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MAXIMAL REPRESENTATIONS OF UNIFORM COMPLEX HYPERBOLIC LATTICES IN EXCEPTIONAL HERMITIAN LIE GROUPS

PIERRE-EMMANUEL CHAPUT, JULIEN MAUBON

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1. INTRODUCTION

This paper deals with maximal representations of complex hyperbolic lattices in semisimple Hermitian Lie groups with no compact factors.

A *complex hyperbolic lattice* Γ is a lattice in the Lie group $SU(1, n)$, a finite cover of the group of biholomorphisms of the n -dimensional complex hyperbolic space $\mathbb{H}_{\mathbb{C}}^n = SU(1, n)/U(n)$. Unless otherwise specified, we shall always assume that our lattice Γ is *uniform*, meaning that the quotient $X := \Gamma \backslash \mathbb{H}_{\mathbb{C}}^n$ is compact, and that it is *torsion free*, so that X is also a manifold. The Kähler form on X induced by the $SU(1, n)$ -invariant Kähler form on $\mathbb{H}_{\mathbb{C}}^n$ with constant holomorphic sectional curvature -1 will be denoted by ω .

A semisimple Lie group with no compact factors $G_{\mathbb{R}}$ is said to be *Hermitian* if its associated symmetric space $M = G_{\mathbb{R}}/K_{\mathbb{R}}$ is a Hermitian symmetric space (of the non-compact type). This means that the symmetric space M admits a $G_{\mathbb{R}}$ -invariant complex

structure, which makes it a Kähler manifold. We will call ω_M the $G_{\mathbb{R}}$ -invariant Kähler form of M , normalized so that its holomorphic sectional curvatures lie between -1 and $-\frac{1}{\text{rk}M}$, where $\text{rk}M$ is the rank of the symmetric space M , or equivalently the real rank $\text{rk}_{\mathbb{R}}G_{\mathbb{R}}$ of $G_{\mathbb{R}}$. We will also assume that the complexification G of $G_{\mathbb{R}}$ is simply connected.

If ρ is a representation (a group homomorphism) from Γ to $G_{\mathbb{R}}$, we can define its *Toledo invariant* $\tau(\rho)$ as follows:

$$\tau(\rho) = \frac{1}{n!} \int_X f^* \omega_M \wedge \omega^{n-1},$$

where $f : \mathbb{H}_{\mathbb{C}}^n \rightarrow M$ is a C^∞ and ρ -equivariant map and $f^* \omega_M$ is seen as a 2-form on X by Γ -invariance. The Toledo invariant does not depend on the choice of the map f , it depends only on ρ , and in fact, only on the connected component of ρ in $\text{Hom}(\Gamma, G_{\mathbb{R}})$. Moreover, it satisfies the following Milnor-Wood type inequality:

$$|\tau(\rho)| \leq \text{rk}M \text{vol}(X),$$

a fundamental property established in full generality in [BI07].

Maximal representations $\rho : \Gamma \rightarrow G_{\mathbb{R}}$ are those representations for which the Milnor-Wood inequality is an equality.

In [KM17], Koziarz and the second named author classified maximal representations when $G_{\mathbb{R}}$ is a classical group. In the present work, we extend this classification to all Hermitian groups, and we prove:

Theorem A. *Let Γ be a torsion free uniform lattice in $\text{SU}(1, n)$, $n \geq 2$, and let ρ be a maximal representation of Γ in a Hermitian Lie group $G_{\mathbb{R}}$.*

Then there exists a unique ρ -equivariant harmonic map f from $\mathbb{H}_{\mathbb{C}}^n$ to the associated symmetric space M . If $\tau(\rho) > 0$, f is holomorphic and satisfies $f^ \omega_M = \text{rk}M \omega$. If $\tau(\rho) < 0$, f is antiholomorphic and satisfies $f^* \omega_M = -\text{rk}M \omega$. Moreover f is a totally geodesic embedding.*

It follows quite easily from this that the map f of the theorem is *tight*. Such maps between Hermitian symmetric spaces were classified in [Ham13], and from his classification we deduce:

Corollary B. *Under the assumptions of Theorem A, each simple factor of $G_{\mathbb{R}}$ is either isogenous to $\text{SU}(p, q)$ for some (p, q) with $q \geq np$, or to the exceptional group $E_{6(-14)}$, the latter being possible only if $n = 2$.*

We can also deduce a structure result for the representation ρ . The map f of Theorem A is in fact equivariant w.r.t. a morphism of Lie groups $\varphi : \text{SU}(1, n) \rightarrow G_{\mathbb{R}}$. We denote by $H_{\mathbb{R}}$ the image of φ in $G_{\mathbb{R}}$ and by $Z_{\mathbb{R}}$ the centralizer of $H_{\mathbb{R}}$ in $G_{\mathbb{R}}$.

Corollary C. *Under the assumptions of Theorem A, the representation ρ is reductive, discrete, faithful, and acts cocompactly on the image of f in M . The centralizer $Z_{\mathbb{R}}$ is compact and there exists a group morphism $\rho_{\text{cpt}} : \Gamma \rightarrow Z_{\mathbb{R}}$ such that*

$$\forall \gamma \in \Gamma, \rho(\gamma) = \varphi(\gamma) \rho_{\text{cpt}}(\gamma) = \rho_{\text{cpt}}(\gamma) \varphi(\gamma).$$

Moreover, Lemma 5.11 says that $Z_{\mathbb{R}}$ is exactly the subgroup of $G_{\mathbb{R}}$ fixing all the points of $f(\mathbb{H}_{\mathbb{C}}^n)$. Its isomorphism class is described in Lemma 5.9.

Remark 1.1. The assumption that Γ is torsion free is technical and can be removed by passing to a normal finite index subgroup of Γ . Indeed, using Selberg Lemma, one can always find a torsion free normal finite index subgroup Γ' of Γ . Theorem A and

Corollaries B and C are therefore applicable to Γ' . We have chosen to assume that the complexification G of $G_{\mathbb{R}}$ is simply-connected to simplify the exposition, but this assumption is not necessary: see Remark 5.12.

The global strategy we adopt here is the same as in [KM17]: we consider a Higgs bundle $(\bar{E}, \bar{\theta})$ on the quotient $X = \Gamma \backslash \mathbb{H}_{\mathbb{C}}^n$ associated to a (reductive) representation $\rho : \Gamma \rightarrow G_{\mathbb{R}}$ (see 3.1) and we translate the Milnor-Wood inequality into an inequality involving degrees of subbundles of \bar{E} (see 4.2). This inequality is then proved (in 4.4) using the Higgs-stability properties of $(\bar{E}, \bar{\theta})$, or rather the *leafwise* Higgs-stability properties of the pull-back (E, θ) of $(\bar{E}, \bar{\theta})$ to the projectivized tangent bundle $\mathbb{P}T_X$ of X with respect to the *tautological foliation* on $\mathbb{P}T_X$ (see Subsections 3.2 and 3.3).

Although classical target groups were already treated in [KM17], we decided not to focus immediately on the exceptional cases and instead to provide a more unified perspective, as independent as possible of the classification of the simple Hermitian Lie groups, in the spirit of [BGPR15]. To achieve this, instead of considering the Higgs vector bundle associated to the standard representation of the complexification G of $G_{\mathbb{R}}$ (which is only defined in the classical cases), we work with the Higgs vector bundle (E, θ) associated to the *cominuscul* representation \mathbb{E} of G such that the dual compact symmetric space $M^{\vee} = G/P$ of M is embedded in the projectivization $\mathbb{P}\mathbb{E}$ of \mathbb{E} : this is sometimes called the *first canonical embedding* of M^{\vee} , see [NT76, p. 651].

On the algebraic side, we present in Section 2 a general construction of a graded submodule of \mathbb{E} associated to an element of \mathfrak{u}_{-} if $\mathfrak{g} = \mathfrak{k} \oplus (\mathfrak{u}_{+} \oplus \mathfrak{u}_{-})$ is the Cartan decomposition of the Lie algebra of G . On the geometric side, this construction gives leafwise Higgs subsheaves of $(E, \theta) \rightarrow \mathbb{P}T_X$ associated to the components of the Higgs field θ (see Subsection 4.3), whose existence is then used to prove the Milnor-Wood inequality. To be a bit more precise, on a *generic* fiber of $(E, \theta) \rightarrow \mathbb{P}T_X$, the leafwise Higgs subsheaves we produce admit purely representation theoretic descriptions. The algebraic counterparts of the generic objects are first introduced and studied in Section 2. This is then used in 4.3 to define the subsheaves and prove that they have the desired properties.

This unified approach allows to exclude the possibility of maximal representations in *tube type* target Lie groups, and in particular in $E_{7(-25)}$. Indeed in this case the representation ρ satisfies an inequality stronger than the Milnor-Wood inequality (Proposition 5.1). Maximal representations in $E_{6(-14)}$ are treated in Subsection 5.2 where we prove that they can exist only if $n = 2$, in which case they are essentially induced by a homomorphism $SU(2, 1) \rightarrow E_{6(-14)}$, see Theorem 5.7.

In [BIW09], the authors introduced the notion of *tight* representations. By [BI07, Lemma 5.3], any maximal representation is tight, and by [BIW09, Theorem 3], a tight representation is *reductive*, meaning that the Zariski closure of its image in $G_{\mathbb{R}}$ is a reductive subgroup of $G_{\mathbb{R}}$. This reduces the study of maximal representations to the reductive ones. Furthermore, by e.g. [KM17, Lemma 4.11], any representation can be deformed to a reductive one with the same Toledo invariant. This also reduces the proof of the Milnor-Wood inequality to the case of reductive representations. From now on, we therefore assume without loss of generality that

Assumption 1.2. The representation $\rho : \Gamma \rightarrow G_{\mathbb{R}}$ is reductive.

2. SUBMODULE OF A COMINUSCULE REPRESENTATION ASSOCIATED WITH A NILPOTENT ELEMENT

Here we develop the algebraic material that we will need in Section 4 to give a new and unified proof of the Milnor-Wood inequality.

Let $G_{\mathbb{R}}$ be a simple noncompact Hermitian Lie group and $K_{\mathbb{R}}$ a maximal compact subgroup of $G_{\mathbb{R}}$. The associated irreducible Hermitian symmetric space $G_{\mathbb{R}}/K_{\mathbb{R}}$ will be denoted by the letter M .

Let also $G = G_{\mathbb{R}} \otimes \mathbb{C}$ and $K = K_{\mathbb{R}} \otimes \mathbb{C}$ be the complexifications of the real algebraic groups $G_{\mathbb{R}}$ and $K_{\mathbb{R}}$. We assume that G is simply connected. Let $T \subset K$ be a maximal torus of G . We denote by $\mathfrak{g}, \mathfrak{k}, \mathfrak{t}$ the corresponding complex Lie algebras. Let R be the set of roots of G , Π be a basis of R , and W the Weyl group of R . The center $\mathfrak{z} \subset \mathfrak{t}$ of \mathfrak{k} is 1-dimensional and we let $z \neq 0$ be an element in this center. We have the Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{u}$, where

$$\mathfrak{k} = \mathfrak{t} \oplus \bigoplus_{\alpha: \langle \alpha, z \rangle = 0} \mathfrak{g}_{\alpha} \quad , \quad \mathfrak{u} = \bigoplus_{\alpha: \langle \alpha, z \rangle \neq 0} \mathfrak{g}_{\alpha}.$$

Now the adjoint action of z on \mathfrak{u} gives the complex structure of the Hermitian symmetric space $M = G_{\mathbb{R}}/K_{\mathbb{R}}$: $\text{ad}(z)|_{\mathfrak{u}}$ has therefore exactly two opposite eigenvalues and, up to scaling z , we assume that these are ± 2 . The corresponding eigenspaces \mathfrak{u}_+ and \mathfrak{u}_- are Abelian, their root space decompositions are

$$\mathfrak{u}_+ = \bigoplus_{\alpha: \langle \alpha, z \rangle = 2} \mathfrak{g}_{\alpha} \quad , \quad \mathfrak{u}_- = \bigoplus_{\alpha: \langle \alpha, z \rangle = -2} \mathfrak{g}_{\alpha}.$$

Notation 2.1. The *rank* of the symmetric space M , or equivalently the real rank of $G_{\mathbb{R}}$, will be denoted by p . A root α of \mathfrak{g} is *noncompact* if $\langle \alpha, z \rangle \neq 0$. Linearly independent positive noncompact roots $\alpha_1, \dots, \alpha_r$ are said to be *strongly orthogonal* if for all $i \neq j$, $\alpha_i \pm \alpha_j$ is not a root. All maximal sets of strongly orthogonal roots of \mathfrak{u}_+ have cardinality p ([HC56], or [Hel01, Ch. VIII, §7]).

For a root $\alpha \in R$, we let α^{\vee} be the associated coroot and ϖ_{α} the fundamental weight corresponding to α . We denote by $R^{\vee} = \{\alpha^{\vee} \mid \alpha \in R\}$ the dual root system. The set of simple roots Π is a basis of \mathfrak{t}^* , $\Pi^{\vee} = \{\beta^{\vee} \mid \beta \in \Pi\}$ is a basis of \mathfrak{t} and $\{\varpi_{\alpha} \mid \alpha \in \Pi\}$ is the dual basis of Π^{\vee} . If $\alpha \in R$, we write $\alpha = \sum_{\beta \in \Pi} n_{\beta}(\alpha) \beta$ the expression of α in terms of the simple roots.

In this paper, we use the convention that if the root system R of \mathfrak{g} is simply laced, then all the roots are long. Therefore short roots exist only if R is not simply laced. We recall that there can be at most two root lengths in R and that if there are two root lengths the ratio $\frac{\text{long root length}}{\text{short root length}}$ equals $\sqrt{2}$ (Hermitian symmetric spaces exist only in type A_n, B_n, C_n, D_n, E_6 or E_7).

2.1. The cominuscule representation and its grading.

There exists a unique simple noncompact root $\zeta \in \Pi$. It follows from the classification that ζ is long, see Table 1. The root ζ , or equivalently z , defines the parabolic subalgebra

$$\mathfrak{p} = \mathfrak{k} \oplus \mathfrak{u}^+ = \mathfrak{t} \oplus \bigoplus_{\alpha: n_{\zeta}(\alpha) \geq 0} \mathfrak{g}_{\alpha}$$

of \mathfrak{g} and hence a parabolic subgroup P of G . The projective variety $M^{\vee} = G/P$ is a Hermitian symmetric space of compact type called the compact dual of $M = G_{\mathbb{R}}/K_{\mathbb{R}}$.

Let $\varpi := \varpi_{\zeta}$ be the fundamental weight associated to the noncompact simple root ζ and let \mathbb{E} be the irreducible representation of G whose highest weight is ϖ . Let \mathbb{E}_{ϖ} be the ϖ -eigenspace of \mathbb{E} . Then \mathbb{E}_{ϖ} is 1-dimensional and one can check that its

stabilizer in G is P . This gives a G -equivariant holomorphic and isometric embedding of $M^\vee = G/P$ in the projective space $\mathbb{P}\mathbb{E}$. It is called the *first canonical embedding* of M^\vee . See e.g. [NT76] for more details.

By [Mur59], since G is simply connected, the Picard group $\text{Pic}(G/P)$ is isomorphic to the group of characters $\mathcal{X}(P)$ of P . Since $\mathfrak{p} = \mathfrak{z} \oplus [\mathfrak{p}, \mathfrak{p}]$, $\mathcal{X}(P)$ is isomorphic to \mathbb{Z} and thus it is generated by the smallest positive character of P , namely ϖ . Moreover, the isomorphism $\iota : \mathcal{X}(P) \simeq \text{Pic}(G/P)$ is given by $\lambda \mapsto (G \times \mathbb{C}_\lambda)/P$, where \mathbb{C}_λ denotes the 1-dimensional P -module defined by λ . Since $\mathbb{E}_\varpi \simeq \mathbb{C}_\varpi$ as P -modules, we see that $\mathcal{L} := \iota(\varpi) \simeq \mathcal{O}_{\mathbb{P}\mathbb{E}}(1)|_{M^\vee}$ is a generator of $\text{Pic}(M^\vee)$.

Definition 2.2. A fundamental weight ϖ_β , $\beta \in \Pi$, is *minuscule* w.r.t. the root system R if $\langle \varpi_\beta, \alpha^\vee \rangle \in \{-1, 0, 1\}$ for all $\alpha \in R$ (cf. e.g. [Bou68, Chapter VI, Exercise 24]). It is *cominuscule* if the fundamental coweight ϖ_{β^\vee} associated to the coroot $\beta^\vee \in \Pi^\vee$ is minuscule w.r.t. the dual root system R^\vee .

An irreducible representation of G whose highest weight is a (co)minuscule fundamental weight of R is also called (co)minuscule.

Remark 2.3. If R is simply laced, then it is isomorphic to R^\vee and hence the cominuscule property is equivalent to the minuscule property.

Since the fundamental coweights $\{\varpi_{\beta^\vee} \mid \beta \in \Pi\}$ form by definition the dual basis of the coroots of Π^\vee , i.e. of Π , a fundamental weight ϖ_β is cominuscule if and only if $n_\beta(\alpha) \in \{-1, 0, 1\}$ for all $\alpha \in R$.

We conclude immediately that the weight $\varpi := \varpi_\zeta$, and hence the G -representation \mathbb{E} , are cominuscule. Let indeed α be a root. As we saw, $\langle \alpha, z \rangle \in \{-2, 0, 2\}$ and ζ is the only simple noncompact root, so that $\langle \alpha, z \rangle = n_\zeta(\alpha) \langle \zeta, z \rangle = 2n_\zeta(\alpha)$. Hence the coefficient $n_\zeta(\alpha)$ belongs to $\{-1, 0, 1\}$. Equivalently, we can observe that the coweight ϖ_{ζ^\vee} is $\frac{1}{2}z$, which gives the result.

We now begin our study of the cominuscule representation \mathbb{E} of G .

Notation 2.4. We denote by μ_0 the lowest weight of \mathbb{E} and by $X(\mathbb{E})$ the set of weights of \mathbb{E} . For $\chi \in X(\mathbb{E})$, we write \mathbb{E}_χ for the corresponding weight space. Recall that ϖ is the highest weight of \mathbb{E} .

