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► To cite this version:

Hagen Neidhardt, Artur Stephan, Valentin Zagrebnov. Remarks on the operator-norm convergence of the Trotter product formula. *Integral Equations and Operator Theory*, 2018, 90 (2), pp.UNSP 15. 10.1007/s00020-018-2424-z . hal-01497909

HAL Id: hal-01497909

<https://hal.science/hal-01497909>

Submitted on 29 Mar 2017

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Remarks on the operator-norm convergence of the Trotter product formula

Hagen Neidhardt, Artur Stephan and Valentin A. Zagrebnov

Abstract. We revise the operator-norm convergence of the Trotter product formula for a pair $\{A, B\}$ of generators of semigroups on a Banach space. Operator-norm convergence holds true if the dominating operator A generates a holomorphic contraction semigroup and B is a A -infinitesimally small generator of a contraction semigroup, in particular, if B is a bounded operator. Inspired by studies of evolution semigroups it is shown in the present paper that the operator-norm convergence generally fails even for bounded operators B if A is not a holomorphic generator. Moreover, it is shown that operator norm convergence of the Trotter product formula can be arbitrary slow.

Keywords. Semigroups, bounded perturbations, Trotter product formula, Darboux-Riemann sums, operator-norm convergence.

1. Introduction and main results

Recall that the product formula

$$e^{-\tau C} = \lim_{n \rightarrow \infty} \left(e^{-\tau A/n} e^{-\tau B/n} \right)^n, \quad \tau \geq 0,$$

was established by S. Lie (in 1875) for matrices where $C := A + B$. The proof is based on the telescopic representation

$$\begin{aligned} & \left(e^{-\tau A/n} e^{-\tau B/n} \right)^n - e^{-\tau C} \\ &= \sum_{k=0}^{n-1} \left(e^{-\tau A/n} e^{-\tau B/n} \right)^{n-1-k} \left(e^{-\tau A/n} e^{-\tau B/n} - e^{-\tau C/n} \right) e^{-k\tau C/n}, \end{aligned} \quad (1.1)$$

$n \in \mathbb{N}$, and expansion

$$e^{-\tau X} = I - \tau X + O(\tau^2), \quad \tau \rightarrow 0,$$

for a matrix X in the operator-norm topology $\|\cdot\|$. Indeed, using this expansion one obtains the estimate:

$$\|e^{-\tau A/n} e^{-\tau B/n} - e^{-\tau C/n}\| = O((\tau/n)^2).$$

Then from (1.1) we get the existence of a constant $c_0 > 0$ such that the following estimate holds

$$\begin{aligned} & \left\| (e^{-\tau A/n} e^{-\tau B/n})^n - e^{-\tau C} \right\| \\ & \leq c_0 \frac{\tau^2}{n^2} \sum_{k=0}^{n-1} e^{\tau \frac{n-1-k}{n} \tau \|A\|} e^{\tau \frac{n-1-k}{n} \tau \|B\|} e^{\tau \frac{k}{n} \|C\|} . \end{aligned}$$

Since $\|C\| \leq \|A\| + \|B\|$, one obtains inequality

$$\begin{aligned} & \left\| (e^{-\tau A/n} e^{-\tau B/n})^n - e^{-\tau C} \right\| \\ & \leq c_0 \frac{\tau^2}{n^2} \sum_{k=0}^{n-1} e^{\tau \frac{n-1}{n} (\|A\| + \|B\|)} \leq c_0 \frac{\tau^2}{n} e^{\tau (\|A\| + \|B\|)} , \end{aligned}$$

which yields that

$$\sup_{\tau \in [0, T]} \left\| (e^{-\tau A/n} e^{-\tau B/n})^n - e^{-\tau C} \right\| = O(1/n) , \quad (1.2)$$

as $n \rightarrow \infty$ for any $T > 0$. Note that this proof carries through verbatim for bounded operators A and B on Banach spaces.

H. Trotter [7] has extended this result to unbounded operators A and B on Banach spaces, but in the strong operator topology. He proved that if A and B are generators of contractions semigroups on a separable Banach space such that the algebraic sum $A + B$ is a densely defined closable operator and the closure $C = \overline{A + B}$ is a generator of a contraction semigroup, then

$$e^{-\tau C} = \text{s-}\lim_{n \rightarrow \infty} (e^{-\tau A/n} e^{-\tau B/n})^n , \quad (1.3)$$

uniformly in $\tau \in [0, T]$ for any $T > 0$. It is obvious that this result holds if B is a bounded operator.

Considering the Trotter product formula on a Hilbert space \mathcal{H} , Kato has shown in [4] that for non-negative operators A and B the Trotter formula (1.3) holds in the *strong* operator topology if $\text{dom}(\sqrt{A}) \cap \text{dom}(\sqrt{B})$ is dense in the Hilbert space and $C = A \dot{+} B$ is the form-sum of operators A and B . Later on it was shown in [3] that the relation (1.2) holds if the algebraic sum $C = A + B$ is already a self-adjoint operator. Therefore, (1.2) is valid in particular, if B is a bounded self-adjoint operator.

The historically first result concerning the operator-norm convergence of the Trotter formula in a Banach space is due to [1]. Since the concept of self-adjointness is missing for Banach spaces it was assumed that the *dominating* operator A is a generator of a *contraction holomorphic* semigroup and B is a generator of a contraction semigroup. In Theorem 3.6 of [1] it was shown that if $0 \in \rho(A)$ and if there is a $\alpha \in [0, 1)$ such that $\text{dom}(A^\alpha) \subseteq \text{dom}(B)$ and $\text{dom}(A^*) \subseteq \text{dom}(B^*)$, then for any $T > 0$ one has

$$\sup_{\tau \in [0, T]} \left\| (e^{-\tau A/n} e^{-\tau B/n})^n - e^{-\tau C} \right\| = O(\ln(n)/n^{1-\alpha}) . \quad (1.4)$$

Note that the assumption $0 \in \rho(A)$ was made for simplicity and that the assumption $\text{dom}(A^\alpha) \subseteq \text{dom}(B)$ yields that the operator B is infinitesimally

small with respect to A . Taking into account [5, Corollary IX.2.5] one gets that the well-defined algebraic sum $C = A + B$ is a generator of a contraction holomorphic semigroup. By Theorem 3.6 of [1] the convergence rate (1.4) improves if B is a bounded operator, i.e. $\alpha = 0$. Then for any $T > 0$ one gets

$$\sup_{\tau \in [0, T]} \| (e^{-\tau A/n} e^{-\tau B/n})^n - e^{-\tau C} \| = O((\ln(n))^2/n) .$$

Summarizing, the question arises whether the Trotter product formula converges in the operator-norm if A is a generator of a contraction (but not holomorphic) semigroup and B is a bounded operator? The aim of the present paper is to give an answer to this question for a certain class of generators.

