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► **To cite this version:**

Zeya Jia, Jiang Zeng. Expansions in Askey-Wilson polynomials via Bailey transform. Journal of Mathematical Analysis and Applications, 2017, 10.1016/j.jmaa.2017.03.056 . hal-01498439

**HAL Id: hal-01498439**

**<https://hal.science/hal-01498439v1>**

Submitted on 30 Mar 2017

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# EXPANSIONS IN ASKEY-WILSON POLYNOMIALS VIA BAILEY TRANSFORM

ZEYA JIA AND JIANG ZENG

ABSTRACT. We prove a general expansion formula in Askey-Wilson polynomials using Bailey transform and Bressoud inversion. As applications, we give new proofs and generalizations of some recent results of Ismail-Stanton and Liu. Moreover, we prove a new  $q$ -beta integral formula involving Askey-Wilson polynomials, which includes the Nassrallah-Rahman integral as a special case. We also give a bootstrapping proof of Ismail-Stanton's recent generating function of Askey-Wilson polynomials.

## 1. INTRODUCTION

Andrews [2] demonstrates that  $q$ -orthogonal polynomials can play an important role in the theory of mock theta functions by applying the following expansion of a terminating, balanced  ${}_5\phi_4$  in a series of Askey-Wilson polynomials, [2, (1.3)],

$$(1.1) \quad {}_5\phi_4 \left[ \begin{matrix} q^{-N}, \rho_1, \rho_2, b, c \\ \rho_1 \rho_2 q^{-N}/a, e, f, g \end{matrix}; q, q \right] = \frac{(aq/\rho_1, aq/\rho_2; q)_N}{(aq, aq/\rho_1 \rho_2; q)_N} \\ \times \sum_{n=0}^{\infty} \frac{(q^{-N}, \rho_1, \rho_2, a; q)_n (1 - aq^{2n})}{(q, aq/\rho_1, aq/\rho_2, aq^{N+1}; q)_n (1 - a)} \left( \frac{aq^{N+1}}{\rho_1 \rho_2} \right)^n {}_4\phi_3 \left[ \begin{matrix} q^{-n}, aq^n, b, c \\ e, f, g \end{matrix}; q, q \right],$$

where  $N$  is a non-negative integer, and  $qabc = efg$ .

As a follow-up to [2], Ismail and Stanton [12] show that Andrews' formula (1.1) is one of many similar expansion formulae in the Askey-Wilson polynomials. In particular, they prove the transformation formula:

$$(1.2) \quad {}_{p+1}\phi_p \left[ \begin{matrix} a_1, \dots, a_{p-1}, t_4/z, t_4 z \\ t_1 t_4, t_2 t_4, t_3 t_4, b_1, \dots, b_{p-3} \end{matrix}; q, \delta \right] \\ = \sum_{k=0}^{\infty} P_k(x; \mathbf{t} | q) \frac{(a_1, \dots, a_{p-1}; q)_k}{(t_1 t_4, t_2 t_4, t_3 t_4, b_1, \dots, b_{p-3}; q)_k} \\ \times \frac{(-t_4 \delta)^k q^{\binom{k}{2}}}{(q, t_1 t_2 t_3 t_4 q^{k-1}; q)_k} {}_{p-1}\phi_{p-2} \left[ \begin{matrix} a_1 q^k, \dots, a_{p-1} q^k \\ b_1 q^k, \dots, b_{p-3} q^k, t_1 t_2 t_3 t_4 q^{2k} \end{matrix}; q, \delta \right],$$

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*Date:* March 30, 2017.

*Mathematics Subject Classification(2010)* 33D45, 33D05, 05A15.

*Keywords:* Andrews formula, Askey-Wilson polynomials, Bailey transform, Bressoud inversion, Nassrallah-Rahman integral, expansion formulae.

where  $x = \cos \theta$  and  $z = e^{i\theta}$ , the Askey-Wilson polynomials are defined by

$$(1.3) \quad P_n(x; \mathbf{t}|q) = t_1^{-n} (t_1 t_2, t_1 t_3, t_1 t_4; q)_{n4} \phi_3 \left[ \begin{matrix} q^{-n}, t_1 t_2 t_3 t_4 q^{n-1}, t_1 e^{i\theta}, t_1 e^{-i\theta} \\ t_1 t_2, t_1 t_3, t_1 t_4 \end{matrix} ; q, q \right].$$

Note that taking  $p = 4$ ,  $a_1 = q^{-N}$ ,  $a_2 = \rho_1$ ,  $a_3 = \rho_2$ ,  $b_1 = \rho_1 \rho_2 q^{-N}/a$ ,  $u = 1$  and  $\delta = z$  in (1.2) the  ${}_{p-1}\phi_{p-2}$ , namely  ${}_3\phi_2$ , series at the right-hand side of the transformation can be summed by  $q$ -Pfaff-Saalschütz sum (7.4), we obtain the following result of Liu [17, Theorem 10.1], for any non-negative  $N$  and  $|z| < 1$ ,

$$(1.4) \quad {}_5\phi_4 \left[ \begin{matrix} q^{-N}, \rho_1, \rho_2, b, c \\ \rho_1 \rho_2 q^{-N}/a, e, f, g \end{matrix} ; q, z \right] = \frac{(aq/\rho_1, aq/\rho_2; q)_N}{(aq, aq/\rho_1 \rho_2; q)_N} \\ \times \sum_{n=0}^N \frac{(q^{-N}, \rho_1, \rho_2, a; q)_n (1 - aq^{2n})}{(q, aq/\rho_1, aq/\rho_2, aq^{N+1}; q)_n (1 - a)} \left( \frac{aq^{N+1}}{\rho_1 \rho_2} \right)^n {}_4\phi_3 \left[ \begin{matrix} q^{-n}, aq^n, b, c \\ e, f, g \end{matrix} ; q, z \right].$$

The above formula is an extension of Watson's transformation (7.9). Moreover, the  $z = q$  case corresponds to Andrews's result (1.1) if  $aqbc = efg$ .

This paper arose from the desire to understand the Ismail-Stanton formula (1.2) through Bailey's machinery. Actually, Ismail-Stanton derived (1.2) from an expansion formula due to Ismail-Rahman [9], see also [11], which was proved using the orthogonality relation of Askey-Wilson polynomials, while Andrews' original proof of (1.1) used Bailey's transform with a special Bailey pair, which is equivalent to an inversion relation [3, (12.2.8)]. Looking at Ismail-Stanton's formula through Bailey's glance and using an inversion formula due to Bressoud [5], we are able to generalize formula (1.2) in several ways, see Proposition 2.3, Proposition 2.5 and Theorem 2.6.

A fundamental result about Askey-Wilson polynomials is the Askey-Wilson  $q$ -beta integral,

$$(1.5) \quad \int_0^\pi \frac{h(\cos 2\theta; 1)}{h(\cos \theta; t_1, t_2, t_3, t_4)} d\theta = \frac{2\pi (t_1 t_2 t_3 t_4; q)_\infty}{(q; q)_\infty \prod_{1 \leq r < s \leq 4} (t_r t_s; q)_\infty},$$

where  $\max\{|t_1|, |t_2|, |t_3|, |t_4|\} < 1$  and

$$h(\cos \theta; t_1, \dots, t_r) = \prod_{j=1}^r (t_j e^{i\theta}, t_j e^{-i\theta}; q)_\infty.$$

Nassrallah-Rahman [18] obtained the following important generalization of (1.5)

$$(1.6) \quad \int_0^\pi \frac{h(\cos 2\theta; 1) h(\cos \theta; t_6)}{h(\cos \theta; t_1, t_2, t_3, t_4, t_5)} d\theta = \frac{2\pi (t_6/t_1, t_6 t_1, t_1 t_3 t_4 t_5, t_1 t_2 t_3 t_5, t_1 t_2 t_3 t_4, t_1 t_2 t_4 t_5; q)_\infty}{\prod_{1 \leq r < s \leq 5} (t_r t_s; q)_\infty (q, t_1^2 t_2 t_3 t_4 t_5; q)_\infty} \\ \times {}_8W_7 \left( t_1^2 t_2 t_3 t_4 t_5 / q, t_1 t_5, t_1 t_2, t_1 t_3, t_1 t_4, t_1 t_2 t_3 t_4 t_5 / t_6 ; q, t_6 / t_1 \right),$$

where  $\max\{|t_1|, |t_2|, |t_3|, |t_4|, |t_5|, |t_6|\} < 1$ .

When  $t_6 = t_1 t_2 t_3 t_4 t_5$ , the above  ${}_8W_7$  reduces to 1 and (1.6) becomes the following appealing formula [3, 19]:

$$(1.7) \quad \int_0^\pi \frac{h(\cos 2\theta; 1) h(\cos \theta; t_1 t_2 t_3 t_4 t_5)}{h(\cos \theta; t_1, t_2, t_3, t_4, t_5)} d\theta = \frac{2\pi (t_1 t_2 t_3 t_4, t_1 t_2 t_3 t_5, t_1 t_2 t_4 t_5, t_1 t_3 t_4 t_5, t_2 t_3 t_4 t_5; q)_\infty}{(q; q)_\infty \prod_{1 \leq r < s \leq 5} (t_r t_s; q)_\infty}.$$

By combining Theorem 2.6 and (1.7) we will generalize the Nassrallah-Rahman integral (1.6) in Theorem 2.7, which includes also two integrals of Liu [17, Theorem 1.6] and Zhang-Wang [21, Theorem 4.3].

This paper is organized as follows. In Section 2, we first state and prove a general transformation, Proposition 2.3, and then derive two interesting expansions in Theorems 2.5 and 2.6. Moreover, we give a generalization of Nassrallah-Rahman integral (1.6) in Theorem 2.7. In Section 3, we derive some recent known results in [9, 10, 12] from our main results. In Section 4, we show how to derive some important known  $q$ -integrals from (2.10). In Section 5, we give a “bootstrapping proof” of Ismail-Stanton’s generating function for Askey-Wilson polynomials. In Section 6, we give two general transformations and show how to recover two transformations of Ismail-Stanton and Verma [12, 20].

Throughout this paper, we assume that  $q$  is a complex number such that  $0 < |q| < 1$  and use standard  $q$ -notations in [7, 8]. Moreover, in Section 7, for the reader’s convenience, we list all summation and transformation formulae used in our proofs.

## 2. MAIN RESULTS

Our starting point is the Bailey transform, see [3, Chap. 12] for a gentle introduction.

**Lemma 2.1** (Bailey transform). *Subject to conditions on the four sequences  $\alpha_n$ ,  $\beta_n$ ,  $\gamma_n$  and  $\delta_n$  which make all the infinite series absolutely convergent, if*

$$(2.1) \quad \beta_n = \sum_{r=0}^n \alpha_r \nu_{n-r} \nu_{n+r},$$

and

$$(2.2) \quad \gamma_n = \sum_{r=n}^{\infty} \delta_r \nu_{r-n} \nu_{r+n},$$

then

$$(2.3) \quad \sum_{n=0}^{\infty} \alpha_n \gamma_n = \sum_{n=0}^{\infty} \beta_n \delta_n.$$

For our purpose we need to choose suitable sequences  $(\nu_n, \nu_n)$  so that (2.1) can be inverted. First, we recall the following matrix inversion due to Bressoud [5].