The fact that \mathbb{E} is cominuscule has the following consequence on the weights of \mathbb{E} :

Lemma 2.5. *For any weight χ of \mathbb{E} , and any root $\alpha \in R$, $|\langle \chi, \alpha^\vee \rangle| \leq 2$, and the equality $|\langle \chi, \alpha^\vee \rangle| = 2$ implies that α is short.*

Proof. For the highest weight ϖ of \mathbb{E} , the result follows from the fact that $\langle \varpi, \alpha^\vee \rangle = n_\zeta(\alpha) \frac{\|\zeta\|^2}{\|\alpha\|^2}$. The result still holds if ϖ is replaced by $w \cdot \varpi$, where $w \in W$ is arbitrary, and since any weight of \mathbb{E} is in the convex hull of $W \cdot \varpi$, it holds for any weight. \square

We deduce that the structure of \mathbb{E} with respect to the action of \mathfrak{g}_α for α a long root is particularly simple:

Lemma 2.6. *Let α be a long root and let χ be a weight of \mathbb{E} . We have*

$$\mathfrak{g}_{-\alpha} \cdot \mathbb{E}_\chi = \begin{cases} \mathbb{E}_{\chi-\alpha} & \text{if } \langle \chi, \alpha^\vee \rangle = 1, \\ \{0\} & \text{otherwise.} \end{cases}$$

Proof. Let α be long and let $\mathfrak{sl}_2(\alpha)$ be the Lie subalgebra of \mathfrak{g} corresponding to α . Let $M \subset \mathbb{E}$ be the $\mathfrak{sl}_2(\alpha)$ -submodule generated by \mathbb{E}_χ . By Lemma 2.5, any irreducible component V of M is a $\mathfrak{sl}_2(\alpha)$ -module of dimension 1 or 2.

We therefore have only three possibilities. The first case is when $V = V_\chi$, $\mathfrak{g}_\alpha \cdot V_\chi = \{0\}$ and $\mathfrak{g}_{-\alpha} \cdot V_\chi = \{0\}$. In this case, $\langle \chi, \alpha^\vee \rangle = 0$. The second case is when $V = V_\chi \oplus V_{\chi-\alpha}$, $\mathfrak{g}_\alpha \cdot V_\chi = \{0\}$ and $\mathfrak{g}_{-\alpha} \cdot V_\chi = V_{\chi-\alpha}$. In this case, $\langle \chi, \alpha^\vee \rangle = 1$ (and $\langle \chi - \alpha, \alpha^\vee \rangle = -1$). The third (symmetric) case is when $V = V_\chi \oplus V_{\chi+\alpha}$, $\mathfrak{g}_\alpha \cdot V_\chi = V_{\chi+\alpha}$ and $\mathfrak{g}_{-\alpha} \cdot V_\chi = \{0\}$. In this case, $\langle \chi, \alpha^\vee \rangle = -1$ (and $\langle \chi + \alpha, \alpha^\vee \rangle = 1$).

If $\langle \chi, \alpha^\vee \rangle = 1$, we deduce that $M = M_\chi \oplus M_{\chi-\alpha}$ and that $\mathfrak{g}_{-\alpha} \cdot M_\chi = M_{\chi-\alpha}$. We have $M_\chi = \mathbb{E}_\chi$ so $\dim(\mathbb{E}_\chi) \leq \dim(\mathbb{E}_{\chi-\alpha})$. Arguing with the $\mathfrak{sl}_2(\alpha)$ -submodule M' generated by $\mathbb{E}_{\chi-\alpha}$, we see that the dimensions are equal which implies $M_{\chi-\alpha} = \mathbb{E}_{\chi-\alpha}$. The lemma is proved in this case. If $\langle \chi, \alpha^\vee \rangle \leq 0$, we see that $\mathfrak{g}_{-\alpha} \cdot \mathbb{E}_\chi = \{0\}$. \square

Notation 2.7. We denote by z_{\max} the integer $\langle \varpi, z \rangle$. One can observe that $z_{\max} = 2 \frac{\dim M^\vee}{c_1(M^\vee)}$ (see, e.g., [KM10, Section 2]).

Proposition 2.8. *The set $\{\langle \chi, z \rangle \mid \chi \in X(\mathbb{E})\}$ is the set $\{z_{\max}, z_{\max} - 2, \dots, z_{\max} - 2p\}$.*

Proof. It follows from [RRS92, Theorem 2.1] that the W -orbit of the weight ϖ is exactly the set of weights of the form $\varpi - \sum_{i=1}^k \alpha_i$, where $(\alpha_i)_{1 \leq i \leq k}$ is a family of long strongly orthogonal roots. For any i , we have $\langle \alpha_i, z \rangle = 2$, thus we have the equality of sets

$$\{\langle \mu, z \rangle \mid \mu \in W \cdot \varpi\} = \{z_{\max}, z_{\max} - 2, \dots, z_{\max} - 2p\}.$$

In particular, $\langle \mu_0, z \rangle = z_{\max} - 2p$ and for $\chi \in X(\mathbb{E})$, we have $z_{\max} - 2p \leq \langle \chi, z \rangle \leq z_{\max}$. The result of the proposition now follows from the fact that 2 is a divisor of $\langle \alpha, z \rangle$ for any root α . \square

Now we can introduce the grading of \mathbb{E} :

Definition 2.9. For a relative integer i , let $\mathbb{E}_i := \bigoplus_{\chi: \langle \chi, z \rangle = z_{\max} - 2i} \mathbb{E}_\chi$.

This grading corresponds to the decomposition of \mathbb{E} into irreducible K -modules:

Proposition 2.10. *The K -modules \mathbb{E}_i are irreducible.*

Proof. This might be well-known to experts, but we include a proof for completeness. We give a case by case argument. In all types except in type C_n , the representation \mathbb{E} is well enough understood, so that we get readily the result. In fact, in type A_{n-1} , we have $\mathbb{E} = \wedge^p(\mathbb{C}^p \oplus \mathbb{C}^{n-p})$ and thus $\mathbb{E}_i = \wedge^{p-i}\mathbb{C}^p \otimes \wedge^i\mathbb{C}^{n-p}$: this is an irreducible $S(\mathrm{GL}_p \times \mathrm{GL}_{n-p})$ -module. In type B_n , we have $\mathbb{E} = \mathbb{C}^{2n+1} = \mathbb{C} \oplus \mathbb{C}^{2n-1} \oplus \mathbb{C}$, and each summand is an irreducible Spin_{2n-1} -module, hence an irreducible K -module. In type (D_n, ϖ_1) the situation is similar.

In type (D_n, ϖ_n) , \mathbb{E} is the spinor representation of the spin group and, according to [Che97], we have $\mathbb{E} = \wedge^0\mathbb{C}^n \oplus \wedge^2\mathbb{C}^n \oplus \dots \oplus \wedge^{2p}\mathbb{C}^n$ (where $p = \lfloor n/2 \rfloor$). Thus $\mathbb{E}_i = \wedge^{2i}\mathbb{C}^n$, and this is an irreducible GL_n -module. For the types E_6 and E_7 , we use models of these exceptional Lie algebras and their minuscule representations, as given for example in [Man06]. In type E_6 , we have $\mathbb{E} = \mathbb{C} \oplus V_{16} \oplus V_{10}$, where V_{16} is a spinor representation and V_{10} the vector representation of the spin group Spin_{10} . In type E_7 , we have $\mathbb{E} = \mathbb{C} \oplus V_{27} \oplus V'_{27} \oplus \mathbb{C}$, where V_{27} and V'_{27} are the two minuscule representations of a group of type E_6 . In both cases, the K -modules \mathbb{E}_i are irreducible.

We now deal with the case of type C_n . We denote by $\mathbb{C}^{2n} = \mathbb{C}_a^n \oplus \mathbb{C}_b^n$ a symplectic $2n$ -dimensional space, with \mathbb{C}_a^n and \mathbb{C}_b^n supplementary isotropic subspaces. We then have $\mathbb{E} = (\wedge^n \mathbb{C}^{2n})_\omega$, where the symbol ω means that we take in $\wedge^n \mathbb{C}^{2n}$ the irreducible Sp_{2n} -submodule containing the highest weight line $\wedge^n \mathbb{C}_a^n$.

We first claim that for any i , the variety $M^\vee \cap \mathbb{P}\mathbb{E}_i$ generates $\mathbb{P}\mathbb{E}_i$ as a projective space. To prove this claim, we set $\mathbb{F}_i \subset \mathbb{E}_i$ to be the space generated by the affine

cone over $M^\vee \cap \mathbb{E}_i$. We denote by β_1, \dots, β_n the base of the root system. We consider $\mathbb{F} = \bigoplus_i \mathbb{F}_i$. Then \mathbb{F} is obviously K -stable. Moreover, if $x \in M^\vee \cap \mathbb{P}\mathbb{E}_i$, then applying Lemma 2.5, since β_n is long, the $\mathrm{SL}_2(\beta_n)$ -orbit of x is $\{x\}$ itself or a line $\langle x, y \rangle$ joining x and a point y in \mathbb{E}_{i-1} or \mathbb{E}_{i+1} . In either case, y belongs to M^\vee , and this implies that \mathbb{F} is $\mathrm{SL}_2(\beta_n)$ -stable.

This implies that \mathbb{F} is Sp_{2n} -stable, so $\mathbb{F} = \mathbb{E}$, and $\mathbb{F}_i = \mathbb{E}_i$. To prove that $\mathbb{E}_i = \mathbb{F}_i$ is an irreducible K -module, it is enough to show that $M^\vee \cap \mathbb{P}\mathbb{E}_i$ is a single K -orbit. Note that the previous part of the argument would be valid for any cominuscule representation, but we now give a specific argument in the case of Sp_{2n} to show that $M^\vee \cap \mathbb{P}\mathbb{E}_i$ is a single K -orbit. In fact, \mathbb{E}_i is a submodule of $\wedge^{n-i} \mathbb{C}_a^n \otimes \wedge^i \mathbb{C}_b^n$, and $M^\vee \cap \mathbb{P}\mathbb{E}_i$ represents the set of Lagrangian subspaces of \mathbb{C}^{2n} which meet \mathbb{C}_a^n in dimension $n-i$ and \mathbb{C}_b^n in dimension i . Since \mathbb{C}_b^n is the dual space to \mathbb{C}_a^n , via the symplectic form ω , such a space Λ is equal to the direct sum of $\Lambda \cap \mathbb{C}_a^n$ and $(\Lambda \cap \mathbb{C}_a^n)^\perp$ (with $(\Lambda \cap \mathbb{C}_a^n)^\perp \subset \mathbb{C}_b^n$). We deduce that $M^\vee \cap \mathbb{P}\mathbb{E}_i$ is isomorphic to the Grassmannian of $(n-i)$ -subspaces in \mathbb{C}_a^n , and thus it is a single GL_n -orbit. \square

Proposition 2.11. *We have the following properties:*

- (a) $\mathbb{E} = \mathbb{E}_0 \oplus \mathbb{E}_1 \oplus \dots \oplus \mathbb{E}_p$.
- (b) $\mathbb{E}_0 = \mathbb{E}_\varpi$.
- (c) $\mathbb{E}_{i+1} = \mathfrak{u}_- \cdot \mathbb{E}_i$.
- (d) *The map $\mathbb{E}_0 \otimes \mathfrak{u}_- \rightarrow \mathbb{E}_1$ is an isomorphism.*

Proof. Only the last two points need a proof. Let $U(\mathfrak{u}_-)$ denote the envelopping algebra of \mathfrak{u}_- . The third point follows from the fact that $\mathbb{E} = U(\mathfrak{u}_-) \cdot \mathbb{E}_\varpi$ and the fact that for α a root of \mathfrak{u}_- , we have $\langle \alpha, z \rangle = -2$. The last point follows by Schur's lemma since \mathfrak{u}_- and \mathbb{E}_1 are irreducible \mathfrak{k} -modules. \square

2.2. Rank of a nilpotent element, dominant orthogonal sequences.

We now consider an element $y \in \mathfrak{u}_-$ and we describe a particularly nice representative of its orbit under K .

The K -orbits in \mathfrak{u}_- are parametrized by integers $r \in \{0, \dots, p\}$ and a representative of each orbit is $y_{\alpha_1} + \dots + y_{\alpha_r}$, where the roots α_i are strongly orthogonal and long, and y_{α_i} is a fixed element in the root space of α_i (see e.g. [HC56], [Hel01, Ch. VIII, §7], or [Wol72]). If $y \in \mathfrak{u}_-$ is in the orbit corresponding to the integer r , r is called the *rank* of y .

Remark 2.12. In case \mathfrak{g} has type A , \mathfrak{u}_- identifies with a space of matrices, and the rank as defined above of an element in \mathfrak{u}_- coincides with its rank as a matrix.

The following proposition is a characterization of the rank r of $y \in \mathfrak{u}_-$ in terms of its action on \mathbb{E} :

Proposition 2.13. *Let $y = y_{\alpha_1} + \dots + y_{\alpha_r}$ with $\alpha_1, \dots, \alpha_r$ a family of strongly orthogonal long roots, and let $\varphi(y) : \mathbb{E} \rightarrow \mathbb{E}$ be the corresponding linear map. We have $\varphi(y)^{r+1} = 0$ and $\varphi(y)^r(\mathbb{E}_0) = \mathbb{E}_{\varpi - \alpha_1 - \dots - \alpha_r}$, so in particular $\varphi(y)^r(\mathbb{E}_0) \neq \{0\}$.*

Proof. We have $y_{\alpha_i} \in \mathfrak{g}_{-\alpha_i}$. For any pair (χ, α) with $\chi \in X(\mathbb{E})$ and α a long root, $\chi - 2\alpha$ cannot be a weight of \mathbb{E} by Lemma 2.5 and its proof. We deduce that $\varphi(y_{\alpha_i})^2 = 0$. On the other hand, for any i, j with $i \neq j$, the maps $\varphi(y_{\alpha_i})$ and $\varphi(y_{\alpha_j})$ commute since $\alpha_i + \alpha_j$ is not a root. Thus we get

$$\varphi(y)^k = k! \sum \varphi(y_{\alpha_{j_1}}) \circ \dots \circ \varphi(y_{\alpha_{j_k}}),$$

where the sum is over the increasing sequences $1 \leq j_1 < \dots < j_k \leq r$.

In particular $\varphi(y)^{r+1} = 0$, and $\varphi(y)^r = r! \varphi(y_{\alpha_1}) \circ \cdots \circ \varphi(y_{\alpha_r})$. By Lemma 2.6, we have $\varphi(y_{\alpha_1}) \circ \cdots \circ \varphi(y_{\alpha_r}) \cdot \mathbb{E}_\varpi = \mathbb{E}_{\varpi - \alpha_1 - \cdots - \alpha_r}$, so the proposition is proved. \square

In [Kos12], Kostant introduced his so-called “chain cascade” of orthogonal roots. Here we will need a version of his algorithm where we impose that all the roots of the chain cascade have a positive coefficient on ζ . Note that a similar algorithm is used in [BM15].

Definition 2.14. We define an integer q and, for any integer i such that $1 \leq i \leq q$, a root α_i together with a subset $\Pi_i \subset \Pi$, by the following inductive process:

- We let $\Pi_1 = \Pi$.
- Assuming that $\alpha_1, \dots, \alpha_{i-1}$ and Π_1, \dots, Π_i have been defined, we let α_i be the highest root of the root system $R(\Pi_i)$ generated by Π_i .
- We let $\Sigma_i \subset \Pi_i$ be the set of simple roots β such that

$$(1) \quad \langle \alpha_i^\vee, \beta \rangle \neq 0.$$

- If $\zeta \in \Sigma_i$, then $q = i$ and the algorithm terminates. Otherwise, Π_{i+1} is the connected component of $\Pi_i \setminus \Sigma_i$ containing ζ .

If $(\alpha_i)_{1 \leq i \leq q}$ is the sequence defined by this process, we say that it is the *maximal dominant orthogonal sequence* for ϖ . More generally, the sequences $(\alpha_i)_{1 \leq i \leq r}$ for $1 \leq r \leq q$ are called the *dominant orthogonal sequences*.

The following proposition essentially adapts the results of [Kos12] to our context and explains our terminology.

Proposition 2.15. *Let $(\alpha_i)_{1 \leq i \leq q}$ be the maximal dominant orthogonal sequence for ϖ . Then*

- (1) *The roots α_i are long and strongly orthogonal.*
- (2) *$q = p$.*
- (3) *For any integer $i \leq p$, $\alpha_1^\vee + \cdots + \alpha_i^\vee$ is a dominant coweight.*
- (4) *$\varpi - \alpha_1 - \cdots - \alpha_p$ is the lowest weight μ_0 of the irreducible G -module \mathbb{E} .*

Proof. The root α_i is the highest root of $R(\Pi_i)$, and Π_i contains the long root ζ , so α_i is long. By construction, $\langle \alpha_i^\vee, \alpha_j \rangle = 0$ if $j > i$, so α_i and α_j are orthogonal. Since they are both long, they are strongly orthogonal. Kostant [Kos12, Lemma 1.6] also proved the strong orthogonality. This proves (1).

For (2), let us prove that $(\alpha_1, \dots, \alpha_q)$ is a maximal sequence of orthogonal roots (see also [Kos12, Theorem 1.8]). Let $\alpha \in R$ be such that $\langle \alpha_i^\vee, \alpha \rangle = 0$ holds for all i , and let us assume that $\alpha > 0$. Let i be the greatest integer such that $\alpha \in R(\Pi_i)$. By maximality of i there exists a simple root β in $\text{Supp}(\alpha) \cap \Sigma_i$. Since α_i is dominant on $R(\Pi_i)$, we have $\langle \alpha_i^\vee, \alpha \rangle \geq \langle \alpha_i^\vee, \beta \rangle > 0$, a contradiction to the existence of α .

For the third point, let $i \leq p$ and let $\beta \in \Pi$.

- If $\beta \in \Pi_i$, by construction, $\langle \alpha_j^\vee, \beta \rangle = 0$ for $j < i$. Since α_i is the highest root of Π_i , $\langle \alpha_i^\vee, \beta \rangle \geq 0$. Thus $\langle \alpha_1^\vee + \cdots + \alpha_i^\vee, \beta \rangle = \langle \alpha_i^\vee, \beta \rangle \geq 0$.
- If $\beta \in \Sigma_{i-1}$, then $\langle \alpha_{i-1}^\vee, \beta \rangle \geq 1$ and $\langle \alpha_j^\vee, \beta \rangle = 0$ for $j < i + 1$. Since α_i is long, $\langle \alpha_i^\vee, \beta \rangle \geq -1$. Thus, $\langle \alpha_1^\vee + \cdots + \alpha_i^\vee, \beta \rangle \geq 0$.
- If $\beta \in \Sigma_j$ for $j < i - 1$, then, by construction of Π_i , $\langle \alpha_i^\vee, \beta \rangle = 0$. By induction on i , $\langle \alpha_1^\vee + \cdots + \alpha_{i-1}^\vee, \beta \rangle \geq 0$, so $\langle \alpha_1^\vee + \cdots + \alpha_i^\vee, \beta \rangle \geq 0$.

