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# Laderman matrix multiplication algorithm can be constructed using Strassen algorithm and related tensor's isotropies

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## 1 Introduction

In [19], V. Strassen improves the classical  $2 \times 2$  matrix multiplication algorithm. The current upper bound for  $3 \times 3$  matrix multiplication was reached by J.B. Laderman in [16]. This note presents a geometric relationship between Strassen and Laderman algorithms.

### 1.1 Disclaimer: there is no improvement in this note

We do not improve any practical algorithm or prove any theoretical bound in this short note but focus on effective manipulation of tensor associated to matrix multiplication algorithm. To do so, we present only the minimal number of needed definitions and thus leave many facts outside our scope. We refer to [17] for a complete description of the field and to [1] for a state of the art presentation of theoretical complexity issues.

### 1.2 So, why writing (or reading) it?

We follow the geometric spirit of [12, 8, 4, 6, 5] and related papers: symmetries could be used in practical design of matrix multiplication algorithms. Hence, this note presents another example of this philosophy by giving a precise meaning to the following statement:

the Laderman matrix multiplication algorithm is composed by four  $2 \times 2$  optimal matrix multiplication algorithms and an half of the classical  $2 \times 2$  matrix multiplication algorithm.

## 2 Framework

To do so, we have to present a small part of the classical framework (see [13, 14, 17] ) mainly because we do not take it literally.

Let us start by some basic definitions and notations as the following generic matrices:

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}, B = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix}, C = \begin{pmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{pmatrix}, \quad (1)$$

that will be used in the sequel. Furthermore, as we also consider their  $2 \times 2$  submatrices, let us introduce some associated notations.

**Notations 2.1** We denote by  $\text{Id}_{n \times n}^j$  the identity  $n \times n$  matrix where the  $j$ th diagonal term is 0. Given a  $n \times n$  matrix  $A$ , we denote by  $P_{n \times n}^{jk}A$  the matrix  $\text{Id}_{n \times n}^j \cdot A \cdot \text{Id}_{n \times n}^k$ . For example, the matrix  $P_{3 \times 3}^{33}A$  and  $P_{3 \times 3}^{23}A$  are:

$$\begin{pmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} a_{11} & a_{12} & 0 \\ 0 & 0 & 0 \\ a_{31} & a_{32} & 0 \end{pmatrix}. \quad (2)$$

**Remark 2.1** Given  $M$  a  $3 \times 3$  matrix  $A$ , we sometimes consider implicitly  $P_{3 \times 3}^{ij}A$  as a  $2 \times 2$  matrix in the sequel.

## 2.1 Strassen multiplication algorithm

Considered as  $2 \times 2$  matrices, the matrix product  $P_{3 \times 3}^{33}C = P_{3 \times 3}^{33}A \cdot P_{3 \times 3}^{33}B$  could be computed using Strassen algorithm (see [19]) by performing the following computations:

$$\begin{aligned} t_1 &= (a_{11} + a_{22})(b_{11} + b_{22}), \\ t_2 &= (a_{12} - a_{22})(b_{21} + b_{22}), \\ t_3 &= (-a_{11} + a_{21})(b_{11} + b_{12}), \\ t_4 &= (a_{11} + a_{12})b_{22}, \\ t_5 &= a_{11}(b_{12} - b_{22}), \\ t_6 &= a_{22}(-b_{11} + b_{21}), \\ t_7 &= (a_{21} + a_{22})b_{11}, \\ c_{11} &= t_1 + t_2 - t_4 + t_6, \\ c_{12} &= t_6 + t_7, \\ c_{21} &= t_4 + t_5, \\ c_{22} &= t_1 + t_3 + t_5 - t_7. \end{aligned} \quad (3)$$

In order to consider above algorithm under a geometric standpoint, it is usually presented as a tensor.

## 2.2 Bilinear mappings seen as tensors and associated trilinear forms

**Definitions 2.1** Given a tensor:

$$\mathcal{T} = \sum_{i=1}^r M_{i1} \otimes M_{i2} \otimes M_{i3}, \quad (4)$$

where  $M_{ij}$  are  $n \times n$  matrices:

- the integer  $r$  is the tensor rank of tensor  $\mathcal{T}$ ;
- the unordered list  $[(\text{rank } M_{ij})_{j=1..3}]_{i=1..r}$  is called the type of tensor  $\mathcal{T}$  (rank  $M$  being the classical rank of the matrix  $M$ ).