**Lemma 2.2** (Bressoud’s inversion). *For  $n, k \geq 0$  let*

$$(2.4) \quad C_{n,k}(a, b) = \frac{(1 - aq^{2k})(b; q)_{n+k}(b/a; q)_{n-k}(b/a)^k}{(1 - a)(aq; q)_{n+k}(q; q)_{n-k}}.$$

The following inversion formula holds true

$$(2.5) \quad \beta'_n = \sum_{k=0}^n C_{n,k}(a, b) \alpha'_k \iff \alpha'_n = \sum_{k=0}^n C_{n,k}(b, a) \beta'_k.$$

**Proposition 2.3.** *We have*

$$(2.6) \quad \sum_{n=0}^{\infty} \beta_n \delta_n = \sum_{n=0}^{\infty} \frac{(1 - aq^{2n})(a; q)_n (a/b; q)_n (b/a)^n}{(1 - a)(bq; q)_n (q; q)_n} \\ \times \sum_{k=0}^n \frac{(1 - bq^{2k})(aq^n; q)_k (q^{-n}; q)_k}{(1 - b)(bq^{n+1}; q)_k (bq^{1-n}/a; q)_k} q^k \beta_k \cdot \sum_{r=0}^{\infty} \frac{(b/a; q)_r (b; q)_{r+2n}}{(q; q)_r (aq; q)_{r+2n}} \delta_{r+n},$$

subject to conditions on the two sequences  $\beta_n, \delta_n$  which make all the infinite series absolutely convergent.

*Proof.* Substituting  $\alpha'_k$  by  $\frac{(1-a)(a/b)^k}{(1-aq^{2k})}\alpha_k$  and  $\beta'_n$  by  $\beta_n$  in (2.5) we obtain

$$(2.7) \quad \beta_n = \sum_{k=0}^n \frac{(b/a; q)_{n-k} (b; q)_{n+k}}{(q; q)_{n-k} (aq; q)_{n+k}} \alpha_k.$$

Inverting (2.7) using (2.5) we obtain

$$\alpha_n = \frac{(1 - aq^{2n})(b/a)^n}{(1 - a)} \sum_{k=0}^n \frac{(1 - bq^{2k})(a; q)_{n+k} (a/b; q)_{n-k} (a/b)^k}{(1 - b)(bq; q)_{n+k} (q; q)_{n-k}} \beta_k \\ = \frac{(1 - aq^{2n})(a; q)_n (a/b; q)_n (b/a)^n}{(1 - a)(bq; q)_n (q; q)_n} \sum_{k=0}^n \frac{(1 - bq^{2k})(aq^n; q)_k (q^{-n}; q)_k}{(1 - b)(bq^{n+1}; q)_k (bq^{1-n}/a; q)_k} q^k \beta_k.$$

In view of (2.7) and (2.1), we choose two sequences  $(v_n, \nu_n)$  as

$$v_n = \frac{(b/a; q)_n}{(q; q)_n} \quad \text{and} \quad \nu_n = \frac{(b; q)_n}{(aq; q)_n}.$$

Then, we can compute  $\gamma_n$  by (2.2)

$$\gamma_n = \sum_{r=n}^{\infty} \delta_r \nu_{r-n} v_{r+n} \\ = \sum_{r=0}^{\infty} \delta_{r+n} \frac{(b/a; q)_r (b; q)_{r+2n}}{(q; q)_r (aq; q)_{r+2n}}.$$

Plugging the four sequences  $\alpha_n, \beta_n, \gamma_n$  and  $\delta_n$  into the Bailey transform (2.3) yields (2.6).  $\square$

**Remark 2.4.** *The pair  $(\alpha_n, \beta_n)$  satisfying (2.7) is called a WP-Bailey pair, see [1]. When  $b = 0$  a WP-Bailey pair is called a Bailey pair.*

Setting  $\delta_n = \frac{(a_1, \dots, a_{p-1}; q)_n}{(b_1, \dots, b_{p-1}; q)_n} \delta^n$  in Proposition 2.3, we obtain the following general transformation.

**Theorem 2.5.** Let  $\delta, a_i, b_i$  be any complex numbers such that  $|a_i| < 1, |b_i| < 1$  ( $1 \leq i \leq p-1$ ) and  $|\delta| < 1$ . Under suitable convergence conditions, for any complex sequence  $\{\beta_n\}$ , we have

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{(a_1, \dots, a_{p-1}; q)_n}{(b_1, \dots, b_{p-1}; q)_n} \delta^n \beta_n &= \sum_{n=0}^{\infty} \frac{(a, a/b; q)_n (1 - aq^{2n})(b; q)_{2n}}{(q, bq; q)_n (1 - a)(aq; q)_{2n}} \\ &\times \frac{(a_1, \dots, a_{p-1}; q)_n}{(b_1, \dots, b_{p-1}; q)_n} (b\delta/a)^n \sum_{k=0}^n \frac{(1 - bq^{2k})(aq^n; q)_k (q^{-n}; q)_k}{(1 - b)(bq^{n+1}; q)_k (bq^{1-n}/a; q)_k} q^k \beta_k \\ &\times {}_{p+1}\phi_p \left[ \begin{matrix} a_1 q^n, \dots, a_{p-1} q^n, b/a, bq^{2n} \\ b_1 q^n, \dots, b_{p-1} q^n, aq^{2n+1} \end{matrix} ; q, \delta \right]. \end{aligned}$$

If we choose

$$b = 0, \quad \beta_n = \frac{(g, h; q)_n}{(q, c, d, e; q)_n} u^n, \quad b_{p-1} = b_{p-2} = 0,$$

in Theorem 2.5, then we obtain the following generalisation of (1.2), which is our first main result.

**Theorem 2.6.** Let  $\delta, u, c, d, e, g, h, b_i, a_i$  ( $i \in \mathbb{N}$ ) be any complex numbers such that  $|\delta| < 1, |u| < 1, |a_i| < 1, |b_i| < 1$  ( $1 \leq i \leq p-1$ ). Then the following identity holds

$$(2.8) \quad {}_{p+1}\phi_p \left[ \begin{matrix} a_1, \dots, a_{p-1}, g, h \\ c, d, e, b_1, \dots, b_{p-3} \end{matrix} ; q, \delta u \right] = \sum_{n=0}^{\infty} \frac{(-1)^n q^{\binom{n}{2}} (a_1, \dots, a_{p-1}; q)_n \delta^n}{(q, aq^n, b_1, \dots, b_{p-3}; q)_n} \\ \times {}_4\phi_3 \left[ \begin{matrix} q^{-n}, aq^n, g, h \\ c, d, e \end{matrix} ; q, qu \right] {}_{p-1}\phi_{p-2} \left[ \begin{matrix} a_1 q^n, \dots, a_{p-1} q^n \\ b_1 q^n, \dots, b_{p-3} q^n, aq^{2n+1} \end{matrix} ; q, \delta \right].$$

We recover Ismail-Stanton's result (1.2) by choosing, in the above transformation,

$$u = 1, \quad a = t_1 t_2 t_3 t_4 / q, \quad g = t_4 / z, \quad h = t_4 z, \quad c = t_1 t_4, \quad d = t_2 t_4, \quad e = t_3 t_4$$

(thus  $aqgh = cde$ ) and then applying Sears' transformation (7.5).

By using the expansion formula (2.8) and integral formula (1.7) we can derive a generalization of Nassrallah-Rahman integral (1.6) in Theorem 2.7, which is our second main result. For convenience we shall use the following compact notation

$$(2.9) \quad A(\mathbf{t}) := \frac{2\pi(t_1 t_2 t_3 t_4, t_1 t_2 t_3 t_5, t_1 t_2 t_4 t_5, t_1 t_3 t_4 t_5; q)_{\infty}}{(q, \alpha q; q)_{\infty} \prod_{1 \leq r < s \leq 5} (t_r t_s; q)_{\infty}}.$$

**Theorem 2.7.** Let  $\alpha q = t_1^2 t_2 t_3 t_4 t_5$ . If  $|g| \neq |h|$  and  $\max\{|t_i|\} < 1$  ( $1 \leq i \leq 5$ ), then

$$(2.10) \quad \int_0^{\pi} \frac{h(\cos 2\theta; 1)}{h(\cos \theta; t_1, t_2, t_3, t_4, t_5)} {}_4\phi_3 \left[ \begin{matrix} g, h, t_1 e^{i\theta}, t_1 e^{-i\theta} \\ c, d, \alpha qgh/cd \end{matrix} ; q, t_2 t_3 t_4 t_5 \right] d\theta \\ = A(\mathbf{t}) \sum_{n=0}^{\infty} \frac{(1 - \alpha q^{2n})}{(1 - \alpha)} \frac{(\alpha, t_1 t_2, t_1 t_3, t_1 t_4, t_1 t_5; q)_n (-1)^n q^{\binom{n}{2}} (t_2 t_3 t_4 t_5)^n}{(q, t_1 t_2 t_3 t_4, t_1 t_2 t_3 t_5, t_1 t_2 t_4 t_5, t_1 t_3 t_4 t_5; q)_n} \\ \times {}_4\phi_3 \left[ \begin{matrix} q^{-n}, \alpha q^n, g, h \\ c, d, \alpha qgh/cd \end{matrix} ; q, q \right].$$

Taking  $g = s_4/z, h = s_4 z, c = s_1 s_4, d = s_2 s_4$  and  $\alpha q = s_1 s_2 s_3 s_4$  in Theorem 2.7, we get the following formula involving Askey-Wilson polynomials.

**Corollary 2.8.** *If  $\max\{|t_i|, |s_j|\} < 1$  ( $1 \leq i \leq 5, 1 \leq j \leq 4$ ) and  $\alpha q = t_1^2 t_2 t_3 t_4 t_5$ , then*

$$(2.11) \quad \int_0^\pi \frac{h(\cos 2\theta; 1)}{h(\cos \theta; t_1, t_2, t_3, t_4, t_5)} {}_4\phi_3 \left[ \begin{matrix} s_4 z, s_4/z, t_1 e^{i\theta}, t_1 e^{-i\theta} \\ s_1 s_4, s_2 s_4, s_3 s_4 \end{matrix}; q, t_2 t_3 t_4 t_5 \right] d\theta \\ = A(\mathbf{t}) \sum_{n=0}^{\infty} \frac{(1 - \alpha q^{2n})}{(1 - \alpha)} \frac{(\alpha, t_1 t_2, t_1 t_3, t_1 t_4, t_1 t_5; q)_n (-1)^n q^{\binom{n}{2}} (t_2 t_3 t_4 t_5 s_4)^n P_n(y; \mathbf{s}|q)}{(q, t_1 t_2 t_3 t_4, t_1 t_2 t_3 t_5, t_1 t_2 t_4 t_5, t_1 t_3 t_4 t_5, s_1 s_4, s_2 s_4, s_3 s_4; q)_n},$$

where  $z = e^{i\varphi}$ ,  $y = \cos \varphi$  and  $P_n(y; \mathbf{s}|q)$  are Askey-Wilson polynomials.

