



**HAL**  
open science

## Corrections to "A Global High-Gain Finite-Time Observer"

Tomas Menard, Emmanuel Moulay, Wilfrid Perruquetti

► **To cite this version:**

Tomas Menard, Emmanuel Moulay, Wilfrid Perruquetti. Corrections to "A Global High-Gain Finite-Time Observer". IEEE Transactions on Automatic Control, 2017, 62 (1), pp.509 - 510. 10.1109/TAC.2016.2518742 . hal-01497178

**HAL Id: hal-01497178**

**<https://hal.science/hal-01497178>**

Submitted on 28 Mar 2017

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# Corrections to "A Global High-Gain Finite-Time Observer"

Tomas Ménard, Emmanuel Moulay and Wilfrid Perruquetti

**Abstract**—This note fix the proof of Theorem 2 in the article [2].

Equations from the original paper will be denoted with a star (for example (1\*)) whereas equations of the present corrected paper will be denoted without a star (for example (1)).

## I. THE ERROR

The function  $\tilde{V}_\alpha$  used in the proof of Theorem 2 in [2], and derived from Theorem 10 in [3], is not  $C^1$  with respect to  $(e, \alpha)$ . Indeed, one has

$$\frac{\partial \left( [e_k]^{\frac{1}{\alpha_k q - 1}} \right)}{\partial e_k} = \frac{1}{\alpha_k q} [e_k]^{\frac{1}{\alpha_k q} - 1}. \quad (1)$$

Hence, when  $\alpha \rightarrow 1$ ,  $\frac{1}{\alpha_k q} \rightarrow 1$  and when one of the component of  $e$  goes to zero, the limit  $\lim_{(\alpha, e) \rightarrow (1, e_0)} \frac{1}{\alpha_k q} [e_k]^{\frac{1}{\alpha_k q} - 1}$  does not exist. Thus the function  $\tilde{V}_\alpha$  cannot be used as a candidate Lyapunov function.

## II. THE FIX

Let us first recall Theorem 2 from [2].

**Theorem 1.** *Let us consider system (3\*) with a bounded input  $u$ . Then there exists  $\theta^* \geq 1$  such that for all  $\theta > \theta^*$  there exists  $\epsilon > 0$  such that system (3\*) admits the following global finite-time observer:*

$$\begin{cases} \dot{\hat{x}}_1 = \hat{x}_2 + k_1([e_1]^{\alpha_1} + \rho e_1) + \sum_{j=1}^m g_{1,j}(\hat{x}_1)u_j \\ \vdots \\ \dot{\hat{x}}_n = k_n([e_1]^{\alpha_n} + \rho e_1) + \varphi(\hat{x}) + \sum_{j=1}^m g_{n,j}(\hat{x})u_j \end{cases}$$

for all  $\alpha \in ]1 - \epsilon, 1[$ , where  $e_1 = x_1 - \hat{x}_1$ , the powers  $\alpha_i$  are defined by (5\*), the gains  $k_i$  by (6\*) and  $\rho = \left( \frac{n^2 \theta^{\frac{2}{3}} S_1 + 1}{2} \right)$

where  $S_1$  is defined by (8\*).

In addition, the settling time of the error dynamics is bounded by  $T_1(e_0) + T_2(e_0)$  (with  $e_0 = x_0 - \hat{x}_0$ ), where  $T_1, T_2$  are respectively given by (18\*) and (6).

The statement of Theorem 2 in [2] remains correct, except for the settling time which has to be corrected.

We can define the function  $V_1(e) = e^T S_\infty(1)e$ , for  $e \in \mathbb{R}^n$ , where  $S_\infty(1)$  is the solution of (7\*) for  $\theta = 1$ . This choice

GREYC, UMR-CNRS 6072, Université de Caen, 6 Bd du Maréchal Juin, BP 5186-14032 Caen Cedex, France. (e-mail: tomas.menard@unicaen.fr)

Xlim, UMR-CNRS 7252, Université de Poitiers, 11 bd Marie et Pierre Curie, 86962 Futuroscope Chasseneuil Cedex, France. (e-mail: emmanuel.moulay@univ-poitiers.fr)

NON-A, INRIA Lille Nord Europe and CRISTAL UMR-CNRS 9189, Ecole Centrale de Lille, BP 48, 59651 Villeneuve D'Ascq, France. (e-mail: wilfrid.perruquetti@ec-lille.fr)

corresponds to the linear case, that is  $\alpha = 1$ . Proceeding as in [4], [5], one can construct a candidate Lyapunov function with properties stated next.

