



Convex super-resolution detection of lines in images

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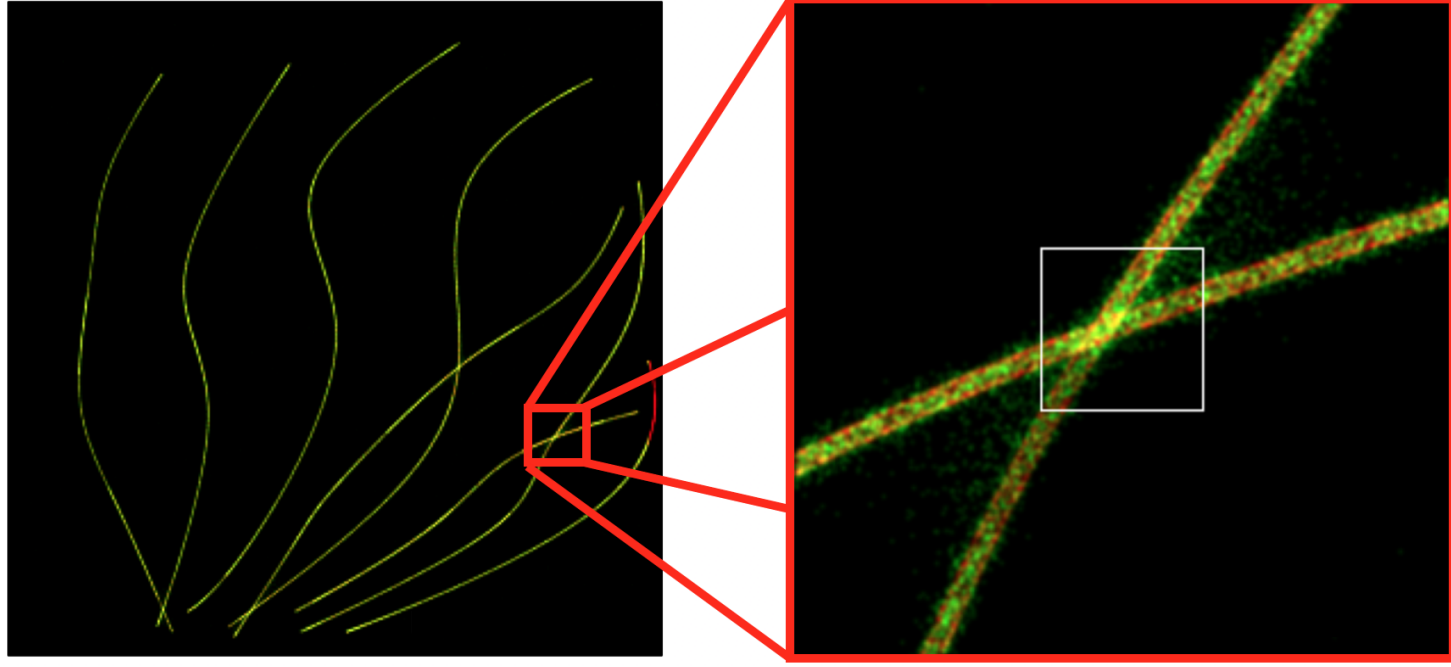
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OBJECTIVES

We present a new convex formulation for the problem of recovering lines in degraded images. Following the recent paradigm of super-resolution, we formulate a dedicated atomic norm penalty and solve this optimization problem by a primal-dual algorithm. Then, a spectral estimation method recovers the line parameters, with subpixel accuracy.



ATOMIC NORM FRAMEWORK

Let $z \in \mathbb{C}^N$ be a vector such as $z = \sum_{k=1}^K c_k a(\omega_k)$ with $c_k \in \mathbb{C}$ and atoms $a(\omega) \in \mathbb{C}^N$ continuously indexed in a dictionary \mathcal{A} by a parameter ω in a compact set Ω . The atomic norm, which enforces sparsity with respect this set \mathcal{A} , is defined as

$$\|z\|_{\mathcal{A}} = \inf_{c'_k, \omega'_k} \left\{ \sum_k |c'_k| : z = \sum_k c'_k a(\omega'_k) \right\}.$$

Consider the dictionary

$$\mathcal{A} = \{a(f, \phi) \in \mathbb{C}^{|I|}, f \in [0, 1], \phi \in [0, 2\pi)\},$$

in which the atoms are the vectors of components $[a(f, \phi)]_i = e^{j(2\pi f i + \phi)}$, $i \in I$, and simply $[a(f)]_i = e^{j2\pi f i}$, $i \in I$, if $\phi = 0$. The atomic norm writes:

$$\|z\|_{\mathcal{A}} = \inf_{c'_k > 0, f'_k, \phi'_k} \left\{ \sum_k c'_k : z = \sum_k c'_k a(f'_k, \phi'_k) \right\}.$$