In Theorem 2.7 choosing  $g = s$ ,  $h = adt/q$ ,  $c = st$  and then letting  $d \rightarrow \infty$ , we obtain

**Theorem 2.9.** *If  $\max\{|t_i|, |at_1^2 t_2 t_3 t_4 t_5/q|\} < 1$ , ( $1 \leq i \leq 5$ ) and  $\alpha q = t_1^2 t_2 t_3 t_4 t_5$ , then*

$$(2.12) \quad \int_0^\pi \frac{h(\cos 2\theta; 1)}{h(\cos \theta; t_1, t_2, t_3, t_4, t_5)} {}_3\phi_2 \left[ \begin{matrix} s, t_1 e^{i\theta}, t_1 e^{-i\theta} \\ st, at_1^2 t_2 t_3 t_4 t_5/q \end{matrix}; q, at_2 t_3 t_4 t_5/q \right] d\theta \\ = A(\mathbf{t}) \sum_{n=0}^{\infty} \frac{(1 - \alpha q^{2n})}{(1 - \alpha)} \frac{(\alpha, t_1 t_2, t_1 t_3, t_1 t_4, t_1 t_5; q)_n (-1)^n q^{\binom{n}{2}} (t_2 t_3 t_4 t_5)^n}{(q, t_1 t_2 t_3 t_4, t_1 t_2 t_3 t_5, t_1 t_2 t_4 t_5, t_1 t_3 t_4 t_5; q)_n} \\ \times {}_3\phi_2 \left[ \begin{matrix} q^{-n}, \alpha q^n, s \\ st, \alpha a \end{matrix}; q, at \right]$$

$$(2.13) \quad = A(\mathbf{t}) \sum_{n=0}^{\infty} \frac{(1 - \alpha q^{2n})}{(1 - \alpha)} \frac{(\alpha, q/a, t_1 t_2, t_1 t_3, t_1 t_4, t_1 t_5; q)_n (\alpha at_2 t_3 t_4 t_5)^n}{(q, \alpha a, t_1 t_2 t_3 t_4, t_1 t_2 t_3 t_5, t_1 t_2 t_4 t_5, t_1 t_3 t_4 t_5; q)_n} \\ \times {}_3\phi_2 \left[ \begin{matrix} q^{-n}, \alpha q^n, t \\ st, q/a \end{matrix}; q, q \right].$$

Note that (2.13) follows from applying the transformation (7.6) to the last  ${}_3\phi_2$  in (2.12) with  $c = \alpha q^n$ ,  $b = t$ ,  $d = st$ ,  $e = q/a$ . A proof and further applications of Theorem 2.7 will be given in section 4.

### 3. APPLICATIONS OF THEOREM 2.6

In this section, we show that Theorem 2.6 encompasses some results of Ismail-Rahman and Ismail-Stanton in [9, 12].

**Theorem 3.1.** *For any non-negative  $N$ , we have*

$$(3.1) \quad \frac{(b_1/b, b_1/c; q)_N}{(b_1/bc, b_1; q)_N} = \sum_{n=0}^N \frac{(q^{-N}, e, f, g; q)_n q^n (-1)^n q^{\binom{n}{2}}}{(q, aq^n, b_1, bcq^{1-N}/b_1; q)_n} {}_4\phi_3 \left[ \begin{matrix} q^{-n}, aq^n, b, c \\ e, f, g \end{matrix}; q, q \right] \\ \times {}_4\phi_3 \left[ \begin{matrix} q^{-N+n}, eq^n, fq^n, gq^n \\ b_1 q^n, bcq^{1-N+n}/b_1, aq^{2n+1} \end{matrix}; q, q \right],$$

and, for  $|b_1/bc| < 1$ ,

$$(3.2) \quad \frac{(b_1/b, b_1/c; q)_\infty}{(b_1/bc, b_1; q)_\infty} = \sum_{n=0}^{\infty} \frac{(e, f, g; q)_n (b_1/bc)^n (-1)^n q^{\binom{n}{2}}}{(q, aq^n, b_1; q)_n} {}_4\phi_3 \left[ \begin{matrix} q^{-n}, aq^n, b, c \\ e, f, g \end{matrix}; q, q \right] \\ \times {}_3\phi_2 \left[ \begin{matrix} eq^n, fq^n, gq^n \\ b_1q^n, aq^{2n+1} \end{matrix}; q, b_1/bc \right].$$

*Proof.* Taking  $p = 5$ ,  $a_1 = q^{-N}$ ,  $a_2 = e$ ,  $a_3 = f$  and  $a_4 = g$  in Theorem 2.6, we have

$${}_3\phi_2 \left[ \begin{matrix} q^{-N}, b, c \\ b_1, b_2 \end{matrix}; q, \delta u \right] = \sum_{n=0}^N \frac{(q^{-N}, e, f, g; q)_n \delta^n (-1)^n q^{\binom{n}{2}}}{(q, aq^n, b_1, b_2; q)_n} {}_4\phi_3 \left[ \begin{matrix} q^{-n}, aq^n, b, c \\ e, f, g \end{matrix}; q, qu \right] \\ \times {}_4\phi_3 \left[ \begin{matrix} q^{-N+n}, eq^n, fq^n, gq^n \\ b_1q^n, b_2q^n, aq^{2n+1} \end{matrix}; q, \delta \right].$$

When  $b_2 = bcq^{1-N}/b_1$ ,  $\delta = q$  and  $u = 1$ , the above  ${}_3\phi_2$  series can be summed by  $q$ -Pfaff-Saalschütz sum (7.4) and we obtain (3.1). Letting  $N \rightarrow \infty$  in (3.1) yields (3.2), where taking the limit inside the sum is justified by Tannery's theorem, the discrete analogue of the Lebesgue dominated convergence theorem. We omit the details.  $\square$

The following connection formula (3.3) was first proved by Ismail-Rahman-Stanton [11] though the limit case (3.4) appeared in a recent paper of Ismail and Stanton [12, Theorem 3.1].

**Theorem 3.2** (Ismail-Rahman-Stanton). *For any non-negative  $n$ , we have*

$$(3.3) \quad (be^{i\theta}, be^{-i\theta}; q)_n = \sum_{k=0}^n f_{n,k}(b, \mathbf{t}) P_k(x, \mathbf{t}|q)$$

where

$$f_{n,k}(b, \mathbf{t}) = \frac{(-b)^k q^{\binom{k}{2}} (q; q)_n (b/t_4, bt_4q^k; q)_{n-k}}{(q, t_1t_2t_3t_4q^{k-1}; q)_k (q; q)_{n-k}} {}_4\phi_3 \left[ \begin{matrix} q^{k-n}, t_2t_4q^k, t_1t_4q^k, t_3t_4q^k \\ bt_4q^k, t_4q^{1-n+k}/b, t_1t_2t_3t_4q^{2k} \end{matrix}; q, q \right],$$

and

$$(3.4) \quad \frac{(be^{i\theta}, be^{-i\theta}; q)_\infty}{(bt_4, b/t_4; q)_\infty} = \sum_{k=0}^n P_k(x, \mathbf{t}|q) \frac{(-b)^k q^{\binom{k}{2}}}{(q, bt_4, t_1t_2t_3t_4q^{k-1}; q)_k} \\ \times {}_3\phi_2 \left[ \begin{matrix} t_2t_4q^k, t_1t_4q^k, t_3t_4q^k \\ bt_4q^k, t_1t_2t_3t_4q^{2k} \end{matrix}; q, \frac{b}{t_4} \right].$$

*Proof.* Let  $a = t_1t_2t_3t_4/q$ ,  $b = t_4z$ ,  $c = t_4/z$ ,  $b_1 = bt_4$ ,  $e = t_2t_4$ ,  $f = t_3t_4$  and  $g = t_1t_4$  and  $z = e^{i\theta}$  in (3.1). Then, Sears' transformation (7.5) infers that

$${}_4\phi_3 \left[ \begin{matrix} q^{-n}, aq^n, b, c \\ e, f, g \end{matrix}; q, q \right] = \frac{(t_1t_3, t_1t_2; q)_n q^{-\binom{n}{2}}}{(t_2t_4, t_3t_4; q)_n} {}_4\phi_3 \left[ \begin{matrix} q^{-n}, t_1t_2t_3t_4q^{n-1}, t_1z, t_1/z \\ t_1t_4, t_1t_3, t_1t_2 \end{matrix}; q, q \right].$$

Now, replacing  $n$  by  $k$  and  $N$  by  $n$  in (3.1) we get (3.3) after simplification. Clearly (3.4) is the limit  $n \rightarrow \infty$  case of (3.3).  $\square$



We can also derive a transformation of Ismail-Rahman-Suslov [10, Theorem 5.3] from Theorem 2.6.

**Theorem 3.3** (Ismail-Rahman-Suslov). *We have*

$$(3.5) \quad \frac{(\alpha, \alpha ab/q; q)_\infty}{(\alpha a, \alpha b; q)_\infty} {}_3\phi_2 \left[ \begin{matrix} q/a, q/b, s \\ st, \alpha c \end{matrix}; q, \alpha abct/q^2 \right] \\ = \sum_{n=0}^{\infty} \frac{(1 - \alpha q^{2n})(\alpha abc/q^2)^n (\alpha, q/a, q/b, q/c; q)_n}{(q, \alpha a, \alpha b, \alpha c; q)_n} {}_3\phi_2 \left[ \begin{matrix} q^{-n}, \alpha q^n, t \\ st, q/c \end{matrix}; q, q \right].$$

*Proof.* In (2.8), setting  $p = 3$  and substituting  $a \rightarrow \alpha$ ,  $a_1 \rightarrow q/a$ ,  $a_2 \rightarrow q/b$ ,  $e \rightarrow \alpha qgh/dc$ ,  $\delta \rightarrow \alpha ab/q$  and  $u = 1$  we can sum the  ${}_2\phi_1$  by the Gauss sum (7.1) and obtain

$$(3.6) \quad \frac{(\alpha, \alpha ab/q; q)_\infty}{(\alpha a, \alpha b; q)_\infty} {}_4\phi_3 \left[ \begin{matrix} q/a, q/b, g, h \\ c, d, \alpha qgh/dc \end{matrix}; q, \alpha ab/q \right] \\ = \sum_{n=0}^{\infty} \frac{(1 - \alpha q^{2n})(-1)^n q^{\binom{n}{2}} (\alpha ab/q)^n (\alpha, q/a, q/b; q)_n}{(q, \alpha a, \alpha b; q)_n} {}_4\phi_3 \left[ \begin{matrix} q^{-n}, \alpha q^n, g, h \\ d, c, \alpha qgh/dc \end{matrix}; q, q \right].$$

By Sears' transformation (7.5)

$$(3.7) \quad {}_4\phi_3 \left[ \begin{matrix} q^{-n}, \alpha q^n, g, h \\ d, c, \alpha qgh/dc \end{matrix}; q, q \right] = \frac{(cq^{-n}/\alpha, q^{1-n}gh/dc; q)_n}{(c, \alpha qgh/dc; q)_n} (\alpha q^n)^n \\ \times {}_4\phi_3 \left[ \begin{matrix} q^{-n}, \alpha q^n, d/g, d/h \\ d, \alpha q/c, dc/gh \end{matrix}; q, q \right].$$

