

Remarks on the operator-norm convergence of the Trotter product formula

Hagen Neidhardt, Artur Stephan and Valentin A. Zagrebnov

Abstract. We revise the operator-norm convergence of the Trotter product formula for a pair $\{A, B\}$ of generators of semigroups on a Banach space. Operator-norm convergence holds true if the dominating operator A generates a holomorphic contraction semigroup and B is a A -infinitesimally small generator of a contraction semigroup, in particular, if B is a bounded operator. Inspired by studies of evolution semigroups it is shown in the present paper that the operator-norm convergence generally fails even for bounded operators B if A is not a holomorphic generator. Moreover, it is shown that operator norm convergence of the Trotter product formula can be arbitrary slow.

Keywords. Semigroups, bounded perturbations, Trotter product formula, Darboux-Riemann sums, operator-norm convergence.

1. Introduction and main results

Recall that the product formula

$$e^{-\tau C} = \lim_{n \rightarrow \infty} \left(e^{-\tau A/n} e^{-\tau B/n} \right)^n, \quad \tau \geq 0,$$

was established by S. Lie (in 1875) for matrices where $C := A + B$. The proof is based on the telescopic representation

$$\begin{aligned} & \left(e^{-\tau A/n} e^{-\tau B/n} \right)^n - e^{-\tau C} \\ &= \sum_{k=0}^{n-1} \left(e^{-\tau A/n} e^{-\tau B/n} \right)^{n-1-k} \left(e^{-\tau A/n} e^{-\tau B/n} - e^{-\tau C/n} \right) e^{-k\tau C/n}, \end{aligned} \quad (1.1)$$

$n \in \mathbb{N}$, and expansion

$$e^{-\tau X} = I - \tau X + O(\tau^2), \quad \tau \rightarrow 0,$$

for a matrix X in the operator-norm topology $\|\cdot\|$. Indeed, using this expansion one obtains the estimate:

$$\|e^{-\tau A/n} e^{-\tau B/n} - e^{-\tau C/n}\| = O((\tau/n)^2).$$

Then from (1.1) we get the existence of a constant $c_0 > 0$ such that the following estimate holds

$$\begin{aligned} & \left\| (e^{-\tau A/n} e^{-\tau B/n})^n - e^{-\tau C} \right\| \\ & \leq c_0 \frac{\tau^2}{n^2} \sum_{k=0}^{n-1} e^{\tau \frac{n-1-k}{n} \tau \|A\|} e^{\tau \frac{n-1-k}{n} \tau \|B\|} e^{\tau \frac{k}{n} \|C\|} . \end{aligned}$$

Since $\|C\| \leq \|A\| + \|B\|$, one obtains inequality

$$\begin{aligned} & \left\| (e^{-\tau A/n} e^{-\tau B/n})^n - e^{-\tau C} \right\| \\ & \leq c_0 \frac{\tau^2}{n^2} \sum_{k=0}^{n-1} e^{\tau \frac{n-1}{n} (\|A\| + \|B\|)} \leq c_0 \frac{\tau^2}{n} e^{\tau (\|A\| + \|B\|)} , \end{aligned}$$

which yields that

$$\sup_{\tau \in [0, T]} \left\| (e^{-\tau A/n} e^{-\tau B/n})^n - e^{-\tau C} \right\| = O(1/n) , \quad (1.2)$$

as $n \rightarrow \infty$ for any $T > 0$. Note that this proof carries through verbatim for bounded operators A and B on Banach spaces.

H. Trotter [7] has extended this result to unbounded operators A and B on Banach spaces, but in the strong operator topology. He proved that if A and B are generators of contractions semigroups on a separable Banach space such that the algebraic sum $A + B$ is a densely defined closable operator and the closure $C = \overline{A + B}$ is a generator of a contraction semigroup, then

$$e^{-\tau C} = \text{s-}\lim_{n \rightarrow \infty} (e^{-\tau A/n} e^{-\tau B/n})^n , \quad (1.3)$$

uniformly in $\tau \in [0, T]$ for any $T > 0$. It is obvious that this result holds if B is a bounded operator.

Considering the Trotter product formula on a Hilbert space \mathbb{T} . Kato has shown in [4] that for non-negative operators A and B the Trotter formula (1.3) holds in the *strong* operator topology if $\text{dom}(\sqrt{A}) \cap \text{dom}(\sqrt{B})$ is dense in the Hilbert space and $C = A \dot{+} B$ is the form-sum of operators A and B . Later on it was shown in [3] that the relation (1.2) holds if the algebraic sum $C = A + B$ is already a self-adjoint operator. Therefore, (1.2) is valid in particular, if B is a bounded self-adjoint operator.

The historically first result concerning the operator-norm convergence of the Trotter formula in a Banach space is due to [1]. Since the concept of self-adjointness is missing for Banach spaces it was assumed that the *dominating* operator A is a generator of a *contraction holomorphic* semigroup and B is a generator of a contraction semigroup. In Theorem 3.6 of [1] it was shown that if $0 \in \rho(A)$ and if there is a $\alpha \in [0, 1)$ such that $\text{dom}(A^\alpha) \subseteq \text{dom}(B)$ and $\text{dom}(A^*) \subseteq \text{dom}(B^*)$, then for any $T > 0$ one has

$$\sup_{\tau \in [0, T]} \left\| (e^{-\tau A/n} e^{-\tau B/n})^n - e^{-\tau C} \right\| = O(\ln(n)/n^{1-\alpha}) . \quad (1.4)$$

Note that the assumption $0 \in \rho(A)$ was made for simplicity and that the assumption $\text{dom}(A^\alpha) \subseteq \text{dom}(B)$ yields that the operator B is infinitesimally

small with respect to A . Taking into account [5, Corollary IX.2.5] one gets that the well-defined algebraic sum $C = A + B$ is a generator of a contraction holomorphic semigroup. By Theorem 3.6 of [1] the convergence rate (1.4) improves if B is a bounded operator, i.e. $\alpha = 0$. Then for any $T > 0$ one gets

$$\sup_{\tau \in [0, T]} \left\| \left(e^{-\tau A/n} e^{-\tau B/n} \right)^n - e^{-\tau C} \right\| = O((\ln(n))^2/n).$$

Summarizing, the question arises whether the Trotter product formula converges in the operator-norm if A is a generator of a contraction (but not holomorphic) semigroup and B is a bounded operator? The aim of the present paper is to give an answer to this question for a certain class of generators.

