Isotropic diffeomorphisms: solutions to a differential system for a deformed random fields study

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Abstract This Note presents the resolution of a differential system on the plane that translates a geometrical problem about isotropic deformations of area and length. The system stems from a probability study on deformed random fields [1], which are the composition of a random field with invariance properties defined on the plane with a deterministic diffeomorphism. The explicit resolution of the differential system allows to prove that a weak notion of isotropy of the deformed field, linked to its excursion sets, in fact coincides with the strong notion of isotropy. The present Note first introduces the probability framework that gave rise to the geometrical issue and then proposes its resolution.

The motivation for the result featured in the present article originates from a probability problem about deformed random fields. Indeed, one of the main results of [1] needed a complete characterization of isotropic deformed fields and it turned out such a description was given by solutions to a system of nonlinear partial differential equations. The resolution of that system is a major step in the proof, yet it is completely independent. Moreover, its analytical flavour as well as the geometric classification it contains makes it interesting on its own and out of step with the probability nature of [1].

Geometrically, the aim is to investigate the class of planar transformations $F : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ that are C^2 and transform isotropically areas of rectangles and lengths of segments :

$$\forall \varphi \in SO(2),\ \forall i \in \{1, 2\}, \quad l_i(F \circ \varphi(E)) = l_i(F(E)) \tag{1}$$

where l_i stands for the Lebesgue measure in $\mathbb{R}^1$ or $\mathbb{R}^2$ depending on E being a segment (embedded in $\mathbb{R}^1$) or a rectangle (viewed as a surface embedded in

$\mathbb{R}^2$). Such a property boils down to the fact that both the norms of each of the columns of the cartesian Jacobian matrix of the polar form of F and its determinant are radial : introducing $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$ the one-dimensional torus,

$$\forall (r,\theta) \in \mathbb{R}^{+*} \times \mathbb{T}, \quad \left\| \mathrm{Jac}_F(r,\theta) \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\| = g(r), \quad \left\| \mathrm{Jac}_F(r,\theta) \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\| = h(r) \quad \text{and} \quad \det\left(\mathrm{Jac}_F\right)(r,\theta) = f(r), \tag{2}$$

where f, g and h are $C^2(\mathbb{R}^{+*}, \mathbb{R})$.

In this Note we prove that the class (1) is exactly the family of "spiral deformations" described in polar coordinates by $F(r,\theta) = (R(r,\theta), \Theta(r,\theta))$ with :

$$\forall (r,\theta) \in \mathbb{R}^+ \times \mathbb{T}, \quad R(r,\theta) = \mathcal{R}(r) \quad \text{and} \quad \Theta(r,\theta) = \pm\theta + \overline{\Theta}(r). \tag{3}$$

The main goal of the present Note is therefore to introduce the background and one of the main results of [1] in order to motivate the analytic problem and then to solve it.

1 Characterization of isotropy in deformed random *via* excursion sets

All the random fields mentioned in this introduction are defined on $\mathbb{R}^2$, take real values and we furthermore assume that they are Gaussian.

A deformed random field is constructed with a regular, stationary and isotropic random field X composed with a deterministic diffeomorphism F such that $F(0) = 0$. The result of this composition is a random field $X \circ F$. Stationarity, respectively isotropy (refered to in the following as strong isotropy), consists in an invariance of the law of a random field under translations, respectively under rotations in $\mathbb{R}^2$. Even though the underlying field X is isotropic, the deformed random fields constructed with X are generally not. It is however possible to characterize explicitely a diffeomorphism F such that for any underlying field X, the deformed field $X \circ F$ is strongly isotropic. Such diffeomorphisms are exactly the spiral diffeomorphisms introduced above (3).

The objective in [1] is to study a deformed field using sparse information, that is, the information provided by excursion sets of the field over some basic subsets in $\mathbb{R}^2$. If a real number u is fixed, the excursion set of the field $X \circ F$ above level u over a compact set T is the random set

$$A_u(X \circ F, T) := \{t \in T \,/\, X(F(t)) \geqslant u\}.$$

One useful functional to study the topology of sets is the Euler characteristic, denoted by χ. Heuristically, the Euler characteristic of a one-dimensional compact regular set is simply the number of intervals in this set ; the Euler characteristic of a two-dimensional compact regular set is the number of connected components minus the number of holes in this set.