Theorem 1 [Caratheodory]. Let $z = (z_n)_{n=-N+1}^{N-1}$ be a vector with Hermitian symmetry $z_{-n} = z_n^*$. z is a positive combination of $K \leq N + 1$ atoms $a(f_k)$ if and only if $\mathbf{T}_N(z_+) \succcurlyeq 0$ and of rank K , where $z_+ = (z_0, \dots, z_{N-1})$ and \mathbf{T}_N is the Toeplitz operator

$$\mathbf{T}_N(z_+) = \begin{pmatrix} z_0 & z_1^* & \cdots & z_{N-1}^* \\ z_1 & z_0 & \cdots & z_{N-2}^* \\ \vdots & \vdots & \ddots & \vdots \\ z_{N-1} & z_{N-2} & \cdots & z_0 \end{pmatrix}.$$

Moreover, this decomposition is unique, if $K \leq N$.

Proposition 1. The atomic norm $\|z\|_{\mathcal{A}}$ can be characterized by this semidefinite program $\text{SDP}(z)$ [2]:

$$\|z\|_{\mathcal{A}} = \min_{q \in \mathbb{C}^N} \left\{ q_0 : \mathbf{T}'_N(z, q) = \begin{pmatrix} \mathbf{T}'_N(q) & z \\ z^* & q_0 \end{pmatrix} \succcurlyeq 0 \right\}.$$

- $l_{n_2}^\# = \hat{x}^\#[:, n_2] = \sum_{k=1}^K c_k a(f_{n_2, k})$
- $t_m^\# = \hat{x}^\#[m, :] = \sum_{k=1}^K c_k a(f_{m, k}, \phi_{m, k})^T$ with amplitude $c_k = \frac{\alpha_k}{\cos \theta_k}$, phase $\phi_{m, k} = -\frac{2\pi \eta_k m}{W}$, frequency $f_{n_2, k} = \frac{\tan \theta_k n_2 - \eta_k}{W}$, $f_{m, k} = \frac{\tan \theta_k m}{W}$.
- $\|l_{n_2}^\#\|_{\mathcal{A}} = \sum_{k=1}^K c_k = \hat{x}^\#[0, n_2]$ by **Theorem 1**.
- $\|t_m^\#\|_{\mathcal{A}} = \text{SDP}(t_m^\#) \leq \sum_{k=1}^K c_k$ by **Proposition 1**.

