ON NAVIER-STOKES-KORTEWEG AND EULER-KORTEWEG SYSTEMS: APPLICATION TO QUANTUM FLUIDS MODELS

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Abstract. In this paper, the main objective is to generalize to the Navier-Stokes-Korteweg (with density dependent viscosities satisfying the BD relation) and Euler-Korteweg systems a recent relative entropy [proposed by D. Bresch, P. Noble and J.–P. Vila, (2016)] introduced for the compressible Navier-Stokes equations with a linear density dependent shear viscosity and a zero bulk viscosity. As a concrete application, this helps to justify mathematically the convergence between global weak solutions of the quantum Navier-Stokes system [recently obtained by I. Lacroix-Violet and A. Vasseur (2016)] and dissipative solutions of the quantum Euler system when the viscosity coefficient tends to zero. This selects a dissipative solution as the limit of a viscous system. Our results are based on the fact that Euler-Korteweg systems and corresponding Navier–Stokes-Korteweg systems can be reformulated through an augmented system as the compressible Navier-Stokes system with density dependent viscosities satisfying the BD algebraic relation. This was also observed recently [by D. Bresch, F. Couderc, P. Noble and J.–P. Vila, (2016)] for the Euler-Korteweg system. As a by-product of our analysis, we show that this augmented formulation helps to define relative entropy estimates for the Euler-Korteweg systems in a simplest way compared to recent works [See D. Donatelli, E. Feireisl, P. Marcati (2015) and J. Giesselmann, C. Lattanzio, A.-E. Tzavaras (2017)] and with less hypothesis required on the capillary coefficient: no concavity assumption needed in our result.

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1. Introduction

Quantum fluid models have attracted a lot of attention in the last decades due to te variety of applications. Indeed, such models can be used to describe superfluids [37], quantum semiconductors [24], weakly interacting Bose gases [29] and quantum trajectories of Bohmian mechanics [44]. Recently some dissipative quantum fluid models have been derived. In particular, under some assumptions and using a Chapman-Enskog expansion in Wigner equation, the authors have obtained in [16] the so-called quantum Navier-Stokes model. Roughly speaking, it corresponds to the classical Navier-Stokes equations with a quantum correction term. The main difficulties of such models lie in the highly nonlinear structure of the third order quantum term and the proof of positivity (or non-negativity) of the particle density. Note that formally, the quantum Euler system corresponds to the limit of the quantum Navier-Stokes model when the viscosity coefficient tends to zero. This type of models belong to more general classes of models: the Navier-Stokes-Korteweg and the Euler-Korteweg systems. Readers interested by Korteweg type systems are referred to the following articles and books: [34, 41, 17, 19, 39, 38, 31] and references cited therein.

The goal of this paper is to extend to these two Korteweg systems a recent relative entropy proposed in [13] introduced for the compressible Navier-Stokes equations with a linear density dependent shear viscosity and a zero bulk viscosity. This leads for each system to the definition of what we call a dissipative solution following the concept introduced by P.-L. Lions in the incompressible setting (see [36]) and later extended to the compressible framework (see [23, 22, 5, 40] for constant viscosities and [30, 13] for density dependent viscosities). As a consequence we obtain some weak-strong uniqueness results and as an application, we can use it to show that a global weak solution (proved in [4, 35], which is also a disspative one) of the quantum Navier-Stokes system converges to a dissipative solution of the quantum Euler system. Our results will be compared to recent results in [20, 26] showing that we relax one hypothesis on the capillarity coefficient.

Let us now present in more details the models of interest here. Note that for the convenience of the reader all the operators are defined in Section 6.2. Let $\Omega = \mathbb{T}^d$ be the torus in dimension d (in this article $1 \le d \le 3$).

Euler-Korteweg system. Following the framework of the paper, we first present the Euler-Kortewg system and then the Navier-Stokes Korteweg one. Note that in all the paper, the systems are supplemented with the following initial conditions

$$\rho|_{t=0} = \rho_0, \quad (\rho\, u)|_{t=0} = \rho_0 u_0 \quad \text{for } x \in \Omega. \tag{1}$$

with the regularity $\rho_0 \ge 0$, $\rho_0 \in L^\gamma(\Omega)$, $\rho_0\, |u_0|^2 \in L^1(\Omega)$, $\sqrt{K(\rho_0)}\nabla\rho_0 \in L^2(\Omega)$. The Euler-Korteweg system describe the time evolution, for $t > 0$ of the density $\rho = \rho(t,x)$ and the momentum $J = J(t,x) = \rho(t,x)u(t,x)$ (with u the velocity), for $x \in \Omega$, of an inviscid fluid. The equations can be written in the form ([20]):

$$\partial_t \rho + \operatorname{div} J = 0, \tag{2}$$

$$\partial_t J + \operatorname{div}\left(\frac{J \otimes J}{\rho}\right) + \nabla(p(\rho)) = \varepsilon^2\, \rho \nabla\left(K(\rho)\Delta\rho + \frac{1}{2}K'(\rho)|\nabla\rho|^2\right), \tag{3}$$

where $K : (0,\infty) \to (0,\infty)$ is a smooth function and p is the pressure function given by $p(\rho) = \rho^\gamma$ for $\gamma > 1$. Note that it could be interesting to consider non-monotone pressure laws as in [27] and [26]. The coefficient ε stands for the Planck constant. In this paper we will consider a function $K(\rho)$ which behaves as ρ^s with $s \in \mathbb{R}$. As mentioned in [20],

$$\rho\nabla\left(K(\rho)\Delta\rho + \frac{1}{2}K'(\rho)|\nabla\rho|^2\right) = \operatorname{div}(\mathbb{K}),$$

with

$$\mathbb{K} = \left(\rho \operatorname{div}(K(\rho)\nabla\rho) + \frac{1}{2}(K(\rho) - \rho K'(\rho))|\nabla\rho|^2\right)\mathbb{I}_{\mathbb{R}^d} - K(\rho)\nabla\rho \otimes \nabla\rho.$$

Following the ideas of [8] with

$$\mu'(\rho) = \sqrt{\rho\, K(\rho)}, \tag{4}$$

we can define the drift velocity v by

$$v = \sqrt{\frac{K(\rho)}{\rho}}\nabla\rho = \frac{\nabla(\mu(\rho))}{\rho}$$

and show the following generalization of the Bohm identity:

$$\operatorname{div}(\mathbb{K}) = \operatorname{div}(\mu(\rho)\nabla v) + \frac{1}{2}\nabla(\lambda(\rho)\operatorname{div} v)$$

with

$$\lambda(\rho) = 2(\mu'(\rho)\rho - \mu(\rho)).$$

Remark 1. *Note that the relation between λ and μ is exactly the BD relation found in* [9] *in the Navier-Stokes setting: see the Navier-Stokes-Korteweg part below.*

We will choose $K(\rho)$ as:

$$K(\rho) = \frac{(s+3)^2}{4}\rho^s \text{ with } s \in \mathbb{R} \qquad \text{in order to get} \qquad \mu(\rho) = \rho^{(s+3)/2}.$$

This multiplicative constant in the definition of K does not affect any generality, it suffices to change the definition of ε. Then we obtain the following augmented formulation for the Euler-Korteweg Equations (2)-(3):

$$\partial_t \rho + \operatorname{div}(\rho\, u) = 0, \tag{5}$$

$$\partial_t(\rho\, u) + \operatorname{div}(\rho\, u \otimes u) + \nabla p(\rho) = \varepsilon^2 \left[\operatorname{div}(\mu(\rho)\nabla v) + \frac{1}{2}\nabla(\lambda(\rho)\operatorname{div} v)\right], \tag{6}$$

$$\partial_t(\rho\, v) + \operatorname{div}(\rho\, v \otimes u) + \operatorname{div}(\mu(\rho)\, {}^t\nabla u) + \frac{1}{2}\nabla(\lambda(\rho)\operatorname{div} u) = 0, \tag{7}$$

with

$$\lambda(\rho) = 2(\rho\, \mu'(\rho) - \mu(\rho)), \qquad v = \nabla\mu(\rho)/\rho.$$

System (5)-(7) is called the Euler-Korteweg augmented system in all the sequel. It has been firstly introduced in this conservative form in [8] to propose a useful construction of a numerical scheme with entropy stability property under a hyperbolic CFL condition for such dispersive PDEs. Remark that in the Euler-Korteweg augmented system, the second order operator matrix is skew-symetric.

The Quantum Euler Equations. Note that the choice $K(\rho) = 1/\rho$ (which gives $\mu(\rho) = \rho$ and $\lambda(\rho) = 0$) leads to the Bohm identity

$$\rho\nabla(K(\rho)\Delta\rho + \frac{1}{2}K'(\rho)|\nabla\rho|^2) = \operatorname{div}(\rho\nabla v) = 2\rho\nabla\left(\frac{\Delta\sqrt{\rho}}{\sqrt{\rho}}\right).$$

In that case the system (5)-(7) becomes

$$\partial_t \rho + \operatorname{div}(\rho\, u) = 0, \tag{8}$$

$$\partial_t(\rho\, u) + \operatorname{div}(\rho\, u \otimes u) + \nabla(p(\rho)) = \varepsilon^2 \operatorname{div}(\rho\, \nabla v), \tag{9}$$

$$\partial_t(\rho\, v) + \operatorname{div}(\rho\, v \otimes u) + \operatorname{div}(\rho\, {}^t\nabla u) = 0, \tag{10}$$

with $v = \nabla \log \rho$ which corresponds to the augmented formulation of the quantum Euler system:

$$\partial_t \rho + \operatorname{div}(\rho u) = 0, \tag{11}$$

$$\partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) + \nabla p(\rho) = 2\,\varepsilon^2\, \rho\, \nabla\left(\frac{\Delta\sqrt{\rho}}{\sqrt{\rho}}\right). \tag{12}$$

Then such a choice gives rise the so called quantum fluid system for which the global existence of weak solutions has been shown in [2, 3] and more recently in [18] assuming the initial velocity irrotational namely $\operatorname{curl}(\rho_0 u_0) = 0$. The existence of local strong solutions has also been proved (see [6]) and global well-posedness for small irrotational data has been performed recently in [1] assuming a natural stability condition on the pressure. We refer to (8)-(10) as the quantum Euler augmented system in all the paper.

Note that the quantum correction $(\Delta\sqrt{\rho})/\sqrt{\rho}$ can be interpreted as a quantum potential, the so-called Bohm potential, which is well known in quantum mechanics. This Bohm potential arises from the fluid dynamical formulation of the single-state Schrödinger equation. The non-locality of quantum mechanics is approximated by the fact that the equations of state do not only depend on the particle density but also on its gradient. These equations were employed to model field emissions from metals and steady-state tunneling in metal-insulator-metal structures and to simulate ultra-small semiconductor devices.

Navier-Stokes-Korteweg system. Let us consider the compressible Navier-Stokes-Korteweg system with density dependent viscosities $\mu(\rho)$ and $\lambda(\rho)$ satisfying the BD relation

$$\lambda(\rho) = 2(\mu'(\rho)\rho - \mu(\rho)),$$

and with the capillarity coefficient $K(\rho)$ linked to the shear viscosity $\mu(\rho)$ in the following manner

$$K(\rho) = [\mu'(\rho)]^2/\rho \text{ with } \mu(\rho) = \rho^{(s+3)/2} \text{ with } s \in \mathbb{R}.$$

Remark 2. *With this choice of shear viscosity, the relation between the capillarity coefficient and the viscosity gives a capillarity coefficient proportional to* ρ^s.

Then using the identity given in the Euler-Korteweg part, the Navier-Stokes-Korteweg system can be written for $x \in \Omega$ and $t > 0$,

$$\partial_t\rho + \operatorname{div}(\rho u) = 0, \tag{13}$$

$$\begin{aligned}\partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) + \nabla p(\rho) - 2\,\nu\, \operatorname{div}(\mu(\rho)D(u)) - \nu\nabla(\lambda(\rho)\operatorname{div} u)\\ = \varepsilon^2\left[(\operatorname{div}(\mu(\rho)^t\nabla v) + \frac{1}{2}\nabla(\lambda(\rho)\operatorname{div} v)\right],\end{aligned} \tag{14}$$

in which the symmetric part of the velocity gradient is $D(u) = \frac{1}{2}(\nabla u +^t \nabla u)$. The parameter $\nu > 0$ stands for the viscosity constant. Multiplying (13) by $\mu'(\rho)$ and taking the gradient, we have the following equation on v:

$$\partial_t(\rho\, v) + \operatorname{div}(\rho\, v \otimes u) + \operatorname{div}(\mu(\rho)\,{}^t\nabla u) + \frac{1}{2}\nabla(\lambda(\rho)\operatorname{div} u) = 0. \tag{15}$$

Moreover defining the intermediate velocity, called effective velocity, $w = u + \nu\, v$, equations (14) and (15) lead to

$$\begin{aligned}\partial_t(\rho\, w) + \operatorname{div}(\rho\, w \otimes u) + \nabla(p(\rho)) - \nu\, \operatorname{div}(\mu(\rho)\nabla w) - \frac{\nu}{2}\nabla(\lambda(\rho)\operatorname{div} w)\\ = (\varepsilon^2 - \nu^2)\,[\operatorname{div}(\mu(\rho)\nabla v) + \frac{1}{2}\nabla(\lambda(\rho)\operatorname{div} v)].\end{aligned}$$

Then (13)-(14) may be reformulated through the following augmented system:

$$\partial_t\rho + \operatorname{div}(\rho u) = 0, \tag{16}$$

$$\begin{aligned}\partial_t(\rho\, w) + \operatorname{div}(\rho\, w \otimes u) + \nabla(p(\rho)) - \nu\, \operatorname{div}(\mu(\rho)\nabla w) - \frac{\nu}{2}\nabla(\lambda(\rho)\operatorname{div} w)\\ = (\varepsilon^2 - \nu^2)\,[\operatorname{div}(\mu(\rho)\nabla v) + \frac{1}{2}\nabla(\lambda(\rho)\operatorname{div} v)],\end{aligned} \tag{17}$$

$$\partial_t(\rho\, v) + \operatorname{div}(\rho\, v \otimes u) + \operatorname{div}(\mu(\rho)\,{}^t\nabla u) + \frac{1}{2}\nabla(\lambda(\rho)\operatorname{div} u) = 0, \tag{18}$$

with $w = u + \nu\nabla\mu(\rho)/\rho$, $v = \nabla\mu(\rho)/\rho$ which we call the Navier-Stokes-Korteweg augmented system in all the sequel.

The Quantum Navier-Stokes Equations. Note that with the choice $K(\rho) = 1/\rho$, which gives $\mu(\rho) = \rho$ and $\lambda(\rho) = 0$, system (16)-(18) becomes

$$\partial_t \rho + \mathrm{div}(\rho u) = 0, \tag{19}$$

$$\partial_t(\rho\, w) + \mathrm{div}(\rho\, w \otimes u) + \nabla(p(\rho)) - \nu\ \mathrm{div}(\rho \nabla w) = (\varepsilon^2 - \nu^2)\ \mathrm{div}(\rho \nabla v), \tag{20}$$

$$\partial_t(\rho\, v) + \mathrm{div}(\rho\, v \otimes u) + \mathrm{div}(\rho\, {}^t\nabla u) = 0, \tag{21}$$

with $w = u + \nu \nabla \log \rho$, $v = \nabla \log \rho$ which is the augmented formulation of the compressible barotropic quantum Navier-Stokes system:

$$\partial_t \rho + \mathrm{div}(\rho u) = 0, \tag{22}$$

$$\partial_t(\rho u) + \mathrm{div}(\rho u \otimes u) + \nabla p(\rho) - 2\,\nu\ \mathrm{div}(\rho D(u)) = 2\,\varepsilon^2\,\rho\,\nabla\left(\frac{\Delta\sqrt{\rho}}{\sqrt{\rho}}\right). \tag{23}$$

In [21, 32, 33], the global existence of weak solutions to (22)-(23) has been shown following the idea introduced in [11] by testing the momentum equation by $\rho\,\phi$ with ϕ a test function. The problem of such formulation is that it requires $\gamma > 3$ for $d = 3$ which is not a suitable assumption for physical cases. In [12] the authors show the existence of solutions for (22)-(23) without quantum term (*i.e.* for $\varepsilon = 0$) by adding a cold pressure term in the momentum equation. The cold pressure is a suitable increasing function p_c satisfying $\lim_{n\to 0} p_c(n) = +\infty$. The key element of the proof is a κ-entropy estimate. In [28], using the same strategy and a κ-entropy with $\kappa = 1/2$, the existence of global weak solutions for (22)-(23) is proven without any extra assumption on γ and the semi-classical limit ε tends to zero is performed. In [42], A. Vasseur and C. Yu consider the compressible barotropic quantum Navier-Stokes equations with damping *i.e.* system (22)-(23) with additional terms in the right hand side of (23): $-r_0 u - r_1 \rho |u|^2 u$. They prove the global-in-time existence of weak solutions and their result is still valuable in the case $r_1 = 0$. Their proof is based on a Faedo-Galerkin approximation (following the ideas of [33]) and a Bresch-Desjardins entropy (see [10, 11]). In [43], the authors use the result obtained in [42] and pass to the limits ε, r_0, r_1 tend to zero to prove the existence of global-in-time weak solutions to degenerate compressible Navier-Stokes equations. Note that to prove such a result they need uniform (with respect to r_0, r_1) estimates to pass to the limit r_0, r_1 tend to 0. To this end they have to firstly pass to the limit ε tends to 0. Recently in [35], global existence of weak solutions for the quantum Navier-Stokes equations (22)-(23) has been proved without drag terms and without any cold pressure. The method is based on the construction of weak solutions that are renormalized in the velocity variable. Note that the construction being uniform with respect to the Planck constant, the authors also perform the semi-classical limit to the associated compressible Navier-Stokes equations. Note also the recent paper [4] concerning the global existence for the quantum Navier-Stokes system. It is important to remark that a global weak solutions of the quantum Navier-Stokes equations in the sense of [35] is also weak solution of the augmented system (due to the regularity which is envolved allowing to write the equation on the drift velocity v). Remark also that there exists no global existence result of weak solutions for the compressible Navier-Stokes-Korteweg system with constant viscosities even in the two-dimensional in space case.

Main objectives of the paper. In this paper, to the author's point of view, there are several interesting and new results. First starting with the global weak solutions of the quantum Navier-Stokes equations constructed in [35] (which is a 1/2-entropy solution in the sense of [12]) we show at the viscous limit the existence of a dissipative solution for the quantum Euler system letting the viscosity goes to zero. This gives the first global existence result of dissipative solution for the quantum Euler system obtained from a quantum Navier-Stokes type system. Note that in [20], it is proved the existence of infinite dissipative solutions of such inviscid quantum system. Here we present a way to select one starting from a Navier-Stokes type system. Secondly, we develop relative entropy estimates for general cases of the Euler-Korteweg and the Navier-Stokes-Korteweg systems extending the augmented formulations introduced recently in [13] and [14]: more general viscosities and third order dispersive terms. This gives a more simple procedure to perform relative entropy than the one developped in [26, 20] for the Euler-Korteweg system. This also helps to get rid the concavity assumption on $K(\rho)$ which is strongly used in [26]. For the interested readers, we provide a comparison of the quantities appearing in our relative entropy to the ones introduced in [26] and remark that they are equivalent under the assumptions made in [26]. Note that to perform our calculations for the Navier-Stokes-Korteweg system, we need to generalize in a non-trivial way the identity (5) in [13]: see Proposition 30 for the generalized identity.

For reader's convenience, let us explain the simple idea behind all the calculations. The kinetic energy corresponding to the Euler-Korteweg system reads

$$\int_\Omega \left(\frac{1}{2}\rho|u|^2 + H(\rho) + K(\rho)|\nabla\rho|^2\right)$$

with

$$H(\rho) = \rho\int_1^\rho \frac{p(z)}{z^2}dz.$$

In [26], they consider that it is an energy written in terms of $(\rho, u, \nabla\rho)$ and they write a relative entropy playing with these unknowns. In our calculations, we write the kinetic energy as follows

$$\int_\Omega \left(\frac{1}{2}\rho|u|^2 + H(\rho) + \rho|v|^2\right)$$

with $v = \sqrt{K(\rho)}\nabla\rho/\sqrt{\rho}$ and we consider three quantities ρ, u and v. This motivates to write an augmented system (ρ, u, v) and to modulate the energy through these three unknowns. This gives a simplest way to define an appropriate relative entropy quantity compared to [26] and [20] and allows to relax the concavity assumption on $K(\rho)$ made in the part concerning Euler-Korteweg system in [26]. Our result covers capillarity coefficient under the form

$$K(\rho) \approx \rho^s \text{ with } s+2 \leq \gamma \text{ and } s \geq -1.$$

Finally our result makes the link between Euler-Korteweg system and Navier-Stokes-Korteweg system. After proving the global existence of 1/2-entropy solutions of the general Navier-Stokes-Korteweg system (this is the subject of a forthcoming paper [15] still in progress: the case $K(\rho) = 1/\rho$ has been recently proved in [35]), this could give the mathematical justification of a physical dissipative solution of the Euler-Korteweg equations obtained from 1/2-entropy solutions of the Navier-Stokes-Korteweg equations in the spirit of [12]. Let us also mention that our relative entropies could be helpful for other singular limits as explained in the book [23] in the case of constant viscosities.