A rotational invariance condition of the mean Euler characteristic of the excursion sets of $X \circ F$ over rectangles is then introduced as a weak isotropy property. More precisely, a random field $X\circ F$ is said to satisfy this weak isotropy property if for any real u, for any rectangle T in $\mathbb{R}^2$ and for rotation φ,

$$\mathbb{E}[\chi(A_u(X\circ F,\varphi(T)))] = \mathbb{E}[\chi(A_u(X\circ F,T))]. \tag{4}$$

The latter condition is in particular clearly true if the deformed field $X\circ F$ is strongly isotropic or, in other words, if F is a spiral diffeomorphism. Provided that we add some assumptions on X, for any rectangle T in $\mathbb{R}^2$, the expectation of $\chi(A_u(X\circ F,T))$ can be expressed as a linear combination of the area of the set $F(T)$ and the length of its frontier, with coefficients depending on u only and not on the precise law of X.

Consequently, it occurs that Condition (4) is equivalent to Condition (1) and therefore to Condition (2) on F in the present Note. Theorem 2.1 that we are going to demonstrate in this Note therefore implies that the spiral diffeomorphisms are the only solutions. This means that the associated deformed field is strongly isotropic, as explained before. As a result, the weak definition of isotropy coincides with the strong definition, as far as deformed fields are concerned.

From a practical point of view, a major consequence is that we only need information contained in the excursion sets of a deformed random field $X\circ F$ (more precisely, Condition (4) fulfilled) to decide the issue of isotropy.

2 Planar deformations modifying lengths and areas isotropically

We now turn to the study of the class of C^2 planar transformations satisfying (1). Using a polar representation for such F we translate the rotational invariant property (2) into the following system of non-linear partial differential equations.

Theorem 2.1 *Let two functions $R:\mathbb{R}^+\times\mathbb{T}\longrightarrow\mathbb{R}^+$ and $\Theta:\mathbb{R}^+\times\mathbb{T}\longrightarrow\mathbb{T}$ be continuous on $\mathbb{R}^+\times\mathbb{T}$ and C^2 in $\mathbb{R}^{+*}\times\mathbb{T}$ that satisfy : $R(\cdot,\cdot)$ is surjective and $R(0,\cdot)$ is a constant function. Let f, g and h be C^1 functions from $\mathbb{R}^{+*}$ to $\mathbb{R}$ such that f does not vanish. Then the following differential equalities hold*

$$\forall(r,\theta)\in\mathbb{R}^{+*}\times\mathbb{T},\quad f(r) = R\partial_r R\partial_\theta\Theta - R\partial_\theta R\partial_r\Theta \tag{5}$$

$$g(r) = (\partial_r R)^2 + (R\partial_r\Theta)^2 \tag{6}$$

$$h(r) = (\partial_\theta R)^2 + (R\partial_\theta\Theta)^2 \tag{7}$$

if and only if there exist ε_1 and ε_2 in $\{-1,1\}$ and Θ_0 in $\mathbb{T}$ such that

(i) h is strictly increasing and continuous on $\mathbb{R}^+$ with $h(0)=0$;

(ii) for all $r>0$, $f(r)=\varepsilon_1\frac{h'(r)}{2}$ and $g(r)h(r)\geqslant f^2(r)$;

(iii) the functions R and Θ are given by

$$\forall(r,\theta)\in\mathbb{R}^+\times\mathbb{T},\quad R(r,\theta)=\sqrt{h(r)}\quad \text{and}\quad \Theta(r,\theta)=\varepsilon_1\theta+\Theta_0+\varepsilon_2\int_0^r\frac{\sqrt{h(r_*)g(r_*)-f^2(r_*)}}{h(r_*)}dr_*.$$

Of important note is the fact that the assumptions made on R and Θ before the differential system are here to ensure that they indeed describe a polar representant of a planar deformation F. Also note that the solutions obtained above are indeed spiral deformations (3).

The rest of this section is devoted to the proof of the theorem above. We first find an equivalent version of $(5)-(6)-(7)$ that is not quadratic. Second, we prove that this equivalent problem can be seen as a specific case of a hyperbolic system of equations solely constraint to (7). Finally we show that these two constraints necessarily imply Theorem 2.1.