Now, plugging (3.7) into (3.6) and substituting  $g \rightarrow s$ ,  $d \rightarrow st$ ,  $c \rightarrow \alpha c$  we obtain

$$(3.8) \quad \frac{(\alpha, \alpha ab/q; q)_\infty}{(\alpha a, \alpha b; q)_\infty} {}_4\phi_3 \left[ \begin{matrix} q/a, q/b, s, h \\ st, \alpha c, qh/ct \end{matrix}; q, \alpha ab/q \right] \\ = \sum_{n=0}^{\infty} \frac{(1 - \alpha q^{2n})(\alpha abc/q^2)^n (\alpha, q/a, q/b, q/c; q)_n (q^{1-n}h/\alpha ct; q)_n (\alpha q^n)^n}{(q, \alpha a, \alpha b, \alpha c; q)_n (qh/ct; q)_n} \\ \times {}_4\phi_3 \left[ \begin{matrix} q^{-n}, \alpha q^n, t, st/h \\ st, q/c, \alpha ct/h \end{matrix}; q, q \right].$$

Now, replace  $h$  by  $q^{-m}$ , for a positive integer  $m$ , then let  $m \rightarrow \infty$  and apply Tannery's theorem. The result is (3.5).  $\square$

When  $st = q/a$ , the  ${}_3\phi_2$  at the left-hand side of (3.5) reduces to a  ${}_2\phi_1$ , which can be summed by (7.1) and we get the following summation formula,

$$(3.9) \quad \sum_{n=0}^{\infty} \frac{(1 - \alpha q^{2n})(\alpha abc/q^2)^n (\alpha, q/a, q/b, q/c; q)_n}{(q, \alpha a, \alpha b, \alpha c; q)_n} {}_3\phi_2 \left[ \begin{matrix} q^{-n}, \alpha q^n, t \\ q/a, q/c \end{matrix}; q, q \right] \\ = \frac{(\alpha, \alpha ab/q, \alpha bc/q, \alpha act/q; q)_\infty}{(\alpha a, \alpha b, \alpha c, \alpha abct/q^2; q)_\infty}.$$

Applying the transformation (7.6) to the above  ${}_3\phi_2$  we obtain another result of Ismail-Rahman-Suslov [10, Theorem 5.1].

**Corollary 3.4** (Ismail-Rahman-Suslov). *We have*

$$(3.10) \quad \sum_{n=0}^{\infty} \frac{(1 - \alpha q^{2n})(\alpha abc/q^2)^n (\alpha, q/a, q/b, q/c; q)_n}{(q, \alpha a, \alpha b, \alpha c; q)_n} {}_3\phi_2 \left[ \begin{matrix} q^{-n}, \alpha q^n, t \\ q/a, q/c \end{matrix}; q, q \right] \\ = \frac{(\alpha, \alpha ab/q, \alpha bc/q, \alpha act/q; q)_{\infty}}{(\alpha a, \alpha b, \alpha c, \alpha abct/q^2; q)_{\infty}}.$$

Ismail-Rahman-Suslov derived the above two results from their main theorem [10, Theorem 1.1], which expresses a double sum as a linear combination of two  ${}_5\phi_4$  sums. We notice that if we make the substitution  $(a, b, c, d, e, f) \rightarrow (\alpha, q/b, q/c, q/d, q, \alpha)$  in their Theorem 1.1, then  $qa/ef = 1$ , which annihilates the factor in front of the first  ${}_5\phi_4$  and reduces the second  ${}_5\phi_4$  to 1 in [10, (1.4)], and we obtain immediately the following remarkable extension of (3.10).

**Theorem 3.5.**

$$(3.11) \quad \sum_{n=0}^{\infty} \frac{(1 - \alpha q^{2n})(abcd/q^2)^n (\alpha, q/b, q/c, q/d; q)_n}{(q, \alpha b, \alpha c, \alpha d; q)_n} {}_4\phi_3 \left[ \begin{matrix} q^{-n}, \alpha q^n, g, h \\ q/b, q/c, abcgh/q \end{matrix}; q, q \right] \\ = \frac{(\alpha, \alpha bd/q, \alpha cd/q, \alpha bcg/q, \alpha bch/q, \alpha bcdgh/q^2; q)_{\infty}}{(\alpha b, \alpha c, \alpha d, \alpha bcgh/q, \alpha bcdg/q^2, \alpha bcdh/q^2; q)_{\infty}}.$$

It seems that (3.11) was first published by Liu [16, Theorem 3].

#### 4. PROOF OF THEOREM 2.7 AND ITS APPLICATIONS

**4.1. Proof of Theorem 2.7.** Choosing  $p = 3$ ,  $u = 1$ ,  $\delta = \alpha a_1 a_2 / q$ ,

$$a_1 = (q/t_1)e^{i\theta}, \quad a_2 = (q/t_1)e^{-i\theta}, \quad e = \alpha qgh/cd$$

in Theorem 2.5, we can sum the  ${}_{p-1}\phi_{p-2}$  by the  $q$ -Gauss sum (7.1) and rewrite (2.8) as

$$(4.1) \quad \frac{(\alpha q, \alpha q/t_1^2; q)_{\infty}}{h(\cos \theta; \alpha q/t_1) {}_4\phi_3 \left[ \begin{matrix} t_1 e^{i\theta}, t_1 e^{-i\theta}, g, h \\ c, d, \alpha qgh/dc \end{matrix}; q, \alpha q/t_1^2 \right]} \\ = \sum_{n=0}^{\infty} \frac{(1 - \alpha q^{2n})(-1)^n q^{\binom{n}{2}} (\alpha q/t_1^2)^n (\alpha, t_1 e^{i\theta}, t_1 e^{-i\theta}; q)_n}{(1 - \alpha)(q, \alpha q e^{i\theta}/t_1, \alpha q e^{-i\theta}/t_1; q)_n} {}_4\phi_3 \left[ \begin{matrix} q^{-n}, \alpha q^n, g, h \\ c, d, \alpha qgh/dc \end{matrix}; q, q \right].$$

It is clear that the series at the left-hand side is convergent if  $|\alpha q/t_1^2| < 1$ . The convergence of the right-hand side can be justified as follows: if  $|h| < |g|$ , then one can show (see [14, (1.11)]) that the terminate  ${}_4\phi_3$  series has the asymptotic formula

$$(4.2) \quad {}_4\phi_3 \left[ \begin{matrix} q^{-n}, \alpha q^n, g, h \\ c, d, \alpha qgh/dc \end{matrix}; q, q \right] \sim \frac{(h, d/g, c/g, qah/dc; q)_{\infty} g^n}{(c, d, h/g, \alpha qgh/dc; q)_{\infty}}, \quad n \rightarrow \infty,$$

N.B. This formula is also given in [10, (1.5)] without the factor  $(h/g; q)_{\infty}$  in the denominator. Hence, in view of the factor  $q^{\binom{n}{2}}$ , the series on the right-hand side of (4.1) converges if  $|g| \neq |h|$ . Hence a sufficient condition of convergence of the infinite series on the two sides of (4.1) is

$$(4.3) \quad |g| \neq |h|, \quad |\alpha q/t_1^2| < 1.$$

Since  $\alpha q/t_1 = t_1 t_2 t_3 t_4 t_5$ , we have  $h(\cos \theta; t_1 t_2 t_3 t_4 t_5) = (\alpha q e^{i\theta}/t_1, \alpha q e^{-i\theta}/t_1; q)_\infty$  and

$$\frac{h(\cos \theta; t_1 t_2 t_3 t_4 t_5)(t_1 e^{i\theta}, t_1 e^{-i\theta}; q)_n}{h(\cos \theta; t_1)(\alpha q e^{i\theta}/t_1, \alpha q e^{-i\theta}/t_1; q)_n} = \frac{h(\cos \theta; t_1 t_2 t_3 t_4 t_5 q^n)}{h(\cos \theta; t_1 q^n)}.$$

Multiplying both sides of (4.1) by

$$\frac{h(\cos 2\theta; 1)h(\cos \theta; t_1 t_2 t_3 t_4 t_5)}{h(\cos \theta; t_1, t_2, t_3, t_4, t_5)}$$

and integrating over  $0 \leq \theta \leq \pi$ , we have

$$(4.4) \quad \int_0^\pi \frac{h(\cos 2\theta; 1)}{h(\cos \theta; t_1, t_2, t_3, t_4, t_5)} {}_4\phi_3 \left[ \begin{matrix} t_1 e^{i\theta}, t_1 e^{-i\theta}, g, h \\ c, d, \alpha q g h / d c \end{matrix}; q, \alpha q / t_1^2 \right] d\theta \\ = \frac{1}{(\alpha q, \alpha q / t_1^2; q)_\infty} \sum_{n=0}^{\infty} \frac{(1 - \alpha q^{2n})(-1)^n q^{\binom{n}{2}} (\alpha q / t_1^2)^n (\alpha; q)_n}{(1 - \alpha)(q; q)_n} {}_4\phi_3 \left[ \begin{matrix} q^{-n}, \alpha q^n, g, h \\ c, d, \alpha q g h / d c \end{matrix}; q, q \right] \\ \times \int_0^\pi \frac{h(\cos 2\theta; 1)h(\cos \theta; t_1 t_2 t_3 t_4 t_5 q^n)}{h(\cos \theta; t_1 q^n, t_2, t_3, t_4, t_5)} d\theta.$$

The last integral can be evaluated by rescaling  $t_1 \rightarrow t_1 q^n$  in (1.7),

$$\int_0^\pi \frac{h(\cos 2\theta; 1)h(\cos \theta; t_1 t_2 t_3 t_4 t_5 q^n)}{h(\cos \theta; t_1 q^n, t_2, t_3, t_4, t_5)} d\theta = \frac{2\pi(t_1 t_2 t_3 t_4, t_1 t_2 t_3 t_5, t_1 t_2 t_4 t_5, t_1 t_3 t_4 t_5, t_2 t_3 t_4 t_5; q)_\infty}{(q; q)_\infty \prod_{1 \leq r < s \leq 5} (t_r t_s; q)_\infty} \\ \times \frac{(t_1 t_2, t_1 t_3, t_1 t_4, t_1 t_5; q)_n}{(t_1 t_2 t_3 t_4, t_1 t_2 t_3 t_5, t_1 t_2 t_4 t_5, t_1 t_3 t_4 t_5; q)_n}.$$

Substituting this in (4.4), we obtain (2.10).  $\square$

**4.2. Nassrallah-Rahman integrals.** We show how to get Nassrallah-Rahman integral (1.6) from our Theorem 2.7. Let  $h = c$  and  $g \rightarrow \infty$  in (2.10), we have

$$(4.5) \quad \int_0^\pi \frac{h(\cos 2\theta; 1)}{h(\cos \theta; t_1, t_2, t_3, t_4, t_5)} {}_2\phi_1 \left[ \begin{matrix} t_1 e^{i\theta}, t_1 e^{-i\theta} \\ d \end{matrix}; q, d/t_1^2 \right] d\theta \\ = \frac{2\pi(t_6/t_1, t_6 t_1, t_1 t_3 t_4 t_5, t_1 t_2 t_3 t_5, t_1 t_2 t_3 t_4, t_1 t_2 t_4 t_5; q)_\infty}{\prod_{1 \leq r < s \leq 5} (t_r t_s; q)_\infty (q, t_1^2 t_2 t_3 t_4 t_5; q)_\infty} \\ \times \sum_{n=0}^{\infty} \frac{(1 - \alpha q^{2n})}{(1 - \alpha)} \frac{(\alpha, t_1 t_2, t_1 t_3, t_1 t_4, t_1 t_5; q)_n}{(q, t_1 t_2 t_3 t_4, t_1 t_2 t_3 t_5, t_1 t_2 t_4 t_5, t_1 t_3 t_4 t_5; q)_n} \\ \times (-1)^n q^{\binom{n}{2}} (t_2 t_3 t_4 t_5)^n {}_2\phi_1 \left[ \begin{matrix} q^{-n}, \alpha q^n \\ d \end{matrix}; q, d/\alpha \right].$$

Now, the above two  ${}_2\phi_1$  series can be summed by  $q$ -Gauss summation (7.1) and  $q$ -Chu-Vandermonde sum (7.3), respectively,

$${}_2\phi_1 \left[ \begin{matrix} t_1 e^{i\theta}, t_1 e^{-i\theta} \\ d \end{matrix}; q, d/t_1^2 \right] = \frac{(d/t_1 e^{i\theta}, d/t_1 e^{-i\theta}; q)_\infty}{(d, d/t_1^2; q)_\infty}, \\ {}_2\phi_1 \left[ \begin{matrix} q^{-n}, \alpha q^n \\ d \end{matrix}; q, d/\alpha \right] = \frac{(\alpha q/d; q)_n (-d/\alpha q)^n q^{-\binom{n}{2}}}{(d; q)_n}.$$

Plugging these into (4.5), and then taking  $d \rightarrow t_6 t_1$ , we get the Nassrallah-Rahman integral (1.6).