It turns out that an appropriate class for that is the class of generators of *evolution* semigroups. To proceed further we need the notion of a *propagator*, or a *solution operator* [6].

A strongly continuous map $U(\cdot, \cdot) : \Delta \rightarrow \mathcal{B}(X)$, where $\Delta := \{(t, s) : 0 < s \leq t \leq T\}$ and $\mathcal{B}(X)$ is the set of bounded operators on the separable Banach space X , is called a *propagator* if the conditions

- (i) $\sup_{(t,s) \in \Delta} \|U(t, s)\|_{\mathcal{B}(X)} < \infty$,
- (ii) $U(t, s) = U(t, r)U(r, s)$, $0 < s \leq r \leq t \leq T$,

are satisfied. Let us consider the Banach space $L^p(\mathcal{I}, X)$, $\mathcal{I} := [0, T]$, $p \in [1, \infty)$. The operator \mathcal{K} is an evolution generator of the evolution semigroup $\{e^{-\tau \mathcal{K}}\}_{\tau \geq 0}$ if there is a propagator such that the representation

$$(e^{-\tau \mathcal{K}} f)(t) = U(t, t - \tau) \chi_{\mathcal{I}}(t - \tau) f(t - \tau), \quad f \in L^p(\mathcal{I}, X), \quad (1.5)$$

holds for a.e. $t \in \mathcal{I}$ and $\tau \geq 0$ [6]. Since $e^{-\tau \mathcal{K}} f = 0$ for $\tau \geq T$, the evolution generator \mathcal{K} can never be a generator of a holomorphic semigroup.

A simple example of an evolution generator is the differentiation operator:

$$\begin{aligned} (D_0 f)(t) &:= \partial_t f(t), \\ f \in \text{dom}(D_0) &:= \{f \in H^{1,p}(\mathcal{I}, X) : f(0) = 0\}. \end{aligned} \quad (1.6)$$

Then by (1.6) one obviously gets the contraction shift semigroup:

$$(e^{-\tau D_0} f)(t) = \chi_{\mathcal{I}}(t - \tau) f(t - \tau), \quad f \in L^p(\mathcal{I}, X), \quad (1.7)$$

for a.e. $t \in \mathcal{I}$ and $\tau \geq 0$. Hence, (1.5) implies that the corresponding propagator of the non-holomorphic evolution semigroup $\{e^{-\tau D_0}\}_{\tau \geq 0}$ is given by $U_{D_0}(t, s) = I$, $(t, s) \in \Delta$.

Note that in [6] we considered the operator $\mathcal{K}_0 := \overline{D_0 + \mathcal{A}}$, where \mathcal{A} is the multiplication operator induced by a generator A of a holomorphic contraction semigroup on X . More precisely

$$\begin{aligned} (\mathcal{A} f)(t) &:= A f(t), \text{ and } (e^{-\tau \mathcal{A}} f)(t) = e^{-\tau A} f(t), \\ f \in \text{dom}(\mathcal{A}) &:= \{f \in L^p(\mathcal{I}, X) : A f(\cdot) \in L^p(\mathcal{I}, X)\}. \end{aligned}$$

Then the perturbation of the shift semigroup (1.7) by \mathcal{A} corresponds to the semigroup with generator \mathcal{K}_0 . One easily checks that \mathcal{K}_0 is an evolution generator of a contraction semigroup on $L^p(\mathcal{I}, X)$ that is never holomorphic. Indeed, since the generators D_0 and \mathcal{A} commute, the representation (1.5) for evolution semigroup $\{e^{-\tau\mathcal{K}_0}\}_{\tau \geq 0}$ takes the form:

$$(e^{-\tau\mathcal{K}_0} f)(t) = e^{-\tau A} \chi_{\mathcal{I}}(t - \tau) f(t - \tau), \quad f \in L^p(\mathcal{I}, X),$$

for a.e. $t \in \mathcal{I}$ and $\tau \geq 0$ with propagator $U_0(t, s) = e^{-(t-s)A}$. Therefore, again $e^{-\tau\mathcal{K}_0} f = 0$ for $\tau \geq T$.

Furthermore, if $B(\cdot)$ is a *strongly measurable* family of generators of contraction semigroups on X , i.e. $B(\cdot) : \mathcal{I} \rightarrow \mathcal{G}(1, 0)$ (see [4], Ch.IX, §1.4), then the induced multiplication operator \mathcal{B} :

$$(\mathcal{B}f)(t) := B(t)f(t), \quad (1.8)$$

$$f \in \text{dom}(\mathcal{B}) := \left\{ f \in L^p(\mathcal{I}, X) : \begin{array}{l} f(t) \in \text{dom}(B(t)) \text{ for a.e. } t \in \mathcal{I} \\ B(t)f(t) \in L^p(\mathcal{I}, X) \end{array} \right\},$$

is a generator of a contraction semigroup on $L^p(\mathcal{I}, X)$.

