Explicit expression for the stationary distribution of reflected brownian motion in a wedge

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EXPLICIT EXPRESSION FOR THE STATIONARY DISTRIBUTION OF REFLECTED BROWNIAN MOTION IN A WEDGE

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Abstract. For Brownian motion in a (two-dimensional) wedge with negative drift and oblique reflection on the axes, we derive an explicit formula for the Laplace transform of its stationary distribution (when it exists), in terms of Cauchy integrals and generalized Chebyshev polynomials. To that purpose we solve a Carleman-type boundary value problem on a hyperbola, satisfied by the Laplace transforms of the boundary stationary distribution.

1. Introduction and main results

Since its introduction in the eighties by Harrison, Reiman, Varadhan and Williams [24, 25, 37, 38, 39], reflected Brownian motion in the quarter plane has received a lot of attention from the probabilistic community. However, and surprisingly, finding a general explicit expression of the stationary distribution has been left as an open problem. The present paper solves this problem in a complete and unified way.

Reflected Brownian motion in two-dimensional cones. The reflected Brownian motion with drift in the quarter plane $\mathbb{R}_+^2$ (or equivalently in arbitrary convex wedges, by performing a simple linear transformation, cf. Appendix A) can be written as

$$Z(t) = Z_0 + B(t) + \mu \cdot t + R \cdot L(t), \qquad \forall t \geqslant 0, \tag{1}$$

where

- Z_0 is any initial point in the quadrant,
- B is a Brownian motion with covariance $\Sigma = \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{pmatrix}$ starting from the origin,
- $\mu = \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}$ denotes the interior drift,
- $R = (R^1, R^2) = \begin{pmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{pmatrix}$ is the reflection matrix,
- $L = \begin{pmatrix} L^1 \\ L^2 \end{pmatrix}$ is the local time.

For $i \in \{1, 2\}$, $L^i(t)$ is a continuous non-decreasing process, increasing only at time t such that $Z^i(t) = 0$, viz., $\int_0^t \mathbb{1}_{\{Z^i(s) \neq 0\}} \mathrm{d}L^i(s) = 0$, for all $t \geqslant 0$. The columns R^1 and R^2 represent the directions in which the Brownian motion is pushed when the axes are reached, see Figure 1.

The reflected Brownian motion $(Z(t))_{t \geqslant 0}$ associated with (Σ, μ, R) is well defined [37, 39], and is a fondamental stochastic process. It has been studied in depth, with focuses on its definition and semimartingale properties [37, 38, 40], its recurrence or transience [38, 9, 28], the possible particular (e.g., product) form of its stationary distribution [27, 13], its Lyapunov functions [16], its links with other stochastic processes [31, 15], its use to approximate large queuing networks [19, 1, 26], the asymptotics of its stationary distribution [23, 11, 21], numerical methods to compute the stationary distribution [9, 10], links with complex analysis [19, 1, 5, 22], etc.

Key words and phrases. Reflected Brownian motion in a wedge; Stationary distribution; Laplace transform; Carleman-type boundary value problem; Boundary value problem with shift; Conformal mapping.

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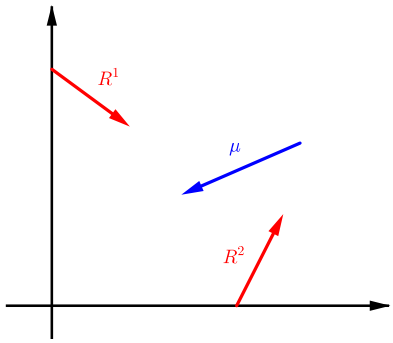

FIGURE 1. Drift μ and reflection vectors R^1 and R^2

The main contribution of the present paper is to find an exact expression for the stationary distribution (via its Laplace transforms, to be introduced in (3) and (4)), thanks to the theory of boundary value problems (BVPs), see our Theorem 1. This is one of the first attempts to apply boundary value techniques to diffusions in the quadrant, after [20] (under the symmetry conditions $\mu_1 = \mu_2$, $\sigma_{11} = \sigma_{22}$, and symmetric reflection vectors in (1)), [19] (which concerns very specific cases of the covariance matrix, essentially the identity), [1] (on diffusions with special behavior on the boundary), [22] (orthogonal reflections, solved by Tutte's invariant approach [36, 2]).

Main results. Under the assumption that

$$r_{11} > 0, \quad r_{22} > 0, \quad r_{11}r_{22} - r_{12}r_{21} > 0, \quad r_{22}\mu_1 - r_{12}\mu_2 < 0, \quad r_{11}\mu_2 - r_{21}\mu_1 < 0, \tag{2}$$

the stationary distribution exists and is absolutely continuous w.r.t. the Lebesgue measure [27, 9], with density denoted by $\pi(x) = \pi(x_1, x_2)$. Let the Laplace transform of π be defined by

$$\varphi(\theta) = \mathbb{E}_\pi[\exp\langle\theta|Z\rangle] = \iint_{\mathbb{R}_+^2} \exp\langle\theta|x\rangle\pi(x)\mathrm{d}x. \tag{3}$$

The integral (3) converges (at least) for $\theta = (\theta_1, \theta_2) \in \mathbb{C}^2$ such that $\Re\theta_1 \leqslant 0$ and $\Re\theta_2 \leqslant 0$. Furthermore we define two finite boundary measures ν_1 and ν_2 such that, for $A \subset \mathbb{R}_+$,

$$\nu_1(A) = \mathbb{E}_\pi\left[\int_0^1 \mathbb{1}_{\{Z(t)\in\{0\}\times A\}}\mathrm{d}L^1(t)\right], \qquad \nu_2(A) = \mathbb{E}_\pi\left[\int_0^1 \mathbb{1}_{\{Z(t)\in A\times\{0\}\}}\mathrm{d}L^2(t)\right].$$

The ν_i have their supports on the axes and may be viewed as boundary invariant measures. They are continuous w.r.t. the Lebesgue measure, see [27]. We define their Laplace transform by (a priori for values of the argument with non-positive real parts)

$$\varphi_1(\theta_2) = \int_{\mathbb{R}_+} \exp(\theta_2 x_2)\nu_1(x_2)\mathrm{d}x_2, \qquad \varphi_2(\theta_1) = \int_{\mathbb{R}_+} \exp(\theta_1 x_1)\nu_2(x_1)\mathrm{d}x_1. \tag{4}$$

There is a simple functional equation relating the Laplace transforms φ, φ_1 and φ_2, see (7) in Section 2.

In addition to (2), which guarantees existence and uniqueness of the stationary distribution, we shall assume that both coordinates of the drift are negative:

$$\mu_1 < 0, \qquad \mu_2 < 0. \tag{5}$$

This hypothesis (also done in [20, 19, 21]) is only technical, and allows us to reduce the number of cases to handle. In Section 3.6 we comment on the case of a drift μ with one non-negative coordinate (having two non-negative coordinates is obviously incompatible with (2)).

Our main result can be stated as follows.

Theorem 1. *Under the assumptions (2) and (5), the Laplace transform φ_1 is equal to*

$$\varphi_1(\theta_2) = \nu_1(\mathbb{R}_+)\left(\frac{w(0)-w(p)}{w(\theta_2)-w(p)}\right)^{-\chi}\exp\left\{\frac{1}{2i\pi}\int_{\mathcal{R}^-}\log G(\theta)\left[\frac{w'(\theta)}{w(\theta)-w(\theta_2)}-\frac{w'(\theta)}{w(\theta)-w(0)}\right]\mathrm{d}\theta\right\}, \quad (6)$$

where

- $\nu_1(\mathbb{R}_+) = \frac{r_{12}\mu_2 - r_{22}\mu_1}{\det R}$, *see Lemma 2,*
- *the function w, related to generalized Chebyshev polynomials, is defined in (22),*
- *the pole p is defined in (11),*
- *the index χ equals 0 or -1, according to the values of the parameters, see (31),*
- *the curve $\mathcal{R}^-$, a (half) branch of hyperbola oriented from the vertex to infinity, is defined in (16), see also Figure 3,*
- *the function G, which only depends on the reflection matrix, is introduced in (17),*
- *to define the function $\log G(\theta)$, we use the determination of the logarithm taking a value in $(-i\pi, i\pi]$ at the vertex of $\mathcal{R}^-$ and varying continuously over the curve $\mathcal{R}^-$, see Section 3.5.*

There is an analogous expression for $\varphi_2(\theta_1)$, and the functional equation (7) finally gives an explicit formula for the bivariate Laplace transform φ. Let us now give some comments around Theorem 1.

- Theorem 1 completely generalizes the results of [20] (with symmetry conditions), [19] (with the identity covariance matrix Σ) and [22] (with orthogonal reflection on the axes). It offers the first explicit expression of the Laplace transforms, covering all the range of (non-degenerate) parameters (Σ, μ, R), thereby solving an old open problem.
- It is worth remarking that the expression (6) is intrinsically non-continuous in terms of the parameters. Indeed, the index χ can take two different values (namely, 0 and 1). For this reason, (6) actually contains two different formulas. In addition, in the subcase $\chi = 0$, the function G may take two different values (-1 and $+1$) at the vertex of $\mathcal{R}^-$. See Remarks 2 and 6 for further related comments.
- The paper [21] obtains the exact asymptotics of the stationary distribution along any direction in the quarter plane, see [21, Theorems 22–28]. Constants in these asymptotics involve the functions φ_1 and φ_2 in (4), and can thus be made explicit with Theorem 1.
- The statement of Theorem 1 (namely, an expression of the Laplace transform as a Cauchy integral), as well as the techniques we shall employ to prove it (viz., reduction to BVPs with shift), are reminiscent of the results and methods used for discrete random walks in the quarter plane, see [18] for a modern reference, and [33, 17] for historical breakthroughs.
- Altogether, Theorem 1 illustrates that the analytic approach consisting in solving quarter plane problems via BVPs is better suited for diffusions than for discrete random walks. We can actually treat any wedge, covariance matrix, drift vector and reflection vectors, whereas in the discrete case, hypotheses should be done on the boundedness of the jumps (only small steps are considered in [18, 4, 2]) and on the cone (typically, half and quarter planes only).

Structure of the paper.

- Section 2: statement of the kernel functional equation, analytic preliminaries, continuation of the Laplace transforms and definition of an important hyperbola
- Section 3: statement and proof that the Laplace transforms satisfy BVP of Carleman-type on branches of hyperbolas, transformation of the Carleman BVP with shift into a (more classical) Riemann BVP, study of the conformal mapping allowing this transformation, resolution of the BVP
- Section 4: asymptotics of the stationary distribution, links with Dai and Miyazawa's asymptotic results

- Section 5: simplifications of the integral expression of Theorem 1 for finite group models, then for orthogonal reflections (leading to a new proof of the results of [22]), uniformization of the kernel, definition of the group of the model
- Appendix A: equivalence between Brownian motion in the quarter plane and Brownian motion in convex wedges

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2. A kernel functional equation and continuation of the Laplace transforms

2.1. **Functional equation.** The following functional equation relates the Laplace transforms:

$$-\gamma(\theta)\varphi(\theta) = \gamma_1(\theta)\varphi_1(\theta_2) + \gamma_2(\theta)\varphi_2(\theta_1), \tag{7}$$

where we have noted

$$\begin{cases} \gamma(\theta) = \frac{1}{2}\langle\theta|\Sigma\theta\rangle + \langle\theta|\mu\rangle = \frac{1}{2}(\sigma_{11}\theta_1^2 + \sigma_{22}\theta_2^2 + 2\sigma_{12}\theta_1\theta_2) + \mu_1\theta_1 + \mu_2\theta_2, \\ \gamma_1(\theta) = \langle R^1|\theta\rangle = r_{11}\theta_1 + r_{21}\theta_2, \\ \gamma_2(\theta) = \langle R^2|\theta\rangle = r_{12}\theta_1 + r_{22}\theta_2. \end{cases} \tag{8}$$

Equation (7) holds at least for values of $\theta = (\theta_1, \theta_2)$ with $\Re\,\theta_1 \leqslant 0$ and $\Re\,\theta_2 \leqslant 0$. To prove this functional equation, the main idea is to use an identity called basic adjoint relationship (BAR); see [22, Section 2.1] and [11, 19] for details.

The kernel of (7) is γ, a second degree polynomial in both variables θ_1 and θ_2. The equation $\gamma(\theta_1, \theta_2) = 0$ defines (two-valued) algebraic functions $\Theta_1^{\pm}(\theta_2)$ and $\Theta_2^{\pm}(\theta_1)$, by

$$\gamma(\Theta_1^{\pm}(\theta_2), \theta_2) = \gamma(\theta_1, \Theta_2^{\pm}(\theta_1)) = 0.$$

Solving these equations readily yields

$$\begin{cases} \Theta_1^{\pm}(\theta_2) = \dfrac{-(\sigma_{12}\theta_2 + \mu_1) \pm \sqrt{\theta_2^2(\sigma_{12}^2 - \sigma_{11}\sigma_{22}) + 2\theta_2(\mu_1\sigma_{12} - \mu_2\sigma_{11}) + \mu_1^2}}{\sigma_{11}}, \\ \Theta_2^{\pm}(\theta_1) = \dfrac{-(\sigma_{12}\theta_1 + \mu_2) \pm \sqrt{\theta_1^2(\sigma_{12}^2 - \sigma_{11}\sigma_{22}) + 2\theta_1(\mu_2\sigma_{12} - \mu_1\sigma_{22}) + \mu_2^2}}{\sigma_{22}}. \end{cases} \tag{9}$$

The polynomials under the square roots in (9) have two zeros (sometimes called branch points), real and of opposite signs. They are denoted by $\theta_2^{\pm}$ and $\theta_1^{\pm}$, respectively:

$$\begin{cases} \theta_2^{\pm} = \dfrac{(\mu_1\sigma_{12} - \mu_2\sigma_{11}) \pm \sqrt{(\mu_1\sigma_{12} - \mu_2\sigma_{11})^2 + \mu_1^2 \det\Sigma}}{\det\Sigma}, \\ \theta_1^{\pm} = \dfrac{(\mu_2\sigma_{12} - \mu_1\sigma_{22}) \pm \sqrt{(\mu_2\sigma_{12} - \mu_1\sigma_{22})^2 + \mu_2^2 \det\Sigma}}{\det\Sigma}. \end{cases} \tag{10}$$

Lemma 2. *One has* $\begin{pmatrix} \varphi_1(0) \\ \varphi_2(0) \end{pmatrix} = \begin{pmatrix} \nu_1(\mathbb{R}_+) \\ \nu_2(\mathbb{R}_+) \end{pmatrix} = -R^{-1}\mu.$

Proof. Equation (4) evaluated at 0 gives $\varphi_i(0) = \nu_i(\mathbb{R}_+)$. We now evaluate (7) at $\theta_2 = 0$, divide by θ_1 and finally evaluate at $\theta_1 = 0$. This yields $-\mu_1 = r_{11}\varphi_1(0) + r_{12}\varphi_2(0)$. In a similar way, we obtain $-\mu_2 = r_{21}\varphi_1(0) + r_{22}\varphi_2(0)$, thereby concluding the proof. □

Note that solving Lemma 2 gives, as announced in Theorem 1,

$$\nu_1(\mathbb{R}_+) = \frac{r_{12}\mu_2 - r_{22}\mu_1}{\det R}, \qquad \nu_2(\mathbb{R}_+) = \frac{r_{21}\mu_1 - r_{11}\mu_2}{\det R},$$

which by (2) are positive.

2.2. **Continuation of the Laplace transforms.** In Section 3.1 we shall state a boundary condition for φ_1, on a curve lying outside its natural domain of definition (namely, the half plane with negative real part). The statement hereafter (straightforward consequence of the functional equation (7), see Proposition 11 for an extended version) proposes a meromorphic continuation on a domain containing the latter curve. Let p be the (unique, when it exists) non-zero point such that (cf. Figure 2)

$$\gamma_1(\Theta_1^-(p), p) = 0. \tag{11}$$

Lemma 3. *The Laplace transform $\varphi_1(\theta_2)$ can be extended meromorphically to the open and simply connected set*

$$\{\theta_2 \in \mathbb{C} : \Re\,\theta_2 \leqslant 0 \text{ or } \Re\,\Theta_1^-(\theta_2) < 0\}, \tag{12}$$

by mean of the formula

$$\varphi_1(\theta_2) = \frac{\gamma_2}{\gamma_1}(\Theta_1^-(\theta_2), \theta_2)\varphi_2(\Theta_1^-(\theta_2)). \tag{13}$$

Observe that the only possible pole of φ_1 in the domain (12) is (simple and) at the point p defined in (11).

Remark 1. *The continuation formula (13) uses the branch Θ_1^- in (9) and not Θ_1^+. The reason is that Θ_1^- is the small branch of the algebraic function $\Theta_1^\pm$, taking the value 0 at 0. Note, this is peculiar to the case of a drift with negative coordinates (our hypothesis (5)). More details can be found in the proof of Theorem 11 in [21].*

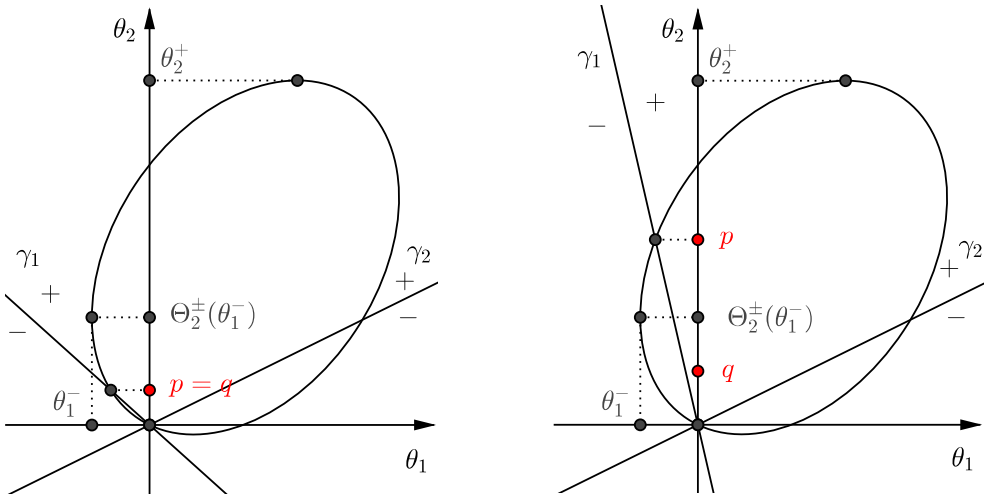

FIGURE 2. Location of p (see (11)) and q (see (21)) according to the sign of $\gamma_1(\theta_1^-, \Theta_2^\pm(\theta_1^-))$. The ellipse and the two straight lines are the sets of real points (θ_1, θ_2) which cancel γ, γ_1 and γ_2, respectively

2.3. **An important hyperbola.** For further use, we need to introduce the curve

$$\mathcal{R} = \{\theta_2 \in \mathbb{C} : \gamma(\theta_1, \theta_2) = 0 \text{ and } \theta_1 \in (-\infty, \theta_1^-)\} = \Theta_2^\pm((-\infty, \theta_1^-)). \tag{14}$$

It is symmetrical w.r.t. the real axis, see Figure 3. Indeed, the discriminant of $\Theta_2^\pm$ (i.e., the polynomial under the square root in (9)) is positive on (θ_1^-, θ_1^+) and negative on $\mathbb{R} \setminus [\theta_1^-, \theta_1^+]$. Accordingly, the branches $\Theta_2^\pm$ take respectively real and complex conjugate values on the sets above. Furthermore, $\mathcal{R}$ has a simple structure, as shown by the following elementary result:

Lemma 4 (Lemma 9 in [1]). *The curve $\mathcal{R}$ in (14) is a branch of hyperbola, given by the equation*

$$\sigma_{22}(\sigma_{12}^2 - \sigma_{11}\sigma_{22})x^2 + \sigma_{12}^2\sigma_{22}y^2 - 2\sigma_{22}(\sigma_{11}\mu_2 - \sigma_{12}\mu_1)x = \mu_2(\sigma_{11}\mu_2 - 2\sigma_{12}\mu_1). \tag{15}$$

We denote the negative imaginary part of $\mathcal{R}$ by

$$\mathcal{R}^- = \{\theta_2 \in \mathcal{R} : \Im\,\theta_2 \leqslant 0\}, \tag{16}$$

see Figure 3. We further denote by $\mathcal{G}_{\mathcal{R}}$ the open domain of $\mathbb{C}$ containing 0 and bounded by $\mathcal{R}$, see again Figure 3. The closure of $\mathcal{G}_{\mathcal{R}}$ is equal to $\mathcal{G}_{\mathcal{R}} \cup \mathcal{R}$ and will be noted $\overline{\mathcal{G}_{\mathcal{R}}}$. Notice that the domain in (12) strictly contains $\mathcal{G}_{\mathcal{R}}$.

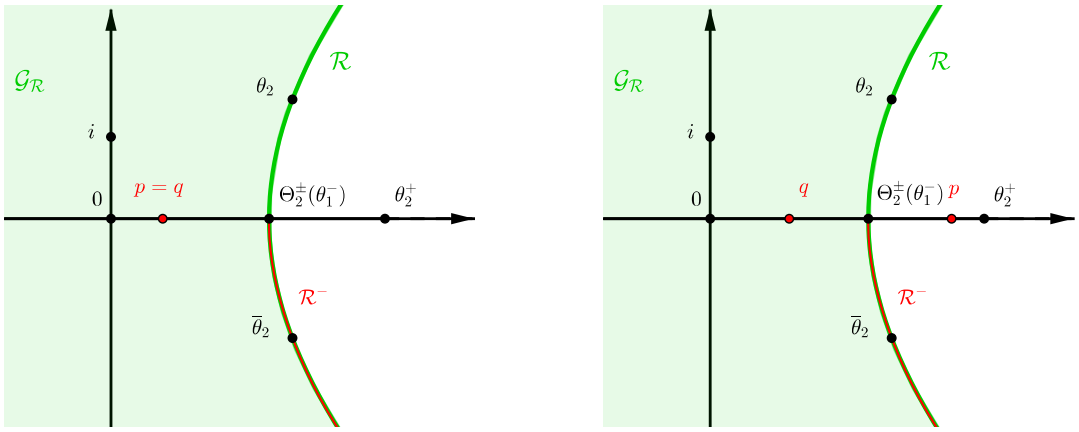

FIGURE 3. The curve $\mathcal{R}$ in (14) is symmetric w.r.t. the horizontal axis, and $\mathcal{G}_{\mathcal{R}}$ is the domain in green. The curve $\mathcal{R}^-$ is the half branch of $\mathcal{R}$ with negative imaginary part. The points p and q are used to define the conformal mapping W, see (11) and (23). If the pole p is in $\mathcal{G}_{\mathcal{R}}$ then $q = p$ (figure on the left), otherwise $q = \frac{1}{2}\Theta_2^{\pm}(\theta_1^-)$ (figure on the right), see (21)

3. A proof of Theorem 1 via reduction to BVPs

Our reasoning for the proof of Theorem 1 reads as follows: we first state (Section 3.1) a Carleman BVP with shift, satisfied by the Laplace transforms; then (Section 3.2) we introduce a conformal mapping, allowing to transform the latter BVP into a more classical one, a Riemann BVP, see Figure 4; we state it in Section 3.3; in the way of solving the Riemann BVP, the index (denoted χ in Theorem 1) turns out to have a crucial role, it is studied in Section 3.4; finally the BVP is solved in Section 3.5.

3.1. Carleman BVP.

For $\theta_2 \in \mathcal{R}$, define

$$G(\theta_2) = \frac{\gamma_1}{\gamma_2}(\Theta_1^-(\theta_2), \theta_2)\frac{\gamma_2}{\gamma_1}(\Theta_1^-(\theta_2), \overline{\theta_2}). \tag{17}$$

Let us also recall that p is defined in (11).

Proposition 5 (Carleman BVP with shift). *The function φ_1 in (4)*

(1) *is meromorphic on $\mathcal{G}_{\mathcal{R}}$,*
- *without pole on $\mathcal{G}_{\mathcal{R}}$ if $\gamma_1(\theta_1^-, \Theta_2^{\pm}(\theta_1^-)) < 0$,*
- *with a single pole on $\mathcal{G}_{\mathcal{R}}$ at p of order one if $\gamma_1(\theta_1^-, \Theta_2^{\pm}(\theta_1^-)) > 0$,*
- *without pole on $\mathcal{G}_{\mathcal{R}}$ and with a single pole of order one on the boundary $\mathcal{R}$ of $\mathcal{G}_{\mathcal{R}}$, at $p = \Theta_2^{\pm}(\theta_1^-)$, if $\gamma_1(\theta_1^-, \Theta_2^{\pm}(\theta_1^-)) = 0$,*

(2) *is continuous on* $\overline{\mathcal{G}_{\mathcal{R}}} \setminus \{p\}$ *and bounded at infinity,*
(3) *satisfies the boundary condition*

$$\varphi_1(\overline{\theta_2}) = G(\theta_2)\varphi_1(\theta_2), \qquad \forall \theta_2 \in \mathcal{R}. \tag{18}$$

It is worth mentioning that the condition on the sign of $\gamma_1(\theta_1^-, \Theta_2^{\pm}(\theta_1^-))$ has a clear geometric meaning: indeed, $\gamma_1(\theta_1^-, \Theta_2^{\pm}(\theta_1^-))$ is negative (resp. positive) if and only if the straight line corresponding to γ_1 crosses the ellipse below (resp. above) the ordinate $\Theta_2^{\pm}(\theta_1^-)$; see Figure 2, left (resp. right).

Remark 2. *Item (1) of Proposition 5 shows that according to the values of the parameters, various cases exist regarding the singularities of the Laplace transform in the domain* $\mathcal{G}_{\mathcal{R}}$*. This is the reason why there isn't a unique expression for the Laplace transform in our main Theorem 1, but two different expressions. These expressions depend in particular on the index* χ *in (31), which is an intrinsically non-continuous function of the parameters.*

Proof of Proposition 5. First of all, it follows from Lemma 3 that the function φ_1 is meromorphic in $\mathcal{G}_{\mathcal{R}}$ and may have a pole of order one at p. Indeed, due to the continuation formula (13), the only potential pole p of φ_1 in $\mathcal{G}_{\mathcal{R}}$ should be a zero of γ_1. It is then on the real line and characterized by (11). Moreover, p defined by (11) is smaller than $\Theta_2^{\pm}(\theta_1^-)$ (i.e., $p \in \mathcal{G}_{\mathcal{R}}$) if and only if the geometric condition $\gamma_1(\theta_1^-, \Theta_2^{\pm}(\theta_1^-)) > 0$ is satisfied, see Figure 2. This demonstrates the first item of Proposition 5.

The second item (in particular the fact that φ_1 is bounded at infinity) comes from Lemma 3 together with the fact that (4) implies that φ_1 (resp. φ_2) is bounded on the set $\{\theta_2 \in \mathbb{C} : \Re\,\theta_2 \leqslant 0\}$ (resp. $\{\theta_2 \in \mathbb{C} : \Re\,\theta_1 \leqslant 0\}$).

To prove the boundary condition (18) (that we announced in [22, Proposition 7]), we consider θ_1 such that $\Re\,\theta_1 < 0$, and evaluate the functional equation (7) at $(\theta_1, \Theta_2^{\pm}(\theta_1))$. This implies

$$\frac{\gamma_1}{\gamma_2}(\theta_1, \Theta_2^{\pm}(\theta_1))\varphi_1(\Theta_2^{\pm}(\theta_1)) + \varphi_2(\theta_1) = 0,$$

which in turn yields

$$\frac{\gamma_1}{\gamma_2}(\theta_1, \Theta_2^{+}(\theta_1))\varphi_1(\Theta_2^{+}(\theta_1)) = \frac{\gamma_1}{\gamma_2}(\theta_1, \Theta_2^{-}(\theta_1))\varphi_1(\Theta_2^{-}(\theta_1)). \tag{19}$$

Restricting (19) to values of $\theta_1 \in (-\infty, \theta_1^-)$, for which $\Theta_2^+(\theta_1)$ and $\Theta_2^-(\theta_1)$ are complex conjugate (see Section 2), we reach the conclusion that

$$\varphi_1(\overline{\theta_2}) = \frac{\gamma_1}{\gamma_2}(\theta_1, \theta_2)\frac{\gamma_2}{\gamma_1}(\theta_1, \overline{\theta_2})\varphi_1(\theta_2),$$

which, by definition (17) of G, exactly coincides with the boundary condition (18). □

The BVP established in Proposition 5 belongs to the class of homogeneous Carleman (or Riemann-Carleman) BVPs with shift, see [32], the shift being here the complex conjugation.

In some cases, the function G in (18) can be factorized, leading to an interesting particular case, that we comment below:

Remark 3. *If there exists a rational function* F *such that*

$$G(\theta_2) = \frac{F(\theta_2)}{F(\overline{\theta_2})},$$

one can transform the boundary condition (18) for φ_1 *with* $G \neq 1$ *into a boundary condition for* $\varphi \cdot F$ *with* $G = 1$*, namely,*

$$(\varphi \cdot F)(\overline{\theta_2}) = (\varphi \cdot F)(\theta_2).$$

Then the associated BVP could be solved using Tutte's invariants [2]. This is what has been done in [22], for the particular case of orthogonal reflections (corresponding to $F(\theta_2) = \frac{1}{\theta_2}$*).*

The existence of a rational function F *factorizing* G *as above is reminiscent of the notion of decoupling functions, introduced in [2].*

However, a rational factorization term F as in Remark 3 does not exist in general. More precisely, in the case $\frac{\beta}{\pi} \in \mathbb{Q}$, with

$$\beta = \arccos -\frac{\sigma_{12}}{\sqrt{\sigma_{11}\sigma_{22}}}, \tag{20}$$

we prove in Section 5.3 that the existence of F is equivalent for the quantity (46) (a norm) to take the value 1, and then we compute a possible expression for F, see (48).

On the other hand, if $\frac{\beta}{\pi} \notin \mathbb{Q}$, it is still an open problem to characterize the parameters (Σ, μ, R) for which F exists. (Notice that β is the angle of the wedge in which after linear transformation, the covariance matrix of the Brownian motion $B(t)$ in (1) is the identity, see Appendix A, in particular Figure 9.)

As a consequence, we cannot systematically use Tutte's invariants technique: we are left with transforming the BVP of Proposition 5 into a more classical one, using a certain conformal mapping having a very convenient gluing property.

3.2. **Conformal gluing.** Our main result in this section is to prove that the function W defined by (23) satisfies the properties of Lemma 6 below, allowing to transform the Carleman BVP with shift on the curve $\mathcal{R}$ of Proposition 5 into a classical BVP on the segment $[0, 1]$.

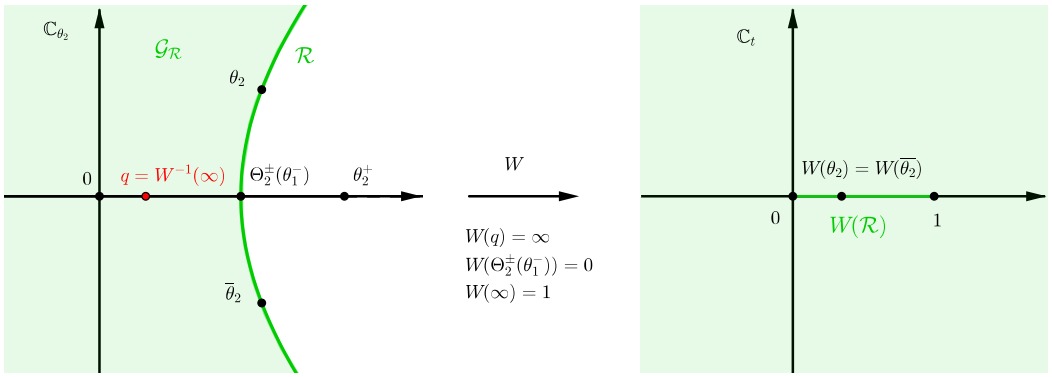

FIGURE 4. Domains, curves and points related to the Carleman BVP with shift on $\mathcal{R}$ (left) and the standard BVP on $[0, 1]$ (right)

First we need to define q by

$$q = \begin{cases} p & \text{if } \gamma_1(\theta_1^-, \Theta_2^{\pm}(\theta_1^-)) > 0, \text{ i.e., if } \varphi_1 \text{ has the pole } p \text{ in } \mathcal{G}_{\mathcal{R}}, \\ \frac{1}{2}\Theta_2^{\pm}(\theta_1^-) & \text{otherwise.} \end{cases} \tag{21}$$

Note, the choice $\frac{1}{2}\Theta_2^{\pm}(\theta_1^-)$ is arbitrary: any point in $\mathcal{G}_{\mathcal{R}}$ would have been suitable. See Figures 2 and 3.

The function W is built on the function w below (note, w is introduced in [22, Theorem 1]; under the symmetry conditions $\mu_1 = \mu_2$, $\sigma_{11} = \sigma_{22}$, and symmetric reflection vectors in (1), Foschini [20] also obtained an expression for the conformal mapping w, see [20, Figure 3]):

$$w(\theta_2) = T_{\frac{\pi}{\beta}}\left(-\frac{2\theta_2 - (\theta_2^+ + \theta_2^-)}{\theta_2^+ - \theta_2^-}\right), \tag{22}$$

which itself uses the branch points (10), the generalized Chebyshev polynomial ($a \geqslant 0$)

$$T_a(x) = \cos(a \arccos(x)) = \frac{1}{2}\Big\{\big(x + \sqrt{x^2-1}\big)^a + \big(x - \sqrt{x^2-1}\big)^a\Big\},$$

and the angle β as in (20). Then we define

$$W(\theta_2) = \frac{w(\theta_2) - w(\Theta_2^{\pm}(\theta_1^-))}{w(\theta_2) - w(q)} = \frac{w(\theta_2) + 1}{w(\theta_2) - w(q)}. \tag{23}$$

Lemma 6. *The function W in (23)*

(i) *is analytic in $\mathcal{G}_{\mathcal{R}} \setminus \{q\}$, continuous in $\overline{\mathcal{G}_{\mathcal{R}}} \setminus \{q\}$ and bounded at infinity,*
(ii) *is one-to-one from $\mathcal{G}_{\mathcal{R}} \setminus \{q\}$ onto $\mathbb{C} \setminus [0,1]$,*
(iii) *satisfies $W(\theta_2) = W(\overline{\theta_2})$ for all $\theta_2 \in \mathcal{R}$.*

Proof. It can be found in [22, Lemma 6] that w in (22)

(i') is analytic in $\mathcal{G}_{\mathcal{R}}$, continuous in $\overline{\mathcal{G}_{\mathcal{R}}}$ and unbounded at infinity (T_a admits an analytic continuation on $\mathbb{C} \setminus (-\infty, -1)$, and even on $\mathbb{C}$ if a is a non-negative integer),
(ii') is one-to-one from $\mathcal{G}$ onto $\mathbb{C} \setminus (-\infty, -1]$,
(iii') satisfies $w(\theta_2) = w(\overline{\theta_2})$ for all $\theta_2 \in \mathcal{R}$.

Here we want to define another conformal gluing function, which glues together the upper part and the lower part of the hyperbola onto the segment $[0,1]$, and which sends the point q in (21) at infinity, see Figure 4. For this reason we set W as in (23): by construction, $W(\Theta_2^{\pm}(\theta_1^-)) = 0$, $W(\infty) = 1$ and $W(q) = \infty$. The proof of Lemma 6 follows from the above-mentioned properties (i')–(iii') of w together with the definition (23) of W. □

Remark 4. *The algebraic nature of the mapping w in (22) (or equivalently W in (23)) is directly related to the rationality of $\frac{\beta}{\pi}$. Precisely, as shown in [22, Proposition 13], the following behaviors are possible:*

- *If $\frac{\beta}{\pi} \in \mathbb{Q}$, then w is algebraic;*
- *If in addition $\frac{\pi}{\beta} \in \mathbb{N}$ (and only in this case), then w is a polynomial;*
- *If $\frac{\beta}{\pi} \notin \mathbb{Q}$, then w is non-algebraic.*

3.3. **Reduction to a standard BVP.** Thanks to the gluing function W in (23), we are able to reformulate the Carleman BVP as a standard BVP for an open contour. See Figure 4 for a compact view of the two complex planes associated to Carleman's and Riemann's BVPs. Define ψ_1 by

$$\psi_1(t) = \varphi_1 \circ W^{-1}(t), \qquad \forall t \in \mathbb{C} \setminus [0,1] \tag{24}$$

(note, ψ_1 is meromorphic on $\mathbb{C} \setminus [0,1]$). Equivalently we have $\varphi_1(\theta_2) = \psi_1 \circ W(\theta_2)$ for $\theta_2 \in \mathcal{G}_{\mathcal{R}}$. Obviously W^{-1} is not well defined on $[0,1]$. However, it does admit upper and lower limits for $t \in [0,1]$:

$$(W^{-1})^+(t) = \lim_{\substack{u \to t \\ \Im u > 0}} W^{-1}(u), \qquad (W^{-1})^-(t) = \lim_{\substack{u \to t \\ \Im u < 0}} W^{-1}(u),$$

and similarly for $\psi_1^+(t)$ and $\psi_1^-(t)$. Then for $\theta_2 \in \mathcal{R}$ and $t = W(\theta_2) = W(\overline{\theta_2})$, we have

$$\varphi_1(\theta_2) = \begin{cases} \psi_1^+(t) & \text{if } \Im\,\theta_2 > 0, \\ \psi_1^-(t) & \text{if } \Im\,\theta_2 < 0, \end{cases} \qquad \varphi_1(\overline{\theta_2}) = \begin{cases} \psi_1^-(t) & \text{if } \Im\,\theta_2 > 0, \\ \psi_1^+(t) & \text{if } \Im\,\theta_2 < 0. \end{cases}$$

Define further

$$H(t) = G((W^{-1})^-(t)), \qquad \forall t \in [0,1]. \tag{25}$$

Then Proposition 5 becomes:

Proposition 7 (Riemann BVP)**.** *The function ψ_1 in (24)*

(1) *is analytic in $\mathbb{C} \setminus [0,1]$, bounded at infinity if $\gamma_1(\theta_1^-, \Theta_2^{\pm}(\theta_1^-)) \leqslant 0$ and admitting a simple pole at infinity otherwise,*
(2) *is continuous on $[0,1]$ from below (with limits ψ_1^-) and above (with limits ψ_1^+), bounded at 0 and 1 (except if $\gamma_1(\theta_1^-, \Theta_2^{\pm}(\theta_1^-)) = 0$: in this case it has a pole of order one at 0),*
(3) *satisfies, with H defined in (25), the boundary condition*

$$\psi_1^+(t) = H(t)\psi_1^-(t), \qquad \forall t \in [0,1]. \tag{26}$$

Proof. Items (1) and (2) directly follow from the corresponding items in Proposition 5. With the above definitions, the boundary equation (18) becomes

$$\begin{cases} \psi_1^-(t) = G(\theta_2)\psi_1^+(t) & \text{if } \Im\,\theta_2 > 0, \\ \psi_1^+(t) = G(\theta_2)\psi_1^-(t) & \text{if } \Im\,\theta_2 < 0. \end{cases}$$

Since $\frac{1}{G(\theta_2)} = G(\overline{\theta_2}) = H(t)$ if $\Im\,\theta_2 > 0$, and $G(\theta_2) = H(t)$ if $\Im\,\theta_2 < 0$, the last item follows. □

3.4. **Index of the BVP.** The resolution of BVPs as in Proposition 7 heavily depends on the *index* χ (see, e.g., [32, Section 5.2]), which is related to the variation of argument of H on $[0,1]$:

$$\Delta = [\arg H]_0^1, \qquad \delta = \arg H(0) \in (-\pi, \pi], \qquad \chi = \left\lfloor \frac{\delta + \Delta}{2\pi} \right\rfloor. \tag{27}$$

Δ quantifies the variation of argument of H on $[0,1]$ and $\arg H(1) = \delta + \Delta$. Since $(W^{-1})^-([0,1]) = \mathcal{R}^-$, Δ in (27) can be equivalently written $[\arg G]_{\mathcal{R}^-}$ (from the vertex to infinity).

Remark 5. *It is important to notice that $\delta \in (-\pi, \pi]$ in (27) corresponds to an arbitrary choice. Any other choice would eventually lead to the same Theorem 1 (though written slightly differently).*

First, we compute δ in (27).

Lemma 8. *We have*

$$\delta = \begin{cases} 0 & \text{if } \gamma_1(\theta_1^-, \Theta_2^\pm(\theta_1^-)) \neq 0, \\ \pi & \text{if } \gamma_1(\theta_1^-, \Theta_2^\pm(\theta_1^-)) = 0. \end{cases} \tag{28}$$

The angle $\delta + \Delta \in (-2\pi, 2\pi)$ and we have

$$\tan\frac{\delta + \Delta}{2} = \frac{\det R \det \Sigma}{\sigma_{12}(r_{11}r_{22} + r_{12}r_{21}) - \sigma_{22}r_{11}r_{12} - \sigma_{11}r_{22}r_{21}}. \tag{29}$$

Note that the denominator of (29) can be negative, zero or positive, depending on the parameters.

Proof of Lemma 8. If $\gamma_1(\theta_1^-, \Theta_2^\pm(\theta_1^-)) \neq 0$, $H(0) = G(\Theta_2^\pm(\theta_1^-)) = 1$ (since $\Theta_2^\pm(\theta_1^-) \in \mathbb{R}$ simplifies the quotient (17)) and then $\delta = 0$. In the other case the limit of H at 0 is -1 and then $\delta = \pi$.

We start by remarking that for $\theta_2 \in \mathcal{R}$, $G(\theta_2) = 1$ if and only if $\theta_2 \in \mathbb{R}$. Accordingly, for $t \in [0,1]$, $H(t) = 1$ only at $t = 0$. Since $|H| = 1$ on $[0,1]$, then necessarily $\delta + \Delta \in [-2\pi, 2\pi]$. We now calculate $H(1) = \lim_{\substack{\theta_2 \to \infty \\ \theta_2 \in \mathcal{R}^-}} G(\theta_2)$. Using the value of the limit (see (9))

$$\lim_{\substack{\theta_2 \to \infty \\ \theta_2 \in \mathcal{R}^-}} \frac{\Theta_1^-(\theta_2)}{\theta_2} = \frac{-\sigma_{12} - i\sqrt{\det \Sigma}}{\sigma_{11}}$$

in the definition (17) of G

$$G(\theta_2) = \frac{\left(r_{11}\frac{\Theta_1^-(\theta_2)}{\theta_2} + r_{21}\right)\left(r_{12}\frac{\Theta_1^-(\theta_2)}{\overline{\theta_2}} + r_{22}\right)}{\left(r_{12}\frac{\Theta_1^-(\theta_2)}{\theta_2} + r_{22}\right)\left(r_{11}\frac{\Theta_1^-(\theta_2)}{\overline{\theta_2}} + r_{21}\right)},$$

we obtain that

$$\begin{aligned} H(1) &= \frac{\left(r_{11}(-\sigma_{12} - i\sqrt{\det\Sigma}) + r_{21}\sigma_{11}\right)\left(r_{12}(-\sigma_{12} + i\sqrt{\det\Sigma}) + r_{22}\sigma_{11}\right)}{\left(r_{12}(-\sigma_{12} - i\sqrt{\det\Sigma}) + r_{22}\sigma_{11}\right)\left(r_{11}(-\sigma_{12} + i\sqrt{\det\Sigma}) + r_{21}\sigma_{11}\right)} \\ &= \frac{\sigma_{22}r_{11}r_{12} + \sigma_{11}r_{22}r_{21} - \sigma_{12}(r_{11}r_{22} + r_{12}r_{21}) - i\det R\det\Sigma}{\sigma_{22}r_{11}r_{12} + \sigma_{11}r_{22}r_{21} - \sigma_{12}(r_{11}r_{22} + r_{12}r_{21}) + i\det R\det\Sigma} = \exp(i(\delta + \Delta)). \end{aligned}$$

It gives (29) and clearly, $\delta + \Delta$ cannot be equal to $\pm 2\pi$ because $\det R \det \Sigma \neq 0$. □

We now compute χ in (27).

Lemma 9. *The index χ can take only the values 0 and -1, and we have the dichotomy:*

- $\chi = 0 \iff \gamma_1(\theta_1^-, \Theta_2^\pm(\theta_1^-)) \leqslant 0 \iff \psi_1$ *has no pole at infinity,*
- $\chi = -1 \iff \gamma_1(\theta_1^-, \Theta_2^\pm(\theta_1^-)) > 0 \iff \psi_1$ *has a simple pole at infinity.*

Note that a simple pole at infinity means that for some non-zero constant c, $\psi_1(t) \underset{\infty}{\sim} ct$.

Proof. We have already seen in Proposition 5 that the sign of $\gamma_1(\theta_1^-, \Theta_2^\pm(\theta_1^-))$ determines whether φ_1 has a pole in $\mathcal{G}_\mathcal{R}$ or not, and thus if ψ_1 has a pole at infinity. This shows the two equivalences on the right in the statement of Lemma 9. We are thus left with proving the first two equivalent conditions.

First, if $\gamma_1(\theta_1^-, \Theta_2^\pm(\theta_1^-)) = 0$, $\delta = \pi$ and we have seen that in this case $H(t) \neq 1$ for all $t \in [0,1]$. By (27), we deduce that $\chi = 0$.

If now $\gamma_1(\theta_1^-, \Theta_2^\pm(\theta_1^-)) \neq 0$ we notice that

$$\chi = \left\lfloor \frac{\Delta}{2\pi} \right\rfloor = 0 \text{ or } -1.$$

Indeed, we have proved in Lemma 8 that $\Delta \in (-2\pi, 2\pi)$. In particular, the sign of Δ determines χ: if $\operatorname{sgn}\Delta \geqslant 0$ then $\chi = 0$ and if $\operatorname{sgn}\Delta < 0$, $\chi = -1$. In the rest of the proof, we show that $\operatorname{sgn}\Delta = -\operatorname{sgn}\gamma_1(\theta_1^-, \Theta_2^\pm(\theta_1^-))$. First, Δ can be computed as

$$\Delta = \arg H(1) = [\arg G]_{\mathcal{R}^-} = \left[\arg \frac{\gamma_1\overline{\gamma_2}}{\gamma_2\overline{\gamma_1}}(\Theta_1^-(\theta_2), \theta_2)\right]_{\mathcal{R}^-}.$$

Let $\theta_2 = a - ib \in \mathcal{R}^-$ (we must have $b \geqslant 0$ and $a \geqslant \Theta_2^\pm(\theta_1^-) > 0$, see Figure 3) and $\theta_1 = \Theta_1^-(\theta_2) \in (-\infty, \theta_1^-]$. Using the expression (8) of γ_1 and γ_2, we obtain

$$\gamma_1\overline{\gamma_2}(\theta_1, \theta_2) = \gamma_1(\theta_1, a)\gamma_2(\theta_1, a) + r_{21}r_{22}b^2 + ib\theta_1 \det R,$$

from where we deduce that

$$\arg \frac{\gamma_1\overline{\gamma_2}}{\gamma_2\overline{\gamma_1}}(\theta_1, \theta_2) = 2\arctan \frac{b\theta_1 \det R}{\gamma_1(\theta_1, a)\gamma_2(\theta_1, a) + r_{21}r_{22}b^2}. \tag{30}$$

We now look for the sign of (30) when $\theta_2 \to \Theta_2^\pm(\theta_1^-)$, while remaining in $\mathcal{R}^-$. This is sufficient to give the sign of Δ, because (30) does not change sign on $\mathcal{R}^-$ due to the fact that $G(\theta_2) = 1$ on $\mathcal{R}^-$ if and only if $\theta_2 = \Theta_2^\pm(\theta_1^-)$.

When $\theta_2 \to \Theta_2^\pm(\theta_1^-)$ we have $b \to 0$, $a \to \Theta_2^\pm(\theta_1^-)$ and $\theta_1 \to \theta_1^-$. We thus have

$$\begin{aligned}
\operatorname{sgn}\arg \frac{\gamma_1\overline{\gamma_2}}{\gamma_2\overline{\gamma_1}}(\theta_1, \theta_2) &= \operatorname{sgn} b \cdot \operatorname{sgn}\theta_1 \cdot \operatorname{sgn}\det R \cdot \operatorname{sgn}\gamma_2(\theta_1^-, \Theta_2^\pm(\theta_1^-)) \cdot \operatorname{sgn}\gamma_1(\theta_1^-, \Theta_2^\pm(\theta_1^-)) \\
&= (+1)(-1)(+1)(+1)\operatorname{sgn}\gamma_1(\theta_1^-, \Theta_2^\pm(\theta_1^-)) \\
&= -\operatorname{sgn}\gamma_1(\theta_1^-, \Theta_2^\pm(\theta_1^-)),
\end{aligned}$$

because $b \geqslant 0$, $\theta_1 < 0$, $\det R > 0$ by (2), and $\gamma_2(\theta_1^-, \Theta_2^\pm(\theta_1^-)) > 0$ (because $r_{22} > 0$ and $r_{22}\mu_1 - r_{12}\mu_2 < 0$, see Figure 2 to visualize this geometric condition). Then $\operatorname{sgn}\Delta = \operatorname{sgn}\arg\frac{\gamma_1\overline{\gamma_2}}{\gamma_2\overline{\gamma_1}}(\theta_1, \theta_2) = -\operatorname{sgn}\gamma_1(\theta_1^-, \Theta_2^\pm(\theta_1^-))$, concluding the proof. □

3.5. **Resolution of the BVP.** Reformulating Proposition 7, the function ψ_1 in (24)

- is sectionally analytic in $\mathbb{C} \setminus [0,1]$,
- is continuous on $[0,1]$ from below (with limits ψ_1^-) and above (with limits ψ_1^+),
  - is bounded at the vicinities of $[0,1]$ if $\gamma_1(\theta_1^-, \Theta_2^\pm(\theta_1^-)) \neq 0$,
  - has a pole of order one at 0 and bounded at 1 if $\gamma_1(\theta_1^-, \Theta_2^\pm(\theta_1^-)) = 0$,
- is bounded at infinity if there is no pole before $\Theta_2^\pm(\theta_1^-)$ (then taking the value $\varphi_1(q)$), and with a pole of order one (see Lemma 3) at infinity if not (in short, it has a pole of order $-\chi$ at infinity),

- satisfies $\psi_1^+(t) = H(t)\psi_1^-(t)$ for all $t \in [0,1]$, with index χ given by Lemma 9:

$$\chi = \begin{cases} 0 & \text{if } \gamma_1(\theta_1^-, \Theta_2^\pm(\theta_1^-)) \leqslant 0, \\ -1 & \text{if } \gamma_1(\theta_1^-, \Theta_2^\pm(\theta_1^-)) > 0. \end{cases} \tag{31}$$

End of proof of Theorem 1. Our main reference for the resolution of the above so-called homogeneous BVP on an open contour is the book [34] of Muskhelishvili, see in particular [34, §79].

First of all, we prove that there exists a non-zero constant c such that

$$\psi_1(t) = c(t-1)^{-\chi} \exp \Gamma(t), \tag{32}$$

where Γ is the following function, sectionally analytic on $\mathbb{C} \setminus [0,1]$:

$$\Gamma(t) = \frac{1}{2i\pi} \int_0^1 \frac{\log H(z)}{z-t} \mathrm{d}z. \tag{33}$$

To make precise the definition (33), we define $\log H(z)$ by the facts that it should vary continuously over $[0,1]$, and its initial value is such that $\log H(0) = i\delta$ (i.e., 0 if $H(0) = 1$ and $i\pi$ if $H(0) = -1$, see (28)).

At the vicinities 0 and 1, we have by [34, §29 and §79] that

$$\exp \Gamma(t) = t^{-\frac{\delta}{2\pi}} \Omega_0(t), \qquad \exp \Gamma(t) = (t-1)^{\frac{\delta+\Delta}{2\pi}} \Omega_1(t),$$

for some function Ω_0 (resp. Ω_1) analytic in a neighborhood of 0 (resp. 1) and non-zero at 0 (resp. 1). Then we set

$$X(t) = t^{\frac{\delta}{\pi}} (t-1)^{-\chi} \exp \Gamma(t). \tag{34}$$

The function X in (34) is sectionally analytic in $\mathbb{C} \setminus [0,1]$, and by construction of Γ and the Sokhotski-Plemelj formulas, it satisfies the boundary condition (26) (see [34, §79] for more details). Furthermore it has a pole of order $-\chi + \frac{\delta}{\pi}$ at infinity and is bounded at 0 and 1: indeed, $\frac{\delta}{\pi}$ and $-\chi$ are both equal to 0 or 1, see Lemmas 8 and 9. Then we consider two cases separately.

- First case: $\gamma_1(\theta_1^-, \Theta_2^\pm(\theta_1^-)) \neq 0$. Then $\delta = 0$, and the function X in (34) simplifies into

$$X(t) = (t-1)^{-\chi} \exp \Gamma(t),$$

and satisfies the exact same boundary condition as (26). Looking at the ratio $\frac{\psi_1}{X}$, the boundary condition (26) gives that on $[0,1]$,

$$\frac{\psi_1^+}{X^+} = \frac{\psi_1^-}{X^-}.$$

The above ratio is then analytic in the entire plane, even at the vicinities 0 and 1. The point 0 is a regular point and 1 is a removable singularity. Indeed, 1 is an isolated singular point, at which $\frac{\psi_1}{X}$ might be infinite with degree less than unity (namely, $-\chi + \frac{\Delta}{2\pi}$). Moreover, the function $\frac{\psi_1}{X}$ is bounded at infinity, because both X and ψ_1 have a pole of the same order $-\chi$. Thanks to Liouville's theorem, we deduce that $\frac{\psi_1}{X}$ is constant. In conclusion, the formula (32) holds in this case.

- Second case: $\gamma_1(\theta_1^-, \Theta_2^\pm(\theta_1^-)) = 0$. Then $\delta = \pi$, $\chi = 0$ and $X(t) = t \exp \Gamma(t)$ in (34). Firstly, we notice that the function $t\psi_1$ satisfies the boundary condition (26), is bounded at 0 and 1, and has a pole of order one at infinity. Moreover, the function X has a pole of order 1 at infinity. Considering then the ratio $\frac{t\psi_1}{X}$, the boundary condition (26) implies that on $[0,1]$,

$$\frac{t\psi_1^+}{X^+} = \frac{t\psi_1^-}{X^-}.$$

The above ratio function is thus analytic in the entire complex plane, including the vicinities 0 and 1. It is indeed bounded at 1, and has a removable singularity at 0: the point 0 is an isolated singular point, at which $\frac{t\psi_1}{X}$ might be infinite with degree less than $\frac{1}{2}$. Using again Liouville's theorem, we deduce that the function $\frac{t\psi_1}{X}$ is a constant. Formula (32) therefore also holds.

We now deduce from (32) the formula (6) stated in Theorem 1. Going from the t-plane back to the θ_2-plane (see (24)), one has that for some constant c,

$$\varphi_1(\theta_2) = \psi_1(W(\theta_2)) = c(W(\theta_2)-1)^{-\chi} \exp\left\{ \frac{1}{2i\pi} \int_{\mathcal{R}^-} \log G(\theta) \frac{W'(\theta)}{W(\theta)-W(\theta_2)} \mathrm{d}\theta \right\}. \tag{35}$$

Using the equation (23) relating W and w, we easily obtain

$$W(\theta_2) - 1 = \frac{w(q)+1}{w(\theta_2)-w(q)}$$

as well as

$$\frac{W'(\theta)}{W(\theta)-W(\theta_2)} = \frac{w'(\theta)}{w(\theta)-w(\theta_2)} - \frac{w'(\theta)}{w(\theta)-w(q)}.$$

Remembering that in the case $\chi = -1$ one has $q = p$, see (21), we finally obtain that for some constant c',

$$\varphi_1(\theta_2) = c' \left(\frac{1}{w(\theta_2)-w(p)} \right)^{-\chi} \exp\left\{ \frac{1}{2i\pi} \int_{\mathcal{R}^-} \log G(\theta) \frac{w'(\theta)}{w(\theta)-w(\theta_2)} \mathrm{d}\theta \right\}.$$

By definition (4) of the Laplace transform we have $\varphi_1(0) = \nu_1(\mathbb{R}_+)$. To find the exact constant c' (and thereby our main result (6)), we simply evaluate the above formula at $\theta_2 = 0$. □

Clearly, the integral expression (6) of φ_1 is meromorphic in the domain $\mathcal{G}_\mathcal{R}$. In Section 4 we shall see that it can be meromorphically continued on the much larger domain $\mathbb{C} \setminus [\theta_2^+, \infty)$.

To conclude this part, let us make Remark 5 more precise. In the case $\gamma_1(\theta_1^-, \Theta_2^\pm(\theta_1^-)) = 0$ (i.e., $H(0) = -1$), we have chosen $\delta = \pi$ in (27). (Recall that choosing δ is tantamount to fixing a determination of the arg (or log) function.) Remarkably, any other choice of δ would have led to the same explicit expression for φ_1, though written differently. For instance, if we had taken $\delta = -\pi$ instead, the index χ would have been -1 (instead of 0), and the two determinations of the logarithm would differ by $-2i\pi$. With our notation in the proof of Theorem 1, we would have obtained for some constant c'',

$$\psi_1(t) = c'' \frac{t-1}{t} \exp \Gamma(t).$$

This actually corresponds to the formula of Theorem 1 associated with $\chi = -1$.

3.6. **Generalizations.** Our main Theorem 1 is derived under the hypothesis (5) that the coordinates (μ_1, μ_2) of the drift are negative. However, the conditions (2) (equivalent to the existence of a stationary distribution) allow the drift to have one non-negative coordinate. In the next few lines, we assume that $\mu_1 \geqslant 0$ or $\mu_2 \geqslant 0$, and we comment on the slight differences which would have arisen in the analytic treatment of the functional equation (7).

In the case of a drift having one non-negative coordinate, the reflected Brownian motion in the quadrant has a pathwise behavior which is quite different (it spends most of its time near the axis towards which the drift is directed). However, from the point of view of our analysis, the differences are only technical. For instance, the continuation formula (13) of Lemma 3 may use the other branch Θ_1^+, as if $\mu_1 > 0$, the small branch (taking value 0 at 0) is Θ_1^+ and not Θ_1^- anymore, see (9). As a result, the poles of φ_1 may be located at places different from those described in Proposition 5. Despite these few differences, there exists a Cauchy-type integral expression for the Laplace transform φ_1, similar to the one stated in Theorem 1.

Let us also very briefly mention here the case of reflected Brownian motion in higher dimension [24, 27]. Compared to its two-dimensional analogue, much less is known. However, an analogue of the functional equation (7) can still be stated (since the BAR exists in any dimension, see [26, 10]). Clearly, our techniques (based on complex analysis) use the dimension 2, and in our opinion, generalizing in higher dimension these BVP techniques is a difficult open problem. In the discrete case too, the case of dimension 3 is less understood. One can mention [8] for some ideas to state a BVP in dimension 3, as well as [3] for more combinatorial techniques.

4. Singularity analysis and asymptotics of the boundary distribution

The asymptotics (up to a constant) of the boundary measures has been obtained by Dai and Miyazawa in [12], see also [11, 21] for the interior measure. In this section we show that our expression (6) for the Laplace transform φ_1 stated in Theorem 1 is perfectly suited for singularity analysis, and accordingly to study the asymptotics (including the computation of the constant) of the boundary stationary distributions ν_1 and ν_2. We shall first recall the result of [12] and express it in terms of our notations. Then we will explain how, thanks to Theorem 1, we could obtain a new proof of this result and make the constants explicit.

4.1. Asymptotic results.

Theorem 10 (Theorem 6.1 of [12]). *The following asymptotics holds:*

$$\lim_{x\to\infty} \frac{\nu_1(x)}{x^\kappa e^{-\tau_2 x}} = b, \tag{36}$$

where $\kappa \in \{-\frac{3}{2}, -\frac{1}{2}, 0, 1\}$ and $\tau_2 \in \{p, p', \theta_2^+\}$ are given in Table 1, and b is some positive constant. See Figures 5 and 6 to visualize geometrically the different cases.

We now propose a series of three remarks on Theorem 10.

• We have already introduced p, see (11), and p' is as follows: it is the (unique, when it exists) non-zero point $p' = \Theta_2^+(r)$, with r defined by $\gamma_2(r, \Theta_2^-(r)) = 0$ and $r \leqslant \Theta_1^\pm(\theta_2^+)$. It will be convenient to adopt the following notation: we will write $p > \theta_2^+$ (resp. $p' > \theta_2^+$) when p (resp. p') does not exist.

• Theorem 6.1 of [12] deals with the asymptotics of $\nu_1(x, \infty)$, and not with that of $\nu_1(x)$, as stated in Theorem 10 below. However, the two statements are equivalent: when the asymptotics of the density has the form $x^\kappa e^{-\tau_2 x}$, the corresponding tail probability is given by the exact same asymptotics (with another constant b), see [11, Lemma D.5].

• Before stating Table 1, which gives the values of κ and τ_2 of Theorem 10, we briefly recall Dai and Miyazawa's notations: $\theta^{(2,\max)}$ is the (unique) point of the ellipse $\gamma = 0$ such that $\theta_2^{(2,\max)} = \theta_2^+$, $\tau_2 = \sup\{\theta_2 > 0 : \varphi(\theta_1, \theta_2) < \infty\}$ and $\theta^{(2,r)}$ is the intersection point of the straight line $\gamma_1 = 0$ and the ellipse $\gamma = 0$. Notice that the definition of τ_2 does not rely on an analytic continuation of φ.

Cases			Dai and Miyazawa's categories		κ	τ_2
p or $p' \in (0, \theta_2^+)$	1.a	$p < \Theta_2^\pm(\theta_1^-)$	$\tau_2 < \theta_2^{(2,\max)}$	Categories I or II	0	p
	1.b	$\Theta_2^\pm(\theta_1^-) \leqslant p < p'$		Category I	0	p
	1.c	$\Theta_2^\pm(\theta_1^-) \leqslant p' < p$		Category III, $\tau_2 \neq \theta_2^{(2,r)}$	0	p'
	1.d	$\Theta_2^\pm(\theta_1^-) \leqslant p = p'$		Category III, $\tau_2 = \theta_2^{(2,r)}$	1	p
p and $p \geqslant \theta_2^+$	2.a	p and $p' > \theta_2^+$	$\tau_2 = \theta_2^{(2,\max)}$	Category I, $\theta^{(2,r)} \neq \theta^{(2,\max)}$	$-\frac{3}{2}$	θ_2^+
	2.b	$\theta_2^+ = p$		Category I, $\theta^{(2,r)} = \theta^{(2,\max)}$	$-\frac{1}{2}$	θ_2^+
	2.c	$\theta_2^+ = p'$		Category II, $\theta^{(2,r)} \neq \theta^{(2,\max)}$	$-\frac{1}{2}$	θ_2^+
	2.d	$\theta_2^+ = p = p'$		Category II, $\theta^{(2,r)} = \theta^{(2,\max)}$	0	θ_2^+

Table 1. Dai and Miyazawa's categories expressed with our notations

4.2. Sketch of the proof of Theorem 10.

The idea is to study the singularities of φ_1 and to use transfer theorems (such as [14, Theorem 37.1]) relating the asymptotics of a function and the singularities of its Laplace transform. In our case, the singularity closest to 0 will determine the asymptotics. Our aim here is not to propose a complete proof of Theorem 10 (for this we refer to [12]), but rather to illustrate that Theorem 1 indeed implies Theorem 10, with the additional

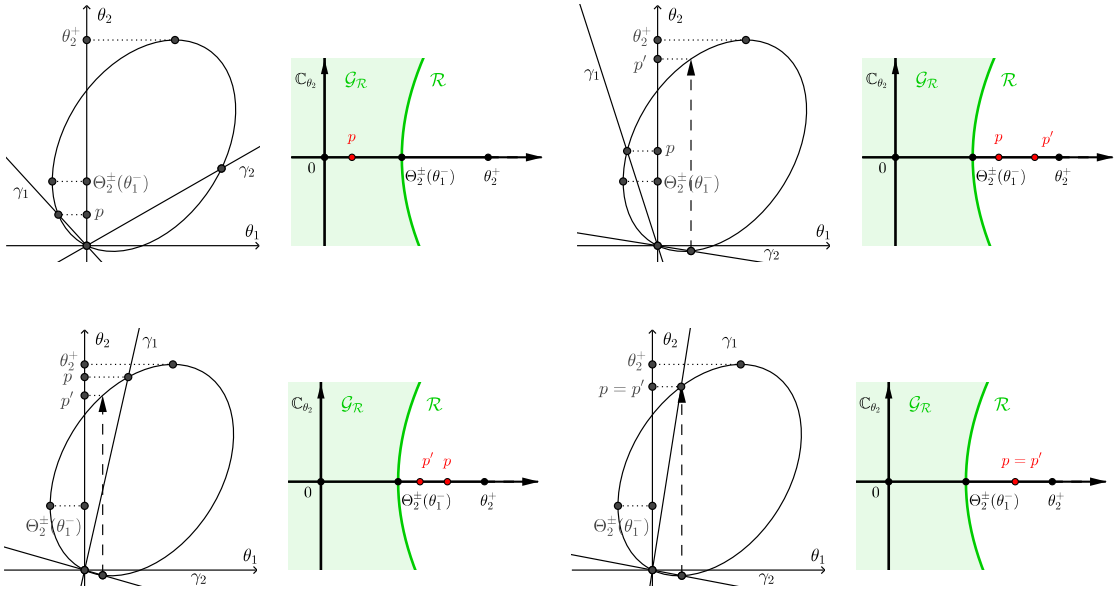

FIGURE 5. Ellipses and curves in the cases 1.a, 1.b, 1.c and 1.d of Table 1

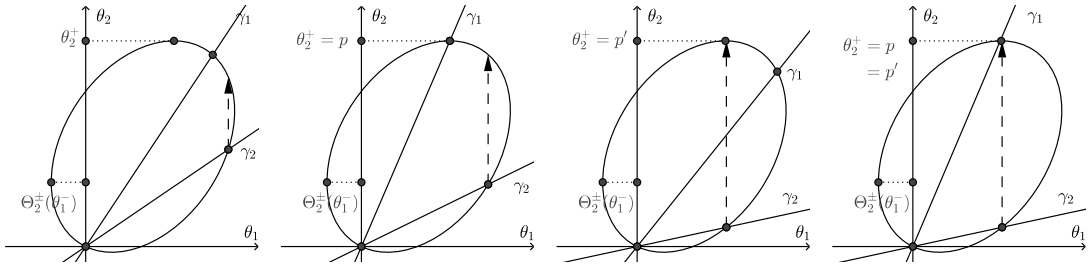

FIGURE 6. Ellipses in the cases 2.a, 2.b, 2.c and 2.d of Table 1

feature of providing an exact expression for the constant b in (36). We give details for the case 1.a of Table 1, and only sketch the difficulties that would arise in the remaining cases.

Proof of Theorem 10 and computation of b in case 1.a. In that case, the index χ equals -1 and the formula (6) gives

$$\varphi_1(\theta_2) = \nu_1(\mathbb{R}_+)\left(\frac{w(0)-w(p)}{w(\theta_2)-w(p)}\right)\exp\left\{\frac{1}{2i\pi}\int_{\mathcal{R}^-}\log G(\theta)\left[\frac{w'(\theta)}{w(\theta)-w(\theta_2)}-\frac{w'(\theta)}{w(\theta)-w(0)}\right]\mathrm{d}\theta\right\}.$$

Recall that w is analytic, one-to-one on $\mathcal{G}_\mathcal{R}$ and further satisfies $w(\theta_2)\neq w(\theta)$, for all $\theta_2\in\mathcal{G}_\mathcal{R}$ and $\theta\in\mathcal{R}^-$. As a first result, the integral part (thus also its exponential) is analytic in the domain $\mathcal{G}_\mathcal{R}$. A second consequence is that $\frac{1}{w(\theta_2)-w(p)}$ has a simple pole at p: $\varphi_1(\theta_2)=\frac{b+o(1)}{\theta_2-p}$, with

$$b = \nu_1(\mathbb{R}_+)\left(\frac{w(0)-w(p)}{w'(p)}\right)\exp\left\{\frac{1}{2i\pi}\int_{\mathcal{R}^-}\log G(\theta)\left[\frac{w'(\theta)}{w(\theta)-w(p)}-\frac{w'(\theta)}{w(\theta)-w(0)}\right]\mathrm{d}\theta\right\}.$$

Theorem 37.1 of [14] gives the announced asymptotics $\nu_1(x)=e^{-px}(b+o(1))$ as $x\to\infty$. □

In the other cases, the singularities are not in $\mathcal{G}_\mathcal{R}$ and we thus need to extend meromorphically φ_1 in a larger domain:

Proposition 11 (Theorem 11 of [21]). *The Laplace transform φ_1 can be meromorphically continued on the domain $\mathbb{C} \setminus [\theta_2^+, \infty)$.*

Proposition 11 has already been proved in [21], see Theorem 11 there. Note that the formula (6) of Theorem 1 provides an alternative, direct, analytic proof, which can be sketched as follows: the equation (6) is valid a priori only for θ_2 in $\mathcal{G}_{\mathcal{R}}$. However, considering a Hankel contour similar to that of Figure 8, surrounding $[\theta_2^+, \infty)$, we could write φ_1 as an integral over the cut $[\theta_2^+, \infty)$. The study of the so-obtained formula would give the singularities p, p' and θ_2^+ as in Table 1, and would lead to the precise asymptotics (we could even obtain the full asymptotic development) and the computation of b.

Remark 6. *We have already commented on the fact within a single formula, Theorem 1 actually captures two different expressions, depending on the value of χ in (31). From an asymptotic viewpoint, Section 4 shows that different cases exist as well, depending on various parameters. It should be noted that these cases are all different, i.e., δ and χ do not govern the asymptotic behavior of the boundary measures.*

5. Algebraic nature and simplification of the Laplace transforms

In this section we are interested in the following question: in which extent is it possible to simplify the expressions of the Laplace transforms given in Theorem 1? For instance, is this possible that these functions be algebraic or even rational?

This is of paramount importance: first, simplified expressions would lead to an easier analysis, in particular for asymptotic analysis or for taking inverse Laplace transforms; second, understanding the parameters (Σ, μ, R) for which the Laplace transforms are rational should reveal intrinsic structure of the model. In the particular case of the identity covariance matrix Σ, some attempts of simplifications may be found in [19, Chapter 4].

In the literature, this question has received much interest in the discrete setting. One can first think at the famous Jackson's networks [29] and their product form solutions. In a closer context, Latouche and Miyazawa [30], Chen, Boucherie and Goseling [6], obtain geometric necessary and sufficient conditions for the stationary distribution of random walks in the quarter plane to be sums of geometric terms. Such criteria can be applied, e.g., to derive an approximation scheme to error bounds for performance measures of random walks in the quarter plane [7].

In our context of reflected diffusions in the quadrant, analogues of these results are obtained by Dieker and Moriarty [13]: a simple condition (involving the single angle (37)) for the stationary density to be a sum of exponential terms is derived. We will discuss the links with our results in Section 5.1.

Another main reason which can lead to simplified expressions comes from the group of the model, which is a group of symmetries naturally associated with the model (cf. Section 5.3). Before being more precise, let us mention that this reason is deeply different than the first one: the group only depends on the covariance matrix and drift vector, and is therefore independent of the reflection matrix, while Dieker and Moriarty's condition also depends on the reflection angles, see (37). In the discrete setting, this group was shown to have a decisive influence on the D-finiteness (a function is D-finite if it satisfies a linear differential equation with polynomial coefficients on $\mathbb{Q}$) of the generating functions, see [18, 4]. For reflected Brownian motion in the quadrant with orthogonal reflections, it is shown in [22] that the Laplace transform is algebraic (note, any algebraic function is D-finite) if and only if the group is finite. We present a structural result in Section 5.2.

In Section 5.3 we prove a criterion for the algebraicity of the Laplace transform, peculiar to the finite group case. Namely, we show that φ_1 is algebraic if and only if a certain norm is equal to 1, which in turn can be reformulated as a condition on a single angle, this way obtaining a result very close to Dieker and Moriarty's condition.

Finally, we focus in Section 5.4 on the case of orthogonal reflections, and derive a new proof of the main result of [22], as a consequence of Theorem 1.

5.1. **Dieker and Moriarty's criterion.** For the sake of completeness, let us mention the following result:

Theorem 12 (Theorem 1 in [13]). *The stationary density is a sum of exponentials if and only if*

$$\frac{\varepsilon+\delta-\pi}{\beta} \in -\mathbb{N} = -\{0,1,2,\ldots\}, \tag{37}$$

with ε and δ in $(0,\pi)$ and

$$\tan\varepsilon = \frac{\sin\beta}{\frac{r_{21}}{r_{11}}\sqrt{\frac{\sigma_{11}}{\sigma_{22}}}+\cos\beta}, \qquad \tan\delta = \frac{\sin\beta}{\frac{r_{12}}{r_{22}}\sqrt{\frac{\sigma_{22}}{\sigma_{11}}}+\cos\beta}. \tag{38}$$

Notice that this is not the exact statement of [13, Theorem 1], as in Dieker and Moriarty's paper, the Brownian motion is assumed to have an identity covariance matrix and evolves in a wedge with an arbitrary opening angle, whereas we consider Brownian motion with arbitrary covariance matrix but in the quarter plane. A simple linear transform, which is made explicit in Appendix A, makes both statements equivalent. The expression (38) of the angles ε and δ follows from this transformation.

5.2. **Structural form of the Laplace transforms and finite group.** By finite group models, we mean that $\frac{\beta}{\pi} \in \mathbb{Q}$, where β is defined in (20), see Section 5.3 for more details on the group. In this case, and in this case only, the function W in (23) is algebraic (as the generalized Chebyshev polynomial (22) is, see Remark 4), yielding the following structural result:

Proposition 13. *If $\frac{\beta}{\pi} \in \mathbb{Q}$, the Laplace transform φ_1 of Theorem 1 is the product of an algebraic function by the exponential of a D-finite function.*

Proof. This easily follows from the fact that the Cauchy integral of a D-finite function is D-finite, see, e.g., [35]. □

However, it is not true in general that the exponential of a D-finite function is still D-finite.

5.3. **An algebraicity criterion for finite group models.** This section is inspired by [18, Chapter 4], and is independent of the rest of the paper. We present purely algebraic manipulations, thanks to which we can study rational and algebraic solutions of the functional equation (7). This question is directly related to Dieker and Moriarty's condition (37), as the Laplace transform of sums of exponential terms is obviously rational.

Theorem 14. *Assume that $\frac{\beta}{\pi} \in \mathbb{Q}$. If the functional equation (7) admits algebraic solutions (and in particular rational solutions), then necessarily*

$$\frac{\varepsilon+\delta-\pi}{\beta} \in \mathbb{Z} + \frac{\pi}{\beta}\mathbb{Z}. \tag{39}$$

Before proving Theorem 14, let us do some comments, in particular a comparison with [13]. Our criterion involves the same angle as [13, Theorem 1], namely $\frac{\varepsilon+\delta-\pi}{\beta}$. However our condition is less restrictive ($-\mathbb{N}$ is included in $\mathbb{Z}+\frac{\pi}{\beta}\mathbb{Z}$), as we allow not only rational but also algebraic solutions. On the other hand, notice that we are not able to characterize the parameters (Σ,μ,R) for which our algebraic solutions are rational, nor to say whether the Laplace transform φ_1 is one of these algebraic solutions. Finally, our proof does not reveal if Theorem 14 still holds in the case $\frac{\beta}{\pi} \notin \mathbb{Q}$ (although our condition (39) does not need $\frac{\beta}{\pi}$ to be rational). Theorem 14 should rather be considered as a structural result on the solutions of the functional equation.

To prove Theorem 14, we shall combine two points of view on the functional equation (7). Firstly, following [18, Chapter 4], we shall perform computations in the quotient ring (below, γ is the kernel (8))

$$\mathbb{C}(\theta_1,\theta_2)/\gamma(\theta_1,\theta_2), \tag{40}$$

which contains all functions of the form $f(\theta_1)+g(\theta_1)\theta_2$, with rational f and g. Secondly, we will also work on a uniformization (or parametrization) of the kernel:

$$\mathcal{S}=\{(\theta_1,\theta_2)\in(\mathbb{C}\cup\{\infty\})^2:\gamma(\theta_1,\theta_2)=0\}=\{(\theta_1(s),\theta_2(s)):s\in\mathbb{C}\cup\{\infty\}\},\tag{41}$$

where

$$\begin{cases}\theta_1(s)=\dfrac{\theta_1^+ +\theta_1^-}{2}+\dfrac{\theta_1^+ -\theta_1^-}{4}\left(s+\dfrac{1}{s}\right),\\ \theta_2(s)=\dfrac{\theta_2^+ +\theta_2^-}{2}+\dfrac{\theta_2^+ -\theta_2^-}{4}\left(\dfrac{s}{e^{i\beta}}+\dfrac{e^{i\beta}}{s}\right).\end{cases}\tag{42}$$

Equations (41) and (42) are easily verified, by evaluating the kernel at $(\theta_1(s),\theta_2(s))$ and using the values of the branch points (10) and β in (20); see also [22, Section 5].

In this context, the group of the model is the dihedral group $\langle\xi,\eta\rangle$ acting on the Riemann sphere $\mathbb{C}\cup\{\infty\}$, where

$$\xi(s)=\frac{1}{s},\qquad \eta(s)=\frac{e^{2i\beta}}{s},\qquad \delta(s)=\eta\xi(s)=e^{2i\beta}s.\tag{43}$$

It is finite if and only if δ in (43) has finite order, i.e., if and only if $\frac{\beta}{\pi}\in\mathbb{Q}$.

Finally, we define s_0 such that $(\theta_1(s_0),\theta_2(s_0))=(0,0)$; the equations in (42) give

$$s_0=-\frac{\theta_1^+ +\theta_1^- +2i\sqrt{-\theta_1^+\theta_1^-}}{\theta_1^+ -\theta_1^-}=-\frac{\theta_2^+ +\theta_2^- -2i\sqrt{-\theta_2^+\theta_2^-}}{\theta_2^+ -\theta_2^-}e^{i\beta}.\tag{44}$$

Proof of Theorem 14. We shall mainly work on the Riemann sphere $\mathcal{S}$ introduced in (41), and we use the following notation: any function f of the complex variable θ_1 can be lifted on $\mathcal{S}$, by setting

$$\widetilde{f}(s)=f(\theta_1(s)).$$

Most of the time we will lighten the notation, writing $f(s)$ instead.

The starting point is to translate the boundary condition (18) to $\mathcal{S}$:

$$\varphi_1(\delta s)=G(s)\varphi_1(s),\tag{45}$$

where δ is as in (43), and by definition (17) of G one has

$$G(s)=\frac{\gamma_1(s)/\gamma_2(s)}{\gamma_1(\eta s)/\gamma_2(\eta s)}.$$

Assuming that the group has order $2n$ ($\delta^n=1$, with $\delta=\eta\xi$) and iterating n times (45), we get

$$\varphi_1(s)=\varphi_1(\delta^n s)=G(s)\cdot G(\delta s)\cdots G(\delta^{n-1}s)\cdot\varphi_1(s).$$

As an obvious consequence, for a solution in the quotient ring (40) to exist, it is necessary for the following quantity to equal 1:

$$N(G):=G(s)\cdot G(\delta s)\cdots G(\delta^{n-1}s)=\frac{\gamma_1(s)\cdot\gamma_1(\delta s)\cdots\gamma_1(\delta^{n-1}s)\cdot\gamma_2(\eta s)\cdot\gamma_2(\delta\eta s)\cdots\gamma_2(\delta^{n-1}\eta s)}{\gamma_2(s)\cdot\gamma_2(\delta s)\cdots\gamma_2(\delta^{n-1}s)\cdot\gamma_1(\eta s)\cdot\gamma_1(\delta\eta s)\cdots\gamma_1(\delta^{n-1}\eta s)}.\tag{46}$$

(Our notation $N(G)$ comes from the fact that (46) is interpreted as a norm, see [18, Chapter 4].) The key point is that by Lemma 15 (whose proof is technical, and can be found below), $N(G)=1$ if and only if the condition on the angles (39) holds. Condition (39) is therefore necessary for the existence of solutions in the quotient ring (40), and a fortiori for rational solutions of (7).

Conversely, let us assume that $N(G)=1$. We shall prove the existence of algebraic solutions of the functional equation (7). Unfortunately, this does not exclude a priori the existence of non-algebraic solutions, and thus this does not characterize the algebraic nature of φ_1 (clearly there is not uniqueness of the solutions to the fundamental functional equation). Introduce

$$F=1+G+G\cdot G(\delta)+\cdots+G\cdot G(\delta)\cdots G(\delta^{n-2}).\tag{47}$$

It is easily seen that G can be factorized as

$$G = \frac{F}{F(\delta)}. \tag{48}$$

In particular, (48) offers a solution to the factorization problem of Remark 3, solution which is peculiar to the case $N(G) = 1$.

Equation (48) enables us to reformulate (45) as $(\varphi_1 \cdot F)(\delta) = \varphi_1 \cdot F$. Such a function invariant by δ must be a function of s^n, and we deduce that the solutions of (45) in the quotient ring have the form $H(s^n)/F(s)$, for an arbitrary function H and F as in (47). □

Lemma 15. *The norm $N(G)$ in (46) equals 1 if and only if the geometric condition (39) holds.*

Proof. We start from the expression (46) of the norm. With (8) one has $\gamma_1(\theta) = r_{11}\theta_1 + r_{21}\theta_2$ and $\gamma_2(\theta) = r_{12}\theta_1 + r_{22}\theta_2$. By convenience and also by homogeneity of $N(G)$ we shall equivalently write, with obvious notations,

$$\gamma_1(\theta) = \theta_1 + \rho_1\theta_2, \qquad \gamma_2(\theta) = \theta_1 + \rho_2\theta_2.$$

For some fixed value of $\rho \in \mathbb{R}$ (not necessarily positive), consider now the equation

$$\theta_1(s) + \rho \cdot \theta_2(s) = 0, \tag{49}$$

where θ_1 and θ_2 are the coordinates of the parametrization (42). Due to the particular form of θ_1 and θ_2, Equation (49) has two solutions. One of them is s_0 (and in particular is independent of ρ), where by definition s_0 is the unique value $s \in \mathbb{C}$ such that $\theta_1(s) = \theta_2(s) = 0$, see (44). We call the other root $s(\rho)$. Notice that $|s_0| = |s(\rho)| = 1$. Going back to the variable θ, $s(\rho_1)$ (resp. $s(\rho_2)$) corresponds on Figure 7 to the point θ^* (resp. θ^{**}).

The norm $N(G)$ in (46) can thus be rewritten as

$$\begin{aligned} N(G) &= \frac{(s - s(\rho_1)) \cdot (s - \delta^{-1}s(\rho_1)) \cdots (s - \delta^{-(n-1)}s(\rho_1)) \cdot (s - \eta s(\rho_2)) \cdots (s - \eta\delta^{-(n-1)}s(\rho_2))}{(s - s(\rho_2)) \cdot (s - \delta^{-1}s(\rho_2)) \cdots (s - \delta^{-(n-1)}s(\rho_2)) \cdot (s - \eta s(\rho_1)) \cdots (s - \eta\delta^{-(n-1)}s(\rho_1))} \\ &= \frac{(s - s(\rho_1)) \cdot (s - \delta s(\rho_1)) \cdots (s - \delta^{n-1}s(\rho_1)) \cdot (s - \eta s(\rho_2)) \cdot (s - \delta\eta s(\rho_2)) \cdots (s - \delta^{n-1}\eta s(\rho_2))}{(s - s(\rho_2)) \cdot (s - \delta s(\rho_2)) \cdots (s - \delta^{n-1}s(\rho_2)) \cdot (s - \eta s(\rho_1)) \cdot (s - \delta\eta s(\rho_1)) \cdots (s - \delta^{n-1}\eta s(\rho_1))}. \end{aligned} \tag{50}$$

Using the root-coefficient relationships in (49), we obtain

$$s(\rho) = \frac{1}{s_0} \frac{(\theta_1^+ - \theta_1^-) + (\theta_2^+ - \theta_2^-)\rho e^{i\beta}}{(\theta_1^+ - \theta_1^-) + (\theta_2^+ - \theta_2^-)\rho e^{-i\beta}} = e^{i(2\omega(\rho) - \arg s_0)},$$

with β as in (20) and

$$\tan\omega(\rho) = \frac{(\theta_2^+ - \theta_2^-)\rho \sin\beta}{(\theta_1^+ - \theta_1^-) + (\theta_2^+ - \theta_2^-)\rho\cos\beta}.$$

Note that it follows from (10) that

$$\frac{\theta_1^+ - \theta_1^-}{\theta_2^+ - \theta_2^-} = \sqrt{\frac{\sigma_{22}}{\sigma_{11}}}.$$

In particular, $\tan\omega(\rho)$ can be rewritten as

$$\tan\omega(\rho) = \frac{\sin\beta}{\frac{1}{\rho}\sqrt{\frac{\sigma_{22}}{\sigma_{11}}} + \cos\beta} \tag{51}$$

and we easily obtain, with δ and ε as in (56),

$$\omega(\rho_2) = \delta, \qquad \omega(\rho_1) = \beta - \varepsilon. \tag{52}$$

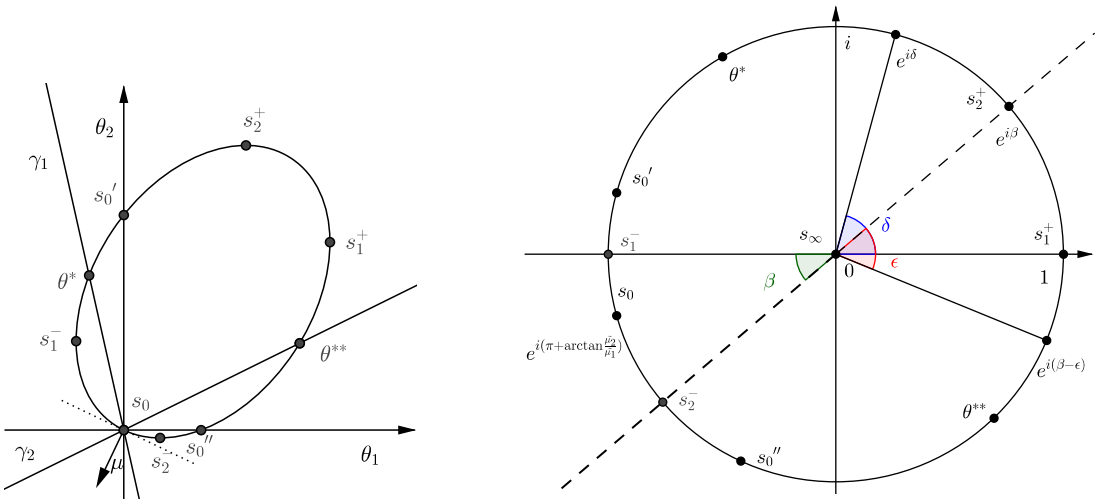

FIGURE 7. The ellipse $\{(\theta_1,\theta_2)\in\mathbb{R}^2 : \gamma(\theta_1,\theta_2)=0\}$ (left) becomes a circle (right) on $\mathcal{S}$ after uniformization (42)

In the norm $N(G)$ in (50), the arguments of the zeros of the numerator (resp. denominator) are at $\mathcal{F}(\rho_1)\cup\mathcal{G}(\rho_2)$ (resp. $\mathcal{F}(\rho_2)\cup\mathcal{G}(\rho_1)$), where

$$\begin{aligned}\mathcal{F}(\rho) &= \{2\omega(\rho)-\arg s_0, 2\omega(\rho)-\arg s_0+2\beta,\dots,2\omega(\rho)-\arg s_0+2(n-1)\beta\}\\ &= 2\omega(\rho)-\arg s_0+2\beta\times\{0,\dots,n-1\}\\ &= 2\omega(\rho)-\arg s_0+2\beta\mathbb{Z}\end{aligned}$$

and

$$\begin{aligned}\mathcal{G}(\rho) &= \{-(2\omega(\rho)-\arg s_0)+2\beta, -(2\omega(\rho)-\arg s_0)+4\beta,\dots,-(2\omega(\rho)-\arg s_0)+2n\beta\}\\ &= -(2\omega(\rho)-\arg s_0)+2\beta\times\{1,\dots,n\}\\ &= -(2\omega(\rho)-\arg s_0)+2\beta\mathbb{Z}.\end{aligned}$$

(We can replace $\{0,\dots,n-1\}$ and $\{1,\dots,n\}$ by $\mathbb{Z}$, because the group is finite and of order $2n$ if and only if $\beta=k\pi/n$, for some integer k relatively prime with n.)

The zeros of $\mathcal{F}(\rho_1)\cup\mathcal{G}(\rho_2)$ compensate with those of $\mathcal{F}(\rho_2)\cup\mathcal{G}(\rho_1)$ if and only if the family $\mathcal{F}(\rho_1)$ is canceled by $\mathcal{F}(\rho_2)$, and $\mathcal{G}(\rho_1)$ by $\mathcal{G}(\rho_2)$. Observing that $e^{i\mathcal{F}(\rho_1)}=e^{i\mathcal{F}(\rho_2)}$ (resp. $e^{i\mathcal{G}(\rho_1)}=e^{i\mathcal{G}(\rho_2)}$) if and only if $2\omega(\rho_1)-2\omega(\rho_2)\in\frac{2\pi}{n}\mathbb{Z}$, we conclude by (52), writing that $\frac{\pi}{n}\in\mathbb{Z}+\frac{\pi}{\beta}\mathbb{Z}$. □

Remark 7. *It is not possible to have $\omega(\rho_1)=\omega(\rho_2)$, since with (51) this would imply $\rho_1=\rho_2$ and thus the two reflection vectors would be parallel, which is excluded by our assumptions (2).*

5.4. **Orthogonal reflections.** Here we consider the case of orthogonal reflections, which is equivalent for the reflection matrix R in (1) to be the identity matrix; see also Figure 1. By developing the theory of Tutte's invariants (introduced by Tutte in [36] for the enumeration of properly colored triangulations, and used in [2] for the enumeration of quadrant walks) for the Brownian motion, we proved in [22] the following result:

Theorem 16 (Theorem 1 in [22])**.** *Let R be the identity matrix in (1). The Laplace transform φ_1 is equal to*

$$\varphi_1(\theta_2)=\frac{-\mu_1 w'(0)}{w(\theta_2)-w(0)}\theta_2. \tag{53}$$

In this section we derive a new proof of this result, as a consequence of Theorem 1. More generally, the proof below would work for any parameters (Σ,μ,R) such that $G(\theta_2)=\frac{F(\theta_2)}{F(\overline{\theta_2})}$ (cf. Remark 3), yielding a rational expression of $\varphi_1(\theta_2)$ in terms of $w(\theta_2)$ and θ_2.

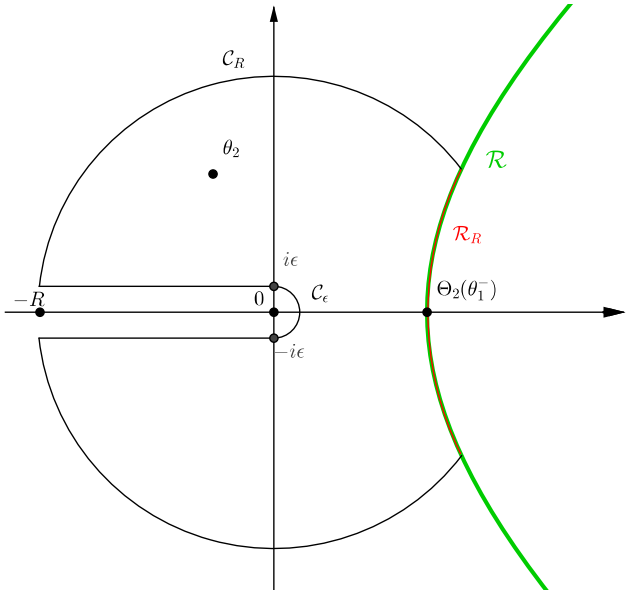

FIGURE 8. Integration contour used in the proof of Theorem 16

Proof. Let us first notice that the index $\chi = 0$. Indeed, since $G(\theta_2) = \frac{\overline{\theta_2}}{\theta_2}$, we have $\arg G(\theta_2) = -2\arg\theta_2 > 0$ for $\theta_2 \in \mathcal{R}$, and thus $\Delta > 0$, see the proof of Lemma 9.

Starting from the formula (35), we have for some constant C

$$\varphi_1(\theta_2) = C\exp\left\{\frac{1}{2i\pi}\int_{\mathcal{R}^-}(\log\overline{\theta} - \log\theta)\frac{W'(\theta)}{W(\theta)-W(\theta_2)}\mathrm{d}\theta\right\} = C\exp\left\{\frac{1}{2i\pi}\int_{\mathcal{R}}\log\theta\frac{W'(\theta)}{W(\theta)-W(\theta_2)}\mathrm{d}\theta\right\}.$$

To compute the above integral, we first integrate on the contour represented on Figure 8. The residue theorem gives

$$\frac{1}{2i\pi}\left\{\int_{\mathcal{R}_R}+\int_{\mathcal{C}_R}+\int_{\mathcal{C}_\varepsilon}+\int_{-R+i\varepsilon}^{i\varepsilon}+\int_{-i\varepsilon}^{-R-i\varepsilon}\right\}\log\theta\frac{W'(\theta)}{W(\theta)-W(\theta_2)}\mathrm{d}\theta = \log\theta_2 - \log q. \tag{54}$$

It is easy to see that

$$\lim_{R\to\infty}\frac{1}{2i\pi}\int_{\mathcal{R}_R}\log\theta\frac{W'(\theta)}{W(\theta)-W(\theta_2)}\mathrm{d}\theta = \frac{1}{2i\pi}\int_{\mathcal{R}}\log\theta\frac{W'(\theta)}{W(\theta)-W(\theta_2)}\mathrm{d}\theta$$

and that in the limits when $\varepsilon\to 0$ and $R\to\infty$, the contributions on $\mathcal{C}_\varepsilon$ and $\mathcal{C}_R$ both converge to 0, because W is analytic at 0 and ∞, respectively.

Furthermore,

$$\begin{aligned}
\lim_{\varepsilon\to 0}\lim_{R\to\infty}\frac{1}{2i\pi}\left\{\int_{-R+i\varepsilon}^{i\varepsilon}+\int_{-i\varepsilon}^{-R-i\varepsilon}\right\}&\log\theta\frac{W'(\theta)}{W(\theta)-W(\theta_2)}\mathrm{d}\theta\\
&= \lim_{\varepsilon\to 0}\frac{1}{2i\pi}\int_{-\infty}^{0}(\log(t+i\varepsilon)-\log(t-i\varepsilon))\frac{W'(\theta)}{W(\theta)-W(\theta_2)}\mathrm{d}\theta\\
&= \frac{1}{2i\pi}\int_{-\infty}^{0}2i\pi\frac{W'(\theta)}{W(\theta)-W(\theta_2)}\mathrm{d}\theta\\
&= \log\frac{W(0)-W(\theta_2)}{W(\infty)-W(\theta_2)}\\
&= \log\frac{w(\theta_2)-w(0)}{w(q)-w(0)}.
\end{aligned}$$

Above we have used the dominated convergence and the fact that the principal determination of the logarithm gives us $\lim_{\varepsilon\to 0}(\log(t+i\varepsilon)-\log(t-i\varepsilon))=2i\pi$. Letting $R\to\infty$ and then $\varepsilon\to 0$ in (54), we have, for some constants C and C',

$$\varphi_1(\theta_2)=C\exp\left\{\frac{1}{2i\pi}\int_{\mathcal{R}}\log\theta\frac{W'(\theta)}{W(\theta)-W(\theta_2)}\mathrm{d}\theta\right\}=C'\frac{\theta_2}{w(\theta_2)-w(0)}.$$

Since by Lemma 2 one has $\varphi_1(0)=-\mu_1$, this eventually gives the right constant in (53). □

Appendix A. Equivalence between Brownian motion in wedges and Brownian motion in the quarter plane

We use the notation of Section 1. Up to an isomorphism, studying Brownian motion in the quarter plane with arbitrary covariance matrix Σ is equivalent to studying Brownian motion in a cone of angle $\beta=\arccos-\frac{\sigma_{12}}{\sqrt{\sigma_{11}\sigma_{22}}}$, with covariance identity. In this short section we relate the key parameters (angles of the reflection vectors and drift) before and after the linear transformation.

Let us define the linear transforms

$$T=\begin{pmatrix}\frac{1}{\sin\beta} & \cot\beta\\ 0 & 1\end{pmatrix}\begin{pmatrix}\frac{1}{\sqrt{\sigma_{11}}} & 0\\ 0 & \frac{1}{\sqrt{\sigma_{22}}}\end{pmatrix},\qquad T^{-1}=\begin{pmatrix}\sqrt{\sigma_{11}} & 0\\ 0 & \sqrt{\sigma_{22}}\end{pmatrix}\begin{pmatrix}\sin\beta & -\cos\beta\\ 0 & 1\end{pmatrix}. \tag{55}$$

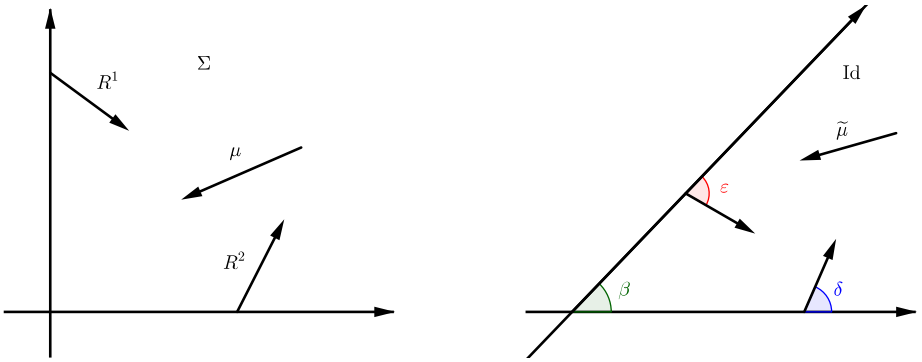

Figure 9. The linear transformation T in (55) from the quadrant to the wedge of opening angle β

Obviously the reflected Brownian motion associated to (Σ,μ,R) becomes a Brownian motion (with covariance identity) in a wedge of angle β and with parameters $(\mathrm{Id},T\mu,TR)$. The new angles of reflection are δ and ε (cf. Figure 9), such that

$$\begin{cases}\tan\delta=\dfrac{\sin\beta}{a+\cos\beta}, & \cos\delta=\dfrac{a+\cos\beta}{\sqrt{a^2+2a\cos\beta+1}}, & \sin\delta=\dfrac{\sin\beta}{\sqrt{a^2+2a\cos\beta+1}},\\ \tan\varepsilon=\dfrac{\sin\beta}{b+\cos\beta}, & \cos\varepsilon=\dfrac{b+\cos\beta}{\sqrt{b^2+2b\cos\beta+1}}, & \sin\varepsilon=\dfrac{\sin\beta}{\sqrt{b^2+2b\cos\beta+1}},\end{cases} \tag{56}$$

where $a=\frac{r_{12}}{r_{22}}\sqrt{\frac{\sigma_{22}}{\sigma_{11}}}$ and $b=\frac{r_{21}}{r_{11}}\sqrt{\frac{\sigma_{11}}{\sigma_{22}}}$. The new drift is $\widetilde{\mu}=T\mu$, where

$$\widetilde{\mu}_1=\frac{\mu_1}{\sqrt{\sigma_{11}}}\frac{1}{\sin\beta}+\frac{\mu_2}{\sqrt{\sigma_{22}}}\cot\beta\quad\text{ and }\quad\widetilde{\mu}_2=\frac{\mu_2}{\sqrt{\sigma_{22}}}.$$

Thanks to (10) and (44), we have $\tan\arg s_0=\frac{\widetilde{\mu}_2}{\widetilde{\mu}_1}$, see Figure 7.

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