In the following, we record some other well-known special cases of Theorem 2.7.

- **Askey-Wilson integral** When  $t_5 = 0$ , Theorem 2.7 immediately reduces to the Askey-Wilson integral (1.5).
- **Rahman Integral** (1.7) When  $c = g$  and  $h = d$  in (2.10), in the left hand, the  ${}_4\phi_3$  series reduces to a  ${}_2\phi_1$  series which can be summed by using  $q$ -Gauss summation (7.1),

$${}_2\phi_1 \left[ \begin{matrix} t_1 e^{i\theta}, t_1 e^{-i\theta} \\ \alpha q \end{matrix} ; q, t_2 t_3 t_4 t_5 \right] = \frac{(t_1 t_2 t_3 t_4 t_5 e^{i\theta}, t_1 t_2 t_3 t_4 t_5 e^{-i\theta}; q)_\infty}{(\alpha q, t_2 t_3 t_4 t_5; q)_\infty}.$$

On the other hand, using  $q$ -Chu-Vandermonde sums (7.2), we have

$${}_2\phi_1 \left[ \begin{matrix} q^{-n}, \alpha q^n \\ \alpha q \end{matrix} ; q, q \right] = \frac{(q^{1-n}; q)_n}{(\alpha q; q)_n} (\alpha q^n)^n,$$

which is zero for  $n \geq 1$ . After some simplification, this integral reduces to (1.7).

- **Ismail integral** Ismail [8, p. 442] uses the following integral to derive Nassrallah-Rahman formula via analytic prolongation.

$$(4.6) \quad \int_0^\pi \frac{h(\cos 2\theta; 1)(\alpha e^{i\theta}, \alpha e^{-i\theta}; q)_n}{h(\cos \theta; t_1, t_2, t_3, t_4)} d\theta = \frac{2\pi(\alpha/t_4, \alpha t_4; q)_n (t_1 t_2 t_3 t_4; q)_\infty}{(q; q)_\infty \prod_{1 \leq r < s \leq 4} (t_r t_s; q)_\infty} \\ \times {}_4\phi_3 \left[ \begin{matrix} q^{-n}, t_1 t_4, t_2 t_4, t_3 t_4 \\ \alpha t_4, t_1 t_2 t_3 t_4, q^{1-n} t_4 / \alpha \end{matrix} ; q, q \right].$$

When  $h = c$ ,  $g \rightarrow \infty$ ,  $d = t_6 t_1$  and  $t_1 \leftrightarrow t_5$  in (2.10), in the right hand side, the inner summation can be written as

$$(4.7) \quad {}_8W_7 \left( t_5^2 t_1 t_2 t_3 t_4 / q, t_1 t_5, t_2 t_5, t_5 t_3, t_5 t_4, t_1 t_2 t_3 t_4 t_5 / t_6 ; q, t_6 / t_5 \right).$$

Replacing  $a = t_5^2 t_1 t_2 t_3 t_4 / q$ ,  $b = t_5 t_4$ ,  $c = t_1 t_5$ ,  $d = t_2 t_5$ ,  $e = t_5 t_3$  and  $f = t_1 t_2 t_3 t_4 t_5 / t_6$  into  ${}_8W_7$  transformation (7.10), the above factor is equal to

$$(4.8) \quad \frac{(t_5^2 t_1 t_2 t_3 t_4, t_5 t_4, t_1 t_6, t_2 t_6, t_3 t_6, t_1 t_2 t_3 t_4; q)_\infty}{(t_2 t_3 t_4 t_5, t_1 t_3 t_4 t_5, t_1 t_2 t_4 t_5, t_5 t_6, t_1 t_2 t_3 t_6, t_6 / t_5; q)_\infty} \\ \times {}_8W_7 \left( t_1 t_2 t_3 t_6 / q, t_2 t_3, t_1 t_3, t_1 t_2, t_6 / t_4, t_6 / t_5 ; q, t_4 t_5 \right).$$

Applying  ${}_8W_7$  transformation (7.11) once more (with  $a = t_1 t_2 t_3 t_6 / q$ ,  $b = t_2 t_3$ ,  $c = t_1 t_3$ ,  $d = t_1 t_2$ ,  $e = t_6 / t_4$ ,  $f = t_6 / t_5$ ), the  ${}_8W_7$  series of (4.8) is equal to

$$(4.9) \quad \frac{(t_1 t_2 t_3 t_6, t_1 t_2 t_3 t_4 t_5 / t_6, t_4 t_6, t_5 t_6; q)_\infty}{(t_1 t_2 t_3 t_4, t_1 t_2 t_3 t_5, t_6^2, t_4 t_5; q)_\infty} \\ \times {}_8W_7 \left( t_6^2 / q, t_6 / t_1, t_6 / t_2, t_6 / t_3, t_6 / t_4, t_6 / t_5 ; q, t_1 t_2 t_3 t_4 t_5 / t_6 \right).$$

Replacing (4.8) and (4.9) into (4.7), we have another form of Nassrallah-Rahman integral (1.6),

$$\int_0^\pi \frac{h(\cos 2\theta; 1) h(\cos \theta; t_6)}{h(\cos \theta; t_1, t_2, t_3, t_4, t_5)} d\theta = \frac{2\pi (\prod_{j=1}^5 (t_6 t_j; q)_\infty (t_1 t_2 t_3 t_4 t_5 / t_6; q)_\infty}{(q, t_6^2; q)_\infty \prod_{1 \leq r < s \leq 5} (t_r t_s; q)_\infty} \\ \times {}_8W_7 \left( t_6^2 / q, t_6 / t_1, t_6 / t_2, t_6 / t_3, t_6 / t_4, t_6 / t_5 ; q, t_1 t_2 t_3 t_4 t_5 / t_6 \right),$$

Taking  $t_1 = t_6 q^n$  in the above result, we have

$$(4.10) \quad \int_0^\pi \frac{h(\cos 2\theta; 1)(t_6 e^{i\theta}, t_6 e^{-i\theta}; q)_n}{h(\cos \theta; t_2, t_3, t_4, t_5)} d\theta \\ = \frac{2\pi(t_2 t_3 t_4 t_5 q^n; q)_\infty (t_6^2 q^n; q)_\infty \prod_{j=2}^5 (t_j t_6; q)_\infty}{(q, t_6^2; q)_\infty \prod_{j=2}^5 (t_j t_6 q^n; q)_\infty \prod_{2 \leq r < s \leq 5} (t_r t_s; q)_\infty} \\ \times {}_8W_7 \left( t_6^2/q, q^{-n}, t_6/t_2, t_6/t_3, t_6/t_4, t_6/t_5; q, t_2 t_3 t_4 t_5 q^n \right).$$

By using Watson's transformation ( $a = t_6^2/q$ ,  $b = t_6/t_2$ ,  $c = t_6/t_3$ ,  $d = t_6/t_4$  and  $e = t_6/t_5$  in (7.9)), the  ${}_8W_7$  series can be reduced to

$${}_8W_7 \left( t_6^2/q, q^{-n}, t_6/t_2, t_6/t_3, t_6/t_4, t_6/t_5; q, t_2 t_3 t_4 t_5 q^n \right) \\ = \frac{(t_6^2, t_4 t_5; q)_n}{(t_4 t_6, t_5 t_6; q)_n} {}_4\phi_3 \left[ \begin{matrix} q^{-n}, t_6/t_4, t_6/t_5, t_2 t_3 \\ t_2 t_6, t_3 t_6, q^{1-n}/t_4 t_5 \end{matrix}; q, q \right] \\ = \frac{(t_6^2, t_4 t_5, t_6/t_2, t_2 t_3 t_4 t_5; q)_n}{(t_4 t_6, t_5 t_6, t_3 t_6, t_4 t_5; q)_n} {}_4\phi_3 \left[ \begin{matrix} q^{-n}, t_2 t_3, t_2 t_4, t_2 t_5 \\ t_2 t_6, t_2 t_3 t_4 t_5, q^{1-n} t_2/t_6 \end{matrix}; q, q \right].$$

The second step is obtained by Sears' transformation (7.5) ( $a = t_2 t_3$ ,  $b = t_6/t_4$ ,  $c = t_6/t_5$ ,  $d = t_2 t_6$ ,  $e = t_3 t_6$ ,  $f = q^{1-n}/t_4 t_5$ ). Replacing the above formula into (4.10), we get (4.6) after taking  $t_2 \leftrightarrow t_4$ ,  $t_1 \leftrightarrow t_5$  and  $t_6 \rightarrow \alpha$ .

- **Ismail-Stanton-Viennot integral** It is proved in [13] that

$$(4.11) \quad \int_0^\pi \frac{h(\cos 2\theta; 1)}{h(\cos \theta; t_1, t_2, t_3, t_4, t_5)} d\theta = \frac{2\pi(t_1 t_2 t_3 t_4, t_2 t_3 t_4 t_5, t_1 t_5; q)_\infty}{(q; q)_\infty \prod_{1 \leq r < s \leq 5} (t_r t_s; q)_\infty} \\ \times {}_3\phi_2 \left[ \begin{matrix} t_2 t_3, t_2 t_4, t_3 t_4 \\ t_1 t_2 t_3 t_4, t_2 t_3 t_4 t_5 \end{matrix}; q, t_1 t_5 \right],$$

where  $\max\{|t_1|, |t_2|, |t_3|, |t_4|, |t_5|\} < 1$ .