In [6] it was assumed that $\{B(t)\}_{t \in \mathcal{I}}$ is a strongly measurable family of generators of contraction semigroups and that A is a generator of a bounded holomorphic semigroup with $0 \in \rho(A)$ for simplicity. Moreover, we supposed that the following conditions are satisfied:

- (i) $\text{dom}(A^\alpha) \subseteq \text{dom}(B(t))$ for a.e. $t \in \mathcal{I}$ and some $\alpha \in (0, 1)$ such that

$$\text{ess sup}_{t \in \mathcal{I}} \|B(t)A^{-\alpha}\|_{\mathcal{B}(X)} < \infty;$$

- (ii) $\text{dom}(A^*) \subseteq \text{dom}(B(t)^*)$ for a.e. $t \in \mathcal{I}$ such that

$$\text{ess sup}_{t \in \mathcal{I}} \|B(t)^*(A^{-1})^*\|_{\mathcal{B}(X)} < \infty;$$

- (iii) there is a $\beta \in (\alpha, 1)$ and $L_\beta > 0$ such that

$$\|A^{-1}(B(t) - B(s))A^{-\alpha}\|_{\mathcal{B}(X)} \leq L_\beta |t - s|^\beta, \quad t, s \in \mathcal{I}. \quad (1.9)$$

Under these assumptions it turns out that $\mathcal{K} := \mathcal{K}_0 + \mathcal{B}$ is a generator of a contraction evolution semigroup, i.e there is a propagator $\{U(t, s)\}_{(t, s) \in \Delta}$ such that the representation (1.5) is valid. Moreover, we prove in [6] the Trotter product formula converges in the operator norm with convergence rate $O(1/n^{\beta-\alpha})$:

$$\sup_{\tau \geq 0} \left\| \left(e^{-\tau\mathcal{K}_0/n} e^{-\tau\mathcal{B}/n} \right)^n - e^{-\tau\mathcal{K}} \right\|_{\mathcal{B}(L^p(\mathcal{I}, X))} = O(1/n^{\beta-\alpha}).$$

We comment that if $B(\cdot) : \mathcal{I} \rightarrow \mathcal{B}(X)$ is a Hölder continuous function with Hölder exponent $\beta \in (0, 1)$, then the assumptions (i)-(iii) are satisfied for any $\alpha \in (0, \beta)$. Then our results [6] yield that

$$\sup_{\tau \geq 0} \left\| \left(e^{-\tau\mathcal{K}_0/n} e^{-\tau\mathcal{B}/n} \right)^n - e^{-\tau\mathcal{K}} \right\|_{\mathcal{B}(L^p(\mathcal{I}, X))} = O(1/n^\gamma), \quad (1.10)$$

holds for any $\gamma \in (0, \beta)$. Moreover, in this case the perturbation of the shift semigroup (1.7) by a bounded generator (1.8) gives an evolution semigroup

with generator $D_0 + \mathcal{B}$. Then as a corollary of (1.10) for $\mathcal{A} = 0$, we get the Trotter product estimate

$$\sup_{\tau \geq 0} \left\| \left(e^{-\tau D_0/n} e^{-\tau \mathcal{B}/n} \right)^n - e^{-\tau(D_0 + \mathcal{B})} \right\|_{\mathcal{B}(L^p(\mathcal{I}, X))} = O(1/n^\gamma). \quad (1.11)$$

The aim of our note is to show that the convergence rate (1.11) is close to the optimal one. To this end we consider the simple case, when $X = \mathbb{C}$ and we put for simplicity $\mathcal{I} := [0, 1]$.

The *main results* of this paper can be summarized as follows:

If the operator \mathcal{B} is equal to the multiplication operator Q induced by a bounded measurable function $q(\cdot) : \mathcal{I} \rightarrow \mathbb{C}$ in $L^p(\mathcal{I})$, then one can verify that the condition (1.9) is equivalent to $q(\cdot) \in C^{0,\beta}(\mathcal{I})$, see definition below. In this case the convergence rate is

$$\sup_{\tau \geq 0} \left\| e^{-\tau(D_0 + Q)} - \left(e^{-\tau D_0/n} e^{-\tau Q/n} \right)^n \right\|_{\mathcal{B}(L^p(\mathcal{I}, X))} = O(1/n^\beta). \quad (1.12)$$

This result remains true if $q(\cdot)$ is Lipschitz continuous, i.e. $\beta = 1$. But if $q(\cdot)$ is *only* continuous, then

$$\sup_{\tau \geq 0} \left\| e^{-\tau(D_0 + Q)} - \left(e^{-\tau D_0/n} e^{-\tau Q/n} \right)^n \right\|_{\mathcal{B}(L^p(\mathcal{I}, X))} = o(1). \quad (1.13)$$

Moreover, for any convergent to zero sequence $\delta_n > 0$, $n \in \mathbb{N}$, there exists a continuous function $q(\cdot)$ such that

$$\sup_{\tau \geq 0} \left\| e^{-\tau(D_0 + Q)} - \left(e^{-\tau D_0/n} e^{-\tau Q/n} \right)^n \right\|_{\mathcal{B}(L^p(\mathcal{I}, X))} = \omega(\delta_n), \quad (1.14)$$

where the Landau *symbol* $\omega(\cdot)$ is defined below.

Finally, there is an example of a bounded measurable function $q(\cdot)$ such that

$$\limsup_{n \rightarrow \infty} \sup_{\tau \geq 0} \left\| e^{-\tau(D_0 + Q)} - \left(e^{-\tau D_0/n} e^{-\tau Q/n} \right)^n \right\|_{\mathcal{B}(L^p(\mathcal{I}, X))} > 0. \quad (1.15)$$

Hence, in contrast to the holomorphic case, when the *dominating* operator is a generator of a holomorphic semigroup (1.4), the Trotter product formula (1.15) with dominating generator D_0 , may *not* converge in the operator-norm.

The paper is organized as follows. In Section 2 we reformulate the convergence of the Trotter product formula in terms of the corresponding evolutions semigroups. In Section 3 we prove the results (1.12)-(1.15).

We conclude this section by few remarks concerning **notation** used in this paper.

1. We use a definition of the *generator* C of a semigroup (1.3), which differs from the standard one by a *minus* [5].

2. Furthermore, we widely use the so-called *Landau symbols*:

$$g(n) = O(f(n)) \iff \limsup_{n \rightarrow \infty} \left| \frac{g(n)}{f(n)} \right| < \infty ,$$

$$g(n) = o(f(n)) \iff \limsup_{n \rightarrow \infty} \left| \frac{g(n)}{f(n)} \right| = 0 ,$$

$$g(n) = \Theta(f(n)) \iff 0 < \liminf_{n \rightarrow \infty} \left| \frac{g(n)}{f(n)} \right| \leq \limsup_{n \rightarrow \infty} \left| \frac{g(n)}{f(n)} \right| < \infty ,$$

$$g(n) = \omega(f(n)) \iff \limsup_{n \rightarrow \infty} \left| \frac{g(n)}{f(n)} \right| = \infty .$$

3. We use the notation $C^{0,\beta}(\mathcal{I}) = \{f : \mathcal{I} \rightarrow \mathbb{C} : \text{there is some } K > 0 \text{ such that } |f(x) - f(y)| \leq K|x - y|^\beta\}$ for $\beta \in (0, 1]$.