**Proposition 1.** *Let  $a \in C^\infty(\mathbb{R}, \mathbb{R})$  be such that*

$$a = \begin{cases} 0 & \text{on } (-\infty, 1] \\ 1 & \text{on } [2, +\infty) \end{cases} \quad \text{and } a' \geq 0 \text{ on } \mathbb{R}. \quad (2)$$

*There exists  $\epsilon > 0$  such that for all  $\alpha \in ]1 - \epsilon, 1 + \epsilon[$ , the function  $\bar{V}_\alpha$  defined as*

$$\bar{V}_\alpha(e) = \int_0^{+\infty} \frac{1}{t^3} (a \circ V_1)(t^{r_1(\alpha)} e_1, \dots, t^{r_n(\alpha)} e_n) dt \quad (3)$$

*if  $e \in \mathbb{R}^n \setminus \{0\}$  and  $\bar{V}_\alpha(0) = 0$  is well defined, radially unbounded, of class  $C^1(\mathbb{R}^n, \mathbb{R})$ , and satisfies*

- $\bar{V}_\alpha(\delta_\lambda^{r(\alpha)} e) = \lambda^2 \bar{V}_\alpha(e)$ , for all  $e \in \mathbb{R}^n$  and  $\lambda > 0$ .
- $\langle \nabla \bar{V}_\alpha(e), Ae - F(S_\infty^{-1}(1)C^T, e) \rangle \leq -\gamma(\bar{V}_\alpha(e))^{\frac{1+\alpha}{2}}$ , for all  $e \in \mathbb{R}^n$ , where  $\gamma > 0$ .

*where  $F, C$  and  $\delta_\lambda^{r(\alpha)}$  are defined in [2].*

*Proof of proposition 1.* Let,  $\alpha \in ]1 - \frac{1}{n}, +\infty[$ , proceeding as in [4], one directly shows that  $\bar{V}_\alpha$  is well defined, radially unbounded,  $C^1$  on  $\mathbb{R}^n$ , and homogeneous of degree 2 with respect to the weights  $r(\alpha)$ . Then, only point b) remains to prove.

Following the same lines as in [4], there exists  $l, L > 0$  such that for all  $e \in \{e \in \mathbb{R}^n \mid \bar{V}_\alpha(e) = 1\}$ , one has

$$\begin{aligned} \langle \nabla \bar{V}_\alpha(e), Ae - F(S_\infty^{-1}(1)C^T, e) \rangle &= \int_l^L \frac{1}{t^{\alpha+2}} a' \left( V_1 \left( \delta_t^{r(\alpha)} e \right) \right) \times \\ &\left\langle \nabla V_1 \left( \delta_t^{r(\alpha)} e \right), A \delta_t^{r(\alpha)} e - F \left( S_\infty^{-1}(1)C^T, \delta_t^{r(\alpha)} e \right) \right\rangle dt \end{aligned}$$

Consider the function  $g(e, t, \alpha) = \left\langle \nabla V_1 \left( \delta_t^{r(\alpha)} e \right), A \delta_t^{r(\alpha)} e - F \left( S_\infty^{-1}(1)C^T, \delta_t^{r(\alpha)} e \right) \right\rangle$ , where  $(e, t, \alpha) \in \{e \in \mathbb{R}^n, \bar{V}_\alpha(e) = 1\} \times \{t \in [l, L]\} \times ]1 - \frac{1}{n}, +\infty[$ . The function  $g$  is continuous,  $(e, t)$  belongs to a compact set and there exists  $\gamma_1 > 0$  such that the image of  $g$  is included in  $]-\infty, -\gamma_1[$  for  $(e, t) \in \{e \in \mathbb{R}^n, \bar{V}_\alpha(e) = 1\} \times \{t \in [l, L]\}$  and  $\alpha = 1$  (since it corresponds to the linear case). We can then apply Lemma 26.8 in [1] (tube lemma) which gives the existence of  $\epsilon > 0$  such that for all  $(e, t, \alpha) \in \{e \in \mathbb{R}^n, \bar{V}_\alpha(e) = 1\} \times \{t \in [l, L]\} \times ]1 - \epsilon, 1 + \epsilon[$ ,  $g(e, t, \alpha) \leq -\gamma_1$ .

Then we have

$$\begin{aligned} \langle \nabla \bar{V}_\alpha(e), Ae - F(S_\infty^{-1}(1)C^T, e) \rangle \\ \leq -\gamma_1 \int_l^L \frac{1}{t^{\alpha+2}} a' \left( V_1 \left( \delta_t^{r(\alpha)} e \right) \right) dt \leq -\gamma (\bar{V}_\alpha(e))^{\frac{2+\alpha-1}{2}} \end{aligned} \quad (4)$$

where  $\gamma > 0$  is a lower bound of  $\gamma_1 \int_l^L \frac{1}{t^{\alpha+2}} a' \left( V_1 \left( \delta_t^{r(\alpha)} e \right) \right) dt$  for  $(e, \alpha) \in \{e \in \mathbb{R}^n, \bar{V}_\alpha(e) = 1\} \times ]1 - \epsilon, 1 + \epsilon[$ . Since  $\bar{V}_\alpha$  is homogeneous of degree 2 with respect to the weights  $r(\alpha)$ , inequality (4) is valid for all  $e \in \mathbb{R}^n$ .  $\square$