It turns out that an appropriate class for that is the class of generators of *evolution* semigroups. To proceed further we need the notion of a *propagator*, or a *solution operator* [6].

A strongly continuous map $U(\cdot, \cdot) : \Delta \longrightarrow \mathcal{B}(X)$, where $\Delta := \{(t, s) : 0 < s \leq t \leq T\}$ and $\mathcal{B}(X)$ is the set of bounded operators on the separable Banach space X , is called a *propagator* if the conditions

- (i) $\sup_{(t,s) \in \Delta} \|U(t, s)\|_{\mathcal{B}(X)} < \infty$,
- (ii) $U(t, s) = U(t, r)U(r, s)$, $0 < s \leq r \leq t \leq T$,

are satisfied. Let us consider the Banach space $L^p(\mathcal{I}, X)$, $\mathcal{I} := [0, T]$, $p \in [1, \infty)$. The operator \mathcal{K} is an evolution generator of the evolution semigroup $\{e^{-\tau \mathcal{K}}\}_{\tau \geq 0}$ if there is a propagator such that the representation

$$(e^{-\tau \mathcal{K}} f)(t) = U(t, t - \tau) \chi_{\mathcal{I}}(t - \tau) f(t - \tau), \quad f \in L^p(\mathcal{I}, X), \quad (1.5)$$

holds for a.e. $t \in \mathcal{I}$ and $\tau \geq 0$ [6]. Since $e^{-\tau \mathcal{K}} f = 0$ for $\tau \geq T$, the evolution generator \mathcal{K} can never be a generator of a holomorphic semigroup.

A simple example of an evolution generator is the differentiation operator:

$$(D_0 f)(t) := \partial_t f(t), \quad f \in \text{dom}(D_0) := \{f \in H^{1,p}(\mathcal{I}, X) : f(0) = 0\}. \quad (1.6)$$

Then by (1.6) one obviously gets the contraction shift semigroup:

$$(e^{-\tau D_0} f)(t) = \chi_{\mathcal{I}}(t - \tau) f(t - \tau), \quad f \in L^p(\mathcal{I}, X), \quad (1.7)$$

for a.e. $t \in \mathcal{I}$ and $\tau \geq 0$. Hence, (1.5) implies that the corresponding propagator of the non-holomorphic evolution semigroup $\{e^{-\tau D_0}\}_{\tau \geq 0}$ is given by $U_{D_0}(t, s) = I$, $(t, s) \in \Delta$.

Note that in [6] we considered the operator $\mathcal{K}_0 := \overline{D_0 + \mathcal{A}}$, where \mathcal{A} is the multiplication operator induced by a generator A of a holomorphic contraction semigroup on X . More precisely

$$(\mathcal{A}f)(t) := Af(t), \text{ and } (e^{-\tau \mathcal{A}} f)(t) = e^{-\tau A} f(t), \\ f \in \text{dom}(\mathcal{A}) := \{f \in L^p(\mathcal{I}, X) : Af(\cdot) \in L^p(\mathcal{I}, X)\} .$$

Then the perturbation of the shift semigroup (1.7) by \mathcal{A} corresponds to the semigroup with generator \mathcal{K}_0 . One easily checks that \mathcal{K}_0 is an evolution generator of a contraction semigroup on $L^p(\mathcal{I}, X)$ that is never holomorphic. Indeed, since the generators D_0 and \mathcal{A} commute, the representation (1.5) for evolution semigroup $\{e^{-\tau\mathcal{K}_0}\}_{\tau \geq 0}$ takes the form:

$$(e^{-\tau\mathcal{K}_0} f)(t) = e^{-\tau A} \chi_{\mathcal{I}}(t - \tau) f(t - \tau), \quad f \in L^p(\mathcal{I}, X),$$

for a.e. $t \in \mathcal{I}$ and $\tau \geq 0$ with propagator $U_0(t, s) = e^{-(t-s)A}$. Therefore, again $e^{-\tau\mathcal{K}_0} f = 0$ for $\tau \geq T$.

Furthermore, if $B(\cdot)$ is a *strongly measurable* family of generators of contraction semigroups on X , i.e. $B(\cdot) : \mathcal{I} \rightarrow \mathcal{G}(1, 0)$ (see [4], Ch.IX, §1.4), then the induced multiplication operator \mathcal{B} :

$$(\mathcal{B}f)(t) := B(t)f(t), \quad (1.8)$$

$$f \in \text{dom}(\mathcal{B}) := \left\{ f \in L^p(\mathcal{I}, X) : \begin{array}{l} f(t) \in \text{dom}(B(t)) \text{ for a.e. } t \in \mathcal{I} \\ B(t)f(t) \in L^p(\mathcal{I}, X) \end{array} \right\},$$

is a generator of a contraction semigroup on $L^p(\mathcal{I}, X)$.