The paper is organized as follows. Section 2 is devoted to notations and preliminary results. In section 3, we give the definition of the relative entropy formula and we established the associated estimate. This one is used to define what we call a dissipative solution for the Euler-Korteweg system and we established a weak/strong uniqueness result. The same results are obtained for the Navier-Stokes-Korteweg system in section 4. In section 5 we use the previous results to show the limit when the viscosity tends to zero in the quantum Navier-Stokes system. Finally we give in Appendix, a comparaison between the formula used here and the one used in [26, 20], and we state the definitions used for the operators.

2. Notations and preliminary results

In this section we give some notations, technical lemmas and energy estimates which will be used in the next sections.

2.1. **Notations and technical lemmas.**

Let us first give some notations that will be used all along the paper.

Enthalpy modulation. First of all, let us define the function H called the enthalpy by

$$H(\rho) = \rho e(\rho) = \rho \int_1^\rho \frac{p(z)}{z^2}\, dz.$$

Namely we have:

$$\rho H'(\rho) - H(\rho) = p(\rho), \qquad H''(\rho) = \frac{p'(\rho)}{\rho}.$$

To be more precise, since $p(\rho) = \rho^\gamma$ with $\gamma > 1$, this yields to $H(\rho) = \dfrac{1}{\gamma - 1} p(\rho)$. As usually for compressible flows, linked to H we define the modulated enthalpy

$$H(\rho|r) = H(\rho) - H(r) - H'(r)(\rho - r),$$

with r a smooth bounded function of (t,x) on $\mathbb{R}^+ \times \Omega$.

Drift velocities. As mentioned in the introduction, we define the drift velocity by

$$v = \frac{\nabla(\mu(\rho))}{\rho} = \sqrt{\frac{K(\rho)}{\rho}} \nabla\rho.$$

We add to this definition, the following one

$$V = \frac{\nabla(\mu(r))}{r} = \sqrt{\frac{K(r)}{r}} \nabla r,$$

still for r a smooth bounded function of (t,x) on $\mathbb{R}^+ \times \Omega$. Note that by definition ∇v and ∇V are symmetric matrices because v, V are gradient of functions.

Some important properties on various modulated quantities. We introduce the function ϕ defined by

$$\phi(\tau) = \int_0^\tau \frac{p'(\mu^{-1}(s))}{\mu'(\mu^{-1}(s))} ds, \tag{24}$$

and the two functions

$$\phi_1(\rho|r) = \phi(\mu(\rho)) - \phi(\mu(r)) - \phi'(\mu(r))(\mu(\rho) - \mu(r)), \tag{25}$$

$$\phi_2(\rho|r) = \phi''(\mu(r))(\mu(\rho) - \mu(r))\, r - \rho\, (\phi'(\mu(\rho)) - \phi'(\mu(r))). \tag{26}$$

Remark 3. *Note that in the case $K(\rho) = 1/\rho$, which gives (using (4)) $\mu(\rho) = \rho$, these two functions are directly linked to $H(\rho|r)$. Indeed, in this case we have*

$$\begin{aligned}\phi_1(\rho|r) &= p(\rho) - p(r) - p'(r)(\rho - r) = (\gamma - 1)H(\rho|r),\\ \phi_2(\rho|r) &= \rho p'(r) - \rho p'(\rho) + rp''(r)(\rho - r) = -\gamma(\gamma - 1)H(\rho|r). \square\end{aligned}$$

As usually in compressible flows (see [23]) let us define the set $\mathcal{F}$ by

$$\mathcal{F} = \left\{\rho \leq \frac{r}{2} \text{ or } \rho \geq 2r\right\}.$$

Let us now give some technical lemmas which will be used in the following. First of all, following (4.1) of [22] we have

Lemma 4. *Assuming p smooth, $p(0) = 0$, $p'(\rho) > 0 \quad \forall \rho > 0$, $\lim_{\rho\to\infty} \dfrac{p'(\rho)}{\rho^{\alpha-1}} = a > 0$ for $\alpha > 3/2$, we have:*

$$H(\rho|r) \geq C(r)(\rho - r)^2 \text{ if } \rho \in \mathcal{F}^c \quad \text{and} \quad H(\rho|r) \geq C(r)(1 + \rho)^\gamma \text{ otherwise},$$

with $C(r)$ uniformly bounded for r belonging to compact sets in $\mathbb{R}^+ \times \Omega$.

Concerning the functions ϕ_1 and ϕ_2, we can show

Lemma 5. *Let us assume that $\mu(\rho) = \rho^{(s+3)/2}$ with $\gamma \geq s + 2$ and $s \geq -1$. Assume ϕ_i with $i = 1, 2$ defined by (24)–(26). Then*

$$|\phi_i(\rho|r)| \leq C(r)|\rho - r|^2 \text{ if } \rho \in \mathcal{F}^c \quad \text{and} \quad |\phi_i(\rho|r)| \leq C(r)(1 + \rho)^\gamma \text{ otherwise},$$

with $C(r)$ uniformly bounded for r belonging to compact sets in $\mathbb{R}^+ \times \Omega$.

Remark 6. *Let us remark that the choice $\mu(\rho) = \rho^{(s+3)/2}$ with $s \in \mathbb{R}$ and the assumption $\gamma \geq 2 + s$ correspond to the case considered in [26] because $K(\rho)$ is of order ρ^s. Moreover, for the particular case of interest in this paper $K(\rho) = 1/\rho$ (i.e. $s = -1$), the assumption $2 + s \leq \gamma$ is trivially satisfied since we have $\gamma > 1$.*

Proof of the lemma for ϕ_1. Using Taylor expansions and the fact that $\phi''(\mu(c))$, $\mu'(c)$ are bounded with c in a compact we easily obtain

$$|\phi_1(\rho|r)| \quad \leq \quad C(r)|\mu(\rho) - \mu(r)|^2 \leq C(r)|\rho - r|^2 \text{ on } \mathcal{F}^c.$$

Moreover, since

$$p(\rho) = \rho^\gamma,\ K(\rho) = \frac{(s+3)^2}{4}\rho^s,\ \mu'(\rho) = \sqrt{\rho K(\rho)},$$

we have $\phi(\tau) = \tau^{2\gamma/(s+3)}$ and then by definition

$$|\phi_1(\rho|r)| \quad = \quad \left|\rho^\gamma - r^\gamma - \frac{2\gamma}{s+3} r^{\frac{2\gamma-(s+3)}{2}}(\rho^{\frac{s+3}{2}} - r^{\frac{s+3}{2}})\right|,$$

which gives

$$|\phi_1(\rho|r)| \leq C(r)(1 + \rho)^\gamma \text{ on } \mathcal{F},$$

since by assumption $2\gamma \geq 2(s + 2) \geq s + 3$ with $s \geq -1$.

Proof of the lemma for ϕ_2. Let us write $\theta = \dfrac{s+3}{2}$ then $\mu(\rho) = \rho^\theta$ and $\phi(\rho) = \rho^{\gamma/\theta}$. Then

$$\begin{aligned}
\phi_2(\rho|r) &= \frac{2\gamma}{s+3}\left[\left(\frac{2\gamma}{s+3}-1\right) r^{\gamma-s-3}(\rho^\theta - r^\theta)r - \rho(\rho^{\gamma-\theta} - r^{\gamma-\theta})\right] \\
&= \frac{2\gamma}{s+3}\left[\left(\frac{2\gamma}{s+3}-1\right) r^{\gamma-s-2}(\rho^\theta - r^\theta) - \rho^{1+\gamma-\theta} + \rho\, r^{\gamma-\theta}\right] \\
&= \frac{2\gamma}{s+3}\left[\left(\frac{2\gamma}{s+3}-1\right) r^{\gamma-s-2}(\rho^\theta - r^\theta) - f(\mu(\rho)) + \rho\, r^{\gamma-\theta}\right]
\end{aligned}$$

with $f(\rho) = \rho^{\frac{\gamma+1}{\theta}-1}$. Note that we have

$$\begin{aligned}
f(\rho|r) &= f(\mu(\rho)) - f(\mu(r)) - f'(\mu(r))(\mu(\rho)-\mu(r)) \\
&= f(\mu(\rho)) - (r^\theta)^{\frac{\gamma+1}{\theta-1}} - \left(\frac{\gamma+1}{\theta}-1\right)(r^\theta)^{\frac{\gamma+1}{\theta}-2}(\rho^\theta - r^\theta).
\end{aligned}$$

Then

$$\begin{aligned}
\phi_2(\rho|r) &= \frac{2\gamma}{s+3}\left[\left(\frac{2\gamma}{s+3}-1\right) r^{\gamma-s-2}(\rho^\theta - r^\theta) - f(\rho|r) - (r^\theta)^{\frac{\gamma+1}{\theta}-1}\right] \\
&\quad -\frac{2\gamma}{s+3}\left[\left(\frac{2(\gamma+1)}{s+3}-1\right)(r^\theta)^{\frac{\gamma+1}{\theta}-2}(\rho^\theta - r^\theta) - \rho\, r^{\gamma-\theta}\right] \\
&= \frac{2\gamma}{s+3}\left[\left(\frac{2\gamma}{s+3}-1\right) r^{\gamma-s-2}(\rho^\theta - r^\theta) - f(\rho|r) - r^{1+\gamma-\theta}\right] \\
&\quad -\frac{2\gamma}{s+3}\left[\left(\frac{2(\gamma+1)}{s+3}-1\right)(r^{\gamma-s-2}(\rho^\theta - r^\theta) - \rho\, r^{\gamma-\theta}\right] \\
&= \frac{2\gamma}{s+3}\left[-\frac{1}{\theta} r^{\gamma-s-2}(\rho^\theta - r^\theta) - f(\rho|r) + \rho\, r^{\gamma-\theta} - r^{1+\gamma-\theta}\right].
\end{aligned}$$

This can be written $\phi_2(\rho|r) = \dfrac{2\gamma}{s+3}(-f(\rho|r) + g(\rho|r))$ with

$$\begin{aligned}
g(\rho|r) &= r^{\gamma-\theta}\left[\rho - r - \frac{1}{\theta} r^{\tau-s-2}(\rho^\theta - r^\theta)\right] \\
&= r^{\gamma-\theta}\left[(\rho^\theta)^{1/\theta} - (r^\theta)^{1/\theta} - \frac{1}{\theta}(r^\theta)^{1/\theta-1}(\rho^\theta - r^\theta)\right].
\end{aligned}$$

In the case $\rho \in \mathcal{F}^c$, using Taylor expansions this leads to

$$|f(\rho|r)| \le C(r)|\mu(\rho) - \mu(r)]^2 \le C(r)|\rho - r|^2,$$
$$|g(\rho|r)| \le C(r)|\rho^\theta - r^\theta|^2 \le C(r)|\rho - r|^2,$$

and then

$$|\phi_2(\rho|r)| \le C(r)|\rho - r|^2.$$

When $\rho \in \mathcal{F}$, since $2\gamma \ge 2s+4 \ge s+3$ and $s+3 \ge 2$,

$$|\phi_2(\rho|r)| \;\le\; C(r)|r^{\gamma-(s+3)}(\rho^{\frac{s+3}{2}} - r^{\frac{s+3}{2}}) - \rho(\rho^{\gamma-\frac{s+3}{2}} - r^{\gamma-\frac{s+3}{2}})| \le C(r)(1+\rho)^\gamma.$$

This completes the proof of Lemma 5. □

Using Lemmas 4 and 5, we directly obtain

Lemma 7. *Let us assume that $\mu(\rho) = \rho^{(s+3)/2}$ with $\gamma \geq s+2$ and $s \geq -1$. We have*

$$|\phi_1(\rho|r)| \leq C(r)H(\rho|r) \quad \text{and} \quad |\phi_2(\rho|r)| \leq C(r)H(\rho|r),$$

with $C(r)$ uniformly bounded for r belonging to compact sets in $\mathbb{R}^+ \times \Omega$.

Let us now prove the following lemma

Lemma 8. *Let us assume that $\mu(\rho) = \rho^{(s+3)/2}$ with $\gamma \geq s+2$ and $s \geq -1$. We have*

$$\rho|\mu'(\rho) - \mu'(r)|^2 \leq C(r)H(\rho|r),$$

with $C(r)$ uniformly bounded for r belonging to compact sets in $\mathbb{R}^+ \times \Omega$.

Proof.

$$\rho|\mu'(\rho) - \mu'(r)|^2 = \rho|\mu'(\rho) - \mu'(r)|^2 1_{\mathcal{F}} + \rho|\mu'(\rho) - \mu'(r)|^2 1_{\mathcal{F}^c}.$$

We have

$$\rho|\mu'(\rho) - \mu'(r)|^2 1_{\mathcal{F}} \leq 2\rho(|\mu'(\rho)|^2 + |\mu'(r)|^2)1_{\mathcal{F}} \leq \frac{(s+3)^2}{2}\rho^{s+2}1_{\mathcal{F}} + 2C(r)\rho\, 1_{\mathcal{F}}.$$

Using $\rho^{s+2} \leq (1+\rho)^{s+2}$ and the assumption $\gamma \geq s+2$ in the first term, and, the assumption $\gamma > 1$ in the second one, we obtain:

$$\rho|\mu'(\rho) - \mu'(r)|^2 1_{\mathcal{F}} \leq \frac{(s+3)^2}{2}(1+\rho)^{\gamma}1_{\mathcal{F}} + 2C(r)(1+\rho)^{\gamma}1_{\mathcal{F}} \leq C(r)(1+\rho)^{\gamma}1_{\mathcal{F}}.$$

Moreover,

$$\rho|\mu'(\rho) - \mu'(r)|^2 1_{\mathcal{F}^c} = \frac{s+3}{2}\rho\left|\rho^{\frac{s+1}{2}} - r^{\frac{s+1}{2}}\right|^2 1_{\mathcal{F}^c} = \frac{s+3}{2}\rho\frac{|\rho^{s+1} - r^{s+1}|^2}{|\rho^{\frac{s+1}{2}} + r^{\frac{s+1}{2}}|^2}1_{\mathcal{F}^c},$$

and then

$$\rho|\mu'(\rho) - \mu'(r)|^2 1_{\mathcal{F}^c} \leq \frac{s+3}{2}\rho\frac{|\rho^{s+1} - r^{s+1}|^2}{|r^{\frac{s+1}{2}}|^2}1_{\mathcal{F}^c} \leq C(r)|\rho^{s+1} - r^{s+1}|^2 1_{\mathcal{F}^c} \leq C(r)|\rho - r|^2 1_{\mathcal{F}^c}.$$

Using lemma 4, we finally obtain the result. □

An important relation. The last technical and important lemma is

Lemma 9. *Let us assume that $\mu(\rho) = \rho^{(s+3)/2}$ with $\gamma \geq s+2$ and $s \geq -1$. We have*

$$\rho\left(\frac{p'(\rho)}{\mu'(\rho)}v - \frac{p'(r)}{\mu'(r)}V\right) \cdot (v - V) = [\nabla\phi_1(\rho|r) + \phi_2(\rho|r)V] \cdot V + \rho\frac{p'(\rho)}{\mu'(\rho)}|V - v|^2$$

with ϕ_1 and ϕ_2 defined by (24)–(26).

Remark 10. *This lemma generalizes to general $\mu(\rho)$ the relation (5) established in* [13] *when $\mu(\rho) = \rho$. This is an important lemma which helps to control the terms coming from the pressure in the relative entropy at the Navier-Stokes level.*

Proof. Remark first that

$$\rho\left(\frac{p'(\rho)}{\mu'(\rho)}v - \frac{p'(r)}{\mu'(r)}V\right) \cdot (v - V) = \rho\frac{p'(\rho)}{\mu'(\rho)}|V - v|^2 + \rho\left(\frac{p'(\rho)}{\mu'(\rho)} - \frac{p'(r)}{\mu'(r)}\right)(v - V) \cdot V$$

We have

$$\begin{aligned}
&\rho\left(\frac{p'(\rho)}{\mu'(\rho)}-\frac{p'(r)}{\mu'(r)}\right)(v-V)\\
=\ &\left(\frac{p'(\rho)}{\mu'(\rho)}-\frac{p'(r)}{\mu'(r)}\right)(\rho\, v-\rho\, V)\\
=\ &\left(\frac{p'(\rho)}{\mu'(\rho)}-\frac{p'(r)}{\mu'(r)}\right)\nabla(\mu(\rho))-\frac{\rho}{r}\left(\frac{p'(\rho)}{\mu'(\rho)}-\frac{p'(r)}{\mu'(r)}\right)\nabla(\mu(r)).
\end{aligned}$$

Moreover, it is easy to see that by definition

$$\begin{aligned}
\nabla(\phi_1(\rho|r)) &= \phi'(\mu(\rho))\nabla\mu(\rho)-\phi''(\mu(r))(\mu(\rho)-\mu(r))\nabla\mu(r)-\phi'(\mu(r))\nabla\mu(\rho)\\
&= \left(\frac{p'(\rho)}{\mu'(\rho)}-\frac{p'(r)}{\mu'(r)}\right)\nabla\mu(\rho)-\phi''(\mu(r))(\mu(\rho)-\mu(r))\nabla\mu(r),
\end{aligned}$$

and then using the definition of $\phi_2(\rho|r)$,

$$\rho\left(\frac{p'(\rho)}{\mu'(\rho)}-\frac{p'(r)}{\mu'(r)}\right)(v-V)=\nabla\phi_1(\rho|r)+\phi_2(\rho|r)V.$$

□

2.2. **Energy estimates.** In this subsection we give the energy equalities for the augmented Euler-Korteweg and Navier-Stokes-Korteweg systems. They will be used in the following to establish the estimates for the relative entropy associated to each one.

Euler-Korteweg system. For the augmented Euler-Korteweg system we can show the following proposition.

Proposition 11. *All weak solution (ρ, u, v) of system (5)–(7) satisfies:*

$$\frac{dE_{EuK}(\rho,u,v)}{dt}=0,$$

where E_{EuK} is the natural energy density given by

$$E_{EuK}(t)=E_{EuK}(\rho,u,v)=\int_\Omega\left(\frac{1}{2}\,\rho\,|u|^2+\frac{1}{2}\,\varepsilon^2\,K(\rho)|\nabla\rho|^2+H(\rho)\right). \tag{27}$$

Proof. We remark that

$$2\rho K(\rho)|\nabla\sqrt{\rho}|^2=\frac{1}{2}K(\rho)|\nabla\rho|^2=\frac{1}{2}\rho|v|^2$$

and refer to [20] for the proof of this proposition. □

Navier-Stokes-Korteweg system. Concerning the augmented Navier-Stokes-Korteweg system, defining the energy

$$E_{NSK}(t)=E_{NSK}(\rho,v,w) = \int_\Omega\left(\frac{\nu^2+\varepsilon^2}{2}\rho\,|v|^2+\frac{\rho}{2}|w|^2+H(\rho)\right),$$

we have the following energy-dissipation result:

Proposition 12. *Let (ρ, v, w) be a weak solution of* (16)-(18) *we have*

$$\frac{dE_{NSK}}{dt}(\rho,v,w)+\nu\int_\Omega\left(\mu(\rho)\left(|\nabla u|^2+\varepsilon^2|\nabla v|^2\right)+\mu'(\rho)H''(\rho)|\nabla\rho|^2\right)$$
$$+\nu\int_\Omega\left(\frac{\lambda(\rho)}{2}\left((\operatorname{div}(u))^2+\varepsilon^2(\operatorname{div}(v))^2\right)\right)=0.$$

Before proving the result, let us remark that

$$|\nabla u|^2=|D(u)|^2+|A(w)|^2.$$

Let us point out that this estimate corresponds to the 1/2-entropy estimate introduced in [12] and is linked to the so called BD-entropy introduced in [9]. We recall how to get it for reader's convenience.

