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On Curved Manifolds and Deformed Spaces

Shlomo Barak

Taga Innovations 28 Beit Hillel St. Tel Aviv 670017 Israel

Corresponding author: shlomo@tagapro.com

Abstract

Riemannian geometry is the geometry of bent manifolds. However, as this paper shows in a tangible rather than formal way, it is also the geometry of deformed zones of space. Applying Riemannian geometry to zones of space enables us to understand General Relativity (GR) almost intuitively, and inspires our imagination.

Space in GR is considered a continuous manifold, curved by energy/momentum. Both Einstein (1933) [1] and Feynman (1963) [2], however, considered the option of space being a deformed continuum rather than a bent continuous manifold. In the GDM (2017) [3], though, space is a 3D **deformed lattice** rather than a bent continuous manifold.

Keywords: Curvature; Space; Manifold; Riemannian geometry

1. Introduction

We relate to space not as a passive static arena for fields and particles but as an active elastic entity. The gravitational waves that General Relativity (GR) predicts, and have been observed recently, support our understanding. The mathematical objects of GR are n -dimensional manifolds in hyper-spaces with more dimensions than n . These are not necessarily the physical objects that GR accounts for. Einstein was led to GR by arguments that were unrelated to a possible elastic 3D space. However, in n -dimensional elastic **deformed** spaces, Euclidian geometry **is not** valid and we are compelled to use Riemannian geometry. Hence, GR is also the theory of our elastic 3D space. Steane in his recent book “Relativity Made Relatively Easy” [4] elaborates and clarifies these points. Rindler [5] uses elastic spaces to enable visualization of bent manifolds, whereas Callahan [6] declares: “...**in physics we associate curvature with stretching rather than bending.**”

We define space density ρ , for an elastic space lattice, as the number of space cells per unit volume. The density of an un-deformed, uniform, space is denoted ρ_0 . If space is uniform, all of its elementary cells are of the same size, and Euclidian geometry is valid. When the density is not uniform, i.e., cells are of different sizes, an internal observer discovers that Euclidian geometry **is not** valid. When this internal observer measures circles, he will find an **Excess Radius** δr that differs from zero. This happens since he and his yardstick are both deformed exactly as the background.

2. Intrinsic Curvature at a Point P on a 2D Manifold

2.1 2D Curved Surfaces in a 3D Euclidian Space

To obtain the curvature of a two-dimensional surface (manifold) at a point, P, that is intrinsic to the surface, consider the length C of a closed path that is the locus of all points that are at the same geodesic distance, r from the point.

For a sphere of radius r_0 , the above circumference C for small r/r_0 , as Fig. (1) shows, is:

$$C = 2\pi r_0 \sin\left(\frac{r}{r_0}\right) \approx 2\pi\left(r - \frac{r^3}{6r_0^2}\right) = 2\pi\left(r - \frac{Kr^3}{6}\right) \quad (1)$$

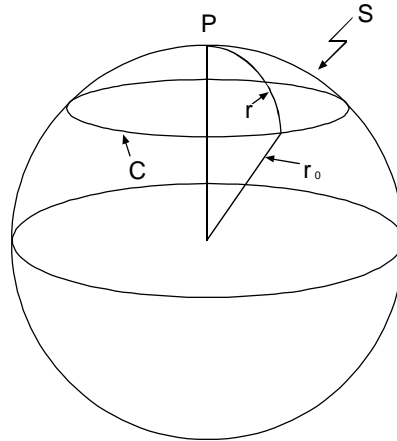
Here $K = \frac{1}{r_0^2}$ is a natural **definition** for the curvature of a sphere of radius r_0 .

By extracting K from (1), we obtain the **curvature** K and re-define it for a more general case:

$$K = \frac{3}{\pi} \lim_{r \rightarrow 0} \frac{2\pi r - C}{r^3} \quad (2)$$

This K depends only on the intrinsic properties of the metric of the two-dimensional surface with no reference to the embedding space. The sign of the curvature is positive if $C < 2\pi r$, negative if $C > 2\pi r$, and zero if $C = 2\pi r$.

