}{1+s} ds &= \left[\frac{\mu_i}{1+s} \right]_{t'}^t + \int_{t'}^t \frac{\mu_i}{(1+s)^2} ds + \int_{t'}^t \frac{\mu_i}{1+s} ds \\ &\leq \frac{1}{1+t} + \frac{1}{1+t'} + \log \frac{1+t}{1+t'} \\ &\leq \frac{2}{1+t'} + \log \frac{1+t}{1+t'}, \end{aligned}$$

hence integrating (68) for $t \geq t' \geq t_0$, it follows that

$$\tilde{L}(t) - \tilde{L}(t') + c \int_{t'}^t \tilde{L}(s)^2 ds \leq \frac{C}{1+t'} + C \log \frac{1+t}{1+t'}.$$

We already know that $\tilde{L}(t) \geq 0$ for all $t \geq t_0$. Let us now prove that $\limsup_{t \rightarrow \infty} \tilde{L}(t) = 0$. Firstly, for all $h > 0$ such that $t \geq t_0(1+h)$, we have

$$\tilde{L}(t) - \tilde{L}\left(\frac{t}{1+h}\right) \leq \frac{C(1+h)}{1+t} + C \log(1+h) \leq C \left(\frac{(1+h)}{1+t} + h \right).$$

hence,

$$\tilde{L}\left(\frac{t}{1+h}\right) \geq \tilde{L}(t) - \frac{C(1+h)}{1+t} - Ch.$$

Assume that $\limsup_{t \rightarrow \infty} \tilde{L}(t) > 0$, then there exists $\varepsilon > 0$ and a sequence $t_n \rightarrow +\infty$ such that $\tilde{L}(t_n) \geq \varepsilon > 0$. In such a case applying the previous inequality with $t = t_n$ with n large enough and with $0 \leq h \leq \frac{\varepsilon}{4C}$, it yields that

$$\forall t \in \left[\frac{t_n}{1 + \frac{\varepsilon}{4C}}, t_n \right], \quad \tilde{L}(t) \geq \varepsilon/2,$$

hence, denoting by $t'_n = \frac{t_n}{1 + \frac{\varepsilon}{4C}}$, we have

$$c \frac{\varepsilon^2}{4} (t_n - t'_n) \leq \tilde{L}(t_n) + c \int_{t'_n}^{t_n} \tilde{L}(s)^2 ds \leq \varepsilon/2 + L_\infty + C \log(1+t'_n).$$

In particular we see that

$$t_n \times c \frac{\varepsilon^3}{4} \leq \varepsilon/2 + L_\infty + C \log(1+t_n),$$

which contradicts $t_n \rightarrow +\infty$.

10 Self similar convergence

For the extension of the results of section 6 to nonsymmetric initial data, we first notice that some results are immediate consequence of the comparison principle. Moreover we see that

$$\forall \alpha < 1, \quad \int_0^\infty \tilde{\mu}_i e^{\alpha \tau} d\tau < +\infty.$$

We use the same notations than before except that now

$$\mathcal{L} = \mathcal{L}^1 + \mathcal{L}^2,$$

the superscript denoting the fact that $u^i, \tilde{\mu}_i$ replaces $u, \tilde{\mu}$. Similarly, the bounds on the moments remain valid. Then, we have

$$\forall j = 1, 2, \quad i = 2, 3, \quad \sup_\tau \mathcal{L}_i^j e^{\alpha \tau} < +\infty.$$

Therefore, following the lines of the computations (52), by replacing $\tilde{\mu}^2$ by $\tilde{\mu}_1 \tilde{\mu}_2$, we obtain

$$\frac{d}{d\tau} \mathcal{L}(\tau) \leq - \sum_{i=1,2} \left(\frac{1}{2} (I(u_i|G) + I(u_i|G_\mu)) + O(\tilde{\mu}_i^2 e^\tau + \tilde{\mu}_i^3 e^{2\tau}) \right).$$

As a consequence, using again the logarithmic Sobolev inequality, it yields that

$$\frac{d}{d\tau} (e^{\alpha \tau} \mathcal{L}(\tau)) \leq (\alpha - 1)(\mathcal{L}_1^1 + \mathcal{L}_1^2) + h,$$

where h denotes the rest which belongs to L^1 . Consequently, we conclude that

$$\sup_\tau e^{\alpha \tau} \mathcal{L}(\tau) < +\infty, \quad \forall \alpha < 1.$$

11 Linear stability. Proof of Proposition 1.1

In this section we prove Proposition 1.1. We first compute the steady states and then we perform a linear stability analysis.

Lemma 11.1 *The steady states $(\bar{n}_i, \bar{\mu}_i)_{i=1,2}$ of the system (1)–(2)–(3) with (5) satisfy $\bar{n}_i = \bar{n}$ and $\bar{\mu}_i = \bar{\mu}$, for $i = 1, 2$, where $(\bar{n}, \bar{\mu})$ is solution to*

$$\begin{cases} \bar{n}(x) = \bar{\mu} \exp(-\chi \bar{\mu}^2 x), & x \geq 0, \\ 0 = \chi \bar{\mu}^2 - \chi \bar{\mu} + 1. \end{cases} \quad (69)$$

Such a steady state exists iff $\chi \geq 4$. Moreover, in the case where $\chi > 4$, there are two steady states, $(\mu_- e^{-\chi \mu_-^2 x}, \mu_-)$ and $(\mu_+ e^{-\chi \mu_+^2 x}, \mu_+)$ with $0 < \mu_- < \mu_+ < 1$.

Proof The only part to prove is that any steady state has identical values for the two cells. A straightforward computation yields that $(\bar{n}_i, \bar{\mu}_i)_{i=1,2}$ satisfies, for $i = 1$ and 2 , the following system:

$$\begin{cases} \bar{n}_i(x) = \bar{\mu}_i \exp(-\chi \bar{\mu}_1 \bar{\mu}_2 x), & x \geq 0, \\ 0 = \chi \bar{\mu}_1 \bar{\mu}_2 - \chi \bar{\mu}_i + 1. \end{cases}$$

From the second equation in the previous system it follows that $\bar{\mu}_1 = \bar{\mu}_2$, hence it yields to (69) which admits a real solution $\bar{\mu}$ iff $\chi \geq 4$.

Lemma 11.2 Assume that $\chi > 4$, and let $\bar{\mu} \in \{\mu_-, \mu_+\}$. The linearized system associated to (1)–(2)–(3) with (5) does not admit 0 as an eigenvalue. Moreover, a complex $\lambda \neq 0$ satisfying $Re(\lambda) \geq 0$ is an eigenvalue of the linear system if and only if it satisfies

$$(\chi \bar{\mu}^2 + \beta_1) \left(\lambda + 2 \frac{\chi^2 \bar{\mu}^4}{\lambda} + 1 \right) + 2 \chi \bar{\mu}^2 - \lambda = 0, \quad (70)$$

where β_1 is the unique root of the equation $X^2 + \chi \bar{\mu}^2 X - \lambda = 0$ satisfying $Re(\beta_1) < -\chi \bar{\mu}^2$.

Proof of Lemma 11.2 Let us consider zero mass perturbations around the steady state $(\bar{n}, \bar{\mu})$ solution to (69). For $i = 1$ or 2, we define

$$\begin{cases} n_i(t, x) = \bar{n}(x) + \tilde{n}_i(x) \exp(\lambda t), & x \geq 0, \\ \mu_i(t) = \bar{\mu} + \tilde{\mu}_i \exp(\lambda t), \\ 0 = \int_0^{+\infty} \tilde{n}_i(t, x) dx + \tilde{\mu}_i \exp(\lambda t), \end{cases} \quad (71)$$

where we assume that $\lambda \in \mathbb{C}$. We linearize (1)–(2)–(3) and we obtain two systems for $i = 1, 2$:

$$\begin{cases} \lambda \tilde{n}_i(x) = \tilde{n}_i''(x) + \chi \bar{\mu}^2 \tilde{n}_i'(x) - \chi^2 \bar{\mu}^4 \exp(-\chi \bar{\mu}^2 x) (\tilde{\mu}_1 + \tilde{\mu}_2), \\ \lambda \tilde{\mu}_i = \tilde{n}_i(0) - \tilde{\mu}_i, \\ \lambda \tilde{\mu}_i = \tilde{n}_i'(0) + \chi \bar{\mu}^2 (\tilde{n}_i(0) + \tilde{\mu}_1 + \tilde{\mu}_2). \end{cases} \quad (72)$$