For (4), we observe that $s_{\alpha_i}(\varpi) = \varpi - \alpha_i$ since α_i is long, by Lemma 2.5. Thus $s_{\alpha_1} \cdots s_{\alpha_p}(\varpi) = \varpi - \alpha_1 - \cdots - \alpha_p$ is a weight of \mathbb{E} . We prove that it is a lowest weight. Let $\beta \in \Pi$. If $\beta \neq \zeta$, then $\langle \varpi - \alpha_1 - \cdots - \alpha_p, \beta^\vee \rangle = \langle -\alpha_1 - \cdots - \alpha_p, \beta^\vee \rangle$, and this is non-positive by (3) since all the roots α_i have the same length. If $\beta = \zeta$, then we compute that $\langle \varpi - \alpha_1 - \cdots - \alpha_p, \zeta^\vee \rangle = 1 - \langle \alpha_p, \zeta^\vee \rangle \leq 0$ since $\zeta \in \Sigma_p$. \square

For the convenience of the reader and later use, we recall in Table 1 below what we obtain applying this recursive construction in all cases. We don't indicate all the roots α_i , but rather the sum of the corresponding coroots $\alpha_1^\vee + \dots + \alpha_r^\vee$, by indicating in the shape of a Dynkin diagram the values $\langle \alpha_1^\vee + \dots + \alpha_r^\vee, \beta \rangle$ for all simple roots β : in other words, $\alpha_1^\vee + \dots + \alpha_r^\vee$ is expressed as an integer combination of fundamental coweights. In the last column, we express the smallest root θ such that $\langle \alpha_1^\vee + \dots + \alpha_r^\vee, \theta \rangle = 2$.

(G, ϖ)	Condition	$\alpha_1^\vee + \dots + \alpha_r^\vee$	θ
$(A_{p+q-1}, \varpi_{p-1})$	$r < p$ or $r < q$	$0 \dots 010 \dots 010 \dots 0$	$0 \dots 011 \dots 110 \dots 0$
$(A_{p+q-1}, \varpi_{p-1})$	$r = p = q$	$0 \dots 020 \dots 0$	$0 \dots 010 \dots 0$
(B_n, ϖ_1)	$r = 1$	$010 \dots 0$	$12 \dots 22$
(B_n, ϖ_1)	$r = 2$	$200 \dots 0$	$10 \dots 00$
(C_n, ϖ_n)	$r < n$	$0 \dots 010 \dots 00$	$0 \dots 022 \dots 21$
(C_n, ϖ_n)	$r = n$	$0 \dots 02$	$0 \dots 01$
(D_n, ϖ_1)	$r = 1$	$010 \dots 0 \dots 0 \begin{smallmatrix} 0 \\ 0 \end{smallmatrix}$	$122 \dots 2 \dots 2 \begin{smallmatrix} 1 \\ 1 \end{smallmatrix}$
(D_n, ϖ_1)	$r = 2$	$200 \dots 0 \dots 0 \begin{smallmatrix} 0 \\ 0 \end{smallmatrix}$	$100 \dots 0 \dots 0 \begin{smallmatrix} 0 \\ 0 \end{smallmatrix}$
(D_n, ϖ_n)	$2r \leq n - 2$	$0 \dots 010 \dots 0 \begin{smallmatrix} 0 \\ 0 \end{smallmatrix}$	$0 \dots 122 \dots 2 \begin{smallmatrix} 1 \\ 1 \end{smallmatrix}$
(D_n, ϖ_n)	$2r = n - 1$	$0 \dots 000 \dots 0 \begin{smallmatrix} 1 \\ 1 \end{smallmatrix}$	$0 \dots 000 \dots 1 \begin{smallmatrix} 1 \\ 1 \end{smallmatrix}$
(D_n, ϖ_n)	$2r = n$	$0 \dots 000 \dots 0 \begin{smallmatrix} 2 \\ 0 \end{smallmatrix}$	$0 \dots 000 \dots 0 \begin{smallmatrix} 1 \\ 0 \end{smallmatrix}$
(E_6, ϖ_1)	$r = 1$	$0 \ 0 \ 0 \ 0 \ 0 \begin{smallmatrix} 1 \\ 0 \end{smallmatrix}$	$1 \ 2 \ 3 \ 2 \ 1 \begin{smallmatrix} 2 \\ 2 \end{smallmatrix}$
(E_6, ϖ_1)	$r = 2$	$1 \ 0 \ 0 \ 0 \ 1 \begin{smallmatrix} 0 \\ 0 \end{smallmatrix}$	$1 \ 1 \ 1 \ 1 \ 1 \begin{smallmatrix} 0 \\ 0 \end{smallmatrix}$
(E_7, ϖ_7)	$r = 1$	$1 \ 0 \ 0 \ 0 \ 0 \ 0 \begin{smallmatrix} 0 \\ 0 \end{smallmatrix}$	$2 \ 3 \ 4 \ 3 \ 2 \ 1 \begin{smallmatrix} 2 \\ 2 \end{smallmatrix}$
(E_7, ϖ_7)	$r = 2$	$0 \ 0 \ 0 \ 0 \ 1 \ 0 \begin{smallmatrix} 0 \\ 0 \end{smallmatrix}$	$0 \ 1 \ 2 \ 2 \ 2 \ 1 \begin{smallmatrix} 1 \\ 1 \end{smallmatrix}$
(E_7, ϖ_7)	$r = 3$	$0 \ 0 \ 0 \ 0 \ 0 \ 2 \begin{smallmatrix} 0 \\ 0 \end{smallmatrix}$	$0 \ 0 \ 0 \ 0 \ 0 \ 1 \begin{smallmatrix} 0 \\ 0 \end{smallmatrix}$

TABLE 1. Dominant orthogonal sequences

Remark 2.16. *Tube type cominuscule modules.* The symmetric space $G_{\mathbb{R}}/K_{\mathbb{R}}$ has tube type for (A_{p+q-1}, ϖ_p) when $p = q$, for (B_n, ϖ_1) , for (C_n, ϖ_n) , for (D_n, ϖ_n) when n is even, for (D_n, ϖ_1) , and for (E_7, ϖ_7) . Glancing at Table 1, one can readily check that $G_{\mathbb{R}}/K_{\mathbb{R}}$ has tube type if and only if $z = \alpha_1^\vee + \dots + \alpha_p^\vee$.

By abuse of notation, we will say that the G -module \mathbb{E} itself has tube type if the corresponding symmetric space $G_{\mathbb{R}}/K_{\mathbb{R}}$ has tube type.

If \mathbb{E} has tube type it follows that $\langle \alpha_1^\vee + \cdots + \alpha_p^\vee, \beta \rangle = 2\delta_{\beta, \zeta}$ because by definition z is equal to $2\varpi_{\zeta^\vee}$. Since ζ and all the roots α_i are long, we have $\langle \alpha_i^\vee, \zeta \rangle = \langle \alpha_i, \zeta^\vee \rangle$, so $\alpha_1 + \cdots + \alpha_p = 2\varpi$. We get $\varpi - (\alpha_1 + \cdots + \alpha_p) = -\varpi$ and this is the lowest weight μ_0 by Proposition 2.15(4). This implies that $\langle \mu_0, z \rangle = -\langle \varpi, z \rangle$, so $z_{\max} = p$ and $\mathbb{E}_p = \mathbb{E}_{\mu_0}$. Since the lowest weight is the opposite of the highest weight, the tube type representation \mathbb{E} is autodual: $\mathbb{E} \simeq \mathbb{E}^\vee$. Moreover, for any weight χ , we have an isomorphism of T -modules $\mathbb{E}_{-\chi} \simeq \mathbb{E}_\chi^\vee$. Since the characters of K inject into the characters of T , we get in particular that $\mathbb{E}_p = \mathbb{E}_{-\varpi} \simeq \mathbb{E}_\varpi^\vee = \mathbb{E}_0^\vee$ as K -modules.

We make the following observations:

Proposition 2.17. *Let $(\alpha_1, \dots, \alpha_r)$ be a dominant orthogonal sequence and $h = \alpha_1^\vee + \cdots + \alpha_r^\vee$. Then:*

- *h is dominant: for any positive root α , we have $\langle \alpha, h \rangle \geq 0$.*
- *For any root α , we have $\langle \alpha, h \rangle \leq 2$.*
- *If a root α satisfies $\langle \alpha, h \rangle = 2$, then $\langle \alpha, z \rangle = 2$.*
- *We have $\langle \varpi, h \rangle = r$.*

Proof. Recall the table 1. The first point has been proved in Proposition 2.15(3). Let Θ be the highest root of the root system of G , which can be found for example in [Bou68]. Since h is dominant, the second item follows from the fact that $\langle \Theta, h \rangle = 2$ in all cases. For the third item, we have indicated in the last column of the array the smallest root θ such that $\langle \theta, h \rangle = 2$. It is thus enough to check that $\langle \theta, z \rangle = 2$, or equivalently that θ has coefficient 1 on the simple root corresponding to ϖ . This is again readily checked.

To prove that $\langle \varpi, h \rangle = r$, recall that if $(\alpha_1, \dots, \alpha_p)$ is the maximal dominant orthogonal sequence, the weight $\varpi - \sum_1^p \alpha_i$ is the lowest weight μ_0 by Proposition 2.15(4). Thus, $\langle \varpi - \mu_0, h \rangle = \langle \alpha_1 + \cdots + \alpha_p, \alpha_1^\vee + \cdots + \alpha_r^\vee \rangle = 2r$, since $\langle \alpha_i, \alpha_j^\vee \rangle = 2\delta_{i,j}$ by orthogonality of the roots α_i . Moreover, h is part of some \mathfrak{sl}_2 -triple (x, h, y) , so $\langle \mu_0, h \rangle = -\langle \varpi, h \rangle$, by the representation theory of \mathfrak{sl}_2 . This proves that $\langle \varpi, h \rangle = r$. \square

Remark 2.18. The four points of the proposition fail in most cases if we perform the algorithm starting with a weight which is not cominuscule.

2.3. The submodule associated with a nilpotent element.

We explain now that an element $y \in \mathfrak{u}_-$ defines a graded subspace \mathbb{V} in \mathbb{E} . From now on we assume that $y = y_{\alpha_1} + \cdots + y_{\alpha_r}$, where $(\alpha_1, \dots, \alpha_r)$ is the dominant orthogonal sequence obtained by the algorithm of Definition 2.14 and we denote by h the element $\alpha_1^\vee + \cdots + \alpha_r^\vee$ explicited in Table 1.

We begin with the following consequence of Proposition 2.11:

Corollary 2.19. *For $\chi \in X(\mathbb{E}_i)$, we have $\langle \chi, h \rangle \geq r - 2i$.*

Proof. This follows from Proposition 2.11 and the fact that for any root α , we have $\langle \alpha, h \rangle \leq 2$ (Proposition 2.17). \square

We define the subspace $\mathbb{V} \subset \mathbb{E}$ associated to y by the equality condition in the inequality of this last corollary:

Definition 2.20. Let $\mathbb{V} \subset \mathbb{E}$ be defined by $\mathbb{V} = \bigoplus \mathbb{V}_i$ with

$$\mathbb{V}_i = \bigoplus_{\chi \in X(\mathbb{E}_i) : \langle \chi, h \rangle = r - 2i} \mathbb{E}_\chi.$$

Observe that the subspaces \mathbb{V}_i are non trivial exactly when $i \in \{0, \dots, r\}$, since by Proposition 2.17, we have $\langle \chi, h \rangle \in \{-r, -r+1, \dots, r\}$ for $\chi \in X(\mathbb{E})$.

The elements y and h fit in a \mathfrak{sl}_2 -triple (x, h, y) for some $x \in \mathfrak{u}_+$. Define a descending filtration $(\mathbb{F}_k)_{r \geq k \geq -r}$ of \mathbb{E} by

$$\mathbb{F}_k := \bigoplus_{\chi \in X(\mathbb{E}) : \langle \chi, h \rangle \leq k} \mathbb{E}_\chi.$$

The centralizer of y stabilizes each subspace \mathbb{F}_k , and since any two \mathfrak{sl}_2 -triples are congruent under this centralizer [McG02, Theorem 3.8], this filtration only depends on y , and not on h : see also the argument given for the canonical parabolic subalgebra at the end of [McG02, Paragraph 3.2]. Using the representation theory of \mathfrak{sl}_2 , we can see that \mathbb{F}_k depends only on y , since it can be described as follows:

$$(2) \quad \mathbb{F}_k = \sum_{\substack{\ell \geq 0 \\ k + \ell + 1 \geq 0}} \text{Ker } y^{k+\ell+1} \cap \text{Im } y^\ell$$

Moreover, for all $k \geq 0$,

$$y^k : \mathbb{F}_k / \mathbb{F}_{k-1} \xrightarrow{\sim} \mathbb{F}_{-k} / \mathbb{F}_{-k-1}$$

(where $\mathbb{F}_{-r-1} := \{0\}$).

It is plain from Corollary 2.19 that the \mathbb{V}_i 's can alternatively be defined by $\mathbb{V}_i = \mathbb{E}_i \cap \mathbb{F}_{r-2i}$, for all $i = 0, 1, \dots, r$.

This implies the following equality on dimensions:

Lemma 2.21. *We have $\dim \mathbb{V}_{r-i} = \dim \mathbb{V}_i$.*

Proof. We know that $\mathbb{V}_i = \mathbb{E}_i \cap \mathbb{F}_{r-2i}$, $\mathbb{V}_{r-i} = \mathbb{E}_{r-i} \cap \mathbb{F}_{2i-r}$, and that for $0 \leq i \leq r/2$, y^{r-2i} is an isomorphism between $\mathbb{F}_{r-2i} / \mathbb{F}_{r-2i-1}$ and $\mathbb{F}_{2i-r} / \mathbb{F}_{2i-r-1}$. By Proposition 2.11, y^{r-2i} maps \mathbb{E}_i to \mathbb{E}_{r-i} . Since $\mathbb{E}_k \cap \mathbb{F}_{r-2k-1} = \{0\}$ for all k by Corollary 2.19, we get that y^{r-2i} is an isomorphism between \mathbb{V}_i and \mathbb{V}_{r-i} . \square

The following algebraic fact is at the heart of the construction of Higgs subsheaves in Section 4:

Proposition 2.22. *The following inclusions hold:*

- $y \cdot \mathbb{V}_i \subset \mathbb{V}_{i+1}$.
- $\mathfrak{u}_+ \cdot \mathbb{V}_i \subset \mathbb{V}_{i-1}$.

Proof. The first statement holds because $y \in \bigoplus_i \mathfrak{g}_{-\alpha_i}$ and each α_i satisfies $\langle -\alpha_i, z \rangle = \langle -\alpha_i, h \rangle = -2$. The second statement holds because for α a root of \mathfrak{u}_+ , we have $\langle \alpha, z \rangle = 2$ (and $\langle \alpha, h \rangle \leq 2$ by Proposition 2.17). \square

Definition 2.23. Let $\mathfrak{q} \subset \mathfrak{g}$ be the parabolic subalgebra defined by the coweight $z - h$:

$$\mathfrak{q} = \mathfrak{t} \oplus \bigoplus_{\alpha : \langle \alpha, h \rangle \leq \langle \alpha, z \rangle} \mathfrak{g}_\alpha.$$

A Levi factor of \mathfrak{q} is

$$\mathfrak{l} = \mathfrak{t} \oplus \bigoplus_{\alpha : \langle \alpha, h \rangle = \langle \alpha, z \rangle} \mathfrak{g}_\alpha.$$

Let $\mathfrak{l}_\pm \subset \mathfrak{l}$ be the nilpotent subalgebras of \mathfrak{l} defined by

$$\mathfrak{l}_\pm = \bigoplus_{\alpha : \langle \alpha, h \rangle = \langle \alpha, z \rangle = \pm 2} \mathfrak{g}_\alpha.$$

A Levi factor of $\mathfrak{k} \cap \mathfrak{q}$ is

$$\mathfrak{h} = \mathfrak{t} \oplus \bigoplus_{\alpha : \langle \alpha, h \rangle = \langle \alpha, z \rangle = 0} \mathfrak{g}_\alpha.$$

We denote by Q , resp. H , the subgroups of G with Lie algebra \mathfrak{q} , resp. \mathfrak{h} . We observe that H is a Levi subgroup of $K \cap Q$: in fact, we have $K \cap Q = H \cdot R(K \cap Q)$, where $R(K \cap Q)$ denotes the radical of $K \cap Q$.

For the convenience of the reader, the conditions that define the different Lie subalgebras of \mathfrak{g} we are considering are displayed in Table 2 (we abbreviate the condition $\langle \alpha, z \rangle = 0$, resp. $\langle \alpha, h \rangle = 0$, on a root α as $z = 0$, resp. $h = 0$):

Lie algebra	\mathfrak{k}	\mathfrak{u}_\pm	\mathfrak{q}	\mathfrak{l}	\mathfrak{l}_\pm	\mathfrak{h}
Condition	$z = 0$	$z = \pm 2$	$h \leq z$	$h = z$	$h = z = \pm 2$	$h = z = 0$

TABLE 2. Lie subalgebras under consideration

We have the following lemmas concerning the \mathbb{V}_i 's:

Lemma 2.24. *We have $\mathbb{V}_{i+1} = \mathfrak{l}_- \cdot \mathbb{V}_i$ and $\mathbb{V}_{i-1} = \mathfrak{l}_+ \cdot \mathbb{V}_i$.*

Proof. Let us denote by $X(\mathbb{E}_i)$ the set of weights of \mathbb{E}_i . We know by Proposition 2.11 that $\mathbb{E}_{i+1} = \mathfrak{u}_- \mathbb{E}_i$. Thus,

$$\mathbb{E}_{i+1} = \left(\bigoplus_{\alpha: \langle \alpha, z \rangle = -2} \mathfrak{g}_\alpha \right) \cdot \left(\bigoplus_{\chi \in X(\mathbb{E}_i)} \mathbb{E}_\chi \right) = \bigoplus_{\chi \in X(\mathbb{E}_i), \alpha \in \Phi(\mathfrak{u}_-)} \mathbb{E}_{\chi + \alpha}.$$

Given $\chi \in X(\mathbb{E}_i)$ and $\alpha \in \Phi(\mathfrak{u}_-)$, we can make two observations:

- If $\langle \chi, h \rangle > r - 2i$ then we have $\langle \chi + \alpha, h \rangle > r - 2(i + 1)$ and thus $\mathbb{E}_{\chi + \alpha} \not\subset \mathbb{V}_{i+1}$.
- If $\langle \alpha, h \rangle > -2$ then $\langle \chi + \alpha, h \rangle > r - 2(i + 1)$.