In [6] it was assumed that $\{B(t)\}_{t \in \mathcal{I}}$ is a strongly measurable family of generators of contraction semigroups and that A is a generator of a bounded holomorphic semigroup with $0 \in \rho(A)$ for simplicity. Moreover, we supposed that the following conditions are satisfied:

- (i) $\text{dom}(A^\alpha) \subseteq \text{dom}(B(t))$ for a.e. $t \in \mathcal{I}$ and some $\alpha \in (0, 1)$ such that

$$\text{ess sup}_{t \in \mathcal{I}} \|B(t)A^{-\alpha}\|_{\mathcal{B}(X)} < \infty;$$

- (ii) $\text{dom}(A^*) \subseteq \text{dom}(B(t)^*)$ for a.e. $t \in \mathcal{I}$ such that

$$\text{ess sup}_{t \in \mathcal{I}} \|B(t)^*(A^{-1})^*\|_{\mathcal{B}(X)} < \infty;$$

- (iii) there is a $\beta \in (\alpha, 1)$ and $L_\beta > 0$ such that

$$\|A^{-1}(B(t) - B(s))A^{-\alpha}\|_{\mathcal{B}(X)} \leq L_\beta |t - s|^\beta, \quad t, s \in \mathcal{I}. \quad (1.9)$$

Under these assumptions it turns out that $\mathcal{K} := \mathcal{K}_0 + \mathcal{B}$ is a generator of a contraction evolution semigroup, i.e there is a propagator $\{U(t, s)\}_{(t,s) \in \Delta}$ such that the representation (1.5) is valid. Moreover, we prove in [6] the Trotter product formula converges in the operator norm with convergence rate $O(1/n^{\beta-\alpha})$:

$$\sup_{\tau \geq 0} \left\| \left(e^{-\tau\mathcal{K}_0/n} e^{-\tau\mathcal{B}/n} \right)^n - e^{-\tau\mathcal{K}} \right\|_{\mathcal{B}(L^p(\mathcal{I}, X))} = O(1/n^{\beta-\alpha}).$$

We comment that if $B(\cdot) : \mathcal{I} \rightarrow \mathcal{B}(X)$ is a Hölder continuous function with Hölder exponent $\beta \in (0, 1)$, then the assumptions (i)-(iii) are satisfied for any $\alpha \in (0, \beta)$. Then our results [6] yield that

$$\sup_{\tau \geq 0} \left\| \left(e^{-\tau\mathcal{K}_0/n} e^{-\tau\mathcal{B}/n} \right)^n - e^{-\tau\mathcal{K}} \right\|_{\mathcal{B}(L^p(\mathcal{I}, X))} = O(1/n^\gamma), \quad (1.10)$$

holds for any $\gamma \in (0, \beta)$. Moreover, in this case the perturbation of the shift semigroup (1.7) by a bounded generator (1.8) gives an evolution semigroup

with generator $D_0 + \mathcal{B}$. Then as a corollary of (1.10) for $\mathcal{A} = 0$, we get the Trotter product estimate

$$\sup_{\tau \geq 0} \left\| \left(e^{-\tau D_0/n} e^{-\tau \mathcal{B}/n} \right)^n - e^{-\tau(D_0 + \mathcal{B})} \right\|_{\mathcal{B}(L^p(\mathcal{I}, X))} = O(1/n^\gamma) . \quad (1.11)$$

The aim of our note is to show that the convergence rate (1.11) is close to the optimal one. To this end we consider the simple case, when $X = \mathbb{C}$ and we put for simplicity $\mathcal{I} := [0, 1]$.

The *main results* of this paper can be summarized as follows: If the operator \mathcal{B} is equal to the multiplication operator Q induced by a bounded measurable function $q(\cdot) : \mathcal{I} \rightarrow \mathbb{C}$ in $L^p(\mathcal{I})$, then one can verify that the condition (1.9) is equivalent to $q(\cdot) \in C^{0,\beta}(\mathcal{I})$, see definition below. In this case the convergence rate is

$$\sup_{\tau \geq 0} \left\| e^{-\tau(D_0 + Q)} - \left(e^{-\tau D_0/n} e^{-\tau Q/n} \right)^n \right\|_{\mathcal{B}(L^p(\mathcal{I}, X))} = O(1/n^\beta) . \quad (1.12)$$

This result remains true if $q(\cdot)$ is Lipschitz continuous, i.e. $\beta = 1$. But if $q(\cdot)$ is *only* continuous, then

$$\sup_{\tau \geq 0} \left\| e^{-\tau(D_0 + Q)} - \left(e^{-\tau D_0/n} e^{-\tau Q/n} \right)^n \right\|_{\mathcal{B}(L^p(\mathcal{I}, X))} = o(1) . \quad (1.13)$$

Moreover, for any convergent to zero sequence $\delta_n > 0$, $n \in \mathbb{N}$, there exists a continuous function $q(\cdot)$ such that

$$\sup_{\tau \geq 0} \left\| e^{-\tau(D_0 + Q)} - \left(e^{-\tau D_0/n} e^{-\tau Q/n} \right)^n \right\|_{\mathcal{B}(L^p(\mathcal{I}, X))} = \omega(\delta_n) , \quad (1.14)$$

where the Landau *symbol* $\omega(\cdot)$ is defined below.

Finally, there is an example of a bounded measurable function $q(\cdot)$ such that

$$\limsup_{n \rightarrow \infty} \sup_{\tau \geq 0} \left\| e^{-\tau(D_0 + Q)} - \left(e^{-\tau D_0/n} e^{-\tau Q/n} \right)^n \right\|_{\mathcal{B}(L^p(\mathcal{I}, X))} > 0 . \quad (1.15)$$

Hence, in contrast to the holomorphic case, when the *dominating* operator is a generator of a holomorphic semigroup (1.4), the Trotter product formula (1.15) with dominating generator D_0 , may *not* converge in the operator-norm.

The paper is organized as follows. In Section 2 we reformulate the convergence of the Trotter product formula in terms of the corresponding evolutions semigroups. In Section 3 we prove the results (1.12)-(1.15).

We conclude this section by few remarks concerning **notation** used in this paper.