When  $g = 1$ , the integral in (2.10) becomes

$$(4.12) \quad \int_0^\pi \frac{h(\cos 2\theta; 1)}{h(\cos \theta; t_1, t_2, t_3, t_4, t_5)} d\theta = \frac{2\pi(t_1 t_2 t_3 t_4, t_1 t_2 t_3 t_5, t_1 t_2 t_4 t_5, t_1 t_3 t_4 t_5; q)_\infty}{(q, \alpha q; q)_\infty \prod_{1 \leq r < s \leq 5} (t_r t_s; q)_\infty} \\ \times \sum_{n=0}^{\infty} \frac{(1 - \alpha q^{2n})}{(1 - \alpha)} \frac{(\alpha, t_1 t_2, t_1 t_3, t_1 t_4, t_1 t_5; q)_n (-1)^n q^{\binom{n}{2}} (t_2 t_3 t_4 t_5)^n}{(q, t_1 t_2 t_3 t_4, t_1 t_2 t_3 t_5, t_1 t_2 t_4 t_5, t_1 t_3 t_4 t_5; q)_n}.$$

In the right-hand side of (4.12), the summation becomes

$$\sum_{n=0}^{\infty} \frac{(\alpha, \alpha^{1/2}, \alpha^{-1/2}, t_1 t_2, t_1 t_3, t_1 t_4, t_1 t_5; q)_n}{(q, \alpha^{1/2}, \alpha^{-1/2}, t_1 t_2 t_3 t_4, t_1 t_2 t_3 t_5, t_1 t_2 t_4 t_5, t_1 t_3 t_4 t_5; q)_n} (-1)^n q^{\binom{n}{2}} (t_2 t_3 t_4 t_5)^n.$$

Setting  $a = \alpha$ ,  $b = t_1 t_2$ ,  $c = t_1 t_3$ ,  $d = t_1 t_4$ ,  $e = t_1 t_5$  and  $N \rightarrow 0$  in Watson's transformation (7.9), the above factor is equal to

$$(4.13) \quad \frac{(\alpha q, t_2 t_3; q)_\infty}{(t_1 t_2 t_3 t_5, t_1 t_2 t_3 t_4; q)_\infty} {}_3\phi_2 \left[ \begin{matrix} t_1 t_4, t_1 t_5, t_4 t_5 \\ t_1 t_3 t_4 t_5, t_1 t_2 t_4 t_5 \end{matrix}; q, t_2 t_3 \right].$$

Using  ${}_3\phi_2$  transformation (7.8) ( $a = t_1t_4$ ,  $b = t_1t_5$ ,  $c = t_4t_5$ ,  $d = t_1t_3t_4t_5$ ,  $e = t_1t_2t_4t_5$ ), the above  ${}_3\phi_2$  series is equal to

$$(4.14) \quad \frac{(t_1t_5, t_2t_3t_4t_5, t_1t_2t_3t_4; q)_\infty}{(t_1t_3t_4t_5, t_1t_2t_4t_5, t_2t_3; q)_\infty} {}_3\phi_2 \left[ \begin{matrix} t_2t_4, t_3t_4, t_2t_3 \\ t_2t_3t_4t_5, t_1t_2t_3t_4 \end{matrix} ; q, t_1t_5 \right].$$

Substituting (4.13) and (4.14) into the integral (4.12), we get (4.11).

**Remark 4.1.** *In the next section, we will give another proof of (4.11) as an application of (5.1).*

**4.3. Two integrals of Liu and Zhang-Wang.** When  $h = d$ ,  $c = \alpha u$  and  $g = \alpha uv/q$ , the  ${}_3\phi_2$  series at the right-hand side of (2.10) can be summed by  $q$ -Pfaff-Saalschütz sum (7.4). Thus we recover Liu's result [17, Theorem 1.6].

**Theorem 4.2** (Liu).

$$(4.15) \quad \int_0^\pi \frac{h(\cos 2\theta; 1)}{h(\cos \theta; t_1, t_2, t_3, t_4, t_5)} {}_3\phi_2 \left[ \begin{matrix} \alpha uv/q, t_1e^{i\theta}, t_1e^{-i\theta} \\ \alpha u, \alpha v \end{matrix} ; q, t_2t_3t_4t_5 \right] d\theta \\ = \frac{2\pi(t_1t_2t_3t_4, t_1t_2t_3t_5, t_1t_2t_4t_5, t_1t_3t_4t_5; q)_\infty}{(q, \alpha q; q)_\infty \prod_{1 \leq r < s \leq 5} (t_r t_s; q)_\infty} \\ \times \sum_{n=0}^{\infty} \frac{(1 - \alpha q^{2n})}{(1 - \alpha)} \frac{(\alpha, q/u, q/v, t_1t_2, t_1t_3, t_1t_4, t_1t_5; q)_n (-1)^n q^{\binom{n}{2}} (\alpha^2 uv/t_1^2)^n}{(q, t_1t_2t_3t_4, \alpha u, \alpha v, t_1t_2t_3t_5, t_1t_2t_4t_5, t_1t_3t_4t_5; q)_n},$$

where  $\alpha q = t_1^2 t_2 t_3 t_4 t_5$  and  $\max\{|t_1|, |t_2|, |t_3|, |t_4|, |t_5|\} < 1$ .

Taking  $u = t_1g/\alpha$  and  $v = t_1t_5/\alpha$ , then (4.15) reduces to

$$(4.16) \quad \int_0^\pi \frac{h(\cos 2\theta; 1)}{h(\cos \theta; t_1, t_2, t_3, t_4, t_5)} {}_3\phi_2 \left[ \begin{matrix} g/t_2t_3t_4, t_1e^{i\theta}, t_1e^{-i\theta} \\ gt_1, t_1t_5 \end{matrix} ; q, t_2t_3t_4t_5 \right] d\theta \\ = \frac{2\pi(t_1t_2t_3t_4, t_1t_2t_3t_5, t_1t_2t_4t_5, t_1t_3t_4t_5; q)_\infty}{(q, \alpha q; q)_\infty \prod_{1 \leq r < s \leq 5} (t_r t_s; q)_\infty} \\ \times \sum_{n=0}^{\infty} \frac{(1 - \alpha q^{2n})}{(1 - \alpha)} \frac{(\alpha, t_1t_2, t_1t_3, t_1t_4, \alpha q/gt_1; q)_n (-1)^n q^{\binom{n}{2}} (gt_5)^n}{(q, gt_1, t_1t_2t_3t_5, t_1t_2t_4t_5, t_1t_3t_4t_5; q)_n}.$$

Using the limit  $N \rightarrow \infty$  case of Watson's transformation (7.9) with  $a = \alpha$ ,  $b = t_1t_2$ ,  $c = t_1t_3$ ,  $d = t_1t_4$ ,  $e = \alpha q/gt_1$ , the summation at the right-hand side of (4.16) is transformed to

$$\frac{(t_1^2 t_2 t_3 t_4 t_5, g/t_4; q)_\infty}{(t_1 t_2 t_3 t_5, gt_1; q)_\infty} {}_3\phi_2 \left[ \begin{matrix} t_4 t_5, t_1 t_4, \alpha q/gt_1 \\ t_1 t_2 t_4 t_5, t_1 t_3 t_4 t_5 \end{matrix} ; q, g/t_4 \right].$$

Hence (4.16) is equivalent to

$$(4.17) \quad \int_0^\pi \frac{h(\cos 2\theta; 1)}{h(\cos \theta; t_1, t_2, t_3, t_4, t_5)} {}_3\phi_2 \left[ \begin{matrix} g/t_2t_3t_4, t_1e^{i\theta}, t_1e^{-i\theta} \\ gt_1, t_1t_5 \end{matrix} ; q, t_2t_3t_4t_5 \right] d\theta \\ = \frac{2\pi(t_1t_2t_3t_4, t_1t_2t_4t_5, t_1t_3t_4t_5, g/t_4; q)_\infty}{(q, gt_1; q)_\infty \prod_{1 \leq r < s \leq 5} (t_r t_s; q)_\infty} {}_3\phi_2 \left[ \begin{matrix} t_4 t_5, t_1 t_4, \alpha q/gt_1 \\ t_1 t_2 t_4 t_5, t_1 t_3 t_4 t_5 \end{matrix} ; q, g/t_4 \right].$$

Applying the  ${}_3\phi_2$  transformations (7.7) and (7.8) to the above two  ${}_3\phi_2$  in (4.17), respectively, we obtain

$$\begin{aligned} {}_3\phi_2 \left[ \begin{matrix} g/t_2t_3t_4, t_1e^{i\theta}, t_1e^{-i\theta} \\ gt_1, t_1t_5 \end{matrix} ; q, t_2t_3t_4t_5 \right] \\ = \frac{(t_1t_2t_3t_4t_5/g, gt_5; q)_\infty}{(t_1t_5, t_2t_3t_4t_5; q)_\infty} {}_3\phi_2 \left[ \begin{matrix} g/t_2t_3t_4, ge^{i\theta}, ge^{-i\theta} \\ gt_1, gt_5 \end{matrix} ; q, t_1t_2t_3t_4t_5/g \right], \end{aligned}$$

and

$$\begin{aligned} {}_3\phi_2 \left[ \begin{matrix} t_4t_5, t_1t_4, \alpha q/gt_1 \\ t_1t_3t_4t_5, t_1t_2t_4t_5 \end{matrix} ; q, g/t_4 \right] \\ = \frac{(\alpha q/gt_1, gt_1, gt_5; q)_\infty}{(t_1t_3t_4t_5, t_1t_2t_4t_5, g/t_4; q)_\infty} {}_3\phi_2 \left[ \begin{matrix} g/t_2, g/t_3, g/t_4 \\ gt_1, gt_5 \end{matrix} ; q, \alpha q/gt_1 \right]. \end{aligned}$$

Plugging these into (4.17) and taking  $t_1 = a$ ,  $t_2 = b$ ,  $t_3 = c$ ,  $t_4 = d$ ,  $t_5 = f$ , we get the following integral formula of Zhang and Wang [21, Theorem 4.3].

**Theorem 4.3** (Zhang and Wang).

$$(4.18) \quad \int_0^\pi \frac{h(\cos 2\theta; 1)}{h(\cos \theta; a, b, c, d, f)} {}_3\phi_2 \left[ \begin{matrix} g/bcd, ge^{i\theta}, ge^{-i\theta} \\ ag, fg \end{matrix} ; q, \frac{abcdf}{g} \right] d\theta \\ = \frac{2\pi(abcd, bcdf; q)_\infty}{(q, ab, ac, ad, bc, bd, cd, bf, cf, df; q)_\infty} {}_3\phi_2 \left[ \begin{matrix} g/b, g/c, g/d \\ af, fg, \end{matrix} ; q, \frac{abcdf}{g} \right],$$

provided  $|abcdf/g| < 1$ .

## 5. ISMAIL-STANTON'S GENERATING FUNCTION OF ASKEY-WILSON POLYNOMIALS

Ismail and Stanton [12] use the orthogonality relation of Askey-Wilson polynomials and (4.11) to prove the following generating function of Askey-Wilson polynomials.

**Theorem 5.1** (Ismail-Stanton). *The Askey-Wilson polynomials have the generating function*

$$(5.1) \quad \sum_{n=0}^{\infty} P_n(x, \mathbf{t}|q) c_n(\mathbf{t}, b) = \frac{1}{(be^{i\theta}, be^{-i\theta})_\infty},$$

where

$$(5.2) \quad c_n(\mathbf{t}, b) = \frac{b^n(t_2t_3t_4bq^n; q)_\infty}{(q, t_1t_2t_3t_4q^{n-1}; q)_n \prod_{j=2}^4 (t_j b; q)_\infty} {}_3\phi_2 \left[ \begin{matrix} t_2t_3q^n, t_2t_4q^n, t_3t_4q^n \\ t_1t_2t_3t_4q^{2n}, t_2t_3t_4bq^n \end{matrix} ; q, t_1b \right].$$

Prior to Ismail-Stanton's work, Kim and Stanton [15] proved the following special case of (5.1).