Fig. (1)
Curvature of a Sphere



Note that for any two-dimensional surface (manifold) **Gaussian curvature** can also be defined as:

$$K = \frac{1}{(r_0)_1 (r_0)_2} \quad (3)$$

where $(r_0)_1$ and $(r_0)_2$ are the radii of the largest and smallest osculating circles of sections formed by the corresponding planes perpendicular to a third plane, tangential to the surface at

P. If the two osculating circles are on the same side of this third plane, we define the curvature as **positive**, and if on opposite sides - as **negative**. This definition ensures compatibility with the definition expressed by equation (2).

For a saddle surface, see Fig. (5), the two osculating circles at point P in the middle of the saddle are on opposite sides of the third tangential plane to the surface at P, and thus the saddle at P has a negative curvature.

2.2 The Excess Radius δr of Curved Manifolds

The **Excess Radius** δr of curved manifolds [4], around a point P is:

$$\delta r \equiv r_{\text{measured}} - c_{\text{measured}}/2\pi . \quad (4)$$

See Fig. (2).

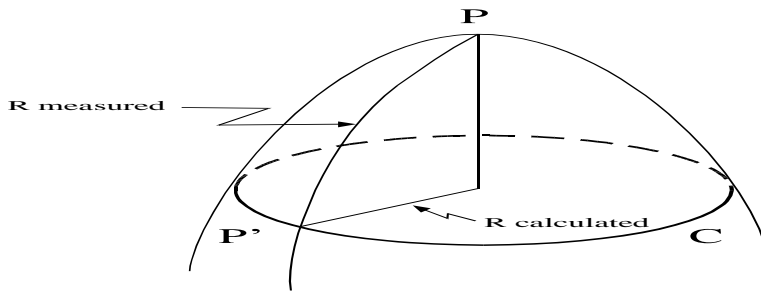


Fig. (2) Excess Radius

Measuring distances with an intrinsic local yardstick gives the radius, r_{measured} from point P to point P', and the circumference, c_{measured} , of a circle centered at P passing through P'. The calculated radius is:

$$r_{\text{calculated}} = c_{\text{measured}}/2\pi . \quad (5)$$

Here r_{measured} is notated as simply r . According to [4] equation (10.7), for a 2D bent surface (2D manifold) the Gaussian curvature is:

$$K = \lim_{r \rightarrow 0} \frac{6\delta r}{r^3} \quad (6)$$

R_c is the radius of curvature at the point P on the surface, and:

$$K = 1/R_c^2 \quad (7)$$

According to [4], for the case of spherical symmetry, (6) for small r is:

$$K \approx 6\delta r/r^3 \quad (8)$$

For no radial symmetry, K is taken as the average of the largest and smallest curvatures, k_1 , k_2 , of two circles.

Note [7] that, R_c , the radius of curvature of space in the Schwarzschild solution depends on the coordinate distance r from the center of the mass M, and is:

$$R_c = r_s^{-1/2} r^{3/2} \quad (9)$$

where $r_s = 2GM/c^2$ is the Schwarzschild radius.

From (9) we get:

$$K = r_s/r^3 \quad (10)$$

From (8) and (10) we get:

$$\delta r \sim 1/6 r_s \quad (11)$$

More on the subject appears in [8].