1. We use a definition of the *generator* C of a semigroup (1.3), which differs from the standard one by a *minus* [5].

2. Furthermore, we widely use the so-called *Landau symbols*:

$$\begin{aligned} g(n) = O(f(n)) &\iff \limsup_{n \rightarrow \infty} \left| \frac{g(n)}{f(n)} \right| < \infty, \\ g(n) = o(f(n)) &\iff \limsup_{n \rightarrow \infty} \left| \frac{g(n)}{f(n)} \right| = 0, \\ g(n) = \Theta(f(n)) &\iff 0 < \liminf_{n \rightarrow \infty} \left| \frac{g(n)}{f(n)} \right| \leq \limsup_{n \rightarrow \infty} \left| \frac{g(n)}{f(n)} \right| < \infty, \\ g(n) = \omega(f(n)) &\iff \limsup_{n \rightarrow \infty} \left| \frac{g(n)}{f(n)} \right| = \infty. \end{aligned}$$

3. We use the notation $C^{0,\beta}(\mathcal{I}) = \{f : \mathcal{I} \rightarrow \mathbb{C} : \text{there is some } K > 0 \text{ such that } |f(x) - f(y)| \leq K|x - y|^\beta\}$ for $\beta \in (0, 1]$.

2. Trotter product formula and evolution semigroups

Below we consider the Banach space $L^p(\mathcal{I}, X)$ for $\mathcal{I} := [0, T]$, $p \in [1, \infty)$. Recall that semigroup $\{\mathcal{U}(\tau)\}_{\tau \geq 0}$, on the Banach space $L^p(\mathcal{I}, X)$ is called an *evolution semigroup* if there is a propagator $\{U(t, s)\}_{(t,s) \in \Delta}$ such that the representation (1.5) holds.

Let \mathcal{K}_0 be the generator of an evolution semigroup $\{\mathcal{U}_0(\tau)\}_{\tau \geq 0}$ and let \mathcal{B} be a multiplication operator induced by a measurable family $\{B(t)\}_{t \in \mathcal{I}}$ of generators of contraction semigroups. Note that in this case the multiplication operator \mathcal{B} (1.8) is a generator of a contraction semigroup $(e^{-\tau \mathcal{B}} f)(t) = e^{-\tau B(t)} f(t)$, on the Banach space $L^p(\mathcal{I}, X)$. Since $\{\mathcal{U}_0(\tau)\}_{\tau \geq 0}$ is an evolution semigroup, then by definition (1.5) there is a propagator $\{U_0(t, s)\}_{(t,s) \in \Delta}$ such that the representation

$$(\mathcal{U}_0(\tau)f)(t) = U_0(t, t - \tau)\chi_{\mathcal{I}}(t - \tau)f(t - \tau), \quad f \in L^p(\mathcal{I}, X),$$

is valid for a.e. $t \in \mathcal{I}$ and $\tau \geq 0$. Then we define

$$G_j(t, s; n) := U_0\left(s + j \frac{(t-s)}{n}, s + (j-1) \frac{(t-s)}{n}\right) e^{-\frac{(t-s)}{n} B\left(s + (j-1) \frac{(t-s)}{n}\right)}$$

where $j \in \{1, 2, \dots, n\}$, $n \in \mathbb{N}$, $(t, s) \in \Delta$, and we set

$$V_n(t, s) := \prod_{j=1}^{n \leftarrow} G_j(t, s; n), \quad n \in \mathbb{N}, \quad (t, s) \in \Delta,$$

where the product is increasingly ordered in j from the right to the left. Then a straightforward computation shows that the representation

$$\left(\left(e^{-\tau \mathcal{K}_0/n} e^{-\tau \mathcal{B}/n} \right)^n f \right)(t) = V_n(t, t - \tau)\chi_{\mathcal{I}}(t - \tau)f(t - \tau), \quad (2.1)$$

$f \in L^p(\mathcal{I}, X)$, holds for each $\tau \geq 0$ and a.e. $t \in \mathcal{I}$.

Proposition 2.1. *Let \mathcal{K} and \mathcal{K}_0 be generators of evolution semigroups on the Banach space $L^p(\mathcal{I}, X)$ for some $p \in [1, \infty)$. Further, let $\{B(t) \in \mathcal{G}(1, 0)\}_{t \in \mathcal{I}}$*

be a strongly measurable family of generators of contraction on X semigroups. Then

$$\begin{aligned} \sup_{\tau \geq 0} \left\| e^{-\tau\mathcal{K}} - \left(e^{-\tau\mathcal{K}_0/n} e^{-\tau\mathcal{B}/n} \right)^n \right\|_{\mathcal{B}(L^p(\mathcal{I}, X))} \\ = \operatorname{ess\,sup}_{(t,s) \in \Delta} \|U(t, s) - V_n(t, s)\|_{\mathcal{B}(X)}, \quad n \in \mathbb{N}. \end{aligned} \quad (2.2)$$

Proof. Let $\{L(\tau)\}_{\tau \geq 0}$ be the left-shift semigroup on the Banach space $\mathfrak{X} = L^p(\mathcal{I}, X)$:

$$(L(\tau)f)(t) = \chi_{\mathcal{I}}(t + \tau)f(t + \tau), \quad f \in L^p(\mathcal{I}, X).$$

Using that we get

$$\begin{aligned} & \left(L(\tau) \left(e^{-\tau\mathcal{K}} - \left(e^{-\tau/n\mathcal{K}_0} e^{-\tau\mathcal{B}/n} \right)^n \right) f \right) (t) \\ &= \{U(t + \tau, t) - V_n(t + \tau, t)\} \chi_{\mathcal{I}}(t + \tau) f(t), \end{aligned}$$

for $\tau \geq 0$ and a.e. $t \in \mathcal{I}$. It turns out that for each $n \in \mathbb{N}$ the operator $L(\tau) \left(e^{-\tau\mathcal{K}} - \left(e^{-\tau/n\mathcal{K}_0} e^{-\tau\mathcal{B}/n} \right)^n \right)$ is a multiplication operator induced by $\{(U(t + \tau, t) - V_n(t + \tau, t))\chi_{\mathcal{I}}(t + \tau)\}_{t \in \mathcal{I}}$. Therefore,