The first equality of the lemma follows from these observations. The proof of the second equality is similar. \square

Note that \mathfrak{l}_- is an \mathfrak{h} -module. More precisely, we have:

Lemma 2.25. *The modules \mathbb{V}_i are irreducible \mathfrak{h} -modules. The Lie algebra \mathfrak{l}_- is an irreducible \mathfrak{h} -module.*

Proof. Denote by \mathfrak{k}_+ resp. \mathfrak{h}_+ the sum of the root spaces for positive roots in \mathfrak{k} resp. \mathfrak{h} . Let i be fixed and such that $\mathbb{V}_i \neq \{0\}$. By Proposition 2.11, \mathbb{E}_i is an irreducible \mathfrak{k} -module. Let μ_i be the lowest weight of \mathbb{E}_i . We have $\mathbb{E}_{\mu_i} \subset \mathbb{V}_i$. Since \mathbb{E}_i is irreducible, we have $\mathbb{E}_i = U(\mathfrak{k}_+) \cdot \mathbb{E}_{\mu_i}$ (here U denotes the universal enveloping algebra). Now, as in the proof of the Lemma 2.24, we see that this implies that $\mathbb{V}_i = U(\mathfrak{h}_+) \cdot \mathbb{E}_{\mu_i}$. This proves that \mathbb{V}_i is irreducible. Now, by Lemma 2.24 again, we have $\mathbb{V}_1 = \mathfrak{l}_- \cdot \mathbb{V}_0 \simeq \mathfrak{l}_- \cdot \mathbb{E}_0$: thus \mathfrak{l}_- is also irreducible. \square

This allows to compute the *slope* of the H -modules \mathbb{V}_i . We first need two definitions and a lemma.

Definition 2.26. Let \mathbb{W} be a H -module (here H can be any reductive group). The *slope* of \mathbb{W} is the element $\mu(\mathbb{W}) = \frac{\det(\mathbb{W})}{\dim(\mathbb{W})}$ in $X(H) \otimes_{\mathbb{Z}} \mathbb{Q}$, where $X(H)$ is the character group of H . We say that \mathbb{W} is *equislope* if, for the decomposition $\mathbb{W} = \bigoplus_i \mathbb{W}_i$ as a sum of irreducible submodules, we have $\forall i, j, \mu(\mathbb{W}_i) = \mu(\mathbb{W}_j)$.

Lemma 2.27. *Let \mathbb{W}, \mathbb{W}' be equislope H -modules (here again, H can be any reductive group). Then $\mathbb{W} \otimes \mathbb{W}'$ is equislope.*

Proof. Let Z be the center of H . It is known that restriction to Z yields an injection $X(H) \hookrightarrow X(Z)$. In fact, by [Bor69, Proposition 14.2], we have $H = Z \cdot (H, H)$, and any character of H is trivial on (H, H) .

Let us first assume that \mathbb{W} and \mathbb{W}' are irreducible H -modules. By Schur's lemma, there are characters χ, ψ of Z such that $\forall g \in Z, \forall w \in \mathbb{W}, \forall w' \in \mathbb{W}', g \cdot w = \chi(g)w$ and $g \cdot w' = \psi(g)w'$. Therefore, in $X(Z) \otimes \mathbb{Q}$, we have $\chi = \mu(\mathbb{W})|_Z$ and $\psi = \mu(\mathbb{W}')|_Z$. Let now $\mathbb{U} \subset \mathbb{W} \otimes \mathbb{W}'$ be an irreducible component. We have $\forall u \in \mathbb{U}, \forall g \in Z, g \cdot u = \chi(g)\psi(g)u$. Therefore $\mu(\mathbb{U})|_Z = \chi + \psi$. Thus, $(\mu(\mathbb{W}) + \mu(\mathbb{W}'))|_Z = \mu(\mathbb{U})|_Z$. Since restriction of characters to Z is injective, we have $\mu(\mathbb{U}) = \mu(\mathbb{W}) + \mu(\mathbb{W}')$.

Let now \mathbb{W} and \mathbb{W}' be arbitrary H -modules, and write the decomposition into irreducible submodules: $\mathbb{W} = \bigoplus \mathbb{W}_i$ and $\mathbb{W}' = \bigoplus \mathbb{W}'_j$. Let i, j be fixed and let $\mathbb{U} \subset \mathbb{W}_i \otimes \mathbb{W}'_j$ be an irreducible factor. We have proved that $\mu(\mathbb{U}) = \mu(\mathbb{W}_i) + \mu(\mathbb{W}'_j)$. Since \mathbb{W} and \mathbb{W}' are equislope, we have $\mu(\mathbb{W}_i) = \mu(\mathbb{W})$ and $\mu(\mathbb{W}'_j) = \mu(\mathbb{W}')$. Thus, \mathbb{U} has slope $\mu(\mathbb{W}) + \mu(\mathbb{W}')$ and the lemma is proved. \square

Remark 2.28. Let Γ be the Schur functor associated with a partition λ of weight $|\lambda|$. Let \mathbb{W} be an H -module. Since $\Gamma(\mathbb{W})$ is a direct factor of $\mathbb{W}^{\otimes |\lambda|}$, it follows from the Proposition that $\mu(\Gamma(\mathbb{W})) = |\lambda|\mu(\mathbb{W})$.

We apply this lemma to the H -modules \mathbb{V}_i :

Proposition 2.29. *We have $\mu(\mathbb{V}_i) = \mu(\mathbb{V}_0) + i\mu(\mathbb{V}_0^\vee \otimes \mathbb{V}_1)$.*

Proof. The H -modules \mathbb{V}_i are irreducible by Lemma 2.25, thus equislope. The same holds for the H -module $\mathbb{L}_- \simeq \mathbb{V}_0^\vee \otimes \mathbb{V}_1$, and we have $\mu(\mathbb{L}_-) = \mu(\mathbb{V}_0^\vee \otimes \mathbb{V}_1)$. Let i be fixed. By Lemma 2.27, $\mathbb{V}_i \otimes \mathbb{L}_-$ is equislope. Since, by Lemma 2.24, \mathbb{V}_{i+1} is a direct factor of $\mathbb{V}_i \otimes \mathbb{L}_-$, it follows that $\mu(\mathbb{V}_{i+1}) = \mu(\mathbb{V}_i) + \mu(\mathbb{L}_-)$. \square

Concerning the submodule \mathbb{V} , we have

Proposition 2.30. *(1) The subspace $\mathbb{V} \subset \mathbb{E}$ is stable under \mathfrak{q} .*

(2) \mathbb{V}_i is a $K \cap Q$ -module.

(3) The nilpotent radical of \mathfrak{q} acts trivially on \mathbb{V} .

(4) The subgroup of elements in G which preserve \mathbb{V} is exactly Q .

Proof. Let us prove the first point. The subspace \mathbb{V} is clearly stable under \mathfrak{t} . Let α be such that $\langle \alpha, h \rangle \leq \langle \alpha, z \rangle$. Let $v \in \mathbb{V}_\chi \subset \mathbb{V}_i$, thus we have $\langle \chi, h \rangle = r - 2i$. For $x \in \mathfrak{g}_\alpha$, we have $x \cdot v \in \mathbb{E}_{\chi+\alpha} \subset \mathbb{E}_{i-\langle \alpha, z \rangle/2}$. Since $\langle \alpha, h \rangle \leq \langle \alpha, z \rangle$, we have $\langle \chi + \alpha, h \rangle \leq r - 2i + \langle \alpha, z \rangle$, thus either $x \cdot v = 0$ or $\langle \chi + \alpha, h \rangle = r - 2i + \langle \alpha, z \rangle$, by Corollary 2.19. In the second case, we get $x \cdot v \in \mathbb{V}_{i-\langle \alpha, z \rangle/2}$. The fact that \mathbb{V}_i is a $K \cap Q$ -module follows because $\mathbb{V}_i = \mathbb{E}_i \cap \mathbb{V}$, \mathbb{E}_i is K -stable, and \mathbb{V} is Q -stable.

For the third point, let α be a root of the nilpotent radical of \mathfrak{q} : we have $\langle \alpha, h \rangle < \langle \alpha, z \rangle$. Let $v \in \mathbb{V}_\chi \subset \mathbb{V}_i$, thus we have $\langle \chi, h \rangle = r - 2i$. For $x \in \mathfrak{g}_\alpha$, we have $x \cdot v \in \mathbb{E}_{\chi+\alpha} \subset \mathbb{E}_{i-\langle \alpha, z \rangle/2}$. However, we get $\langle \chi + \alpha, h \rangle > r - 2i + \langle \alpha, z \rangle$. Thus, $x \cdot v = 0$.

Finally, to prove that the stabilizer of \mathbb{V} is exactly Q , let us denote by $\mathbf{stab}(\mathbb{V}) \subset \mathfrak{g}$ the Lie subalgebra preserving the subspace \mathbb{V} in \mathbb{E} . We know by (1) that $\mathbf{stab}(\mathbb{V}) \supset \mathfrak{q}$. On the other hand, let α be a root and let $0 \neq x \in \mathfrak{g}_\alpha$ be such that $x \cdot \mathbb{V} \subset \mathbb{V}$. Let us assume as a first case that $\langle \alpha, z \rangle = -2$. Since, by Proposition 2.11(e), the action of \mathfrak{g} on \mathbb{E} induces an isomorphism $\mathbb{E}_1 \simeq \mathbb{E}_0 \otimes \mathfrak{u}_-$, we have $x \cdot \mathbb{E}_\varpi = \mathbb{E}_{\varpi+\alpha}$. Then $\mathbb{E}_{\varpi+\alpha} \subset \mathbb{V}$, so $\langle \alpha, h \rangle = -2$, and $x \in \mathfrak{q}$. Assume now that $\langle \alpha, z \rangle = 0$, and by contradiction that $\langle \alpha, h \rangle > 0$. Proposition 2.17 then implies that $\langle \alpha, h \rangle = 1$. For any integer i , we cannot have $\langle \alpha, \alpha_i^\vee \rangle = 2$ because this would imply $\alpha = \alpha_i$ and $\langle \alpha, h \rangle = 2$. Let i be such that $\langle \alpha, \alpha_i^\vee \rangle > 0$: then $\langle \alpha, \alpha_i^\vee \rangle = 1$. Therefore, $\alpha - \alpha_i$ is a root. Since $\langle \alpha, z \rangle = 0$,

$\mathfrak{g}_\alpha \cdot \mathbb{E}_0 \subset \mathbb{E}_0 \cap \mathbb{E}_{\varpi+\alpha}$, thus $\mathfrak{g}_\alpha \cdot \mathbb{E}_0 = \{0\}$. It follows that

$$\mathfrak{g}_{\alpha-\alpha_i} \cdot \mathbb{E}_\varpi = [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha_i}] \cdot \mathbb{E}_\varpi = \mathfrak{g}_\alpha \cdot (\mathfrak{g}_{-\alpha_i} \cdot \mathbb{E}_\varpi).$$

Since $\mathfrak{g}_{\alpha-\alpha_i} \cdot \mathbb{E}_\varpi = \mathbb{E}_{\varpi+\alpha-\alpha_i}$ (again by Proposition 2.11(e)), this contradicts $x \cdot \mathbb{V} \subset \mathbb{V}$.

Let now $\text{Stab}(\mathbb{V}) \subset G$ be the subgroup stabilizing \mathbb{V} . We have $Q \subset \text{Stab}(\mathbb{V})$, thus $\text{Stab}(\mathbb{V})$ is parabolic and therefore connected [Hum75, Corollary 23.1.B]. It follows that $Q = \text{Stab}(\mathbb{V})$. \square

Moreover:

Proposition 2.31. *The \mathfrak{l} -module \mathbb{V} is irreducible and, as an \mathfrak{l} -module, it has tube type.*

Proof. Combining Lemmas 2.24 and 2.25, we get the irreducibility of \mathbb{V} . Note that, by definition, any root α of \mathfrak{l} satisfies $\langle \alpha, z \rangle = \langle \alpha, h \rangle$. Now, a dominant orthogonal sequence for the weight ϖ as a weight of \mathfrak{g} is also a dominant orthogonal sequence for ϖ as a weight of \mathfrak{l} . It follows that \mathbb{V} satisfies the assumption of Remark 2.16, and therefore \mathbb{V} has tube type. This means that if L is the subgroup of G with Lie algebra \mathfrak{l} and if we set $L_{\mathbb{R}} = L \cap G_{\mathbb{R}}$ and $H_{\mathbb{R}} = H \cap G_{\mathbb{R}} = L_{\mathbb{R}} \cap K_{\mathbb{R}}$, then the Hermitian symmetric space $L_{\mathbb{R}}/H_{\mathbb{R}}$ has tube type. \square

Example 2.32. We give an example of this construction. We assume that we are in the first arrow of the array (1), namely that G has type A_{p+q-1} for some positive integers $p \leq q$ and that $\varpi = \varpi_{p-1}$.

Let $y \in \mathfrak{u}_-$ be an element of rank r . We have a natural decomposition $\mathbb{C}^{p+q} = N \oplus A \oplus I \oplus B$, with $\ker y = N \oplus I \oplus B$ and $\text{Im } y = I$. We choose a basis (e_i) such that N , resp. A , I , B is generated by (e_1, \dots, e_{p-r}) , resp. (e_{p-r+1}, \dots, e_p) , $(e_{p+1}, \dots, e_{p+r})$, $(e_{p+r+1}, \dots, e_{p+q})$.

The element $h \in \mathfrak{t}$ acts on the Lie algebra of G , and the corresponding weights of block matrices from $N \oplus A \oplus I \oplus B$ to itself are $\begin{pmatrix} 0 & -1 & 1 & 0 \\ 1 & 0 & 2 & 1 \\ -1 & -2 & 0 & -1 \\ 0 & -1 & 1 & 0 \end{pmatrix}$. Beware that

with our choice of base (e_i) , the positive roots do *not* correspond to matrix coordinates above the diagonal.

The weights for the central element z are $\begin{pmatrix} 0 & 0 & 2 & 2 \\ 0 & 0 & 2 & 2 \\ -2 & -2 & 0 & 0 \\ -2 & -2 & 0 & 0 \end{pmatrix}$. Thus the Lie algebras

\mathfrak{q} , \mathfrak{l} and \mathfrak{h} are respectively

$$\begin{pmatrix} * & * & * & * \\ 0 & * & * & * \\ 0 & * & * & * \\ 0 & 0 & 0 & * \end{pmatrix}, \quad \begin{pmatrix} * & 0 & 0 & 0 \\ 0 & * & * & 0 \\ 0 & * & * & 0 \\ 0 & 0 & 0 & * \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} * & 0 & 0 & 0 \\ 0 & * & 0 & 0 \\ 0 & 0 & * & 0 \\ 0 & 0 & 0 & * \end{pmatrix}.$$

Thus we see that Q is exactly the stabilizer of the flag $(N \subset N \oplus A \oplus I)$, that the subgroup of G corresponding to \mathfrak{l} is $S(\text{GL}(N) \times \text{GL}(A \oplus I) \times \text{GL}(B))$ and H is the block diagonal group $S(\text{GL}(N) \times \text{GL}(A) \times \text{GL}(I) \times \text{GL}(B))$, and finally that the intersections of $G_{\mathbb{R}} = \text{SU}(p, q)$ with these two latter groups are isomorphic to $S(\text{U}(p-r) \times \text{U}(r, r) \times \text{U}(q-r))$ and $S(\text{U}(p-r) \times \text{U}(r) \times \text{U}(r) \times \text{U}(q-r))$ respectively.

On the other hand, we have $\mathbb{E} = \wedge^p(N \oplus A \oplus I \oplus B)$ and it is easy to check that $\mathbb{V} = \wedge^{p-r} N \otimes \wedge^r(A \oplus I)$, which confirms that the stabilizer of \mathbb{V} is Q .

Remark 2.33. Given an element y' in \mathfrak{u}_+ instead of \mathfrak{u}_- , one can consider the dual representation \mathbb{E}^\vee of G , and construct as above a graded subspace $\mathbb{V}' = \bigoplus_{i=0}^{\text{rk } y'} \mathbb{V}'_i$ of $\mathbb{E}^\vee = \bigoplus_{i=0}^p \mathbb{E}^\vee_i$ (with $\mathbb{V}'_i \subset \mathbb{E}^\vee_i$). It has the same properties as the subspace $\mathbb{V} \subset \mathbb{E}$ discussed above with the obvious modifications, e.g. the statement of Proposition 2.22 for \mathbb{V}' is that $y' \cdot \mathbb{V}'_i \subset \mathbb{V}'_{i+1}$ and $\mathfrak{u}_- \cdot \mathbb{V}'_i \subset \mathbb{V}'_{i-1}$.

3. HIGGS BUNDLES ON FOLIATED KÄHLER MANIFOLDS

3.1. Harmonic Higgs bundles.

We keep the notation of the previous sections. In particular, $G_{\mathbb{R}}$ is a simple non-compact Hermitian Lie group, $K_{\mathbb{R}}$ its maximal compact subgroup, $M = G_{\mathbb{R}}/K_{\mathbb{R}}$ the associated irreducible Hermitian symmetric space of the noncompact type, G and K are the complexifications of $G_{\mathbb{R}}$ and $K_{\mathbb{R}}$, and $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{u}$ and $\mathfrak{g}_{\mathbb{R}} = \mathfrak{k}_{\mathbb{R}} \oplus \mathfrak{u}_{\mathbb{R}}$ are the associated Cartan decompositions of the Lie algebras of G and $G_{\mathbb{R}}$.

Let now Y be a compact Kähler manifold, $\Gamma = \pi_1(Y)$ its fundamental group, and $\rho : \Gamma \rightarrow G_{\mathbb{R}}$ be a reductive representation (group homomorphism) of Γ into $G_{\mathbb{R}}$.

In this case, by [Cor88], there exists a ρ -equivariant *harmonic* map f from the universal cover \tilde{Y} of Y to the symmetric space $M = G_{\mathbb{R}}/K_{\mathbb{R}}$ associated to $G_{\mathbb{R}}$. The fact that Y is Kähler and the nonpositivity of the complexified sectional curvature of M imply by a Bochner formula due to [Sam78, Siu80] that the map f is *pluriharmonic* (i.e. its restriction to 1-dimensional complex submanifolds of Y is still harmonic), and that the image of its (complexified) differential at every point $y \in Y$ is an Abelian subalgebra of $T_{f(y)}^{\mathbb{C}} M$ identified with \mathfrak{u} .

By the work of C. Simpson, this gives a *harmonic $G_{\mathbb{R}}$ -Higgs principal bundle* (P_K, θ) on Y . We will now briefly describe the construction and the properties of such a Higgs bundle. Details and proofs can be found in the original papers [Sim88, Sim92].

There is a flat principal bundle $P_{G_{\mathbb{R}}} \rightarrow Y$ of group $G_{\mathbb{R}}$ associated to the representation ρ . The ρ -equivariant map $f : \tilde{Y} \rightarrow G_{\mathbb{R}}/K_{\mathbb{R}}$ defines a reduction of its structure group to $K_{\mathbb{R}}$, i.e. a principal $K_{\mathbb{R}}$ bundle $P_{K_{\mathbb{R}}} \subset P_{G_{\mathbb{R}}}$. The differential of f can be seen as a 1-form with values in $P_{K_{\mathbb{R}}}(\mathfrak{u}_{\mathbb{R}})$, the vector bundle associated to the adjoint action of $K_{\mathbb{R}}$ on $\mathfrak{u}_{\mathbb{R}}$.

