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# ON THE OUTLYING EIGENVALUES OF A POLYNOMIAL IN LARGE INDEPENDENT RANDOM MATRICES

SERBAN T. BELINSCHI, HARI BERCOVICI, AND MIREILLE CAPITAINÉ

**ABSTRACT.** Given a selfadjoint polynomial  $P(X, Y)$  in two noncommuting self-adjoint indeterminates, we investigate the asymptotic eigenvalue behavior of the random matrix  $P(A_N, B_N)$ , where  $A_N$  and  $B_N$  are independent Hermitian random matrices and the distribution of  $B_N$  is invariant under conjugation by unitary operators. We assume that the empirical eigenvalue distributions of  $A_N$  and  $B_N$  converge almost surely to deterministic probability measures  $\mu$  and  $\nu$ , respectively. In addition, the eigenvalues of  $A_N$  and  $B_N$  are assumed to converge uniformly almost surely to the support of  $\mu$  and  $\nu$ , respectively, except for a fixed finite number of fixed eigenvalues (spikes) of  $A_N$ . It is known that almost surely the empirical distribution of the eigenvalues of  $P(A_N, B_N)$  converges to a certain deterministic probability measure  $\eta$  (sometimes denoted  $\eta = P^\square(\mu, \nu)$ ) and, when there are no spikes, the eigenvalues of  $P(A_N, B_N)$  converge uniformly almost surely to the support of  $\eta$ . When spikes are present, we show that the eigenvalues of  $P(A_N, B_N)$  still converge uniformly to the support of  $\eta$ , with the possible exception of certain isolated outliers whose location can be determined in terms of  $\mu, \nu, P$ , and the spikes of  $A_N$ . We establish a similar result when  $B_N$  is replaced by a Wigner matrix. The relation between outliers and spikes is described using the operator-valued subordination functions of free probability theory. These results extend known facts from the special case in which  $P(X, Y) = X + Y$ .

## 1. INTRODUCTION

Let  $\mu$  and  $\nu$  be two Borel probability measures with bounded support on  $\mathbb{R}$ . Suppose given, for each positive integer  $N$ , selfadjoint  $N \times N$  independent random matrices  $A_N$  and  $B_N$ , with the following properties:

- (a) the distribution of  $B_N$  is invariant under conjugation by unitary  $N \times N$  matrices;
- (b) the empirical eigenvalue distributions of  $A_N$  and  $B_N$  converge almost surely to  $\mu$  and  $\nu$ , respectively;
- (c) the eigenvalues of  $A_N$  and  $B_N$  converge uniformly almost surely to the supports of  $\mu$  and  $\nu$ , respectively, with the exception of a fixed number  $p$  of *spikes*, that is, fixed eigenvalues of  $A_N$  that lie outside the support of  $\mu$ .

When spikes are absent, that is, when  $p = 0$ , it was shown in [23] that the eigenvalues of  $A_N + B_N$  converge uniformly almost surely to the support of the free additive convolution  $\mu \boxplus \nu$ . When  $p > 0$ , the eigenvalues of  $A_N + B_N$  also converge uniformly almost surely to a compact set  $K \subset \mathbb{R}$  such that  $\text{supp}(\mu \boxplus \nu) \subset K$  and  $K \setminus \text{supp}(\mu \boxplus \nu)$  has no accumulation points in  $\mathbb{R} \setminus \text{supp}(\mu \boxplus \nu)$ . Moreover, if  $t \in K \setminus \text{supp}(\mu \boxplus \nu)$ , then  $\omega(t)$  is one of the spikes of  $A_N$ , where  $\omega$  is

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a certain subordination function arising in free probability. The relative position of the eigenvectors corresponding to spikes and outliers is also given in terms of subordination functions. We refer to [11] for this result.