2. Trotter product formula and evolution semigroups

Below we consider the Banach space $L^p(\mathcal{I}, X)$ for $\mathcal{I} := [0, T]$, $p \in [1, \infty)$. Recall that semigroup $\{\mathcal{U}(\tau)\}_{\tau \geq 0}$, on the Banach space $L^p(\mathcal{I}, X)$ is called an *evolution semigroup* if there is a propagator $\{U(t, s)\}_{(t,s) \in \Delta}$ such that the representation (1.5) holds.

Let \mathcal{K}_0 be the generator of an evolution semigroup $\{\mathcal{U}_0(\tau)\}_{\tau \geq 0}$ and let \mathcal{B} be a multiplication operator induced by a measurable family $\{B(t)\}_{t \in \mathcal{I}}$ of generators of contraction semigroups. Note that in this case the multiplication operator \mathcal{B} (1.8) is a generator of a contraction semigroup $(e^{-\tau \mathcal{B}} f)(t) = e^{-\tau B(t)} f(t)$, on the Banach space $L^p(\mathcal{I}, X)$. Since $\{\mathcal{U}_0(\tau)\}_{\tau \geq 0}$ is an evolution semigroup, then by definition (1.5) there is a propagator $\{U_0(t, s)\}_{(t,s) \in \Delta}$ such that the representation

$$(\mathcal{U}_0(\tau)f)(t) = U_0(t, t - \tau)\chi_{\mathcal{I}}(t - \tau)f(t - \tau), \quad f \in L^p(\mathcal{I}, X),$$

is valid for a.e. $t \in \mathcal{I}$ and $\tau \geq 0$. Then we define

$$G_j(t, s; n) := U_0\left(s + j \frac{(t-s)}{n}, s + (j-1) \frac{(t-s)}{n}\right) e^{-\frac{(t-s)}{n} B\left(s + (j-1) \frac{(t-s)}{n}\right)}$$

where $j \in \{1, 2, \dots, n\}$, $n \in \mathbb{N}$, $(t, s) \in \Delta$, and we set

$$V_n(t, s) := \prod_{j=1}^{n \leftarrow} G_j(t, s; n), \quad n \in \mathbb{N}, \quad (t, s) \in \Delta,$$

where the product is increasingly ordered in j from the right to the left. Then a straightforward computation shows that the representation

$$\left(\left(e^{-\tau \mathcal{K}_0/n} e^{-\tau \mathcal{B}/n} \right)^n f \right) (t) = V_n(t, t - \tau)\chi_{\mathcal{I}}(t - \tau)f(t - \tau), \quad (2.1)$$

$f \in L^p(\mathcal{I}, X)$, holds for each $\tau \geq 0$ and a.e. $t \in \mathcal{I}$.

Proposition 2.1. *Let \mathcal{K} and \mathcal{K}_0 be generators of evolution semigroups on the Banach space $L^p(\mathcal{I}, X)$ for some $p \in [1, \infty)$. Further, let $\{B(t) \in \mathcal{G}(1, 0)\}_{t \in \mathcal{I}}$*

be a strongly measurable family of generators of contraction on X semigroups. Then

$$\begin{aligned} & \sup_{\tau \geq 0} \left\| e^{-\tau\mathcal{K}} - \left(e^{-\tau\mathcal{K}_0/n} e^{-\tau\mathcal{B}/n} \right)^n \right\|_{\mathcal{B}(L^p(\mathcal{I}, X))} \\ &= \text{ess sup}_{(t,s) \in \Delta} \|U(t,s) - V_n(t,s)\|_{\mathcal{B}(X)}, \quad n \in \mathbb{N}. \end{aligned} \quad (2.2)$$

Proof. Let $\{L(\tau)\}_{\tau \geq 0}$ be the left-shift semigroup on the Banach space $\mathfrak{X} = L^p(\mathcal{I}, X)$:

$$(L(\tau)f)(t) = \chi_{\mathcal{I}}(t + \tau)f(t + \tau), \quad f \in L^p(\mathcal{I}, X).$$

Using that we get

$$\begin{aligned} & \left(L(\tau) \left(e^{-\tau\mathcal{K}} - \left(e^{-\tau/n\mathcal{K}_0} e^{-\tau\mathcal{B}/n} \right)^n \right) f \right) (t) \\ &= \{U(t + \tau, t) - V_n(t + \tau, t)\} \chi_{\mathcal{I}}(t + \tau) f(t), \end{aligned}$$

for $\tau \geq 0$ and a.e. $t \in \mathcal{I}$. It turns out that for each $n \in \mathbb{N}$ the operator $L(\tau) \left(e^{-\tau\mathcal{K}} - \left(e^{-\tau/n\mathcal{K}_0} e^{-\tau\mathcal{B}/n} \right)^n \right)$ is a multiplication operator induced by $\{(U(t + \tau, t) - V_n(t + \tau, t))\chi_{\mathcal{I}}(t + \tau)\}_{t \in \mathcal{I}}$. Therefore,

$$\begin{aligned} & \left\| L(\tau) \left(e^{-\tau\mathcal{K}} - \left(e^{-\tau\mathcal{K}_0/n} e^{-\tau\mathcal{B}/n} \right)^n \right) \right\|_{\mathcal{B}(\mathfrak{X})} \\ &= \text{ess sup}_{t \in \mathcal{I}} \|U(t + \tau, t) - V_n(t + \tau, t)\|_{\mathcal{B}(X)} \chi_{\mathcal{I}}(t + \tau), \end{aligned}$$

for each $\tau \geq 0$. Note that one has

$$\begin{aligned} & \sup_{\tau \geq 0} \left\| L(\tau) \left(e^{-\tau\mathcal{K}} - \left(e^{-\tau\mathcal{K}_0/n} e^{-\tau\mathcal{B}/n} \right)^n \right) \right\|_{\mathcal{B}(\mathfrak{X})} \\ &= \text{ess sup}_{\tau \geq 0} \left\| L(\tau) \left(e^{-\tau\mathcal{K}} - \left(e^{-\tau\mathcal{K}_0/n} e^{-\tau\mathcal{B}/n} \right)^n \right) \right\|_{\mathcal{B}(\mathfrak{X})}. \end{aligned}$$

