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# On Bent Manifolds and Deformed Spaces

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## Abstract

Riemannian geometry is the geometry of bent manifolds. However, as this paper shows in both a tangible and a formal way, it is also the geometry of deformed spaces. Applying Riemannian geometry to zones of our 3D space enables us to understand General Relativity (GR) almost intuitively, and inspires our imagination.

Space in GR is considered a continuous manifold, bent (curved) by energy/momentum. Both Einstein (1933) [1] and Feynman (1963) [2], however, considered the option of space being a deformed (curved) continuum rather than a bent (curved) continuous manifold. In the GDM [3], though, space is a 3D **deformed lattice** rather than a bent continuous manifold.

**Keywords:** Curvature; Space; Manifold; Riemannian geometry

## 1. Introduction

We relate to space not as a passive static arena for fields and particles but as an active elastic entity. The gravitational waves that General Relativity (GR) predicts, and have been observed recently, support our understanding. The mathematical objects of GR are  $n$ -dimensional manifolds in hyper-spaces with more dimensions than  $n$ . These are not necessarily the physical objects that GR accounts for. Einstein was led to GR by arguments that were unrelated to a possible elastic 3D space. However, in  $n$ -dimensional elastic **deformed** spaces, Euclidian geometry **is not** valid and we are compelled to use Riemannian geometry. Hence, GR is also the theory of our elastic 3D space. Steane in his recent book “Relativity Made Relatively Easy” [4] elaborates and clarifies these points. Rindler [5] uses elastic spaces to enable visualization of bent manifolds, whereas Callahan [6] declares: “...**in physics we associate curvature with stretching rather than bending.**”

We define space density  $\rho$ , for an elastic space lattice, as the number of space cells per unit volume. The density of an un-deformed, uniform, space is denoted  $\rho_0$ . If space is uniform, all of its elementary cells are of the same size, and Euclidian geometry is valid. When the density is not uniform, i.e., cells are of different sizes, an internal observer discovers that Euclidian geometry **is not** valid. When this internal observer measures circles, he will find an **Excess Radius**  $\delta r$  that differs from zero. This happens since he and his yardstick are both deformed exactly as the background.

We will show that, at large, the geometry of bent manifolds and deformed spaces is similar. However, in certain cases they differ, and these un-noticed differences are sometimes the cause of perplexing situations in physics.

Note that the term Curvature and Radius of Curvature are used both for manifolds and spaces.

The following equation, as we show, relates the Gaussian Curvature  $K$  at a point  $P$  in a

deformed space to the square of the relative gradient of space density  $\nabla\rho$  at the point to space density  $\rho$ , at the point: (21) in Section 15:

$$K = \frac{4\pi}{45} \left( \frac{\nabla\rho}{\rho} \right)^2$$

## 2. Intrinsic Curvature at a Point P on a 2D Manifold

On a manifold a yardstick retains its length, but is bent like the manifold.

### 2.1 2D Curved Surfaces in a 3D Euclidian Space

To obtain the curvature of a two-dimensional surface (manifold) at a point, P, that is intrinsic to the surface, consider the length C of a closed path that is the locus of all points that are at the same geodesic distance, r from the point.

For a sphere of radius  $r_0$ , the above circumference C for small  $r/r_0$ , as Fig. (1) shows, is:

$$C = 2\pi r_0 \sin\left(\frac{r}{r_0}\right) \approx 2\pi \left( r - \frac{r^3}{6r_0^2} \right) = 2\pi \left( r - \frac{Kr^3}{6} \right) \quad (1)$$

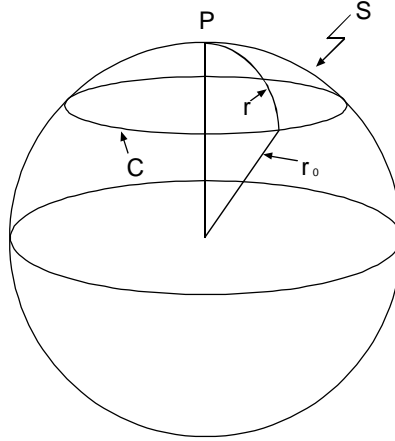
Here  $K = \frac{1}{r_0^2}$  is a natural **definition** for the curvature of a sphere of radius  $r_0$ .

By extracting K from (1), we obtain the **curvature** K and re-define it for a more general case:

$$K = \frac{3}{\pi} \lim_{r \rightarrow 0} \frac{2\pi r - C}{r^3} \quad (2)$$

This K depends only on the intrinsic properties of the metric of the two-dimensional surface with no reference to the embedding space. The sign of the curvature is positive if  $C < 2\pi r$ , negative if  $C > 2\pi r$ , and zero if  $C = 2\pi r$ .

**Fig. (1)**  
**Curvature of a Sphere**



Note that for any two-dimensional surface (manifold) **Gaussian curvature** can also be defined as:

$$K = \frac{1}{(r_0)_1 (r_0)_2} \quad (3)$$

where  $(r_0)_1$  and  $(r_0)_2$  are the radii of the largest and smallest osculating circles of sections formed by the corresponding planes perpendicular to a third plane, tangential to the surface at P. If the two osculating circles are on the same side of this third plane, we define the curvature as **positive**, and if on opposite sides - as **negative**. This definition ensures compatibility with the definition expressed by equation (2).

