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# Laderman matrix multiplication algorithm can be constructed using Strassen algorithm and related tensor's isotropies

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## 1 Introduction

In [21], V. Strassen presented a noncommutative algorithm for multiplication of two  $2 \times 2$  matrices using only 7 multiplications. The current upper bound 23 for  $3 \times 3$  matrix multiplication was reached by J.B. Laderman in [16]. This note presents a *geometric* relationship between Strassen and Laderman algorithms. By doing so, we retrieve a *geometric* formulation of results very similar to those presented by O. Sýkora in [22].

### 1.1 Disclaimer: there is no improvement in this note

We do not improve any practical algorithm or prove any theoretical bound in this short note but focus on effective manipulation of tensor associated to matrix multiplication algorithm. To do so, we present only the minimal number of needed definitions and thus leave many facts outside our scope. We refer to [17] for a complete description of the field and to [1] for a state-of-the-art presentation of theoretical complexity issues.

### 1.2 So, why writing (or reading) it?

We follow the geometric spirit of [12, 8, 4, 6, 5] and related papers: symmetries could be used in practical design of matrix multiplication algorithms. Hence, this note presents another example of this philosophy by giving a precise geometric meaning to the following statement:

Laderman matrix multiplication algorithm is composed by four  $2 \times 2$  optimal matrix multiplication algorithms, a half of the classical  $2 \times 2$  matrix multiplication algorithm and a correction term.

## 2 Framework

To do so, we have to present a small part of the classical framework (for a complete presentation see [13, 14, 17]) mainly because we do not take it literally

and only use a simplified version. Let us start by some basic definitions and notations as the following generic matrices:

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}, B = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix}, C = \begin{pmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{pmatrix}, \quad (1)$$

that will be used in the sequel. Furthermore, as we also consider their  $2 \times 2$  submatrices, let us introduce some associated notations.

**Notations 2.1** Let  $n, i, j$  be positive integers such that  $i \leq n$  and  $j \leq n$ . We denote by  $\text{Id}_{n \times n}^j$  the identity  $n \times n$  matrix where the  $j$ th diagonal term is 0. Given a  $n \times n$  matrix  $A$ , we denote by  $\tilde{A}^{jk}$  the matrix  $\text{Id}_{n \times n}^j \cdot A \cdot \text{Id}_{n \times n}^k$ . For example, the matrix  $\tilde{A}^{33}$ ,  $\tilde{B}^{32}$  and  $\tilde{C}^{23}$  are:

$$\begin{pmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} b_{11} & 0 & b_{13} \\ b_{21} & 0 & b_{23} \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} c_{11} & c_{12} & 0 \\ 0 & 0 & 0 \\ c_{31} & c_{32} & 0 \end{pmatrix}. \quad (2)$$

Given a  $n \times n$  matrix  $A$ , we sometimes consider  $\tilde{A}^{ij}$  as the  $(n-1) \times (n-1)$  matrix  $\overline{A}^{ij}$  where the line and column composed of 0 are removed.

At the opposite, given any  $(n-1) \times (n-1)$  matrix  $A$ , we denote by  $\underline{A}_{ij}$  the  $n \times n$  matrix where a line and column of 0 were added to  $A$  in order to have  $\overline{\underline{A}_{ij}}^{ij} = A$ .

## 2.1 Strassen multiplication algorithm

Considered as  $2 \times 2$  matrices, the matrix product  $\overline{C}^{33} = \overline{A}^{33} \cdot \overline{B}^{33}$  could be computed using Strassen algorithm (see [21]) by performing the following computations:

$$\begin{aligned} t_1 &= (a_{11} + a_{22})(b_{11} + b_{22}), & t_2 &= (a_{12} - a_{22})(b_{21} + b_{22}), \\ t_3 &= (-a_{11} + a_{21})(b_{11} + b_{12}), & t_4 &= (a_{11} + a_{12})b_{22}, \\ t_5 &= a_{11}(b_{12} - b_{22}), & t_6 &= a_{22}(-b_{11} + b_{21}), & t_7 &= (a_{21} + a_{22})b_{11}, \\ c_{11} &= t_1 + t_2 - t_4 + t_6, & c_{12} &= t_6 + t_7, \\ c_{21} &= t_4 + t_5, & c_{22} &= t_1 + t_3 + t_5 - t_7. \end{aligned} \quad (3)$$

In order to consider above algorithm under a geometric standpoint, it is usually presented as a tensor.

## 2.2 Bilinear mappings seen as tensors and associated trilinear forms

**Definitions 2.1** Given a tensor  $\mathcal{T}$  decomposable as sum of rank-one tensors:

$$\mathcal{T} = \sum_{i=1}^r T_{i1} \otimes T_{i2} \otimes T_{i3}, \quad (4)$$

where  $T_{ij}$  are  $n \times n$  matrices:

- the integer  $r$  is the tensor rank of tensor  $\mathcal{T}$ ;
- the unordered list  $[(\text{rank } M_{ij})_{j=1 \dots 3}]_{i=1 \dots r}$  is called the type of tensor  $\mathcal{T}$  (rank  $A$  being the classical rank of the matrix  $A$ ).