3. Intrinsic Positive Curvature in a 2D Deformed Space

Consider Fig. (3), in which the circles represent space cells, or that the circles' diameters represent the lengths of yardsticks. Here, the yardstick at position P is the smallest and from P the yardstick increases in size symmetrically. This situation is analogous to a metallic plate, where the temperature increases from the center of the surface outwards, and thus the density decreases, i.e., the cell size increases. The 2D inside observer in Fig. (3), finds that the ratio of the circumference C of the circle to the radius r, as measured by the intrinsic yardstick, is:

$C/r < 2\pi$. They therefore conclude that there are two possibilities, that they live in a two-dimensional space with variable density, or that they live on a curved two-dimensional surface “manifold” in a three-dimensional hyper space.

$$\begin{aligned}
 r_{\text{measured}} &= 8 \\
 C_{\text{measured}} &= 36 \\
 C/r &= 36/8 = 4.5 < 2\pi \\
 r_{\text{measured}} &> r_{\text{calculated}} \\
 \text{The Excess Radius} \\
 \delta r &= r_{\text{meas}} - r_{\text{cal}} > 0
 \end{aligned}$$

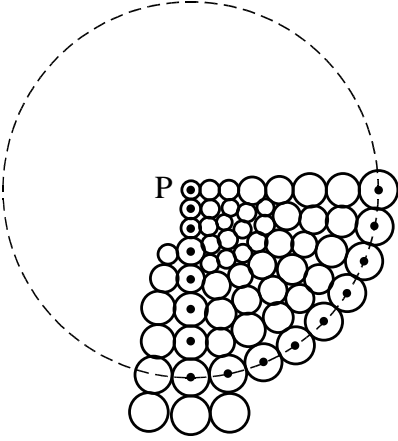


Fig. (3) Positive Curvature

2D observers cannot imagine a three-dimensional space, but may be able to accept the necessary abstraction. In any case, they are not able to decide between the two possibilities.

4. Intrinsic Negative Curvature in a 2D Deformed Space

$$\begin{aligned}
 r_{\text{meas}} &= 8 \\
 C_{\text{meas}} &= 64 \\
 \frac{C}{r} &= 10.2 > 2\pi \\
 r_{\text{meas}} &< r_{\text{cal}} \\
 \text{The Excess Radius} \\
 \delta r &= r_{\text{meas}} - r_{\text{cal}} < 0
 \end{aligned}$$

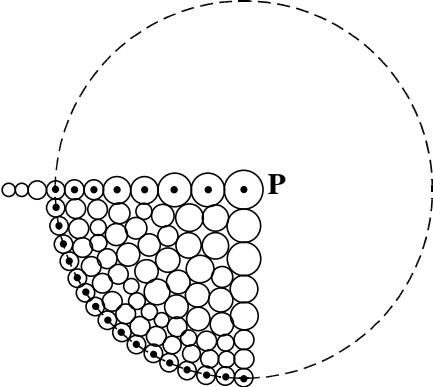


Fig. (4) Negative Curvature

Fig. (4) shows a cell at point P, which is the largest, and from P outwards the cells decrease in size. In this case, $\frac{C}{r} > 2\pi$ and:

$$\delta r = r_{\text{meas}} - r_{\text{cal}} < 0.$$

Fig. (4) is analogous to a metal plate, where the temperature decreases from the point P outwards, as if the point P is heated with a blow-torch.

5. Examples of Curvature in a 2D Deformed Space

5.1 Positive Curvature

To measure circles around a point P, Fig. (3), we take the normalized density at P as $\rho(0) = 1$. Assuming radial symmetry, the density for external observers is $\rho(r)$. Let dr' be the yardstick's length for an internal observer and dr for the external observer. These dr' and dr are related by:

$$dr' = dr/\rho(r).$$

At P these yardsticks are equal since we have chosen $\rho(0) = 1$.

If, at a distance R from P the density is $\rho(R) = 1/2$, then:

$$dr' = dr/\rho(R) = 2dr$$

For external observers the circumference of a circle with radius R is $c = 2\pi R$, whereas for the internal observers it is:

$$c' = 2\pi R (dr/dr')_{\text{at } R} = 2\pi R \rho(R) = \pi R.$$

The radius of this circle, as measured by internal observers, is:

$$R' = \int_0^R dr' = \int_0^R 1/\rho(r) \cdot dr$$

and it should be larger than R. If, for example:

$$\rho(r) = 1/(1 + r/r_0) = 1/(1 + r/18)$$

then indeed $\rho(0) = 1$ and, for $R = r = 9$, $\rho(R) = 1/2$. The radius R' is then:

$$R' = \int_0^R (1 + r/18) \cdot dr = (r + r^2/36) \Big|_0^R = 11.25$$

Thus the Excess Radius is:

$$\delta R' = R' - c'/2\pi = R' - 1/2R \sim 11 - 4.5 \sim 6.5.$$

This criterion, $\delta R' > 0$, for positive curvature, expresses the fact that space around P is contracted.