We first investigate the condition for having $Re(\lambda) \geq 0$ (linear instability). In the case where $\chi > 4$, the roots of $X^2 + \chi \bar{\mu}^2 X - \lambda = 0$ are

$$\beta_1 = \frac{-\chi \bar{\mu}^2 - \sqrt{(\chi \bar{\mu}^2)^2 + 4\lambda}}{2} \quad \text{and} \quad \beta_2 = \frac{-\chi \bar{\mu}^2 + \sqrt{(\chi \bar{\mu}^2)^2 + 4\lambda}}{2}.$$

Here we have abusively denoted by $\sqrt{(\chi \bar{\mu}^2)^2 + 4\lambda}$ the only complex number with positive real part satisfying $z^2 = (\chi \bar{\mu}^2)^2 + 4\lambda$. With these notations, we have $Re(\beta_1) \leq -\chi \bar{\mu}^2$, $Re(\beta_2) \geq 0$, the inequalities being strict as soon as we have $\lambda \neq 0$. Hence, the solution to (72) can be written as:

$$\tilde{n}_i(x) = C_i \exp(\beta_1 x) + D_i \exp(\beta_2 x) - \frac{\chi^2 \bar{\mu}^4}{\lambda} \exp(-\chi \bar{\mu}^2 x) (\tilde{\mu}_1 + \tilde{\mu}_2). \quad (73)$$

The case $\lambda = 0$ leads immediately to $\tilde{\mu}_1 + \tilde{\mu}_2 = 0$ and $\tilde{n}_i(x) = \tilde{\mu}_i e^{-\chi \bar{\mu}^2 x}$ so that the constraint on the mass leads to $\tilde{\mu}_i (1 + \frac{1}{\chi \bar{\mu}^2}) = 0$ and therefore $\tilde{\mu}_i = 0$. Consequently 0 is not an eigenvalue for the linearized system. From now on, we can assume that

$$Re(\lambda) \geq 0, \quad \lambda \neq 0, \quad Re(\beta_1) < -\chi \bar{\mu}^2.$$

Firstly, one can notice that, since we consider integrable perturbations we deduce that $D_i = 0$. The last two equations of (72) now read as the two systems

$$\begin{cases} (\lambda + 1) \tilde{\mu}_i - C_i = -\frac{\chi^2 \bar{\mu}^4}{\lambda} (\tilde{\mu}_1 + \tilde{\mu}_2), \\ \lambda \tilde{\mu}_i - (\chi \bar{\mu}^2 + \beta_1) C_i = \chi \bar{\mu}^2 (\tilde{\mu}_1 + \tilde{\mu}_2). \end{cases} \quad (74)$$

Since $\chi\bar{\mu}^2\beta_1 + \beta_1^2 = \lambda$, we notice that

$$\det \begin{pmatrix} \lambda + 1 & -1 \\ \lambda & -(\chi\bar{\mu}^2 + \beta_1) \end{pmatrix} = \underbrace{(\chi\bar{\mu}^2 + \beta_1)}_{\neq 0} \underbrace{(-\lambda - 1 + \beta_1)}_{\text{Re} < -\chi\bar{\mu}^2 - 1} \neq 0.$$

We are led to the constraint $C_1 = C_2 = C$ and $\tilde{\mu}_1 = \tilde{\mu}_2 := \tilde{\mu}$. Writing back system (74) in terms of C, μ only, the existence of a nonzero solution to (74) is then equivalent to the degeneracy of the obtained system i.e. (70) is fulfilled.

Lemma 11.3 *There exists a positive real eigenvalue for the linearized system if and only if $\chi\bar{\mu}^2 < 1$.*

Proof of lemma 11.3 Denoting by $x = \frac{\lambda}{\chi^2\bar{\mu}^4}$ and $\gamma = \chi\bar{\mu}^2$ and recalling the expression of β_1 , we obtain

$$\frac{\gamma}{2} \left((1 - \sqrt{1 + 4x}) \left(\gamma^2 x + \frac{2}{x} + 1 \right) + 4 - 2\gamma x \right) = 0,$$

hence, after multiplication by $2/\gamma$, we look for a positive root x to

$$f(x) := (1 - \sqrt{1 + 4x}) \left(\gamma^2 x + \frac{2}{x} + 1 \right) + 4 - 2\gamma x.$$