2.1 A non quadratic equivalent

Here we prove the following proposition that gives the shape of the derivatives of R and Θ.

Proposition 2.2 *Let R, Θ, f, g and h be functions as described by Theorem 2.1. Then, they satisfy the system $(5)-(6)-(7)$ if and only if there exist p in $\mathbb{N}$ and a continuous function $\Phi : \mathbb{R}^+ \times \mathbb{T} \longrightarrow \mathbb{T}$ such that*

$$\begin{aligned}\forall (r,\theta) \in \mathbb{R}^+ \times \mathbb{T}, \quad \partial_r R &= \sqrt{g(r)}\cos\left(\Phi(r,\theta)\right)\\ R\partial_r\Theta &= \sqrt{g(r)}\sin\left(\Phi(r,\theta)\right)\\ \partial_\theta R &= (-1)^p\sqrt{\frac{gh-f^2}{g}}(r)\cos\left(\Phi(r,\theta)\right) - \frac{f}{\sqrt{g}}(r)\sin\left(\Phi(r,\theta)\right)\\ R\partial_\theta\Theta &= (-1)^p\sqrt{\frac{gh-f^2}{g}}(r)\sin\left(\Phi(r,\theta)\right) + \frac{f}{\sqrt{g}}(r)\cos\left(\Phi(r,\theta)\right).\end{aligned} \tag{8}$$

Proof of Proposition 2.2 First, functions satisfying (8) are solutions to our original system $(5)-(6)-(7)$ by mere computation.

Now assume that the functions are solutions of $(5)-(6)-(7)$. The key is to see the quantities involved as complex numbers functions : $Z_1 = \partial_r R + iR\partial_r\Theta$ and $Z_2(r,\theta) = R\partial_\theta\Theta - i\partial_\theta R$. Then the system $(5)-(6)-(7)$ translates into

$$\forall (r,\theta) \in \mathbb{R}^+ \times \mathbb{T}, \quad |Z_1(r,\theta)|^2 = g(r) \quad \text{and} \quad |Z_2(r,\theta)|^2 = h(r) \quad \text{and} \quad \mathrm{Re}\left(Z_1(r,\theta)\overline{Z_2(r,\theta)}\right) = f(r).$$

We first note that $f(r)^2 \leqslant g(r)h(r)$ and recalling that f never vanishes on $\mathbb{R}^+$ it follows that neither g nor h can be null on R^+. Therefore since

$$g + h \pm 2f = \left(\partial_r R \pm R\partial_\theta\Theta\right)^2 + \left(\partial_\theta R \mp R\partial_r\Theta\right)^2 \geqslant 0$$

it follows $2\,|f| \leqslant g(r) + h(r)$ and therefore we must in fact have

$$\forall r \in \mathbb{R}^+, \quad f(r)^2 < g(r)h(r). \tag{9}$$

We can thus define the complex numbers

$$W_1(r,\theta) = \frac{Z_1(r,\theta)}{\sqrt{g(r)}} \quad \text{and} \quad W_2(r,\theta) = \sqrt{\frac{f(r)^2}{g(r)\left(g(r)h(r) - f(r)^2\right)}}\left[Z_1(r,\theta) - \frac{g(r)}{f(r)}Z_2(r,\theta)\right].$$

which are of prime importance since they satisfy the following orthonormality property :

$$\forall (r,\theta) \in \mathbb{R}^+ \times \mathbb{T}, \quad |W_1(r,\theta)|^2 = |W_2(r,\theta)|^2 = 1 \quad \text{and} \quad \mathrm{Re}\left(W_1(r,\theta)\overline{W_2(r,\theta)}\right) = 0.$$

We deduce that there exist a continuous function $\Phi : \mathbb{R}^+ \times \mathbb{T} \longrightarrow \mathbb{T}$ and an integer $p \geqslant 0$ such that

$$\forall (r,\theta) \in \mathbb{R}^+ \times \mathbb{T}, \quad W_1(r,\theta) = e^{i\Phi(r,\theta)} \quad \text{and} \quad W_2(r,\theta) = e^{-i\Phi(r,\theta) + (2p+1)\frac{\pi}{2}}.$$

Coming back to the original Z_1, Z_2 and then to R and Θ concludes the proof. ■

2.2 A hyperbolic system under constraint

We now find a more general system of equations satisfied by the functions we are looking for as well as a restrictive property that defines them.