5.2 Negative Curvature

If, at a distance R from the origin P, see Fig. (4), the density is $\rho(R) = 2$, then:

$$dr' = dr/\rho(R) = 1/2 dr$$

For external observers the circumference of a circle with radius R is $c = 2\pi R$, whereas for the internal observers it is:

$$c' = 2\pi R (dr/dr')_{atR} = 2\pi R \rho(R) = 4\pi R$$

The radius of this circle, as measured by the internal observer, is:

$$R' = \int_0^R dr' = \int_0^R 1/\rho(r) \cdot dr$$

and it should be smaller than R. If, for example:

$$\rho(r) = (1 + r/r_0) = (1 + r/9)$$

then indeed $\rho(0) = 1$, and for $R = r = 9$ we get $\rho(R) = 2$. The radius R' is then:

$$R' = \int_0^R 1/(1 + r/9) \cdot dr = [9 \ln(1 + r/9)] \Big|_0^R = 6.24$$

Thus the Excess Radius, in this case, is:

$$\delta R' = R' - c'/2\pi = R' - 2R = 6.24 - 9 \sim -2.8.$$

This criterion, $\delta R' < 0$, for negative curvature, expresses the fact that space around p is dilated.

6. A Saddle

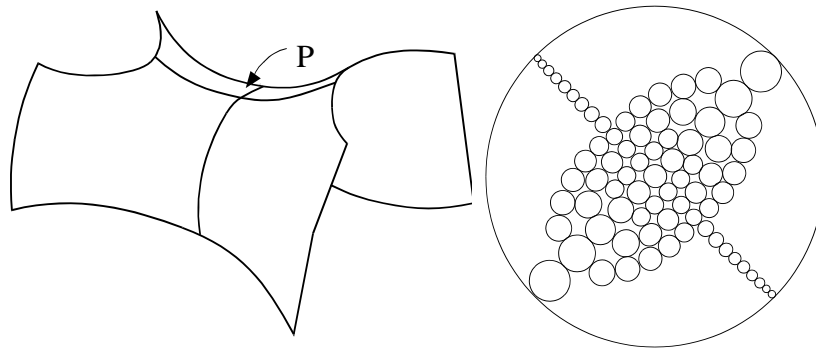


Fig. (5) Saddle-like 2-D Surface in 3-D Space versus an Elastic 2-D Surface in 2-D Space

7. Intrinsic Curvature at a Point P in a 3D Deformed Space

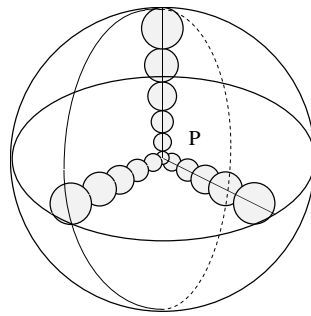


Fig. (6) Three-dimensional Positive Curvature

Fig. (6) shows a small cell located at point P, and cells of increasing size radiating from P with spherical symmetry. It is similar to the 2D space, shown in Fig. (3). If we measure the circumference of a **great circle** whose center is P, in any direction, a measurement of the Excess Radius yields:

$$\delta r = r_{\text{meas}} - r_{\text{cal}} > 0$$

Measurements of circumferences and radii can be taken around any point inside the sphere. But for any point, except point P in Fig, (6), there is no symmetry.

For a deformed 3D space with spherical symmetry around P the Gaussian curvature at P is also expressed by equation (2).