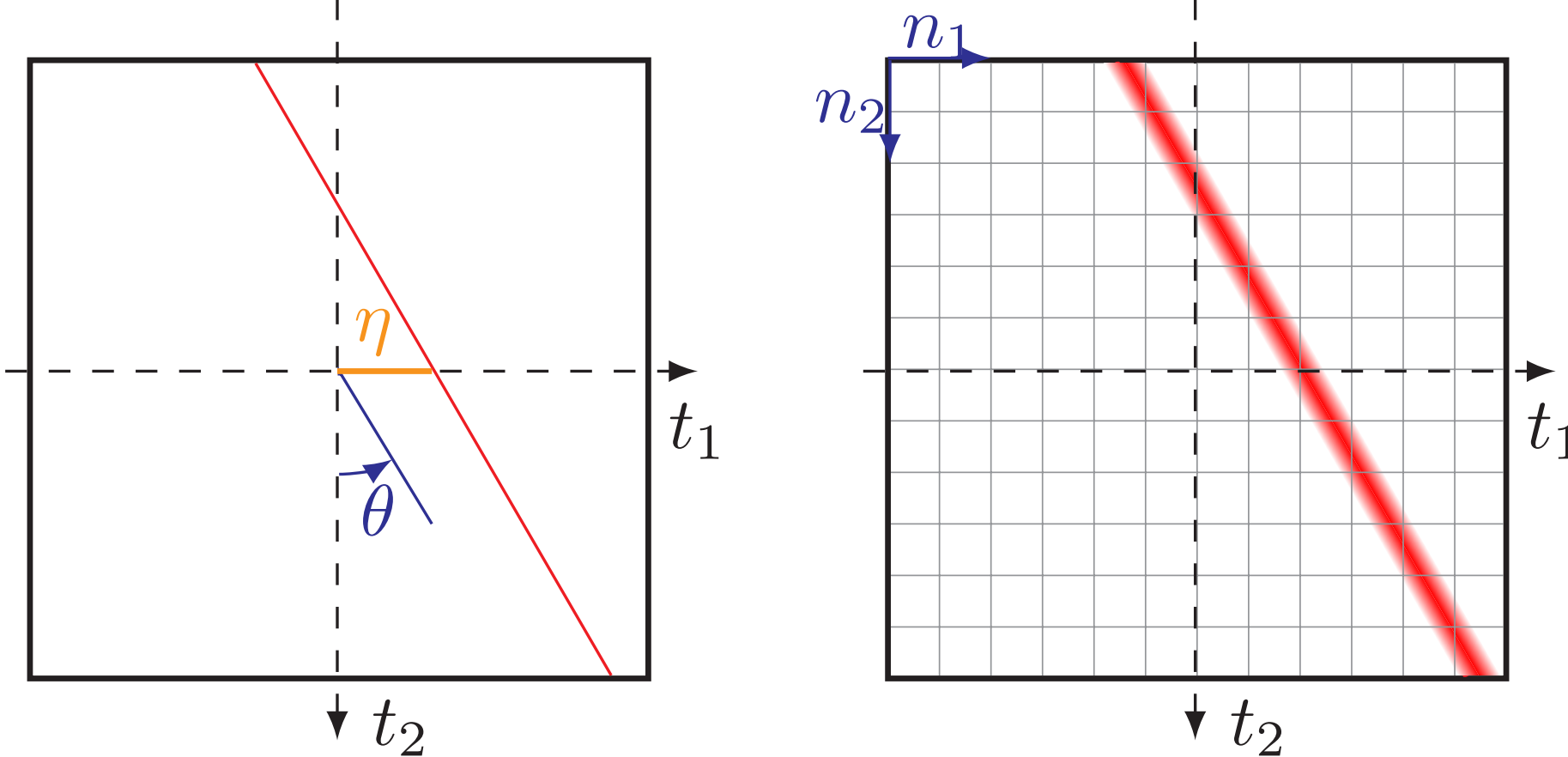
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- [2] B. N. Bhaskar *et al.*, Atomic norm denoising with applications to line spectral estimation, IEEE Transactions on signal processing, 2013.
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MODEL OF NOISY BLURRED LINES

A sum of K perfect lines of infinite length, with angle $\theta_k \in (-\pi/2, \pi/2]$, amplitude $\alpha_k > 0$, and off-set $\eta_k \in \mathbb{R}$, is defined as the distribution