Lemma 2.3 *Let R, Θ, f, g and h be functions as described by Theorem 2.1. Then, they satisfy the system $(5)-(6)-(7)$ if and only if they satisfy (7) and there exist $\alpha, \beta : \mathbb{R}^+ \longrightarrow \mathbb{R}$ continuous with $\beta(r) > 0$ such that*

$$\forall (r,\theta) \in \mathbb{R}^{+*} \times \mathbb{T}, \quad \begin{aligned} \partial_\theta R &= \alpha(r)\partial_r R - \beta(r) R \partial_r \Theta \\ R\partial_\theta \Theta &= \alpha(r) R \partial_r \Theta + \beta(r)\partial_r R. \end{aligned} \tag{10}$$

Remark 1 Even if this set of equations still seems non-linear, it actually is linear in $X = (ln(R), \Theta)$. It indeed satisfies a vectorial transport equation $\partial_\theta X + A(r)\partial_r X = 0$ with $A(r)$ being skew-symmetric and invertible. This equation is however non trivial as even in the case $\alpha(r) = 0$ we are left to solve $\partial_\theta\left[ln(R)\right] = -\beta(r)\partial_\theta\Theta$ and $\partial_\theta\Theta = \beta(r)\partial_\theta\left[ln(R)\right]$. And so $ln(R)$ and Θ are both solutions to $\partial_{\theta\theta} f + \beta(r)^2\partial_{rr} f = 0$. For more on this subject we refer the reader to [2].

Proof of Lemma 2.3 The necessary condition follows directly from the set of equations (8) given in Proposition 2.2, denoting $\alpha(r) = (-1)^p\frac{\sqrt{gh-f^2}}{g}(r)$ and $\beta(r) = f(r)/g(r)$ and dividing by $R > 0$.

The sufficient condition follows by direct computation from (10) and (7), defining $g(r) = \frac{h(r)}{\alpha^2(r)+\beta^2(r)}$ and $f(r) = \beta(r)g(r) > 0$. ■

We now show that solutions to the hyperbolic system that are constraint by (7) must satisfy that R is radially symmetric.

Proposition 2.4 *Let R, Θ be solution to* (10) *with R and Θ verifying the assumptions of Theorem 2.1. Suppose that (R, Θ) also satisfies* (7) *; then R is isotropic : for all (r, θ) in $\mathbb{R}^+ \times \mathbb{T}$, $R(r, \theta) = \mathcal{R}(r)$ with moreover $\mathcal{R}(0) = 0$ and $\mathcal{R}'(r) > 0$ for all $r > 0$.*

Proof of Proposition 2.4 Let us first prove that $R(r, \theta) = 0$ if and only if $r = 0$.

The surjectivity of R implies that there exists (r_0, θ_0) such that $R(r_0, \theta_0) = 0$. If $r_0 \neq 0$ then by Lemma 2.3 R and Θ satisfy (5) at (r_0, θ_0) and thus $f(r_0) = 0$. This contradicts the fact that f does not vanish. Therefore $r_0 = 0$ and since $R(0, \cdot)$ is constant we get that

$$\forall \theta \in \mathbb{T}, \quad R(0, \theta) = 0. \tag{11}$$

Recall that R is positive and it follows

$$\exists r_0 > 0,\ \forall r \in (0, r_0],\ \forall \theta \in \mathbb{T}, \quad \partial_r R(r, \theta) \geqslant 0. \tag{12}$$

We now turn to the study of local extrema of $R(r, \cdot)$. For a fixed $r > 0$, if ϕ is a local extremum of $R(r, \cdot)$ then $\partial_\theta R(r, \phi) = 0$ and also, thanks to (10), $\alpha(r) \partial_r R(r, \phi) = \beta(r) R \partial_r \Theta(r, \phi)$. Plugging these equalities inside the equation satisfied by $R \partial_\theta \Theta$ in (10) and since $\beta(r) \neq 0$ we get

$$\left(R \partial_\theta \Theta\right)(r, \phi) = \frac{\alpha^2(r) + \beta^2(r)}{\beta(r)} \partial_r R(r, \phi).$$