### 2.3 Tensors' contractions

To explicit the relationship between what is done in the sequel and the bilinear mapping associated to matrix multiplication, let us consider the following tensor's contractions:

**Definitions 2.2** Using the notation of definition 2.1 given a tensor  $\mathcal{T}$  and three matrices  $A, B$  and  $C$  with coefficients in the algebra  $\mathbb{K}$ :

- the (1, 2) contraction of  $\mathcal{T} \otimes A \otimes B$  defined by:

$$\sum_{i=1}^r \text{Trace}({}^t M_{i1} \cdot A) \text{Trace}({}^t M_{i2} \cdot B) M_{i3} \quad (5)$$

corresponds to a bilinear application  $\mathbb{K}^{n \times n} \times \mathbb{K}^{n \times n} \mapsto \mathbb{K}^{n \times n}$  with indeterminates  $A$  and  $B$ .

- the (1, 2, 3) (a.k.a full) contraction of  $\mathcal{T} \otimes A \otimes B \otimes C$  defined by:

$$\langle \mathcal{T} | A \otimes B \otimes C \rangle = \sum_{i=1}^r \text{Trace}({}^t M_{i1} \cdot A) \text{Trace}({}^t M_{i2} \cdot B) \text{Trace}({}^t M_{i3} \cdot C) \quad (6)$$

corresponds to a trilinear form  $\mathbb{K}^{n \times n} \times \mathbb{K}^{n \times n} \times \mathbb{K}^{n \times n} \mapsto \mathbb{K}$  with indeterminates  $A, B$  and  $C$ .

**Remarks 2.2** As the studied object is the tensor, its expressions as full or incomplete contractions are equivalent. Thus, even if matrix multiplication algorithm is a bilinear application, we are going to work in the sequel with trilinear forms (see [10] for bibliographic references on this standpoint).

The definition in 2.2 are taken to express the full contraction as a degenerate inner product between tensors; it is not the usual choice made in the literature and so, we have to explicitly recall the notion used in the sequel.

Strassen multiplication algorithm 3 is equivalent to the tensor  $\mathcal{S}$  defined by:

$$\begin{aligned} & \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \\ & \begin{pmatrix} -1 & 0 \\ 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} -1 & 0 \\ 1 & 0 \end{pmatrix} + \\ & \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} -1 & 0 \\ 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} + \\ & \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix}, \end{aligned} \quad (7)$$

and one can check that this tensor defines the matrix multiplication algorithm (3) and that its tensor rank is 7.

### 2.4 $2 \times 2$ matrix multiplication tensors induced by a $3 \times 3$ matrix multiplication tensor

. Given any  $3 \times 3$  matrix multiplication tensor, one can define  $3^3$  induced  $2 \times 2$  matrix multiplication tensors as follow. First, let us introduce the following projection operator:

**Definition 2.3** Using notation introduced in definition 2.1, we defines:

$$P_{ijk}(A \otimes B \otimes C) = P_{3 \times 3}^{ij} A \otimes P_{3 \times 3}^{jk} B \otimes P_{3 \times 3}^{ki} C \quad (8)$$

and we extend the definition of this operator by additivity and thus, it could be applied on any tensor  $\mathcal{T}$  described in definition 2.1.

There is  $3^3$  such projection and given any matrix multiplication tensor  $\mathcal{M}$ , the full contraction satisfying the following property:

$$\langle \mathcal{M} | P_{ijk}(A \otimes B \otimes C) \rangle = \langle P_{ijk} \mathcal{M} | A \otimes B \otimes C \rangle \quad (9)$$

where the projection operator apply on a  $3 \times 3$  matrix multiplication tensor defines explicitly a  $2 \times 2$  matrix multiplication tensor.

The following property holds:

**Lemma 2.1**

$$2^3 \langle \mathcal{M} | A \otimes B \otimes C \rangle = \sum_{1 \leq i, j, k \leq 3} \langle \mathcal{M} | P_{ijk}(A \otimes B \otimes C) \rangle \quad (10)$$

and thus, we have:

$$\langle \mathcal{M} | A \otimes B \otimes C \rangle = \left\langle \frac{1}{2^3} \sum_{1 \leq i, j, k \leq 3} P_{ijk} \mathcal{M} \middle| A \otimes B \otimes C \right\rangle. \quad (11)$$

The obvious facts made in this section underline the relationships between any  $n \times n$  matrix multiplication tensor and the  $n^3$  induced  $(n-1) \times (n-1)$  algorithms.

Considering the Laderman matrix multiplication tensor, we are going to explore further this kind of relationships. First, let us introduce this tensor.