Our purpose is to show that analogous results hold when the sum  $A_N + B_N$  is replaced by an arbitrary selfadjoint polynomial  $P(A_N, B_N)$ . Then, by a comparison procedure to the particular case when  $B_N$  is a G.U.E. (Gaussian unitary ensemble), we are also able to identify the outliers of an arbitrary selfadjoint polynomial  $P(A_N, \frac{X_N}{\sqrt{N}})$  when  $X_N$  is a Wigner matrix independent from  $A_N$ . This extends an earlier result [22] pertaining to additive deformations of Wigner matrices. More precisely we consider a Hermitian matrix  $X_N = [X_{ij}]_{i,j=1}^N$ , where  $[X_{ij}]_{i \geq 1, j \geq 1}$  is an infinite array of random variables such that

- (X0)  $X_N$  is independent from  $A_N$ ,
- (X1)  $X_{ii}$ ,  $\sqrt{2}\Re(X_{ij})$ ,  $i < j$ ,  $\sqrt{2}\Im(X_{ij})$ ,  $i < j$ , are independent, centered with variance 1,
- (X2) there exist  $K, x_0 > 0$ ,  $n_0 \in \mathbb{N}$ , and a random variable  $Z$  with finite fourth moment such that

$$\frac{1}{n^2} \sum_{1 \leq i, j \leq n} \mathbb{P}(|X_{ij}| > x) \leq K \mathbb{P}(|Z| > x) \quad x > x_0, n > n_0.$$

- (X3)  $\sup\{\mathbb{E}(|X_{ij}|^3) : i, j \in \mathbb{N}, i < j\} < +\infty$ .

**Remark 1.1.** The matrix  $X_N$  is called a G.U.E. if the variables  $X_{ii}$ ,  $\sqrt{2}\Re(X_{ij})$ ,  $i < j$ , and  $\sqrt{2}\Im(X_{ij})$ ,  $i < j$ , are independent standard Gaussian. Assumptions (X2) and (X3) obviously hold if these variables are merely independent and identically distributed with a finite fourth moment.

Our result lies in the lineage of recent, and not so recent, works [5, 7, 8, 14, 18, 19, 21, 22, 26, 27, 31, 33, 35, 39, 40, 41] studying the influence of additive or multiplicative perturbations on the extremal eigenvalues of classical random matrix models, the seminal paper being [7], where the so-called BBP phase transition was observed.

We note that Shlyakhtenko [45] considered a framework which makes it possible to understand this kind of result as a manifestation of infinitesimal freeness. In fact, the results of [45] also allow one to detect the presence of spikes from the behaviour of the bulk of the eigenvalues of  $P(A_N, B_N)$ , even when  $P(A_N, B_N)$  has no outlying eigenvalues. In a related result, Collins, Hasebe and Sakuma [24] study the ‘purely spike’ case in which  $\mu = \nu = \delta_0$  and the eigenvalues of  $A_N$  and  $B_N$  accumulate to given sequences  $(a_k)_{k=1}^\infty$  and  $(b_k)_{k=1}^\infty$  of real numbers converging to zero.

## 2. NOTATION AND PRELIMINARIES ON STRONG ASYMPTOTIC FREENESS

We recall that a  $C^*$ -probability space is a pair  $(\mathcal{A}, \tau)$ , where  $\mathcal{A}$  is a  $C^*$ -algebra and  $\tau$  is a state on  $\mathcal{A}$ . We always assume that  $\tau$  is faithful. The elements of  $\mathcal{A}$  are referred to as *random variables*.

If  $(\Omega, \Sigma, P)$  is a classical probability space, then  $(L^\infty(\Omega), \mathbb{E})$  is a  $C^*$ -probability space, where  $\mathbb{E}$  is the usual expected value. Given  $N \in \mathbb{N}$ ,  $(M_N(\mathbb{C}), \text{tr}_N)$  is a  $C^*$ -probability space, where  $\text{tr}_N = \frac{1}{N} \text{Tr}_N$  denotes the normalized trace. More generally, if  $(\mathcal{A}, \tau)$  is an arbitrary  $C^*$ -probability space and  $N \in \mathbb{N}$ , then  $M_N(\mathcal{A}) = M_N(\mathbb{C}) \otimes \mathcal{A}$  becomes a  $C^*$ -probability space with the state  $\text{tr}_N \otimes \tau$ .