### 2.3 Tensors' contractions

To explicit the relationship between what is done in the sequel and the bilinear mapping associated to matrix multiplication, let us consider the following tensor's contractions:

**Definitions 2.2** Using the notation of definition 2.1 given a tensor  $\mathcal{T}$  and three  $n \times n$  matrices  $A, B$  and  $C$  with coefficients in the algebra  $\mathbb{K}$ :

- the (1, 2) contraction of  $\mathcal{T} \otimes A \otimes B$  defined by:

$$\sum_{i=1}^r \text{Trace}({}^tT_{i1} \cdot A) \text{Trace}({}^tT_{i2} \cdot B) T_{i3} \quad (5)$$

corresponds to a bilinear application  $\mathbb{K}^{n \times n} \times \mathbb{K}^{n \times n} \mapsto \mathbb{K}^{n \times n}$  with indeterminates  $A$  and  $B$ .

- the (1, 2, 3) (a.k.a. full) contraction of  $\mathcal{T} \otimes A \otimes B \otimes C$  defined by:

$$\langle \mathcal{T} | A \otimes B \otimes C \rangle = \sum_{i=1}^r \text{Trace}({}^tT_{i1} \cdot A) \text{Trace}({}^tT_{i2} \cdot B) \text{Trace}({}^tT_{i3} \cdot C) \quad (6)$$

corresponds to a trilinear form  $\mathbb{K}^{n \times n} \times \mathbb{K}^{n \times n} \times \mathbb{K}^{n \times n} \mapsto \mathbb{K}$  with indeterminates  $A, B$  and  $C$ .

**Remarks 2.1** As the studied object is the tensor, its expressions as full or incomplete contractions are equivalent. Thus, even if matrix multiplication is a bilinear application, we are going to work in the sequel with trilinear forms (see [10] for bibliographic references on this standpoint).

The definition in 2.2 are taken to express the full contraction as a degenerate inner product between tensors; it is not the usual choice made in the literature and so, we have to explicitly recall some notions used in the sequel.

Strassen multiplication algorithm (3) is equivalent to the tensor  $\mathcal{S}$  defined by:

$$\begin{aligned} & \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \\ & \begin{pmatrix} -1 & 0 \\ 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} -1 & 0 \\ 1 & 0 \end{pmatrix} + \\ & \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} -1 & 0 \\ 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} + \\ & \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix}. \end{aligned} \quad (7)$$

This tensor defines the matrix multiplication algorithm (3) and its tensor rank is 7.

### 2.4 $2 \times 2$ matrix multiplication tensors induced by a $3 \times 3$ matrix multiplication tensor

Given any  $3 \times 3$  matrix multiplication tensor, one can define  $3^3$  induced  $2 \times 2$  matrix multiplication tensors as shown in this section. First, let us introduce the following operators that generalize to tensor the notations 2.1:

**Definitions 2.3** Using notations introduced in definition 2.1, we define:

$$A \otimes \widetilde{B} \otimes C^{ijk} = \widetilde{A}^{ij} \otimes \widetilde{B}^{jk} \otimes \widetilde{C}^{ki}, \quad (8a)$$

$$\overline{A \otimes B \otimes C}^{ijk} = \overline{A}^{ij} \otimes \overline{B}^{jk} \otimes \overline{C}^{ki}, \quad (8b)$$

$$\underline{A \otimes B \otimes C}_{ijk} = \underline{A}_{ij} \otimes \underline{B}_{jk} \otimes \underline{C}_{ki} \quad (8c)$$

and we extend the definitions of these operators by additivity in order to be applied on any tensor  $\mathcal{T}$  described in definition 2.1.

There is  $n^3$  such projections and given any matrix multiplication tensor  $\mathcal{M}$ , the full contraction satisfying the following trivial properties:

$$\left\langle \mathcal{M} \mid A \otimes \widetilde{B} \otimes C^{ijk} \right\rangle = \left\langle \widetilde{\mathcal{M}}^{ijk} \mid A \otimes B \otimes C \right\rangle = \left\langle \overline{\mathcal{M}}^{ijk} \mid \overline{A \otimes B \otimes C}^{ijk} \right\rangle \quad (9)$$

(where the projection operator apply on an  $n \times n$  matrix multiplication tensor); it defines explicitly a  $(n-1) \times (n-1)$  matrix multiplication tensor.

The following property holds:

**Lemma 2.1**

$$(n-1)^3 \langle \mathcal{M} \mid A \otimes B \otimes C \rangle = \sum_{1 \leq i, j, k \leq n} \left\langle \mathcal{M} \mid A \otimes \widetilde{B} \otimes C^{ijk} \right\rangle \quad (10)$$

and thus, we have:

$$\langle \mathcal{M} \mid A \otimes B \otimes C \rangle = \left\langle \frac{1}{(n-1)^3} \sum_{1 \leq i, j, k \leq n} \widetilde{\mathcal{M}}^{ijk} \mid A \otimes B \otimes C \right\rangle. \quad (11)$$

The obvious facts made in this section underline the relationships between any  $n \times n$  matrix multiplication tensor and the  $n^3$  induced  $(n-1) \times (n-1)$  algorithms.