Proof. Multiplying (17) by w, (18) by $\nu^2 v$, integrating with respect to space and summing, we obtain:

$$\nu^2\int_\Omega\partial_t(\rho v)\cdot v+\int_\Omega\partial_t(\rho w)\cdot w+\nu^2\int_\Omega\Big(\operatorname{div}(\rho v\otimes u)+\operatorname{div}(\mu(\rho)^t\nabla u)+\frac{1}{2}\nabla\left(\lambda(\rho)\operatorname{div}(u)\right)\Big)\cdot v$$
$$+\int_\Omega\Big(\operatorname{div}(\rho w\otimes u)+\nabla p(\rho)-\nu\operatorname{div}(\mu(\rho)\nabla w)-\frac{\nu}{2}\nabla(\lambda(\rho)\operatorname{div}(w))\Big)\cdot w$$
$$-(\varepsilon^2-\nu^2)\int_\Omega\left[\operatorname{div}(\mu(\rho)\nabla v)+\frac{1}{2}\nabla(\lambda(\rho)\operatorname{div}(v))\right]\cdot w=0. \tag{28}$$

Using integrations by parts and the definition of w and v,

$$\bullet\int_\Omega\partial_t(\rho w)\cdot w=\int_\Omega\partial_t\left(\frac{\rho}{2}|w|^2\right)-\frac{1}{2}\int_\Omega\operatorname{div}(\rho u)|w|^2,$$

$$\bullet\int_\Omega\operatorname{div}(\rho w\otimes u)\cdot w=\frac{1}{2}\int_\Omega\operatorname{div}(\rho u)|w|^2,$$

$$\bullet\int_\Omega(\partial_t(\rho v))\cdot v=\int_\Omega\partial_t\left(\frac{\rho}{2}|v|^2\right)-\frac{1}{2}\int_\Omega\operatorname{div}(\rho u)|v|^2,$$

$$\bullet\int_\Omega\operatorname{div}(\rho v\otimes u)\cdot v=\frac{1}{2}\int_\Omega\operatorname{div}(\rho u)|v|^2,$$

$$\bullet\int_\Omega\nabla p(\rho)\cdot w=\int_\Omega\partial_t(H(\rho))+\nu\int_\Omega\mu'(\rho)H''(\rho)|\nabla\rho|^2,$$

$$\bullet-\varepsilon^2\int_\Omega\left[\operatorname{div}(\mu(\rho)\nabla v)+\frac{1}{2}\nabla(\lambda(\rho)\operatorname{div}(v))\right]\cdot w=\varepsilon^2\int_\Omega\partial_t\left(\frac{\rho}{2}|v|^2\right)$$
$$+\varepsilon^2\nu\int_\Omega\left(\mu(\rho)|\nabla v|^2+\frac{1}{2}\lambda(\rho)(\operatorname{div}(v))^2\right).$$

To obtain this equality, we use (8) in [8] since v here corresponds to $\nabla\varphi(\rho)$ in [8] and $\mu'(\rho)$ here corresponds to $F'(\rho)$ in [8].

$$\bullet-\nu\int_\Omega\operatorname{div}(\mu(\rho)\nabla w)\cdot w=\nu\int_\Omega\mu(\rho)|\nabla u|^2+2\nu^2\int_\Omega\mu(\rho)\nabla u:\nabla v+\nu^3\int_\Omega\mu(\rho)|\nabla v|^2,$$

$$\bullet \nu^2 \int_\Omega \mathrm{div}(\mu(\rho)^t\nabla u)\cdot v = -\nu^2 \int_\Omega \mu(\rho)^t\nabla u : \nabla v,$$

$$\bullet -\frac{\nu}{2}\int_\Omega \nabla(\lambda(\rho)\,\mathrm{div}(w))\cdot w = \frac{\nu}{2}\int_\Omega \left(\lambda(\rho)(\mathrm{div}(u))^2 + 2\nu\lambda(\rho)(\mathrm{div}(u))(\mathrm{div}(v)) + \nu^2\lambda(\rho)(\mathrm{div}(v))^2\right),$$

$$\bullet \frac{\nu^2}{2}\int_\Omega \nabla(\lambda(\rho)\,\mathrm{div}(u))\cdot v = -\frac{\nu^2}{2}\int_\Omega \lambda(\rho)(\mathrm{div}(u))(\mathrm{div}(v)),$$

$$\bullet \nu^2 \int_\Omega \mathrm{div}(\mu(\rho)\nabla v)\cdot w = -\nu^2\int_\Omega (\mu(\rho)\nabla v : \nabla u + \nu\mu(\rho)|\nabla v|^2),$$

$$\bullet \frac{\nu^2}{2}\int_\Omega \nabla(\lambda(\rho)\,\mathrm{div}(v))\cdot w = -\frac{\nu^2}{2}\int_\Omega (\lambda(\rho)\,\mathrm{div}(v)\,\mathrm{div}(u) + \nu\lambda(\rho)(\mathrm{div}(v))^2).$$

Using all the previous equalities in (28), we obtain

$$\begin{aligned}\frac{dE_{NSK}}{dt}(\rho,v,w) + \nu\int_\Omega \left(\varepsilon^2\mu(\rho)|\nabla v|^2 + \mu'(\rho)H''(\rho)|\nabla\rho|^2 + \frac{\lambda(\rho)}{2}(\mathrm{div}(u))^2 + \frac{\varepsilon^2\lambda(\rho)}{2}(\mathrm{div}(v))^2\right)\\ +\nu\int_\Omega \left(\mu(\rho)|\nabla u|^2 + \nu\mu(\rho)\nabla u : \nabla v - \nu\mu(\rho)^t\nabla u : \nabla v\right) = 0.\end{aligned}$$

Using the fact that ∇v is symmetric, we show :

$$\nabla u : \nabla v - {}^t\nabla u : \nabla v = 0$$

and we obtain the proposition. $\square$

Note that, in order to pass to the limit ν tends to zero, the real variable of interest is not v but $\bar v$ defined by: $\bar v = \sqrt{\varepsilon^2-\nu^2}\, v$ under the assumption $\varepsilon > \nu$. With this new variable the augmented System (16)-(18) becomes:

$$\partial_t\rho + \mathrm{div}\,(\rho\, u) = 0, \tag{29}$$

$$\begin{aligned}\partial_t(\rho w) + \mathrm{div}(\rho\, w\otimes u) + \nabla p(\rho) - \nu\,\mathrm{div}(\mu(\rho)\nabla w) - \frac{\nu}{2}\nabla(\lambda(\rho)\,\mathrm{div}\, w)\\ = \sqrt{\varepsilon^2-\nu^2}\left(\mathrm{div}(\mu(\rho)\nabla\bar v) + \frac{1}{2}\nabla(\lambda(\rho)\,\mathrm{div}\,\bar v)\right),\end{aligned} \tag{30}$$

$$\begin{aligned}\partial_t(\rho\bar v) + \mathrm{div}(\rho\,\bar v\otimes u) - \nu\,\mathrm{div}(\mu(\rho)\nabla\bar v) - \frac{\nu}{2}\nabla(\lambda(\rho)\,\mathrm{div}\,\bar v)\\ +\sqrt{\varepsilon^2-\nu^2}\left(\mathrm{div}(\mu(\rho)\,{}^t\nabla w) + \frac{1}{2}\nabla(\lambda(\rho)\,\mathrm{div}\, w)\right) = 0.\end{aligned} \tag{31}$$

The energy associated to this new formulation is

$$E^{NF}_{NSK}(t) = \int_\Omega \left(\frac{\rho}{2}|\bar v|^2 + \frac{\rho}{2}|w|^2 + H(\rho)\right),$$

and we have the following estimate:

Proposition 13. *Let $(\rho, \bar{v}, w)$ be a weak solution of* (29)-(31) *we have*

$$\sup_{\tau\in[0,t]} E^{NF}_{NSK}(\tau) + \nu \int_0^t \int_\Omega \mu'(\rho)H''(\rho)|\nabla\rho|^2$$
$$+\nu \int_0^t \int_\Omega \Big(\mu(\rho)(|\nabla w|^2 + |\nabla \bar{v}|^2) + \frac{\lambda(\rho)}{2}\left(|\operatorname{div}\bar{v}|^2 + |\operatorname{div} w|^2\right)\Big)$$
$$\leq E^{NF}_{NSK}(0).$$

Proof : We multiply (30) by w and (31) by $\bar{v}$ and we integrate with respect to the space variable. Summing and using analogous computations than the ones did for the proof of Proposition 12, we obtain the result.

Remark 14. *Note that ε doesn't appear explicitly in this new formulation of the energy. Indeed it is hidden in the terms containing $\bar{v}$.*

3. The Euler-Kortewg System : relative entropy and dissipative solution

In this section, we consider the problem (2)–(3) through its augmented formulation (5)–(7). The goal of this section is to give the definition of what we call a dissipative solution for this problem. To this end we have to establish a relative entropy inequality.

3.1. Relative entropy inequality.

In [22], E. Feireisl, B.-J. Jin and A. Novotny have introduced relative entropies, suitable weak solutions and weak-strong uniqueness properties for the compressible Navier-Stokes equations with constant viscosities. The goal of this subsection is to establish a relative entropy inequality for the Euler-Korteweg System using the augmented formulation introduced in [8] and extending the ideas in [13] and [14] to such system in order to be able to define what is called a dissipative solution.

Let us consider the following relative entropy functional, denoted $\mathcal{E}_{EuK}(\rho, u, v|r, U, V)$ and defined by

$$\mathcal{E}_{EuK}(t) = \mathcal{E}_{EuK}(\rho, u, v|r, U, V)(t)$$
$$(32)\qquad = \frac{1}{2}\int_\Omega \rho\left(|u-U|^2 + \varepsilon^2\left|\sqrt{\frac{K(\rho)}{\rho}}\nabla\rho - \sqrt{\frac{K(r)}{r}}\nabla r\right|^2\right) + \int_\Omega H(\rho|r)$$
$$= \frac{1}{2}\int_\Omega \rho\left(|u-U|^2 + \varepsilon^2\,|v-V|^2\right) + \int_\Omega H(\rho|r),$$

where (ρ, u, v) is a weak solution of System (5)–(7). Note that the definition used here is different from the one used in [26] but we can show that the twice are equivalent in some sense. We refer to appendix 6.1 for more details. Let us just say that such an energy measures the distance between an entropic weak solution (ρ, u, v) to any smooth enough test function (r, U, V). The goal here is to prove an inequality of type

$$\mathcal{E}_{EuK}(t) - \mathcal{E}_{EuK}(0) \leq C\int_0^t \mathcal{E}_{EuK}(\xi)d\xi,$$

with C a positive constant. To this end let us first prove the following proposition.

Proposition 15. *Let (ρ, u, v) be a weak solution to System* (5)-(7)*. We have:*

$$\begin{aligned}
\mathcal{E}_{EuK}(t)-\mathcal{E}_{EuK}(0) &= \int_0^t\int_\Omega \rho\,(U-u)\cdot\partial_t U + \int_0^t\int_\Omega \rho(\nabla U\,u)\cdot(U-u) + \varepsilon^2\int_0^t\int_\Omega \rho\,(V-v)\cdot\partial_t V \\
&\quad +\varepsilon^2\int_0^t\int_\Omega \rho(\nabla V\,u)\cdot(V-v) + \varepsilon^2\int_0^t\int_\Omega \mu(\rho)\nabla v:\nabla U - \varepsilon^2\int_0^t\int_\Omega \mu(\rho)\nabla V : {}^t\nabla u \\
&\quad +\frac{\varepsilon^2}{2}\int_0^t\int_\Omega \lambda(\rho)\,\left(\operatorname{div} v\,\operatorname{div} U - \operatorname{div} u\,\operatorname{div} V\right) - \int_0^t\int_\Omega p(\rho)\operatorname{div} U \\
&\quad -\int_0^t\int_\Omega \left[\partial_t(H'(r))(\rho-r)+\rho\nabla H'(r)\cdot u\right],
\end{aligned} \tag{33}$$

for all $t\in[0,T]$ and for all smooth test functions (r,U,V) with

$$r\in\mathcal{C}^1([0,T]\times\bar\Omega),\quad r>0,\quad U,V\in\mathcal{C}^1([0,T]\times\bar\Omega).$$

Proof. Thanks to Proposition 11 we have

$$\begin{aligned}
\mathcal{E}_{EuK}(t)-\mathcal{E}_{EuK}(0) &= \int_\Omega\left(\frac{\rho}{2}|U|^2-\rho\,u\cdot U+\frac12\rho\varepsilon^2\frac{K(r)}{r}|\nabla r|^2-\rho\varepsilon^2\sqrt{\frac{K(\rho)}{\rho}}\nabla\rho\cdot\sqrt{\frac{K(r)}{r}}\nabla r\right)(t) \\
&\quad -\int_\Omega\left(\frac{\rho}{2}|U|^2-\rho\,u\cdot U+\frac12\rho\varepsilon^2\frac{K(r)}{r}|\nabla r|^2-\rho\varepsilon^2\sqrt{\frac{K(\rho)}{\rho}}\nabla\rho\cdot\sqrt{\frac{K(r)}{r}}\nabla r\right)(0) \\
&\quad -\int_\Omega\big(H(r)+H'(r)(\rho-r)\big)(t)+\int_\Omega\big(H(r)+H'(r)(\rho-r)\big)(0)
\end{aligned}$$

i.e.

$$\begin{aligned}
\mathcal{E}_{EuK}(t)-\mathcal{E}_{EuK}(0) &= \int_0^t\int_\Omega\frac{d}{dt}\left(\frac{\rho}{2}|U|^2-\rho\,u\cdot U+\frac12\rho\,\varepsilon^2\,|V|^2-\rho\ \varepsilon^2\,v\cdot V\right) \\
&\quad -\int_0^t\int_\Omega\frac{d}{dt}\big(H(r)+H'(r)(\rho-r)\big).
\end{aligned} \tag{34}$$

We multiply (6) by U, (7) by $\varepsilon^2\,V$ and we integrate with respect to time and space. Writting

$$\partial_t(\rho\,u\cdot U)=\partial_t(\rho\,u)\cdot U+\rho\,u\cdot\partial_t U,\quad \partial_t(\rho\,v\cdot V)=\partial_t(\rho v)\cdot V+\rho\,v\cdot\partial_t V$$

and thanks to integrations by parts, we obtain

$$\begin{aligned}
\mathcal{E}_{EuK}(t)-\mathcal{E}_{EuK}(0) &= \int_0^t\int_\Omega\partial_t\left(\frac{\rho}{2}|U|^2\right)-\int_0^t\int_\Omega\rho\,u\cdot\partial_t U-\int_0^t\int_\Omega\rho(\nabla U\,u)\cdot u \\
&\quad +\varepsilon^2\int_0^t\int_\Omega\mu(\rho)\,\nabla v:\nabla U+\int_0^t\int_\Omega\frac{\varepsilon^2}{2}\lambda(\rho)\operatorname{div} v\,\operatorname{div} U \\
&\quad +\varepsilon^2\int_0^t\int_\Omega\partial_t\left(\frac{\rho}{2}|V|^2\right)-\varepsilon^2\int_0^t\int_\Omega\rho\,v\cdot\partial_t V \\
&\quad -\varepsilon^2\int_0^t\int_\Omega\rho\,(\nabla V\ u)\cdot v-\varepsilon^2\int_0^t\int_\Omega\mu(\rho)\nabla V:{}^t\nabla u \\
&\quad -\int_0^t\int_\Omega\frac{\varepsilon^2}{2}\lambda(\rho)\operatorname{div} u\,\operatorname{div} V-\int_0^t\int_\Omega p(\rho)\operatorname{div} U \\
&\quad -\int_0^t\int_\Omega\partial_t(H(r)+H'(r)(\rho-r)).
\end{aligned}$$

Using (5) and

$$\partial_t\left(\frac{\rho}{2}|U|^2\right) = \frac{1}{2}\partial_t\rho\,|U|^2 + \rho\,U\cdot\partial_t U, \quad \partial_t\left(\frac{\rho}{2}|V|^2\right) = -\frac{1}{2}\mathrm{div}(\rho u)|V|^2 + \rho\,V\cdot\partial_t V,$$

thanks to integrations by parts we have

$$\begin{aligned}\mathcal{E}_{EuK}(t) - \mathcal{E}_{EuK}(0) \quad = \quad & \int_0^t\int_\Omega \rho\,(U-u)\cdot\partial_t U + \varepsilon^2\int_0^t\int_\Omega \rho\,(V-v)\cdot\partial_t V \\ & + \int_0^t\int_\Omega \rho\,(\nabla U\,u)\cdot(U-u) + \varepsilon^2\int_0^t\int_\Omega \rho\,(\nabla V\,u)\cdot(V-v) \\ & + \varepsilon^2\int_0^t\int_\Omega \mu(\rho)\,\nabla v : \nabla U - \varepsilon^2\int_0^t\int_\Omega \mu(\rho)\,\nabla V : {}^t\nabla u \\ & + \frac{\varepsilon^2}{2}\int_0^t\int_\Omega \lambda(\rho)\,(\mathrm{div}\,v\;\mathrm{div}\,U - \mathrm{div}\,u\;\mathrm{div}\,V) \\ & - \int_0^t\int_\Omega p(\rho)\;\mathrm{div}\,U - \int_0^t\int_\Omega \partial_t(H(r) + H'(r)(\rho - r)).\end{aligned}$$

This last inequality gives the result since with Equation (5) we have:

$$\int_\Omega \partial_t(H'(r)(\rho-r)) = \int_\Omega \left(\partial_t(H'(r))(\rho - r) + \rho\nabla(H'(r))\cdot u\right).$$

$\square$

Proposition 16. *Let (ρ, u, v) be a weak solution of (5)–(7) and (r, U, V) be a strong solution of*

$$\partial_t r + \mathrm{div}\,(r\,U) = 0, \tag{35}$$

$$r\,(\partial_t U + \nabla U\,U) + \nabla p(r) - \varepsilon^2\left[\mathrm{div}(\mu(r)\nabla V) + \frac{1}{2}\nabla(\lambda(r)\,\mathrm{div}\,V)\right] = 0, \tag{36}$$

$$r\,(\partial_t V + \nabla V\,V) + \mathrm{div}(\mu(r)\,{}^t\nabla U) + \frac{1}{2}\nabla(\lambda(r)\,\mathrm{div}\,U) = 0. \tag{37}$$

We have

$$\begin{aligned}\mathcal{E}_{EuK}(t) - \mathcal{E}_{EuK}(0) \quad = \quad & \int_0^t\int_\Omega \rho(u-U)\cdot(\nabla U(U-u)) - \varepsilon^2\int_0^t\int_\Omega \rho(V-v)\cdot(\nabla V(U-u)) \\ & - \int_0^t\int_\Omega (p(\rho) - p(r) - (\rho - r)p'(r))\,\mathrm{div}\,U \\ & - \varepsilon^2\int_0^t\int_\Omega \rho\,(v-V)\cdot\nabla U(v-V) + \varepsilon^2\int_0^t\int_\Omega \rho(v-V)\cdot\nabla V(u-U) \\ & - \varepsilon^2\int_0^t\int_\Omega \rho\left(\mu''(\rho)\nabla\rho - \mu''(r)\nabla(r)\right)\cdot((v-V)\;\mathrm{div}\,U + (U-u)\;\mathrm{div}\,V) \\ & - \varepsilon^2\int_0^t\int_\Omega \rho\,(\mu'(\rho) - \mu'(r))\,((v-V)\;\cdot\nabla(\mathrm{div}\,U) + (U-u)\;\cdot\nabla(\mathrm{div}\,V)).\end{aligned}$$

Proof. Multiplying (36) by $\dfrac{\rho}{r}\,(U-u)$ and (37) by $\varepsilon^2\,\dfrac{\rho}{r}\,(V-v)$ and integrating with respect to time and space we have:

$$\begin{aligned}\mathcal{E}_{EuK}(t)-\mathcal{E}_{EuK}(0) &= -\int_0^t\int_\Omega \rho\,(\nabla U\,(U-u))\cdot(U-u)\\ &\quad -\varepsilon^2\int_0^t\int_\Omega \rho\,(\nabla V\,(U-u))\cdot(V-v)\\ &\quad +\varepsilon^2\,(I_1^{Eu}+I_2^{Eu})+I_3^{Eu}\end{aligned}$$

with

$$\begin{aligned}I_1^{Eu} &= \int_0^t\int_\Omega \Big(\frac{\rho}{r}\,\mathrm{div}(\mu(r)\nabla V)\cdot(U-u)-\frac{\rho}{r}\,\mathrm{div}(\mu(r)^t\nabla U)\cdot(V-v)\Big)\\ &\quad +\int_0^t\int_\Omega \mu(\rho)\left(\nabla v:\nabla U-\nabla V:{}^t\nabla u\right),\end{aligned}$$

$$\begin{aligned}2\,I_2^{Eu} &= \int_0^t\int_\Omega \frac{\rho}{r}(U-u)\cdot\nabla\left(\lambda(r)\,\mathrm{div}\,V\right)-\int_0^t\int_\Omega\frac{\rho}{r}(V-v)\cdot\nabla\left(\lambda(r)\,\mathrm{div}\,U\right)\\ &\quad +\int_0^t\int_\Omega \lambda(\rho)\left(\mathrm{div}\,v\;\mathrm{div}\,U-\mathrm{div}\,u\;\mathrm{div}\,V\right),\end{aligned}$$

$$I_3^{Eu} = \int_0^t\int_\Omega\Big(-p(\rho)\,\mathrm{div}\,U-\frac{\rho}{r}\,\nabla p(r)\cdot(U-u)-\partial_t(H'(r))(\rho-r)-\rho\,\nabla(H'(r))\cdot u\Big).$$