Now that a new candidate Lyapunov function has been defined, we explain how it will be used to correct the proof

of Theorem 2 in [2]. Please note that part 1 of the proof is correct, then it has already been proved that every trajectory starting from  $e_0 \in \mathbb{R}^n$  enter the ball  $\mathcal{B}_{\|\cdot\|_{S_\infty(\theta)}}(1)$  after time  $T_1(e_0) = \log(1/V(e_0))/\kappa(\theta)$  (see equation (18\*)).

Denote  $\bar{e} = \Delta_\theta e$ , where  $\Delta_\theta = \text{diag} \left[ 1 \quad \frac{1}{\theta} \quad \dots \quad \frac{1}{\theta^{n-1}} \right]$ , in the remaining, we will show that for every  $\theta \geq \theta_2 \triangleq \frac{2}{\gamma}(M_1 + 2)$ , there exists  $\epsilon > 0$  such that the following inequality

$$\dot{\bar{V}}_\alpha(\bar{e}) \leq - \left( \frac{\gamma}{2}\theta - 1 \right) (\bar{V}_\alpha(\bar{e}))^{\frac{2+\alpha-1}{2}} + M_1 \bar{V}_\alpha(\bar{e}) \quad (5)$$

holds for every  $\bar{e} \in \mathcal{B}_{\|\cdot\|_{S_\infty(1)}}(1)$ ,  $\alpha \in ]1 - \epsilon, 1[$ , where  $M_1 > 0$  is a constant independent of  $\theta$ . This inequality replaces inequality (19\*). Inequality (5) directly implies that the error system (11\*) is finite time stable on  $\mathcal{B}_{\|\cdot\|_{S_\infty(\theta)}}(1)$ . Thus, after time  $T_1(e_0)$ , the error enters  $\mathcal{B}_{\|\cdot\|_{S_\infty(\theta)}}(1)$  and after time  $T_1(e_0) + T_2(e_0)$  the error reaches the origin, where the settling time  $T_2(e_0)$  is bounded as follows

$$T_2(e_0) \leq \frac{\ln \left( 1 - \frac{M_1}{\frac{\gamma}{2}\theta - 1} \bar{V}_\alpha(e_0)^{1 - \frac{2+\alpha-1}{2}} \right)}{M_1 \left( \frac{2+\alpha-1}{2} - 1 \right)}. \quad (6)$$

The remaining of the corrected proof is very similar to the original one. The dynamics of  $\bar{e}$  is given by

$$\begin{aligned} \dot{\bar{e}} &= \theta (A\bar{e} - F(S_\infty^{-1}(1)C^T, \bar{e}) - \rho S_\infty^{-1}(1)C^T C\bar{e}) \\ &\quad + \Delta_\theta D(x, \hat{x}, u). \end{aligned}$$

One has

$$\dot{\bar{V}}_\alpha(\bar{e}) \triangleq \theta \bar{W}_1 + \bar{W}_2 \quad (7)$$

with  $\bar{W}_1 = \langle \nabla \bar{V}_\alpha(\bar{e}), A\bar{e} - F(S_\infty^{-1}(1)C^T, \bar{e}) - \rho S_\infty^{-1}(1)C^T C\bar{e} \rangle$  and  $\bar{W}_2 = \langle \nabla \bar{V}_\alpha(\bar{e}), \Delta_\theta D(x, \hat{x}, u) \rangle$ .

Following the same lines as in [2], one can show that there exists  $\theta_2 \geq 0$  such that for every  $\theta \geq \theta_2$ , there exists  $\epsilon > 0$  such that for all  $\alpha \in ]1 - \epsilon, 1[$  inequality (5) holds true.

## REFERENCES

- [1] J. Munkres. *Topology*. Prentice Hall, 2 edition, 1999.
- [2] T. Ménard, E. Moulay, and W. Perruquetti. A global high-gain finite-time observer. *IEEE Transactions on Automatic Control*, 55(6):1500–1506, 2010.
- [3] W. Perruquetti, T. Floquet, and E. Moulay. Finite-time observers: application to secure communication. *IEEE Transactions on Automatic Control*, 53(1):356–360, 2008.
- [4] L. Rosier. Homogeneous lyapunov function for homogeneous continuous vector field. *Systems & Control Letters*, 19(6):467–473, 1992.
- [5] H. Yu, Y. Shen, and X. Xia. Adaptive finite-time consensus in multi-agent networks. *Systems & Control Letters*, 62(10):880 – 889, 2013.