**Proposition 5.2** (Kim-Stanton). *We have the following generating function of continuous dual  $q$ -Hahn polynomials  $P_n(x; 0, t_2, t_3, t_4|q)$ ,*

$$(5.3) \quad \sum_{k=0}^{\infty} \frac{P_k(x; 0, t_2, t_3, t_4|q)}{(q, bt_2t_3t_4; q)_k} b^k = \frac{(bt_2, bt_3, bt_4; q)_\infty}{(bt_2t_3t_4, be^{i\theta}, be^{-i\theta}; q)_\infty}.$$

Kim and Stanton's idea is to use the "bootstrapping method": they derive (5.3) from the generating function of Al-Salam-Chihara polynomials  $P_n(x; 0, 0, t_3, t_4|q)$  by using the connection formula, [4, 8],

$$(5.4) \quad \frac{P_n(x; A, t_2, t_3, t_4|q)}{(q, t_2t_3, t_2t_4, t_3t_4; q)_n} = \sum_{k=0}^n \frac{P_k(x, \mathbf{t}|q)(At_2t_3t_4q^{n-1}; q)_k}{(q, t_2t_3, t_2t_4, t_3t_4, t_1t_2t_3t_4q^{k-1}; q)_k} \times \frac{t_1^{n-k}(A/t_1; q)_{n-k}}{(q, t_1t_2t_3t_4q^{2k}; q)_{n-k}}.$$

We show that the same idea works for Ismail-Stanton's formula (5.1), that is, one can derive (5.1) from (5.3) by using (5.4).

**Proof of Theorem 5.1** Letting  $A = 0$  and summing the two sides of (5.4), multiplied with  $\frac{(t_2t_3, t_2t_4, t_3t_4; q)_n b^n}{(bt_2t_3t_4; q)_n}$ , over  $n \geq 0$ , we obtain

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{P_n(x; 0, t_2, t_3, t_4|q)}{(q, bt_2t_3t_4; q)_n} b^n &= \sum_{n=0}^{\infty} \frac{b^n}{(q, bt_2t_3t_4; q)_n} \sum_{k=0}^n P_k(x, \mathbf{t}|q) \frac{(q, t_2t_3, t_2t_4, t_3t_4; q)_n}{(q, t_2t_3, t_2t_4, t_3t_4; q)_k} \\ &\quad \times \frac{t_1^{n-k}}{(t_1t_2t_3t_4q^{k-1}; q)_k (q, t_1t_2t_3t_4q^{2k}; q)_{n-k}} \\ &= \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} \frac{b^{n+k} P_k(x, \mathbf{t}|q) (t_2t_3, t_2t_4, t_3t_4; q)_{n+k}}{(bt_2t_3t_4; q)_{n+k} (q, t_2t_3, t_2t_4, t_3t_4; q)_k} \\ &\quad \times \frac{t_1^n}{(t_1t_2t_3t_4q^{k-1}; q)_k (q, t_1t_2t_3t_4q^{2k}; q)_n} \\ &= \sum_{k=0}^{\infty} \frac{b^k P_k(x, \mathbf{t}|q)}{(q, bt_2t_3t_4, t_1t_2t_3t_4q^{k-1}; q)_k} \sum_{n=0}^{\infty} \frac{(t_2t_3q^k, t_2t_4q^k, t_3t_4q^k; q)_n (bt_1)^n}{(q, t_1t_2t_3t_4q^{2k}, bt_2t_3t_4q^k; q)_n}. \end{aligned}$$

In view of (5.3), we can rewrite the above equation as:

$$\frac{(bt_2, bt_3, bt_4; q)_{\infty}}{(bt_2t_3t_4, be^{i\theta}, be^{-i\theta}; q)_{\infty}} = \sum_{k=0}^{\infty} \frac{b^k P_k(x, \mathbf{t}|q)}{(q, bt_2t_3t_4, t_1t_2t_3t_4q^{k-1}; q)_k} \times {}_3\phi_2 \left[ \begin{matrix} t_2t_3q^k, t_2t_4q^n, t_3t_4q^k \\ t_1t_2t_3t_4q^{2k}, bt_2t_3t_4q^k \end{matrix}; q, bt_1 \right].$$

The result follows then after some simplification.  $\square$

As an application of (5.1), we give another proof of Ismail-Stanton-Viennot integral (4.11).

**Another Proof of (4.11)** In view of (5.1) with  $b \rightarrow t_5$  the left-hand side of (4.11) is

$$\sum_{n=0}^{\infty} c_n(\mathbf{t}, t_5) \frac{(t_1t_4, t_1t_3, t_1t_4; q)_n}{t_1^n} \sum_{k=0}^n \frac{(q^{-n}, t_1t_2t_3t_4q^{n-1}; q)_k}{(q, t_1t_2, t_1t_3, t_1t_4; q)_k} q^k \int_0^{\pi} \frac{h(\cos 2\theta; 1)(t_1e^{i\theta}, t_1e^{-i\theta}; q)_k}{h(\cos \theta; t_1, t_2, t_3, t_4)} d\theta.$$

The inner integral can be evaluated by replacing  $t_1 \rightarrow t_1q^k$  in the Askey-Wilson integral (1.5),

$$(5.5) \quad \frac{2\pi(t_1t_2t_3t_4; q)_{\infty}}{(q; q)_{\infty} \prod_{1 \leq r < s \leq 4} (t_r t_s; q)_{\infty}} \sum_{n=0}^{\infty} c_n(\mathbf{t}, t_5) \frac{(t_1t_4, t_1t_3, t_1t_4; q)_n}{t_1^n} {}_2\phi_1 \left[ \begin{matrix} q^{-n}, t_1t_2t_3t_4q^{n-1} \\ t_1t_2t_3t_4 \end{matrix}; q, q \right].$$

The inner  ${}_2\phi_1$  series can be summed by  $q$ -Chu-Vandemonde (7.2)

$${}_2\phi_1 \left[ \begin{matrix} q^{-n}, t_1t_2t_3t_4q^{n-1} \\ t_1t_2t_3t_4 \end{matrix}; q, q \right] = \frac{(q^{1-n}; q)_n (t_1t_2t_3t_4q^{n-1})^n}{(t_1t_2t_3t_4; q)_n}.$$



Since  $(q^{1-n}; q)_n = 0$  for  $n \geq 1$ , (5.5) reduces to

$$\frac{2\pi(t_1 t_2 t_3 t_4; q)_\infty}{(q; q)_\infty \prod_{1 \leq r < s \leq 4} (t_r t_s; q)_\infty} c_0(\mathbf{t}, t_5),$$

which is clearly equal to the right-hand side of (4.11) in view of (5.2).  $\square$

## 6. MORE TRANSFORMATION FORMULAE

Ismail-Stanton [12, §5-6] proved the following expansion formula:

$$(6.1) \quad \sum_{n=0}^{\infty} \frac{(az, a/z; q)_n}{(q; q)_n} A_n B_n \delta^n = \sum_{n=0}^{\infty} \frac{(-\delta)^n q^{\binom{n}{2}}}{(q, Cq^{k-1}; q)_k} \\ \times \sum_{k=0}^n \frac{(q^{-n}, Cq^{n-1}, az, a/z; q)_k q^k A_k}{(q; q)_k} \sum_{r=0}^{\infty} \frac{\delta^r B_{r+n}}{(q, Cq^{2n}; q)_r}.$$

They also derive several interesting results from the above identity. We note that (6.1) follows from Proposition 2.3. Indeed, substituting  $\beta_n \rightarrow \frac{A_n u^n}{(q, \alpha, \beta; q)_n}$ ,  $\delta_n \rightarrow (\alpha, \beta; q)_n B_n \delta^n$  and  $b \rightarrow \gamma$  in (2.6) yields the following result.

**Theorem 6.1.**

$$(6.2) \quad \sum_{n=0}^{\infty} A_n B_n \frac{(\delta u)^n}{(q; q)_n} = \sum_{n=0}^{\infty} \frac{(1 - aq^{2n})(a; q)_n (a/\gamma; q)_n (\gamma/a)^n}{(1-a)(\gamma q; q)_n (q; q)_n} \\ \times \sum_{k=0}^n \frac{(1 - \gamma q^{2k})(aq^n; q)_k (q^{-n}; q)_k}{(1-\gamma)(\gamma q^{n+1}; q)_k (\gamma q^{1-n}/a; q)_k} \frac{(uq)^k A_k}{(q, \alpha, \beta; q)_k} \\ \times \sum_{r=0}^{\infty} \frac{(\gamma/a; q)_r (\gamma; q)_{r+2n} (\alpha, \beta; q)_{r+n} \delta^{r+n} B_{r+n}}{(q; q)_r (aq; q)_{r+2n}}.$$

Letting  $\gamma = 0$  and  $A_n \rightarrow A_n(b, c; q)_n$ , the above formula reduces to

$$(6.3) \quad \sum_{n=0}^{\infty} \frac{(b, c; q)_n}{(q; q)_n} A_n B_n (\delta u)^n = \sum_{n=0}^{\infty} \frac{(a; q)_n (1 - aq^{2n})(-1)^n q^{\binom{n}{2}}}{(q; q)_n (1-a)(aq; q)_{2n}} \\ \times \sum_{k=0}^n \frac{(q^{-n}, aq^n, b, c; q)_k}{(q, \alpha, \beta; q)_k} (uq)^k A_k \sum_{r=0}^{\infty} \frac{\delta^{r+n} B_{r+n} (\alpha, \beta; q)_{n+r}}{(q, aq^{2n+1}; q)_r}.$$

Obviously (6.3) reduces to (6.1) when  $\alpha = \beta = 0$ ,  $b \rightarrow az$ ,  $c \rightarrow a/z$ ,  $u \rightarrow 1$  and  $a \rightarrow C/q$ .

Verma [20] (see also [7, p. 84]) proved the following important expansion formula

$$(6.4) \quad \sum_{n=0}^{\infty} A_n B_n \frac{(xw)^n}{(q; q)_n} \\ = \sum_{n=0}^{\infty} \frac{(-x)^n}{(q, \gamma q^n; q)_n} q^{\binom{n}{2}} \sum_{j=0}^n \frac{(q^{-n}, \gamma q^n; q)_j}{(q, \alpha, \beta; q)_j} (wq)^r A_j \sum_{k=0}^{\infty} \frac{(\alpha, \beta; q)_{n+k}}{(q, \gamma q^{2n+1}; q)_k} x^k B_{k+n}.$$

We note that the above formula corresponds to (6.3) with  $b = c = 0$ ,  $a = \gamma$ ,  $u = w$  and  $\delta = x$ .

Besides, the special  $\beta_n = \frac{(g, h, y; q)_n}{(q, e, f, t; q)_n} u^n$  case of Proposition 2.5 gives the following result.