Considering the Laderman matrix multiplication tensor, we are going to explore further this kind of relationships. First, let us introduce this tensor.

## 2.5 Laderman matrix multiplication tensor

The Laderman tensor  $\mathcal{L}$  described below by giving its full contraction:

$$\begin{aligned}
& (a_{11} - a_{21} + a_{12} - a_{22} - a_{32} + a_{13} - a_{33}) b_{22} c_{21} & + \\
& a_{22} (-b_{11} + b_{21} - b_{31} + b_{12} - b_{22} - b_{23} + b_{33}) c_{12} & + \\
& a_{13} b_{31} (c_{11} + c_{21} + c_{31} + c_{12} + c_{32} + c_{13} + c_{23}) & + \\
& (a_{11} - a_{31} + a_{12} - a_{22} - a_{32} + a_{13} - a_{23}) b_{23} c_{31} & + \\
& a_{32} (-b_{11} + b_{21} - b_{31} - b_{22} + b_{32} + b_{13} - b_{23}) c_{13} & + \\
& a_{11} b_{11} (c_{11} + c_{21} + c_{31} + c_{12} + c_{22} + c_{13} + c_{33}) & + \\
& (-a_{11} + a_{31} + a_{32}) (b_{11} - b_{13} + b_{23}) (c_{31} + c_{13} + c_{33}) & + \\
& (a_{22} - a_{13} + a_{23}) (b_{31} + b_{23} - b_{33}) (c_{31} + c_{12} + c_{32}) & + \\
& (-a_{11} + a_{21} + a_{22}) (b_{11} - b_{12} + b_{22}) (c_{21} + c_{12} + c_{22}) & + \\
& (a_{32} - a_{13} + a_{33}) (b_{31} + b_{22} - b_{32}) (c_{21} + c_{13} + c_{23}) & + \quad (12) \\
& (a_{21} + a_{22}) (-b_{11} + b_{12}) (c_{21} + c_{22}) & + \\
& (a_{31} + a_{32}) (-b_{11} + b_{13}) (c_{31} + c_{33}) & + \\
& (a_{13} - a_{33}) (b_{22} - b_{32}) (c_{13} + c_{23}) & + \\
& (a_{11} - a_{21}) (-b_{12} + b_{22}) (c_{12} + c_{22}) & + \\
& (a_{32} + a_{33}) (-b_{31} + b_{32}) (c_{21} + c_{23}) & + \\
& (-a_{11} + a_{31}) (b_{13} - b_{23}) (c_{13} + c_{33}) & + \\
& (a_{13} - a_{23}) (b_{23} - b_{33}) (c_{12} + c_{32}) & + \\
& (a_{22} + a_{23}) (-b_{31} + b_{33}) (c_{31} + c_{32}) & + \\
& a_{12} b_{21} c_{11} + a_{23} b_{32} c_{22} + a_{21} b_{13} c_{32} + a_{31} b_{12} c_{23} + a_{33} b_{33} c_{33} & 
\end{aligned}$$

and was introduced in [16] (we do not study in this note any other *inequivalent* algorithm of same tensor rank e.g. [15, 9, 19, 20], etc). Considering the projections introduced in definition 2.3, we notice that:

**Remark 2.2** *Considering definitions introduced in Section 2.4, we notice that Laderman matrix multiplication tensor defines 4 optimal  $2 \times 2$  matrix multiplication tensors  $\overline{\mathcal{L}}^{ijk}$  with  $(i, j, k)$  in  $\{(2, 1, 3), (2, 3, 2), (3, 1, 2), (3, 3, 3)\}$  and 23 other with tensor rank 8.*

Further computations show that:

**Remark 2.3** *The type of the Laderman matrix multiplication tensor is*

$$[(2, 2, 2)|4, ((1, 3, 1), (3, 1, 1), (1, 1, 3))|2, (1, 1, 1)|13] \quad (13)$$

where  $m|n$  indicates that  $m$  is repeated  $n$  times.

## 2.6 Tensors' isotropies

We refer to [13, 14] for a complete presentation of automorphism group operating on varieties defined by algorithms for computation of bilinear mappings and as a reference for the following theorem:

**Theorem 2.1** *The isotropy group of the  $n \times n$  matrix multiplication tensor is*

$$\mathrm{PGL}(\mathbb{C}^n)^{\times 3} \rtimes \mathfrak{S}_3, \quad (14)$$

where  $\mathrm{PGL}$  stands for the projective linear group and  $\mathfrak{S}_3$  for the symmetric group on 3 elements.