If we enlarge the structure group of $P_{K_{\mathbb{R}}}$ to K and complexify the whole situation, the pluriharmonicity of f implies that the K -principal bundle $P_K \rightarrow Y$ is a holomorphic bundle. Moreover, the (complexified) differential $d^{1,0}f : T^{1,0}Y \rightarrow T^{\mathbb{C}}M$ of the harmonic map f defines a holomorphic section θ of $P_K(\mathfrak{u}) \otimes \Omega_Y^1$, where $P_K(\mathfrak{u})$ is the holomorphic vector bundle associated to the principal bundle P_K and the adjoint representation of K on \mathfrak{u} . The section θ is called the *Higgs field* and satisfies the integrability condition $[\theta, \theta] = 0$ as a section of $P_K(\mathfrak{u}) \otimes \Omega_Y^2$. The couple (P_K, θ) is called a *$G_{\mathbb{R}}$ -Higgs principal bundle* on Y .

If now \mathbb{E} is a (complex) representation of G we can construct the associated holomorphic vector bundle $E := P_K(\mathbb{E})$ over Y . Since \mathbb{E} is a representation of G and not only of K , the Higgs field θ can be seen as a holomorphic 1-form with values in the endomorphisms of E , i.e. a section of $\text{End}(E) \otimes \Omega_Y^1$. The couple (E, θ) is called a *$G_{\mathbb{R}}$ -Higgs vector bundle* on Y . The harmonic map f , seen as a reduction of the structure group of P_K to the compact subgroup $K_{\mathbb{R}}$, together with a $K_{\mathbb{R}}$ -invariant metric on \mathbb{E} , gives a Hermitian metric on (E, θ) called the *harmonic metric*.

The existence of this harmonic metric and the fact that P_K comes from a flat principal $G_{\mathbb{R}}$ bundle imply that for any representation \mathbb{E} of G , the associated Higgs vector bundle $(E, \theta) \rightarrow Y$ is *Higgs polystable* of degree 0, see [Sim88]. To explain what Higgs

polystability means, we first define *Higgs subsheaves* of the Higgs bundle (E, θ) . A coherent subsheaf \mathcal{F} of $\mathcal{O}_Y(E)$ is a Higgs subsheaf if it is invariant by the Higgs field, i.e. it satisfies $\theta(\mathcal{F}) \subset \mathcal{F} \otimes \Omega_Y^1$. The Higgs vector bundle (E, θ) is said to be Higgs stable if for any Higgs subsheaf \mathcal{F} of (E, θ) such that $0 < \text{rk} \mathcal{F} < \text{rk} E$, we have $\mu(\mathcal{F}) < \mu(E)$, where $\mu(\mathcal{F})$ is the slope of \mathcal{F} , i.e. its degree (computed w.r.t. the Kähler form of Y) divided by its rank. The Higgs bundle (E, θ) is Higgs polystable if it is a direct sum of Higgs stable Higgs vector bundles of the same slope. Note that here E is flat as a C^∞ bundle, so that its degree is zero.

Remark 3.1. Since moreover we assumed that $G_{\mathbb{R}}$ is a Hermitian group, then as a K -representation we have $\mathfrak{u} = \mathfrak{u}_+ \oplus \mathfrak{u}_-$ and the Higgs field θ on the principal bundle P_K (or on any associated vector bundle E) has two components $\beta \in P_K(\mathfrak{u}_-) \otimes \Omega_Y^1$ and $\gamma \in P_K(\mathfrak{u}_+) \otimes \Omega_Y^1$. The vanishing of one of these components means that the harmonic map f is holomorphic or antiholomorphic.

3.2. Harmonic Higgs bundles on foliated Kähler manifolds.

Assume now that the base Kähler manifold Y of the harmonic $G_{\mathbb{R}}$ -Higgs vector bundle $(E, \theta) \rightarrow Y$ of degree 0 admits a smooth holomorphic foliation by complex curves and that this foliation \mathcal{T} admits an invariant transverse measure. Our goal in this section is to explain the interplay between the Higgs bundle and the foliation. Details can be found in [KM17, §2.2].

We first weaken the notion of Higgs subsheaves of (E, θ) to *leafwise Higgs subsheaves* by asking only an invariance by the Higgs field along the leaves. More precisely we now consider the Higgs field as a section of $\text{End}(E) \otimes L^\vee$, where L^\vee is the dual of the holomorphic line subbundle L of T_Y tangent of the foliation \mathcal{T} . A leafwise Higgs subsheaf \mathcal{F} of (E, θ) is then a subsheaf of E such that $\theta(\mathcal{F} \otimes L) \subset \mathcal{F}$.

The invariant transverse measure gives a current of integration along the leaves of \mathcal{T} . This allows to define the *foliated degree* $\deg_{\mathcal{T}} \mathcal{F}$ of a coherent sheaf \mathcal{F} on Y by applying this current to the first Chern class of \mathcal{F} .

The Higgs bundle enjoys the following leafwise polystability property w.r.t. the foliated degree ([KM17, Proposition 2.2]):

Proposition 3.2. *Assume that the invariant transverse measure comes from an invariant transverse volume form. Then the Higgs bundle (E, θ) on the foliated Kähler manifold Y is weakly polystable along the leaves in the following sense:*

- (1) *it is semistable along the leaves of \mathcal{T} : if \mathcal{F} is a leafwise Higgs subsheaf of (E, θ) , then $\deg_{\mathcal{T}} \mathcal{F} \leq 0$.*
- (2) *if \mathcal{F} is a saturated leafwise Higgs subsheaf of (E, θ) such that $\deg_{\mathcal{T}} \mathcal{F} = 0$, then the singular locus $\mathcal{S}(\mathcal{F})$ of \mathcal{F} is saturated under the foliation \mathcal{T} . Moreover, on $Y \setminus \mathcal{S}(\mathcal{F})$, and if F denotes the holomorphic subbundle of E such that $\mathcal{F} = \mathcal{O}_{Y \setminus \mathcal{S}(\mathcal{F})}(F)$ and F^\perp its C^∞ orthogonal complement w.r.t. the harmonic metric, then $\theta(F^\perp \otimes L) \subset F^\perp$ and for each leaf \mathcal{L} of \mathcal{T} such that $\mathcal{L} \cap \mathcal{S}(\mathcal{F}) = \emptyset$, F^\perp is holomorphic on \mathcal{L} and $(E, \theta) = (F, \theta|_F) \oplus (F^\perp, \theta|_{F^\perp})$ is an holomorphic orthogonal decomposition on \mathcal{L} .*

(In the proposition the *singular locus* $\mathcal{S}(\mathcal{F})$ of a subsheaf \mathcal{F} of $\mathcal{O}_Y(E)$ is the complement of the subset of Y where \mathcal{F} is the sheaf of sections of a subbundle F of E . Equivalently, it is the subset of Y where $\mathcal{O}_Y(E)/\mathcal{F}$ is not locally free. If \mathcal{F} is saturated $\mathcal{S}(\mathcal{F})$ has codimension at least 2 in Y .)

3.3. The tautological foliation on the projectivized tangent bundle of a complex hyperbolic manifold.

A n -dimensional complex hyperbolic manifold X is a quotient of the complex hyperbolic n -space $\mathbb{H}_{\mathbb{C}}^n = \mathrm{SU}(1, n)/\mathrm{U}(n)$ by a discrete torsion free subgroup Γ of $\mathrm{SU}(1, n)$. It is a Hermitian locally symmetric space of rank 1.

The complex hyperbolic space $\mathbb{H}_{\mathbb{C}}^n$ is an open subset in the projective space \mathbb{CP}^n : it's the subset of negative lines in \mathbb{C}^{n+1} for an Hermitian form of signature $(n, 1)$. Intersections of lines of \mathbb{CP}^n with $\mathbb{H}_{\mathbb{C}}^n$ are totally geodesic complex subspaces of $\mathbb{H}_{\mathbb{C}}^n$ isometric to the Poincaré disc. They are called *complex geodesics*. The space \mathcal{G} of complex geodesics is the complex homogeneous space $\mathrm{SU}(1, n)/\mathrm{S}(\mathrm{U}(1, 1) \times \mathrm{U}(n - 1))$.

The *projectivized tangent bundle* of $\mathbb{H}_{\mathbb{C}}^n$ is the complex homogeneous space $\mathbb{PT}_{\mathbb{H}_{\mathbb{C}}^n} = \mathrm{SU}(1, n)/\mathrm{S}(\mathrm{U}(1) \times \mathrm{U}(1) \times \mathrm{U}(n - 1))$. The natural $\mathrm{SU}(1, n)$ -equivariant fibration $\pi_{\mathcal{G}} : \mathbb{PT}_{\mathbb{H}_{\mathbb{C}}^n} \rightarrow \mathcal{G}$ which associates to a tangent line to $\mathbb{H}_{\mathbb{C}}^n$ the complex geodesic it defines is a disc bundle over \mathcal{G} .

By $\mathrm{SU}(1, n)$ -equivariance, this fibration endows the projectivized tangent bundle $\mathbb{PT}_X = \Gamma \backslash \mathbb{PT}_{\mathbb{H}_{\mathbb{C}}^n}$ of $X = \Gamma \backslash \mathbb{H}_{\mathbb{C}}^n$ with a smooth holomorphic foliation \mathcal{T} by complex curves, whose leaves are given by the tangent spaces of the (immersed) complex geodesics in X . This foliation is called the *tautological foliation* of \mathbb{PT}_X because the tangent line bundle L to the leaves is naturally isomorphic to the tautological line bundle $\mathcal{O}_{\mathbb{PT}_X}(-1)$ on \mathbb{PT}_X .

The space \mathcal{G} of complex geodesics of $\mathbb{H}_{\mathbb{C}}^n$ is a pseudo-Kähler manifold: it admits a non-degenerate but indefinite Kähler form $\omega_{\mathcal{G}}$. This form defines a transverse invariant volume form for the tautological foliation \mathcal{T} on \mathbb{PT}_X , and the associated notion of foliated degree $\deg_{\mathcal{T}}$ for sheaves on \mathbb{PT}_X has the following fundamental property [KM17, Proposition 3.1]:

Proposition 3.3. *Assume that $X = \Gamma \backslash \mathbb{H}_{\mathbb{C}}^n$ is compact and let $\pi : \mathbb{PT}_X \rightarrow X$ be the projectivized tangent bundle of X . If \mathcal{F} is a coherent \mathcal{O}_X -sheaf, then $\deg_{\mathcal{T}}(\pi^* \mathcal{F}) = \deg \mathcal{F}$, where $\deg \mathcal{F}$ is the usual degree of \mathcal{F} computed w.r.t. the Kähler form on X induced by the $\mathrm{SU}(1, n)$ -invariant Kähler form on $\mathbb{H}_{\mathbb{C}}^n$.*

4. THE MILNOR-WOOD INEQUALITY

Let $X = \Gamma \backslash \mathbb{H}_{\mathbb{C}}^n$ be a compact complex hyperbolic manifold of (complex) dimension n and ρ be a representation of Γ in a simple Hermitian Lie group $G_{\mathbb{R}}$, whose associated symmetric space is called M . In this section we use the material developped or recalled in Sections 2 and 3 to prove the Milnor-Wood inequality

$$|\tau(\rho)| \leq \mathrm{rk} M \mathrm{vol}(X).$$

Recall that we may and do assume that the representation ρ is reductive (see the discussion just before Assumption 1.2).

4.1. Setup.

Consider the representation \mathbb{E} of G defined in §2.1. As explained in §3.1, this gives rise to a flat harmonic $G_{\mathbb{R}}$ -Higgs vector bundle $(\bar{E}, \bar{\theta})$ over X . As a representation of K , $\mathbb{E} = \oplus_{i=0}^p \mathbb{E}_i$ where p is the real rank of $G_{\mathbb{R}}$. This means that the Higgs bundle \bar{E} admits the holomorphic decomposition $\bar{E} = \oplus_{i=0}^p \bar{E}_i$. Moreover, the components $\bar{\beta} \in P_K(\mathfrak{u}_-)$ and $\bar{\gamma} \in P_K(\mathfrak{u}_+)$ of the Higgs field $\bar{\theta} \in \mathrm{Hom}(\bar{E}, \bar{E}) \otimes \Omega_X^1$ (see Remark 3.1) satisfy

$$\bar{\beta} \in \bigoplus_{i=0}^{p-1} \mathrm{Hom}(\bar{E}_i, \bar{E}_{i+1}) \otimes \Omega_X^1 \text{ and } \bar{\gamma} \in \bigoplus_{i=0}^{p-1} \mathrm{Hom}(\bar{E}_{i+1}, \bar{E}_i) \otimes \Omega_X^1.$$

We pull-back the harmonic Higgs bundle $(\bar{E}, \bar{\theta}) \rightarrow X$ to the projectivized tangent bundle \mathbb{PT}_X of X to obtain a harmonic Higgs bundle that we denote by $(E, \theta) \rightarrow \mathbb{PT}_X$.

We restrict the Higgs field θ to the tangent space L of the tautological foliation on $\mathbb{P}T_X$, so that its components β and γ satisfy

$$\beta \in \bigoplus_{i=0}^{p-1} \text{Hom}(E_i, E_{i+1}) \otimes L^\vee \text{ and } \gamma \in \bigoplus_{i=0}^{p-1} \text{Hom}(E_{i+1}, E_i) \otimes L^\vee$$

Definition 4.1. For $\xi \in \mathbb{P}T_X$, the rank $\text{rk}\beta_\xi$ of β_ξ is the largest value of k such that $(\beta_\xi)^k : E_0 \otimes L^k \rightarrow E_k$ is not zero.

The generic rank $\text{rk}\beta$ of β is the maximum of the ranks of β_ξ for $\xi \in \mathbb{P}T_X$.

The singular locus of β is the following subset of $\mathbb{P}T_X$:

$$\mathcal{S}(\beta) := \{\xi \in \mathbb{P}T_X \mid \text{rk}\beta_\xi < \text{rk}\beta\} = \{\xi \in \mathbb{P}T_X \mid (\beta_\xi)^{\text{rk}\beta} : E_0 \otimes L^r \rightarrow E_r \text{ vanishes}\}$$

The regular locus of β is $\mathcal{R}(\beta) := \mathbb{P}T_X \setminus \mathcal{S}(\beta)$.

The singular locus $\mathcal{S}(\bar{\beta})$ of $\bar{\beta}$ is the projection to X of $\mathcal{S}(\beta)$.

The regular locus $\mathcal{R}(\bar{\beta})$ of $\bar{\beta}$ is $X \setminus \mathcal{S}(\bar{\beta})$.

Definition 4.2. We define similarly $\text{rk}\gamma$, $\mathcal{S}(\gamma)$, $\mathcal{R}(\gamma)$, $\mathcal{S}(\bar{\gamma})$ and $\mathcal{R}(\bar{\gamma})$, except that we consider the dual representation to define the rank of γ_ξ for $\xi \in \mathbb{P}T_X$: $\text{rk}\gamma_\xi$ is the largest value of k such that $({}^t\gamma_\xi)^k : E_0^\vee \otimes L^k \rightarrow E_k^\vee$ is not zero.

Observe that while $\mathcal{S}(\beta)$ and $\mathcal{S}(\gamma)$ are analytic subsets of $\mathbb{P}T_X$ of codimension at least 1, $\mathcal{S}(\bar{\beta})$ and $\mathcal{S}(\bar{\gamma})$ are analytic subsets of X (because $\pi : \mathbb{P}T_X \rightarrow X$ is a proper map) but they might be equal to X .

4.2. Rewording of the inequality.

Since the Hermitian symmetric space M associated to $G_{\mathbb{R}}$ is a Kähler-Einstein manifold, the first Chern form $c_1(T_M)$ of its tangent bundle is a constant multiple of the $G_{\mathbb{R}}$ -invariant Kähler form ω_M : $c_1(T_M) = -\frac{1}{4\pi} c_M \omega_M$ for some positive constant c_M . On the other hand, the line bundle \mathcal{L} associated to the K -representation \mathbb{E}_0 is a generator of the Picard group of the compact dual M^\vee of M and it can be checked that the canonical bundle K_{M^\vee} of M^\vee is precisely given by $\mathcal{L}^{\otimes c_M}$, see e.g. [KM10, Section 2]. Therefore the pull-back $f^*\omega_M$ is 4π times the first Chern form of the line bundle $f^*\mathcal{L} = \bar{E}_0$, so that the Toledo invariant of ρ is

$$(3) \quad \tau(\rho) = 4\pi \deg(\bar{E}_0) = 4\pi \deg \mathcal{T}(E_0),$$

where the last equality follows from Proposition 3.3. Similarly, we get that

$$(4) \quad \deg(K_X) = \frac{n+1}{4\pi} \text{vol}(X).$$

On the other hand, if L is the tangential line bundle to the tautological foliation \mathcal{T} on the projectivized tangent bundle $\mathbb{P}T_X$, and if L^\vee is the dual line bundle, one can compute as explained in [KM17, Section 4.3.1] that

$$(5) \quad \deg \mathcal{T}(L^\vee) = \frac{1}{2\pi} \text{vol}(X).$$

Therefore, the Milnor-Wood inequality can be rephrased as an inequality between the foliated degrees of the line bundles E_0 and L^\vee on $\mathbb{P}T_X$:

$$(6) \quad |\deg \mathcal{T}(E_0)| \leq \frac{p}{2} \deg \mathcal{T}(L^\vee),$$

where p is the rank of the symmetric space M .

4.3. Leafwise Higgs subsheaves associated to the components of the Higgs field.

We now define a subsheaf \mathcal{V} of $\mathcal{E} := \mathcal{O}(E)$ associated associated to β in the same way we defined the submodule \mathbb{V} of \mathbb{E} associated to the nilpotent element $y \in \mathfrak{u}_-$ in Definition 2.20 (for all $\xi \in L$, $\beta(\xi) \in P_K(\mathfrak{u}_-)$ is a nilpotent endomorphism of the bundle E). This subsheaf will be shown to be a leafwise Higgs subsheaf of the Higgs bundle (E, θ) on $\mathbb{P}T_X$. In Section 4.4 this will be used to prove the Milnor-Wood inequality.

More precisely, we follow the alternative definition of \mathbb{V} given after Definition 2.20 and for $k = r, r-1, \dots, -r+1, -r$, we consider the following subsheaves of \mathcal{E} :

$$\mathcal{F}_k := \sum_{\substack{\ell \geq 0 \\ k + \ell + 1 \geq 0}} \text{Ker } \beta^{k+\ell+1} \cap \text{Im } \beta^\ell$$

where in order to define $\text{Ker } \beta^j$ we see β as a sheaf morphism from \mathcal{E} to $\mathcal{E} \otimes (L^\vee)^j$ and to define $\text{Im } \beta^j$ we see β as a sheaf morphism from $\mathcal{E} \otimes L^j$ to \mathcal{E} .

For $k = 0, 1, \dots, r$, let \mathcal{V}_k be the saturation in $\mathcal{E}_k := \mathcal{O}(E_k)$ of the subsheaf $\mathcal{E}_k \cap \mathcal{F}_{r-2k}$ and let $\mathcal{V} = \oplus_{0 \leq k \leq r} \mathcal{V}_k$.

Since the sheaves \mathcal{V}_k are saturated subsheaves of $\mathcal{O}(E_k)$, they exits a *big* open subset \mathcal{U} of $\mathbb{P}T_X$ (an open subset \mathcal{U} of $\mathbb{P}T_X$ is big if $\text{codim } \mathbb{P}T_X \setminus \mathcal{U} \geq 2$) and subbundles V_k of E_k defined on \mathcal{U} such that the restriction of the \mathcal{V}_k 's to \mathcal{U} are the sheaves of sections of the V_k 's. On \mathcal{U} we let V be the subbundle $\oplus_{0 \leq k \leq r} V_k$, so that $\mathcal{V}|_{\mathcal{U}} = \mathcal{O}_{\mathcal{U}}(V)$.

Observe that on the regular locus $\mathcal{R}(\beta)$ of β , the rank of β^k , as a vector bundle morphism from $E \otimes L^k$ to E , is constant. Hence on this open subset the formulas used above to define the subsheaves \mathcal{F}_k of \mathcal{E} in fact define subbundles F_k of E such that $\mathcal{F}_k|_{\mathcal{R}(\beta)} = \mathcal{O}_{\mathcal{R}(\beta)}(F_k)$. Therefore, on $\mathcal{R}(\beta)$, the subbundles V_k such that $\mathcal{V}_k|_{\mathcal{R}(\beta)} = \mathcal{O}(V_k)$ are given by $V_k = E_k \cap F_{r-2k}$ and we may assume that $\mathcal{R}(\beta)$ is contained and dense in \mathcal{U} .

Lemma 4.3. *On the big open set \mathcal{U} , the subsheaf \mathcal{V} defines a reduction $P_{K \cap Q}$ of the structure group of P_K to the subgroup $K \cap Q \subset K$.*

Proof. We begin by working on $\mathcal{R}(\beta) \subset \mathcal{U}$. We view an element p of P_K above $\xi \in \mathbb{P}T_X$ as an isomorphism between the fiber E_ξ of $E = P_K(\mathbb{E})$ and the model space \mathbb{E} . The component β of the Higgs field is a section of $P_K(\mathfrak{u}_-) \otimes L^\vee \subset P_K(\text{End}(\mathbb{E})) \otimes L^\vee$.

Since for all $\xi \in \mathcal{R}(\beta)$ and all $\eta \in L_\xi$, $\eta \neq 0$, we have that $\beta_\xi(\eta)$ has rank r , there exists $p \in (P_K)_\xi$ such that $p \circ \beta_\xi(\eta) \circ p^{-1} = y \in \mathfrak{u}_- \subset \text{End}(\mathbb{E})$, so that $p(V_\xi) = \mathbb{V}$. Therefore, on $\mathcal{R}(\beta)$, by Proposition 2.30 (4), the subbundle V of E defines a (holomorphic) reduction $P_{K \cap Q}$ of the structure group of P_K to the subgroup $K \cap Q$ of K (Q is the normalizer in G of the parabolic subalgebra \mathfrak{q} of Definition 2.20). Explicitely $P_{K \cap Q} = \{p \in P_K \mid p(V_{\pi(p)}) = \mathbb{V}\}$.

We now work on \mathcal{U} . Enlarge the structure group of P_K to $\text{GL}(\mathbb{E})$. The subbundle $V = \oplus_{k=0}^r V_k$ of E defines a reduction P_S of the structure group of $P_{\text{GL}(\mathbb{E})}$ to the stabilizer S of \mathbb{V} in $\text{GL}(\mathbb{E})$ by setting $P_S = \{p \in P_{\text{GL}(\mathbb{E})} \mid p(V_{\pi(p)}) = \mathbb{V}\}$.

Let $B \subset \mathcal{U}$ be an open ball on which P_K is trivial. Then the reductions $P_{K \cap Q}$ of P_K on $B \cap \mathcal{R}(\beta)$ and P_S of $P_{\text{GL}(\mathbb{E})}$ on B are respectively given by holomorphic maps $\sigma : B \cap \mathcal{R}(\beta) \rightarrow K/(K \cap Q)$ and $s : B \rightarrow \text{GL}(\mathbb{E})/S$. Moreover, if ι denotes the natural map $K/(K \cap Q) \rightarrow \text{GL}(\mathbb{E})/S$, which is injective, then we have $s = \iota \circ \sigma$ on $B \cap \mathcal{R}(\beta)$. Since $K/(K \cap Q)$ is compact, its image by ι is closed in $\text{GL}(\mathbb{E})/S$. Therefore, since $B \cap \mathcal{R}(\beta)$ is dense in B , s maps B to $\iota(K/(K \cap Q))$. This means that the reduction $P_{K \cap Q}$ initially defined on $\mathcal{R}(\beta)$ extends to \mathcal{U} . \square

We deduce that

Proposition 4.4. *The subsheaf \mathcal{V} is a leafwise Higgs subsheaf of the Higgs bundle (E, θ) on $\mathbb{P}T_X$.*

Proof. By Proposition 2.22, we know that y and u_+ stabilize \mathbb{V} . Therefore, on $\mathcal{R}(\beta)$, the two components β and γ of the Higgs field stabilize the subsheaf \mathcal{V} since it is the sheaf of sections of the subbundle $V = P_{K \cap Q}(\mathbb{V})$ of $E = P_{K \cap Q}(\mathbb{E})$. By continuity, this still holds on \mathcal{U} since on this big open set \mathcal{V} is also the sheaf of section of $V = P_{K \cap Q}(\mathbb{V})$. Now, \mathcal{V} is a saturated, hence normal, subsheaf of $\mathcal{O}(E)$ by definition. Hence the restriction map $\mathcal{V}(\mathbb{P}T_X) \rightarrow \mathcal{V}(\mathcal{U})$ is an isomorphism since $\text{codim } \mathbb{P}T_X \setminus \mathcal{U} \geq 2$. Therefore \mathcal{V} is indeed a leafwise Higgs subsheaf of (E, θ) on $\mathbb{P}T_X$. \square

Instead of working with β on the Higgs bundle (E, θ) , we can consider ${}^t\gamma$ on the dual Higgs bundle $(E^\vee, {}^t\theta)$ and exactly the same reasoning yields a leafwise Higgs subsheaf \mathcal{V}' of $(E^\vee, {}^t\theta)$, see Remark 2.33.

4.4. Proof of the Milnor-Wood inequality.

Together with the computation of the slopes of the H -submodules \mathbb{V}_k of \mathbb{E} , the construction of the leafwise Higgs subsheaves \mathcal{V} of (E, θ) and \mathcal{V}' of $(E^\vee, {}^t\theta)$ gives the Milnor Wood inequality:

Theorem 4.5. *We have the inequalities $\deg_{\mathcal{T}}(E_0) + \frac{\text{rk}\beta}{2} \deg_{\mathcal{T}}(L) \leq \mu_{\mathcal{T}}(\mathcal{V}) \leq 0$ and $\deg_{\mathcal{T}}(E_0^\vee) + \frac{\text{rk}\gamma}{2} \deg_{\mathcal{T}}(L) \leq \mu_{\mathcal{T}}(\mathcal{V}') \leq 0$, therefore $|\deg_{\mathcal{T}}(E_0)| \leq \frac{p}{2} \deg_{\mathcal{T}}(L^\vee)$.*

Proof. We begin with the inequality $\deg_{\mathcal{T}}(E_0) + \frac{\text{rk}\beta}{2} \deg_{\mathcal{T}}(L) \leq \mu_{\mathcal{T}}(\mathcal{V})$. Let $n_k := \dim \mathbb{V}_k$. First, recall that Proposition 2.30 (3) states that the unipotent radical of Q acts trivially on \mathbb{V} . Therefore so does the unipotent radical of $K \cap Q$. Thus, in fact, \mathbb{V} is a $(K \cap Q)/R_u(K \cap Q)$ -module, and $(K \cap Q)/R_u(K \cap Q) \simeq H$ is reductive. Thus we may apply Proposition 2.29 and deduce that the $K \cap Q$ -representations $(\det \mathbb{V}_{k+1})^{n_1 n_k}$ and $(\det \mathbb{V}_k)^{n_1 n_{k+1}} \otimes (\det \mathbb{V}_1)^{n_k n_{k+1}} \otimes (\mathbb{V}_0^*)^{n_1 n_k n_{k+1}}$ are isomorphic. On the big open set $\mathcal{U} \subset \mathbb{P}T_X$, we have $\mathcal{V}_k = \mathcal{O}(V_k)$ where $V_k = P_{K \cap Q}(\mathbb{V}_k)$. Therefore on \mathcal{U} , and hence on $\mathbb{P}T_X$, the line bundles $(\det \mathcal{V}_{k+1})^{n_1 n_k}$ and $(\det \mathcal{V}_k)^{n_1 n_{k+1}} \otimes (\det \mathcal{V}_1)^{n_k n_{k+1}} \otimes (\mathcal{V}_0^*)^{n_1 n_k n_{k+1}}$ are isomorphic. This implies that $\mu_{\mathcal{T}}(\mathcal{V}_{k+1}) = \mu_{\mathcal{T}}(\mathcal{V}_k) + \mu_{\mathcal{T}}(\mathcal{V}_0^\vee \otimes \mathcal{V}_1)$, i.e. that $\mu_{\mathcal{T}}(\mathcal{V}_k) = \deg_{\mathcal{T}}(\mathcal{V}_0) + k \mu_{\mathcal{T}}(\mathcal{V}_0^\vee \otimes \mathcal{V}_1)$.

Let r be short for $\text{rk}\beta$. Since $\beta^r : \mathcal{V}_0 \otimes L^r \rightarrow \mathcal{V}_r$ is not zero, we also have $\mu_{\mathcal{T}}(\mathcal{V}_r) \geq \mu_{\mathcal{T}}(\mathcal{V}_0) + r \mu_{\mathcal{T}}(L)$ so that $\mu_{\mathcal{T}}(\mathcal{V}_0^\vee \otimes \mathcal{V}_1) \geq \deg_{\mathcal{T}}(L)$.

Finally, remembering that $n_k = n_{r-k}$ by Lemma 2.21 and that $\mathcal{V}_0 = E_0$, we get

$$\begin{aligned} 2 \deg_{\mathcal{T}}(\mathcal{V}) &= \sum_{k=0}^r \deg_{\mathcal{T}}(\mathcal{V}_k) + \sum_{k=0}^r \deg_{\mathcal{T}}(\mathcal{V}_{r-k}) \\ &= \sum_{k=0}^r (n_k \mu_{\mathcal{T}}(\mathcal{V}_k) + n_{r-k} \mu_{\mathcal{T}}(\mathcal{V}_{r-k})) \\ &\geq \sum_{k=0}^r n_k (\deg_{\mathcal{T}}(\mathcal{V}_0) + k \deg_{\mathcal{T}}(L) + \deg_{\mathcal{T}}(\mathcal{V}_0) + (r-k) \deg_{\mathcal{T}}(L)) \\ &\geq (\dim \mathbb{V}) (2 \deg_{\mathcal{T}}(E_0) + r \deg_{\mathcal{T}}(L)) \end{aligned}$$

The inequality $\mu_{\mathcal{T}}(\mathcal{V}) \leq 0$ follows from Propositions 4.4 and 3.2.

Finally the inequalities involving E_0^\vee , γ and \mathcal{V}' are proved exactly in the same way. The conclusion follows since $\text{rk}\beta \leq p$ and $\text{rk}\gamma \leq p$. \square

In case of equality in Theorem 4.5, we have:

Proposition 4.6. *Assume that $\deg \tau(E_0) + \frac{\text{rk} \beta}{2} \deg \tau(L) = 0$. Then*

- (1) *on the regular locus $\mathcal{R}(\beta) = \mathbb{P}T_X \setminus \mathcal{S}(\beta)$ of β , the orthogonal complement $V^\perp = \oplus V_i^\perp$ of the subbundle $V = \oplus V_i$ of E w.r.t. the harmonic metric is stable under the Higgs field $\theta : E \otimes L \rightarrow E$;*
- (2) *the regular locus $\mathcal{R}(\bar{\beta}) \subset X$ of $\bar{\beta}$ is (open and) dense in X .*

Similarly, if $\deg \tau(E_0^\vee) + \frac{\text{rk} \gamma}{2} \deg \tau(L) = 0$, then the orthogonal complement of $V' \subset E^\vee$ is stable by ${}^t\theta : E^\vee \otimes L \rightarrow E^\vee$ on the regular locus $\mathcal{R}(\gamma) \subset \mathbb{P}T_X$ of γ and $\mathcal{R}(\bar{\gamma}) \subset X$ is (open and) dense in X .

Proof. The first point follows from the discussion after the definition of the subsheaves \mathcal{V}_k and the polystability property (2) in Proposition 3.2, since our hypothesis implies that $\deg \tau \mathcal{V} = 0$ by Proposition 4.5. Proposition 3.2 (2) implies that the singular locus $\mathcal{S}(\beta) \subset \mathbb{P}T_X$ is saturated under the tautological foliation \mathcal{T} , see the proof of [KM17, Lemma 4.5]. This, together with M. Ratner's results on unipotent flows, in turn implies the second point of the proposition, see [KM17, Proposition 3.6]. \square

5. MAXIMAL REPRESENTATIONS

Maximal representations $\rho : \Gamma \rightarrow G_{\mathbb{R}}$, where Γ is a uniform lattice in $\text{SU}(1, n)$ with $n \geq 2$ and $G_{\mathbb{R}}$ is a *classical* Lie group of Hermitian type, were classified in [KM17]. Therefore we focus here on *exceptional* targets, namely $G_{\mathbb{R}}$ is either $E_{6(-14)}$, which is not of tube type, or $E_{7(-25)}$, which is.

In Section 5.1 we exclude the possibility of maximal representations in $E_{7(-25)}$. In fact, our uniform approach allows to easily prove that maximal representations in tube type target groups $G_{\mathbb{R}}$ do not exist. The case of $E_{6(-14)}$ is treated in Section 5.2.

5.1. Tube type targets.

We prove that whenever $G_{\mathbb{R}}$ has tube type and $n \geq 2$, representations from Γ to $G_{\mathbb{R}}$ satisfy an inequality stronger than the Milnor-Wood inequality, preventing any representation in such a group to be maximal:

Proposition 5.1. *Let Γ be a uniform lattice in $\text{SU}(1, n)$, with $n \geq 2$, and let $X = \Gamma \backslash \mathbb{H}_{\mathbb{C}}^n$. Assume that $G_{\mathbb{R}}$ has tube type and let p be the real rank of $G_{\mathbb{R}}$. Let ρ be a representation $\Gamma \rightarrow G_{\mathbb{R}}$. Then*

$$|\tau(\rho)| \leq \max \left(p - 1, \frac{p}{2} \cdot \frac{n + 1}{n} \right) \text{vol}(X) < p \text{vol}(X).$$

Proof. We may assume that $\tau(\rho) > 0$. Recall that we assumed without loss of generality that ρ is reductive (Assumption 1.2). Then, the constructions of §3 and §4 are valid and the inequality of the Proposition is equivalent to the inequality

$$\deg \tau(E_0) \leq \max \left(\frac{p - 1}{2}, \frac{p}{2} \cdot \frac{n + 1}{2n} \right) \deg \tau(L^\vee) < \frac{p}{2} \deg \tau(L^\vee).$$

We use freely the notation of §4. If the generic rank of β on the projectivized tangent bundle $\mathbb{P}T_X$ of X is $\leq p - 1$ then we are done by Theorem 4.5. Therefore we may assume that the generic rank of β on $\mathbb{P}T_X$ is p .

We come back to the Higgs bundle $(\bar{E}, \bar{\theta})$ on X and we consider $\bar{\beta} : \bar{E} \otimes T_X \rightarrow \bar{E}$. The fact that $\text{rk} \beta = p$ implies that $\bar{\beta}^p$, seen as a morphism from $\bar{E}_0 \otimes \bar{E}_p^\vee$ to the p -th symmetric power $S^p \Omega_X^1$ of Ω_X^1 , is not zero. Since Ω_X^1 is a semistable bundle over X (X is Kähler-Einstein), so is $S^p \Omega_X^1$. On the other hand, $\bar{E}_0 \otimes \bar{E}_p^\vee$ is also semistable because it is a line bundle by Remark 2.16. Therefore $\mu(\bar{E}_0 \otimes \bar{E}_p^\vee) \leq \mu(S^p \Omega_X^1)$, so that $\deg \bar{E}_0 - \deg \bar{E}_p \leq p \mu(\Omega_X^1)$. Now, as explained in Remark 2.16, the K -modules \mathbb{E}_p and

\mathbb{E}_0^\vee are isomorphic, so that $\deg \bar{E}_p = -\deg \bar{E}_0$. Hence the result, since by equations (4) and (5), $\deg(\Omega_X^1) = \frac{n+1}{2} \deg \tau(L^\vee)$. \square

5.2. Target group $E_{6(-14)}$.

5.2.1. Algebraic preliminaries. In the case $G_{\mathbb{R}} = E_{6(-14)}$, the minuscule representation of $G = E_6$ is the standard representation of E_6 on the 27 dimensional complex exceptional Jordan algebra $\mathbb{E} = \mathfrak{J}_{\mathbb{C}}$. The real rank of $E_{6(-14)}$ is 2 and $\mathbb{E} = \mathbb{E}_0 \oplus \mathbb{E}_1 \oplus \mathbb{E}_2$ with \mathbb{E}_0 , \mathbb{E}_1 , and \mathbb{E}_2 of dimension 1, 16 and 10 respectively.

We start with a description of the Spin_{10} -representations \mathbb{E}_1 and \mathbb{E}_2 in terms of octonions. More precisely, by Proposition 2.11(e), there is a Spin_{10} -equivariant isomorphism $\mathbb{E}_1 \simeq \mathbb{E}_0 \otimes \mathfrak{u}_-$. Choosing a non-zero vector in \mathbb{E}_0 , this yields an isomorphism $\alpha : \mathbb{E}_1 \rightarrow \mathfrak{u}_-$. We consider the quadratic map $\kappa : \mathbb{E}_1 \rightarrow \mathbb{E}_2$ defined by $\kappa(x) = \alpha(x) \cdot x$. This is a Spin_{10} -equivariant quadratic map $\mathbb{E}_1 \rightarrow \mathbb{E}_2$. As the next proposition shows, there is, up to a scale, only one such map. It is certainly well-known to specialists, however we could not find an adequate reference:

Proposition 5.2. *There is an identification of \mathbb{E}_1 with $\mathbb{O} \oplus \mathbb{O}$ and \mathbb{E}_2 with $\mathbb{C} \oplus \mathbb{O} \oplus \mathbb{C}$ such that $\kappa(u, v) = (N(u), uv, N(v))$.*

Proof. We consider the Spin_{10} half-spin representation \mathbb{E}_1^* . According to [Che97], when we restrict to Spin_8 , this representation splits as $\mathcal{S}^+ \oplus \mathcal{S}^-$, where \mathcal{S}^\pm denote the two half-spin representations of Spin_8 . Similarly, the Spin_{10} vector representation \mathbb{E}_2 splits as $\mathbb{C} \oplus \mathbb{V} \oplus \mathbb{C}$, where \mathbb{V} denotes the 8-dimensional vector representation.