This is based on the fact that if $F(\cdot) : \mathbb{R}_+ \rightarrow \mathcal{B}(\mathfrak{X})$ is strongly continuous, then $\sup_{\tau \geq 0} \|F(\tau)\|_{\mathcal{B}(\mathfrak{X})} = \text{ess sup}_{\tau \geq 0} \|F(\tau)\|_{\mathcal{B}(\mathfrak{X})}$. Hence, we find

$$\begin{aligned} & \sup_{\tau \geq 0} \left\| L(\tau) \left(e^{-\tau\mathcal{K}} - \left(e^{-\tau\mathcal{K}_0/n} e^{-\tau\mathcal{B}/n} \right)^n \right) \right\|_{\mathcal{B}(\mathfrak{X})} \\ &= \text{ess sup}_{\tau \geq 0} \text{ess sup}_{t \in \mathcal{I}} \|U(t + \tau, t) - V_n(t + \tau, t)\|_{\mathcal{B}(X)} \chi_{\mathcal{I}}(t + \tau). \end{aligned}$$

Further, if $\Phi(\cdot, \cdot) : \mathbb{R}_+ \times \mathcal{I} \rightarrow \mathcal{B}(X)$ is a strongly measurable function, then

$$\text{ess sup}_{(\tau,t) \in \mathbb{R}_+ \times \mathcal{I}} \|\Phi(\tau, t)\|_{\mathcal{B}(X)} = \text{ess sup}_{\tau \geq 0} \text{ess sup}_{t \in \mathcal{I}} \|\Phi(\tau, t)\|_{\mathcal{B}(X)}.$$

Then, taking into account two last equalities, one obtains

$$\begin{aligned} & \sup_{\tau \geq 0} \left\| L(\tau) \left(e^{-\tau\mathcal{K}} - \left(e^{-\tau\mathcal{K}_0/n} e^{-\tau\mathcal{B}/n} \right)^n \right) \right\|_{\mathcal{B}(\mathfrak{X})} \\ &= \text{ess sup}_{(\tau,t) \in \mathbb{R}_+ \times \mathcal{I}} \|U(t + \tau, t) - V_n(t + \tau, t)\|_{\mathcal{B}(X)} \chi_{\mathcal{I}}(t + \tau) = \\ &= \text{ess sup}_{(t,s) \in \Delta} \|U(t,s) - V_n(t,s)\|_{\mathcal{B}(X)}, \end{aligned}$$

that proves (2.2) □

3. Bounded perturbations of the shift semigroup generator

3.1. Basic facts

We study bounded perturbations of the evolution generator D_0 (1.6). To do this aim we consider $\mathcal{I} = [0, 1]$, $X = \mathbb{C}$ and we denote by $L^p(\mathcal{I})$ the Banach space $L^p(\mathcal{I}, \mathbb{C})$.

For $t \in \mathcal{I}$, let $q : t \mapsto q(t) \in L^\infty(\mathcal{I})$. Then, q induces a bounded multiplication operator Q on the Banach space $L^p(\mathcal{I})$:

$$(Qf)(t) = q(t)f(t), \quad f \in L^p(\mathcal{I}).$$

For simplicity we assume that $q \geq 0$. Then Q generates on $L^p(\mathcal{I})$ a contraction semigroup $\{e^{-\tau Q}\}_{\tau \geq 0}$. Since generator Q is bounded, the closed operator $\mathcal{A} := D_0 + Q$, with domain $\text{dom}(\mathcal{A}) = \text{dom}(D_0)$, is generator of a semigroup on $L^p(\mathcal{I})$. By [7], the Trotter product formula in the strong topology follows immediately

$$\left(e^{-\tau D_0/n} e^{-\tau Q/n} \right)^n f \rightarrow e^{-\tau(D_0+Q)} f, \quad f \in L^p(\mathcal{I}), \quad (3.1)$$

uniformly in $\tau \in [0, T]$ on bounded time intervals.

Following [2, §5], we define on $X = \mathbb{C}$ a family of bounded operators $\{V(t)\}_{t \in \mathcal{I}}$ by

$$V(t) := e^{-\int_0^t ds q(s)}.$$

Note that for almost every $t \in \mathcal{I}$ these operators are positive. Then $V^{-1}(t)$ exists and it has the form

$$V^{-1}(t) = e^{\int_0^t ds q(s)}.$$

The operator families $\{V(t)\}_{t \in \mathcal{I}}$ and $\{V^{-1}(t)\}_{t \in \mathcal{I}}$ induce two bounded multiplication operators \mathcal{V} and \mathcal{V}^{-1} on $L^p(\mathcal{I})$, respectively. Then invertibility implies that $\mathcal{V} \mathcal{V}^{-1} = \mathcal{V}^{-1} \mathcal{V} = \text{Id}|_{L^p}$. Using the operator \mathcal{V} one easily verifies that $D_0 + Q$ is similar to D_0 , i.e. one has

$$\mathcal{V}^{-1}(D_0 + Q)\mathcal{V} = D_0, \quad \text{or} \quad D_0 + Q = \mathcal{V}D_0\mathcal{V}^{-1}.$$

Hence, the semigroup generated on $L^p(\mathcal{I})$ by $D_0 + Q$ gets the explicit form:

$$\begin{aligned} \left(e^{-\tau(D_0+Q)} f \right) (t) &= (\mathcal{V}e^{-\tau D_0}\mathcal{V}^{-1} f) (t) = \\ &= e^{-\int_{t-\tau}^t q(y)dy} f(t-\tau) \chi_{\mathcal{I}}(t-\tau). \end{aligned} \quad (3.2)$$

Since by (1.5) the propagator $U(t, s)$ that corresponds to evolution semigroup (3.2) is defined by

$$\left(e^{-\tau(D_0+Q)} \right) f(t) = U(t, t-\tau) f(t-\tau) \chi_{\mathcal{I}}(t-\tau),$$

we deduce that it is equal to $U(t, s) = e^{-\int_s^t dy q(y)}$.