## 2.5 Laderman matrix multiplication tensor

The following tensor — described here by giving the full contraction  $\langle \mathcal{L} | A \otimes B \otimes C \rangle$ :

$$\begin{aligned}
& (a_{11} - a_{21} + a_{12} - a_{22} - a_{32} + a_{13} - a_{33}) b_{22} c_{21} & + \\
& a_{22} (-b_{11} + b_{21} - b_{31} + b_{12} - b_{22} - b_{23} + b_{33}) c_{12} & + \\
& a_{13} b_{31} (c_{11} + c_{21} + c_{31} + c_{12} + c_{32} + c_{13} + c_{23}) & + \\
& (a_{11} - a_{31} + a_{12} - a_{22} - a_{32} + a_{13} - a_{23}) b_{23} c_{31} & + \\
& a_{32} (-b_{11} + b_{21} - b_{31} - b_{22} + b_{32} + b_{13} - b_{23}) c_{13} & + \\
& a_{11} b_{11} (c_{11} + c_{21} + c_{31} + c_{12} + c_{22} + c_{13} + c_{33}) & + \\
& (-a_{11} + a_{31} + a_{32}) (b_{11} - b_{13} + b_{23}) (c_{31} + c_{13} + c_{33}) & + \\
& (a_{22} - a_{13} + a_{23}) (b_{31} + b_{23} - b_{33}) (c_{31} + c_{12} + c_{32}) & + \\
& (-a_{11} + a_{21} + a_{22}) (b_{11} - b_{12} + b_{22}) (c_{21} + c_{12} + c_{22}) & + \\
& (a_{32} - a_{13} + a_{33}) (b_{31} + b_{22} - b_{32}) (c_{21} + c_{13} + c_{23}) & + \\
& (a_{21} + a_{22}) (-b_{11} + b_{12}) (c_{21} + c_{22}) & + \\
& (a_{31} + a_{32}) (-b_{11} + b_{13}) (c_{31} + c_{33}) & + \\
& (a_{13} - a_{33}) (b_{22} - b_{32}) (c_{13} + c_{23}) & + \\
& (a_{11} - a_{21}) (-b_{12} + b_{22}) (c_{12} + c_{22}) & + \\
& (a_{32} + a_{33}) (-b_{31} + b_{32}) (c_{21} + c_{23}) & + \\
& (-a_{11} + a_{31}) (b_{13} - b_{23}) (c_{13} + c_{33}) & + \\
& (a_{13} - a_{23}) (b_{23} - b_{33}) (c_{12} + c_{32}) & + \\
& (a_{22} + a_{23}) (-b_{31} + b_{33}) (c_{31} + c_{32}) & + \\
& a_{12} b_{21} c_{11} + a_{23} b_{32} c_{22} + a_{21} b_{13} c_{32} + a_{31} b_{12} c_{23} + a_{33} b_{33} c_{33} & 
\end{aligned} \quad (12)$$

was introduced in [16] (we do not study in this note any other *inequivalent* algorithm of same tensor rank (e.g. [15, 9, 18], etc.). Considering the projections introduced in definition 2.3, we notice that:

**Remark 2.3** *Laderman matrix multiplication tensor defines 4 optimal  $2 \times 2$  matrix multiplication tensors and 19 other such tensors of tensor rank 8.*

Further computations show that:

**Remark 2.4** *The type of the Laderman matrix multiplication tensor is*

$$[(2, 2, 2)|4, ((1, 3, 1), (3, 1, 1), (1, 1, 3))|2, (1, 1, 1)|13] \quad (13)$$

where  $m|n$  indicates that  $m$  is repeated  $n$  times.

## 2.6 Tensors' isotropies

We refer to [13, 14] for a complete presentation of automorphism group operating on varieties defined by algorithms for computation of bilinear mappings and as a reference for the following theorem:

**Theorem 2.1** *The isotropy group of the  $n \times n$  matrix multiplication tensor is*

$$\mathrm{PGL}(\mathbb{C}^n)^{\times 3} \rtimes \mathfrak{S}_6, \quad (14)$$

where  $\mathrm{PGL}$  stands for the projective linear group and  $\mathfrak{S}_6$  for the symmetric group on 6 elements.

Even if we do not explicit the concrete action of this isotropy group on matrix multiplication tensor, let us precise some terminologies:

**Definitions 2.4** *Given a tensor defining matrix multiplication computations, the orbit of this tensor is called the multiplication algorithm and any of the points composing this orbit is a variant of this algorithm.*

**Remark 2.5** *As shown in [11], matrix multiplication is characterised by its isotropy group.*

**Remark 2.6** *We only need the  $\mathrm{PGL}(\mathbb{C}^n)^{\times 3}$  part of this group (a.k.a. sandwiching) and thus focus on it in the sequel.*

As our framework and notations differ slightly from the framework classically found in the literature, we have to explicitly define several well-known notions for the sake of clarity. Hence, let us recall the *sandwiching* action:

**Definition 2.5** *Given  $g = (G_1 \times G_2 \times G_3)$  an element of  $\mathrm{PGL}(\mathbb{C}^n)^{\times 3}$ , its action on a tensor  $\mathcal{T}$  is given by:*

$$g\mathcal{T} = \sum_{i=1}^r ({}^tG_1^{-1}M_{i1}{}^tG_2) \otimes ({}^tG_2^{-1}M_{i2}{}^tG_3) \otimes ({}^tG_3^{-1}M_{i3}{}^tG_1). \quad (15)$$

**Example 2.1** Let us consider the action of the following isotropy

$$\rho = \begin{pmatrix} 0 & 1/\lambda \\ -1 & 0 \end{pmatrix} \times \begin{pmatrix} 1/\lambda & -1/\lambda \\ 0 & 1 \end{pmatrix} \times \begin{pmatrix} -1/\lambda & 0 \\ 1 & -1 \end{pmatrix} \quad (16)$$

on the Strassen variant of the Strassen algorithm. The resulting tensor  $\mathcal{W}$  is:

$$\begin{aligned} & \begin{pmatrix} -1 & \lambda \\ -\frac{1}{\lambda} & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & -\lambda \\ \frac{1}{\lambda} & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & -\lambda \\ \frac{1}{\lambda} & 0 \end{pmatrix} + \begin{pmatrix} -1 & 1 \\ -\frac{1}{\lambda} & 1 \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + \\ & \begin{pmatrix} 1 & 0 \\ \frac{1}{\lambda} & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ \frac{1}{\lambda} & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ \frac{1}{\lambda} & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + \\ & \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} -1 & \lambda \\ -\frac{1}{\lambda} & 1 \end{pmatrix} + \begin{pmatrix} 1 & -\lambda \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & -\lambda \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & -\lambda \\ 0 & 0 \end{pmatrix} + \\ & \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} -1 & \lambda \\ -\frac{1}{\lambda} & 1 \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \end{aligned} \quad (17)$$

that is the well-known Winograd variant of Strassen algorithm.

**Remarks 2.7** We keep the parameter  $\lambda$  useless in our presentation as a tribute to the construction made in [7] that gives an elegant and elementary (i.e. based on matrix eigenvalues) construction of Winograd variant of Strassen matrix multiplication algorithm.

This variant is remarkable in its own as shown in [3] because it is optimal w.r.t. multiplicative and additive complexity.

**Remark 2.8** Tensor's type is an invariant of isotropy's action. Hence, two tensor in the same orbit share the same type. Or equivalently, two tensors with the same type are two variants that represent the same matrix multiplication algorithm.

This remark will allows us to recognise the tensor constructed below as a variant of the Laderman matrix multiplication algorithm.

### 3 A tensor's construction

Let us now present the construction of a variant of Laderman matrix multiplication algorithm based on Winograd variant of Strassen matrix multiplication algorithm.

First, let us give the full contraction of the tensor  $\mathcal{W} \otimes A \otimes B \otimes C$ :

$$\left(-a_{22} - \frac{a_{32}}{\lambda} + \lambda a_{23}\right) \left(b_{22} + \frac{b_{32}}{\lambda} - \lambda b_{23}\right) \left(c_{22} + \frac{c_{32}}{\lambda} - \lambda c_{23}\right) + \quad (18a)$$

$$(a_{22} - \lambda a_{23}) (b_{22} - \lambda b_{23}) (c_{22} - \lambda c_{23}) + \quad (18b)$$

$$\left(a_{22} + \frac{a_{32}}{\lambda}\right) \left(b_{22} + \frac{b_{32}}{\lambda}\right) \left(c_{22} + \frac{c_{32}}{\lambda}\right) + \quad (18c)$$

$$a_{23} \left(-b_{22} - \frac{b_{32}}{\lambda} + \lambda b_{23} + b_{33}\right) c_{32} + \quad (18d)$$

$$\left(-a_{22} - \frac{a_{32}}{\lambda} + \lambda a_{23} + a_{33}\right) b_{32} c_{23} + \quad (18e)$$

$$a_{32} b_{23} \left(-c_{22} - \frac{c_{32}}{\lambda} + \lambda c_{23} + c_{33}\right) + \quad (18f)$$

$$a_{33} b_{33} c_{33} \quad (18g)$$

### 3.1 A Klein four-group of isotropies

Let us introduce now the following notations,

$$\text{Id}_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad P_{(12)} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (19)$$

used to defined the following group of isotropies:

$$G = \left\{ \begin{array}{ll} g_1 = \text{Id}_3^{\times 3}, & g_2 = (\text{Id}_3 \times P_{(12)} \times P_{(12)}), \\ g_3 = (P_{(12)} \times P_{(12)} \times \text{Id}_3), & g_4 = (P_{(12)} \times \text{Id}_3 \times P_{(12)}) \end{array} \right\}. \quad (20)$$

that is isomorphic to the Klein four-group.

### 3.2 Its action on Winograd variant of Strassen algorithm

In the sequel, we are interested in the action of Klein four-group (20) on our Winograd variant of Strassen algorithm:

$$G\mathcal{W} = \sum_{i=1} g_i \mathcal{W}. \quad (21)$$

As we have,

$$\langle g_i \mathcal{W} | A \otimes B \otimes C \rangle = \langle \mathcal{W} | g_i(A \otimes B \otimes C) \rangle, \quad (22)$$

the action of isotropies  $g_i$  is just a permutation of our generic matrix coefficients. Hence, we have the following full contraction of the tensor  $g_2 \mathcal{W} \otimes A \otimes B \otimes C$ :

$$\left(-a_{21} - \frac{a_{31}}{\lambda} + \lambda a_{23}\right) \left(b_{11} + \frac{b_{31}}{\lambda} - \lambda b_{13}\right) \left(c_{12} + \frac{c_{32}}{\lambda} - \lambda c_{13}\right) + \quad (23a)$$

$$(a_{21} - \lambda a_{23}) (b_{11} - \lambda b_{13}) (c_{12} - \lambda c_{13}) + \quad (23b)$$

$$\left(a_{21} + \frac{a_{31}}{\lambda}\right) \left(b_{11} + \frac{b_{31}}{\lambda}\right) \left(c_{12} + \frac{c_{32}}{\lambda}\right) + \quad (23c)$$

$$a_{23} \left(-b_{11} - \frac{b_{31}}{\lambda} + \lambda b_{13} + b_{33}\right) c_{32} + \quad (23d)$$

$$\left(-a_{21} - \frac{a_{31}}{\lambda} + \lambda a_{23} + a_{33}\right) b_{31} c_{13} + \quad (23e)$$

$$a_{31} b_{13} \left(-c_{12} - \frac{c_{32}}{\lambda} + \lambda c_{13} + c_{33}\right) + \quad (23f)$$

$$a_{33} b_{33} c_{33}, \quad (23g)$$



the full contraction of the tensor  $g_3\mathcal{W} \otimes A \otimes B \otimes C$ :

$$\left(-a_{11} - \frac{a_{31}}{\lambda} + \lambda a_{13}\right) \left(b_{12} + \frac{b_{32}}{\lambda} - \lambda b_{13}\right) \left(c_{21} + \frac{c_{31}}{\lambda} - \lambda c_{23}\right) + \quad (24a)$$

$$(a_{11} - \lambda a_{13}) (b_{12} - \lambda b_{13}) (c_{21} - \lambda c_{23}) + \quad (24b)$$

$$\left(a_{11} + \frac{a_{31}}{\lambda}\right) \left(b_{12} + \frac{b_{32}}{\lambda}\right) \left(c_{21} + \frac{c_{31}}{\lambda}\right) + \quad (24c)$$

$$a_{13} \left(-b_{12} - \frac{b_{32}}{\lambda} + \lambda b_{13} + b_{33}\right) c_{31} + \quad (24d)$$

$$\left(-a_{11} - \frac{a_{31}}{\lambda} + \lambda a_{13} + a_{33}\right) b_{32} c_{23} + \quad (24e)$$

$$a_{31} b_{13} \left(-c_{21} - \frac{c_{31}}{\lambda} + \lambda c_{23} + c_{33}\right) + \quad (24f)$$

$$a_{33} b_{33} c_{33} \quad (24g)$$

and the full contraction of the tensor  $g_4\mathcal{W} \otimes A \otimes B \otimes C$ :