For the case of **no** symmetry, we have to determine the degree of deformation of circles around point P, in three orthogonal planes through P. For each of the orthogonal planes we have to determine the largest and smallest curvatures, k_1 , k_2 , called the **principle curvatures**. Thus to specify a deformed three-dimensional space around a point P we need 3×2 numbers. The average of these six principle curvatures is the average Gaussian curvature of the deformed space at the locality of P.

According to the above, a complete definition of curvature in close proximity to a point in three-dimensional space requires six “curvature numbers”. These represent three pairs of curvature numbers for each of the three intersecting planes perpendicular to each other. These curvature numbers are components of a symmetric tensor of 2nd rank called the contracted **Riemannian tensor of curvature**, or the **Ricci tensor**, see Appendix A and B.

7.1 Geodesics

Fig. (7) shows the shortest distance between two points, A and B, in a deformed two-dimensional space. This figure shows that the shortest distance between points A and B for an internal observer, with their changing yardstick, is the solid line path that passes through the centers of seven cells, and not the “straight” dashed line through A and B, that passes through nine cells.

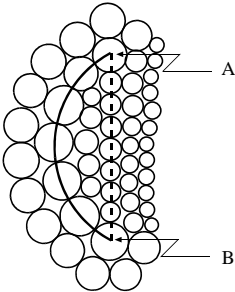


Fig. (7) Geodesics

The shortest distance between points A and B, solid line, is the **geodesic line**. There is no difficulty in imagining this in three dimensions.

8. Local versus Global Curvature and the Flat Universe

If a measurement of the radius and circumference of a circle at **any point**, on a surface, produces the Excess Radius result $\delta r = r_{\text{meas}} - r_{\text{cal}} > 0$, the observers will conclude that they live on a finite two-dimensional **closed** surface in a three-dimensional space, and not in a deformed space. Thus we realize that **curving and deforming are equivalent only locally**.

If space density of the universe is, by and large, uniform, and only changes locally around stars and galaxies, then on the scale of the universe, Euclidian geometry is valid. In this case, we can say that we are living in a **Flat Universe**. In other words, our universe is curved only locally, but not globally. According to observations made by cosmologists our universe is flat. These cosmologists assume that the universe is a 3D manifold bent in a hyper-space with an extra dimension. Hence they wonder why, from all possible values, the curvature of the universe happens to be zero.

There is no need to wonder; **if** we consider ourselves 3D internal observers in a universe whose space is 3D and elastic, with local deformations. In our motion from place to place, both ourselves and our yardstick change. The only way for us to learn about our deformed space is thus to use Riemannian geometry. We are part of some cellular design and, in motion, **it is the design and the energy that moves, not the cells**. This is possible if elementary particles are not alien to space but simply wavepackets in motion.

9. The Metric, Ricci tensor and General Relativity (GR)

The metric tensor g_{ij} and Ricci tensor R_{ij} appear in the left hand side of the Einstein equation of GR (Rindler 2004) [5]:

$$R_{ij} - \frac{1}{2}Rg_{ij} = 8\pi G/c^4 \cdot T_{ij}$$

A **metric tensor** represents local deformations, around a given point, of both a continuum and a lattice. For a lattice, this representation is legitimate only if the sizes of its cells are orders of

magnitude smaller than the scale of the deformation in this locality since we are using continuous differential equations. An example of such a metric is the Schwarzschild metric [5]. A metric tensor can also represent a global deformation like the FRW metric [5].

The **Ricci curvature tensor** (Appendix B) is the corresponding matrix of traces of the **Riemannian curvature tensor** (Appendix A). An **element R_{ij} of the Ricci tensor** is an average of sectional curvatures, around a given point, of the intersection line of a manifold with the $x_i x_j$ plane. The **scalar curvature R** is defined as the trace of the Ricci curvature.

The reason why Einstein added the term $-\frac{1}{2}Rg_{ij}$ to his equation is explained by Feynman [8] and Penrose [9] pages 461-2.