Now we study the corresponding Trotter product formula. For a fixed $\tau \geq 0$ and $n \in \mathbb{N}$, we define approximation V_n by

$$\left(\left(e^{-\tau D_0/n} e^{-\tau Q/n} \right)^n f \right) (t) =: V_n(t, t-\tau) \chi_{\mathcal{I}}(t-\tau) f(t-\tau).$$

Then by straightforward calculations, similar to (2.1), one finds that

$$V_n(t, s) = e^{-\frac{t-s}{n} \sum_{k=0}^{n-1} q(s+k\frac{t-s}{n})}, \quad (t, s) \in \Delta .$$

Proposition 3.1. *Let $q \in L^\infty(\mathcal{I})$ be non-negative. Then*

$$\begin{aligned} & \sup_{\tau \geq 0} \left\| e^{-\tau(D_0+Q)} - \left(e^{-\tau D_0/n} e^{-\tau Q/n} \right)^n \right\|_{\mathcal{B}(L^p(\mathcal{I}))} \\ &= \Theta \left(\operatorname{ess\,sup}_{(t,s) \in \Delta} \left| \int_s^t q(y) dy - \frac{t-s}{n} \sum_{k=0}^{n-1} q\left(s+k\frac{t-s}{n}\right) \right| \right) \end{aligned}$$

as $n \rightarrow \infty$, where Θ is the Landau symbol defined in Section 1.

Proof. First, by Proposition 2.1 and by $U(t, s) = e^{-\int_s^t dy q(y)}$ we obtain

$$\begin{aligned} & \sup_{\tau \geq 0} \left\| e^{-\tau(D_0+Q)} - \left(e^{-\tau D_0/n} e^{-\tau Q/n} \right)^n \right\|_{\mathcal{B}(L^p(\mathcal{I}))} \quad (3.3) \\ &= \operatorname{ess\,sup}_{(t,s) \in \Delta} \left| e^{-\int_s^t dy q(y)} - e^{-\frac{t-s}{n} \sum_{k=0}^{n-1} q(s+k\frac{t-s}{n})} \right| . \end{aligned}$$

Then, using the inequality

$$e^{-\max\{x,y\}} |x-y| \leq |e^{-x} - e^{-y}| \leq |x-y|, \quad 0 \leq x, y ,$$

for $0 \leq s < t \leq 1$ one finds the estimates

$$e^{-\|q\|_{L^\infty}} R_n(t, s; q) \leq \left| e^{-\int_s^t dy q(y)} - e^{-\frac{t-s}{n} \sum_{k=0}^{n-1} q(s+k\frac{t-s}{n})} \right| \leq R_n(t, s; q) ,$$

where

$$R_n(t, s, q) := \left| \int_s^t dy q(y) - \frac{t-s}{n} \sum_{k=0}^{n-1} q\left(s+k\frac{t-s}{n}\right) \right|, \quad (t, s) \in \Delta . \quad (3.4)$$

Hence, for the left-hand side of (3.3) we get the estimate

$$e^{-\|q\|_{L^\infty}} R_n(q) \leq \sup_{\tau \geq 0} \left\| e^{-\tau(D_0+Q)} - \left(e^{-\tau D_0/n} e^{-\tau Q/n} \right)^n \right\|_{\mathcal{B}(L^p)} \leq R_n(q) ,$$

where $R_n(q) := \operatorname{ess\,sup}_{(t,s) \in \Delta} R_n(t, s; q)$, $n \in \mathbb{N}$. These estimates together with definition of Θ prove the assertion. \square

Note that by virtue of (3.4) and Proposition 3.1 the operator-norm convergence rate of the Trotter product formula for the pair $\{D_0, Q\}$ coincides with the convergence rate of the integral Darboux-Riemann sum approximation of the Lebesgue integral.

3.2. Examples

First we consider the case of a real Hölder-continuous function $q \in C^{0,\beta}(\mathcal{I})$.

Theorem 3.2. *If $q \in C^{0,\beta}(\mathcal{I})$ is non-negative, then*

$$\sup_{\tau \geq 0} \left\| e^{-\tau(D_0+Q)} - \left(e^{-\tau D_0/n} e^{-\tau Q/n} \right)^n \right\| = O(1/n^\beta) ,$$

as $n \rightarrow \infty$.

Proof. One has

$$\begin{aligned} \int_s^t dy q(y) - \frac{t-s}{n} \sum_k^{n-1} q\left(s + \frac{k}{n}(t-s)\right) \\ = \sum_{k=0}^{n-1} \int_{\frac{k}{n}(t-s)}^{\frac{k+1}{n}(t-s)} dy \left(q(s+y) - q\left(s + \frac{k}{n}(t-s)\right) \right) , \end{aligned}$$

which yields the estimate

$$\begin{aligned} \left| \int_s^t dy q(y) - \frac{t-s}{n} \sum_k^{n-1} q\left(s + \frac{k}{n}(t-s)\right) \right| \\ \leq \sum_{k=0}^{n-1} \int_{\frac{k}{n}(t-s)}^{\frac{k+1}{n}(t-s)} dy \left| q(s+y) - q\left(s + \frac{k}{n}(t-s)\right) \right| . \end{aligned}$$