Then we can apply the constraint on the angular derivatives (7) to conclude

$$\forall (r, \phi) \text{ such that } \partial_\theta R(r, \phi) = 0, \quad \left(\partial_r R(r, \phi)\right)^2 = \frac{\beta^2(r)}{(\alpha^2(r) + \beta^2(r))^2} h(r). \tag{13}$$

Finally, for any $r > 0$, $R(r, \cdot)$ is 2π-periodic and thus has a global maximum and a global minimum on $\mathbb{T}$. Let us choose r_1 in $(0, r_0)$ where r_0 has been defined in (12). Functions R and Θ are C^2 in $\mathbb{R}^{+*} \times \mathbb{T}$ and for any θ, the determinant of the Jacobian of (R, θ) is equal to $f(r_1)/R(r_1)$, by (5). This determinant does not vanish so by the local inverse function theorem there exist $[r_1 - \delta_1, r_1 + \delta_1] \subset [0, r_0]$ and two functions $\phi_m,\ \phi_M : [r_1 - \delta_1, r_1 + \delta_1] \longrightarrow \mathbb{T}$ that are C^1 and such that

$$\forall r > 0, \quad R(r, \phi_m(r)) = \min_{\theta \in [-\pi, \pi]} R(r, \theta) \quad \text{and} \quad R(r, \phi_M(r)) = \max_{\theta \in [-\pi, \pi]} R(r, \theta).$$

By definition, for all r in $[r_1 - \delta_1, r_1 + \delta_1]$, $\partial_\theta R(r, \phi_{m/M}(r)) = 0$ and because ϕ_m and ϕ_M are C^1 this implies :

$$\forall r \in [r_1 - \delta_1, r_1 + \delta_1], \quad \partial_r \left(R(r, \phi_{m/M}(r))\right) = \partial_r R(r, \phi_{m/M}(r)). \tag{14}$$

Thanks to (14), (13) and (12) we thus obtain $\partial_r\left(R(r,\phi_m(r))\right)=\partial_r\left(R(r,\phi_M(r))\right)$ on $[r_1-\delta_1,r_1+\delta_1]$. This implies

$$\forall r\in[r_1-\delta_1,r_1+\delta_1],\qquad \min_{\theta\in[-\pi,\pi]}R(r,\theta)=\max_{\theta\in[-\pi,\pi]}R(r,\theta)+\left[\min_{\theta\in[-\pi,\pi]}R(r_1-\delta_1,\theta)-\max_{\theta\in[-\pi,\pi]}R(r_1-\delta_1,\theta)\right].$$

To conclude we iterate : either $r_1-\delta_1=0$ and we define $r_2=0$ or we can start our argument again with $r_2=r_1-\delta_1$. Iterating the process we construct a sequence $(r_n)_{n\in\mathbb{N}^*}$ either strictly decreasing or reaching 0 at a certain step and such that

$$\forall r\in[r_{n+1},r_1],\qquad \min_{\theta\in[-\pi,\pi]}R(r,\theta)=\max_{\theta\in[-\pi,\pi]}R(r,\theta)+\min_{\theta\in[-\pi,\pi]}R(r_{n+1},\theta)-\max_{\theta\in[-\pi,\pi]}R(r_{n+1},\theta). \tag{15}$$

This sequence thus converges to $r_\infty\geqslant 0$. If $r_\infty\neq 0$ then we could start our process again at r_∞ and construct another decreasing sequence still satisfying (15). In the end we will construct a sequence converging to 0 so without loss of generality we assume that $r_\infty=0$.

Hence, since R is continuous on $\mathbb{R}^+\times\mathbb{T}$ and (11) holds true, it follows by taking the limit as n tends to ∞ in (15) :

$$\forall r\in[0,r_1],\qquad \min_{\theta\in[-\pi,\pi]}R(r,\theta)=\max_{\theta\in[-\pi,\pi]}R(r,\theta).$$

The equality above implies that $\theta\mapsto R(r,\theta)$ is constant for any $r\leqslant r_1$.