Now, the quadratic map κ is given by a Spin_{10} -equivariant injection $\mathbb{E}_2 \subset \mathbb{E}_1^* \otimes \mathbb{E}_1^*$. Since there are equivariant maps $\mathcal{S}^+ \oplus \mathcal{S}^- \rightarrow \mathbb{V}$, $\mathcal{S}^+ \otimes \mathcal{S}^+ \rightarrow \mathbb{C}$ and $\mathcal{S}^- \otimes \mathcal{S}^- \rightarrow \mathbb{V}$, and no equivariant maps from other factors in this tensor product to \mathbb{E}_2 , κ is of the form $\kappa(s_+, s_-) = (\psi_+(s_+), \varphi(s_+, s_-), \psi_-(s_-))$, for some equivariant maps ψ_+, φ and ψ_- . None of these maps can vanish, otherwise the image of κ would be degenerate. Moreover, by triality, there are, up to scale, only one such map, which can be given, once $\mathcal{S}^+, \mathcal{S}^-$ and \mathbb{V} are identified with the space of octonions \mathbb{O} , by the formulas: $\psi_+(s_+) = N(s_+)$, $\varphi(s_+, s_-) = s_+ s_-$ and $\psi_-(s_-) = N(s_-)$. The proposition follows. \square

Given $x \in \mathfrak{u}_-$ resp. $y \in \mathfrak{u}_+$, x resp. y defines linear maps $\mathbb{E}_0 \rightarrow \mathbb{E}_1$ and $\mathbb{E}_1 \rightarrow \mathbb{E}_2$ resp. $\mathbb{E}_1 \rightarrow \mathbb{E}_0$ and $\mathbb{E}_2 \rightarrow \mathbb{E}_1$. We denote these maps by $\lambda_1(x), \lambda_2(x)$ resp. $\mu_1(y), \mu_2(y)$. We thus have maps $\lambda_1(x) : \mathbb{E}_0 \rightarrow \mathbb{E}_1$, $\lambda_2(x) : \mathbb{E}_1 \rightarrow \mathbb{E}_2$ and $\mu_1(y) : \mathbb{E}_1 \rightarrow \mathbb{E}_0$, $\mu_2(y) : \mathbb{E}_2 \rightarrow \mathbb{E}_1$.

We can deduce from the explicit formula above some information about maps $\lambda_2(x)$:

Proposition 5.3. *Let $x, y \in \mathbb{E}_1 \simeq \mathfrak{u}_-$. Assume $x \neq 0$ and $y \neq 0$.*

- (a) *x has rank one if and only if $\kappa(x) = 0$.*
- (b) *x has rank one if and only if $\lambda_2(x)$ has rank 5.*
- (c) *x has rank two if and only if $\lambda_2(x)$ has rank 9.*
- (d) *Assume that any non trivial linear combination of x and y has rank 2. Then $\dim(\text{Ker } \lambda_2(x) \cap \text{Ker } \lambda_2(y)) \leq 3$.*
- (e) *Assume that x and y have rank 1 and $\dim(\text{Im } \lambda_2(x) \cap \text{Im } \lambda_2(y)) \geq 4$. Then x and y are proportional.*

Proof. We use the above isomorphism $\mathbb{E}_1 \simeq \mathbb{O} \oplus \mathbb{O}$. According to [Igu70], there are exactly 3 orbits in \mathbb{E}_1 under $\text{Spin}_{10} \times \mathbb{C}^*$. Let $u \in \mathbb{O}$ such that $N(u) = 0$. We have $\kappa(u, 0) = (0, 0, 0)$ and $\kappa(1, 0) = (1, 0, 0)$. Thus, $(u, 0)$ and $(1, 0)$ cannot be in the same orbit. It follows that $(u, 0)$ has rank 1 and $(1, 0)$ has rank 2 and statement (a) of the proposition is proved.

Let $\tilde{\kappa} : \mathbb{E}_1 \times \mathbb{E}_1 \rightarrow \mathbb{E}_2$ be the polarization of κ , namely, the unique symmetric bilinear map such that $\tilde{\kappa}(x, x) = \kappa(x)$ for all x in \mathbb{E}_1 . We have $\lambda_2(x) = \tilde{\kappa}(x, \cdot)$. Thus the image of $\lambda_2(u, 0)$ is the set of triples (t, z, t') with $t \in \mathbb{C}$ arbitrary, z a right multiple of u in \mathbb{O} , and $t' = 0$: this space has dimension 5. On the other hand, the image of $\lambda_2(1, 0)$ is the set of triples (t, z, t') with t and z arbitrary and $t' = 0$. It has dimension 9. Points (b) and (c) are proved.

For point (d), let us assume that any non trivial linear combination of x and y has rank 2. Thanks to the result of Igusa, we may assume that $x = (1, 0)$. Writing $y = (a, b)$, the assumption implies that $b \neq 0$ (in fact, if $y = (a, 0)$, then some linear combination of x and y will be of the form $(u, 0)$ with $N(u) = 0$). The kernel of $\lambda_2(x)$ is the space of elements of the form $(u, 0)$ with $\langle u, 1 \rangle = 0$. If such an element is in the kernel of $\lambda_2(y)$ then $bu = 0$ and so $N(u) = 0$. Thus, the intersection of the kernels of $\lambda_2(x)$ and $\lambda_2(y)$ is isomorphic to an isotropic subspace of the space of octonions u with $\langle u, 1 \rangle = 0$. Such an isotropic subspace can have at most dimension 3.

Finally, let us assume that x and y have rank 1 and that $\dim(\text{Im } \lambda_2(x) \cap \text{Im } \lambda_2(y)) \geq 4$. Then we may assume that $x = (u, 0)$ with $N(u) = 0$ as above. The image of $\lambda_2(x)$ is then the set of triples $(t, z, 0)$ with t arbitrary and z of the form uw for some octonion w . Thus, this space is an isotropic subspace of \mathbb{E}_2 of maximal dimension 5. Using the Spin_{10} -action, it follows that for any $x \in \mathbb{E}_1$ of rank 1, the image of $\lambda_2(x)$ is an isotropic subspace of dimension 5. Since two maximal isotropic subspaces in the same family can intersect only in odd dimension, it follows from the hypothesis on x and y that the images of $\lambda_2(x)$ and $\lambda_2(y)$ are equal. One can check that this implies that y is proportional to $(u, 0) = x$. \square

We constructed a quadratic Spin_{10} -equivariant map $\kappa : \mathbb{E}_1 \rightarrow \mathbb{E}_2$ identifying \mathbb{E}_1 with \mathfrak{u}_- and using the linear map $\mathbb{E}_1 \rightarrow \mathbb{E}_2$ given by $x \in \mathfrak{u}_-$. Similarly, let $\iota : \mathbb{E}_1^* \rightarrow \mathbb{E}_2^*$ be the quadratic equivariant map obtained identifying \mathbb{E}_1^* with \mathfrak{u}_+ and using the linear map ${}^t\mu_2(w) : E_1^* \rightarrow E_2^*$ given by $w \in \mathfrak{u}_+$. With the same proof, we get information about \mathfrak{u}_+ and the linear maps $\mu_2(w)$:

Proposition 5.4. *Let $w, z \in \mathbb{E}_1^* \simeq \mathfrak{u}_+$. Assume $w \neq 0$ and $z \neq 0$.*

- (a) *w has rank one if and only if $\iota(w) = 0$.*
- (b) *w has rank one if and only if $\mu_2(w)$ has rank 5.*
- (c) *w has rank two if and only if $\mu_2(w)$ has rank 9.*
- (d) *Assume that any non trivial linear combination of w and z has rank 2. Then $\dim(\text{Im } \mu_2(w) \cap \text{Im } \mu_2(z)) \leq 3$.*
- (e) *Assume that w and z have rank 1 and $\dim(\text{Ker } \mu_2(w) \cap \text{Ker } \mu_2(z)) \geq 4$. Then w and z are proportional.*

5.2.2. Maximal representations. Let $\rho : \Gamma \rightarrow E_{6(-14)}$ be a reductive representation. We may therefore consider the the Higgs bundle $(\bar{E}, \bar{\theta})$ on X and its pull-back (E, θ) on $\mathbb{P}T_X$ associated to ρ and the representation of E_6 on $\mathbb{E} = \mathbb{E}_0 \oplus \mathbb{E}_1 \oplus \mathbb{E}_2$ as in Section 4.

Recall that the components of the Higgs field $\bar{\theta}$ are

$$P_K(\mathfrak{u}_-) \ni \bar{\beta} =: (\bar{\beta}_1, \bar{\beta}_2) \in \left(\text{Hom}(\bar{E}_0, \bar{E}_1) \otimes \Omega_X^1 \right) \oplus \left(\text{Hom}(\bar{E}_1, \bar{E}_2) \otimes \Omega_X^1 \right)$$

and

$$P_K(\mathfrak{u}_+) \ni \bar{\gamma} =: (\bar{\gamma}_1, \bar{\gamma}_2) \in \left(\text{Hom}(\bar{E}_1, \bar{E}_0) \otimes \Omega_X^1 \right) \oplus \left(\text{Hom}(\bar{E}_2, \bar{E}_1) \otimes \Omega_X^1 \right)$$

To lighten the notation, the fibers of the bundles \bar{E} , \bar{E}_0 , \bar{E}_1 and \bar{E}_2 above some $x \in X$ will also be denoted by \bar{E} , \bar{E}_0 , \bar{E}_1 and \bar{E}_2 .

Propositions 5.3 and 5.4 immediately imply the following:

Lemma 5.5. *Let $x \in X$ and ξ be a holomorphic tangent vector at x .*

As an element of $\text{Hom}(\bar{E}_1, \bar{E}_2)$, $\bar{\beta}_2(\xi)$ has rank 0, 5 or 9. Moreover:

- (i) *If $\bar{\beta}(\xi)$ has rank 1, i.e. $\bar{\beta}_1(\xi) \neq 0$ but $\bar{\beta}_2(\xi)\bar{\beta}_1(\xi) = 0$, then $\bar{\beta}_2(\xi) : \bar{E}_1 \rightarrow \bar{E}_2$ has rank 5;*
- (ii) *If $\bar{\beta}(\xi)$ has rank 2, i.e. if $\bar{\beta}_2(\xi)\bar{\beta}_1(\xi) \neq 0$, then $\bar{\beta}_2(\xi) : \bar{E}_1 \rightarrow \bar{E}_2$ has rank 9;*
- (iii) *If any non trivial linear combination of $\bar{\beta}(\xi)$ and $\bar{\beta}(\eta)$ has rank 2, then we have $\dim(\text{Ker } \bar{\beta}_2(\xi) \cap \text{Ker } \bar{\beta}_2(\eta)) \leq 3$.*

Similarly, as an element of $\text{Hom}(\bar{E}_2, \bar{E}_1)$, $\bar{\gamma}_2(\xi)$ has rank 0, 5 or 9. Moreover:

- (a) *If $\bar{\gamma}(\xi)$ has rank 1, i.e. $\bar{\gamma}_2(\xi) \neq 0$ but $\bar{\gamma}_1(\xi)\bar{\gamma}_2(\xi) = 0$, then $\bar{\gamma}_2(\xi) : \bar{E}_2 \rightarrow \bar{E}_1$ has rank 5;*
- (b) *If $\bar{\gamma}(\xi)$ has rank 2, i.e. if $\bar{\gamma}_1(\xi)\bar{\gamma}_2(\xi) \neq 0$, then $\bar{\gamma}_2(\xi) : \bar{E}_2 \rightarrow \bar{E}_1$ has rank 9;*
- (c) *If $\bar{\gamma}(\xi)$ and $\bar{\gamma}(\eta)$ have rank 1 and $\dim(\text{Ker } \bar{\gamma}_2(\xi) \cap \text{Ker } \bar{\gamma}_2(\eta)) \geq 4$, then $\bar{\gamma}(\xi)$ and $\bar{\gamma}(\eta)$ are colinear.*

Thanks to this lemma, in case of equality in the Milnor-Wood inequality, we may prove

Proposition 5.6. *If $\deg \tau(E_0) = \deg \tau(L^\vee)$ and $x \in \mathcal{R}(\bar{\beta})$, then for all $\xi \in T_{X,x}$, $\bar{\gamma}(\xi) = 0$.*

If $\deg \tau(E_0) = -\deg \tau(L^\vee)$ and $x \in \mathcal{R}(\bar{\gamma})$, then for all $\xi \in T_{X,x}$, $\bar{\beta}(\xi) = 0$.

Proof. We prove only the first assertion, the proof of the second one follows exactly the same lines. The letters ξ and η will denote (holomorphic) tangent vectors at x .

The equality $\deg \tau(E_0) = \deg \tau(L^\vee)$ and Theorem 4.5 imply that the generic rank of β on $\mathbb{P}T_X$ is 2. Therefore, since x belongs to the regular locus $\mathcal{R}(\bar{\beta})$ of $\bar{\beta}$, for all $\xi \neq 0$ in $T_{X,x}$, the rank of $\bar{\beta}(\xi)$ is 2, so that the rank of $\bar{\beta}_2(\xi)$ is 9 by Lemma 5.5.

We will make a crucial use of the integrability relation $[\bar{\theta}, \bar{\theta}] = 0$ of the Higgs field $\bar{\theta}$. This relation is equivalent to the following three conditions:

$$\begin{cases} \bar{\gamma}_1(\xi)\bar{\beta}_1(\eta) = \bar{\gamma}_1(\eta)\bar{\beta}_1(\xi) & \text{in } \text{End}(\bar{E}_0) \\ \bar{\beta}_1(\xi)\bar{\gamma}_1(\eta) + \bar{\gamma}_2(\xi)\bar{\beta}_2(\eta) = \bar{\beta}_1(\eta)\bar{\gamma}_1(\xi) + \bar{\gamma}_2(\eta)\bar{\beta}_2(\xi) & \text{in } \text{End}(\bar{E}_1) \\ \bar{\beta}_2(\xi)\bar{\gamma}_2(\eta) = \bar{\beta}_2(\eta)\bar{\gamma}_2(\xi) & \text{in } \text{End}(\bar{E}_2) \end{cases}$$

which hold for all ξ, η .

Suppose first that there exists ξ such that $\bar{\gamma}_2(\xi) : \bar{E}_2 \rightarrow \bar{E}_1$ has rank 9. Consider the subspace $W := \text{Ker } \bar{\gamma}_1(\xi) \cap \text{Ker } \bar{\beta}_2(\xi) \subset \bar{E}_1$. Since $\dim \bar{E}_1 = 16$, $\dim \text{Ker } \bar{\gamma}_1(\xi) = 15$ and $\dim \text{Ker } \bar{\beta}_2(\xi) = 7$, we have $\dim W \geq 6$. On this subspace, the second integrability condition reads $\bar{\beta}_1(\xi)\bar{\gamma}_1(\eta) + \bar{\gamma}_2(\xi)\bar{\beta}_2(\eta) = 0$ for all η . Therefore $\bar{\gamma}_2(\xi)\bar{\beta}_2(\eta)(W) \subset \bar{E}_1$ is 1-dimensional. Because of our assumption on the rank of $\bar{\gamma}_2(\xi)$, $\bar{\beta}_2(\eta)(W)$ is of dimension at most 2, and this implies that $\dim W \cap \text{Ker } \bar{\beta}_2(\eta) \geq 4$, hence that $\dim \text{Ker } \bar{\beta}_2(\xi) \cap \text{Ker } \bar{\beta}_2(\eta) \geq 4$. We get a contradiction with Lemma 5.5(iii).

Suppose now that for all $\xi \neq 0$, $\bar{\gamma}_2(\xi)$ has rank 5. Fix $\xi \neq 0$, and let $[\xi]$ be the class of ξ in the fiber of $\mathbb{P}T_X$ above x . Let $V(\xi) = V_0(\xi) \oplus V_1(\xi) \oplus V_2(\xi)$ be the fiber above $[\xi]$ of the subbundle V of the Higgs bundle (E, θ) on $\mathbb{P}T_X$. We have

$$\begin{cases} V_0(\xi) = E_0 = \bar{E}_0 \\ V_1(\xi) = E_1 \cap F_0 = E_1 \cap (\text{Ker } \beta_{[\xi]} \oplus \text{Ker } \beta_{[\xi]}^2 \cap \text{Im } \beta_{[\xi]}) = \text{Ker } \bar{\beta}_2(\xi) \oplus \text{Im } \bar{\beta}_1(\xi) \\ V_2(\xi) = E_2 \cap F_{-2} = E_2 \cap (\text{Ker } \beta_{[\xi]} \cap \text{Im } \beta_{[\xi]}^2) = \text{Im } \bar{\beta}_2(\xi)\bar{\beta}_1(\xi) \end{cases}$$

and we know by Proposition 4.6 (1) that the orthogonal complement $V_1(\xi)^\perp \oplus V_2(\xi)^\perp$ of $V_0(\xi) \oplus V_1(\xi) \oplus V_2(\xi)$ is invariant by $\bar{\theta}(\xi)$, in particular that $\bar{\gamma}_2(\xi)$ maps $V_2(\xi)^\perp$ to $V_1(\xi)^\perp$.

By the third integrability condition, $\bar{\gamma}_2(\xi)$ maps $\text{Ker } \bar{\gamma}_2(\eta)$ in $\text{Ker } \bar{\beta}_2(\eta)$. Hence $\bar{\gamma}_2(\xi)$ maps $V_2(\xi)^\perp \cap \text{Ker } \bar{\gamma}_2(\eta)$ to $\text{Ker } \bar{\beta}_2(\eta) \cap V_1(\xi)^\perp$.

But $\bar{\beta}_2(\xi)$ is injective on $V_1(\xi)^\perp$ because $\text{Ker } \bar{\beta}_2(\xi) \subset V_1(\xi)$. Hence for η close to ξ , $\bar{\beta}_2(\eta)$ is also injective on $V_1(\xi)^\perp$, so that $V_2(\xi)^\perp \cap \text{Ker } \bar{\gamma}_2(\eta) \subset \text{Ker } \bar{\gamma}_2(\xi)$. Now, $\dim V_2(\xi)^\perp = 9$ and $\text{rk } \bar{\gamma}_2(\eta) = 5$, thus $V_2(\xi)^\perp \cap \text{Ker } \bar{\gamma}_2(\eta)$ is at least 4 dimensional, and so is $\text{Ker } \bar{\gamma}_2(\xi) \cap \text{Ker } \bar{\gamma}_2(\eta)$. Again, this implies by Lemma 5.5(c) that $\bar{\gamma}_2(\xi)$ and $\bar{\gamma}_2(\eta)$ are colinear, a contradiction since we assumed that all the $\bar{\gamma}_2(\zeta)$, $\zeta \neq 0$, have rank 5.