Since $q \in C^{0,\beta}(\mathcal{I})$, there is a constant $L_\beta > 0$ such that for $y \in [\frac{k}{n}(t-s), \frac{k+1}{n}(t-s)]$ one has

$$\left| q(s+y) - q\left(s + \frac{k}{n}(t-s)\right) \right| \leq L_\beta |y - \frac{k}{n}(t-s)|^\beta \leq L_\beta \frac{(t-s)^\beta}{n^\beta} .$$

Hence, we find

$$\left| \int_s^t q(y) dy - \frac{t-s}{n} \sum_k^{n-1} q\left(s + \frac{k}{n}(t-s)\right) \right| \leq L_\beta \frac{(t-s)^{1+\beta}}{n^\beta} \leq L_\beta \frac{1}{n^\beta} ,$$

which proves

$$\operatorname{ess\,sup}_{(t,s) \in \Delta} \left| \int_s^t q(y) dy - \frac{t-s}{n} \sum_k^{n-1} q\left(s + \frac{k}{n}(t-s)\right) \right| = O\left(\frac{1}{n^\beta}\right) .$$

Applying now Proposition 3.1 one completes the proof. \square

It is a natural question: what happens, when q is only continuous?

Theorem 3.3. *If $q : \mathcal{I} \rightarrow \mathbb{C}$ is continuous and non-negative, then*

$$\left\| e^{-\tau(D_0+Q)} - \left(e^{-\tau D_0/n} e^{-\tau Q/n} \right)^n \right\| = o(1) , \quad (3.5)$$

as $n \rightarrow \infty$.

Proof. Since $q(\cdot)$ is continuous, then for any $\varepsilon > 0$ there is $\delta > 0$ such that for $|y-x| < \delta$ we have $|q(y) - q(x)| < \varepsilon$, $y, x \in \mathcal{I}$. Therefore, if $1/n < \delta$, then for $y \in (\frac{k}{n}(t-s), \frac{k+1}{n}(t-s))$ we have

$$\left| q(s+y) - q\left(s + \frac{k}{n}(t-s)\right) \right| < \varepsilon , \quad (t, s) \in \Delta .$$

Hence,

$$\left| \int_s^t q(y) dy - \frac{t-s}{n} \sum_k^{n-1} q\left(s + \frac{k}{n}(t-s)\right) \right| \leq \varepsilon(t-s) \leq \varepsilon ,$$

which yields

$$\operatorname{ess\,sup}_{(t,s) \in \Delta} \left| \int_s^t q(y) dy - \frac{t-s}{n} \sum_k^{n-1} q\left(s + \frac{k}{n}(t-s)\right) \right| = o(1) .$$

Now it remains only to apply Proposition 3.1. □

We comment that for a general continuous q one can say nothing about the convergence rate. Indeed, it can be shown that in (3.5) the convergence to zero can be arbitrary slow.

Theorem 3.4. *Let $\delta_n > 0$ be a sequence with $\delta_n \rightarrow 0$ as $n \rightarrow \infty$. Then there exists a continuous function $q : \mathcal{I} = [0, 1] \rightarrow \mathbb{R}$ such that*

$$\sup_{\tau \geq 0} \left\| e^{-\tau(D_0+Q)} - \left(e^{-\tau D_0/n} e^{-\tau Q/n} \right)^n \right\|_{\mathcal{B}(L^p(\mathcal{I}))} = \omega(\delta_n) \quad (3.6)$$

as $n \rightarrow \infty$, where ω is the Landau symbol defined in Section 1.

Proof. Taking into account Theorem 6 of [8], we find that for any sequence $\{\delta_n\}_{n \in \mathbb{N}}$, $\delta_n > 0$ satisfying $\lim_{n \rightarrow \infty} \delta_n = 0$ there exists a continuous function $f(\cdot) : [0, 2\pi] \rightarrow \mathbb{R}$ such that

$$\left| \int_0^{2\pi} f(x) dx - \frac{2\pi}{n} \sum_{k=1}^n f(2k\pi/n) \right| = \omega(\delta_n) ,$$

as $n \rightarrow \infty$. Setting $q(y) := f(2\pi(1-y))$, $y \in [0, 1]$, we get a continuous function $q(\cdot) : [0, 1] \rightarrow \mathbb{R}$, such that

$$\left| \int_0^1 q(y) dy - \frac{1}{n} \sum_{k=0}^{n-1} q(k/n) \right| = \omega(\delta_n) .$$

Because $q(\cdot)$ is continuous we find

$$\begin{aligned} \operatorname{ess\,sup}_{(t,s) \in \Delta} \left| \int_s^t q(y) dy - \frac{t-s}{n} \sum_{n=0}^{n-1} q\left(s + k \frac{t-s}{n}\right) \right| \\ \geq \left| \int_0^1 q(y) dy - \frac{1}{n} \sum_{k=0}^{n-1} q(k/n) \right| , \end{aligned}$$

which yields

$$\operatorname{ess\,sup}_{(t,s) \in \Delta} \left| \int_s^t q(y) dy - \frac{t-s}{n} \sum_{n=0}^{n-1} q\left(s + k \frac{t-s}{n}\right) \right| = \omega(\delta_n) .$$

Applying now Proposition 3.1 we prove (3.6). □

Our final comment concerns the case when q is only *measurable*. Then it can happen that the Trotter product formula for that pair $\{D_0, Q\}$ does not converge in the operator-norm topology.