$$\begin{aligned} & \left\| L(\tau) \left(e^{-\tau\mathcal{K}} - \left(e^{-\tau\mathcal{K}_0/n} e^{-\tau\mathcal{B}/n} \right)^n \right) \right\|_{\mathcal{B}(\mathfrak{X})} \\ &= \operatorname{ess\,sup}_{t \in \mathcal{I}} \|U(t + \tau, t) - V_n(t + \tau, t)\|_{\mathcal{B}(X)} \chi_{\mathcal{I}}(t + \tau), \end{aligned}$$

for each $\tau \geq 0$. Note that one has

$$\begin{aligned} \sup_{\tau \geq 0} \left\| L(\tau) \left(e^{-\tau\mathcal{K}} - \left(e^{-\tau\mathcal{K}_0/n} e^{-\tau\mathcal{B}/n} \right)^n \right) \right\|_{\mathcal{B}(\mathfrak{X})} \\ = \operatorname{ess\,sup}_{\tau \geq 0} \left\| L(\tau) \left(e^{-\tau\mathcal{K}} - \left(e^{-\tau\mathcal{K}_0/n} e^{-\tau\mathcal{B}/n} \right)^n \right) \right\|_{\mathcal{B}(\mathfrak{X})}. \end{aligned}$$

This is based on the fact that if $F(\cdot) : \mathbb{R}_+ \rightarrow \mathcal{B}(\mathfrak{X})$ is strongly continuous, then $\sup_{\tau \geq 0} \|F(\tau)\|_{\mathcal{B}(\mathfrak{X})} = \operatorname{ess\,sup}_{\tau \geq 0} \|F(\tau)\|_{\mathcal{B}(\mathfrak{X})}$. Hence, we find

$$\begin{aligned} \sup_{\tau \geq 0} \left\| L(\tau) \left(e^{-\tau\mathcal{K}} - \left(e^{-\tau\mathcal{K}_0/n} e^{-\tau\mathcal{B}/n} \right)^n \right) \right\|_{\mathcal{B}(\mathfrak{X})} \\ = \operatorname{ess\,sup}_{\tau \geq 0} \operatorname{ess\,sup}_{t \in \mathcal{I}} \|U(t + \tau, t) - V_n(t + \tau, t)\|_{\mathcal{B}(X)} \chi_{\mathcal{I}}(t + \tau). \end{aligned}$$

Further, if $\Phi(\cdot, \cdot) : \mathbb{R}_+ \times \mathcal{I} \rightarrow \mathcal{B}(X)$ is a strongly measurable function, then

$$\operatorname{ess\,sup}_{(\tau, t) \in \mathbb{R}_+ \times \mathcal{I}} \|\Phi(\tau, t)\|_{\mathcal{B}(X)} = \operatorname{ess\,sup}_{\tau \geq 0} \operatorname{ess\,sup}_{t \in \mathcal{I}} \|\Phi(\tau, t)\|_{\mathcal{B}(X)}.$$

Then, taking into account two last equalities, one obtains

$$\begin{aligned} \sup_{\tau \geq 0} \left\| L(\tau) \left(e^{-\tau\mathcal{K}} - \left(e^{-\tau\mathcal{K}_0/n} e^{-\tau\mathcal{B}/n} \right)^n \right) \right\|_{\mathcal{B}(\mathfrak{X})} \\ = \operatorname{ess\,sup}_{(\tau, t) \in \mathbb{R}_+ \times \mathcal{I}} \|U(t + \tau, t) - V_n(t + \tau, t)\|_{\mathcal{B}(X)} \chi_{\mathcal{I}}(t + \tau) = \\ = \operatorname{ess\,sup}_{(t, s) \in \Delta} \|U(t, s) - V_n(t, s)\|_{\mathcal{B}(X)}, \end{aligned}$$

that proves (2.2) □

3. Bounded perturbations of the shift semigroup generator

3.1. Basic facts

We study bounded perturbations of the evolution generator D_0 (1.6). To do this aim we consider $\mathcal{I} = [0, 1]$, $X = \mathbb{C}$ and we denote by $L^p(\mathcal{I})$ the Banach space $L^p(\mathcal{I}, \mathbb{C})$.

For $t \in \mathcal{I}$, let $q : t \mapsto q(t) \in L^\infty(\mathcal{I})$. Then, q induces a bounded multiplication operator Q on the Banach space $L^p(\mathcal{I})$:

$$(Qf)(t) = q(t)f(t), \quad f \in L^p(\mathcal{I}).$$

For simplicity we assume that $q \geq 0$. Then Q generates on $L^p(\mathcal{I})$ a contraction semigroup $\{e^{-\tau Q}\}_{\tau \geq 0}$. Since generator Q is bounded, the closed operator $\mathcal{A} := D_0 + Q$, with domain $\text{dom}(\mathcal{A}) = \text{dom}(D_0)$, is generator of a semigroup on $L^p(\mathcal{I})$. By [7], the Trotter product formula in the strong topology follows immediately

$$\left(e^{-\tau D_0/n} e^{-\tau Q/n}\right)^n f \rightarrow e^{-\tau(D_0+Q)} f, \quad f \in L^p(\mathcal{I}), \quad (3.1)$$

uniformly in $\tau \in [0, T]$ on bounded time intervals.

Following [2, §5], we define on $X = \mathbb{C}$ a family of bounded operators $\{V(t)\}_{t \in \mathcal{I}}$ by

$$V(t) := e^{-\int_0^t ds q(s)}.$$

Note that for almost every $t \in \mathcal{I}$ these operators are positive. Then $V^{-1}(t)$ exists and it has the form

$$V^{-1}(t) = e^{\int_0^t ds q(s)}.$$

The operator families $\{V(t)\}_{t \in \mathcal{I}}$ and $\{V^{-1}(t)\}_{t \in \mathcal{I}}$ induce two bounded multiplication operators \mathcal{V} and \mathcal{V}^{-1} on $L^p(\mathcal{I})$, respectively. Then invertibility implies that $\mathcal{V} \mathcal{V}^{-1} = \mathcal{V}^{-1} \mathcal{V} = \text{Id}|_{L^p}$. Using the operator \mathcal{V} one easily verifies that $D_0 + Q$ is similar to D_0 , i.e. one has

$$\mathcal{V}^{-1}(D_0 + Q)\mathcal{V} = D_0, \text{ or } D_0 + Q = \mathcal{V}D_0\mathcal{V}^{-1}.$$

Hence, the semigroup generated on $L^p(\mathcal{I})$ by $D_0 + Q$ gets the explicit form:

$$\begin{aligned} \left(e^{-\tau(D_0+Q)} f\right)(t) &= (\mathcal{V}e^{-\tau D_0}\mathcal{V}^{-1}f)(t) = \\ &= e^{-\int_{t-\tau}^t q(y)dy} f(t-\tau) \chi_{\mathcal{I}}(t-\tau). \end{aligned} \quad (3.2)$$

Since by (1.5) the propagator $U(t, s)$ that corresponds to evolution semigroup (3.2) is defined by

$$\left(e^{-\tau(D_0+Q)}\right) f(t) = U(t, t-\tau) f(t-\tau) \chi_{\mathcal{I}}(t-\tau),$$

we deduce that it is equal to $U(t, s) = e^{-\int_s^t dy q(y)}$.

Now we study the corresponding Trotter product formula. For a fixed $\tau \geq 0$ and $n \in \mathbb{N}$, we define approximation V_n by

$$\left(\left(e^{-\tau D_0/n} e^{-\tau Q/n}\right)^n f\right)(t) =: V_n(t, t-\tau) \chi_{\mathcal{I}}(t-\tau) f(t-\tau).$$

Then by straightforward calculations, similar to (2.1), one finds that

$$V_n(t, s) = e^{-\frac{t-s}{n} \sum_{k=0}^{n-1} q(s+k\frac{t-s}{n})}, \quad (t, s) \in \Delta.$$

Proposition 3.1. *Let $q \in L^\infty(\mathcal{I})$ be non-negative. Then*

$$\begin{aligned} & \sup_{\tau \geq 0} \left\| e^{-\tau(D_0+Q)} - \left(e^{-\tau D_0/n} e^{-\tau Q/n} \right)^n \right\|_{\mathcal{B}(L^p(\mathcal{I}))} \\ &= \Theta \left(\operatorname{ess\,sup}_{(t,s) \in \Delta} \left| \int_s^t q(y) dy - \frac{t-s}{n} \sum_{k=0}^{n-1} q\left(s+k\frac{t-s}{n}\right) \right| \right) \end{aligned}$$

as $n \rightarrow \infty$, where Θ is the Landau symbol defined in Section 1.

Proof. First, by Proposition 2.1 and by $U(t, s) = e^{-\int_s^t dy q(y)}$ we obtain

$$\begin{aligned} & \sup_{\tau \geq 0} \left\| e^{-\tau(D_0+Q)} - \left(e^{-\tau D_0/n} e^{-\tau Q/n} \right)^n \right\|_{\mathcal{B}(L^p(\mathcal{I}))} \\ &= \operatorname{ess\,sup}_{(t,s) \in \Delta} \left| e^{-\int_s^t dy q(y)} - e^{-\frac{t-s}{n} \sum_{k=0}^{n-1} q(s+k\frac{t-s}{n})} \right|. \end{aligned} \quad (3.3)$$

Then, using the inequality

$$e^{-\max\{x,y\}} |x - y| \leq |e^{-x} - e^{-y}| \leq |x - y|, \quad 0 \leq x, y,$$

for $0 \leq s < t \leq 1$ one finds the estimates

$$e^{-\|q\|_{L^\infty}} R_n(t, s; q) \leq \left| e^{-\int_s^t dy q(y)} - e^{-\frac{t-s}{n} \sum_{k=0}^{n-1} q(s+k\frac{t-s}{n})} \right| \leq R_n(t, s; q),$$

where

$$R_n(t, s, q) := \left| \int_s^t dy q(y) - \frac{t-s}{n} \sum_{k=0}^{n-1} q\left(s+k\frac{t-s}{n}\right) \right|, \quad (t, s) \in \Delta. \quad (3.4)$$

Hence, for the left-hand side of (3.3) we get the estimate

$$e^{-\|q\|_{L^\infty}} R_n(q) \leq \sup_{\tau \geq 0} \left\| e^{-\tau(D_0+Q)} - \left(e^{-\tau D_0/n} e^{-\tau Q/n} \right)^n \right\|_{\mathcal{B}(L^p)} \leq R_n(q),$$

where $R_n(q) := \operatorname{ess\,sup}_{(t,s) \in \Delta} R_n(t, s; q)$, $n \in \mathbb{N}$. These estimates together with definition of Θ prove the assertion. \square

Note that by virtue of (3.4) and Proposition 3.1 the operator-norm convergence rate of the Trotter product formula for the pair $\{D_0, Q\}$ coincides with the convergence rate of the integral Darboux-Riemann sum approximation of the Lebesgue integral.

3.2. Examples

First we consider the case of a real Hölder-continuous function $q \in C^{0,\beta}(\mathcal{I})$.

Theorem 3.2. *If $q \in C^{0,\beta}(\mathcal{I})$ is non-negative, then*

$$\sup_{\tau \geq 0} \left\| e^{-\tau(D_0+Q)} - \left(e^{-\tau D_0/n} e^{-\tau Q/n} \right)^n \right\| = O(1/n^\beta),$$

as $n \rightarrow \infty$.

Proof. One has

$$\begin{aligned} \int_s^t dy q(y) - \frac{t-s}{n} \sum_k^{n-1} q\left(s + \frac{k}{n}(t-s)\right) \\ = \sum_{k=0}^{n-1} \int_{\frac{k}{n}(t-s)}^{\frac{k+1}{n}(t-s)} dy \left(q(s+y) - q\left(s + \frac{k}{n}(t-s)\right) \right) , \end{aligned}$$

which yields the estimate

$$\begin{aligned} \left| \int_s^t dy q(y) - \frac{t-s}{n} \sum_k^{n-1} q\left(s + \frac{k}{n}(t-s)\right) \right| \\ \leq \sum_{k=0}^{n-1} \int_{\frac{k}{n}(t-s)}^{\frac{k+1}{n}(t-s)} dy \left| q(s+y) - q\left(s + \frac{k}{n}(t-s)\right) \right| . \end{aligned}$$

Since $q \in C^{0,\beta}(\mathcal{I})$, there is a constant $L_\beta > 0$ such that for $y \in [\frac{k}{n}(t-s), \frac{k+1}{n}(t-s)]$ one has

$$\left| q(s+y) - q\left(s + \frac{k}{n}(t-s)\right) \right| \leq L_\beta |y - \frac{k}{n}(t-s)|^\beta \leq L_\beta \frac{(t-s)^\beta}{n^\beta} .$$

Hence, we find

$$\left| \int_s^t q(y) dy - \frac{t-s}{n} \sum_k^{n-1} q\left(s + \frac{k}{n}(t-s)\right) \right| \leq L_\beta \frac{(t-s)^{1+\beta}}{n^\beta} \leq L_\beta \frac{1}{n^\beta} ,$$

which proves

$$\operatorname{ess\,sup}_{(t,s) \in \Delta} \left| \int_s^t q(y) dy - \frac{t-s}{n} \sum_k^{n-1} q\left(s + \frac{k}{n}(t-s)\right) \right| = O\left(\frac{1}{n^\beta}\right) .$$

Applying now Proposition 3.1 one completes the proof. \square

It is a natural question: what happens, when q is only continuous?

Theorem 3.3. *If $q : \mathcal{I} \rightarrow \mathbb{C}$ is continuous and non-negative, then*

$$\left\| e^{-\tau(D_0+Q)} - \left(e^{-\tau D_0/n} e^{-\tau Q/n} \right)^n \right\| = o(1) , \quad (3.5)$$

as $n \rightarrow \infty$.

Proof. Since $q(\cdot)$ is continuous, then for any $\varepsilon > 0$ there is $\delta > 0$ such that for $|y-x| < \delta$ we have $|q(y) - q(x)| < \varepsilon$, $y, x \in \mathcal{I}$. Therefore, if $1/n < \delta$, then for $y \in (\frac{k}{n}(t-s), \frac{k+1}{n}(t-s))$ we have

$$|q(s+y) - q\left(s + \frac{k}{n}(t-s)\right)| < \varepsilon, \quad (t, s) \in \Delta .$$

Hence,

$$\left| \int_s^t q(y) dy - \frac{t-s}{n} \sum_k^{n-1} q\left(s + \frac{k}{n}(t-s)\right) \right| \leq \varepsilon(t-s) \leq \varepsilon ,$$

which yields

$$\operatorname{ess\,sup}_{(t,s) \in \Delta} \left| \int_s^t q(y) dy - \frac{t-s}{n} \sum_{k=0}^{n-1} q\left(s + k \frac{t-s}{n}\right) \right| = o(1) .$$

Now it remains only to apply Proposition 3.1. \square

We comment that for a general continuous q one can say nothing about the convergence rate. Indeed, it can be shown that in (3.5) the convergence to zero can be arbitrary slow.

Theorem 3.4. *Let $\delta_n > 0$ be a sequence with $\delta_n \rightarrow 0$ as $n \rightarrow \infty$. Then there exists a continuous function $q : \mathcal{I} = [0, 1] \rightarrow \mathbb{R}$ such that*

$$\sup_{\tau \geq 0} \left\| e^{-\tau(D_0+Q)} - \left(e^{-\tau D_0/n} e^{-\tau Q/n} \right)^n \right\|_{\mathcal{B}(L^p(\mathcal{I}))} = \omega(\delta_n) \quad (3.6)$$

as $n \rightarrow \infty$, where ω is the Landau symbol defined in Section 1.

Proof. Taking into account Theorem 6 of [8], we find that for any sequence $\{\delta_n\}_{n \in \mathbb{N}}$, $\delta_n > 0$ satisfying $\lim_{n \rightarrow \infty} \delta_n = 0$ there exists a continuous function $f(\cdot) : [0, 2\pi] \rightarrow \mathbb{R}$ such that

$$\left| \int_0^{2\pi} f(x) dx - \frac{2\pi}{n} \sum_{k=1}^n f(2k\pi/n) \right| = \omega(\delta_n) ,$$

as $n \rightarrow \infty$. Setting $q(y) := f(2\pi(1-y))$, $y \in [0, 1]$, we get a continuous function $q(\cdot) : [0, 1] \rightarrow \mathbb{R}$, such that

$$\left| \int_0^1 q(y) dy - \frac{1}{n} \sum_{k=0}^{n-1} q(k/n) \right| = \omega(\delta_n) .$$

Because $q(\cdot)$ is continuous we find

$$\begin{aligned} \operatorname{ess\,sup}_{(t,s) \in \Delta} \left| \int_s^t q(y) dy - \frac{t-s}{n} \sum_{n=0}^{n-1} q\left(s + k \frac{t-s}{n}\right) \right| \\ \geq \left| \int_0^1 q(y) dy - \frac{1}{n} \sum_{k=0}^{n-1} q(k/n) \right| , \end{aligned}$$

which yields

$$\operatorname{ess\,sup}_{(t,s) \in \Delta} \left| \int_s^t q(y) dy - \frac{t-s}{n} \sum_{n=0}^{n-1} q\left(s + k \frac{t-s}{n}\right) \right| = \omega(\delta_n) .$$

Applying now Proposition 3.1 we prove (3.6). \square

Our final comment concerns the case when q is only *measurable*. Then it can happen that the Trotter product formula for that pair $\{D_0, Q\}$ does not converge in the operator-norm topology.

Theorem 3.5. *There is a non-negative function $q \in L^\infty([0, 1])$ such that*

$$\limsup_{n \rightarrow \infty} \sup_{\tau \geq 0} \left\| e^{-\tau(D_0+Q)} - \left(e^{-\tau D_0/n} e^{-\tau Q/n} \right)^n \right\|_{B(L^p(\mathcal{I}))} > 0. \quad (3.7)$$

Proof. Let us introduce the open intervals

$$\begin{aligned} \Delta_{0,n} &:= (0, \frac{1}{2^{2n+2}}), \\ \Delta_{k,n} &:= (t_{k,n} - \frac{1}{2^{2n+2}}, t_{k,n} + \frac{1}{2^{2n+2}}), \quad k = 1, 2, \dots, 2^n - 1, \\ \Delta_{2^n,n} &:= (1 - \frac{1}{2^{2n+2}}, 1), \end{aligned}$$

$n \in \mathbb{N}$, where

$$t_{k,n} = \frac{k}{2^n}, \quad k = 0, \dots, n, \quad n \in \mathbb{N}.$$

Notice that $t_{0,n} = 0$ and $t_{2^n,n} = 1$. One easily checks that the intervals $\Delta_{k,n}$, $k = 0, \dots, 2^n$, are mutually disjoint. We introduce the open sets

$$\mathcal{O}_n = \bigcup_{k=0}^{2^n} \Delta_{k,n} \subseteq \mathcal{I}, \quad n \in \mathbb{N}.$$

and

$$\mathcal{O} = \bigcup_{n \in \mathbb{N}} \mathcal{O}_n \subseteq \mathcal{I}.$$

Then it is clear that

$$|\mathcal{O}_n| = \frac{1}{2^{n+1}}, \quad n \in \mathbb{N}, \quad \text{and} \quad |\mathcal{O}| \leq \frac{1}{2}.$$

Therefore, the Lebesgue measure of the closed set $\mathcal{C} := \mathcal{I} \setminus \mathcal{O} \subseteq \mathcal{I}$ can be estimated by

$$|\mathcal{C}| \geq \frac{1}{2}.$$

Using the characteristic function $\chi_{\mathcal{C}}(\cdot)$ of the set \mathcal{C} we define

$$q(t) := \chi_{\mathcal{C}}(t), \quad t \in \mathcal{I}.$$

The function $q(\cdot)$ is measurable and it satisfies $0 \leq q(t) \leq 1$, $t \in \mathcal{I}$.

Let $\varepsilon \in (0, 1)$. We choose $s \in (0, \varepsilon)$ and $t \in (1 - \varepsilon, 1)$ and we set

$$\xi_{k,n}(t, s) := s + k \frac{t-s}{2^n}, \quad k = 0, \dots, 2^n - 1, \quad n \in \mathbb{N}, \quad (t, s) \in \Delta.$$

Note that $\xi_{k,n}(t, s) \in (0, 1)$, $k = 0, \dots, 2^n - 1$, $n \in \mathbb{N}$. Moreover, we have

$$t_{k,n} - \xi_{k,n}(t, s) = k \frac{1}{2^n} - s - k \frac{t-s}{2^n} = k \frac{1-t+s}{2^n} - s,$$

which leads to the estimate

$$|t_{k,n} - \xi_{k,n}(t, s)| \leq \varepsilon \left(\frac{k}{2^{n-1}} + 1 \right), \quad k = 0, \dots, 2^n - 1, \quad n \in \mathbb{N}.$$

Hence

$$|t_{k,n} - \xi_{k,n}(t, s)| \leq 3\varepsilon, \quad k = 0, \dots, 2^n - 1, \quad n \in \mathbb{N}.$$

Let $\varepsilon_n := 1/(3 \cdot 2^{2n+2})$ for $n \in \mathbb{N}$. Then we get that $\xi_{k,n}(t, s) \in \Delta_{k,n}$ for $k = 0, \dots, 2^n - 1$, $n \in \mathbb{N}$, $s \in (0, \varepsilon_n)$ and for $t \in (1 - \varepsilon_n, 1)$.

Now let

$$S_n(t, s; q) := \frac{t-s}{n} \sum_{k=0}^{n-1} q(s + k \frac{t-s}{n}), \quad n \in \mathbb{N}, \quad (t, s) \in \Delta.$$

We consider

$$S_{2^n}(t, s; q) = \frac{t-s}{n} \sum_{k=0}^{2^n-1} q(s + k \frac{t-s}{2^n}) = \frac{t-s}{n} \sum_{k=0}^{2^n-1} q(\xi_{k,n}(t, s)),$$

$n \in \mathbb{N}$, $(t, s) \in \Delta$. If $s \in (0, \varepsilon_n)$ and $t \in (1 - \varepsilon_n, 1)$, then $S_{2^n}(t, s; q) = 0$, $n \in \mathbb{N}$ and

$$\left| \int_s^t q(y) dy - S_{2^n}(t, s; q) \right| = \int_s^t q(y) dy, \quad n \in \mathbb{N},$$

for $s \in (0, \varepsilon_n)$ and $t \in (1 - \varepsilon_n, 1)$. In particular, this yields

$$\operatorname{ess\,sup}_{(t,s) \in \Delta} \left| \int_s^t q(y) dy - S_{2^n}(t, s; q) \right| \geq \operatorname{ess\,sup}_{(t,s) \in \Delta} \int_s^t q(y) dy \geq \int_{\mathcal{I}} \chi_C(y) dy \geq \frac{1}{2}.$$

Hence, we obtain

$$\limsup_{n \rightarrow \infty} \operatorname{ess\,sup}_{(t,s) \in \Delta} \left| \int_s^t q(y) dy - S_{2^n}(t, s; q) \right| \geq \frac{1}{2},$$

and applying Proposition 3.1 we finish the prove of (3.7). \square

We note that Theorem 3.5 does not exclude the convergence of the Trotter product formula for the pair $\{D_0, Q\}$ in the *strong* operator topology. Examples of this dichotomy are known for the Trotter-Kato product formula in Hilbert spaces [3]. By virtue of (3.1) and (3.7), Theorem 3.5 yields an example of this dichotomy in Banach spaces.

Acknowledgments

The preparation of the paper was supported by the European Research Council via ERC-2010-AdG no 267802 (“Analysis of Multiscale Systems Driven by Functionals”). V.A.Z. thanks WIAS for hospitality.

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