We first note that $f(0) = 0$. Differentiating f we obtain:

$$f'(x) = -\frac{2}{\sqrt{1 + 4x}} \left(\gamma^2 x + \frac{2}{x} + 1 \right) + (1 - \sqrt{1 + 4x}) \left(\gamma^2 - \frac{2}{x^2} \right) - 2\gamma,$$

and we see that $f'(x) \sim 2 - 2\gamma$ when $x \rightarrow 0$. The function is decreasing from 0 in the case where $\gamma \geq 1$ while it increases initially and then goes to $-\infty$ in the case $\gamma < 1$. There exists a positive root if and only if $\gamma < 1$.

Lemma 11.4 *Assume that $\chi\bar{\mu}^2 > 1$, then there does not exist $\lambda \in \{z \neq 0, \text{Re}(z) \geq 0\}$ satisfying (70).*

Proof of Lemma 11.4 We keep the notation $\gamma = \chi\bar{\mu}^2$ and $x = \lambda\gamma^{-2}$. The function f can be extended on $\{z \neq 0, \text{Re}(z) \geq 0\}$ keeping the (abusive) notation $z = \sqrt{1 + 4x}$ in the definition of f (since there is a unique root of $z^2 = 1 + 4x$ with $\text{Re}(z) \geq 1$). We notice that cancelling f implies cancelling

$$A(z) = (1 - z) \left(\gamma^2 \frac{z^2 - 1}{4} + \frac{8}{z^2 - 1} + 1 \right) + 4 - \gamma \frac{z^2 - 1}{2}.$$

Since we are looking only for roots in $\{z \neq 1, \text{Re}(z) \geq 1\}$ it is easier to look for zeros of the function

$$B(z) := (z + 1)A(z) = - \left(\gamma^2 \frac{(z^2 - 1)^2}{4} + 8 + z^2 - 1 \right) + 4(z + 1) - \gamma \frac{(z^2 - 1)(z + 1)}{2}.$$

B is a four degree polynomial. As f does for $x = 0$, B vanishes for $z = 1$.

$$\begin{aligned} B(X) &= (X - 1) \left(-\frac{\gamma^2}{4} (X - 1)(X + 1)^2 - (X + 1) + 4 - \frac{\gamma}{2} (X + 1)^2 \right) \\ &=: (X - 1)C(X). \end{aligned} \tag{75}$$

Now, finding an eigenvalue with positive real part is equivalent to finding a root $z \neq 1$ of C with $\text{Re}(z) \geq 1$.

Lemma 11.5 Assume $\gamma > 1$, then the polynomial C , defined by (75), does not have any root satisfying $Re(z) > 1$.

Proof of lemma 11.5 Since $\gamma > 1$, we already know that such a root cannot be real. It has to be complex. It might be convenient to rewrite the polynome C as $C(X) = D(X + 1)$ where

$$D(X) = -\frac{\gamma^2}{4}(X - 2)X^2 - X + 4 - \frac{\gamma}{2}X^2 = -\frac{\gamma^2}{4}X^3 + \left(\frac{\gamma^2}{2} - \frac{\gamma}{2}\right)X^2 - X + 4.$$

Finding a root z of C such that $Re(z) > 1$ is equivalent to finding a root z of D such that $Re(z) > 2$. We already know that no real root can satisfy this property. Therefore, we assume that D , which is of degree 3, has two complex conjugate roots and one real one, which we denote respectively by z, \bar{z} and x . Since $D(0) = 4$ and $D < 0$ on $[2, \infty)$, we know that $x \in (0, 2)$. Using the relations between roots and coefficients, we obtain

$$2Re(z) + x = 2 - \frac{2}{\gamma}, \quad Re(z) = 1 - \frac{x}{2} - \frac{1}{\gamma} < 2.$$

This ends the proof of lemma 11.5, hence the proof of lemma 11.4.

To achieve the proof of the proposition 11.1, we only need now to notice the following property

$$\gamma_- = \chi\mu_-^2 < 1 < \gamma_+ = \chi\mu_+^2.$$

The stability results stated in Proposition 1.1 are deduced from Lemma 11.1 and Lemma 11.5.