Theorem 3.5. *There is a non-negative function $q \in L^\infty([0, 1])$ such that*

$$\limsup_{n \rightarrow \infty} \sup_{\tau \geq 0} \left\| e^{-\tau(D_0+Q)} - \left(e^{-\tau D_0/n} e^{-\tau Q/n} \right)^n \right\|_{B(L^p(\mathcal{I}))} > 0. \quad (3.7)$$

Proof. Let us introduce the open intervals

$$\begin{aligned} \Delta_{0,n} &:= \left(0, \frac{1}{2^{2n+2}}\right), \\ \Delta_{k,n} &:= \left(t_{k,n} - \frac{1}{2^{2n+2}}, t_{k,n} + \frac{1}{2^{2n+2}}\right), \quad k = 1, 2, \dots, 2^n - 1, \\ \Delta_{2^n,n} &:= \left(1 - \frac{1}{2^{2n+2}}, 1\right), \end{aligned}$$

$n \in \mathbb{N}$, where

$$t_{k,n} = \frac{k}{2^n}, \quad k = 0, \dots, n, \quad n \in \mathbb{N}.$$

Notice that $t_{0,n} = 0$ and $t_{2^n,n} = 1$. One easily checks that the intervals $\Delta_{k,n}$, $k = 0, \dots, 2^n$, are mutually disjoint. We introduce the open sets

$$\mathcal{O}_n = \bigcup_{k=0}^{2^n} \Delta_{k,n} \subseteq \mathcal{I}, \quad n \in \mathbb{N}.$$

and

$$\mathcal{O} = \bigcup_{n \in \mathbb{N}} \mathcal{O}_n \subseteq \mathcal{I}.$$

Then it is clear that

$$|\mathcal{O}_n| = \frac{1}{2^{n+1}}, \quad n \in \mathbb{N}, \quad \text{and} \quad |\mathcal{O}| \leq \frac{1}{2}.$$

Therefore, the Lebesgue measure of the closed set $\mathcal{C} := \mathcal{I} \setminus \mathcal{O} \subseteq \mathcal{I}$ can be estimated by

$$|\mathcal{C}| \geq \frac{1}{2}.$$

Using the characteristic function $\chi_{\mathcal{C}}(\cdot)$ of the set \mathcal{C} we define

$$q(t) := \chi_{\mathcal{C}}(t), \quad t \in \mathcal{I}.$$

The function $q(\cdot)$ is measurable and it satisfies $0 \leq q(t) \leq 1$, $t \in \mathcal{I}$.

Let $\varepsilon \in (0, 1)$. We choose $s \in (0, \varepsilon)$ and $t \in (1 - \varepsilon, 1)$ and we set

$$\xi_{k,n}(t, s) := s + k \frac{t-s}{2^n}, \quad k = 0, \dots, 2^n - 1, \quad n \in \mathbb{N}, \quad (t, s) \in \Delta.$$

Note that $\xi_{k,n}(t, s) \in (0, 1)$, $k = 0, \dots, 2^n - 1$, $n \in \mathbb{N}$. Moreover, we have

$$t_{k,n} - \xi_{k,n}(t, s) = k \frac{1}{2^n} - s - k \frac{t-s}{2^n} = k \frac{1-t+s}{2^n} - s,$$

which leads to the estimate

$$|t_{k,n} - \xi_{k,n}(t, s)| \leq \varepsilon \left(\frac{k}{2^{n-1}} + 1 \right), \quad k = 0, \dots, 2^n - 1, \quad n \in \mathbb{N}.$$

Hence

$$|t_{k,n} - \xi_{k,n}(t, s)| \leq 3\varepsilon, \quad k = 0, \dots, 2^n - 1, \quad n \in \mathbb{N}.$$

Let $\varepsilon_n := 1/(3 \cdot 2^{2n+2})$ for $n \in \mathbb{N}$. Then we get that $\xi_{k,n}(t, s) \in \Delta_{k,n}$ for $k = 0, \dots, 2^n - 1$, $n \in \mathbb{N}$, $s \in (0, \varepsilon_n)$ and for $t \in (1 - \varepsilon_n, 1)$.

Now let

$$S_n(t, s; q) := \frac{t-s}{n} \sum_{k=0}^{n-1} q\left(s + k \frac{t-s}{n}\right), \quad n \in \mathbb{N}, \quad (t, s) \in \Delta.$$

We consider

$$S_{2^n}(t, s; q) = \frac{t-s}{n} \sum_{k=0}^{2^n-1} q\left(s + k \frac{t-s}{2^n}\right) = \frac{t-s}{n} \sum_{k=0}^{2^n-1} q(\xi_{k,n}(t, s)),$$

$n \in \mathbb{N}$, $(t, s) \in \Delta$. If $s \in (0, \varepsilon_n)$ and $t \in (1 - \varepsilon_n, 1)$, then $S_{2^n}(t, s; q) = 0$, $n \in \mathbb{N}$ and

$$\left| \int_s^t q(y) dy - S_{2^n}(t, s; q) \right| = \int_s^t q(y) dy, \quad n \in \mathbb{N},$$

for $s \in (0, \varepsilon_n)$ and $t \in (1 - \varepsilon_n, 1)$. In particular, this yields

$$\operatorname{ess\,sup}_{(t,s) \in \Delta} \left| \int_s^t q(y) dy - S_{2^n}(t, s; q) \right| \geq \operatorname{ess\,sup}_{(t,s) \in \Delta} \int_s^t q(y) dy \geq \int_{\mathcal{I}} \chi_C(y) dy \geq \frac{1}{2}.$$

Hence, we obtain

$$\limsup_{n \rightarrow \infty} \operatorname{ess\,sup}_{(t,s) \in \Delta} \left| \int_s^t q(y) dy - S_{2^n}(t, s; q) \right| \geq \frac{1}{2},$$

and applying Proposition 3.1 we finish the prove of (3.7). \square

We note that Theorem 3.5 does not exclude the convergence of the Trotter product formula for the pair $\{D_0, Q\}$ in the *strong* operator topology. Examples of this dichotomy are known for the Trotter-Kato product formula in Hilbert spaces [3]. By virtue of (3.1) and (3.7), Theorem 3.5 yields an example of this dichotomy in Banach spaces.

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Hagen Neidhardt
WIAS Berlin
Mohrenstr. 39, D-10117 Berlin, Germany
e-mail: hagen.neidhardt@wias-berlin.de

Artur Stephan
Humboldt Universität zu Berlin
Institut für Mathematik
Unter den Linden 6, D-10099 Berlin, Germany
e-mail: stephan@math.hu-berlin.de

Valentin A. Zagrebnov
Aix-Marseille Université, CNRS, Centrale Marseille, I2M,
Marseille, France
e-mail: valentin.zagrebnov@univ-amu.fr