**Theorem 6.2.** *Let  $\delta, u, y, h, t, e, f, g, b_i, a_i$  ( $i \in \mathbb{N}$ ) be any complex numbers. Then the following formal power series in  $\zeta$  and  $u$  holds*

$$\begin{aligned} & {}_{p+1}\phi_p \left[ \begin{matrix} a_1, \dots, a_{p-1}, y, h, g \\ t, e, f, b_1, \dots, b_{p-1} \end{matrix} ; q, u\delta \right] = \sum_{n=0}^{\infty} \frac{(1 - aq^{2n})(a, a/b, a_1, \dots, a_{p-1}; q)_n (b\delta a^{-1})^n (b; q)_{2n}}{(1 - a)(bq, q, b_1, \dots, b_{p-1}; q)_n (aq; q)_{2n}} \\ & \times {}_8\phi_7 \left[ \begin{matrix} b, b^{1/2}q, -b^{1/2}q, y, h, g, aq^n, q^{-n} \\ b^{1/2}, -b^{1/2}, bq^{n+1}, bq^{1-n}/a, t, e, f \end{matrix} ; q, qu \right] {}_{p+1}\phi_p \left[ \begin{matrix} a_1q^n, \dots, a_{p-1}q^n, b/a, bq^{2n} \\ b_1q^n, \dots, b_{p-1}q^n, aq^{2n+1} \end{matrix} ; q, \delta \right]. \end{aligned}$$

Finally, we record two special cases of Theorem 6.2 when the above  ${}_8\phi_7$  is summable in closed form.

- Taking  $u = 1, t = bq/y, e = bq/h, f = bq/g$  and  $b^2q = ayhg$  in Theorem 6.2, the  ${}_8\phi_7$  series can be summed by Jackson's summation (7.13)

$$\begin{aligned} (6.5) \quad & {}_{p+1}\phi_p \left[ \begin{matrix} a_1, \dots, a_{p-1}, y, h, g \\ bq/y, bq/h, bq/g, b_1, \dots, b_{p-1} \end{matrix} ; q, \delta \right] \\ & = \sum_{n=0}^{\infty} \frac{(1 - aq^{2n})(a, a/b; q)_n (a_1, \dots, a_{p-1}; q)_n (b; q)_{2n}}{(1 - a)(q; q)_n (b_1, \dots, b_{p-1})_n (aq; q)_{2n}} \\ & \times \frac{(bq/gh, bq/yg, bq/gg; q)_n}{(bq/y, bq/h, bq/g, by/ygh; q)_n} {}_{p+1}\phi_p \left[ \begin{matrix} a_1q^n, \dots, a_{p-1}q^n, b/a, bq^{2n} \\ b_1q^n, \dots, b_{p-1}q^n, aq^{2n+1} \end{matrix} ; q, \delta \right]. \end{aligned}$$

- Taking  $y = t, h = e, u = b/ag$  and  $f = bq/g$  in Theorem 6.2, the  ${}_6\phi_5$  series can be summed by (7.12)

$$\begin{aligned} (6.6) \quad & {}_{p+1}\phi_p \left[ \begin{matrix} a_1, \dots, a_{p-1}, g \\ bq/g, b_1, \dots, b_{p-1} \end{matrix} ; q, b\delta/ag \right] \\ & = \sum_{n=0}^{\infty} \frac{(1 - aq^{2n})(a, a_1, \dots, a_{p-1}, ag/b; q)_n (bg\delta/a)^n (b; q)_{2n}}{(1 - a)(q, b_1, \dots, b_{p-1}, bq/g; q)_n (aq; q)_{2n}} \\ & \times {}_{p+1}\phi_p \left[ \begin{matrix} a_1q^n, \dots, a_{p-1}q^n, b/a, bq^{2n} \\ b_1q^n, \dots, b_{p-1}q^n, aq^{2n+1} \end{matrix} ; q, \delta \right]. \end{aligned}$$

**Acknowledgement.** This work was supported by the LABEX MILYON (ANR-10-LABX-0070) of Universit de Lyon, within the program "Investissements d'Avenir" (ANR-11-IDEX-0007) operated by the French National Research Agency (ANR). The first author would like to thank Institut Camille Jordan and Universit Claude Bernard Lyon 1 for hosting her for the academic year 2014–2015 so that this research could take place. This visit was also supported by a grant of CSC Ph.D. student visiting program.

## 7. APPENDIX

The following formulae are taken from [7, Appendices II and III].

The  $q$ -Gauss sum,

$$(7.1) \quad {}_2\phi_1 \left[ \begin{matrix} a, b \\ c \end{matrix}; q, c/ab \right] = \frac{(c/a, c/b; q)_\infty}{(c, c/ab; q)_\infty}, \quad (|c/ab| < 1).$$

The  $q$ -Chu-Vandermonde sums,

$$(7.2) \quad {}_2\phi_1 \left[ \begin{matrix} a, q^{-n} \\ c \end{matrix}; q, q \right] = \frac{(c/a; q)_n a^n}{(c; q)_n},$$

and, reversing the order of summation,

$$(7.3) \quad {}_2\phi_1 \left[ \begin{matrix} a, q^{-n} \\ c \end{matrix}; q, cq^n/a \right] = \frac{(c/a; q)_n}{(c; q)_n}.$$

The  $q$ -Pfaff-Saalschütz sum,

$$(7.4) \quad {}_3\phi_2 \left[ \begin{matrix} a, b, q^{-n} \\ c, abq^{1-n}/c \end{matrix}; q, q \right] = \frac{(c/a, c/b; q)_n}{(c, c/ab; q)_n}.$$

Sears' transformation,

$$(7.5) \quad {}_4\phi_3 \left[ \begin{matrix} a, b, c, q^{-n} \\ d, e, f \end{matrix}; q, q \right] = \frac{(e/a, f/a; q)_n a^n}{(e, f; q)_n} {}_4\phi_3 \left[ \begin{matrix} a, d/b, d/c, q^{-n} \\ d, aq^{1-n}/e, aq^{1-n}/f \end{matrix}; q, q \right],$$

where  $def = abcq^{1-n}$ .

Transformations of finite  ${}_3\phi_2$  series (by sending  $c, f \rightarrow 0$  in (7.5)),

$$(7.6) \quad {}_3\phi_2 \left[ \begin{matrix} q^{-n}, a, b \\ d, e \end{matrix}; q, q \right] = \frac{(e/a; q)_n a^n}{(e; q)_n} {}_3\phi_2 \left[ \begin{matrix} q^{-n}, a, d/b \\ d, aq^{1-n}/e \end{matrix}; q, bq/e \right].$$

Transformations of  ${}_3\phi_2$  series,

$$(7.7) \quad {}_3\phi_2 \left[ \begin{matrix} a, b, c \\ d, e \end{matrix}; q, de/abc \right] = \frac{(e/a, de/bc; q)_\infty}{(e, de/abc; q)_\infty} {}_3\phi_2 \left[ \begin{matrix} a, d/b, d/c \\ d, de/bc \end{matrix}; q, e/a \right]$$

$$(7.8) \quad = \frac{(b, de/ab, de/bc; q)_\infty}{(d, e, de/abc; q)_\infty} {}_3\phi_2 \left[ \begin{matrix} e/b, d/b, de/abc \\ de/ab, de/bc \end{matrix}; q, b \right],$$

where  $\max\{|de/abc|, |e/a|, |b|\} < 1$ .

Watson's transformation,

$$(7.9) \quad {}_8\phi_7 \left[ \begin{matrix} a, a^{1/2}q, -a^{1/2}q, b, c, d, e, q^{-n} \\ a^{1/2}, -a^{1/2}, aq/b, aq/c, aq/d, aq/e, aq^{n+1} \end{matrix}; q, \frac{a^2q^{n+2}}{bcde} \right] \\ = \frac{(aq, aq/de; q)_n}{(aq/d, aq/e; q)_n} {}_4\phi_3 \left[ \begin{matrix} aq/bc, d, e, q^{-n} \\ aq/b, aq/c, deq^{-n}/a \end{matrix}; q, q \right].$$

Transformations of very-well-poised  ${}_8\phi_7$  series,

$$\begin{aligned}
 (7.10) \quad & {}_8\phi_7 \left[ \begin{matrix} a, a^{1/2}q, -a^{1/2}q, b, c, d, e, f \\ a^{1/2}, -a^{1/2}, aq/b, aq/c, aq/d, aq/e, aq/f \end{matrix} ; q, \frac{a^2q^2}{bcdef} \right] \\
 &= \frac{(aq, b, bc\mu/a, bd\mu/a, be\mu/a, bf\mu/a; q)_\infty}{(aq/c, aq/d, aq/e, aq/f, \mu q, b\mu/a; q)_\infty} \\
 (7.11) \quad & \times {}_8\phi_7 \left[ \begin{matrix} \mu, \mu^{1/2}q, -\mu^{1/2}q, aq/bc, aq/bd, aq/be, aq/bf, b\mu/a \\ \mu^{1/2}, -\mu^{1/2}, bc\mu/a, bd\mu/a, be\mu/a, bf\mu/a, aq/b \end{matrix} ; q, b \right] \\
 &= \frac{(aq, aq/ef, \lambda q/e, \lambda q/f; q)_\infty}{(aq/e, aq/f, \lambda q, \lambda q/ef; q)_\infty} \\
 & \times {}_8\phi_7 \left[ \begin{matrix} \lambda, \lambda^{1/2}q, -\lambda^{1/2}q, \lambda b/a, \lambda c/a, \lambda d/a, e, f \\ \lambda^{1/2}, -\lambda^{1/2}, aq/b, aq/c, aq/d, \lambda q/e, \lambda q/f \end{matrix} ; q, \frac{aq}{ef} \right],
 \end{aligned}$$

where  $\lambda = a^2q/bcd$ ,  $\mu = a^3q^3/b^2cdef$  and  $\max\{|a^2q^2/bcdef|, |aq/ef|, |b|\} < 1$ .

Rogers'  ${}_6\phi_5$  summation,

$$(7.12) \quad {}_6\phi_5 \left[ \begin{matrix} a, a^{1/2}q, -a^{1/2}q, b, c, q^{-n} \\ a^{1/2}, -a^{1/2}, aq/b, aq/c, aq^{n+1} \end{matrix} ; q, \frac{aq^{n+1}}{bc} \right] = \frac{(aq, aq/bc; q)_n}{(aq/b, aq/c; q)_n}.$$

Jackson's  ${}_8\phi_7$  summation,

$$(7.13) \quad {}_8\phi_7 \left[ \begin{matrix} a, a^{1/2}q, -a^{1/2}q, b, c, d, e, q^{-n} \\ a^{1/2}, -a^{1/2}, aq/b, aq/c, aq/d, aq/e, aq^{n+1} \end{matrix} ; q, q \right] = \frac{(aq, aq/bc, aq/bd, aq/cd; q)_n}{(aq/b, aq/c, aq/d, aq/bcd; q)_n},$$

where  $a^2q = bcdeq^{-n}$ .

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EAST CHINA NORMAL UNIVERSITY, DEPARTMENT OF MATHEMATICS, 500 DONGCHUAN ROAD, SHANGHAI 200241, P. R. CHINA

*E-mail address:* [jiawei163jzy@163.com](mailto:jiawei163jzy@163.com);

UNIV LYON, UNIVERSITÉ CLAUDE BERNARD LYON 1, CNRS UMR 5208, INSTITUT CAMILLE JORDAN, 43 BLVD. DU 11 NOVEMBRE 1918, F-69622 VILLEURBANNE CEDEX, FRANCE

*E-mail address:* [zeng@math.univ-lyon1.fr](mailto:zeng@math.univ-lyon1.fr)