10. Summary

Tangibility inspires imagination. “Imagination is more important than knowledge” A Einstein.

Acknowledgments

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Appendix A Sectional and Riemannian Curvature

This section, based on F. Morgan (1998) [10] Chapter 5, discusses the Riemannian and Ricci tensors. We leave it to textbooks in differential geometry to show that these tensors are actually intrinsic.

The sectional curvature, k_{ij} , of a surface sliced by the x_i, x_j plane at a point P is:

$$k_{ij}(P) = a_{ii}a_{jj} - a_{ij}a_{ij} \quad (A1)$$

where:

$$a_{ij} = \frac{\partial^2 f(x_i, x_j)}{\partial x_i \partial x_j} \quad (A2)$$

$f(x_i, x_j)$ is the line of intersection of the surface with the plane.

If we choose coordinates that diagonalize the form (A1) we get:

$$k_{ij}(P) = k_i k_j \quad (A3)$$

which is the Gauss curvature. The **scalar sectional curvature** at P is then the weighted average of the sectional curvatures $k_i k_j$:

$$k(p) = \sum_{1 \leq i < j \leq m} P_{ij}^2 k_i k_j \quad (A4)$$

The Riemannian tensor is:

$$R_{ijkl} = a_{ik}a_{jl} - a_{jk}a_{il} \quad (A5)$$

Thus the Riemannian curvature tensor elements are just the 2 x 2 minors of the symmetrical **second fundamental tensor**:

$$\begin{bmatrix} a_{11} & a_{12} & \cdot & \cdot & a_{1m} \\ a_{21} & a_{22} & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{m1} & \cdot & \cdot & \cdot & a_{mm} \end{bmatrix} \quad (\text{A6})$$

Immediately:

$$R_{jikl} = -R_{ijkl} \quad \text{and} \quad R_{ijlk} = -R_{ijkl} \quad (\text{A7})$$

$$R_{klij} = R_{ijkl} \quad (\text{A8})$$

$$R_{ijkl} + R_{iklj} + R_{iljk} = 0 \quad (\text{A9})$$

which is **Bianchi's first identity** on permutation of the last three indices. R_{ijkl} can be looked upon as a matrix of matrices:

$$\begin{bmatrix} [R_{i1k1}] & [R_{i1k2}] & \cdot & \cdot & [R_{i1km}] \\ [R_{i2k1}] & [R_{i2k2}] & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ [R_{imk1}] & [R_{imk2}] & \cdot & \cdot & [R_{imkm}] \end{bmatrix} \quad (\text{A10})$$

$$R_{jl} = \sum_i R_{ijil} \quad (\text{A11})$$

Appendix B Ricci curvature

The **Ricci curvature tensor** is the corresponding matrix of traces of the Riemannian curvature tensor. The **Ricci curvature** is thus an **average of sectional curvatures**. The **scalar curvature** R (The contracted Ricci tensor), is defined as the trace of the Ricci curvature:

$$R = \sum_i R_{ii} \quad (\text{B1})$$

which is proportional to the average of all sectional curvatures at a point. For a three-dimensional space $m = 3$. j, l will have the values 1, 2, 3 and i, k the values 1, 2. The matrix, R , is thus:

$$\begin{bmatrix} R_{i1k1} & R_{i1k2} & R_{i1k3} \\ R_{i2k1} & R_{i2k2} & R_{i2k3} \\ R_{i3k1} & R_{i3k2} & R_{i3k3} \end{bmatrix} \quad (B2)$$

$$[R_{i1k1}] = \begin{bmatrix} R_{1111} & R_{1121} \\ R_{1112} & R_{1122} \end{bmatrix} \quad (B3)$$

If we diagonalize, we get six curvature numbers:

$$\begin{bmatrix} k_1 & & & & & \\ & k_2 & & & & \\ & & k_3 & & & \\ & & & k_4 & & \\ & 0 & & & k_5 & \\ & & & & & k_6 \end{bmatrix} \quad (B4)$$