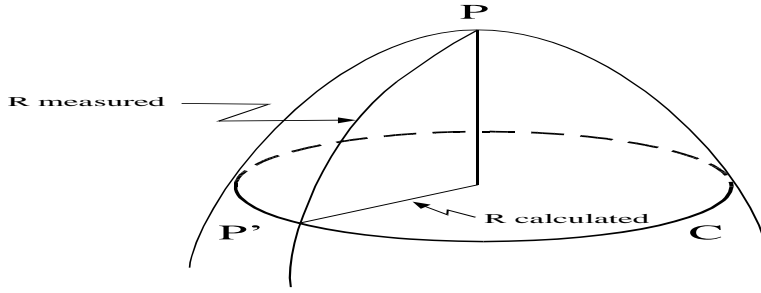
For a saddle surface, see Fig. (5), the two osculating circles at point P in the middle of the saddle are on opposite sides of the third tangential plane to the surface at P, and thus the saddle at P has a negative curvature.

## 2.2 The Excess Radius $\delta r$ of Curved Manifolds

The **Excess Radius**  $\delta r$  of curved manifolds [4], around a point P is:

$$\delta r \equiv r_{\text{measured}} - C_{\text{measured}} / 2\pi . \quad (4)$$

See Fig. (2).



**Fig. (2) Excess Radius**

Measuring distances with an intrinsic local yardstick gives the radius,  $r(\text{measured})$  from point P to point P', and the circumference,  $c(\text{measured})$ , of a circle centered at P passing through P.

The calculated radius is:

$$r_{\text{calculated}} = c_{\text{measured}} / 2\pi. \quad (5)$$

### 2.3 The Gaussian Curvature

Here  $r_{\text{measured}}$  is notated as simply  $r$ . According to [4] equation (10.7), for a 2D bent surface (2D manifold) the Gaussian curvature is:

$$K = \lim_{r \rightarrow 0} \frac{6\delta r}{r^3} \quad (6)$$

See the relation of (6) to (2):

$R_c$  is the radius of curvature at the point P on the surface, and:

$$K = 1/R_c^2 \quad (7)$$

According to [4], for the case of spherical symmetry, (6) for small  $r$  is:

$$K \approx 6\delta r / r^3 \quad (8)$$

For no radial symmetry,  $K$  is taken as the geometric mean (average) of the largest and smallest curvatures,  $k_1, k_2$ , one for each of the osculating circles; as if we have two separated surfaces.

## 2.4 Gaussian Curvature and the Schwarzschild Metric

Note [7] that,  $R_c$ , the radius of curvature of space in the Schwarzschild solution depends on the coordinate distance  $r$  from the center of the mass  $M$ , and is:

$$R_c = r_s^{-1/2} r^{3/2} \quad (9)$$

where  $r_s = 2GM/c^2$  is the Schwarzschild radius.

From (9) we get:

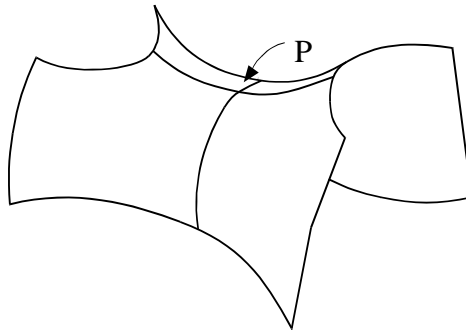
$$K = r_s/r^3 \quad (10)$$

From (8) and (10) we get:

$$\delta r \sim 1/6 r_s \quad (11)$$

More on the subject appears in [8].

## 3. Intrinsic Negative Curvature at a Point P on a 2D Manifold - A Saddle



**Fig. (5) Saddle-like 2-D Manifold in 3-D Space**

An osculating circle, of radius  $r$ , on the saddle around the point  $P$  has a longer circumference than that of a circle on a flat surface, since the radius wiggles up and down [5]. Therefore, according to (4) the Excess Radius  $\delta r$  of our curved manifold is negative. Gaussian curvature is:

$$K = \lim_{r \rightarrow 0} \frac{6\delta r}{r^3} \quad (6)$$

and hence the saddle is curved negatively  $K < 0$ .

Let us draw a plane tangential to the saddle at P. Perpendicular to this plane; we draw two more planes with osculating circles through P from the two sides of the plane tangential to the saddle at P. Let  $r_{c1}$  be the radius of an up circle with the smallest radius, and  $r_{c2}$  be the radius of a down circle with, also, the smallest radius. In this case [like (3) but with a minus sign]:

$$K = -1/(r_{c1} r_{c2}) \quad (8)$$

If the radii are equal and denoted  $r_c$  then:

$$K = -1/r_c^2 \quad (9)$$

#### 4. Intrinsic Positive Curvature in a 2D Deformed Space

In a deformed space a yardstick **does not retain** its length, but is contracted or dilated (stretched) like its local space.

##### 4.1 Positive Symmetric Curvature at a Point P

Consider Fig. (3), in which the circles represent space cells, or that the circles' diameters represent the lengths of yardsticks. Here, the yardstick at position P is the smallest and from P the yardstick increases in size symmetrically. This situation is analogous to a metallic plate, where the temperature increases from the center of the surface outwards, and thus the density decreases, i.e., the cell size increases. The 2D, inside observer, in Fig. (3), finds that the ratio of the circumference C of the circle to the radius r, as measured by the intrinsic yardstick, is:

$$C/r < 2\pi.$$

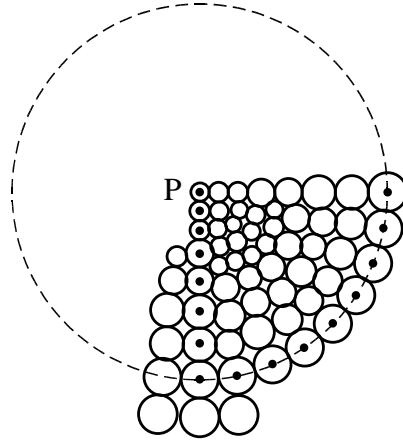
The **Excess Radius**  $\delta r$ , in this case, is the same as for a curved manifold around a point P:

$$\delta r \equiv r_{\text{measured}} - c_{\text{measured}}/2\pi . \quad (4)$$

For a positive curved 2D space, as Fig. (3) shows,  $\delta r = r_{\text{meas}} - r_{\text{cal}} > 0$  .



$$\begin{aligned}
r_{\text{measured}} &= 8 \\
c_{\text{measured}} &= 36 \\
c/r &= 36/8 = 4.5 < 2\pi \\
r_{\text{measured}} &> r_{\text{calculated}} \\
\text{The Excess Radius} \\
\delta r &= r_{\text{meas}} - r_{\text{cal}} > 0
\end{aligned}$$



**Fig. (3) Positive Curvature**

The 2D observer, therefore, concludes that there are two possibilities, that he lives in a two-dimensional space with a variable density, or on a curved two-dimensional surface “manifold” bent in a three-dimensional hyper space.

The 2D observer cannot imagine a three-dimensional space, but may be able to accept the necessary abstraction. In any case, he is not able to decide between the two possibilities just by examining his locality.

## 4.2 The Gaussian Curvature

We define the Gaussian curvature in the same way as in Section 2.3:

$$K = \lim_{r \rightarrow 0} \frac{6\delta r}{r^3} \quad (6)$$

and:

$$K = 1/R_c^2 \quad (7)$$

$R_c$  is defined as the “radius of curvature” at the point P. Note, however, that the term “radius”, in here, does not stand for the radius as measured by an internal or external observer.

According to [4], for the case of symmetry, (6) for small  $r$  gives:

$$K \approx 6\delta r/r^3 \quad (8)$$

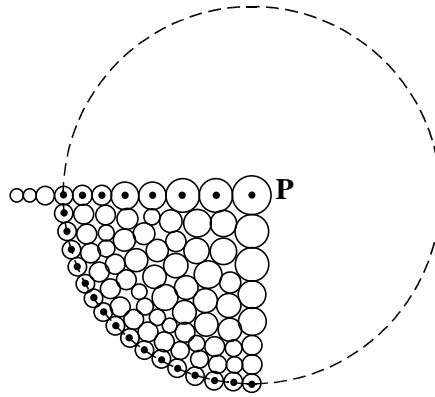
### 4.3 Asymmetric Positive Curvature at a Point P

For no radial symmetry,  $K$  is taken as the geometric mean (average) of the largest and smallest curvatures,  $k_1, k_2$ ; as if we have two surfaces with their corresponding excess radii  $\delta r_1$  and  $\delta r_2$ . A more accurate  $K$  should be taken based on the space density around  $P$ .

## 5. Intrinsic Negative Curvature in a 2D Deformed Space

### 5.1 Symmetric Curvature at a Point P

$$\begin{aligned} r_{\text{meas}} &= 8 \\ C_{\text{meas}} &= 64 \\ \frac{C}{r} &= 10.2 > 2\pi \\ r_{\text{meas}} &< r_{\text{cal}} \\ \text{The Excess Radius} \\ \delta r &= r_{\text{meas}} - r_{\text{cal}} < 0 \end{aligned}$$



**Fig. (4) Negative Curvature**

Fig. (4) shows a cell at point  $P$ , which is the largest, and from  $P$  outwards the cells decrease in

size. In this case,  $\frac{C}{r} > 2\pi$  and  $\delta r = r_{\text{meas}} - r_{\text{cal}} < 0$ . In this case, according to (6):

The curvature is negative  $K < 0$ .

Fig. (4) is analogous to a metal plate, where the temperature decreases from the point  $P$  outwards, as if the point  $P$  is heated with a blow-torch.

## 5.2 Asymmetric Curvature at a Point P

For no radial symmetry, K is taken, with a minus sign, as the geometric mean (average) of the excess radii  $\delta r_1$  and  $\delta r_2$ . A more accurate K should be taken based on the space density around P.