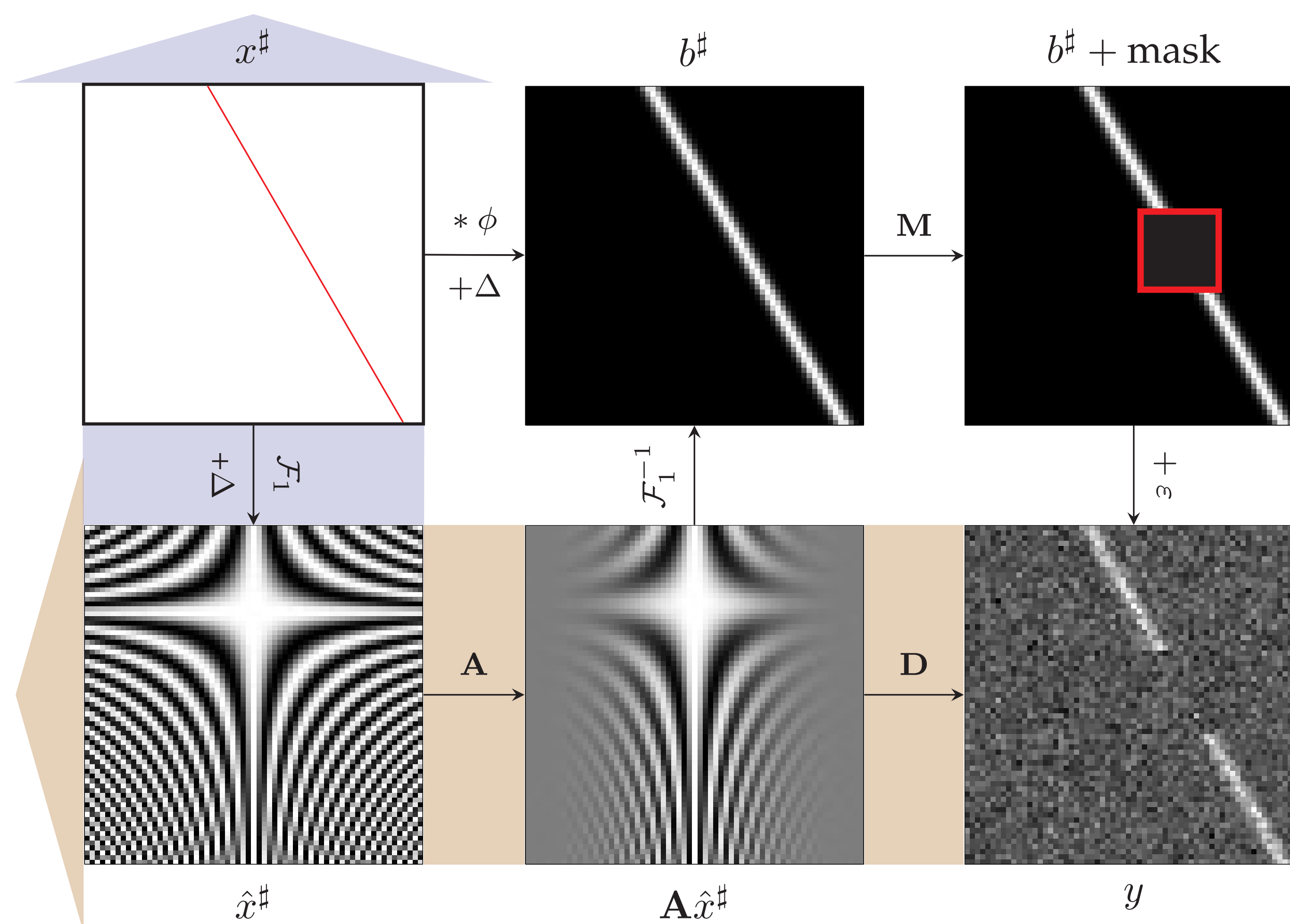
$$x^\#(t_1, t_2) = \sum_{k=1}^K \alpha_k \delta(\cos \theta_k (t_1 - \eta_k) + \sin \theta_k t_2).$$



The image observed $b^\#$ of size $W \times H$ is obtained by the convolution of $x^\#$ with a blur function ϕ , following by a sampling with unit step Δ : $b^\#[n_1, n_2] = (x^\# * \phi)(n_1, n_2)$. The point spread function ϕ is separable, that is $x^\# * \phi$ can be obtained by a first horizontal convolution $u^\# = x^\# * \varphi_1$, where φ_1 is W -periodic and bandlimited, that is its Fourier coefficients $\hat{g}[m]$ are zero for $|m| \geq (W-1)/2 = M+1$, so $\hat{u}^\#[m, n_2] = \hat{g}[m] \hat{x}^\#[m, n_2]$; and then a second vertical convolution with φ_2 , such as the discrete filter $h[n] = (\varphi_2 * \text{sinc})[n]$ has compact support, gives $\hat{b}^\#[m, :] = \hat{u}^\#[m, :] * h = \hat{g}[m] \hat{x}^\#[m, :] * h$, hence $\mathbf{A} \hat{x}^\# = \hat{b}^\#$

$$\hat{x}^\#[m, n_2] = \sum_{k=1}^K \frac{\alpha_k}{\cos \theta_k} e^{j2\pi(\tan \theta_k n_2 - \eta_k)m/W}.$$

SUPER-RESOLUTION AND REGULARIZATION OF LINES



(Observation operator)

$$\mathbf{H} = \mathbf{M} \mathbf{F}_1^{-1} \mathbf{A}$$

\mathbf{A} : blur operator

\mathbf{F}_1 : horiz. Fourier operator

\mathbf{M} : inpainting operator

$$\tilde{x} \in \arg \min_{\hat{x}, q \in \mathcal{X} \times \mathcal{Q}} \frac{1}{2} \|\mathbf{H} \hat{x} - y\|_F^2,$$

$$\begin{cases} \forall n_2, m \\ \hat{x}[0, n_2] = \hat{x}[0, 0] \leq c, \\ q[m, 0] \leq c, \\ \mathbf{T}'_{H_S}(\hat{x}[m, :], q[m, :]) \succcurlyeq 0, \\ \mathbf{T}_{M+1}(\hat{x}[:, n_2]) \succcurlyeq 0, \end{cases}$$