Even if we do not completely explicit the concrete action of this isotropy group on matrix multiplication tensor, let us precise some terminologies:

**Definitions 2.4** *Given a tensor defining matrix multiplication computations, the orbit of this tensor is called the multiplication algorithm and any of the points composing this orbit is a variant of this algorithm.*

**Remark 2.4** *As shown in [11], matrix multiplication is characterised by its isotropy group.*

**Remark 2.5** *In this note, we only need the  $\text{PGL}(\mathbb{C}^n)^{\times 3}$  part of this group (a.k.a. sandwiching) and thus focus on it in the sequel.*

As our framework and notations differ slightly from the framework classically found in the literature, we have to explicitly define several well-known notions for the sake of clarity. Hence, let us recall the *sandwiching* action:

**Definition 2.5** *Given  $\mathbf{g} = (G_1 \times G_2 \times G_3)$  an element of  $\text{PGL}(\mathbb{C}^n)^{\times 3}$ , its action on a tensor  $\mathcal{T}$  is given by:*

$$\mathbf{g} \diamond \mathcal{T} = \sum_{i=1}^r \mathbf{g} \diamond (T_{i1} \otimes T_{i2} \otimes T_{i3}), \quad (15)$$

$$\mathbf{g} \diamond (T_{i1} \otimes T_{i2} \otimes T_{i3}) = ({}^t G_1^{-1} T_{i1} {}^t G_2) \otimes ({}^t G_2^{-1} T_{i2} {}^t G_3) \otimes ({}^t G_3^{-1} T_{i3} {}^t G_1).$$

**Example 2.1** *Let us consider the action of the following isotropy*

$$\begin{pmatrix} 0 & 1/\lambda \\ -1 & 0 \end{pmatrix} \times \begin{pmatrix} 1/\lambda & -1/\lambda \\ 0 & 1 \end{pmatrix} \times \begin{pmatrix} -1/\lambda & 0 \\ 1 & -1 \end{pmatrix} \quad (16)$$

*on the Strassen variant of the Strassen algorithm. The resulting tensor  $\mathcal{W}$  is:*

$$\begin{aligned} \sum_{i=1}^7 w_i &= \begin{pmatrix} -1 & \lambda \\ -\frac{1}{\lambda} & 0 \end{pmatrix} \otimes \begin{pmatrix} \frac{1}{\lambda} & -\lambda \\ \frac{1}{\lambda} & 0 \end{pmatrix} \otimes \begin{pmatrix} \frac{1}{\lambda} & -\lambda \\ \frac{1}{\lambda} & 0 \end{pmatrix} + \begin{pmatrix} -1 & 1 \\ -\frac{1}{\lambda} & 1 \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \\ &+ \begin{pmatrix} \frac{1}{\lambda} & 0 \\ \frac{1}{\lambda} & 0 \end{pmatrix} \otimes \begin{pmatrix} \frac{1}{\lambda} & 0 \\ \frac{1}{\lambda} & 0 \end{pmatrix} \otimes \begin{pmatrix} \frac{1}{\lambda} & 0 \\ \frac{1}{\lambda} & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \\ &+ \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} -1/\lambda & \lambda \\ -\frac{1}{\lambda} & 1 \end{pmatrix} + \begin{pmatrix} 1 & -\lambda \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & -\lambda \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & -\lambda \\ 0 & 0 \end{pmatrix} \\ &+ \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} -1 & \lambda \\ -\frac{1}{\lambda} & 1 \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \end{aligned} \quad (17)$$

that is the well-known Winograd variant of Strassen algorithm.

**Remarks 2.6** *We keep the parameter  $\lambda$  useless in our presentation as a tribute to the construction made in [7] that gives an elegant and elementary (i.e. based on matrix eigenvalues) construction of Winograd variant of Strassen matrix multiplication algorithm.*

*This variant is remarkable in its own as shown in [3] because it is optimal w.r.t. multiplicative and additive complexity.*

**Remark 2.7** *Tensor's type is an invariant of isotropy's action. Hence, two tensors in the same orbit share the same type. Or equivalently, two tensors with the same type are two variants that represent the same matrix multiplication algorithm.*

This remark will allow us in Section 3.4 to recognise the tensor constructed below as a variant of the Laderman matrix multiplication algorithm.

### 3 A tensor's construction

Let us now present the construction of a variant of Laderman matrix multiplication algorithm based on Winograd variant of Strassen matrix multiplication algorithm.

First, let us give the full contraction of the tensor  $\underline{\mathcal{W}}_{111} \otimes A \otimes B \otimes C$ :

$$\left(-a_{22} - \frac{a_{32}}{\lambda} + \lambda a_{23}\right) \left(b_{22} + \frac{b_{32}}{\lambda} - \lambda b_{23}\right) \left(c_{22} + \frac{c_{32}}{\lambda} - \lambda c_{23}\right) + \quad (18a)$$

$$(a_{22} - \lambda a_{23})(b_{22} - \lambda b_{23})(c_{22} - \lambda c_{23}) + \quad (18b)$$

$$\left(a_{22} + \frac{a_{32}}{\lambda}\right) \left(b_{22} + \frac{b_{32}}{\lambda}\right) \left(c_{22} + \frac{c_{32}}{\lambda}\right) + \quad (18c)$$

$$a_{23} \left(-b_{22} - \frac{b_{32}}{\lambda} + \lambda b_{23} + b_{33}\right) c_{32} + \quad (18d)$$

$$\left(-a_{22} - \frac{a_{32}}{\lambda} + \lambda a_{23} + a_{33}\right) b_{32} c_{23} + \quad (18e)$$

$$a_{32} b_{23} \left(-c_{22} - \frac{c_{32}}{\lambda} + \lambda c_{23} + c_{33}\right) + \quad (18f)$$

$$a_{33} b_{33} c_{33} \quad (18g)$$

#### 3.1 A Klein four-group of isotropies

Let us introduce now the following notations:

$$\text{Id}_{3 \times 3} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad P_{(12)} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (19)$$

used to defined the following group of isotropies:

$$\mathbf{K} = \left\{ \begin{array}{ll} \mathbf{g}_1 = \text{Id}_{3 \times 3}^{\times 3}, & \mathbf{g}_2 = (\text{Id}_{3 \times 3} \times P_{(12)} \times P_{(12)}), \\ \mathbf{g}_3 = (P_{(12)} \times P_{(12)} \times \text{Id}_{3 \times 3}), & \mathbf{g}_4 = (P_{(12)} \times \text{Id}_{3 \times 3} \times P_{(12)}) \end{array} \right\} \quad (20)$$

that is isomorphic to the Klein four-group.

#### 3.2 Its action on Winograd variant of Strassen algorithm

In the sequel, we are interested in the action of Klein four-group (20) on our Winograd variant of Strassen algorithm:

$$\mathbf{K} \diamond \underline{\mathcal{W}}_{111} = \sum_{\mathbf{g} \in \mathbf{K}} \mathbf{g} \diamond \underline{\mathcal{W}}_{111} = \sum_{\mathbf{g} \in \mathbf{K}} \sum_{i=1}^7 \mathbf{g} \diamond w_{i111} \quad (21)$$

As we have for any isotropy  $\mathbf{g}$ :

$$\langle \mathbf{g} \diamond \underline{\mathcal{W}}_{111} | A \otimes B \otimes C \rangle = \langle \underline{\mathcal{W}}_{111} | \mathbf{g} \diamond (A \otimes B \otimes C) \rangle, \quad (22)$$



the action of isotropies  $\mathfrak{g}_i$  is just a permutation of our generic matrix coefficients. Hence, we have the full contraction of the tensor  $(\mathfrak{g}_2 \diamond \underline{\mathcal{W}}_{111}) \otimes A \otimes B \otimes C$ :

$$\left(-a_{21} - \frac{a_{31}}{\lambda} + \lambda a_{23}\right) \left(b_{11} + \frac{b_{31}}{\lambda} - \lambda b_{13}\right) \left(c_{12} + \frac{c_{32}}{\lambda} - \lambda c_{13}\right) + \quad (23a)$$

$$(a_{21} - \lambda a_{23}) (b_{11} - \lambda b_{13}) (c_{12} - \lambda c_{13}) + \quad (23b)$$

$$\left(a_{21} + \frac{a_{31}}{\lambda}\right) \left(b_{11} + \frac{b_{31}}{\lambda}\right) \left(c_{12} + \frac{c_{32}}{\lambda}\right) + \quad (23c)$$

$$a_{23} \left(-b_{11} - \frac{b_{31}}{\lambda} + \lambda b_{13} + b_{33}\right) c_{32} + \quad (23d)$$

$$\left(-a_{21} - \frac{a_{31}}{\lambda} + \lambda a_{23} + a_{33}\right) b_{31} c_{13} + \quad (23e)$$

$$a_{31} b_{13} \left(-c_{12} - \frac{c_{32}}{\lambda} + \lambda c_{13} + c_{33}\right) + \quad (23f)$$

$$a_{33} b_{33} c_{33}, \quad (23g)$$

the full contraction of the tensor  $(\mathfrak{g}_3 \diamond \underline{\mathcal{W}}_{111}) \otimes A \otimes B \otimes C$ :

$$\left(-a_{11} - \frac{a_{31}}{\lambda} + \lambda a_{13}\right) \left(b_{12} + \frac{b_{32}}{\lambda} - \lambda b_{13}\right) \left(c_{21} + \frac{c_{31}}{\lambda} - \lambda c_{23}\right) + \quad (24a)$$

$$(a_{11} - \lambda a_{13}) (b_{12} - \lambda b_{13}) (c_{21} - \lambda c_{23}) + \quad (24b)$$

$$\left(a_{11} + \frac{a_{31}}{\lambda}\right) \left(b_{12} + \frac{b_{32}}{\lambda}\right) \left(c_{21} + \frac{c_{31}}{\lambda}\right) + \quad (24c)$$

$$a_{13} \left(-b_{12} - \frac{b_{32}}{\lambda} + \lambda b_{13} + b_{33}\right) c_{31} + \quad (24d)$$

$$\left(-a_{11} - \frac{a_{31}}{\lambda} + \lambda a_{13} + a_{33}\right) b_{32} c_{23} + \quad (24e)$$

$$a_{31} b_{13} \left(-c_{21} - \frac{c_{31}}{\lambda} + \lambda c_{23} + c_{33}\right) + \quad (24f)$$

$$a_{33} b_{33} c_{33} \quad (24g)$$

and the full contraction of the tensor  $(\mathfrak{g}_4 \diamond \underline{\mathcal{W}}_{111}) \otimes A \otimes B \otimes C$ :

$$\left(-a_{12} - \frac{a_{32}}{\lambda} + \lambda a_{13}\right) \left(b_{21} + \frac{b_{31}}{\lambda} - \lambda b_{23}\right) \left(c_{11} + \frac{c_{31}}{\lambda} - \lambda c_{13}\right) + \quad (25a)$$

$$(a_{12} - \lambda a_{13}) (b_{21} - \lambda b_{23}) (c_{11} - \lambda c_{13}) + \quad (25b)$$

$$\left(a_{12} + \frac{a_{32}}{\lambda}\right) \left(b_{21} + \frac{b_{31}}{\lambda}\right) \left(c_{11} + \frac{c_{31}}{\lambda}\right) + \quad (25c)$$

$$a_{13} \left(-b_{21} - \frac{b_{31}}{\lambda} + \lambda b_{23} + b_{33}\right) c_{31} + \quad (25d)$$

$$\left(-a_{12} - \frac{a_{32}}{\lambda} + \lambda a_{13} + a_{33}\right) b_{31} c_{13} + \quad (25e)$$

$$a_{32} b_{23} \left(-c_{11} - \frac{c_{31}}{\lambda} + \lambda c_{13} + c_{33}\right) + \quad (25f)$$

$$a_{33} b_{33} c_{33}. \quad (25g)$$

There is several noteworthy points in theses expressions:

**Remarks 3.1** • *the term (18g) is a fixed point of  $\mathbb{K}$ 's action;*

- the trilinear terms (18d) and (23d), (18e) and (24e), (18f) and (25f), (23e) and (25e), (23f) and (24f), (24d) and (25d) could be added in order to obtain new rank-one tensors without changing the tensor rank. For example (18d)+(23d) is equal to:

$$a_{23} \left( -b_{22} - \frac{b_{32}}{\lambda} + \lambda b_{23} + 2b_{33} - b_{11} - \frac{b_{31}}{\lambda} + \lambda b_{13} \right) c_{32}. \quad (26)$$

The tensor rank of the tensor  $\mathbf{K} \diamond \mathcal{W}_{111} = \sum_{\mathbf{g} \in \mathbf{K}} \mathbf{g} \diamond \mathcal{W}_{111}$  is  $1 + 3 \cdot 4 + 6 = 19$ . Unfortunately, this tensor does not define a matrix multiplication algorithm (otherwise according to the lower bound presented in [2], it would be optimal and this note would have another title and impact).

In the next section, after studying the action of isotropy group  $\mathbf{K}$  on the classical matrix multiplication algorithm, we are going to show how the tensor constructed above take place in construction of matrix multiplication tensor.

### 3.3 How far are we from a multiplication tensor?

Let us consider the classical  $3 \times 3$  matrix multiplication algorithm

$$\mathcal{M} = \sum_{1 \leq i, j, k \leq 3} e_j^i \otimes e_k^j \otimes e_i^k \quad (27)$$

where  $e_j^i$  denotes the matrix with a single non-zero coefficient 1 at the intersection of line  $i$  and column  $j$ . By considering the trilinear monomial:

$$a_{ij} b_{jk} c_{ki} = \left\langle e_j^i \otimes e_k^j \otimes e_i^k \mid A \otimes B \otimes C \right\rangle, \quad (28)$$

we describe below the action of an isotropy  $\mathbf{g}$  on this tensor by the induced action:

$$\begin{aligned} \mathbf{g} \diamond a_{ij} b_{jk} c_{ki} &= \left\langle \mathbf{g} \diamond (e_j^i \otimes e_k^j \otimes e_i^k) \mid A \otimes B \otimes C \right\rangle, \\ &= \left\langle e_j^i \otimes e_k^j \otimes e_i^k \mid \mathbf{g} \diamond (A \otimes B \otimes C) \right\rangle. \end{aligned} \quad (29)$$

**Remark 3.2** *The isotropies in  $\mathbf{K}$  act as a permutation on rank-one component of the tensor  $\mathcal{M}$ : we say that the group  $\mathbf{K}$  is a stabilizer of  $\mathcal{M}$ . More precisely, we have the following 9 orbits represented by the trilinear monomial sums:*

$$\sum_{i=1}^4 \mathfrak{g}_i \diamond a_{11} b_{11} c_{11} = a_{11} b_{11} c_{11} + a_{12} b_{22} c_{21} + a_{22} b_{21} c_{12} + a_{21} b_{12} c_{22}, \quad (30a)$$

$$\sum_{i=1}^4 \mathfrak{g}_i \diamond a_{22} b_{22} c_{22} = a_{22} b_{22} c_{22} + a_{21} b_{11} c_{12} + a_{11} b_{12} c_{21} + a_{12} b_{21} c_{11}, \quad (30b)$$

$$\sum_{i=1}^4 \mathfrak{g}_i \diamond a_{22} b_{23} c_{32} = a_{22} b_{23} c_{32} + a_{21} b_{13} c_{32} + a_{11} b_{13} c_{31} + a_{12} b_{23} c_{31}, \quad (30c)$$

$$\sum_{i=1}^4 \mathfrak{g}_i \diamond a_{23} b_{32} c_{22} = a_{23} b_{32} c_{22} + a_{23} b_{31} c_{12} + a_{13} b_{32} c_{21} + a_{13} b_{31} c_{11}, \quad (30d)$$

$$\sum_{i=1}^4 \mathfrak{g}_i \diamond a_{32} b_{22} c_{23} = a_{32} b_{22} c_{23} + a_{31} b_{11} c_{13} + a_{31} b_{12} c_{23} + a_{32} b_{21} c_{13}, \quad (30e)$$

$$\frac{1}{2} \sum_{i=1}^4 \mathfrak{g}_i \diamond a_{23} b_{33} c_{32} = a_{23} b_{33} c_{32} + a_{13} b_{33} c_{31}, \quad (30f)$$

$$\frac{1}{2} \sum_{i=1}^4 \mathfrak{g}_i \diamond a_{32} b_{23} c_{33} = a_{32} b_{23} c_{33} + a_{31} b_{13} c_{33}, \quad (30g)$$

$$\frac{1}{2} \sum_{i=1}^4 \mathfrak{g}_i \diamond a_{33} b_{32} c_{23} = a_{33} b_{32} c_{23} + a_{33} b_{31} c_{13}, \quad (30h)$$

$$\frac{1}{4} \sum_{i=1}^4 \mathfrak{g}_i \diamond a_{33} b_{33} c_{33} = a_{33} b_{33} c_{33}. \quad (30i)$$

Hence, the action of  $\mathbf{K}$  decomposes the classical matrix multiplication tensor  $\mathcal{M}$  as a transversal action of  $\mathbf{K}$  on the implicit projection  $\widetilde{\mathcal{M}}^{111}$ , its action on the rank-one tensor  $e_1^1 \otimes e_1^1 \otimes e_1^1$  and a correction term also related to orbits under  $\mathbf{K}$ :

$$\begin{aligned} \mathcal{M} &= \mathbf{K} \diamond (e_1^1 \otimes e_1^1 \otimes e_1^1) + \mathbf{K} \diamond \widetilde{\mathcal{M}}^{111} - \mathcal{R}, \\ \mathcal{R} &= (1/2) \mathbf{K} \diamond (e_3^2 \otimes e_3^3 \otimes e_2^2) + (1/2) \mathbf{K} \diamond (e_3^3 \otimes e_2^2 \otimes e_3^2) \\ &\quad + (1/2) \mathbf{K} \diamond (e_2^2 \otimes e_3^2 \otimes e_3^3) + 3 \mathbf{K} \diamond (e_3^3 \otimes e_3^3 \otimes e_3^3). \end{aligned} \quad (31)$$

### 3.4 Resulting matrix multiplication algorithm

The term  $\widetilde{\mathcal{M}}^{111}$  is a  $2 \times 2$  matrix multiplication algorithm that could be replaced by any other one. Choosing  $\underline{\mathcal{W}}_{111}$ , we have the following properties:

- the tensor rank of  $\mathbf{K} \diamond \underline{\mathcal{W}}_{111}$  is 19;
- its addition with the correction term  $\mathcal{R}$  does not change its tensor rank.

Hence, we obtain a matrix multiplication tensor with rank  $23(= 19 + 4)$ . Furthermore, the resulting tensor have the same type than the Laderman matrix multiplication tensor, and thus it is a variant of the same algorithm.

We conclude that the Laderman matrix multiplication algorithm can be constructed using the orbit of an optimal  $2 \times 2$  matrix multiplication algorithm under the action of a given group leaving invariant classical  $3 \times 3$  matrix multiplication variant/algorithm and with a transversal action on one of its projections.

## 4 Concluding remarks

All the observations presented in this short note came from an experimental mathematical approach using the computer algebra system Maple [18]. While implementing effectively (if not efficiently) several tools needed to manipulate matrix multiplication tensor—tensors, their isotropies and contractions, etc.—in order to understand the theory, the relationship between the Laderman matrix multiplication algorithm and the Strassen algorithm became clear by simple computations that will be tedious or impossible by hand.

As already shown in [22], this kind of geometric configuration could be found and used with other matrix size.

The main opinion supported by this work is that symmetries play a central role in effective computation for matrix multiplication algorithm and that only a geometrical interpretation may bring further improvement.

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## A A cyclic isotropy group of order 4 leading to same resulting tensor

Instead of Klein-four group  $K$  presented in Section 3.1, one can also use another cyclic group  $C$  of order 4 that is a stabilizer of  $\mathcal{L}$  but such that its generator  $f$

is not a sandwiching. We do not give any further details here to avoid supplementary definitions but nevertheless, we present an example of the resulting 8 orbits, again represented by trilinear monomial sums:

$$\sum_{i=1}^4 f^i \diamond a_{11} b_{11} c_{11} = a_{11} b_{11} c_{11} + a_{12} b_{22} c_{21} + a_{22} b_{21} c_{12} + a_{21} b_{12} c_{22}, \quad (32a)$$

$$\sum_{i=1}^4 f^i \diamond a_{22} b_{22} c_{22} = a_{22} b_{22} c_{22} + a_{21} b_{11} c_{12} + a_{11} b_{12} c_{21} + a_{12} b_{21} c_{11}, \quad (32b)$$

$$\sum_{i=1}^4 f^i \diamond a_{22} b_{23} c_{32} = a_{22} b_{23} c_{32} + a_{21} b_{13} c_{32} + a_{11} b_{13} c_{31} + a_{12} b_{23} c_{31}, \quad (32c)$$

$$\sum_{i=1}^4 f^i \diamond a_{32} b_{22} c_{23} = a_{32} b_{22} c_{23} + a_{31} b_{12} c_{23} + a_{23} b_{31} c_{12} + a_{13} b_{31} c_{11}, \quad (32d)$$

$$\sum_{i=1}^4 f^i \diamond a_{23} b_{32} c_{22} = a_{23} b_{32} c_{22} + a_{31} b_{11} c_{13} + a_{13} b_{32} c_{21} + a_{32} b_{21} c_{13}, \quad (32e)$$

$$\sum_{i=1}^4 f^i \diamond a_{32} b_{23} c_{33} = a_{32} b_{23} c_{33} + a_{31} b_{13} c_{33} + a_{23} b_{33} c_{32} + a_{13} b_{33} c_{31}, \quad (32f)$$

$$\frac{1}{2} \sum_{i=1}^4 f^i \diamond a_{33} b_{32} c_{23} = a_{33} b_{32} c_{23} + a_{33} b_{31} c_{13}, \quad (32g)$$

$$\frac{1}{4} \sum_{i=1}^4 f^i \diamond a_{33} b_{33} c_{33} = a_{33} b_{33} c_{33}. \quad (32h)$$

Hence, the action of  $\mathbf{C}$  decomposes the classical matrix multiplication tensor  $\mathcal{M}$  as a transversal action of  $\mathbf{C}$  on the implicit projection  $\widetilde{\mathcal{M}}^{111}$ , its action on the rank-one tensor  $e_1^1 \otimes e_1^1 \otimes e_1^1$  and a correction term also related to orbits under  $\mathbf{C}$ :

$$\begin{aligned} \mathcal{M} &= \mathbf{C} \diamond (e_1^1 \otimes e_1^1 \otimes e_1^1) + \mathbf{C} \diamond \widetilde{\mathcal{M}}^{111} - \mathcal{R}, \\ \mathcal{R} &= (1/2) \mathbf{C} \diamond (e_3^3 \otimes e_2^2 \otimes e_3^3) + 3 \mathbf{C} \diamond (e_3^3 \otimes e_3^3 \otimes e_3^3) \\ &\quad + \mathbf{C} \diamond (e_2^3 \otimes e_2^2 \otimes e_3^3). \end{aligned} \quad (33)$$

Even if the groups  $\mathbf{K}$  and  $\mathbf{C}$  are different, the resulting actions on the coefficients of matrices  $A, B$  and  $C$  define the same orbit (in fact, the following identity holds: (32f)=(30f)+(30g), (32d)+(32e)=(30d)+(30e) and the other orbits are identical). Hence, the conclusions done in Section 3.4 remain the same.

## B Stabilizer group of isotropies

It is shown in [6] that the stabilizer group of Laderman matrix multiplication algorithm is isomorphic to  $\mathfrak{S}_4$ . This group is also a stabilizer of classical  $3 \times 3$  matrix multiplication algorithm  $\mathcal{M}$ . Mutatis mutandis, we have with our notations and in the coordinates used in this note:

$$\begin{aligned} 6\mathcal{M} &= \mathfrak{S}_4 \diamond (e_1^1 \otimes e_1^1 \otimes e_1^1) + \mathfrak{S}_4 \diamond \widetilde{\mathcal{M}}^{111} - \mathcal{R}, \\ \mathcal{R} &= (6/4) \mathfrak{S}_4 \diamond (e_3^3 \otimes e_2^2 \otimes e_3^3) + 18 \mathfrak{S}_4 \diamond (e_3^3 \otimes e_3^3 \otimes e_3^3). \end{aligned} \quad (34)$$

This kind of relations holds for any nontrivial subgroup of  $\mathfrak{S}_4$ ; for example with its dihedral subgroup  $D_4$ , we have:

$$\begin{aligned}
2\mathcal{M} &= D_4 \diamond (e_1^1 \otimes e_1^1 \otimes e_1^1) + D_4 \diamond \widetilde{\mathcal{M}}^{111} - \mathcal{R}, \\
\mathcal{R} &= D_4 \diamond (e_3^2 \otimes e_3^3 \otimes e_2^3) + (1/2) D_4 \diamond (e_3^3 \otimes e_2^3 \otimes e_3^2) \\
&\quad + 6 D_4 \diamond (e_3^3 \otimes e_3^3 \otimes e_3^3).
\end{aligned} \tag{35}$$