Using $rH''(r)=p'(r)$, we have

$$\frac{\rho}{r}\nabla p(r)=\rho\nabla(H'(r)).$$

Mutiplying (35) by $H''(r)$ and using $rH''(r)=p'(r)$ we obtain

$$\partial_t H'(r)+\nabla H'(r)\cdot U+p'(r)\,\mathrm{div}\,U=0.$$

Using $rH''(r)=p'(r)$ and an integration by parts, we have

$$\int_0^t\int_\Omega r\nabla H'(r)U=-\int_0^t\int_\Omega p(r)\,\mathrm{div}\,U.$$

Then,

$$I_3^{Eu}=\int_0^t\int_\Omega (p(r)-p(\rho)-(r-\rho)p'(r))\,\mathrm{div}\,U.$$

We have

$$I_1^{Eu}=I_4^{Eu}+I_5^{Eu},$$

where

$$I_4^{Eu} = \int_0^t\int_\Omega \mu(r)\nabla\left(\frac{\rho}{r}\right)\cdot\left(\nabla U(V-v)-\nabla V(U-u)\right),$$

and using the symmetry of ∇v and ∇V,

$$\begin{aligned}
I_5^{Eu} &= \int_0^t \int_\Omega \Big[\frac{\rho}{r}\mu(r)\left({}^t\nabla U : \nabla(V-v) - \nabla V : \nabla(U-u)\right) + \mu(\rho)\left(\nabla v : \nabla U - \nabla V : {}^t\nabla u\right)\Big] \\
&= \int_0^t \int_\Omega \frac{\rho}{r}\mu(r)(\nabla V : \nabla u - {}^t\nabla U : \nabla v) + \int_0^t \int_\Omega \mu(\rho)(\nabla v : \nabla U - \nabla V : {}^t\nabla u) \\
&= \int_0^t \int_\Omega \Big(\frac{\rho}{r}\mu(r) - \mu(\rho)\Big)(\nabla V : \nabla u - {}^t\nabla U : \nabla v) \\
&= \int_0^t \int_\Omega \Big(\mu(\rho) - \frac{\rho}{r}\mu(r)\Big)(-\nabla V : \nabla u + {}^t\nabla U : \nabla v -{}^t \nabla U : \nabla V + \nabla U : \nabla V) \\
&= \int_0^t \int_\Omega \Big(\mu(\rho) - \frac{\rho}{r}\mu(r)\Big)({}^t\nabla U : \nabla(v-V) - \nabla(u-U) : \nabla V) \\
&= \int_0^t \int_\Omega \rho\left(\frac{\mu(\rho)}{\rho} - \frac{\mu(r)}{r}\right)(-(v-V)\cdot \operatorname{div}({}^t\nabla U) + (u-U)\cdot \operatorname{div}(\nabla V)) \\
&- \int_0^t \int_\Omega \nabla\Big(\mu(\rho) - \frac{\rho}{r}\mu(r)\Big)\cdot(\nabla U(v-V) - \nabla V(u-U)).
\end{aligned}$$

Then using the definitions of v and V,

$$\begin{aligned}
I_1^{Eu} &= \int_0^t \int_\Omega \rho\left(\frac{\mu(\rho)}{\rho} - \frac{\mu(r)}{r}\right)(-(v-V)\cdot \operatorname{div}({}^t\nabla U) + (u-U)\cdot \operatorname{div}(\nabla V)) \\
&\quad - \int_0^t \int_\Omega \rho\left(\frac{\nabla\mu(\rho)}{\rho} - \frac{\nabla\mu(r)}{r}\right)(\nabla U(v-V) - \nabla V(u-U)) \\
&= \int_0^t \int_\Omega \rho\left(\frac{\mu(\rho)}{\rho} - \frac{\mu(r)}{r}\right)((v-V)\cdot \operatorname{div}({}^t\nabla U) + (u-U)\cdot \operatorname{div}(\nabla V)) \\
&\quad - \int_0^t \int_\Omega \rho(v-V)\cdot\nabla U(v-V) + \int_0^t \int_\Omega \rho(v-V)\cdot\nabla V(u-U).
\end{aligned}$$

Let us recall that $\lambda(s) = 2(s\,\mu'(s) - \mu(s))$. Then

$$\begin{aligned}
2I_2^{Eu} &= \int_0^t \int_\Omega \frac{\rho}{r}(U-u)\cdot\nabla(\lambda(r)\operatorname{div} V) - \int_0^t \int_\Omega \frac{\rho}{r}(V-v)\cdot\nabla(\lambda(r)\operatorname{div} U) \\
&\quad + \int_0^t \int_\Omega \lambda(\rho)\,(\operatorname{div} v\, \operatorname{div} U - \operatorname{div} u\, \operatorname{div} V) \\
&= 2I_6^{Eu} - \int_0^t \int_\Omega \nabla\left(\frac{\rho}{r}\right)\cdot(U-u)\lambda(r)\operatorname{div} V + \int_0^t \int_\Omega \nabla\left(\frac{\rho}{r}\right)\cdot(V-v)\lambda(r)\operatorname{div} U,
\end{aligned}$$

with

$$
\begin{aligned}
2I_6^{Eu} &= \int_0^t \int_\Omega \lambda(\rho)\,(\operatorname{div}(v)\ \operatorname{div} U - \operatorname{div} u\ \operatorname{div} V) \\
&\quad - \int_0^t \int_\Omega \frac{\rho}{r} \operatorname{div}(U-u)\lambda(r) \operatorname{div} V + \int_0^t \int_\Omega \frac{\rho}{r} \operatorname{div}(V-v)\lambda(r) \operatorname{div} U \\
&= \int_0^t \int_\Omega \lambda(\rho)\,(\operatorname{div}(v-V)\ \operatorname{div} U + \operatorname{div}(U-u)\ \operatorname{div} V) \\
&\quad - \int_0^t \int_\Omega \frac{\rho}{r} \operatorname{div}(U-u)\lambda(r) \operatorname{div} V + \int_0^t \int_\Omega \frac{\rho}{r} \operatorname{div}(V-v)\lambda(r) \operatorname{div} U \\
&= \int_0^t \int_\Omega \Big(\lambda(\rho) - \frac{\rho}{r}\lambda(r)\Big)\,(\operatorname{div}(v-V)\ \operatorname{div} U + \operatorname{div}(U-u)\ \operatorname{div} V) \\
&= -\int_0^t \int_\Omega \nabla \Big(\lambda(\rho) - \frac{\rho}{r}\lambda(r)\Big) \cdot ((v-V)\ \operatorname{div} U + (U-u)\ \operatorname{div} V) \\
&\quad - \int_0^t \int_\Omega \rho \left(\frac{\lambda(\rho)}{\rho} - \frac{\lambda(r)}{r}\right) ((v-V)\cdot \nabla(\operatorname{div} U) + (U-u)\cdot\nabla(\operatorname{div} V)).
\end{aligned}
$$

Since $\lambda'(\rho) = 2\,\rho\,\mu''(\rho)$, which leads to

$$
\nabla \Big(\lambda(\rho) - \frac{\rho}{r}\lambda(r)\Big) = 2\,\rho\,\big(\mu''(\rho)\nabla\rho - \mu''(r)\nabla r\big) - \nabla\Big(\frac{\rho}{r}\Big)\,\lambda(r),
$$

we have

$$
\begin{aligned}
2I_2^{Eu} &= -2\int_0^t \int_\Omega \rho\,\big(\mu''(\rho)\nabla\rho - \mu''(r)\nabla r\big)\cdot((v-V)\ \operatorname{div} U + (U-u)\ \operatorname{div} V) \\
&\quad - \int_0^t \int_\Omega \Big(\lambda(\rho) - \frac{\rho}{r}\lambda(r)\Big)((v-V)\cdot\nabla(\operatorname{div} U) + (U-u)\cdot\nabla(\operatorname{div} V))
\end{aligned}
$$

and

$$
\begin{aligned}
I_1^{Eu} + I_2^{Eu} &= \int_0^t \int_\Omega \rho \left(\frac{\mu(\rho)}{\rho} - \frac{\mu(r)}{r}\right) (-(v-V)\cdot \operatorname{div}({}^t\nabla U) + (u-U)\cdot \operatorname{div}(\nabla V)) \\
&\quad - \int_0^t \int_\Omega \rho\,(v-V)\cdot\nabla U(v-V) + \int_0^t \int_\Omega \rho(v-V)\cdot\nabla V(u-U) \\
&\quad - \int_0^t \int_\Omega \rho\,\big(\mu''(\rho)\nabla\rho - \mu''(r)\nabla(r)\big)\cdot((v-V)\ \operatorname{div} U + (U-u)\ \operatorname{div} V) \\
&\quad - \frac{1}{2}\int_0^t \int_\Omega \Big(\lambda(\rho) - \frac{\rho}{r}\lambda(r)\Big)((v-V)\cdot\nabla(\operatorname{div} U) + (U-u)\cdot\nabla(\operatorname{div} V)) \\
&= -\int_0^t \int_\Omega \rho(v-V)\cdot\nabla U(v-V) + \int_0^t \int_\Omega \rho(v-V)\cdot\nabla V(u-U) \\
&\quad - \int_0^t \int_\Omega \rho\,\big(\mu''(\rho)\nabla\rho - \mu''(r)\nabla(r)\big)\cdot((v-V)\ \operatorname{div} U + (U-u)\ \operatorname{div} V) \\
&\quad - \int_0^t \int_\Omega \rho(\mu'(\rho) - \mu'(r))\,((v-V)\cdot\nabla(\operatorname{div} U) + (U-u)\cdot\nabla(\operatorname{div} V)).
\end{aligned}
$$

This concludes the proof. □

Theorem 17. *Let us assume $\mu(\rho) = \rho^{(s+3)/2}$ with $\gamma \geq s+2$ and $s \geq -1$. Let (ρ, u, v) be a weak solution of* (5)-(7) *and (r, U, V) be a strong solution of* (35)-(37)*. We have*

$$\mathcal{E}_{EuK}(t) - \mathcal{E}_{EuK}(0) \leq C(r, U, V) \int_0^t \int_\Omega \mathcal{E}_{EuK}(\xi) d\xi,$$

where $C(r, U, V)$ is a uniformly bounded constant on $\mathbb{R}^+ \times \Omega$.

Using the Gronwall's Lemma, we directly obtain:

Corollary 18. *Let us assume $\mu(\rho) = \rho^{(s+3)/2}$ with $\gamma \geq s+2$ and $s \geq -1$. Let (ρ, u, v) be a weak solution of* (5)-(7) *and (r, U, V) a strong solution of* (35)-(37)*. Then*

$$\mathcal{E}_{EuK}(t) \leq \mathcal{E}_{EuK}(0) \exp(C\, t),$$

with $C = C(r, U, V)$ a uniformly bounded constant on $\mathbb{R}^+ \times \Omega$.

Note that theorem 17 is a direct consequence of proposition 16 and the following lemma.

Lemma 19. *We assume that $\mu(\rho) = \rho^{(s+3)/2}$ with $s \geq -1$. Let (ρ, u, v) be a weak solution of* (5)–(7) *and (r, U, V) be a strong solution of* (35)–(37)*. Then*

$$\left| \int_0^t \int_\Omega \rho \left(\mu''(\rho)\nabla\rho - \mu''(r)\nabla(r)\right) \cdot \left((v - V) \operatorname{div} U + (U - u) \operatorname{div} V\right) \right|$$
$$\leq C \, \frac{s+1}{2} \int_0^t \int_\Omega \rho(|v - V|^2 + |u - U|^2),$$

and, if $\gamma \geq 2 + s$, we have

$$\left| \int_0^t \int_\Omega \rho(\mu'(\rho) - \mu'(r)) \left((v - V) \cdot \nabla(\operatorname{div} U) + (U - u) \cdot \nabla(\operatorname{div} V)\right) \right|$$

$$\leq C \int_0^t \int_\Omega \left(H(\rho|r) + \rho(|v - V|^2 + |u - U|^2)\right),$$

where $C = C(r, U, V)$ is a uniformly bounded constant on $\mathbb{R}^+ \times \Omega$.

Proof. Since $\mu(\rho) = \rho^{(s+3)/2}$ and $v = \nabla(\mu(\rho))/\rho$, $V = \nabla(\mu(r))/r$, we have

$$\mu''(\rho)\nabla\rho - \mu''(r)\nabla r \;\; = \;\; \frac{s+1}{2}(v - V),$$

which gives the first part of the lemma using Young inequality. For the second one, using Young inequality, we have:

$$\int_0^t \int_\Omega \left|\rho(\mu'(\rho) - \mu'(r)) \left((v - V) \cdot \nabla(\operatorname{div} U) + (U - u) \cdot \nabla(\operatorname{div} V)\right)\right|$$
$$\leq C \left(\frac{1}{2} \int_0^t \int_\Omega \rho|\mu'(\rho) - \mu'(r)|^2 + \int_0^t \int_\Omega \rho|v - V|^2 + \int_0^t \int_\Omega \rho|u - U|^2 \right),$$

with $C = C(U, V)$ a uniformly bounded constant on $\mathbb{R}^+ \times \Omega$. Using Lemma 8 in the first integral, we obtain the result. □

Let U be a given and smooth function. We define r as the strong solution of equation (35) and we introduce the functions $\mathscr{E}_1$ and $\mathscr{E}_2$ such that

$$r\,(\partial_t U + U\cdot\nabla U) + \nabla p(r) - \varepsilon^2\,\mathrm{div}(\mu(r)\,{}^t\nabla V) + \frac{1}{2}\nabla(\lambda(r)\,\mathrm{div}\,V) = \mathscr{E}_1(r,U,V), \tag{38}$$

$$r\,(\partial_t V + U\cdot\nabla V) + \mathrm{div}(\mu(r)\,{}^t\nabla U) + \frac{1}{2}\nabla(\lambda(r)\,\mathrm{div}\,U) = \mathscr{E}_2(r,U,V), \tag{39}$$

with $V = \nabla(\mu(r))/r$.

Using the proof of Theorem 17, it is easy to see that the following proposition is true.

Proposition 20. *Let us assume $\mu(\rho)=\rho^{(s+3)/2}$ with $\gamma \geq s+2$ and $s\geq -1$. Let (ρ,u,v) be a weak solution of (5)–(7) and (r,U,V) a strong solution of (35), (38)– (39). Then*

$$\mathcal{E}_{EuK}(t) \quad\leq\quad \mathcal{E}_{EuK}(0) + C\int_0^t \mathcal{E}_{EuK}(\xi)\,d\xi + b_{EuK}(t),$$

where

$$b_{EuK}(t) = \int_0^t\int_\Omega \frac{\rho}{r}|\,\mathscr{E}_1\cdot(U-u) + \varepsilon^2\,\mathscr{E}_2\cdot(V-v)|,$$

and $C = C(\varepsilon^2, r, U, V)$ is a uniformly bounded constant on $\mathbb{R}^+\times\Omega$.

Using the Gronwall's Lemma we immediately obtain the next corollary.

Corollary 21. *Let us assume $\mu(\rho)=\rho^{(s+3)/2}$ with $\gamma\geq s+2$, $\gamma>1$ and $s\geq -1$. Let (ρ,u,v) be a weak solution of (5)-(7) and (r,U,V) a strong solution of (35), (38)-(39). Then*

$$\mathcal{E}_{EuK}(t) \leq \mathcal{E}_{EuK}(0)\exp(C\,t) + b_{EuK}(t) + C\int_0^t b_{EuK}(\xi)\,\exp(C\,(t-\xi))\,d\xi,$$

where

$$b_{EuK}(t) = \int_0^t\int_\Omega \frac{\rho}{r}|\,\mathscr{E}_1\cdot(U-u) + \varepsilon^2\mathscr{E}_2\cdot(V-v)|,$$

and $C = C(\varepsilon^2, r, U, V)$ is a uniformly bounded constant on $\mathbb{R}^+\times\Omega$.

3.2. **Dissipative solutions and weak-strong uniqueness result.** In this subsection, we give the definition of what we call a dissipative solution for the Euler-Korteweg System. We recall that $\mathcal{E}_{EuK}(t)$ stands for

$$\mathcal{E}_{EuK}(t) = \mathcal{E}_{EuK}(\rho,u,v|r,U,V)(t)$$

defined in (32).

We introduce the function $\mathscr{E}$ given by

$$r\,(\partial_t U + U\cdot\nabla U) + \nabla p(r) - \varepsilon^2\,\mathrm{div}(\mu(r)\,{}^t\nabla V) + \frac{1}{2}\nabla(\lambda(r)\,\mathrm{div}\,V) = \mathscr{E}(r,U), \tag{40}$$

with U a given smooth enough function and r a strong solution of equation (35). Using $r\,V = \nabla(\mu(r))$ and equation (35) we obtain

$$r\,(\partial_t V + U\cdot\nabla V) + \mathrm{div}(\mu(r)\,{}^t\nabla U) + \frac{1}{2}\nabla(\lambda(r)\,\mathrm{div}\,U) = 0. \tag{41}$$

Definition 22. *Let us assume $\mu(\rho) = \rho^{(s+3)/2}$ (i.e. $K(\rho) = \frac{(s+3)^2}{4}\rho^s$) with $\gamma \geq s+2$ and $s \geq -1$. Let ρ_0 and u_0 smooth enough. The pair (ρ, u) is a dissipative solution of the Euler-Korteweg System* (2)-(3) *satisfying the initial conditions* (1)*, if the triplet (ρ, u, v) with $\rho\, v = \nabla(\mu(\rho))$ satisfies*

$$\mathcal{E}_{EuK}(t) \leq \mathcal{E}_{EuK}(0)\exp(C\,t) + b_{EuK}(t) + C\int_0^t b_{EuK}(\xi)\exp(C\,(t-\xi))\,d\xi,$$

with $C = C(\varepsilon^2, r, U, V)$ a uniformly bounded constant on $\mathbb{R}^+ \times \Omega$, and where

$$b_{EuK}(t) = \int_0^t \int_\Omega \frac{\rho}{r} |\mathscr{E}\cdot (U-u)|,$$

for all strong solution (r, U, V) of (35)*,* (40)*-*(41)*.*

Remark 23. *Note that by definition, for $\mu(\rho) = \rho^{(s+3)/2}$ (i.e. $K(\rho) = \dfrac{(s+3)^2}{4}\rho^s$) with $\gamma \geq s+2$ and $s \geq -1$, using Corollary 21, all weak solution of* (2)-(3) *satisfying the initial conditions* (1)*, is also a dissipative solution in the sense of Definition 22.*

As a direct consequence, we can establish the following weak-strong uniqueness property (see [25]).

Theorem 24. *Let us assume $\mu(\rho) = \rho^{(s+3)/2}$ (i.e. $K(\rho) = \dfrac{(s+3)^2}{4}\rho^s$) with $\gamma \geq s+2$ and $s \geq -1$. Let us consider a dissipative solution (ρ, u) to the Euler-Korteweg system* (2)-(3) *satisfying the initial conditions* (1)*. We define $\rho\, v = \nabla(\mu(\rho))$. Let us assume that (r, U) is a strong solution of* (35) *and*

$$r\,(\partial_t U + \nabla U\, U) + \nabla p(r) - \varepsilon^2 r \nabla\left(K(r)\Delta r + \frac{1}{2}K'(r)|\nabla r|^2\right) = 0. \tag{42}$$

We define $r\,V = \nabla(\mu(r))$. If $r|_{t=0} = \rho_0$, $U|_{t=0} = u_0$ then $\rho = r$, $u = U$ and $v = V$, which means that the problem satisfies a weak-strong uniqueness property.

Proof. If (r, U) is a strong solution of (35), (42) then $\mathscr{E} = 0$ and $b_{EuK}(t) = 0$. The function H being convex this gives

$$0 \leq \frac{1}{2}\int_\Omega \rho\left(|u-U|^2 + \varepsilon^2\,|v-V|^2\right)\,dx \leq \mathcal{E}_{EuK}(t) \leq \mathcal{E}_{EuK}(0)\exp(C\,t). \tag{43}$$

If $r(t=0) = \rho_0$, $U(t=0) = u_0$ then $v(t=0) = V(t=0)$ and $\mathcal{E}_{EuK}(0) = 0$. This leads to $u = U$, $v = V$ and $\rho = r$ using (43). $\square$

Note that, as already mentioned before, all the results and definitions of this section are still valid for the compressible quantum Euler System. Indeed it corresponds to the special case $K(\rho) = 1/\rho$ in the Euler-Korteweg System for which the assumption $2 + s \leq \gamma$ is satisfied since $s = -1$ and $\gamma > 1$. In particular we have the following definition of what we call a dissipative solution of the quantum Euler system. This one will be used in section 5.

Definition 25. *Let ρ_0 and u_0 smooth enough. The pair (ρ, u) is a dissipative solution of the quantum Euler system* (11)-(12) *satisfying the initial conditions* (1) *if the triplet (ρ, u, v) with $v = \nabla \log\rho$ satisfies*

$$\mathcal{E}_{EuQ}(t) \leq \mathcal{E}_{EuQ}(0)\exp(C\,t) + b_{EuQ}(t) + C\int_0^t b_{EuQ}(\xi)\exp(C\,(t-\xi))\,d\xi,$$

with a constant $C = C(\varepsilon^2, r, U, V)$ *uniformly bounded on* $\mathbb{R}^+ \times \Omega$*, and*

$$\mathcal{E}_{EuQ}(t) = \mathcal{E}_{EuK}(t), \text{ for } K(\rho) = 1/\rho,$$

$$b_{EuQ}(t) = \int_0^t \int_\Omega \frac{\rho}{r} | \mathscr{E} \cdot (U - u)|,$$

for all strong solution (r, U, V) *of* (35) *and*

$$r\left(\partial_t U + U \cdot \nabla U\right) + \nabla p(r) - \varepsilon^2 \operatorname{div}\left(r \nabla V\right) = \mathscr{E}(r, U), \tag{44}$$

$$r\left(\partial_t V + U \cdot \nabla V\right) + \operatorname{div}\left(r\, {}^t\nabla U\right) = 0, \tag{45}$$

with $V = \nabla \log r$*.*

4. The Navier-Stokes-Korteweg System: relative entropy and dissipative solution

The goal of this section is to define what we call a dissipative solution for the Navier-Stokes-Korteweg System. To this end, we consider the augmented System (16)-(18) and we establish a relative entropy estimate. Here the viscous term adds some difficulties compare to the case of the Euler-Korteweg system.

4.1. Relative entropy inequality.

In this section, we establish a relative entropy inequality for a weak solution $(\rho, \bar{v}, w)$ of the augmented System (29)-(31). This will then be used to give the definition of what is called a dissipative solution for the Navier-Stokes-Korteweg system. We define the following relative entropy functional

$$\begin{aligned}
\mathcal{E}_{NSK}(t) &= \mathcal{E}_{NSK}(\rho, \bar{v}, w | r, \bar{V}, W) \\
&= \frac{1}{2} \int_\Omega \rho \left(|\bar{v} - \bar{V}|^2 + |w - W|^2\right) + \int_\Omega H(\rho | r) \\
&\quad + \nu \int_0^t \int_\Omega \mu(\rho) \left(|\nabla \bar{v} - \nabla \bar{V}|^2 + |\nabla w - \nabla W|^2\right) \\
&\quad + \frac{\nu}{2} \int_0^t \int_\Omega \lambda(\rho) \left((\operatorname{div} \bar{v} - \operatorname{div} \bar{V})^2 + (\operatorname{div} w - \operatorname{div} W)^2\right).
\end{aligned}$$

Proposition 26. *Any weak solution* $(\rho, \bar{v}, w)$ *of System* (29)-(31) *satisfies the following inequality for all* $t \in [0, T]$ *and for any test functions*

$$r \in C^1([0, T] \times \bar{\Omega}), \quad r > 0, \quad \bar{V},\, W \in C^1([0, T] \times \bar{\Omega}),$$

$$
\begin{aligned}
\mathcal{E}_{NSK}(\rho,\bar v,w|r,\bar V,W)(t) \;\le\;& \mathcal{E}_{NSK}(\rho,\bar v,w|r,\bar V,W)(0)\\
&+\int_0^t\int_\Omega \rho\left(\partial_t\bar V\cdot(\bar V-\bar v)+(\nabla\bar V\,u)\cdot(\bar V-\bar v)\right)\\
&+\int_0^t\int_\Omega \rho\left(\partial_t W\cdot(W-w)+(\nabla W\,u)\cdot(W-w)\right)\\
&+\nu\int_0^t\int_\Omega \mu(\rho)\,\left(|\nabla\bar V|^2+|\nabla W|^2-\nabla\bar v:\nabla\bar V-\nabla W:\nabla w\right)\\
&+\sqrt{\varepsilon^2-\nu^2}\int_0^t\int_\Omega \mu(\rho)\left(\nabla\bar v:\nabla W-{}^t\nabla w:\nabla\bar V\right)\\
&+\sqrt{\varepsilon^2-\nu^2}\int_0^t\int_\Omega \frac{\lambda(\rho)}{2}\left(\operatorname{div}\bar v\operatorname{div}W-\operatorname{div}w\operatorname{div}\bar V\right)\\
&+\frac{\nu}{2}\int_0^t\int_\Omega \lambda(\rho)\left((\operatorname{div}\bar V)^2-\operatorname{div}\bar v\operatorname{div}\bar V+(\operatorname{div}W)^2-\operatorname{div}w\operatorname{div}W\right)\\
&-\int_0^t\int_\Omega\left(\partial_t(H'(r))(\rho-r)+\rho\,\nabla(H'(r))\cdot\,u+p(\rho)\operatorname{div}W\right)\\
&-\nu\int_0^t\int_\Omega \mu'(\rho)H''(\rho)|\nabla\rho|^2.
\end{aligned}
$$

Proof. Thanks to Proposition 13, we have

$$
\begin{aligned}
\mathcal{E}_{NSK}(t)-\mathcal{E}_{NSK}(0) \;\le\;& \int_\Omega\left(\frac{\rho}{2}|\bar V|^2-\rho\bar v\cdot\bar V+\frac{\rho}{2}|W|^2-\rho w\cdot W\right)(t)\\
&-\int_\Omega\left(\frac{\rho}{2}|\bar V|^2-\rho\bar v\cdot\bar V+\frac{\rho}{2}|W|^2-\rho w\cdot W\right)(0)\\
&-\int_\Omega\left(H(r)+H'(r)(\rho-r)\right)(t)+\int_\Omega\left(H(r)+H'(r)(\rho-r)\right)(0)\\
&-\nu\int_0^t\int_\Omega \mu'(\rho)H''(\rho)|\nabla\rho|^2+\nu\left(I_1^{NS}+I_2^{NS}\right),
\end{aligned}
\tag{46}
$$

where

$$
2I_1^{NS}=\int_0^t\int_\Omega \lambda(\rho)\left(|\operatorname{div}\bar V|^2-2\operatorname{div}\bar v\operatorname{div}\bar V+|\operatorname{div}W|^2-2\operatorname{div}w\operatorname{div}W\right),
\tag{47}
$$

$$
I_2^{NS}=\int_0^t\int_\Omega \mu(\rho)\,\left(|\nabla\bar V|^2-2\nabla\bar v:\nabla\bar V+|\nabla W|^2-2\nabla w:\nabla W\right).
\tag{48}
$$

Using (30),

$$
\begin{aligned}
\partial_t(\rho\,w\cdot W) &= \partial_t(\rho\,w)\cdot W+\rho\,w\cdot\partial_t W\\
&= \Big(-\operatorname{div}(\rho\,w\otimes u)-\nabla p(\rho)+\nu\operatorname{div}(\mu(\rho)\nabla w)+\sqrt{\varepsilon^2-\nu^2}\operatorname{div}(\mu(\rho)\nabla\bar v)\Big)\cdot W\\
&\quad +A_1\cdot W+\rho\,w\cdot\partial_t W,
\end{aligned}
$$

where

$$
A_1=\frac{\nu}{2}\nabla(\lambda(\rho)\mathrm{div}w)+\frac{\sqrt{\varepsilon^2-\nu^2}}{2}\nabla(\lambda(\rho)\mathrm{div}\bar v).
$$

Using (31),

$$\begin{aligned}
\partial_t(\rho\,\bar v\,\cdot\bar V) &= \partial_t(\rho\,\bar v)\,\cdot\bar V+\rho\,\bar v\,\cdot\partial_t\bar V\\
&= \Big(-\mathrm{div}(\rho\,\bar v\otimes u)+\nu\,\mathrm{div}(\mu(\rho)\nabla\bar v)-\sqrt{\varepsilon^2-\nu^2}\,\mathrm{div}(\mu(\rho)\,{}^t\nabla w)\Big)\cdot\bar V+A_2\cdot\bar V+\rho\,\bar v\,\cdot\partial_t\bar V,
\end{aligned}$$

where

$$A_2=\frac{\nu}{2}\,\nabla(\lambda(\rho)\,\mathrm{div}\,\bar v)-\frac12\,\sqrt{\varepsilon^2-\nu^2}\,\nabla(\lambda(\rho)\,\mathrm{div}\,w).$$

Then,

$$\begin{aligned}
\mathcal{E}_{NSK}(t)-\mathcal{E}_{NSK}(0) \;\le\; & \int_0^t\int_\Omega\Big[\partial_t\left(\frac{\rho}{2}|\bar V|^2\right)-\rho\bar v\cdot\partial_t\bar V+\partial_t\left(\frac{\rho}{2}|W|^2\right)-\rho\,w\cdot\partial_t W\Big]\\
&+\int_0^t\int_\Omega\Big(\mathrm{div}(\rho\,\bar v\otimes u)-\nu\,\mathrm{div}(\mu(\rho)\nabla\bar v)+\sqrt{\varepsilon^2-\nu^2}\,\mathrm{div}(\mu(\rho)\,{}^t\nabla w)\Big)\cdot\bar V\\
&+\int_0^t\int_\Omega\Big(\mathrm{div}(\rho w\otimes u)+\nabla p(\rho)-\nu\,\mathrm{div}(\mu(\rho)\nabla w)-\sqrt{\varepsilon^2-\nu^2}\,\mathrm{div}(\mu(\rho)\nabla\bar v)\Big)\cdot W\\
&-\int_0^t\int_\Omega\partial_t(H(r)+H'(r)(\rho-r))-\nu\int_0^t\int_\Omega\mu'(\rho)H''(\rho)|\nabla\rho|^2\\
&-\int_0^t\int_\Omega(A_1\cdot W+A_2\cdot\bar V)+\nu\left(I_1^{NS}+I_2^{NS}\right).
\end{aligned}$$

Using an integration by parts, we obtain

$$\int_0^t\int_\Omega(A_1\cdot W+A_2\cdot\bar V)=\int_0^t\int_\Omega(-\nu A_3+\sqrt{\varepsilon^2-\nu^2}A_4),$$

where

$$2\,A_3=\lambda(\rho)\,\left(\mathrm{div}\,\bar v\,\mathrm{div}\,\bar V+\mathrm{div}\,w\,\mathrm{div}\,W\right),$$

$$2\,A_4=\lambda(\rho)\,\left(\mathrm{div}\,w\,\mathrm{div}\,\bar V-\mathrm{div}\,\bar v\,\mathrm{div}\,W\right).$$

Using (29) and an integration by parts

$$\begin{aligned}
\int_0^t\int_\Omega\partial_t(H(r)+H'(r)(\rho-r)) &= \int_0^t\int_\Omega\big(H'(r)\,\partial_t r+\partial_t(H'(r))(\rho-r)+H'(r)\,\partial_t\rho-H'(r)\,\partial_t r\big)\\
&= \int_0^t\int_\Omega\big(\partial_t(H'(r))(\rho-r)-H'(r)\mathrm{div}\,(\rho\,u)\big)\\
&= \int_0^t\int_\Omega\big(\partial_t(H'(r))(\rho-r)+\rho\nabla(H'(r))\cdot u\big).
\end{aligned}$$

Since

$$\partial_t\left(\frac{\rho}{2}|\bar V|^2\right)=\frac12\partial_t\rho|\bar V|^2+\rho\bar V\cdot\partial_t\bar V,\quad \partial_t\left(\frac{\rho}{2}|W|^2\right)=\frac12\partial_t\rho|W|^2+\rho W\cdot\partial_t W,$$

and since $\nabla\bar{v}, \nabla\bar{V}$ are symmetric matrices (recall that v and V are gradient of functions), thanks to (29) and integrations by parts we obtain

$$\begin{aligned}
\mathcal{E}_{NSK}(t) - \mathcal{E}_{NSK}(0) \leq & \int_0^t \int_\Omega \rho\ \partial_t \bar{V} \cdot (\bar{V} - \bar{v}) + \int_0^t \int_\Omega \rho\, (\nabla \bar{V}\, u) \cdot (\bar{V} - \bar{v}) \\
& + \int_0^t \int_\Omega \rho\, \partial_t W \cdot (W - w) + \int_0^t \int_\Omega \rho\, (\nabla W\, u) \cdot (W - w) \\
& + \nu \int_0^t \int_\Omega \mu(\rho)\ \big(\nabla \bar{v} : \nabla \bar{V} + \nabla w : \nabla W\big) \\
& + \sqrt{\varepsilon^2 - \nu^2} \int_0^t \int_\Omega \mu(\rho)\ \big(\nabla \bar{v} : \nabla W - {}^t\nabla w : \nabla \bar{V}\big) \\
& - \int_0^t \int_\Omega p(\rho)\, \mathrm{div} W - \nu \int_0^t \int_\Omega \mu'(\rho) H''(\rho) |\nabla \rho|^2 \\
& - \int_0^t \int_\Omega \partial_t (H'(r))(\rho - r) - \int_0^t \int_\Omega \rho \nabla (H'(r)) \cdot u \\
& + \int_0^t \int_\Omega (\nu\, A_3 - \sqrt{\varepsilon^2 - \nu^2}\, A_4) + \nu\, \big(I_1^{NS} + I_2^{NS}\big)\,,
\end{aligned}$$

which gives the proposition. □

Let us introduce the following system

$$\partial_t r + \mathrm{div}\, (r\, U) = 0, \tag{49}$$

$$\begin{aligned}
& r\ (\partial_t W + \nabla W\, U) + \nabla p(r) - \nu\ \mathrm{div}(\mu(r)\, \nabla W) - \frac{\nu}{2} \nabla (\lambda(r)\, \mathrm{div}\, W) \\
& \quad = \sqrt{\varepsilon^2 - \nu^2} \left(\mathrm{div}(\mu(r)\, \nabla \bar{V}) + \frac{1}{2} \nabla (\lambda(r)\, \mathrm{div}\, \bar{V}) \right),
\end{aligned} \tag{50}$$

$$\begin{aligned}
& r\ \big(\partial_t \bar{V} + \nabla \bar{V}\, U\big) - \nu\ \mathrm{div}(\mu(r)\, \nabla \bar{V}) - \frac{\nu}{2} \nabla (\lambda(r)\, \mathrm{div}\, \bar{V}) \\
& + \sqrt{\varepsilon^2 - \nu^2} \left(\mathrm{div}(\mu(r)\, {}^t\nabla W) + \frac{1}{2} \nabla (\lambda(r)\, \mathrm{div}\, W) \right) = 0,
\end{aligned} \tag{51}$$

with

$$U = W - \nu\, V, \quad \bar{V} = \sqrt{\varepsilon^2 - \nu^2}\, V.$$

Defining

$$\begin{aligned}
I_3^{NS} = & \int_0^t \int_\Omega \lambda(\rho)\, \big((\mathrm{div}\, \bar{V})^2 + (\mathrm{div}\, W)^2 - \mathrm{div}\, \bar{v}\, \mathrm{div}\, \bar{V} - \mathrm{div}\, w\, \mathrm{div}\, W\big) \\
+ & \int_0^t \int_\Omega \frac{\rho}{r} \Big(\nabla (\lambda(r)\, \mathrm{div}\, \bar{V}) \cdot (\bar{V} - \bar{v}) + \nabla (\lambda(r)\, \mathrm{div}\, W) \cdot (W - w) \Big),
\end{aligned} \tag{52}$$

$$\begin{aligned}
I_4^{NS} = & \int_0^t \int_\Omega \mu(\rho)\, \big(|\nabla \bar{V}|^2 + |\nabla W|^2 - \nabla \bar{v} : \nabla \bar{V} - \nabla W : \nabla w\big) \\
+ & \int_0^t \int_\Omega \frac{\rho}{r} \Big(\, \mathrm{div}\, \big(\mu(r) \nabla \bar{V}\big) \cdot (\bar{V} - \bar{v}) + \mathrm{div}\, (\mu(r) \nabla W) \cdot (W - w) \Big),
\end{aligned} \tag{53}$$

$$\begin{aligned} I_5^{NS} &= \int_0^t\int_\Omega \mu(r)\left(\nabla\bar V\nabla\left(\frac{\rho}{r}\right)\right)\cdot(w-W) + \int_0^t\int_\Omega \mu(r)\left({}^t\nabla W\nabla\left(\frac{\rho}{r}\right)\right)\cdot(\bar V-\bar v) \\ &\quad + \int_0^t\int_\Omega \mu(\rho)\left(\nabla\bar v:\nabla W - {}^t\nabla w:\nabla\bar V\right) + \int_0^t\int_\Omega \rho\,\frac{\mu(r)}{r}\left(\nabla\bar V:\nabla w - {}^t\nabla W:\nabla\bar v\right), \end{aligned} \tag{54}$$

$$\begin{aligned} 2\,I_6^{NS} &= \int_0^t\int_\Omega \frac{\rho}{r}\left(\nabla(\lambda(r)\,\mathrm{div}\,\bar V)\cdot(W-w) - \nabla(\lambda(r)\,\mathrm{div}\,W)\cdot(\bar V-\bar v)\right) \\ &\quad - \int_0^t\int_\Omega \lambda(\rho)\,\left(\mathrm{div}\,w\,\mathrm{div}\,\bar V - \mathrm{div}\,\bar v\,\mathrm{div}\,W\right), \end{aligned} \tag{55}$$

we have

Proposition 27. *Let $(r,\bar V,W)$ be a strong solution of* (49)-(51)*. Any weak solution $(\rho,\bar v,w)$ of System* (29)–(31) *satisfies the following inequality*

$$\begin{aligned} \mathcal{E}_{NSK}(t)-\mathcal{E}_{NSK}(0) &\le \int_0^t\int_\Omega \rho\,\left[(\nabla\bar V\,(u-U))\cdot(\bar V-\bar v) + (\nabla W\,(u-U))\cdot(W-w)\right] \\ &\quad - \int_0^t\int_\Omega \left(p(\rho)-p(r)-p'(r)\,(\rho-r)\right)\mathrm{div}\,U \\ &\quad + \frac{\nu}{\sqrt{\varepsilon^2-\nu^2}}\int_0^t\int_\Omega \left(\rho\,\nabla(H'(r))\cdot(\bar v-\bar V) - p(\rho)\;\mathrm{div}\,\bar V\right) \\ &\quad -\nu\int_0^t\int_\Omega \mu'(\rho)H''(\rho)|\nabla\rho|^2 \\ &\quad + \frac{\nu}{2}I_3^{NS} + \nu I_4^{NS} + \sqrt{\varepsilon^2-\nu^2}\,\left(I_5^{NS}+I_6^{NS}\right), \end{aligned}$$

where I_i^{NS} for $i=3,4,5,6$ are given by (52)-(55)*.*

Proof. Multiplying (50) by $\dfrac{\rho}{r}(W-w)$ and (51) by $\dfrac{\rho}{r}(\bar V-\bar v)$, integrating with respect to time and space, and using (49), we obtain

$$\begin{aligned}
\mathcal{E}_{NSK}(t)-\mathcal{E}_{NSK}(0) \;\le\;& \nu\int_0^t\int_\Omega \frac{\rho}{r}\left[\mathrm{div}(\mu(r)\nabla\bar V)\cdot(\bar V-\bar v)+\frac12\nabla(\lambda(r)\,\mathrm{div}\,\bar V)\cdot(\bar V-\bar v)\right]\\
&-\sqrt{\varepsilon^2-\nu^2}\int_0^t\int_\Omega\frac{\rho}{r}\left[\mathrm{div}(\mu(r)^t\nabla W)\cdot(\bar V-\bar v)+\frac12\nabla(\lambda(r)\,\mathrm{div}\,W)\cdot(\bar V-\bar v)\right]\\
&+\int_0^t\int_\Omega\rho\,(\nabla\bar V\,(u-U))\cdot(\bar V-\bar v)+\int_0^t\int_\Omega\rho\,(\nabla W\,(u-U))\cdot(W-w)\\
&+\nu\int_0^t\int_\Omega\frac{\rho}{r}\left[\mathrm{div}(\mu(r)\nabla W)\cdot(W-w)+\frac12\,\nabla(\lambda(r)\,\mathrm{div}\,W)\cdot(W-w)\right]\\
&+\sqrt{\varepsilon^2-\nu^2}\int_0^t\int_\Omega\frac{\rho}{r}\left[\mathrm{div}(\mu(r)\nabla\bar V)\cdot(W-w)+\frac12\nabla(\lambda(r)\,\mathrm{div}\,\bar V)\cdot(W-w)\right]\\
&+\nu\int_0^t\int_\Omega\mu(\rho)\,\left(|\nabla\bar V|^2+|\nabla W|^2-\nabla\bar v:\nabla\bar V-\nabla W:\nabla w\right)\\
&+\sqrt{\varepsilon^2-\nu^2}\int_0^t\int_\Omega\mu(\rho)\left(\nabla\bar v:\nabla W-{}^t\nabla w:\nabla\bar V\right)\\
&+\sqrt{\varepsilon^2-\nu^2}\int_0^t\int_\Omega\frac{\lambda(\rho)}{2}\left(\mathrm{div}\,\bar v\,\mathrm{div}\,W-\mathrm{div}\,w\,\mathrm{div}\,\bar V\right)\\
&-\frac{\nu}{2}\int_0^t\int_\Omega\lambda(\rho)\left((\mathrm{div}\,\bar V)^2-\mathrm{div}\,\bar v\,\mathrm{div}\,\bar V+(\mathrm{div}\,W)^2-\mathrm{div}\,w\,\mathrm{div}\,W\right)\\
&+\int_0^t\int_\Omega(p(r)\,\mathrm{div}\,U-p(\rho)\,\mathrm{div}\,W)-\nu\int_0^t\int_\Omega\mu'(\rho)H''(\rho)|\nabla\rho|^2+I_7^{NS},
\end{aligned}$$

where

$$\begin{aligned}
I_7^{NS} \;=\;& -\int_0^t\int_\Omega\frac{\rho}{r}\nabla(p(r))\cdot(W-w)-\int_0^t\int_\Omega\partial_t(H'(r))(\rho-r)\\
&-\int_0^t\int_\Omega\rho\nabla(H'(r))\cdot u+\int_0^t\int_\Omega H'(r)\,\partial_t r.
\end{aligned}\tag{56}$$

Using (49) $H'(r)\partial_t r+H'(r)\mathrm{div}(r\,U)=0$ which leads, with an integration by parts, to

$$\int_0^t\int_\Omega(H'(r)\partial_t r-r\nabla(H'(r))\cdot U)=0.$$

Then

$$\int_0^t\int_\Omega(H'(r)\partial_t r-\nabla p(r)\cdot U)=0,$$

or

$$\int_0^t\int_\Omega(H'(r)\partial_t r+p(r)\mathrm{div}(U))=0.\tag{57}$$

Moreover

$$\partial_t(H'(r))=-p'(r)\mathrm{div}U-H''(r)\nabla r\cdot U=-p'(r)\mathrm{div}U-\nabla(H'(r))\cdot U.$$

Then

$$\begin{aligned} I_7^{NS} &= \int_0^t \int_\Omega \rho\, \nabla(H'(r)) \cdot (-W + w + U - u) + \int_0^t \int_\Omega p'(r) \mathrm{div} U (\rho - r) \\ &= \nu \int_0^t \int_\Omega \rho \nabla(H'(r)) \cdot (v - V) + \int_0^t \int_\Omega p'(r) \mathrm{div} U (\rho - r). \end{aligned}$$

Therefore

$$\begin{aligned} \mathcal{E}_{NSK}(t) - \mathcal{E}_{NSK}(0) \;\le\; & \int_0^t \int_\Omega \rho \, \left[(\nabla \bar V \,(u - U)) \cdot (\bar V - \bar v) + (\nabla W\,(u-U)) \cdot (W - w) \right] \\ & + \int_0^t \int_\Omega [p(r)\,\mathrm{div} U - p(\rho)\,\mathrm{div} U - \nu\, p(\rho)\,\mathrm{div} V] - \nu \int_0^t \int_\Omega \mu'(\rho) H''(\rho) |\nabla \rho|^2 \\ & + \frac{\nu}{2} \int_0^t \int_\Omega \lambda(\rho) \left((\mathrm{div}\, \bar V)^2 - \mathrm{div}\, \bar v\, \mathrm{div}\, \bar V + (\mathrm{div}\, W)^2 - \mathrm{div}\, w\, \mathrm{div}\, W \right) \\ & + I_7^{NS} + \nu\, I_4^{NS} + \nu I_8^{NS} + \sqrt{\varepsilon^2 - \nu^2}\, (I_5^{NS} + I_6^{NS}) \end{aligned}$$

where

$$I_8^{NS} = \frac{1}{2} \int_0^t \int_\Omega \frac{\rho}{r} \left(\nabla(\lambda(r)\,\mathrm{div}\, \bar V) \cdot (\bar V - \bar v) + \nabla(\lambda(r)\,\mathrm{div}\, W) \cdot (W - w) \right), \tag{58}$$

and I_i^{NS} for $i = 4, 5, 6, 7$ are given by (53)-(56). Finally

$$\begin{aligned} \mathcal{E}_{NSK}(t) - \mathcal{E}_{NSK}(0) \;\le\; & \int_0^t \int_\Omega \left[\rho\, (\nabla \bar V \,(u - U)) \cdot (\bar V - \bar v) + \rho\, (\nabla W\,(u-U)) \cdot (W - w) \right] \\ & + \int_0^t \int_\Omega \left[(p'(r)(\rho - r) - p(\rho) + p(r))\ \mathrm{div}\, U - \nu\, p(\rho)\ \mathrm{div}\, V \right] \\ & - \frac{\nu}{\sqrt{\varepsilon^2 - \nu^2}} \int_0^t \int_\Omega \rho \nabla(H'(r)) \cdot (\bar V - \bar v) - \nu \int_0^t \int_\Omega \mu'(\rho) H''(\rho) |\nabla \rho|^2 \\ & + \frac{\nu}{2} \int_0^t \int_\Omega \lambda(\rho) \left((\mathrm{div}\, \bar V)^2 - \mathrm{div}\, \bar v\, \mathrm{div}\, \bar V + (\mathrm{div}\, W)^2 - \mathrm{div}\, w\, \mathrm{div}\, W \right) \\ & + \frac{\nu}{2} \int_0^t \int_\Omega \frac{\rho}{r} \left(\nabla(\lambda(r)\,\mathrm{div}\, \bar V) \cdot (\bar V - \bar v) + \nabla(\lambda(r)\,\mathrm{div}\, W) \cdot (W - w) \right) \\ & + \nu \int_0^t \int_\Omega \mu(\rho) \left(|\nabla \bar V|^2 + |\nabla W|^2 - \nabla \bar v : \nabla \bar V - \nabla W : \nabla w \right) \\ & + \nu \int_0^t \int_\Omega \frac{\rho}{r} \Big(\mathrm{div}\, \left(\mu(r) \nabla \bar V\right) \cdot (\bar V - \bar v) + \mathrm{div}\, (\mu(r) \nabla W) \cdot (W - w) \Big) \\ & + \sqrt{\varepsilon^2 - \nu^2}\ \left(I_5^{NS} + I_6^{NS} \right), \end{aligned}$$

with I_5^{NS} and I_6^{NS} given by (54) and (55). This gives the proposition. □

Lemma 28. *Let I_5^{NS} given by (54) and I_6^{NS} given by (55). Under the assumptions of Proposition 27, we have*

$$\begin{aligned} I_5^{NS} &= -\int_0^t\int_\Omega \rho\left(\frac{\mu(\rho)}{\rho}-\frac{\mu(r)}{r}\right)\Big(\mathrm{div}(\nabla\bar V)\cdot(W-w)+\mathrm{div}({}^t\nabla W)\cdot(\bar v-\bar V)\Big)\\ &\quad -\int_0^t\int_\Omega \rho\Big(\nabla\bar V(W-w)+\nabla W(\bar v-\bar V)\Big)\cdot(v-V), \end{aligned}$$

and

$$\begin{aligned} 2\,I_6^{NS} &= -2\int_0^t\int_\Omega \rho\left(\mu''(\rho)\nabla\rho-\mu''(r)\nabla r\right)\cdot\big((W-w)\ \mathrm{div}\,\bar V+(\bar v-\bar V)\ \mathrm{div}\,W\big)\\ &\quad -\int_0^t\int_\Omega \Big(\lambda(\rho)-\frac{\rho}{r}\lambda(r)\Big)\left((W-w)\ \cdot\nabla(\mathrm{div}\,\bar V)+(\bar v-\bar V)\ \cdot\nabla(\mathrm{div}\,W)\right). \end{aligned}$$

Proof. We recall that

$$I_5^{NS} = \int_0^t\int_\Omega \mu(r)\left[\left(\nabla\bar V\nabla\left(\frac{\rho}{r}\right)\right)\cdot(w-W)+\left({}^t\nabla W\nabla\left(\frac{\rho}{r}\right)\right)\cdot(\bar V-\bar v))\right]+I_9^{NS}$$

where, using the symmetry of $\nabla\bar V$ and $\nabla\bar v$ ($\bar V$ and $\bar v$ being gradients of functions),

$$\begin{aligned} I_9^{NS} &= \int_0^t\int_\Omega \mu(\rho)\left(\nabla\bar v:\nabla W-{}^t\nabla w:\nabla\bar V\right)+\int_0^t\int_\Omega \rho\,\frac{\mu(r)}{r}\left(\nabla\bar V:\nabla w-{}^t\nabla W:\nabla\bar v\right)\\ &= \int_0^t\int_\Omega \rho\left(\frac{\mu(\rho)}{\rho}-\frac{\mu(r)}{r}\right)(-\nabla\bar V:\nabla w+{}^t\nabla W:\nabla\bar v-\nabla\bar V:{}^t\nabla W+\nabla\bar V:\nabla W)\\ &= \int_0^t\int_\Omega \rho\left(\frac{\mu(\rho)}{\rho}-\frac{\mu(r)}{r}\right)(\nabla\bar V:(\nabla W-\nabla w)+{}^t\nabla W:(\nabla\bar v-\nabla\bar V))\\ &= \int_0^t\int_\Omega -\rho\left(\frac{\mu(\rho)}{\rho}-\frac{\mu(r)}{r}\right)(\mathrm{div}(\nabla\bar V)\cdot(W-w)+\mathrm{div}({}^t\nabla W)\cdot(\bar v-\bar V))\\ &\quad -\int_0^t\int_\Omega (\nabla\bar V(W-w)+\nabla W(\bar v-\bar V))\cdot\nabla\left(\mu(\rho)-\frac{\rho}{r}\mu(r)\right)\\ &= -\int_0^t\int_\Omega \rho\left(\frac{\mu(\rho)}{\rho}-\frac{\mu(r)}{r}\right)(\mathrm{div}(\nabla\bar V)\cdot(W-w)+\mathrm{div}({}^t\nabla W)\cdot(\bar v-\bar V))\\ &\quad -\int_0^t\int_\Omega \rho\left(\nabla\bar V(W-w)+\nabla W(\bar v-\bar V)\right)\cdot\left(\frac{\nabla\mu(\rho)}{\rho}-\frac{\nabla\mu(r)}{r}\right)\\ &\quad -\int_0^t\int_\Omega \mu(r)\left[\left(\nabla\bar V\nabla\left(\frac{\rho}{r}\right)\right)\cdot(w-W)+\left({}^t\nabla W\nabla\left(\frac{\rho}{r}\right)\right)\cdot(\bar V-\bar v)\right] \end{aligned}$$

and then

$$\begin{aligned} I_5^{NS} &= -\int_0^t\int_\Omega \rho\left(\frac{\mu(\rho)}{\rho}-\frac{\mu(r)}{r}\right)(\mathrm{div}(\nabla\bar V)\cdot(W-w)+\mathrm{div}({}^t\nabla W)\cdot(\bar v-\bar V))\\ &\quad -\int_0^t\int_\Omega \rho\left(\nabla\bar V(W-w)+\nabla W(\bar v-\bar V)\right)\cdot(v-V). \end{aligned}$$

We recall that

$$
\begin{aligned}
2\, I_6^{NS} \;=\; & \int_0^t\int_\Omega \frac{\rho}{r}\big(\nabla(\lambda(r)\operatorname{div}\bar V)\cdot(W-w)-\nabla(\lambda(r)\operatorname{div}W)\cdot(\bar V-\bar v)\big)\\
& -\int_0^t\int_\Omega \lambda(\rho)\,\big(\operatorname{div}w\operatorname{div}\bar V-\operatorname{div}\bar v\operatorname{div}W\big).
\end{aligned}
$$

An integration by parts gives

$$
2\, I_6^{NS} \;=\; 2\, I_{10}^{NS}-\int_0^t\int_\Omega\nabla\left(\frac{\rho}{r}\right)\lambda(r)\operatorname{div}\bar V\cdot(W-w)+\int_0^t\int_\Omega\nabla\left(\frac{\rho}{r}\right)\lambda(r)\operatorname{div}W\cdot(\bar V-\bar v),
$$

where

$$
\begin{aligned}
2\, I_{10}^{NS} \;=\; & \int_0^t\int_\Omega \lambda(\rho)(\operatorname{div}\bar v\operatorname{div}W-\operatorname{div}w\operatorname{div}\bar V)\\
& +\int_0^t\int_\Omega \frac{\rho}{r}\lambda(r)\,\big(\operatorname{div}W\operatorname{div}(\bar V-\bar v)-\operatorname{div}\bar V\operatorname{div}(W-w)\big)\\
=\; & -\int_0^t\int_\Omega \lambda(\rho)\,\big(\operatorname{div}\bar V\operatorname{div}(w-W)+\operatorname{div}\bar V\operatorname{div}W\big)\\
& +\int_0^t\int_\Omega \lambda(\rho)\,\big(\operatorname{div}W\operatorname{div}(\bar v-\bar V)+\operatorname{div}W\operatorname{div}\bar V\big)\\
& -\int_0^t\int_\Omega \frac{\rho}{r}\lambda(r)\,\big(\operatorname{div}\bar V\operatorname{div}(W-w)-\operatorname{div}W\operatorname{div}(\bar V-\bar v)\big)\\
=\; & \int_0^t\int_\Omega \Big[\Big(\lambda(\rho)-\frac{\rho}{r}\lambda(r)\Big)\operatorname{div}\bar V\operatorname{div}(W-w)+\Big(\lambda(\rho)-\frac{\rho}{r}\lambda(r)\Big)\operatorname{div}W\operatorname{div}(\bar v-\bar V)\Big]\\
=\; & -\int_0^t\int_\Omega \nabla\Big(\lambda(\rho)-\frac{\rho}{r}\lambda(r)\Big)\cdot\big((W-w)\operatorname{div}\bar V+(\bar v-\bar V)\operatorname{div}W\big)\\
& -\int_0^t\int_\Omega \Big(\lambda(\rho)-\frac{\rho}{r}\lambda(r)\Big)\,\big((W-w)\cdot\nabla(\operatorname{div}\bar V)+(\bar v-\bar V)\cdot\nabla(\operatorname{div}W)\big).
\end{aligned}
$$

Since $\lambda(\rho)=2(\rho\mu'(\rho)-\mu(\rho))$, we have $\lambda'(\rho)=2\,\rho\,\mu''(\rho)$, and then

$$
\nabla(\lambda(\rho)-\frac{\rho}{r}\lambda(r)) \;=\; 2\,\rho\mu''(\rho)\nabla\rho-\nabla\left(\frac{\rho}{r}\right)\lambda(r)-2\,\rho\,\mu''(r)\nabla r,
$$

which gives the result for I_6^{NS}. $\square$

Lemma 29. *Let I_5^{NS} given by (54) and I_6^{NS} given by (55). Let us assume $\mu(\rho)=\rho^{(s+3)/2}$ with $\gamma\geq s+2$ and $s\geq -1$. Under the assumptions of Proposition 27 we have*

$$
\left|I_5^{NS}+I_6^{NS}\right| \;\leq\; C\int_0^t\int_\Omega\big(\rho|w-W|^2+\rho|v-V|^2+\rho|\bar v-\bar V|^2+H(\rho|r)\big),
$$

where $C=C(r,\bar V,W)$ is a uniformly bounded constant on $\mathbb{R}^+\times\Omega$.

Proof. By definition of $\lambda(\rho)$, lemma 28 directly leads to

$$\begin{aligned} I_5^{NS} + I_6^{NS} &= -\int_0^t \int_\Omega \rho\,(\nabla \bar V (W-w))\cdot(v-V) + \int_0^t\int_\Omega \rho\,(\nabla W(\bar V - \bar v))\cdot(v-V) \\ &\quad - \int_0^t\int_\Omega \rho\,\big(\mu''(\rho)\nabla\rho - \mu''(r)\nabla(r)\big)\cdot\big((W-w)\,\operatorname{div}\bar V + (\bar v - \bar V)\,\operatorname{div} W\big) \\ &\quad - \int_0^t\int_\Omega \rho(\mu'(\rho)-\mu'(r))\,\big((W-w)\cdot\nabla(\operatorname{div}\bar V) + (\bar v - \bar V)\cdot\nabla(\operatorname{div} W)\big)\,. \end{aligned}$$

Moreover in an analogous way than for lemma 19, we can show that

$$\begin{aligned} &\left|\int_0^t\int_\Omega \rho\,\big(\mu''(\rho)\nabla\rho - \mu''(r)\nabla(r)\big)\cdot\big((W-w)\,\operatorname{div}\bar V + (\bar v - \bar V)\,\operatorname{div} W\big)\right| \\ &\qquad\qquad \le C\int_0^t\int_\Omega \rho\,\big(|W-w|^2 + |\bar v - \bar V|^2\big) \end{aligned}$$

and

$$\begin{aligned} &\left|\int_0^t\int_\Omega \rho(\mu'(\rho)-\mu'(r))\,\big((W-w)\cdot\nabla(\operatorname{div}\bar V) + (\bar v - \bar V)\cdot\nabla(\operatorname{div} W)\big)\right| \\ &\qquad\qquad \le C\int_0^t\int_\Omega \big(H(\rho|r) + \rho(|w-W|^2 + |\bar v - \bar V|^2)\big)\,. \end{aligned}$$

$\square$

Lemma 30. *Let I_3^{NS} given by* (52) *and I_4^{NS} given by* (53)*. Under the assumptions of Proposition 27, we have*

$$\begin{aligned} I_3^{NS} &= -2\int_0^t\int_\Omega \rho(\mu''(\rho)\nabla\rho - \mu''(r)\nabla r)\cdot((\bar V - \bar v)\operatorname{div}\bar V + (W-w)\operatorname{div} W) \\ &\quad - \int_0^t\int_\Omega \Big(\lambda(\rho) - \frac{\rho}{r}\lambda(r)\Big)\,(\nabla(\operatorname{div}\bar V)\cdot(\bar V - \bar v) + \nabla(\operatorname{div} W)\cdot(W-w)), \end{aligned}$$

and

$$\begin{aligned} I_4^{NS} &= -\int_0^t\int_\Omega \rho(v-V)\cdot\big(\nabla\bar V(\bar V - \bar v) + {}^t\nabla W(W-w)\big) \\ &\quad - \int_0^t\int_\Omega \rho\left(\frac{\mu(\rho)}{\rho} - \frac{\mu(r)}{r}\right)(\operatorname{div}(\nabla\bar V)\cdot(\bar V - \bar v) + \operatorname{div}(\nabla W)\cdot(W-w)). \end{aligned}$$

Proof. By definition in Proposition 27, we have

$$\begin{aligned} I_3^{NS} &= \int_0^t\int_\Omega \lambda(\rho)\,\big((\operatorname{div}\bar V)^2 + (\operatorname{div} W)^2 - \operatorname{div}\bar v\operatorname{div}\bar V - \operatorname{div} w\operatorname{div} W\big) \\ &\quad + \int_0^t\int_\Omega \frac{\rho}{r}\,\big(\nabla(\lambda(r)\operatorname{div}\bar V)\cdot(\bar V - \bar v) + \nabla(\lambda(r)\operatorname{div} W)\cdot(W-w)\big)\,. \end{aligned}$$

The first integral can be written

$$\int_0^t\int_\Omega \lambda(\rho)\,\big(\operatorname{div}\bar V\operatorname{div}(\bar V - \bar v) + \operatorname{div} W\operatorname{div}(W-w)\big)\,,$$

and, using integrations by parts, the second one becomes

$$-\int_0^t\int_\Omega \lambda(r)\nabla\left(\frac{\rho}{r}\right)\cdot\big(\operatorname{div}(\bar V)(\bar V-\bar v)+\operatorname{div}(W)(W-w)\big)$$
$$-\int_0^t\int_\Omega \frac{\rho}{r}\lambda(r)\left(\operatorname{div}\bar V\operatorname{div}(\bar V-\bar v)+\operatorname{div}W\operatorname{div}(W-w)\right).$$

Then

$$\begin{aligned} I_3^{NS} \quad &= \quad \int_0^t\int_\Omega \Big(\lambda(\rho)-\frac{\rho}{r}\lambda(r)\Big)\left(\operatorname{div}\bar V\operatorname{div}(\bar V-\bar v)+\operatorname{div}W\operatorname{div}(W-w)\right)\\ &\quad -\int_0^t\int_\Omega \lambda(r)\nabla\left(\frac{\rho}{r}\right)\cdot\left(\operatorname{div}(\bar V)(\bar V-\bar v)+\operatorname{div}(W)(W-w)\right).\end{aligned}$$

Using integrations by parts we obtain

$$\begin{aligned} &\int_0^t\int_\Omega \Big(\lambda(\rho)-\frac{\rho}{r}\lambda(r)\Big)\left(\operatorname{div}\bar V\operatorname{div}(\bar V-\bar v)+\operatorname{div}W\operatorname{div}(W-w)\right)\\ = \quad &-\int_0^t\int_\Omega \nabla\Big(\lambda(\rho)-\frac{\rho}{r}\lambda(r)\Big)\cdot\big((\bar V-\bar v)\operatorname{div}\bar V+(W-w)\operatorname{div}W\big)\\ &-\int_0^t\int_\Omega \Big(\lambda(\rho)-\frac{\rho}{r}\lambda(r)\Big)\big(\nabla(\operatorname{div}\bar V)\cdot(\bar V-\bar v)+\nabla(\operatorname{div}W)\cdot(W-w)\big).\end{aligned}$$

Moreover, using that $\lambda'(\rho)=2\rho\mu''(\rho)$, we have

$$\begin{aligned} &-\int_0^t\int_\Omega \nabla\Big(\lambda(\rho)-\frac{\rho}{r}\lambda(r)\Big)\cdot\big((\bar V-\bar v)\operatorname{div}\bar V+(W-w)\operatorname{div}W\big)\\ &=-2\int_0^t\int_\Omega \rho(\mu''(\rho)\nabla\rho-\mu''(r)\nabla r)\cdot\big((\bar V-\bar v)\operatorname{div}\bar V+(W-w)\operatorname{div}W\big)\\ &\qquad+\int_0^t\int_\Omega \lambda(r)\nabla\left(\frac{\rho}{r}\right)\cdot\left(\operatorname{div}(\bar V)(\bar V-\bar v)+\operatorname{div}(W)(W-w)\right).\end{aligned}$$

Then

$$\begin{aligned} I_3^{NS} \quad &= \quad -\int_0^t\int_\Omega 2\left(\rho\mu''(\rho)\nabla\rho-\rho\mu''(r)\nabla r\right)\cdot\left((\bar V-\bar v)\operatorname{div}\bar V+(W-w)\operatorname{div}W\right)\\ &\quad -\int_0^t\int_\Omega \Big(\lambda(\rho)-\frac{\rho}{r}\lambda(r)\Big)\left(\nabla\operatorname{div}(\bar V)\cdot(\bar V-\bar v)+\nabla(\operatorname{div}W)\cdot(W-w)\right).\end{aligned}$$

This concludes the proof for I_3^{NS}.

For I_4^{NS}, let us recall the definition:

$$\begin{aligned} I_4^{NS} \quad &= \quad \int_0^t\int_\Omega \mu(\rho)(|\nabla\bar V|^2+|\nabla W|^2-\nabla\bar v:\nabla\bar V-\nabla W:\nabla w)\\ &\quad +\int_0^t\int_\Omega \frac{\rho}{r}\left(\operatorname{div}(\mu(r)\nabla\bar V)\cdot(\bar V-\bar v)+\operatorname{div}(\mu(r)\nabla W)\cdot(W-w)\right).\end{aligned}$$

The first integral can be written

$$\int_0^t\int_\Omega \mu(\rho)\left(\nabla\bar V:(\nabla\bar V-\nabla\bar v)+\nabla W:(\nabla W-\nabla w)\right),$$

and, using integrations by parts, the second one becomes

$$-\int_0^t\int_\Omega \mu(r)\nabla\left(\frac{\rho}{r}\right)\cdot\big(\nabla\bar V(\bar V-\bar v)+{}^t\nabla W(W-w)\big)$$
$$-\int_0^t\int_\Omega \frac{\rho}{r}\mu(r)\big(\nabla\bar V:\nabla(\bar V-\bar v)+\nabla W:\nabla(W-w)\big).$$

Then

$$\begin{aligned}I_4^{NS} &= \int_0^t\int_\Omega \Big(\mu(\rho)-\frac{\rho}{r}\mu(r)\Big)\big(\nabla\bar V:\nabla(\bar V-\bar v)+\nabla W:\nabla(W-w)\big)\\ &\quad -\int_0^t\int_\Omega \mu(r)\nabla\left(\frac{\rho}{r}\right)\cdot(\nabla\bar V(\bar V-\bar v)+^t\nabla W(W-w)).\end{aligned}$$

Using integrations by parts and the symmetry of $\nabla\bar V$ ($\bar V$ being a gradient of function), we have

$$\begin{aligned}&\int_0^t\int_\Omega \Big(\mu(\rho)-\frac{\rho}{r}\mu(r)\Big)\big(\nabla\bar V:\nabla(\bar V-\bar v)+\nabla W:\nabla(W-w)\big)\\ = & -\int_0^t\int_\Omega \nabla\Big(\mu(\rho)-\frac{\rho}{r}\mu(r)\Big)\cdot\big(\nabla\bar V(\bar V-\bar v)+{}^t\nabla W(W-w)\big)\\ & -\int_0^t\int_\Omega \rho\left(\frac{\mu(\rho)}{\rho}-\frac{\mu(r)}{r}\right)\big(\mathrm{div}(\nabla\bar V)\cdot(\bar V-\bar v)+\mathrm{div}(\nabla W)\cdot(W-w)\big).\end{aligned}$$

Then

$$\begin{aligned}I_4^{NS} &= -\int_0^t\int_\Omega \rho\left(\frac{\nabla\mu(\rho)}{\rho}-\frac{\nabla\mu(r)}{r}\right)\cdot\big(\nabla\bar V(\bar V-\bar v)+{}^t\nabla W(W-w)\big)\\ &\quad -\int_0^t\int_\Omega \rho\left(\frac{\mu(\rho)}{\rho}-\frac{\mu(r)}{r}\right)(\mathrm{div}(\nabla\bar V)\cdot(\bar V-\bar v)+\mathrm{div}(\nabla W)\cdot(W-w)),\end{aligned}$$

which gives the result. □

Using the previous lemma and the symmetry of $\nabla\bar V$, we obtain the following lemma

Lemma 31. *Let I_3^{NS} given by (52) and I_4^{NS} given by (53). We assume $\mu(\rho)=\rho^{(s+3)/2}$ with $\gamma\geq s+2$ and $s\geq -1$. Under the assumptions of Proposition 27, we have*

$$\left|\frac{1}{2}I_3^{NS}+I_4^{NS}\right| \;\leq\; C\int_0^t\int_\Omega \big(\rho|v-V|^2+\rho|\bar v-\bar V|^2+\rho|w-W|^2+H(\rho|r)\big),$$

where $C=C(r,\bar V,W)$ is a uniformly bounded constant on $\mathbb{R}^+\times\Omega$.

Proof. We have:

$$\begin{aligned}
\frac{1}{2}I_3^{NS}+I_4^{NS} &= -\int_0^t\int_\Omega \rho\,(v-V)\cdot\big(\nabla\bar V(\bar V-\bar v)+{}^t\nabla W(W-w)\big)\\
&-\int_0^t\int_\Omega \rho\,(\mu''(\rho)\nabla\rho-\mu''(r)\nabla r)\cdot((\bar V-\bar v)\,\mathrm{div}\,\bar V+(W-w)\,\mathrm{div}\,W)\\
&-\int_0^t\int_\Omega \rho\,(\mu'(\rho)-\mu'(r))\big(\nabla(\mathrm{div}\,\bar V)\cdot(\bar V-\bar v)+\nabla(\mathrm{div}\,W)\cdot(W-w)\big)\\
&-\int_0^t\int_\Omega \rho\left(\frac{\mu(\rho)}{\rho}-\frac{\mu(r)}{r}\right)(\mathrm{div}(\nabla W)-\nabla(\mathrm{div}\,W))\cdot(W-w).
\end{aligned}$$

In an analogous way than for the lemma 19, we can show

$$\int_0^t\int_\Omega \rho\left(\frac{\mu(\rho)}{\rho}-\frac{\mu(r)}{r}\right)(\mathrm{div}(\nabla W)-\nabla(\mathrm{div}\,W))\cdot(W-w)\le C\int_0^t\int_\Omega\big(H(\rho|r)+\rho\,|W-w|^2\big)\,.$$

Then using an analogous result than the one used in the proof of lemma 29 we obtain the result. □

Let us now define

$$I_{11}^{NS} = -\frac{\nu}{\sqrt{\varepsilon^2-\nu^2}}\int_0^t\int_\Omega\big(\rho\nabla(H'(r))\cdot(\bar V-\bar v)+p(\rho)\,\mathrm{div}\,V\big)-\nu\int_0^t\int_\Omega \mu'(\rho)H''(\rho)|\nabla\rho|^2.$$

Using the definition of H and an integration by parts, we obtain

$$I_{11}^{NS} = -\frac{\nu}{\varepsilon^2-\nu^2}\int_0^t\int_\Omega \rho\left(\frac{p'(r)}{\mu'(r)}\bar V-\frac{p'(\rho)}{\mu'(\rho)}\bar v\right)\cdot(\bar V-\bar v), \tag{59}$$

with $v=\nabla(\mu(\rho))/\rho,\ \bar v=\sqrt{\varepsilon^2-\nu^2}v,\ V=\nabla(\mu(r))/r,\ \bar V=\sqrt{\varepsilon^2-\nu^2}V$. We can show the following proposition.

Proposition 32. *Let I_{11}^{NS} given by (59). Assuming $\mu(\rho)=\rho^{(s+3)/2}$ with $\gamma\ge s+2,\ s\ge -1$ and the hypothesis of Proposition 27, there exists a contant $C=C(r,U,\bar V,W)$ uniformly bounded on $\mathbb{R}^+\times\Omega$ such that*

$$I_{11}^{NS}\le C\frac{\nu}{\varepsilon^2-\nu^2}\int_0^t\int_\Omega H(\rho|r).$$

Proof. Using Lemma 9, we can write

$$I_{11}^{NS}=-\frac{\nu}{\varepsilon^2-\nu^2}\int_0^t\int_\Omega \rho\,\frac{p'(\rho)}{\mu'(\rho)}|\bar V-\bar v|^2-I_{12}^{NS} \tag{60}$$

where

$$I_{12}^{NS}=\frac{\nu}{\varepsilon^2-\nu^2}\int_0^t\int_\Omega\Big(\sqrt{\varepsilon^2-\nu^2}\nabla\phi_1(\rho|r)+\phi_2(\rho|r)\bar V\Big)\cdot\bar V. \tag{61}$$

Using an integration by parts

$$I_{12}^{NS} = -\frac{\nu}{\sqrt{\varepsilon^2-\nu^2}}\int_0^t\int_\Omega\phi_1(\rho|r)\,\mathrm{div}(\bar V)+\frac{\nu}{\varepsilon^2-\nu^2}\int_0^t\int_\Omega\phi_2(\rho|r)\bar V\cdot\bar V.$$

Now using lemma 7 we obtain

$$I_{12}^{NS}\le\frac{C\,\nu}{\sqrt{\varepsilon^2-\nu^2}}\int_0^t\int_\Omega H(\rho|r)+\frac{C\,\nu}{\varepsilon^2-\nu^2}\int_0^t\int_\Omega H(\rho|r)\le\frac{C\,\nu}{\varepsilon^2-\nu^2}\int_0^t\int_\Omega H(\rho|r),$$

which gives the result due to the expression (60) and the sign of the first quantity in the right-hand side. ☐

Theorem 33. *Assuming $\mu(\rho)=\rho^{(s+3)/2}$, $\gamma\geq s+2$ and $s\geq -1$, any weak solution $(\rho,\bar v,w)$ of System* (29)-(31) *satisfies the following inequality*

$$\mathcal{E}_{NSK}(t)-\mathcal{E}_{NSK}(0) \;\leq\; C\left(1+\frac{\nu}{\varepsilon^2-\nu^2}\right)\int_0^t \mathcal{E}_{NSK}(\xi)d\xi \tag{62}$$

where $(r,\bar V,W)$ is a strong solution of (49)-(51) *and where $C=C(r,U,\bar V,W)$ is a constant uniformly bounded on $\mathbb{R}^+\times\Omega$.*

Proof. Thanks to Proposition 27 we have

$$\begin{aligned}
\mathcal{E}_{NSK}(t)-\mathcal{E}_{NSK}(0) \;\leq\; & \int_0^t\int_\Omega \rho\,\left[(\nabla\bar V\,(u-U))\cdot(\bar V-\bar v)+(\nabla W\,(u-U))\cdot(W-w)\right]\\
&-\int_0^t\int_\Omega \big(p(\rho)-p(r)-p'(r)\,(\rho-r)\big)\operatorname{div}U\\
&+I_{11}^{NS}+\frac{\nu}{2}I_3^{NS}+\nu I_4^{NS}+\sqrt{\varepsilon^2-\nu^2}\,\left(I_5^{NS}+I_6^{NS}\right),
\end{aligned}$$

with I_i^{NS} for $i=3,4,5,6$ given by (52)-(55) and I_{11}^{NS} given by (59). This gives with the regularity of U, $\bar V$ and W and the previous lemmas

$$\begin{aligned}
\mathcal{E}_{NSK}(t)-\mathcal{E}_{NSK}(0) \;\leq\; & C\int_0^t\int_\Omega \rho\,\left(|u-U|^2+|v-V|^2+|\bar v-\bar V|^2+|w-W|^2\right)\\
&-\int_0^t\int_\Omega \big(p(\rho)-p(r)-p'(r)\,(\rho-r)\big)\operatorname{div}U\\
&+C\frac{\nu}{\varepsilon^2-\nu^2}\int_0^t\int_\Omega H(\rho|r)\\
\leq\; & C\left(1+\frac{\nu}{\varepsilon^2-\nu^2}\right)\int_0^t \mathcal{E}_{NSK}(\xi)d\xi.
\end{aligned}$$

☐

Corollary 34. *Let $(r,\bar V,W)$ be a strong solution of* (49)-(51). *Assuming $\mu(\rho)=\rho^{(s+3)/2}$, $\gamma\geq s+2$ and $s\geq -1$ any weak solution $(\rho,w,\bar v)$ of* (29)-(31) *satisfies the following inequality*

$$\mathcal{E}_{NSK}(\rho,\bar v,w|r,\bar V,W)(t)\leq \;\mathcal{E}_{NSK}(\rho,\bar v,w|r,\bar V,W)(0)\exp\left(C\left(1+\frac{\nu}{\varepsilon^2-\nu^2}\right)t\right),$$

where $C=C(r,U,\bar V,W)$ is a constant uniformly bounded on $\mathbb{R}^+\times\Omega$.

Proof. Thanks to the previous proposition and the Gronwall's Lemma, we have the inequality. ☐

Let U be a given and smooth function. We define r as the strong solution of (35), and we introduce the functions $\mathscr{E}_1^\nu$ and $\mathscr{E}_2^\nu$ such that

$$\begin{aligned} r\,(\partial_t W + U\cdot\nabla W) + \nabla p(r) - \nu\,\mathrm{div}(\mu(r)\,\nabla W) - \frac{\nu}{2}\nabla(\lambda(r)\,\mathrm{div}\,W) \\ -\sqrt{\varepsilon^2-\nu^2}\left(\mathrm{div}(\mu(r)\,\nabla\bar V) + \frac{1}{2}\nabla(\lambda(r)\,\mathrm{div}\,\bar V)\right) = \mathscr{E}_1^\nu(r,\bar V,W), \end{aligned} \tag{63}$$

$$\begin{aligned} r\,\left(\partial_t\bar V + U\cdot\nabla\bar V\right) - \nu\,\mathrm{div}(\mu(r)\nabla\bar V) - \frac{\nu}{2}\nabla(\lambda(r)\,\mathrm{div}\,\bar V) \\ +\sqrt{\varepsilon^2-\nu^2}\left(\mathrm{div}(\mu(r)\,{}^t\nabla W) + \frac{1}{2}\nabla(\lambda(r)\,\mathrm{div}\,W)\right) = \mathscr{E}_2^\nu(r,\bar V,W). \end{aligned} \tag{64}$$

In a same way than for the proof of Theorem 33, we have the following result.

Proposition 35. *Let us assume $\mu(\rho) = \rho^{(s+3)/2}$ (i.e. $K(\rho) = \dfrac{(s+3)^2}{4}\rho^s$), $\gamma \geq s+2$ and $s \geq -1$. Let $(\rho,\bar v,w)$ be a weak solution of System* (29)-(31) *and $(r,\bar V,W)$ a strong solution of* (35), (63)-(64). *Then*

$$\mathcal{E}_{NSK}(\rho,\bar v,w|r,\bar V,W)(t) - \mathcal{E}_{NSK}(\rho,\bar v,w|r,\bar V,W)(0) \;\leq\; C\left(1+\frac{\nu}{\varepsilon^2-\nu^2}\right)\int_0^t \mathcal{E}_{NSK} + b^\nu(t),$$

with

$$b^\nu(t) = \int_0^t\int_\Omega\left[\frac{\rho}{r}\mathscr{E}_1^\nu\cdot(W-w) + \frac{\rho}{r}\mathscr{E}_2^\nu\cdot(\bar V-\bar v)\right],$$

and where $C = C(r,U,\bar V,W)$ is a constant uniformly bounded on $\mathbb{R}^+\times\Omega$.

Using the Gronwall's Lemma, we immediately obtain the following corollary.

Corollary 36. *Let us assume $\mu(\rho) = \rho^{(s+3)/2}$ (i.e. $K(\rho) = \dfrac{(s+3)^2}{4}\rho^s$) with $\gamma \geq s+2$ and $s \geq -1$. Let $(\rho,\bar v,w)$ be a weak solution of System* (29)-(31) *and $(r,\bar V,W)$ a strong solution of* (35), (63)-(64). *Then*

$$\mathcal{E}_{NSK}(t) \;\leq\; \mathcal{E}_{NSK}(0)\,\exp(F^\nu\,t) + F^\nu\int_0^t b^\nu(\xi)\,\exp(F^\nu\,(t-\xi))\,d\xi + b^\nu(t),$$

where b^ν is defined in Proposition 35 and

$$F^\nu = C\left(1+\frac{\nu}{\varepsilon^2-\nu^2}\right),$$

with $C = C(r,U,\bar V,W)$ a constant uniformly bounded on $\mathbb{R}^+\times\Omega$.

4.2. **Dissipative solution and weak-strong uniqueness result.** Let us now give the definition of what is called a dissipative solution of the compressible Navier-Stokes-Korteweg System. To this end, let us first introduce the function $\mathscr{E}^\nu$ given by

$$\begin{aligned} r\,(\partial_t U + U\cdot\nabla U) + \nabla p(r) - 2\,\nu\,\mathrm{div}(\mu(r)D(U)) - \nu\nabla(\lambda(\rho)\,\mathrm{div}\,U) \\ +\varepsilon^2\left[(\mathrm{div}(\mu(r)^t\nabla V) + \frac{1}{2}\nabla(\lambda(r)\,\mathrm{div}\,V)\right] = \mathscr{E}^\nu(r,U), \end{aligned} \tag{65}$$

with U a given smooth enough function and r a strong solution of equation (35). Using

$$rV = \nabla\mu(r),\;\; \bar V = \sqrt{\varepsilon^2-\nu^2}\,V,\;\; W = U+\nu\,V,$$

and equation (35) we obtain

$$r\ (\partial_t W + U\cdot\nabla W) + \nabla p(r) - \nu\ \mathrm{div}(\mu(r)\,\nabla W) \tag{66}$$

$$-\frac{\nu}{2}\nabla(\lambda(r)\,\mathrm{div}\,W) - \sqrt{\varepsilon^2-\nu^2}\left(\mathrm{div}(\mu(r)\,\nabla\bar V) + \frac{1}{2}\nabla(\lambda(r)\,\mathrm{div}\,\bar V)\right) = \mathscr{E}^\nu(r,U),$$

$$r\ \left(\partial_t\bar V + U\cdot\nabla\bar V\right) + \sqrt{\varepsilon^2-\nu^2}\ \left(\mathrm{div}(\mu(r)\,{}^t\nabla U) + \frac{1}{2}\nabla(\lambda(r)\,\mathrm{div}\,U\right) = 0. \tag{67}$$

Before giving the definition let us recall that $\mathcal{E}_{NSK}(t)$ stands for

$$\mathcal{E}_{NSK}(t) = \mathcal{E}_{NSK}(\rho,\bar v, w|r,\bar V, W)(t).$$

Definition 37. *Let us assume $\mu(\rho)=\rho^{(s+3)/2}$, $\gamma\geq s+2$ and $s\geq -1$. Let ρ_0 and u_0 smooth enough. The pair (ρ,u) is a dissipative solution of (13)-(14), (1) if the triplet $(\rho,\bar v,w)$ (with $\rho\, v=\nabla\mu(\rho)$, $\bar v=\sqrt{\varepsilon^2-\nu^2}\,v$, $w=u+\nu\, v$) satisfies for all strong solution (r,U) of (35), (68) or in other words $(r,\bar V,W)$ strong solution of (35), (69)-(70)*

$$\mathcal{E}_{NSK}(t) \leq \mathcal{E}_{NSK}(0)\,\exp(F^\nu t) + F^\nu\int_0^t b_{NSK}(\xi)\,\exp(F^\nu\,(t-\xi))\,d\xi + b_{NSK}(t),$$

with F^ν given in Corollary 36 and

$$b_{NSK}(t) = \int_0^t\int_\Omega\left[\frac{\rho}{r}\mathscr{E}^\nu\cdot(W-w)\right].$$

As in the case of the Euler-Korteweg system, a direct consequence of the method is the following weak-strong uniqueness result.