The problem can be rewritten in this way:

$$\tilde{X} = \arg \min_{X \in \mathcal{H}} \left\{ F(X) + G(X) + \sum_{i=0}^{N-1} H_i(L_i(X)) \right\}$$

with $F(X) = \frac{1}{2} \|\mathbf{H} \hat{x} - y\|_F^2$, $X = (\hat{x}, q)$, ∇F a β -Lipschitz gradient, a proximal indicator $G = \iota_{\mathcal{B}}$ where \mathcal{B} are the two first boundary constraints, and $N = M + 1 + H_S$ linear composite terms, where $H_i = \iota_{\mathcal{C}}$ with \mathcal{C} the cone of semidefinite positive matrices, and $L_i \in \{L_m^{(1)}, L_{n_2}^{(2)}\}$, defined by $L_m^{(1)}(X) = \mathbf{T}'_{H_S}(\hat{x}[m, :], q[m, :])$ and $L_{n_2}^{(2)}(X) = \mathbf{T}_{M+1}(\hat{x}[:, n_2])$. \mathbf{L} denotes the concatenation of the L_i operators.

Let $\tau > 0$ and $\sigma > 0$ such that $\frac{1}{\tau} - \sigma \|\mathbf{L}\|^2 \geq \frac{\beta}{2}$.

Algorithm: Primal-dual splitting method [Condat]

Input: The blurred and noisy data image y

Output: \tilde{x} solution of the optimization problem

- 1: Initialize all primal and dual variables to zero
- 2: **for** $n = 1$ **to** Number of iterations **do**
- 3: $X_{n+1} = \text{prox}_{\tau G}(X_n - \tau \nabla F(X_n) - \tau \sum_i L_i^* \xi_{i,n})$,
- 4: **for** $i = 0$ **to** $N - 1$ **do**
- 5: $\xi_{i,n+1} = \text{prox}_{\sigma H_i^*}(\xi_{i,n} + \sigma L_i(2X_{n+1} - X_n))$,
- 6: **end for**
- 7: **end for**

SPECTRAL ESTIMATION BY A PRONY-LIKE METHOD

Let be $d_k \in \mathbb{C}$, $f_k \in [-1/2, 1/2)$, $\zeta_k = e^{j2\pi f_k}$ and

$$z_i = \sum_{k=1}^K d_k (e^{j2\pi f_k})^i, \quad \forall i = 0, \dots, |I| - 1,$$

The **annihilating polynomial filter** is defined by:

$$H(\zeta) = \prod_{l=1}^K (\zeta - \zeta_l) = \sum_{l=0}^K h_l \zeta^{K-l} \text{ with } h_0 = 1,$$

$$\sum_{l=0}^K h_l z_{r-l} = \sum_{k=1}^K d_k \zeta_k^{r-K} \underbrace{\left(\sum_{l=0}^K h_l \zeta_k^{K-l} \right)}_{H(\zeta_k)=0} = 0.$$

$$\mathbf{P}_K(z)h = \begin{pmatrix} z_K & \cdots & z_0 \\ \vdots & \ddots & \vdots \\ z_{|I|-1} & \cdots & z_{|I|-K-1} \end{pmatrix} \begin{pmatrix} h_0 \\ \vdots \\ h_K \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$$

- 1 From $\mathbf{P}_K(z)$, compute h by a SVD. Form H whose roots give access to the frequencies f_k .
- 2 Since $z = \mathbf{U}d$ with $\mathbf{U} = (a(f_1), \dots, a(f_K))$, find amplitudes by LS: $d = (\mathbf{U}^H \mathbf{U})^{-1} \mathbf{U}^H z$.

Procedure for retrieving the line parameters

- 1 For each column $\hat{x}[m, :]$ compute $\{\tilde{f}_{m,k}\}_k$ by 1
- 2 For each column $\hat{x}[m, :]$ compute $\{\tilde{d}_{m,k}\}_k$ by 2
- 3 $\{\tilde{f}_{m,k}\}_m = \{\frac{\tan \theta_k m}{W}\}_m$ lin. regression $\rightarrow \{\tilde{\theta}_k\}$
- 4 $\tilde{\alpha}_{m,k} = |\tilde{d}_{m,k}| \cos(\tilde{\theta}_k)$ and $\{\alpha_k\}_k = \mathbb{E}\{\{\tilde{\alpha}_{m,k}\}_m\}$
- 5 $\tilde{d}_{m,k}/|\tilde{d}_{m,k}| = (e^{-j2\pi \frac{\eta_k}{W}})^m \rightarrow \{\eta_k\}_k$ by 1

This procedure enables to estimate the line parameters from the solution \tilde{x} of the optimization problem:

