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Random Discretization of the Finite Fourier Transform and Related Kernel Random Matrices.

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Abstract— The finite Fourier transform operator, and in particular its singular values, have been extensively studied in relation with band-limited functions. We study here the sequence of singular values of a random discretization of the finite Fourier transform in relation with applications to wireless communication. We prove that, with high probability, this sequence is close to the sequence of singular values of the finite Fourier transform itself. This also leads us to develop ℓ^2 estimates for the spectrum of kernel random matrices. This seems to be new to our knowledge. As applications, we give fairly good approximations of the number of degrees of freedom and the capacity of an approximate model of a MIMO wireless communication network. We provide the reader with some numerical examples that illustrate the theoretical results of this paper.

1 Introduction

In this work, we are interested in the study of the behaviour of the spectrum of the $n \times n$ random matrix A , whose j, k entry is given by

$$a_{j,k} = \frac{\sqrt{m}}{n} \exp(2i\pi m Z_j Y_k),$$

where m is a positive number, $1 \leq m \leq n$ and where the Y_j, Z_k are independent random variables, following the uniform law on $I = (-1/2, +1/2)$. Note that this matrix depends on a variable ω varying in some probability space Ω but we systematically omit this variable in the expressions. Also, the matrix A may be seen as a random discretization of the finite Fourier transform \mathcal{F}_m , which is defined on $L^2(I)$ by

$$\mathcal{F}_m(f)(y) = \sqrt{m} \int_{-1/2}^{+1/2} \exp(2i\pi m y z) f(z) dz, \quad |y| < 1/2.$$

With the normalization constant $\frac{\sqrt{m}}{n}$ given in the coefficients $a_{j,k}$, the Hilbert-Schmidt norm of A is equal to the Hilbert-Schmidt norm of \mathcal{F}_m . Recall that the Hilbert-Schmidt norm of A is given by

$$\|A\|_{HS}^2 = \sum_{j,k=1}^n |a_{j,k}|^2.$$

The random matrix $\frac{n}{\sqrt{m}}A$ was proposed by Desgroseilliers, Lévêque and Preissmann [9, 10] as an approximate model for the channel fading matrix (after some renormalization) in a wireless communication MIMO (Multi Input Multi Output) transmission network. This model is done under the

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assumptions that one has a line of sight propagation and that network nodes are gathered in clusters. They do not add any noise for simplification of the model and studied two main related issues: the first issue is the number of degrees of freedom of the system, the second one is its information capacity. They have partial results and we refer to their papers for the interested reader. We found that the previous issues led us to some interesting mathematical questions, which we discuss in the present paper. As in the previous two references, m and n are assumed to be large, with $m \ll n$. As far as the network capacity is concerned, a reasonable expression for it, with p the total power of the transmitters that is assumed to be equally distributed, is given by

$$C(p) = \log \det \left(I_n + \frac{np}{m} A^* A \right). \quad (1)$$

Here I_n is the identity matrix. We show that, with a large probability and under assumptions on m, n and p , this capacity is well approximated by $m \log(\frac{np}{m})$. We also prove that the number of degrees of freedom is well approximated by m . We only need to assume that m and n/m are large enough, which is much less restrictive than the previous results in this direction obtained in [9, 10]. Succeeding in this approximation has been the guiding thread of this study.

The whole paper is based on a comparison of the singular values of A with those of the integral operator \mathcal{F}_m . More precisely, we have the following result,

Theorem 1. *The singular values of A are close to the singular values of \mathcal{F}_m . More precisely, for any $\xi > 0$, we have*

$$\left(\sum_{j=0}^{n-1} |\lambda_j(A^* A) - \lambda_j(\mathcal{F}_m^* \mathcal{F}_m)|^2 \right)^{1/2} \leq \frac{(2\xi + \sqrt{2})m}{\sqrt{n}}. \quad (2)$$

with probability $1 - 4e^2 e^{-\xi^2/2}$.

Here the sequence of the singular values $\lambda_j(T) = (\lambda_j(T^* T))^{1/2}$, are taken in decreasing order, starting from $j = 0$. The notation is used for singular values (arranged in the decreasing order) of a Hilbert-Schmidt operator with finite or infinite rank. We recall that for a Hermitian positive semi-definite operator, the singular values are eigenvalues of this later. Note that we also estimate the expectation of the left hand side of (2).

Remark that for small values of m , one could use a Taylor approximation based technique to derive such estimate, see for instance [4]. But, in this work, we are interested in the spectrum of $A^* A$ when m and n are large. We will see that m/n small will allow to consider the right hand side of (2) as a rest, so that the spectrum of $A^* A$ can be well approximated by the spectrum of the corresponding integral operator $\mathcal{F}_m^* \mathcal{F}_m$. The behaviour of this last operator, known as the Sinc kernel operator, has been largely explored in the literature, see for example [2, 7, 8, 14]. This gives us precise information on singular values of A . As predicted by our theoretical results, we check numerically that the singular values of A , and \mathcal{F}_m have some similarity when m/n is small. The similarity increases when m^2/n is small. Also, by using some precise behaviour and decay estimates of the sequence of the eigenvalues of the Sinc kernel operator, we give fairly good estimates of the number of degrees of freedom and the capacity of the wireless network studied in [9, 10].

Our strategy in proving the previous estimate (2) is simple and essentially based on the use of a generalized version of McDiarmid's inequality. By taking the expectation of the matrix $A^* A$ in the Z -random variables, we find the matrix

$$H = H_{\kappa_m} = \frac{1}{n} \begin{pmatrix} \kappa_m(Y_1, Y_1) & \kappa_m(Y_1, Y_2) & \cdots & \kappa_m(Y_1, Y_n) \\ \kappa_m(Y_2, Y_1) & \kappa_m(Y_2, Y_2) & \cdots & \kappa_m(Y_2, Y_n) \\ \vdots & \vdots & \vdots & \vdots \\ \kappa_m(Y_n, Y_1) & \kappa_m(Y_n, Y_2) & \cdots & \kappa_m(Y_n, Y_n) \end{pmatrix}, \quad (3)$$

where $\kappa_m(x, y)$ is the positive definite Sinc kernel, given by

$$\kappa_m(x, y) = \text{sinc}(m(x - y)) = \frac{\sin m\pi(x - y)}{\pi(x - y)}. \quad (4)$$

The previous random matrix is a special case of a more general (kernel) Gram matrix H_κ , with entries $\frac{1}{n}\kappa(Y_j, Y_k)$, where the Y_j are sample points drawn randomly according to a probability law P on some input space \mathcal{X} and $\kappa(\cdot, \cdot)$ is a kernel function on $\mathcal{X} \times \mathcal{X}$. More precisely, the random variables Y_j form an i.i.d sequence and the kernel κ is symmetric positive semi-definite. Note that we state a large part of our results in this general case. We give a last definition before describing them. Let T_κ be the integral operator with kernel κ and defined on $L^2(\mathcal{X}, P)$, by

$$T_\kappa(f)(x) = \int_{\mathcal{X}} \kappa(x, y)f(y) dP, \quad x \in \mathcal{X}. \quad (5)$$

It is well-known [16], that with large probability, the eigenvalues $\lambda_j(H_\kappa)$ are comparable with their expectation $\mathbb{E}(\lambda_j(H_\kappa))$. These expectations are themselves comparable with the eigenvalues of the integral operator T_κ , associated with the kernel, see [16, 1]. Here, we prove a stronger result by having an inequality in the ℓ^2 norms. If we assume that $R = \sup_y \kappa(y, y)$ is finite, then we prove the following result,

Theorem 2. *Let κ be a positive semi-definite kernel, T_κ the integral operator and H_κ the random kernel matrix as above. We assume that $R = \sup_y \kappa(y, y)$ is finite. Then, for any $\xi > 0$, we have the inequality*

$$\sum_{j=0}^{n-1} (|\lambda_j(H_\kappa) - \lambda_j(T_\kappa)|^2)^{1/2} \leq \frac{(\xi + \sqrt{2})R}{\sqrt{n}},$$

with probability $1 - 2e^2e^{-\xi^2/2}$.