Theorem 38. *Let us assume $\mu(\rho)=\rho^{(s+3)/2}$, $\gamma\geq s+2$ and $s\geq -1$. Let us consider (ρ,u) an entropic solution to the compressible Navier-Stokes-Korteweg system and define $w=u+\nu\,\dfrac{\nabla\mu(\rho)}{\rho}$ and $\bar v=\sqrt{\varepsilon^2-\nu^2}\,\dfrac{\nabla\mu(\rho)}{\rho}$. Let us assume that there exists (r,U) a strong solution of the compressible Navier-Stokes-Korteweg System and assume that $W=U+\nu\dfrac{\nabla\mu(r)}{r}$ and $\bar V=\sqrt{\varepsilon^2-\nu^2}\,\dfrac{\nabla\mu(r)}{r}$.*

If $(\rho_0,u_0)=(r,U)(t=0)$ then $(\rho,\bar v,w)=(r,\bar V,W)$ or $(\rho,u)=(r,U)$, which corresponds to a weak-strong uniqueness property.

Finally, let us give the definition 37 in the particular case of $K(\rho)=1/\rho$ which corresponds to the quantum Navier-Stokes system. This one will be used in Section 5. To this end we introduce the function $\mathscr{E}^\nu_{NSQ}$ given by

$$r\,(\partial_t U + U\cdot\nabla U) + \nabla p(r) - 2\,\nu\,\mathrm{div}(rD(U)) + \varepsilon^2(\mathrm{div}(r^t\nabla V) = \mathscr{E}^\nu_{NSQ}(r,U), \tag{68}$$

with U a given smooth enough function and r a strong solution of Equation (35). Using

$$rV=\nabla\mu(r),\ \bar V=\sqrt{\varepsilon^2-\nu^2}\,V,\ W=U+\nu\,V,$$

and Equation (35) we obtain

$$r\ (\partial_t W + U\cdot\nabla W) + \nabla p(r) - \nu\ \mathrm{div}(r\,\nabla W) - \sqrt{\varepsilon^2-\nu^2}\,\mathrm{div}(r\,\nabla\bar V) = \mathscr{E}^\nu(r,U), \tag{69}$$

$$r\ \left(\partial_t\bar V + U\cdot\nabla\bar V\right) + \sqrt{\varepsilon^2-\nu^2}\ \mathrm{div}(r\,{}^t\nabla U) = 0. \tag{70}$$

We define $\mathcal{E}_{NSQ}(t)$ by

$$\mathcal{E}_{NSQ}(t) = \mathcal{E}_{NSK}(t) \text{ with } K(\rho)=1/\rho.$$

Definition 39. *Let ρ_0 and u_0 smooth enough. The pair (ρ, u) is a dissipative solution of* (22), (23), (1) *if the triplet $(\rho, \bar{v}, w)$ (with $\rho\, v = \nabla\mu(\rho)$, $\bar{v} = \sqrt{\varepsilon^2 - \nu^2}\, v$, $w = u + \nu\, v$) satisfies for all strong solution (r, U) of* (35), (68) *or in other words $(r, \bar{V}, W)$ strong solution of* (35), (69)-(70) *with $K(\rho) = 1/\rho$,*

$$\mathcal{E}_{NSQ}(t) \le \mathcal{E}_{NSQ}(0)\, \exp(F^\nu t) + F^\nu \int_0^t b_{NSQ}(\xi)\, \exp(F^\nu\,(t-\xi))\, d\xi + b_{NSQ}(t),$$

with F^ν given in Corollary 36 and

$$b_{NSQ}(t) = \int_0^t \int_\Omega \left[\frac{\rho}{r} \mathscr{E}^\nu_{NSQ} \cdot (W - w)\right].$$

Remark 40. *Note that by definition, using Corollary 36, all weak solution of* (22)-(23), (1) *is also a dissipative solution in the sense of Definition 39.*

5. From the quantum Navier-Stokes system to the quantum Euler system: the viscous limit

We can now perform the limit of a dissipative solution of the quantum Navier-Stokes system to one of the quantum Euler system when the viscosity constant ν tends to zero. Thanks to the entropies, we have the following regularities on the global weak solution of the quantum Navier-Stokes equations:

$$\sqrt{\rho^\nu}\, \bar{v}^\nu \in L^\infty(0,T; L^2(\Omega)),\ \sqrt{\rho^\nu}\, w^\nu \in L^\infty(0,T; L^2(\Omega)),\ H(\rho^\nu) \in L^\infty(0,T, L^1(\Omega)),$$

where

$$\bar{v}^\nu = \sqrt{\varepsilon^2 - \nu^2}\, \nabla \log \rho^\nu, \quad w^\nu = u^\nu + \nu\, \nabla \log \rho^\nu.$$

Let (r, U, V, W) such that

$$0 < c \le r,\ \sqrt{r}\, V \in L^\infty(0,T; L^2(\Omega)),\ \sqrt{r}\, W \in L^\infty(0,T; L^2(\Omega)),\ H(r) \in L^\infty(0,T; L^1(\Omega)),$$
$$\nabla W,\ \nabla V \in L^\infty(0,T; L^\infty(\Omega)),\ r \in L^\infty(0,T; W^{2,\infty}(\Omega)).$$

The goal of this section is then to prove the following result:

Theorem 41. *Let ρ_0 and u_0 smooth enough. Let (ρ^ν, u^ν) be a weak entropic solution to the quantum Navier-Stokes system* (22)-(23) *with initial conditions* (1). *Let (ρ, u) be the weak limit of (ρ^ν, u^ν) when ν tends to 0 in the sense*

$$\rho^\nu \rightharpoonup \rho \text{ weakly} \star \text{ in } L^\infty(0,T; L^\gamma(\Omega)),$$
$$\sqrt{\rho^\nu}\, w^\nu \rightharpoonup \sqrt{\rho} u \text{ weakly} \star \text{ in } L^\infty(0,T; L^2(\Omega)),$$
$$\sqrt{\rho^\nu}\, \bar{v}^\nu \rightharpoonup \varepsilon \sqrt{\rho} v \text{ weakly} \star \text{ in } L^\infty(0,T; L^2(\Omega)),$$

with $\rho\, v = \nabla \rho$. Then (ρ, u) is a dissipative solution of the quantum Euler system (11)-(12) *with initial conditions* (1).

Proof : According to Remark 40, the pair (ρ^ν, u^ν) being an entropic weak solution, it is also a dissipative one. We want to prove that (ρ, u), which is the limit of (ρ^ν, u^ν) when ν tends to zero, is a dissipative solution of (11)-(12) satisfying the initial conditions (1). The goal is then to prove that (ρ, u) satisfies Definition 25. Let us define $v = \nabla \log \rho$ (because in this case $\mu(\rho) = \rho$). Let (r, U, V) (with $V = \nabla \log r$) be a strong solution of (35), (44)-(45). We define

$$\bar{V}^\nu = \sqrt{\varepsilon^2 - \nu^2}\, V,\ W^\nu = U + \nu\, V.$$

Then it is easy to see that $(r, \bar{V}^\nu, W^\nu)$ is solution of (35), (69)-(70) with

$$\mathscr{E}^\nu_{NSQ}(r,U) = \mathscr{E}(r,U) - 2\,\nu\,\mathrm{div}(rD(U)).$$

Then using Definition 39, (ρ^ν, u^ν) being a dissipative solution we have

$$\mathcal{E}_{NSQ}(t) \le \mathcal{E}_{NSQ}(0)\,\exp(F^\nu t) + F^\nu \int_0^t b^\nu_{NSQ}(\xi)\,\exp(F^\nu\,(t-\xi))\,d\xi + b^\nu_{NSQ}(t), \tag{71}$$

with

$$F^\nu \;=\; C\left(1 + \frac{\nu}{\varepsilon^2 - \nu^2}\right),$$

and

$$b^\nu_{NSQ}(t) = \int_0^t \int_\Omega \left[\frac{\rho}{r}\,(\mathscr{E} - 2\,\nu\,\mathrm{div}(rD(U)))\cdot(W-w)\right].$$

Since by definition we have

$$\mathcal{E}_{NSQ}(\rho^\nu, \bar{v}^\nu, w^\nu | r, \bar{V}^\nu, W^\nu)(t) = \frac{1}{2}\int_\Omega \rho^\nu\,\left(|\bar{v}^\nu - \bar{V}^\nu|^2 + |w^\nu - W^\nu|^2\right)$$
$$+ \int_\Omega \big(H(\rho^\nu) - H(r) - H'(r)(\rho^\nu - r)\big) + \nu \int_0^t \int_\Omega \rho^\nu\,\left(|\nabla \bar{v}^\nu - \nabla \bar{V}|^2 + |\nabla w^\nu - \nabla W|^2\right),$$

we easily obtain

$$\frac{1}{2}\int_\Omega \rho^\nu\,\left(|\bar{v}^\nu - \bar{V}^\nu|^2 + |w^\nu - W^\nu|^2\right) \;+\; \int_\Omega \big(H(\rho^\nu) - H(r) - H'(r)(\rho^\nu - r)\big)$$
$$\le\; \mathcal{E}_{NSQ}(\rho^\nu, \bar{v}^\nu, w^\nu | r, \bar{V}^\nu, W^\nu)(t)$$

and

$$\mathcal{E}_{NSQ}(\rho^\nu, \bar{v}^\nu, w^\nu | r, \bar{V}^\nu, W^\nu)(0) = \frac{1}{2}\int_\Omega \rho^\nu\,\left(|\bar{v}^\nu - \bar{V}^\nu|^2 + |w^\nu - W^\nu|^2\right)(0)$$
$$+ \int_\Omega \big(H(\rho^\nu) - H(r) - H'(r)(\rho^\nu - r)\big)(0).$$

Then (71) gives

$$\frac{1}{2}\int_\Omega \rho^\nu\,\left(|\bar{v}^\nu - \bar{V}^\nu|^2 + |w^\nu - W^\nu|^2\right)(t) + \int_\Omega \big(H(\rho^\nu) - H(r) - H'(r)(\rho^\nu - r)\big)(t)$$
$$\le\; \mathcal{E}_{NSQ}(\rho^\nu, \bar{v}^\nu, w^\nu | r, \bar{V}^\nu, W^\nu)(0)\exp(F^\nu\,t) + F^\nu \int_0^t b^\nu_{NSQ}(\xi)\,\exp(F^\nu\,(t-\xi))\,d\xi + b^\nu_{NSQ}(t).$$

It remains now to pass to the limit ν tends to zero in this inequality. Clearly, using the lower semi-continuity of the term $\mathcal{E}_{NSQ}(\rho^\nu, \bar{v}^\nu, w^\nu | r, \bar{V}^\nu, W^\nu)$, the left-hand side is greater than

$$\frac{1}{2}\int_\Omega \rho\,\left(\varepsilon^2 |\,v - V|^2 + |u - U|^2\right)(t) + \int_\Omega H(\rho|r)(t),$$

which is $\mathcal{E}_{EuQ}(\rho, u, v | r, U, V)(t)$ (*i.e.* $\mathcal{E}_{EuK}(\rho, u, v | r, U, V)(t)$ given by (32) with $K(\rho) = 1/\rho$). For the right hand side, we use the direct limit of the term $\mathcal{E}_{NSQ}(\rho^\nu, \bar{v}^\nu, w^\nu | r, \bar{V}^\nu, W^\nu)(0)$ (through the expression of the initial data) and b^ν_{NSQ} tends to

$$b_{EuQ}(t) = \int_0^t \int_\Omega \left[\frac{\rho}{r}\mathscr{E}\cdot(U-u)\right],$$

to conclude that

$$\mathcal{E}_{EuQ}(t) \leq \mathcal{E}_{EuQ}(0)\exp(C\,t) + b_{EuQ}(t) + C\int_0^t \exp(C\,(t-\xi))b_{EuQ}(\xi)\,d\xi,$$

where $C = C(\varepsilon^2, r, U, V)$ is a uniformly bounded constant on $\mathbb{R}^+ \times \Omega$. Therefore we finally obtain that (ρ, u) satisfies the Definition 25 and then is a dissipative solution of (11)-(12), (1). $\square$

6. Appendix

6.1. **Equivalence of $\mathcal{E}_{EuK}$ and the relative entropy in** [26]**.** Let us consider the relative entropy functional, denoted $\mathcal{E}_{EuK}(\rho, u, v|r, U, V)$ and defined by (32). The goal of this section is to prove that this relative entropy is equivalent to the relative entropy defined by (2.23) in [26] under the concavity assumption on K with $K(\rho) = \rho^s$. Let us first recall the relative entropy $\mathcal{E}_{EuK}^{GLT}$ defined in [26]. It reads

$$\mathcal{E}_{EuK}^{GLT}(\rho, u, \nabla\rho|r, U, \nabla r) = \frac{1}{2}\int_\Omega \rho|u-U|^2 + \frac{1}{2}\varepsilon^2\int_\Omega I_T + \int_\Omega H(\rho|r), \tag{72}$$

where

$$I_T = K(\rho)|\nabla\rho|^2 - K(r)|\nabla r|^2 - K'(r)|\nabla r|^2(\rho - r) - 2K(r)\nabla r(\nabla\rho - \nabla r).$$

Note that I_T corresponds to the term $K(\rho)|\nabla\rho|^2$ linearized in the variables (ρ, q) where $q = \nabla\rho$. Let us now introduce the quantity

$$I_{EuK}^2 \quad = \quad \rho\left|\sqrt{\frac{K(\rho)}{\rho}}\nabla\rho - \sqrt{\frac{K(r)}{r}}\nabla r\right|^2 = \left|\sqrt{K(\rho)}\nabla\rho - \sqrt{\frac{\rho}{r}}\sqrt{K(r)}\nabla r\right|^2.$$

Then our Euler-Korteweg modulated energy reads

$$\mathcal{E}_{EuK}(\rho, u, v|r, U, V) = \frac{1}{2}\int \rho|u-U|^2 + \frac{1}{2}\varepsilon^2\int I_{EuK}^2 + \int H(\rho|r),$$

where $v = \sqrt{K(\rho)}\nabla\rho/\sqrt{\rho}$ and $V = \sqrt{K(r)}\nabla r/\sqrt{r}$. Let us prove that under the hypothesis on K introduced in [26]

$$\mathcal{E}_{EuK}(\rho, u, v|r, U, V) = 0 \qquad \Leftrightarrow \qquad \mathcal{E}_{EuK}^{GLT}(\rho, u, \nabla\rho|r, U, \nabla r) = 0.$$

If so, we prove by this way that our relative entropy and the one in [26] are equivalent under the hypothesis in [26]. Our convergence result will therefore be more general that the one in [26] because it does not asked for concavity hypothesis on $K(\rho)$. First let us prove the following lemma:

Lemma 42. *We have the equality*

$$I_{\mathrm{EuK}}^2 + K(r)\,|\nabla r|^2\left|\sqrt{\frac{K(r)}{K(\rho)}} - \sqrt{\frac{\rho}{r}}\right|^2 - K(r)^2\,|\nabla r|^2\left(\frac{1}{K(\rho)} - \frac{1}{K(r)} + \frac{K'(r)}{K(r)^2}(\rho - r)\right)$$
$$= I_T + 2\sqrt{K(r)}\nabla r I_{\mathrm{EuK}}\left(\sqrt{\frac{K(r)}{K(\rho)}} - \sqrt{\frac{\rho}{r}}\right)$$

Proof. After computations, we check that

$$\begin{aligned} I_{EuK}^2 &= I_T + K'(r)|\nabla r|^2(\rho - r) + \frac{\rho}{r}K(r)|\nabla r|^2 \\ &\quad -2\sqrt{\frac{\rho}{r}}\sqrt{K(\rho)}\sqrt{K(r)}\nabla\rho\cdot\nabla r + 2K(r)\nabla r\cdot\nabla\rho - K(r)|\nabla r|^2 \\ &= I_T + I_1 \end{aligned}$$

where

$$I_1 = K(r)|\nabla r|^2\left(\frac{\rho}{r} - 1 + \frac{K'(r)}{K(r)}(\rho - r) + 2\frac{\sqrt{K(r)}}{\sqrt{K(\rho)}}\sqrt{\frac{\rho}{r}} - 2\frac{\rho}{r}\right) + I_2,$$

with

$$I_2 = 2\sqrt{K(r)}\,\nabla r\, I_3\, I_{EuK} \qquad \text{and} \qquad I_3 = \sqrt{\frac{K(r)}{K(\rho)}} - \sqrt{\frac{\rho}{r}}.$$

Corollary 43. *Let $K(\rho) = \rho^s$ with $-1 \le s \le 0$, then*

$$\mathcal{E}_{EuK}^{GLT}(\rho, u, \nabla\rho|r, U, \nabla r) = 0 \qquad \Leftrightarrow \qquad \mathcal{E}_{EuK}(\rho, u, v|r, U, V) = 0.$$

Proof. Under the assumption on K, we check that

$$\begin{aligned} I_3^2 &= \left(\sqrt{\frac{K(r)}{K(\rho)}} - \sqrt{\frac{\rho}{r}}\right)^2 \\ &= \left(\sqrt{\left(\frac{\rho}{r}\right)^{-s}} - \sqrt{\frac{\rho}{r}}\right)^2 \\ &\le 2\left(\sqrt{\left(\frac{\rho}{r}\right)^{-s}} - 1\right)^2 + 2\left(1 - \sqrt{\frac{\rho}{r}}\right)^2 \\ &\le 2\frac{1}{r^{-s}}(\sqrt{\rho^{-s}} - \sqrt{r^{-s}})^2 + \frac{2}{r}(\sqrt{r} - \sqrt{\rho})^2 \\ &\le \frac{2}{r^{-s}}|\rho^{-s} - r^{-s}| + \frac{2}{r}|r - \rho| \end{aligned}$$

with $0 \le -s \le 1$. Assume $\mathcal{E}_{EuK}^{GLT}(\rho, u, \nabla\rho|r, U, \nabla r) = 0$, then $I_3 = 0$ and

$$\left(\frac{1}{K(\rho)} - \frac{1}{K(r)} + \frac{K'(r)}{K(r)^2}(\rho - r)\right) = 0.$$

Therefore using Lemma 42 we conclude $\mathcal{E}_{EuK}(\rho, u, v|r, U, V) = 0$ (the inverse follows the same lines). This ends the proof. □

6.2. **Definition of the operators.** For the convenience of the reader we recall in this Section all the definitions of the operators used in this article. The definitions used here are the ones presented in [7] in Appendix A.

Let f be a scalar, u, v two vectors and $\sigma = (\sigma_{ij})_{1\le i,j\le d}$ a tensor field defined on $\Omega \subset \mathbb{R}^d$ smooth enough.

- Denoting by $v_1, \cdots, v_d$ the coordinates of v, we call *divergence* of v the scalar given by:
$$\mathrm{div}(v) = \sum_{i=1}^{d} \frac{\partial v_i}{\partial x_i}.$$
- We call *laplacian* of f the scalar given by:
$$\Delta f = \mathrm{div}(\nabla f) = \sum_{i=1}^{d} \frac{\partial^2 f}{\partial x_i^2}.$$
- We call *gradient* of v the tensor given by:
$$\nabla v = \left(\frac{\partial v_i}{\partial x_j}\right)_{1\le i,j\le d}.$$
- We call *divergence* of σ the vector given by:
$$\mathrm{div}(\sigma) = \left(\sum_{j=1}^{d} \frac{\partial \sigma_{ij}}{\partial x_j}\right)_{1\le i\le d}.$$
- We call *laplacian* of v the vector given by:
$$\Delta v = \mathrm{div}(\nabla v).$$
- We call *tensor product* of u and v the tensor given by:
$$u \otimes v = (u_i v_j)_{1\le i,j\le d}.$$

Proposition 44. *Let u, v, w three smooth enough vectors on Ω and r a scalar smooth enough on Ω. We have the following properties.*

- $(u \otimes v)w = (v \cdot w)u$,
- $\mathrm{div}(u \otimes v) = (\mathrm{div}\, v)u + (v \cdot \nabla)u$,
- $\mathrm{div}(r\, u) = \nabla r \cdot u + r\ \mathrm{div}\, u$,
- $\mathrm{div}(r\, u \otimes v) = (\nabla r \cdot v)u + r(v \cdot \nabla)u + r\ \mathrm{div}(v)u$.

Definition 45. *Let τ and σ be two tensors of order 2. We call scalar product of the two tensors the real defined by:*
$$\sigma : \tau = \sum_{1\le i,j\le d} \sigma_{ij}\tau_{ij}.$$
The norm associated to this scalar product is simply denoted by $|\cdot|$ in such a way that
$$|\sigma|^2 = \sigma : \sigma.$$

Remark 46. *By definition we have*
$$\sigma : \tau = {}^t\sigma : {}^t\tau$$

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