We conclude that there exists $\xi \neq 0$ such that $\bar{\gamma}_2(\xi) = 0$. Then also $\bar{\gamma}_1(\xi) = 0$ and by the second integrability condition, for all η , $\bar{\beta}_1(\xi)\bar{\gamma}_1(\eta) = \bar{\gamma}_2(\eta)\bar{\beta}_2(\xi)$. Therefore $\bar{\gamma}_2(\eta)$ has rank at most 1 on $\text{Im } \bar{\beta}_2(\xi)$ which is 9 dimensional in \bar{E}_2 , so that $\bar{\gamma}_2(\eta)$ has rank at most 2, hence vanishes. Therefore $\bar{\gamma}_2 = 0$ and $\bar{\gamma} = 0$ identically on $T_{X,x}$. \square

Theorem 5.7. *Let Γ be a uniform lattice in $\text{SU}(1, n)$ with $n \geq 2$ and ρ be a maximal representation of Γ in $E_{6(-14)}$. Then $n = 2$ and there exists a holomorphic or anti-holomorphic ρ -equivariant embedding from $\mathbb{H}_{\mathbb{C}}^2$ to the symmetric space M associated to $E_{6(-14)}$.*

Proof. By [BIW09], maximal representations are reductive, and we may apply our previous results. We assume $\tau(\rho) > 0$, the case $\tau(\rho) < 0$ being handled similarly. By Proposition 5.6, $\bar{\gamma}$ vanishes on the regular locus $\mathcal{R}(\bar{\beta})$ of $\bar{\beta}$. By Proposition 4.6 (2), $\mathcal{R}(\bar{\beta})$ is dense in X , so that $\bar{\gamma}$ vanishes identically on X . This means that the ρ -equivariant harmonic map $f : \mathbb{H}_{\mathbb{C}}^n \rightarrow M$ used to define the Higgs bundle (E, θ) is holomorphic. \square

5.3. Proof of the main results.

In this subsection, we give detailed proofs of the theorem and corollaries given in the introduction, although some of the arguments might be well-known to specialists.

We assume $\tau(\rho) > 0$, the other case being similar. We may assume that ρ is reductive by [BIW09]. Let then $f : \mathbb{H}_{\mathbb{C}}^n \rightarrow M$ be a harmonic ρ -equivariant map (such a map exists by [Cor88]). By Theorem 5.7, Proposition 5.1 and [KM17], f is holomorphic. By the Ahlfors-Schwarz lemma (cf. [Roy80]), since the holomorphic sectional curvature is -1 on $\mathbb{H}_{\mathbb{C}}^n$ and bounded above by $-\frac{1}{p}$ on M , we have the pointwise inequality $f^*\omega_M \leq p\omega$. The maximality of ρ then implies that $f^*\omega_M = p\omega$. Since there is equality in the Ahlfors-Schwarz lemma, f is totally geodesic (see e.g. [Roy80]).

These properties imply that f is a so-called *tight* holomorphic totally geodesic map $\mathbb{H}_{\mathbb{C}}^n \rightarrow M$ (as defined in [Ham13]). Tight holomorphic maps between Hermitian symmetric spaces were classified in [Ham13]. If the symmetric space M is not irreducible, the map f is tight if and only if all the induced maps to the irreducible factors of M are tight. We may therefore assume that M is irreducible or equivalently that $G_{\mathbb{R}}$ is simple. In this case, and since $n \geq 2$, tight holomorphic totally geodesic maps $\mathbb{H}_{\mathbb{C}}^n \rightarrow M$ only exist when $G_{\mathbb{R}} = \text{SU}(p, q)$ with $q \geq pn$ or when $G_{\mathbb{R}} = E_{6(-14)}$ if $n = 2$. They are deduced one from another by composition by an element of $G_{\mathbb{R}}$.

Remark 5.8. There is a small inaccuracy in [Ham13], where it is said that there are two “tight regular” (in the terminology of this paper) maximal subalgebras of $\mathfrak{e}_{6(-14)}$. In fact, only $\mathfrak{su}(4, 2) \subset \mathfrak{e}_{6(-14)}$ is a maximal subalgebra for these properties. This was confirmed to us by the author.

This proves all the assertions of Theorem A and Corollary B except the uniqueness of the harmonic map $\mathbb{H}_{\mathbb{C}}^n \rightarrow M$ that is ρ -equivariant. To prove it, we need to have a closer look at f . It follows from [Ham13] (see also [KM08, Proposition 3.2]) that f is equivariant with respect to a morphism of Lie groups $\varphi : \text{SU}(1, n) \rightarrow G_{\mathbb{R}}$ and that up to conjugacy of ρ , we may assume that f and φ are as follows:

- for $G_{\mathbb{R}} = \mathrm{SU}(p, q)$ with $q \geq pn$, φ is the composition

$$\mathrm{SU}(1, n) \hookrightarrow \mathrm{SU}(1, n) \times \cdots \times \mathrm{SU}(1, n) \hookrightarrow \mathrm{SU}(p, pn) \hookrightarrow \mathrm{SU}(p, q);$$

- for $G_{\mathbb{R}} = \mathrm{E}_{6(-14)}$ and $n = 2$, φ is the composition

$$\mathrm{SU}(1, 2) \hookrightarrow \mathrm{SU}(1, 2) \times \mathrm{SU}(1, 2) \hookrightarrow \mathrm{SU}(2, 4) \hookrightarrow \mathrm{E}_{6(-14)}.$$

where the last morphism is detailed in the proof of Lemma 5.9 below.

In both cases, the image N of f in $M = G_{\mathbb{R}}/K_{\mathbb{R}}$ is the orbit of $o = K_{\mathbb{R}}$ under $H_{\mathbb{R}} := \varphi(\mathrm{SU}(1, n)) \subset G_{\mathbb{R}}$.

We now describe the centralizer $Z_{\mathbb{R}}$ of $H_{\mathbb{R}}$ in $G_{\mathbb{R}}$. In case $G_{\mathbb{R}} = \mathrm{SU}(p, q)$, let $GZ_{\mathbb{R}}$ denote the group $\mathrm{U}(p) \times \mathrm{U}(q - pn)$ and let $\chi : GZ_{\mathbb{R}} \rightarrow \mathrm{U}(1)$ be the character defined by $\chi(x, y) = \det(x)^{n+1} \cdot \det(y)$. In case $G_{\mathbb{R}} = \mathrm{E}_{6(-14)}$, let $GZ_{\mathbb{R}} = \mathrm{U}(2) \times \mathrm{U}(2)$ and let $\chi : GZ_{\mathbb{R}} \rightarrow \mathrm{U}(1)$ be the character defined by $\chi(x, y) = \det(x)^{21} \cdot \det(y)^6$. Then:

Lemma 5.9. *The centraliser $Z_{\mathbb{R}}$ of $H_{\mathbb{R}}$ in $G_{\mathbb{R}}$ is a subgroup of $K_{\mathbb{R}}$ (hence it is compact). It is isomorphic to the kernel of χ in $GZ_{\mathbb{R}}$.*

Proof. In the case of $\mathrm{SU}(p, q)$, the description of φ given above shows that the standard representation \mathbb{C}^{p+q} of $\mathrm{SU}(p, q)$, when seen as a representation of $\mathrm{SU}(1, n)$ via φ , splits as

$$\mathbb{C}^{p+q} = \mathbb{C}^{p+pn} \oplus \mathbb{C}^r = \mathbb{C}^{1+n} \otimes \mathbb{C}^p \oplus \mathbb{C}^r,$$

where \mathbb{C}^{1+n} is the standard representation of $\mathrm{SU}(1, n)$ and $r = q - pn$. To conclude, we argue as follows. Let $g \in Z_{\mathbb{R}}$. Then g yields an endomorphism of the $H_{\mathbb{R}}$ -module $\mathbb{C}^{1+n} \otimes \mathbb{C}^p \oplus \mathbb{C}^r$. Since by Schur's lemma such an endomorphism will preserve isotypic factors, we see that g must preserve the factors $\mathbb{C}^{1+n} \otimes \mathbb{C}^p$ and \mathbb{C}^r . Moreover it is known that it must act by an element of $\mathrm{U}(p)$ on the first factor, so that it belongs to $GZ_{\mathbb{R}}$.

In the case of $\mathrm{E}_{6(-14)}$, we use a model of the 27-dimensional representation \mathbb{E} given by Manivel in [Man06, Example 3 p.464]. There is a subgroup in $\mathrm{E}_{6(-14)}$ isomorphic to $\mathrm{SU}(2, 4) \times \mathrm{SU}(2)$ and \mathbb{E} splits as $\wedge^2 \mathbb{U} \oplus \mathbb{U} \otimes \mathbb{A}$, where \mathbb{U} resp. \mathbb{A} have complex dimension 6 resp. 2. Here we restrict further to $\mathrm{SU}(1, 2) \times \mathrm{SU}(2)$, where the first factor $\mathrm{SU}(1, 2)$ is diagonally embedded in $\mathrm{SU}(2, 4)$, meaning that the representation \mathbb{U} splits as $\mathbb{V} \otimes \mathbb{B}$, with $\dim \mathbb{V} = 3$ and $\dim \mathbb{B} = 2$. We get

$$\mathbb{E} \simeq \wedge^2(\mathbb{V} \otimes \mathbb{B}) \oplus \mathbb{V} \otimes \mathbb{A} \otimes \mathbb{B} \simeq \wedge^2 \mathbb{V} \otimes S^2 \mathbb{B} \oplus S^2 \mathbb{V} \otimes \wedge^2 \mathbb{B} \oplus \mathbb{V} \otimes \mathbb{A} \otimes \mathbb{B}.$$

As in the case of $\mathrm{SU}(p, q)$, an element g in the centralizer of $H_{\mathbb{Z}}$ will yield a $H_{\mathbb{Z}}$ -equivariant endomorphism, and will preserve each of these factors. Since it is an element of the group E_6 , one sees that it must be given by an element in $\mathrm{U}(\mathbb{A}) \times \mathrm{U}(\mathbb{B})$.

The computation of the character χ is done as follows. If $f = (x, y) \in \mathrm{U}(\mathbb{A}) \times \mathrm{U}(\mathbb{B})$, then the determinant of the action of f on \mathbb{E} is the product of the determinants of the actions of f on $\wedge^2(\mathbb{V} \otimes \mathbb{B})$ and on $\mathbb{V} \otimes \mathbb{A} \otimes \mathbb{B}$. The action on $\mathbb{V} \otimes \mathbb{A} \otimes \mathbb{B}$ has determinant $\det(x)^6 \det(y)^6$, and the action on $\wedge^2(\mathbb{V} \otimes \mathbb{B})$ has determinant $\det(y)^{15}$. \square

Remark 5.10. The compactness of $Z_{\mathbb{R}}$ is proved in greater generality in [BIW09, Theorem 3].

Lemma 5.11. *The fixator of $N = f(\mathbb{H}_{\mathbb{C}}^n)$ in $G_{\mathbb{R}}$ is exactly $Z_{\mathbb{R}}$. The stabilizer of N in $G_{\mathbb{R}}$ is the almost direct product $H_{\mathbb{R}} \cdot Z_{\mathbb{R}}$.*

Proof. Let $o = K_{\mathbb{R}} \in N$ be the base point of M . Let us denote by $\mathrm{Fix}(N) \subset G_{\mathbb{R}}$ the subgroup of elements which fix all the elements in N . We want to prove that $\mathrm{Fix}(N) = Z_{\mathbb{R}}$. We have an inclusion $Z_{\mathbb{R}} \subset \mathrm{Fix}(N)$. Indeed, if $h \in H_{\mathbb{R}}$ and $z \in Z_{\mathbb{R}}$, then $z \cdot o = o$ since $Z_{\mathbb{R}} \subset K_{\mathbb{R}}$. Thus, since g and h commute, $z \cdot (h \cdot o) = h \cdot (z \cdot o) = h \cdot o$.

The subgroup $H_{\mathbb{R}}$ may be defined referring only to N as follows. Let $\mathfrak{g}_{\mathbb{R}} = \mathfrak{k}_{\mathbb{R}} \oplus \mathfrak{p}_{\mathbb{R}}$ be the Cartan decomposition of $\mathfrak{g}_{\mathbb{R}}$. The tangent space $T_o N$ identifies with a subspace of $\mathfrak{p}_{\mathbb{R}}$ that we denote by $\mathfrak{q}_{\mathbb{R}}$. The space $\mathfrak{q}_{\mathbb{R}}$ defines a Lie triple system, so that $\mathfrak{h}_{\mathbb{R}} := [\mathfrak{q}_{\mathbb{R}}, \mathfrak{q}_{\mathbb{R}}] \oplus \mathfrak{q}_{\mathbb{R}} \subset \mathfrak{g}_{\mathbb{R}}$ is a Lie subalgebra. Then, $H_{\mathbb{R}}$ is the connected Lie group of $G_{\mathbb{R}}$ with Lie algebra $\mathfrak{h}_{\mathbb{R}}$.

For the reverse inclusion we need to prove that $\text{Fix}(N) \subset Z_{\mathbb{R}}$. It follows from the given description of $H_{\mathbb{R}}$ that $H_{\mathbb{R}}$ is normalized by $\text{Fix}(N)$. Let $g \in \text{Fix}(N)$ and $h \in H_{\mathbb{R}}$. Then the commutator $ghg^{-1}h^{-1}$ belongs to $H_{\mathbb{R}}$ and acts trivially on N . Thus, it belongs to the center of $H_{\mathbb{R}}$. Since this center is finite, the connexity of $H_{\mathbb{R}}$ implies that $ghg^{-1}h^{-1}$ is the neutral element.

Since the automorphism group of N is $H_{\mathbb{R}}$, the second assertion of the Lemma follows from the first. \square

Proof of Corollary C: The facts that ρ is discrete and faithful and that $\rho(\Gamma)$ acts cocompactly on N follow from the ρ -equivariance of the totally geodesic embedding f . The reductivity of ρ has been already asserted and the compactness of $Z_{\mathbb{R}}$ was established in Lemma 5.9. Now, given $\gamma \in \Gamma$, the equivariance of f w.r.t. ρ and φ means that $\rho(\gamma)$ and $\varphi(\gamma)$ have the same action on N . We let $\rho_{\text{cpt}}(\gamma) = \rho(\gamma)\varphi(\gamma)^{-1}$. This is an element of the fixator of N , which is equal to the centralizer $Z_{\mathbb{R}}$ of $H_{\mathbb{R}}$ by Lemma 5.11. Since $\varphi(\gamma) \in H_{\mathbb{R}}$ by definition of φ , the elements $\varphi(\gamma)$ and $\rho_{\text{cpt}}(\gamma)$ commute. It follows that $\varphi(\gamma)$ and $\rho(\gamma)$ commute, and that ρ_{cpt} is a morphism of groups. \square

Proof of the uniqueness of f : by the uniqueness statement for tight holomorphic totally geodesic maps $\mathbb{H}_{\mathbb{C}}^n \rightarrow M$, we know that if $f' : \mathbb{H}_{\mathbb{C}}^n \rightarrow M$ is another ρ -equivariant harmonic map, then there exists $g \in G_{\mathbb{R}}$ such that $f' = g \circ f$. By ρ -equivariance of f and f' , we have that

$$\rho(\gamma) \circ g(x) = g \circ \rho(\gamma)(x), \forall \gamma \in \Gamma \text{ and } \forall x \in N.$$

It follows that $g \cdot N$ is $\rho(\Gamma)$ -stable. Thus the map $d_{g \cdot N} : N \rightarrow \mathbb{R}, x \mapsto d(x, g \cdot N)$, where d denotes the distance in M , is invariant under the cocompact action of $\rho(\Gamma)$ on N . It is therefore bounded. Since it is moreover convex ([BH99, p. 178]), it is constant, equal to a , say. In the same way, the map $d_N : g \cdot N \rightarrow \mathbb{R}, y \mapsto d(y, N)$ is also constant equal to a .

If $a > 0$ it follows from the sandwich lemma ([BH99, p. 182]) that the convex hull of $N \cup g \cdot N$ in M is isometric to the product $N \times [0, a]$. This implies that there exists a tangent vector $v \in T_o M \simeq \mathfrak{p}_{\mathbb{R}}$, orthogonal to $T_o N \simeq \mathfrak{q}_{\mathbb{R}}$ such that $[v, u] = 0$ for all $u \in \mathfrak{q}_{\mathbb{R}}$. Indeed the norm (for the Killing form) of $[v, u] \in \mathfrak{g}_{\mathbb{R}}$ is up to a constant the sectional curvature of the plane generated by the tangent vectors u and v , which is 0 since they belong to different factors of a Riemannian product. In this case the 1-parameter group of transvections along the geodesic defined by v is included in the centralizer $Z_{\mathbb{R}}$ of $H_{\mathbb{R}}$, a contradiction since $Z_{\mathbb{R}}$ is compact.

Hence $a = 0$ and $g \cdot N = N$. Therefore there exist $h \in H_{\mathbb{R}}$ and $z \in Z_{\mathbb{R}}$ such that $g = hz = zh$. The above commutation relation between $\rho(\gamma) = \varphi(\gamma)\rho_{\text{cpt}}(\gamma)$ and g on N means that $\rho(\gamma)g\rho(\gamma)^{-1}g^{-1}$ fixes N pointwise and hence belongs to $Z_{\mathbb{R}}$ by Lemma 5.11. Hence for all $\gamma \in \Gamma$ we obtain that $\varphi(\gamma)h\varphi(\gamma)^{-1}h^{-1}$ belongs to $Z_{\mathbb{R}} \cap H_{\mathbb{R}}$ (recall that $\rho_{\text{cpt}}(\gamma) \in Z_{\mathbb{R}}$). Now Γ is Zariski dense in $\text{SU}(1, n)$ by the Borel density theorem and we deduce that $\varphi(x)h\varphi(x)^{-1}h^{-1} \in Z_{\mathbb{R}} \cap H_{\mathbb{R}}$ for all $x \in \text{SU}(1, n)$. Since $Z_{\mathbb{R}} \cap H_{\mathbb{R}}$ is finite and $\text{SU}(1, n)$ is connected, $h \in Z_{\mathbb{R}}$. Therefore $g \in Z_{\mathbb{R}}$ and $f' = f$. \square

Remark 5.12. If we drop the assumption that G is simply-connected, then \mathbb{E} might no longer be a representation of G and our constructions cannot be made. However, in this case, letting \tilde{G} be the simply connected cover of G and \mathbb{E} the cominuscul representation

of \tilde{G} that we have been considering, there is an integer k such that $\mathbb{E}^{\otimes k}$ is a representation of G . The arguments given in the article can be adapted with the representation $\mathbb{E}^{\otimes k}$ instead of \mathbb{E} , and the main results (Theorem A, Corollary B and Corollary C) remain true without the simple-connectedness assumption.

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