Such an inequality has been given in [16] for each eigenvalue separately and up to some slightly better constants. The following theorem is central in this work.

Theorem 3. *Assume that $R = \sup_y \kappa(y, y)$ is finite. Then, under the previous notations, we have*

$$\mathbb{E} \left(\sum_{j=0}^{n-1} |\lambda_j(H_\kappa) - \lambda_j(T_\kappa)|^2 \right) \leq \frac{2R^2}{n}. \quad (6)$$

This seems to be new, even if it is inspired from the work of V. Koltchinskii and E. Giné [11], who have given estimates of this kind.

Let us make a last remark. In kernel principal component analysis (Kernel-PCA), the subspace spanned by the first m eigenvectors of H_κ is used to form a lower m dimensional representation of the given data. For more information on this widely used application, the reader is referred for example to [1] and [16]. Such a representation makes also sense for the matrix A and we give estimates for the reconstruction error in the appendix.

This work is organized as follows. In section 2, we first give two concentration inequalities for functions of an i.i.d sequence of n random variables with values in a Hilbert space \mathcal{H} . Then, we prove some general approximation results in the ℓ^2 -norm of the eigenvalues of a random kernel matrix by the eigenvalues of the associated kernel integral operator. Then, we restrict ourselves to the special interesting case of the random Fourier matrix A . In particular, we prove that the spectrum of A^*A can be well approximated by the spectrum of the corresponding integral operator $\mathcal{F}_m^* \mathcal{F}_m$. In section 3, we use some precise decay and estimate of the eigenvalues of the Sinc kernel operator and give

estimates of the number of degrees of freedom and the capacity, associated with the random matrix A . In section 4, we give some numerical examples that illustrate the different results of this work. Finally, in the appendix, we give the proof of Theorem 3, which is a central result of this work. Also, we give some further inequalities relating the singular values of the random discrete finite Fourier and the finite Fourier transform operators.

We will write \mathbb{P} , \mathbb{P}_Z (resp. \mathbb{P}_Y) depending whether we take the probability on the whole probability space, or only in Z_1, \dots, Z_n (resp. Y_1, \dots, Y_n).

2 Concentration inequalities and approximation of the eigenvalues of kernel random matrices.

We first recall the Hilbert valued version of McDiarmid's concentration inequality, given by the following proposition, which is due to T. Hayes. As it will be seen later on, this inequality plays a central role in our estimates of the eigenvalues of kernel random matrices.

Proposition 1 ([5]). *Let Φ be a measurable function on \mathbb{R}^n with values in a real Hilbert space \mathcal{H} . Assume that*

$$\|\Phi(z_1, \dots, z_\ell, \dots, z_n) - \Phi(z_1, \dots, z'_\ell, \dots, z_n)\|_{\mathcal{H}} \leq R$$

for each sequence $(z_j)_{j \neq \ell}, z_\ell, z'_\ell$. If Z_1, \dots, Z_n are independent random variables, then we have

$$\mathbb{P}(\|\Phi(Z_1, \dots, Z_n) - \mathbb{E}\Phi(Z_1, \dots, Z_n)\|_{\mathcal{H}} > \xi) \leq 2e^2 \exp\left(-\frac{2\xi^2}{nR^2}\right).$$

This is McDiarmid's Inequality when $\mathcal{H} = \mathbb{R}$. The previous general McDiarmid's inequality is a direct consequence of the Azuma-Hoeffding inequality for Hilbert-valued random variables given in [5]. The proof of this last one is very intricate. We will also be interested in expectations. The proof of the next lemma is standard. It is an easy generalization of the way to deduce McDiarmid's inequality from Azuma-Hoeffding inequality in the scalar case.

Lemma 1. *Under the same assumptions as in Proposition 1, we have the inequality*

$$\mathbb{E}(\|\Phi(Z_1, \dots, Z_n) - \mathbb{E}\Phi(Z_1, \dots, Z_n)\|_{\mathcal{H}}^2) \leq nR^2.$$

Proof. We give it for completeness. We write

$$\Phi(Z_1, \dots, Z_n) - \mathbb{E}\Phi(Z_1, \dots, Z_n) = \sum_{k=1}^n (\mathbb{E}\Phi(Z_1, \dots, Z_n | \mathcal{F}_k) - \mathbb{E}\Phi(Z_1, \dots, Z_n | \mathcal{F}_{k-1})) = \sum_{k=1}^n V_k$$

with \mathcal{F}_k the σ -algebra generated by Z_1, \dots, Z_k when $k \geq 1$ and \mathcal{F}_0 the σ -algebra generated by the constants, so that $\mathbb{E}(V | \mathcal{F}_0) = \mathbb{E}(V)$. For each $k > 1$, the random variable is \mathcal{F}_k -measurable and we have the equality $\mathbb{E}(V_k | \mathcal{F}_{k-1}) = 0$. As a consequence, if $j < k$, then

$$\mathbb{E}(\langle V_j, V_k \rangle_{\mathcal{H}}) = \mathbb{E}(\langle V_j, \mathbb{E}(V_k | \mathcal{F}_{k-1}) \rangle) = 0.$$

Consequently, we have

$$\mathbb{E}(\|\Phi(Z_1, \dots, Z_n) - \mathbb{E}\Phi(Z_1, \dots, Z_n)\|_{\mathcal{H}}^2) = \sum_{j=1}^n \|V_j\|_{\mathcal{H}}^2.$$

We then use the fact that V_j is obtained as a mean of differences of values of Φ that involve only the j -th coordinate. So each term $\|V_j\|_{\mathcal{H}}^2$ is bounded by R^2 . This concludes the proof. \square

The following lemma will be used to study the general eigenvalues approximation of kernel random matrices.

Lemma 2. *Let \mathcal{X} be the input space and Y_1, \dots, Y_n i.i.d with values in \mathcal{X} . Let P be their common probability law on \mathcal{X} . Let T_κ be the integral operator with kernel κ and defined on $L^2(\mathcal{X}, P)$. Then*

$$\mathbb{E}(\|H_\kappa\|_{\text{HS}}^2) = \frac{1}{n} \int_{\mathcal{X}} \kappa(x, x)^2 dP + \frac{n-1}{n} \|T_\kappa\|_{\text{HS}}^2.$$

Proof. The proof is straightforward, it suffices to note that

$$\begin{aligned} \mathbb{E}(\|H_\kappa\|_{\text{HS}}^2) &= \mathbb{E} \left(\frac{1}{n^2} \sum_{i=1}^n (\kappa(Y_i, Y_i))^2 \right) + \mathbb{E} \left(\frac{1}{n^2} \sum_{i,j=1, i \neq j}^n (\kappa(Y_i, Y_j))^2 \right) \\ &= \frac{1}{n^2} \sum_{i=1}^n \int_{\mathcal{X}} \kappa(x, x)^2 dP + \frac{1}{n^2} \sum_{i,j=1, i \neq j}^n \int_{\mathcal{X} \times \mathcal{X}} (\kappa(x, y))^2 dP dP \\ &= \frac{1}{n} \int_{\mathcal{X}} \kappa(x, x)^2 dP + \frac{n^2 - n}{n^2} \|T_\kappa\|_{\text{HS}}^2 \end{aligned}$$

□

Let H_κ be the random matrix that we defined in the introduction, namely, the matrix with entries $\kappa(Y_j, Y_k)$. We know, by [16] that its eigenvalues $\lambda_j(H_\kappa)$ are comparable with their expectation $\mathbb{E}(\lambda_j(H_\kappa))$. We prove more by comparing the sum of squares of the differences.

Proposition 2. *Let κ be a positive semi-definite kernel and let H_κ be the corresponding random kernel matrix as in Theorem 2. We assume that $R = \sup_y \kappa(y, y)$ is finite. For any $\xi > 0$, we have the inequality*

$$\left(\sum_{j=0}^{n-1} |\lambda_j(H_\kappa) - \mathbb{E}(\lambda_j(H_\kappa))|^2 \right)^{1/2} \leq \frac{\xi R}{\sqrt{n}}$$

with probability $1 - 2e^2 e^{-\xi^2/2}$.

Proof. We use Proposition 1 for the mapping $(x_1, \dots, x_n) \mapsto \lambda(x_1, \dots, x_n)$, which maps $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ into the ordered spectrum of the matrix with entries $\frac{1}{n} \kappa(x_j, x_k)$. The Hilbert space used here is ℓ_n^2 , the space of finite sequences of length n endowed with the Euclidean scalar product. We have to prove that $\|\lambda(x) - \lambda(x')\|_{\ell^2} \leq \frac{2R}{n}$ when all coordinates of x, x' are identical except one of them. It is sufficient to prove that $\|\lambda(x) - \hat{\lambda}(x)\|_{\ell_n^2} \leq \frac{R}{n}$ where $\hat{\lambda}(x)$ is the ordered sequence of eigenvalues of the matrix \hat{H}_κ , obtained from H_κ by substituting the coefficients of j -th row and the j -th column of H_κ with zeros. Note that $\hat{\lambda}_{n-1}(x) = 0$ and from the Cauchy eigenvalues interlacing property, we have

$$\lambda_i(x) \geq \hat{\lambda}_i(x) \geq \lambda_{i+1}(x), \quad \forall 1 \leq i \leq n-1.$$

Hence, by using the previous inequality, together with the trace of a square matrix, one gets

$$\|\lambda(x) - \hat{\lambda}(x)\|_{\ell_n^1} = \sum_j (\lambda_j(x) - \hat{\lambda}_j(x)) = \frac{1}{n} \kappa(x_j, x_j) \leq \frac{R}{n}.$$

Since the ℓ_n^2 -norm is bounded by the ℓ_n^1 -norm, one gets

$$\|\lambda(x) - \hat{\lambda}(x)\|_{\ell_n^2} \leq \frac{R}{n},$$

which we wanted to prove. □

One may be interested to compare our results with those of [1, 16], which concern separately each eigenvalue and not the norm of the spectrum in ℓ_n^2 . It may look surprising that the inequalities that we get from our estimates by just bounding one term by the ℓ_n^2 are the same as theirs, except for the factor e^2 in the probability. This means that if we are interested in having simultaneously estimates for $|\lambda_j(H_\kappa) - \mathbb{E}(\lambda_j(H_\kappa))| \leq \frac{\xi R}{\sqrt{n}}$ for k different indices j , the bound for the probability given in the previous proposition is better than the one given in [16] as soon as $n \geq 8$.

End of the proofs of Theorems 3 and 2. The proof of Theorem 3, which is more elaborate, is postponed to the appendix. Proposition 2 and Theorem 3 imply Theorem 2. \square

Remark that in general there is no reason that the error, which is of order R/\sqrt{n} , be small compared to the Hilbert-Schmidt norm of T_κ or H_κ . But we will see that it is the case for the Sinc kernel.

In the sequel of this section, we are interested in the behaviour of the spectra of the random matrices A^*A and $H = H_{\kappa_m}$. The input space is $I = (-1/2, +1/2)$ and the law P is the uniform law on I . We claim that H is the expectation in Z of A^*A . Indeed, when considering the (j, k) entry of A^*A , we get

$$\mathbb{E}_Z \left(\frac{m}{n^2} \sum_{\ell=1}^n \exp(2i\pi m Z_\ell (Y_k - Y_j)) \right) = \frac{1}{n} \mathbb{E}_Z (m \exp(2i\pi m Z_1 (Y_k - Y_j))) = \frac{1}{n} \kappa_m (Y_k - Y_j).$$

Consequently, we have

$$\mathbb{E}_Z (A^*A) = H. \quad (7)$$

As it is classical, we note \mathcal{Q}_m the operator T_{κ_m} , which is defined on $L^2(I)$ by

$$\mathcal{Q}_m(f)(x) = \int_I \frac{\sin(m\pi(x-y))}{\pi(x-y)} f(y) dy. \quad (8)$$

There is a huge literature on this operator, or more frequently, on the operator $\widetilde{\mathcal{Q}}_c$ with kernel $\frac{\sin(c(x-y))}{\pi(x-y)}$ on the interval $(-1, +1)$. A simple dilation allows to pass from one operator to the other, and one has the equality

$$\lambda_j(\mathcal{Q}_m) = \lambda_j(\widetilde{\mathcal{Q}}_c) \quad \text{for } m = \frac{2c}{\pi}.$$

It is well known (see [6], Chapter 1) that for $m > 1$, we have

$$m - O(\log m) \leq \|\mathcal{Q}_m\|_{\text{HS}}^2 \leq m. \quad (9)$$

This implies that $\|\mathcal{Q}_m\|_{\text{HS}} \sim \sqrt{m}$ for m tending to ∞ . The same is valid for $\|H\|_{\text{HS}}$ when m and n/m tend to ∞ . Under these assumptions the error terms, which are multiples of $\frac{m}{\sqrt{n}}$, tend to ∞ more slowly than \sqrt{m} .

We recall some other elementary properties of the spectrum of \mathcal{Q}_m . All the eigenvalues of \mathcal{Q}_m are bounded by 1 and

$$\text{Trace}(\mathcal{Q}_m) = \sum_{j=0}^{\infty} \lambda_j(\mathcal{Q}_m) = m.$$

Using (9), we deduce that

$$\sum_{j \geq 0} \lambda_j(\mathcal{Q}_m)(1 - \lambda_j(\mathcal{Q}_m)) \leq C_0 \log m, \quad (10)$$

for some constant C_0 independent of m .

Next, we show that the eigenvalues of the random matrix $B = A^*A$ are well approximated by the corresponding eigenvalues of the matrix H . For this purpose, it suffices to check that the matrices B and H have comparable Hilbert-Schmidt norms.

Lemma 3. *For H as before, depending on Z_1, \dots, Z_n , we have the inequalities*

$$(\mathbb{E}_Z(\|A^*A - H\|_{\text{HS}}^2))^{1/2} \leq \frac{m}{\sqrt{n}}. \quad (11)$$

Moreover, for all $\xi > 0$, we have

$$\mathbb{P}_Z \left(\|A^*A - H\|_{\text{HS}} > \xi \frac{m}{\sqrt{n}} \right) \leq 2e^2 e^{-\frac{\xi^2}{2}}. \quad (12)$$

Proof. Consider the real Hilbert space \mathcal{H} of $n \times n$ Hermitian matrices equipped with the real inner product

$$\langle B_1, B_2 \rangle = \text{Trace}(B_2^* B_1).$$

Note that the norm associated with this inner product is simply given by the Hilbert-Schmidt norm $\|\cdot\|_{\text{HS}}$, that is for $M = [m_{ij}] \in \mathcal{H}$, we have

$$\text{Trace}(M^*M) = \|M\|_{\text{HS}}^2.$$

Let the function $\Phi : \mathbb{R}^n \rightarrow \mathcal{H}$, defined by $\Phi(z_1, \dots, z_n) = \sum_{k=1}^n B(z_k)$, where the coefficients of the matrix $B(z_k)$ are given by

$$b_{j,k} = \frac{m}{n^2} \exp(2i\pi m z_k (Y_k - Y_j)). \quad (13)$$

Hence, we have

$$\|\Phi(z_1, \dots, z_j, \dots, z_n) - \Phi(z_1, \dots, \hat{z}_j, \dots, z_n)\|_{\text{HS}} = \|B(z_j) - B(\hat{z}_j)\|_{\text{HS}} \leq \frac{2m}{n^2} \sqrt{n^2 - n} < \frac{2m}{n} = R.$$

Consequently, by using Proposition 1 and (7), one gets (12). Let us give a direct proof of (11) instead of using Lemma 1, which would give the same result but with a constant 2 in the right hand side. We just remark that

$$\mathbb{E}_Z(\|A^*A - H\|_{\text{HS}}^2) = n\mathbb{E}_Z(\|B(Z_1) - \mathbb{E}(B(Z_1))\|_{\text{HS}}^2) \leq n\mathbb{E}_Z(\|B(Z_1)\|_{\text{HS}}^2) = \frac{m^2}{n}.$$

□

The result given by (12) is equivalent to the fact that, with probability larger than $1 - 2e^2 e^{-\xi^2/2}$, we have the inequality

$$\|A^*A - H\|_{\text{HS}} \leq \frac{\xi m}{\sqrt{n}}. \quad (14)$$

By Hoffman–Wielandt inequality, this implies that

$$\left(\sum_{j=0}^{n-1} |\lambda_j(A^*A) - \lambda_j(H)|^2 \right)^{1/2} \leq \|A^*A - H\|_{\text{HS}} \leq \frac{\xi m}{\sqrt{n}}, \quad (15)$$

with probability larger than $1 - 2e^2 e^{-\xi^2/2}$. This inequality makes sense when $\|H\|_{\text{HS}}$ is large compared to the error term $\frac{\xi m}{n^{1/2}}$. This is the case when n/m is large enough.

End of the proof of Theorem 1. Theorem 2 and (15) imply Theorem 1. \square

As a consequence of Theorem 3, (11) and Hoffman–Wielandt inequality, we have also the following proposition.

Proposition 3. *The following inequality holds.*

$$\left(\mathbb{E} \left(\sum_{j=0}^{n-1} |\lambda_j(A^*A) - \lambda_j(\mathcal{F}_m^* \mathcal{F}_m)|^2 \right) \right)^{1/2} \leq \frac{(1 + \sqrt{2})m}{\sqrt{n}}.$$

3 Degrees of Freedom and Capacity, associated with the random Fourier matrix

In this section, we describe what the previous concentration inequalities imply for the spectra of A^*A and H . In particular, we give estimates of the number of degrees of freedom and the capacity associated with the random Sinc kernel matrix H and the matrix A . For this purpose, we first give the decay behaviour of the spectrum of the integral operator \mathcal{Q}_m .

3.1 Decay of the spectra

It is well known that all the eigenvalues of \mathcal{Q}_m are smaller than 1. Roughly speaking, they are very close to 1 for $j \leq m - c \log m$ and very close to 0 for $j > m + c \log m$, for some constant c . The region in between is called the plunge region. The most complete answer of this behaviour, is an asymptotic formula for m tending to ∞ , which has been given by Landau and Widom (see [6]). More precisely, for $0 < \alpha < 1$, let

$$N_{\mathcal{Q}_m}(\alpha) = \#\{\lambda_j(\mathcal{Q}_m); \lambda_j(\mathcal{Q}_m) > \alpha\},$$

then we have

$$N_{\mathcal{Q}_m}(\alpha) = m + \left[\frac{1}{\pi^2} \log \left(\frac{1 - \alpha}{\alpha} \right) \right] \log(m) + o(\log(m)). \quad (16)$$

For the random Sinc kernel matrix H , let $N_H(\alpha)$ be as defined by (16). Also, we will frequently use the following constant

$$\gamma_\xi = \xi + \sqrt{2}, \quad \xi > 0. \quad (17)$$

We know that with a probability larger than $1 - 2e^2 e^{-\xi^2/2}$, we have the inequality $|\lambda_j(H) - \lambda_j(\mathcal{Q}_m)| \leq \frac{\gamma_\xi m}{\sqrt{n}}$, for each $j \geq 0$ (for instance, if we exclude an event of probability 1%, then $\xi = \sqrt{4 + 2 \log(200)}$, so that $\gamma_\xi = \sqrt{2} + \sqrt{4 + 2 \log(200)} \approx 5$.) Using the elementary inequality

$$N_m \left(\alpha + \frac{\gamma_\xi m}{\sqrt{n}} \right) \leq N_H(\alpha) \leq N_m \left(\alpha - \frac{\gamma_\xi m}{\sqrt{n}} \right),$$

we have with probability larger than $1 - 2e^2 e^{-\xi^2/2}$,

$$N_H(\alpha) = m + \left[\frac{1}{\pi^2} \log \left(\frac{1 - \alpha}{\alpha} \right) \right] \log(m) + o(\log(m)) + O \left(\frac{\gamma_\xi m}{\sqrt{n}} \right) \log m, \quad (18)$$

for $\frac{2\gamma_\xi m}{\sqrt{n}} < \alpha < 1 - \frac{2\gamma_\xi m}{\sqrt{n}}$.

This means that Landau–Widom Formula is also an approximate for H with high probability for m^2/n tending to 0. The same is valid for A^*A , with the two following modifications: “high

probability” means now that the probability is larger than $1 - 4e^2e^{-\xi^2/2}$, and γ_ξ is substituted with $\gamma_{2\xi} = \sqrt{2} + 2\xi$.

We will also make use of Landau’s double inequality [7],

$$\lambda_{\lceil m \rceil}(\mathcal{Q}_m) \leq 1/2 \leq \lambda_{\lfloor m \rfloor - 1}(\mathcal{Q}_m). \quad (19)$$

Here, $\lceil x \rceil$ and $\lfloor x \rfloor$ refers the to integer part and the least integer greater than or equal to x , respectively. The previous inequalities say that, roughly speaking, $\lambda_j(\mathcal{Q}_m)$ goes through the value $1/2$ at m . An approximate result is valid for H and A^*A when $\frac{m^2}{n}$ is small.

Landau-Widom Theorem gives us an approximate asymptotic decay of the eigenvalues in the plunge region after m , that is

$$\lambda_k(\mathcal{Q}_m) \approx \exp\left(-\frac{\pi^2(k-m)}{\log m}\right).$$

One does not know any non asymptotic comparable result. The following result is to the best of our knowledge, the first non asymptotic exponential decay estimate of the $\lambda_n(\mathcal{Q}_m)$. It is a consequence of the estimates given in [2]. This kind of statement, with estimates of constants is developed in a separate paper [3].

Theorem 4. *There exists a uniform constant η such that, for $m > 1$ and $k \geq m + \log m + 10$, one has the inequality*

$$\lambda_k(\mathcal{Q}_m) \leq \frac{1}{2} \exp\left[-\eta\left(\frac{k-m}{\log m}\right)\right]. \quad (20)$$

A better decay, namely a super-exponential decay rate, is obtained after the plunge region (see [2, 3]), but we will not use it here.

3.2 Degrees of Freedom

In this paragraph, we are interested in the number of degrees of freedom of a matrix or an operator, which makes sense in the area of wireless communication networks. Different definitions have been given in this context in [18]. We give here a simple definition in terms of eigenvalues. Similar definitions may be found in approximation theory or in computational complexity theory, where it is known as a complexity number. We aim here to show that it is easily approximated for the integral operator \mathcal{Q}_m . Consequently, this provides us with a good approximation for the corresponding random matrices H and A^*A .

Definition 1. *Let T be a Hilbert-Schmidt positive semi-definite Hermitian operator. We define the numbers of degree of freedom at level ε by*

$$\text{deg}_\infty(T, \varepsilon) = \min\{s; \lambda_s(T) \leq \varepsilon\}. \quad (21)$$

Depending on the application in view, it makes sense to be interested in small values of ε , or values that are close to the largest eigenvalue of the integral operator, that is close to 1 when considering \mathcal{Q}_m . Remark that the difference of $\text{deg}_\infty(\mathcal{Q}_m, 1/2)$ with m is bounded by 1 by Landau’s double inequality. For other values of ε we have the following proposition.

Proposition 4. *Let $m \geq 2$ be a positive real number. For $\varepsilon < 1/2$, the number of degrees of freedom satisfies the inequalities*

$$m - 1 \leq \text{deg}_\infty(\mathcal{Q}_m, \varepsilon) \leq m + O(\varepsilon^{-1} \log m). \quad (22)$$

For $1/2 < \varepsilon < 1$, these inequalities are replaced by

$$m - O((1 - \varepsilon)^{-1} \log m) \leq \text{deg}_\infty(\mathcal{Q}_m, \varepsilon) \leq m + 1. \quad (23)$$

Proof. The left hand side of the first inequality, as well as the right hand side of the second one, follow from Landau's double inequality (19). Moreover, by using (10) and (19), we obtain the two inequalities

$$\sum_{j \leq m-1} (1 - \lambda_j(\mathcal{Q}_m)) \leq 2C_0 \log m \quad (24)$$

$$\sum_{j \geq m+1} \lambda_j(\mathcal{Q}_m) \leq 2C_0 \log m. \quad (25)$$

As a consequence of (24), we have the inequality

$$\#\{j \leq m-1, \lambda_j(\mathcal{Q}_m) < \varepsilon\} \leq \frac{2C_0 \log m}{1 - \varepsilon}.$$

The left hand side of (23) follows at once. The remaining inequality, that is, the right hand side of (22), follows from a similar argument, with (25) used in place of (24). \square

Remark 1. *In the inequality (22), it is possible to replace ε^{-1} by $\log(\varepsilon^{-1})$. This follows from the decay of eigenvalues given in (20).*

Let us now consider the degrees of freedom of the random matrices H and A^*A , which we define as follows.

Definition 2. *Let M be a positive semi-definite random Hermitian matrix. We define the numbers of degree of freedom at level ε and confidence level α by*

$$\deg_\infty(M, \varepsilon, \alpha) = \min\{s; \lambda_s(T) \leq \varepsilon \text{ with probability } \geq \alpha\}. \quad (26)$$

The next lemma allows to prove that with high probability that the degrees of freedom of the random matrices H and A^*A are close to the ones of \mathcal{Q}_m .

Lemma 4. *Let T_1, T_2 be two self-adjoint Hilbert-Schmidt operators on the same Hilbert space \mathcal{H} , and let $0 < \varepsilon_1 < \varepsilon_2$. Then*

$$\deg_\infty(T_2, \varepsilon_2) \leq \deg_\infty(T_1, \varepsilon_1) + \frac{\sum_{j \geq 0} (\lambda_j(T_1) - \lambda_j(T_2))^2}{(\varepsilon_1 - \varepsilon_2)^2}.$$

Proof. It is sufficient to write that

$$\begin{aligned} \#\{j > \deg_\infty(T_1, \varepsilon_1), \lambda_j(T_2) > \varepsilon_2\} &\leq \#\{j > \deg_\infty(T_1, \varepsilon_1), \lambda_j(T_2) - \lambda_j(T_1) > \varepsilon_2 - \varepsilon_1\} \\ &\leq \frac{\sum_{j \geq 0} (\lambda_j(T_1) - \lambda_j(T_2))^2}{(\varepsilon_1 - \varepsilon_2)^2}. \end{aligned}$$

\square

We deduce from the two previous statements the following property.

Proposition 5. *For $\varepsilon > 0$, $0 < \delta = 1 - \alpha < 1$, we have*

$$\deg_\infty(H, \varepsilon, \alpha) = m + \mathcal{E}_H, \quad \deg_\infty(A^*A, \varepsilon, \alpha) = m + \mathcal{E}_A,$$

with $|\mathcal{E}_H|, |\mathcal{E}_A| \leq C_\varepsilon \left(\frac{m}{n} \sqrt{\log(\delta^{-1})} + \log m \right)$. Here, C_ε is a constant depending only on ε .

The errors are small for m large and n/m large. Asymptotically, when these two quantities tend to ∞ , we have $\deg_\infty(H, \varepsilon, \delta) \sim m$ and $\deg_\infty(A^*A, \varepsilon, \delta) \sim m$ for fixed ε and δ .

3.3 Capacity of a random matrix

We proceed as we have done for the degrees of freedom. Let us define, for $m \geq 2$ and $s > 0$, the capacity associated with the Sinc kernel integral operator \mathcal{Q}_m ,

$$C_{\mathcal{Q}_m}(s) = \sum_{k \geq 0} \log(1 + s\lambda_k(\mathcal{Q}_m)).$$

We claim the following.

Proposition 6. *For any $m \geq 2$ and any $s > 0$, we have*

$$C_{\mathcal{Q}_m}(s) = m \log(1 + s) + O((\log m)(\log(1 + s))).$$

The implicit constant in O is independent of s and m .

Proof. Let us first assume that $s \leq 2$. We cut the sum into three parts and write

$$C_{\mathcal{Q}_m}(s) = [m] \log(1 + s) + \sum_{k < [m]} \log\left(\frac{1 + s\lambda_k(\mathcal{Q}_m)}{1 + s}\right) + \sum_{k \geq [m]} \log(1 + s\lambda_k(\mathcal{Q}_m)).$$

The last term is bounded by $2 \sum_{k \geq [m]} \lambda_k(\mathcal{Q}_m)$, which is $O(\log m)$ because of (24). In the second term, all the $\lambda_k(\mathcal{Q}_m)$ are bounded below by $1/2$. Since for $0 \leq x \leq 1/2$, we have $\log(1 - x) \geq -2x$, then one gets

$$\log\left(\frac{1 + s\lambda_k(\mathcal{Q}_m)}{1 + s}\right) = \log\left(1 - \frac{s}{1 + s}(1 - \lambda_k(\mathcal{Q}_m))\right) \geq -2(1 - \lambda_k(\mathcal{Q}_m)).$$

But $\sum_{k < [m]} (1 - \lambda_k(\mathcal{Q}_m)) = [m] - m + \sum_{k \geq [m]} \lambda_k(\mathcal{Q}_m)$ and we conclude as before.

Next, for $s \geq 2$, the proof follows the same lines. Let us consider separately the bounds below and above. For the bound above, we cut the sum at $[m_s]$, where m_s is given by

$$m_s = m + \max(10, m + \frac{1}{\eta} \log s) \log m. \tag{27}$$

Here η is the uniform constant given in (20). With this choice, we have

$$\sum_{k > m_s} \lambda_k(\mathcal{Q}_m) \leq \frac{C}{s} \log m.$$

When we write

$$C_{\mathcal{Q}_m}(s) \leq [m_s] \log(1 + s) + \sum_{k > m_s} \log(1 + s\lambda_k(\mathcal{Q}_m)),$$

the bound above follows at once. For the bound below, we write

$$C_{\mathcal{Q}_m}(s) \geq [m] \log(1 + s) + \sum_{k < [m]} \log\left(\frac{1 + s\lambda_k(\mathcal{Q}_m)}{1 + s}\right),$$

and conclude as in the case when $s \leq 2$. □

We now pass from \mathcal{Q}_m to A^*A .

Proposition 7. *There exist three constants C_1, C_2, C_3 such that, for all values of $m \geq 1, n$ and p , we have the following approximation for the expectation of the capacity defined in (1),*

$$\mathbb{E}(C(p)) = m \log \left(1 + \frac{np}{m} \right) + \mathcal{E}, \quad (28)$$

with

$$-m - \left(C_1 \frac{m^2}{n} + C_2 \log m \right) \log \left(1 + \frac{np}{m} \right) \leq \mathcal{E} \leq m + C_3 \log m \log \left(1 + \frac{np}{m} \right).$$

In particular, when m and n/m tend to ∞ ,

$$\mathbb{E}(C(p)) \sim m \log \left(1 + \frac{np}{m} \right).$$

Proof. Let us first prove the bound above. We cut the sum at m_s as before. We conclude for the last term by using the fact that the expectation of the sum $\sum_{j \geq k} \mathbb{E}(\lambda_j(A^*A))$ is bounded by the corresponding sum for \mathcal{Q}_m , that is $\sum_{j \geq k} \lambda_j(\mathcal{Q}_m)$. This is the inequality (36), which is proved in the appendix. For the first sum, we can bound each term by $\log(1 + \frac{np}{m})$ as soon as $\lambda_k(A^*A)$ is bounded

by 1. For the summation of the terms for which $\lambda_k(A^*A) > 1$, we have to add $\sum_{k=0}^{n-1} (\lambda_k(A^*A) - 1)_+$,

which is bounded by $\sum_{k=0}^{n-1} (\lambda_k(A^*A)) = m$. The same bound holds for the expectation. This concludes the proof for the bound above.

For the bound below, we can use the concavity of the logarithm. It is sufficient to give a bound below for

$$\sum_{k \leq m - C_1 \frac{m^2}{n} - C_2 \log m} \log \left(1 + \frac{np}{m} \mathbb{E}(\lambda_k(A^*A)) \right).$$

We claim that we can choose C_1 and C_2 so that $\mathbb{E}(\lambda_k(A^*A)) \geq 1/2$ for $k \leq m - C_1 \frac{m^2}{n} - C_2 \log m$. Indeed,

$$\begin{aligned} \#\{k < m, |1 - \mathbb{E}(\lambda_k(A^*A))| \geq 1/2\} &\leq 4 \sum_{k < m} (1 - \mathbb{E}(\lambda_k(A^*A)))^2 \\ &\leq 8 \sum_{k < m} (1 - \lambda_k(\mathcal{Q}_m))^2 + 8 \sum_{k < m} (\lambda_k(\mathcal{Q}_m) - \mathbb{E}(\lambda_k(A^*A)))^2. \end{aligned}$$

The first sum is $O(\log m)$ by (24). The second one is bounded by $4m^2/n$ by Proposition 3. From now on, C_1 and C_2 are chosen so that $\mathbb{E}(\lambda_k(A^*A)) \geq 1/2$ for $k \leq m - C_1 \frac{m^2}{n} - C_2 \log m$. Finally, the difference

$$\left(m - C_1 \frac{m^2}{n} - C_2 \log m \right) \log \left(1 + \frac{np}{m} \right) - \sum_{k \leq m - C_1 \frac{m^2}{n} - C_2 \log m} \log \left(1 + \frac{np}{m} \mathbb{E}(\lambda_k(A^*A)) \right)$$

is bounded as before by $2 \sum_{k < m - C_1 \frac{m^2}{n} - C_2 \log m} (1 - \mathbb{E}(\lambda_k(A^*A)))_+$, which is bounded by m . We conclude at once. \square

The previous proposition deals with expectation. It is easy to deduce estimates with high probability. Indeed, it suffices to use Mc Diarmid's inequality for the mapping defined by the capacity. This is done in [12]. From the computations made in [12], it follows that, for $\xi > 0$ and with probability larger than $1 - 2e^{-\xi^2/2}$, we have

$$|C(p) - \mathbb{E}(C(p))| \leq \frac{\xi \log(1 + np)}{\sqrt{n}}.$$

Probability value	lower bound of $\frac{n}{m}$
99%	90
95%	75
90%	70
85%	65
80%	60

Table 1: Lower bound of $\frac{n}{m}$ so that the error given by (2) is smaller than \sqrt{m} , with a given probability.

This gives an error that is very small compared to the estimates that we have given for \mathcal{E} .

Remark 2. In [9], the authors have stated that there exists a constant $K > 0$ such that

$$C(n) = \log \det \left(I_n + \frac{n^2}{m} A^* A \right) \leq Kn \log(n), \quad (29)$$

with high probability as n gets large and $m \geq \sqrt{n}$. Bounds below have been obtained in [12]. The previous proposition generalizes these results, with much more precise estimates.

4 Numerical examples

Let us first discuss the order of magnitude of the errors given in the different theorems if we want them to be valid with high probability. For example, the eigenvalues approximation result given by Theorem 1 is valid with probability 99%, whenever $\xi = 2\sqrt{1 + \log 20} \approx 4$. In this case, the right hand side of the inequality (2) is smaller than \sqrt{m} for $n/m \geq 90$. The following table gives more values for the condition on the ratios n/m , for different probabilities values.

The simulations below are done with a sample of size $2n$ of the uniform law on $(-1/2, +1/2)$ in place of the Z_j 's and the Y_k 's for different values of n .

Example 1: In this first example, we illustrate the results of Proposition 3 and Theorem 1. For this purpose, we have computed the spectra of the random matrices A^*A and H with $n = 200$ and different values of $2 \leq m \leq 20$. Since for each value of m , there is approximately m significant eigenvalues, then we have computed the approximation relative ℓ^2 -errors, given by $\frac{1}{\sqrt{m}} \|\lambda(A^*A) - \lambda(H)\|_{\ell^2}$, $\frac{1}{\sqrt{m}} \|\lambda(H) - \lambda(\mathcal{Q}_m)\|_{\ell^2}$ and $\frac{1}{\sqrt{m}} \|\lambda(A^*A) - \lambda(\mathcal{Q}_m)\|_{\ell^2}$. Also, we have computed the magnitude of the corresponding theoretical relative error, given by the quantity $\sqrt{\frac{m}{n}}$. The obtained numerical results are given by Table 2.

Example 2: In this example, we have considered the random matrix H , with $n = 200$ and different values of the bandwidth $2 \leq m \leq 20$. In Figure 1 (a), we have plotted the eigenvalues $(\lambda_j(A^*A))_{0 \leq j \leq 35}$ of the random matrix A^*A , arranged in the decreasing order, versus the eigenvalues of \mathcal{Q}_m . Then, we have repeated the previous numerical tests with the random matrix H instead of the matrix A^*A . The obtained numerical results are given by Figure 1(b). Note that as predicted by proposition 5, the matrices A^*A and H , each has m significant eigenvalues.

Also in order to check the decay of the eigenvalues of the random matrices A^*A and H , we have plotted in Figures 2 (a) and 2(b), the graphs of $\log(\lambda_j(A^*A))$ and $\log(\lambda_j(H))$. Note that as predicted by

$n = 200$	$\frac{\ \lambda(A^*A) - \lambda(H)\ _{\ell^2}}{\sqrt{m}}$	$\frac{\ \lambda(H) - \lambda(\mathcal{Q}_m)\ _{\ell^2}}{\sqrt{m}}$	$\frac{\ \lambda(A^*A) - \lambda(\mathcal{Q}_m)\ _{\ell^2}}{\sqrt{m}}$	$\sqrt{\frac{m}{n}}$
$m = 2$	24.3%	08.6%	20.7%	10.0%
$m = 4$	17.4%	08.1%	17.5%	14.1%
$m = 6$	14.0%	10.7%	15.0%	17.3%
$m = 10$	10.0%	13.4%	14.6%	22.4%
$m = 20$	14.5%	29.4%	17.6%	31.6%

Table 2: Illustrations of the results of Theorem 1 and Proposition 3.

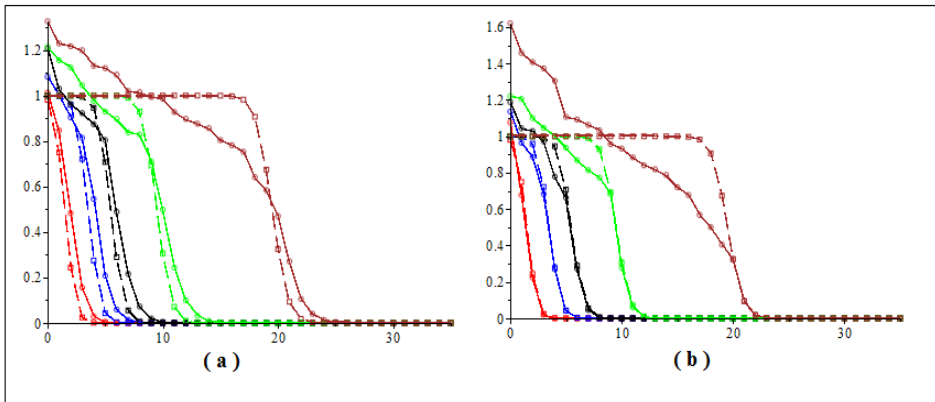


Figure 1: (a) Graphs of $\lambda(A^*A)$ (circles) versus $\lambda(\mathcal{Q}_m)$ (boxes) with $n = 200$ and for the various values of $m = 2, 4, 6, 10, 20$, (from the left to the right), (b) same as (a) with $\lambda(H)$ instead of $\lambda(A^*A)$.

our theoretical results, the eigenvalues of the random matrices A^*A and H have fast decays, starting around $k = m$.

Example 3: In this last example, we illustrate our theoretical estimate for the network capacity. We recall that this capacity is given by equation (1). To illustrate the previous bound estimate of the network capacity, we have considered the value of $n = 200$ and the four values of $m = 2, 4, 6, 10, 20$, then we have computed the eigenvalues $\frac{n^2}{m} \lambda_j(A^*A)$ of the matrices $\frac{n^2}{m} A^*A$. In Table 2, we have listed the values of

$$C_m = \log \det \left(I_n + \frac{n^2}{m} A^*A \right) = \sum_{j=1}^n \log \left(1 + \frac{n^2}{m} \lambda_j(A^*A) \right), \quad (30)$$

as well as the values of the corresponding estimations of the absolute and relative errors, given by

$$E_m = |C_m - \tilde{C}_m|, \quad Er_m = \frac{|C_m - \tilde{C}_m|}{C_m}, \quad \tilde{C}_m = m \log \left(\frac{n^2}{m} \right). \quad (31)$$

Appendices

Appendix A: Proof of Theorem 3.

We will prove the following theorem, which is stronger and in the spirit of the estimates developed in [11].

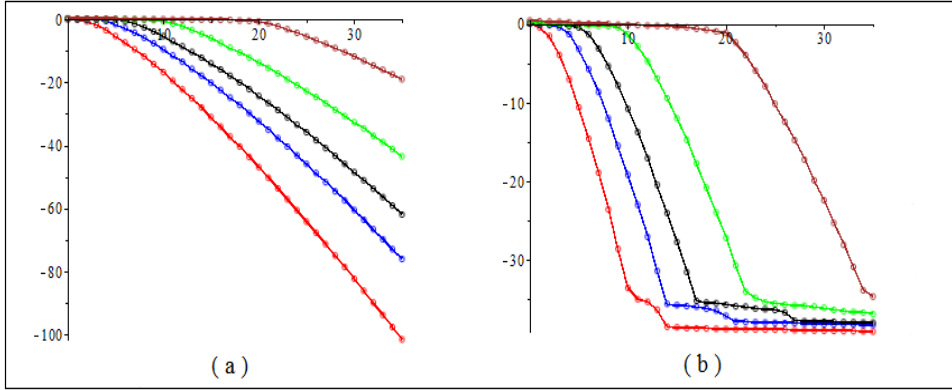


Figure 2: (a) Graphs of $\log(\lambda_j(A^*A))$ with $n = 200$ and for the various values of $m = 2, 4, 6, 10, 20$, (from the left to the right), (b) same as (a) with $\log(\lambda_j(H))$ instead of $\log(\lambda_j(A^*A))$.

m	C_m	$E_m = C_m - \tilde{C}_m $	$Er_m = \frac{ C_m - \tilde{C}_m }{C_m}$
$m = 2$	51.4	31.6	$6.2E - 01$
$m = 4$	70.0	33.1	$4.7E - 01$
$m = 6$	85.5	32.7	$3.8E - 01$
$m = 10$	115.7	32.7	$2.8E - 01$
$m = 20$	181.6	29.6	$1.6E - 01$

Table 3: Illustrations of our bound estimate of the network capacity C_m , given by (30) and for different values of m .

Theorem 5. Let κ be a positive semi-definite kernel as in Theorem 2, T_κ be the integral operator with kernel κ , and H_κ be the associated random kernel matrix. Then, we have

$$\mathbb{E} \left(\sum_{j=0}^{n-1} |\lambda_j(H_\kappa) - \lambda_j(T_\kappa)|^2 \right) \leq \frac{1}{n} \int_{\mathcal{X}} \kappa(x, x)^2 dP + \sum_{j \geq n} \lambda_j(T_\kappa)^2. \quad (32)$$

Proof. The left hand side may be written as

$$\sum_{j=0}^{n-1} \mathbb{E}(\lambda_j(H_\kappa)^2) - \sum_{j=0}^{n-1} \lambda_j(T_\kappa)^2 - 2 \sum_{j=0}^{n-1} \lambda_j(T_\kappa) (\mathbb{E}(\lambda_j(H_\kappa) - \lambda_j(T_\kappa))).$$

We use Lemma 2 for the first term, and find that the sum of the two first terms is bounded by the right hand side of (32). So it is sufficient to prove that

$$\sum_{j=0}^{n-1} \lambda_j(T_\kappa) (\mathbb{E}(\lambda_j(H_\kappa) - \lambda_j(T_\kappa))) \geq 0.$$

We use an Abel transformation and see that this sum is equal to

$$\sum_{j=0}^{n-2} ((\lambda_j(T_\kappa) - \lambda_{j+1}(T_\kappa)) \left(\sum_{k=0}^j \mathbb{E}(\lambda_k(H_\kappa)) - \lambda_k(T_\kappa) \right) + \lambda_{n-1}(T_\kappa) \left(\sum_{k=0}^{n-1} \mathbb{E}(\lambda_k(H_\kappa)) - \lambda_k(T_\kappa) \right)).$$

All terms are non negative in this sum: terms $\lambda_j(T_\kappa) - \lambda_{j+1}(T_\kappa)$ are non negative since the sequence of eigenvalues is non increasing. It is well-known that terms $\sum_{k=0}^j \mathbb{E}(\lambda_k(H_\kappa)) - \lambda_k(T_\kappa)$ are non negative (see [1, 16]). □

To conclude for the proof of Theorem 3, it is sufficient to see that $\int_{\mathcal{X}} \kappa(x, x)^2 dP$ is bounded by R^2 , and to prove the inequality

$$\sum_{j \geq n} \lambda_j(T_\kappa)^2 \leq \frac{R^2}{n}. \quad (33)$$

But since $\sum_{j \geq 0} \lambda_j(T_\kappa) \leq R$ and the sequence $\lambda_j(T_\kappa)$ is decreasing, we have $\lambda_j(T_\kappa) \leq \frac{R}{n}$ for $j \geq n$.

Remark 3. Depending on the decay of the sequence $\lambda_j(T_\kappa)$, the left hand side of (33) may be very small compared to $\frac{R^2}{n}$. In particular, in our example, we have exponential decay for $n > m + \log m + 10$. So this is rapidly negligible and one may replace the constant 2 by 1 in (6), up to a negligible constant.

Appendix B: Further inequalities for the eigenvalues and reconstruction error.

We first recall that the reconstruction error, when approximating an $n \times n$ matrix by its projections $P_V M P_V$ on subspaces V of dimension d , is defined as

$$\mathcal{R}_d(M) = \min \|M - P_V M P_V\|_{\text{HS}}^2,$$

where the minimum is taken over all subspaces V of dimension d . This notion is central in [15]. It is well-known that the minimum is obtained when V is generated by the first d eigenvectors of $M^* M$, so that

$$\mathcal{R}_d(M) = \sum_{j \geq d} \lambda_j(M^* M).$$

Equivalently,

$$\sum_{j < d} \lambda_j(M^* M) = \max \sum_{i=1}^{d-1} \langle M^* M v_i, v_i \rangle,$$

where the supremum is taken over all orthonormal system of $d - 1$ vectors.

We are interested in $\mathcal{R}_d(A)$. Let v_i^* be the system for which the maximum is obtained when $A^* A$ is replaced by H . Clearly

$$\max \sum_{i=1}^d \langle A^* A v_i, v_i \rangle \geq \max \sum_{i=1}^d \langle A^* A v_i^*, v_i^* \rangle.$$

By taking the expectation on both sides, we obtain the inequality

$$\mathbb{E}_Z \left(\sum_{j < d} \lambda_j(A^* A) \right) \geq \sum_{j < d} \lambda_j(H). \quad (34)$$

The inequality

$$\mathbb{E} \left(\sum_{j < d} \lambda_j(H) \right) \geq \sum_{j < d} \lambda_j(\mathcal{Q}_m) \quad (35)$$

can be found in [15]. Its proof follows from the same kind of arguments. As a consequence, we have the inequality

$$\mathbb{E} \left(\sum_{j \geq d} \lambda_j(A^* A) \right) \leq \sum_{j \geq d} \lambda_j(\mathcal{Q}_m). \quad (36)$$

The same authors also use the Bounded Difference inequality to write that, with probability larger than $1 - 2e^{-\xi^2/2}$, we have

$$\left| \sum_{j < d} \lambda_j(H) - \mathbb{E} \left(\sum_{j < d} \lambda_j(H) \right) \right| \leq \frac{\xi m}{\sqrt{n}}. \quad (37)$$

We remark that the trace norm can be used as well as the Hilbert-Schmidt norm in the proof of (3), which allows to use the Bounded Difference Inequality for $\sum_{j < d} \lambda_j(A^* A)$ and the probability P_Z .

With probability larger than $1 - 2e^{-\xi^2/2}$, we have the inequality

$$\left| \sum_{j < d} \lambda_j(A^* A) - \mathbb{E}_Z \left(\sum_{j < d} \lambda_j(A^* A) \right) \right| \leq \frac{\xi m}{\sqrt{n}}. \quad (38)$$

Finally, with probability larger than $1 - 4e^{-\xi^2/2}$, we have the inequality

$$\sum_{j < d} \lambda_j(A)^2 \geq \sum_{j < d} \lambda_j(\mathcal{Q}_m) - \frac{2\xi m}{\sqrt{n}}, \quad (39)$$

and, as a consequence,

$$\mathcal{R}_d(A) = \sum_{j \geq d} \lambda_j(A)^2 \leq \sum_{j < d} \lambda_j(\mathcal{Q}_m) + \frac{2\xi m}{\sqrt{n}}. \quad (40)$$

The same kind of techniques as in [15] can be used to have a reverse inequality.

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