$$\left(-a_{12} - \frac{a_{32}}{\lambda} + \lambda a_{13}\right) \left(b_{21} + \frac{b_{31}}{\lambda} - \lambda b_{23}\right) \left(c_{11} + \frac{c_{31}}{\lambda} - \lambda c_{13}\right) + \quad (25a)$$

$$(a_{12} - \lambda a_{13}) (b_{21} - \lambda b_{23}) (c_{11} - \lambda c_{13}) + \quad (25b)$$

$$\left(a_{12} + \frac{a_{32}}{\lambda}\right) \left(b_{21} + \frac{b_{31}}{\lambda}\right) \left(c_{11} + \frac{c_{31}}{\lambda}\right) + \quad (25c)$$

$$a_{13} \left(-b_{21} - \frac{b_{31}}{\lambda} + \lambda b_{23} + b_{33}\right) c_{31} + \quad (25d)$$

$$\left(-a_{12} - \frac{a_{32}}{\lambda} + \lambda a_{13} + a_{33}\right) b_{31} c_{13} + \quad (25e)$$

$$a_{32} b_{23} \left(-c_{11} - \frac{c_{31}}{\lambda} + \lambda c_{13} + c_{33}\right) + \quad (25f)$$

$$a_{33} b_{33} c_{33}. \quad (25g)$$

There is several noteworthy points in these expressions:

**Remarks 3.1** • *the term (18g) is a fixed point of  $G$ 's action;*

- *the term (18d) and (23d), (18e) and (24e), (18f) and (25f), (23e) and (25e), (23f) and (24f), (24d) and (25d) could be added in order to obtain new tensor parts. For example (18d)+(23d) is equal to:*

$$a_{23} \left(-b_{22} - \frac{b_{32}}{\lambda} + \lambda b_{23} + 2b_{33} - b_{11} - \frac{b_{31}}{\lambda} + \lambda b_{13}\right) c_{32}. \quad (26)$$

The tensor rank of the tensor  $G\mathcal{W} = \sum_{i=1} g_i\mathcal{W}$  is thus  $1 + 3 \cdot 4 + 6 = 19$ . Unfortunately, this tensor does not define a matrix multiplication algorithm (otherwise according to the lower bound presented in [2], it would be optimal and this note would have another title and impact).

### 3.3 How far are we from a multiplication tensor?

However, denoting by  $\mathcal{C}$  the classical  $3 \times 3$  matrix multiplication algorithm, we could compute the expression:

$$\begin{aligned} \langle \mathcal{C} - GW | A \otimes B \otimes C \rangle = & a_{11} b_{11} c_{11} + a_{12} b_{22} c_{21} + a_{22} b_{21} c_{12} + a_{21} b_{12} c_{22} \\ & - c_{31} a_{13} b_{33} - c_{32} a_{23} b_{33} - c_{13} a_{33} b_{31} \\ & - c_{23} a_{33} b_{32} - c_{33} a_{31} b_{13} - c_{33} a_{32} b_{23} \\ & - 3 a_{33} b_{33} c_{33}, \end{aligned} \tag{27}$$

that should be added to  $\langle GW | A \otimes B \otimes C \rangle$  in order to obtain a matrix multiplication tensor.

### 3.4 Resulting matrix multiplication algorithm

**Remark 3.2** *Again, all the negative terms in polynomial (27) could be added to already present terms in  $\langle GW | A \otimes B \otimes C \rangle$  without changing the tensor rank.*

Hence, we obtain a matrix multiplication tensor with rank  $23 = 19 + 4$  positive terms. Furthermore, the resulting tensor have the same type than the Laderman matrix multiplication tensor, and thus it is a variant of the same algorithm.

Thus, we conclude that the Laderman matrix multiplication algorithm is composed by four  $2 \times 2$  optimal matrix multiplication algorithms and an half of the classical  $2 \times 2$  matrix multiplication algorithm (the remaining positive term in 27)).

## 4 Concluding remark

All the observations presented in this short note came from an experimental mathematical approach using the computer algebra system Maple. While implementing effectively (if not efficiently) several tools needed to manipulate matrix multiplication tensor (tensors, their isotropies and contractions, etc.) in order to understand the theory, the relationship between the Laderman matrix multiplication algorithm and the Strassen algorithm became clear by simple computations that will be tedious or impossible by hand.

The main opinion supported by this work is that symmetries play a central role in effective computation for matrix multiplication algorithm and that only a geometrical interpretation may brings further improvement.

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