## 6. Combined Curvature and its Geometric Mean Radius

A point P is located in a zone of space, with an approximately symmetric radius  $R_L$  of curvature at P. By the introduction of a mass M at P, space is curved an additional curvature due to the presence of M. Let this symmetric curvature have a radius  $R_S$ . We can take the overall combined curvature at P as:

$$K = 1/(R_L R_S) \tag{12}$$

The Geometric Mean Radius of the Combined Curvature is thus:

$$R_c = \sqrt{(R_L R_S)} \tag{13}$$

This understanding is related to the long standing issue of Dark Matter. In GR central acceleration is related to the curvature of space [9]. A star at point P curves space locally around it symmetrically. If the star is located at the skirt of a galaxy it is also exposed to the curvature of space around the galaxy, due to the un-homogeny expansion of space around it [10]. This exposure contributes an additional general central acceleration. The geometric mean of the compound local and general accelerations, which is the Milgrom phenomenological equation [10], is related to the Geometric Mean Radius of the Compound Curvature.

## 7. Examples of Curvature in a 2D Deformed Space

### 7.1 Positive Curvature

To measure circles around a point P, Fig. (3), we take the normalized density at P as  $\rho(0) = 1$ .

Assuming radial symmetry, the density for external observers is  $\rho(r)$ . Let  $dr'$  be the yardstick's length for an internal observer and  $dr$  for the external observer. These  $dr'$  and  $dr$  are related by:

$$dr' = dr/\rho(r).$$

At P these yardsticks are equal since we have chosen  $\rho(0) = 1$ .

If, at a distance R from P the density is  $\rho(R) = 1/2$ , then:

$$dr' = dr/\rho(R) = 2dr$$

For external observers the circumference of a circle with radius R is  $c = 2\pi R$ , whereas for the internal observers it is:

$$c' = 2\pi R (dr/dr')_{atR} = 2\pi R \rho(R) = \pi R.$$

The radius of this circle, as measured by internal observers, is:

$$R' = \int_0^R dr' = \int_0^R 1/\rho(r) \cdot dr$$

and it should be larger than R. If, for example:

$$\rho(r) = 1/(1 + r/r_0) = 1/(1 + r/18)$$

then indeed  $\rho(0) = 1$  and, for  $R = r = 9$ ,  $\rho(R) = 1/2$ . The radius  $R'$  is then:

$$R' = \int_0^R (1 + r/18) \cdot dr = (r + r^2/36) \Big|_0^R = 11.25$$

Thus the Excess Radius is:

$$\delta R' = R' - c'/2\pi = R' - 1/2R \sim 11 - 4.5 \sim 6.5.$$

This criterion,  $\delta R' > 0$ , for positive curvature, expresses the fact that space around P is contracted.

## 7.2 Negative Curvature

If, at a distance R from the origin P, see Fig. (4), the density is  $\rho(R) = 2$ , then:

$$dr' = dr/\rho(R) = \frac{1}{2} dr$$

For external observers the circumference of a circle with radius R is  $c = 2\pi R$ , whereas for the internal observers it is:

$$c' = 2\pi R (dr/dr')_{atR} = 2\pi R \rho(R) = 4\pi R$$

The radius of this circle, as measured by the internal observer, is:

$$R' = \int_0^R dr' = \int_0^R \frac{1}{\rho(r)} \cdot dr$$

and it should be smaller than R. If, for example:

$$\rho(r) = (1 + r/r_0) = (1 + r/9)$$

then indeed  $\rho(0) = 1$ , and for  $R = r = 9$  we get  $\rho(R) = 2$ . The radius R' is then:

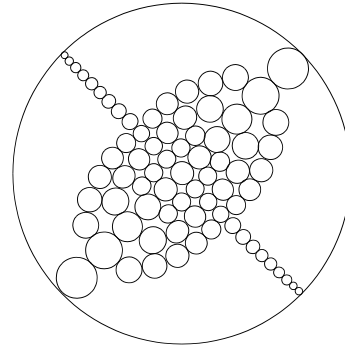
$$R' = \int_0^R \frac{1}{(1 + r/9)} \cdot dr = \left[ 9 \ln(1 + r/9) \right]_0^R = 6.24$$

Thus the Excess Radius, in this case, is:

$$\delta R' = R' - c'/2\pi = R - 2R = 6.24 - 9 \sim -2.8.$$

This criterion,  $\delta R' < 0$ , for negative curvature, expresses the fact that space around p is dilated.

## 8. A Saddle



**Fig. (5) Saddle-like Elastic 2-D Space**

At point P in the center, see Fig. (5), in  $1/4 \pi$  direction, the outward dilation of space means that if this shape appears in all directions the curvature is positive, with a radius of curvature  $R_+$ . If and in the  $3/4 \pi$  direction the outward contraction of space means curvature seems that if this shape appears in all directions the curvature is negative, with a radius of curvature  $R_-$ .

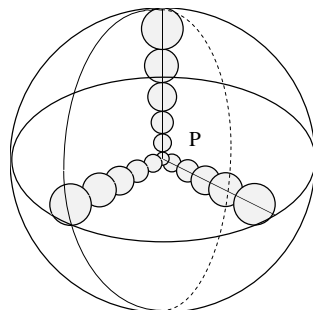
We can **define** the overall curvature K as :

$$K = 1/(R_+ R_-) \quad (14)$$

If  $R_+ - R_- > 0$  then approximately  $K > 0$  , and if  $R_+ - R_- < 0$  then  $K < 0$ .