The rotational invariance of $R(r,\cdot)$ holds for any $r_1<r_0$ where r_0 is such that (12) holds true. Therefore, denoting $r_M=\sup\{r>0:\ \forall\theta\in\mathbb{T},\quad \partial_r R(r,\theta)\geqslant 0\}$ it follows that for any $r\leqslant r_M$, $\theta\mapsto R(r,\theta)$ is constant. Since $R(\cdot,\cdot)$ is C^2 in $\mathbb{R}^{+*}\times\mathbb{T}$ we infer

$$\forall r\in(0,r_M),\ \forall\theta\in\mathbb{T},\quad \partial_\theta R(r,\theta)=0.$$

Suppose that $r_M<+\infty$ then by continuity of $\partial_\theta R$ we get $\partial_\theta R(r_M,\theta)=0$ for all θ. But the definition of r_M implies the existence of θ_M such that $\partial_r R(r_M,\theta_M)=0$. It follows that

$$\partial_\theta R(r_M,\theta_M)=0\quad\text{and}\quad \partial_r R(r_M,\theta_M)=0.$$

Plugging the above inside the constraint (7) yields $f(r_M)=0$ which is a contradiction since $r_M>0$.

We thus conclude that $r_M=+\infty$ and that $\theta\mapsto R(r,\theta)$ is invariant for any $r\geqslant 0$.

■

2.3 Isotropic solutions and proof of Theorem 2.1

Proof of Theorem 2.1 Let us consider R, Θ, f, g and h as in the statement of Theorem 2.1 and satisfying the system $(5)-(6)-(7)$. Thanks to Proposition

2.4, there exists $\mathcal{R} : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ continuous on $\mathbb{R}^+$ and C^2 on $\mathbb{R}^{+*}$ such that $\mathcal{R}(0) = 0$, $\mathcal{R}'(r) > 0$ and $R(r,\theta) = \mathcal{R}(r)$ for any (r,θ) in $\mathbb{R}^+ \times \mathbb{T}$.

First, we recall (9) : $h(r)g(r) > f(r)^2$ for $r > 0$ and since f does not vanish it follows that $h(r) > 0$ and $g(r) > 0$ for any $r > 0$.

Then, thanks to (7) we infer $|\partial_\theta \Theta(r,\theta)| = \sqrt{h(r)}/\mathcal{R}(r)$. The right-hand side does not vanish so neither does $\partial_\theta \Theta$. By continuity it keeps a fixed sign s in $\{-1, 1\}$.

$$\forall (r,\theta) \in \mathbb{R}^{+*} \times \mathbb{T}, \quad \partial_\theta \Theta(r,\theta) = s \frac{\sqrt{h(r)}}{\mathcal{R}(r)}.$$

We now recall that Θ maps $\mathbb{R}^+ \times \mathbb{T}$ to $\mathbb{T}$ and since the right-hand side does not depend on θ such an equality implies that

$$\exists \varepsilon \in \{-1, 0, 1\}, \ \forall r > 0, \quad \frac{\sqrt{h(r)}}{\mathcal{R}(r)} = \varepsilon.$$

Recalling that $\mathcal{R}(0) > 0$ we deduce that $\varepsilon = 1$ and thus

$$\forall r \in \mathbb{R}^+, \quad \mathcal{R}(r) = \sqrt{h(r)} \quad \text{and} \quad \Theta(r,\theta) = \pm\theta + \overline{\Theta}(r) \tag{16}$$

where $\overline{\Theta}$ is a function from $\mathbb{R}^+$ to $\mathbb{T}$.

At last, we use (5) to obtain $f(r) = \pm h'(r)/2$ for any $r > 0$. The hyperbolic system (10), with $\alpha(r) = (-1)^p \frac{\sqrt{gh-f^2}}{g}(r)$ (p an integer defined in Proposition 2.2)) and $\beta(r) = f(r)/g(r)$, yields

$$\forall (r,\theta) \in \mathbb{R}^{+*} \times \mathbb{T}, \quad \partial_r \Theta(r,\theta) = \frac{\alpha(r) R(r,\theta) \partial_\theta \Theta(r,\theta) - \beta(r) \partial_\theta R(r,\theta)}{(\alpha^2(r) + \beta^2(r)) R(r,\theta)}$$

which implies

$$\forall r \geqslant 0, \quad \overline{\Theta}(r) = \pm(-1)^p \int_0^r \frac{\sqrt{g(r_*)h(r_*) - f^2(r_*)}}{h(r_*)} \, dr_* + \overline{\Theta}(0). \tag{17}$$

This concludes the proof because equations (9), (16) and (17) are exactly the conditions stated in Theorem 2.1. The sufficient condition is checked by direct computations. ■

Références

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