Note that an accurate definition is related to space density in the all area.

## 9. Intrinsic Curvature at a Point P in a 3D Deformed Space



**Fig. (6) Three-dimensional Positive Curvature**

Fig. (6) shows a small cell located at point P, and cells of increasing size radiating from P with spherical symmetry. It is similar to the 2D space, shown in Fig. (3). If we measure the circumference of a **great circle** whose center is P, in any direction, a measurement of the Excess Radius yields:

$$\delta r = r_{\text{meas}} - r_{\text{cal}} > 0$$

Measurements of circumferences and radii can be taken around any point inside the sphere. But for any point, except point P in Fig, (6), there is no symmetry.

For a deformed 3D space with spherical symmetry around P the Gaussian curvature at P is also expressed by equation (2).

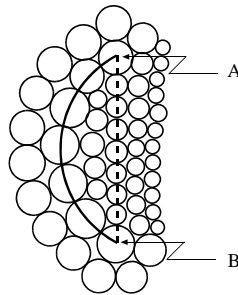
For the case of **no** symmetry, we have to determine the degree of deformation of circles around point P, in three orthogonal planes through P. For each of the orthogonal planes we have to determine the largest and smallest curvatures,  $k_1$ ,  $k_2$ , called the **principle curvatures**. Thus to specify a deformed three-dimensional space around a point P we need 3 x 2 numbers. The average of these six principle curvatures is the average Gaussian curvature of the deformed space at the locality of P.

According to the above, a complete definition of curvature in close proximity to a point in three-dimensional space requires six “curvature numbers”. These represent three pairs of curvature numbers for each of the three intersecting planes perpendicular to each other. These curvature numbers are components of a symmetric tensor of 2<sup>nd</sup> rank called the contracted **Riemannian tensor of curvature**, or the **Ricci tensor**, see Appendix A and B.

## 9.1 Geodesics

Fig. (7) shows the shortest distance between two points, A and B, in a deformed two-dimensional space. This figure shows that the shortest distance between points A and B for an internal observer, with their changing yardstick, is the solid line path that passes through the

centers of seven cells, and not the “straight” dashed line through A and B, that passes through nine cells.



**Fig. (7) Geodesics**

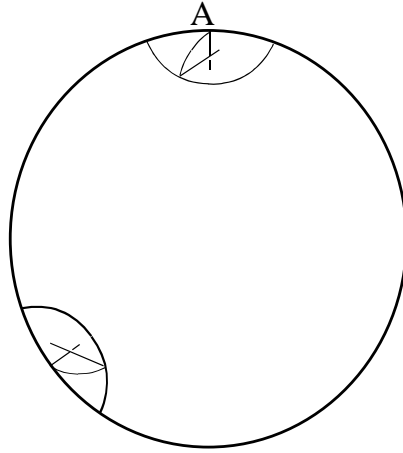
The shortest distance between points A and B, solid line, is the **geodesic line**. There is no difficulty in imagining this in three dimensions.

## **10. Local versus Global Curvature and the Flat Universe**

If a measurement of the radius and circumference of a circle at **any point**, on a surface, produces the Excess Radius result  $\delta r = r_{\text{meas}} - r_{\text{cal}} > 0$ , the observers will conclude that they live on a finite two-dimensional **closed** surface in a three-dimensional space, and not in a deformed space. Thus we realize that **curving and deforming are equivalent only locally**.

If space density of the universe is, by and large, uniform, and only changes locally around stars and galaxies, then on the scale of the universe, Euclidian geometry is valid. In this case, we can say that we are living in a **Flat Universe**. In other words, our universe is curved only locally, but not globally. According to observations made by cosmologists our universe is flat. Cosmologists assume that the universe is a 3D manifold bent in a hyper-space with an extra dimension. Hence they wonder why, from all possible values, the curvature of the universe happens to be zero.





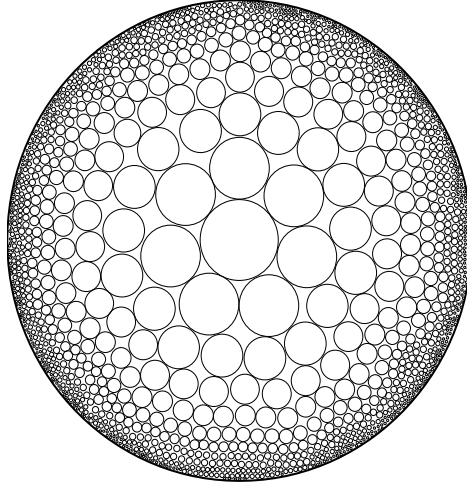
**Fig.(8) Global Curvature**

There is no need to wonder; **if** we consider ourselves 3D internal observers in a universe whose space is 3D and elastic, with local deformations. In our motion from place to place, both ourselves and our yardstick change. The only way for us to learn about our deformed space is thus to use Riemannian geometry. We are part of some cellular design and, in motion, **it is the design and the energy that moves, not the cells.** This is possible if elementary particles are not alien to space but simply wavepackets in motion.

## **11. Infinite Closed Space**

At first sight, space that is both infinite and closed seems a contradiction, but this is not so. Fig. (9) shows, for example, a closed two-dimensional space in a circle of radius  $R$  at whose center is a cell of finite size. From this center outwards, the cell size decreases, tending towards zero as they approach the circumference of the circle. In this case, the number of cells can be infinite, while the area they occupy, the area of a circle of radius  $R$ , is finite.

Obviously, this reasoning can also be applied to a three-dimensional sphere, and many other examples of an infinite closed space exist.



**Fig. (9) Infinite Closed Space**

## 12. The Metric, Ricci tensor and General Relativity (GR)

The metric tensor  $g_{ij}$  and Ricci tensor  $R_{ij}$  appear in the left hand side of the Einstein equation of GR (Rindler 2004) [5]:

$$R_{ij} - \frac{1}{2}Rg_{ij} = 8\pi G/c^4 \cdot T_{ij} \quad (15)$$

A **metric tensor** represents local deformations, around a given point, of both a continuum and a lattice. For a lattice, this representation is legitimate only if the sizes of its cells are orders of magnitude smaller than the scale of the deformation in this locality since we are using continuous differential equations. An example of such a metric is the Schwarzschild metric [5]. A metric tensor can also represent a global deformation like the FRW metric [5].

The **Ricci curvature tensor** (Appendix B) is the corresponding matrix of traces of the **Riemannian curvature tensor** (Appendix A). An **element  $R_{ij}$  of the Ricci tensor** is an average of sectional curvatures, around a given point, of the intersection line of a manifold with the  $x_i x_j$  plane. The **scalar curvature  $R$**  is defined as the trace of the Ricci curvature.

The reason why Einstein added the term  $-\frac{1}{2}Rg_{ij}$  to his equation is explained by Feynman [8] and Penrose [11] pages 461-2.

### 13. 4D Deformed Spacetime and Riemannian geometry

To an outside observer, the universe acts to all intents and purposes as an elastic body and the theory of elasticity is the tool with which to perform measurements and calculations. For an internal observer, in the universe, rulers, clocks and he himself change according to the deformation (space density) at any particular point. This is a change that the internal observer cannot detect directly. The only way an internal observer can detect the distortion, i.e., the change in space density, is by measuring triangles and circles, for example, and detecting any deviation in the sum of the angles in a triangle that should be  $180^\circ$  in undistorted space, or from the ratio of the circumference of a circle to its radius that should be  $2\pi$  in undistorted space. The internal observer interprets any deviation as invalidity of Euclidian geometry, whereas the external observer sees the deviation simply as an expression of a change in space density.

Apparently, the only geometry that the internal observer can adopt in the above case where rulers and clocks change, but do not appear to change, is **Riemannian geometry**.

Clearly, from the above, by adopting the point of view of the external observer it is possible to build a simple and tangible picture of the universe.

This tangible picture relates only to distorted three-dimensional space, which is therefore seen as curved, but, because this distortion determines the speed of light, it anyway determines the rate of clocks at any particular point. Thus, in finding a solution to any dynamic problem of the motion of a beam of light, or a particle, we must deal with an abstract mathematical space with four dimensions. Here we can certainly dispense with the need for tangibility because this is not a real space, but a mathematical space that we build for the purpose of calculations.

## 14. More on Curved Spaces and Riemannian Geometry

### 14.1 Definitions

**Homogenous Space** is a space in which all points are equivalent. An observer in this space is not able to determine their location.

**Isotropic Space** is a space in which all directions are equivalent. An observer in this space is not able, therefore, to determine their direction (orientation).

**Homogenous and Isotropic Space** is, for example, a space with uniform density whose basic space cells are identical and symmetrical.

**Flat Space** is a space that can be described by a rectangular Cartesian coordinate system, where the number of its dimensions is the same as the number of dimensions of the space, and in which the unit matrix represents the metric tensor.

Clearly, all homogenous and isotropic spaces are flat. Not all flat spaces, however, are necessarily, isotropic.

**Curved Space** is a space that is not flat, for example, a space that does not have a uniform density.

### 14.2 F. Morgan: On Riemannian Geometry [12]

*"Introduction*

*The central concept of Riemannian geometry is curvature. It describes the most important geometric features of racetracks and of universes.*

*The curvature  $k$  of a racetrack is defined as the rate at which the direction vector  $T$  of motion is turning, measured in radians or degrees per meter. The curvature is big on sharp curves, zero on straightways.*

*A two-dimensional surface can curve different amounts in different directions, perhaps upward in some directions, downward in others, and along straight lines in between. The principal curvatures  $k_1$  and  $k_2$  are the most upward (positive) and the most downward (negative), respectively.*

*Since  $k_1$  and  $k_2$  measure the amount the surface is curving in space, they cannot be measured by a bug confined to the surface. They are "extrinsic" properties. **Gauss made the astonishing discovery, however, that the Gauss curvature  $G = k_1 k_2$  can, in principle, be measured from within the surface.***

*This result, known as his Theorema Egregium or Remarkable Theorem, says that Gauss curvature is an*

“intrinsic” property. An  $m$ -dimensional hypersurface in  $R^{m+1}$  has  $m$  principal curvatures  $k_1, \dots, k_m$  at each point. For an  $m$ -dimensional surface in  $R^n$ , the situation is still more complicated; it is described not by numbers or by vectors, but by the second fundamental tensor. Still, **Gauss’s Theorema Egregium generalizes to show that an associated “Riemannian curvature tensor” is intrinsic.**

Modern graduate texts in differential geometry strive to give intrinsic curvatures intrinsic definitions, which ignore the ambient  $R^n$  from the outset. In this text, surfaces will start out sitting in  $R^n$ , where we can give, concrete definitions of the second fundamental tensor and the Riemannian curvature tensor. Only later will we prove that the Riemannian curvature tensor actually is intrinsic.”

**3.6. Gauss’s Theorema Egregium.** Gauss curvature  $G$  is intrinsic. Specifically, there are local coordinates  $u_1, u_2$  about any point in  $C^3$  surface  $S$  in  $R^3$  such that the first fundamental form  $g$  at  $p$  is  $I$  to first order. In any such coordinate system, the Gauss curvature is:

$$G = \frac{\partial^2 g_{12}}{\partial u_1 \partial u_2} - \frac{1}{2} \frac{\partial^2 g_{22}}{\partial u_1^2} - \frac{1}{2} \frac{\partial^2 g_{11}}{\partial u_2^2}$$

Remark. To say  $G = I$  to first order means that  $g_{11}(p) = g_{22}(p) = 1, g_{12}(p) = 0$ , and each

$$g_{ij,k}(p) = \frac{\partial g_{ij}}{\partial u_k}(p) = 0$$

Proof. Locally  $S$  is the graph of a function  $f$  over its tangent plane. Orthonormal coordinates on the tangent plane make the metric  $g$  equal to  $I$  to first order. We may assume that  $S$  is tangent to the  $x,y$ -plane at  $p$ . In  $x,y$  coordinates,

Remark. A full appreciation of Gauss’s Teorema Egregium requires the realization that most extrinsic quantities are not intrinsic.

In  $R^3$ , a flat piece of plane can be rolled, or “bent” into a piece of cylinder of radius  $r$  without changing anything a bug on the surface could detect. This bending, however, does change the curvature  $k$  of an arc of latitude of the cylinder from 0 in the plane to  $1/r$  on the cylinder; does change the principal curvatures  $k_1, k_2$  from 0, 0 to  $1/r, 0$ ; and does change the mean curvature  $H$  from 0 to  $1/r$ . The Gauss curvature, however, remains  $G = k_1 k_2 = 0$ , as the theorem guarantees.

“Remark. An intrinsic definition of **the scalar curvature  $R$  at a point  $p$**  in an  $m$ -dimensional surface  $S$  could be based on the formula for the volume of a ball of intrinsic radius  $r$  about  $p$ :

$$\text{volume} = \alpha_m r^m - \alpha_m \frac{R}{3 \cdot (m + 2)} r^{m+2} + \dots, \tag{6.10}$$

where  $\alpha_m$  is the volume of a unit ball in  $R^m$ . When  $m = 2$ , this formula reduces to Equation (3.8). The analogous formula for spheres played a role in R. Schoen’s solution of the Yamabe problem of finding a conformal deformation of a given Riemannian metric to one of constant scalar curvature (see Schoen [Sch, Lemma 2]).”

From equation (6.10) we can obtain the definition of scalar curvature of **two-dimensional** space.

Note that **we notate** the scalar curvature R at a point P in the quotation by the letter K.

$$K = \frac{12}{\pi} \left\{ \frac{\pi r^2 - S}{r^4} \right\} \quad (16)$$

and the scalar curvature of **three-dimensional** space:

$$K = \frac{45}{4\pi} \left\{ \frac{\frac{4\pi}{3} r^3 - V}{r^5} \right\} \quad (17)$$

### 14.3 Space Density and the Volume Change

The relative change in space density is:  $\frac{\delta\rho}{\rho}$

and the relative volume change is:  $\frac{\delta V}{V}$  but:  $\rho \propto \frac{1}{V}$

and therefore:  $\frac{\delta\rho}{\rho} = \frac{\delta \frac{1}{V}}{\frac{1}{V}} = \frac{-\frac{\delta V}{V^2}}{\frac{1}{V}} = -\frac{\delta V}{V}$  and therefore:

$$\frac{\nabla\rho}{\rho} = -\frac{\nabla V}{V} \quad (18)$$

In the case of spherical symmetry:  $\frac{\nabla\rho}{\rho} = -\frac{\frac{\partial V}{\partial r}}{V} \frac{\mathbf{r}}{r}$

## 15. The scalar curvature K and the space density $\rho(\mathbf{r})$

The scalar Riemannian curvature in a three dimensional space (17) is:

$$K = \frac{\frac{4\pi}{3} r^3 - (\text{Volume of Intrinsic Radius})}{5r^5}$$

For the spherical symmetric case:

$$K = \frac{\frac{4\pi}{3}r^3 - \frac{4\pi}{3}(r-u)^3}{5r^5} \cong \frac{4\pi}{5} \frac{u}{r^3} \quad \text{and for the simple case } u = cr:$$

$$K = \frac{\frac{4\pi}{3}r^3 - \frac{4\pi}{3}(1-c)^3r^3}{5r^5} \cong \frac{\frac{4\pi}{3}3c^2r^3}{5r^5} = \frac{4\pi}{5} \frac{c^2}{r^2} \quad (19)$$

On the other hand:

$$\frac{\rho_0 - \rho}{\rho} = \frac{V_0 - V}{V} = \frac{(r+u)^3 - r^3}{(r+u)^3} \cong 3 \frac{u}{r}$$

Therefore:

$$\frac{\nabla\rho}{\rho} = 3 \frac{\nabla u}{r} \quad \text{For the spherical symmetric case: } \frac{\partial\rho}{\rho} = 3 \frac{\partial u}{r} \quad \text{and if:}$$

$$u = cr: \quad \frac{\partial\rho}{\rho} = \frac{3c}{r} \quad \text{then:}$$

$$\left( \frac{\nabla\rho}{\rho} \right)^2 = 9 \frac{c^2}{r^2} \quad (20)$$

Comparing (19) to (20) gives:

$$K = \frac{4\pi}{45} \left( \frac{\nabla\rho}{\rho} \right)^2 \quad (21)$$

**This is the relation between the scalar curvature,  $K$ , and the gradient in space density relative to space density,  $\rho(r)$ .**

## 16. Summary

Tangibility inspires imagination.

“Imagination is more important than knowledge” A Einstein.

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## Appendix A Sectional and Riemannian Curvature

This section, based on F. Morgan (1998) [12] Chapter 5, discusses the Riemannian and Ricci tensors. We leave it to textbooks in differential geometry to show that these tensors are actually intrinsic.

The sectional curvature,  $k_{ij}$ , of a surface sliced by the  $x_i, x_j$  plane at a point P is:

$$k_{ij}(P) = a_{ii}a_{jj} - a_{ij}a_{ij} \quad (A1)$$

where:

$$a_{ij} = \frac{\partial^2 f(x_i, x_j)}{\partial x_i \partial x_j} \quad (A2)$$

$f(x_i, x_j)$  is the line of intersection of the surface with the plane.

If we choose coordinates that diagonalize the form (A1) we get:

$$k_{ij}(P) = k_i k_j \quad (A3)$$

which is the Gauss curvature. The **scalar sectional curvature** at P is then the weighted average of the sectional curvatures  $k_i k_j$ :

$$k(p) = \sum_{1 \leq i < j \leq m} P_{ij}^2 k_i k_j \quad (A4)$$

The Riemannian tensor is:

$$R_{ijkl} = a_{ik}a_{jl} - a_{jk}a_{il} \quad (A5)$$

Thus the Riemannian curvature tensor elements are just the 2 x 2 minors of the symmetrical **second fundamental tensor**:

$$\begin{bmatrix} a_{11} & a_{12} & \cdot & \cdot & a_{1m} \\ a_{21} & a_{22} & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{m1} & \cdot & \cdot & \cdot & a_{mm} \end{bmatrix} \quad (A6)$$

Immediately:

$$R_{jikl} = -R_{ijkl} \quad \text{and} \quad R_{ijlk} = -R_{ijkl} \quad (\text{A7})$$

$$R_{klij} = R_{ijkl} \quad (\text{A8})$$

$$R_{ijkl} + R_{iklj} + R_{iljk} = 0 \quad (\text{A9})$$

which is **Bianchi's first identity** on permutation of the last three indices.  $R_{ijkl}$  can be looked upon as a matrix of matrices:

$$\begin{bmatrix} [R_{i1k1}] & [R_{i1k2}] & \cdot & \cdot & [R_{i1km}] \\ [R_{i2k1}] & [R_{i2k2}] & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ [R_{imk1}] & [R_{imk2}] & \cdot & \cdot & [R_{imkm}] \end{bmatrix} \quad (\text{A10})$$

$$R_{jl} = \sum_i R_{ijil} \quad (\text{A11})$$

## Appendix B Ricci curvature

The **Ricci curvature tensor** is the corresponding matrix of traces of the Riemannian curvature tensor. The **Ricci curvature** is thus an **average of sectional curvatures**. The **scalar curvature**  $R$  (The contracted Ricci tensor), is defined as the trace of the Ricci curvature:

$$R = \sum_i R_{ii} \quad (\text{B1})$$

which is proportional to the average of all sectional curvatures at a point. For a three-dimensional space  $m = 3$ .  $j, l$  will have the values 1, 2, 3 and  $i, k$  the values 1, 2. The matrix,  $R$ , is thus:

$$\begin{bmatrix} R_{i1k1} & R_{i1k2} & R_{i1k3} \\ R_{i2k1} & R_{i2k2} & R_{i2k3} \\ R_{i3k1} & R_{i3k2} & R_{i3k3} \end{bmatrix} \quad (\text{B2})$$

$$[\mathbf{R}_{ikl}] = \begin{bmatrix} \mathbf{R}_{1111} & \mathbf{R}_{1121} \\ \mathbf{R}_{1112} & \mathbf{R}_{1122} \end{bmatrix} \quad (\text{B3})$$

If we diagonalize, we get six curvature numbers:

$$\begin{bmatrix} k_1 & & & & & \\ & k_2 & & & & \\ & & k_3 & & & \\ & & & k_4 & & \\ & 0 & & & k_5 & \\ & & & & & k_6 \end{bmatrix} \quad (\text{B4})$$