The *distribution*  $\mu_a$  of a selfadjoint element  $a$  in a  $C^*$ -probability space  $(\mathcal{A}, \tau)$  is a compactly supported probability measure on  $\mathbb{R}$ , uniquely determined by the requirement that  $\int_{\mathbb{R}} t^n d\mu_a(t) = \tau(a^n)$ ,  $n \in \mathbb{N}$ . The *spectrum* of an element  $a \in \mathcal{A}$  is

$$\sigma(a) = \{\lambda \in \mathbb{C} : \lambda 1 - a \text{ is not invertible in } \mathcal{A}\}.$$

For instance, if  $A \in M_N(\mathbb{C})$  is a selfadjoint matrix, then the distribution of  $A$  relative to  $\text{tr}_N$  is the measure  $\mu_A = \frac{1}{N} \sum_{j=1}^N \delta_{\lambda_j(A)}$ , where  $\{\lambda_1(A), \dots, \lambda_N(A)\}$  is the list of the eigenvalues of  $A$ , repeated according to multiplicity. As usual, the support  $\text{supp}(\mu)$  of a Borel probability measure  $\mu$  on  $\mathbb{R}$  is the smallest closed set  $F \subset \mathbb{R}$  with the property that  $\mu(F) = 1$ . It is known that if  $a = a^* \in \mathcal{A}$  and  $\tau$  is faithful, then  $\sigma(a) = \text{supp}(\mu_a)$ . In the following, we assume that  $\tau$  is a tracial state, that is,  $\tau(ab) = \tau(ba)$ ,  $a, b \in \mathcal{A}$ .

Suppose that we are given  $C^*$ -probability spaces  $\{(\mathcal{A}_N, \tau_N)\}_{N=0}^\infty$  and selfadjoint elements  $a_N \in \mathcal{A}_N$ ,  $N \geq 0$ . We say that  $\{a_N\}_{N=1}^\infty$  *converges in distribution* to  $a_0$  if

$$(2.1) \quad \lim_{N \rightarrow \infty} \tau_N(a_N^k) = \tau_0(a_0^k), \quad k \in \mathbb{N}.$$

We say that  $\{a_N\}_{N=1}^\infty$  *converges strongly in distribution* to  $a_0$  (or to  $\mu_{a_0}$ ) if, in addition to (2.1), the sequence  $\{\text{supp}(\mu_{a_N})\}_{N=1}^\infty$  converges to  $\text{supp}(\mu_{a_0})$  in the Hausdorff metric. This condition simply means that for every  $\varepsilon > 0$  there exists  $N(\varepsilon) \in \mathbb{N}$  such that

$$\text{supp}(\mu_{a_N}) \subset \text{supp}(\mu_{a_0}) + (-\varepsilon, \varepsilon)$$

and

$$\text{supp}(\mu_{a_0}) \subset \text{supp}(\mu_{a_N}) + (-\varepsilon, \varepsilon)$$

for every  $N \geq N(\varepsilon)$ . If all the traces  $\tau_N$  are faithful, strong convergence can be reformulated as follows:

$$\lim_{N \rightarrow \infty} \|P(a_N)\| = \|P(a_0)\|,$$

for every polynomial  $P$  with complex coefficients. This observation allows us to extend the concept of (strong) convergence in distribution to  $k$ -tuples of random variables,  $k \in \mathbb{N}$ . For every  $k \in \mathbb{N}$ , we denote by  $\mathbb{C}\langle X_1, \dots, X_k \rangle$  the algebra of polynomials with complex coefficients in  $k$  noncommuting indeterminates  $X_1, \dots, X_k$ . This is a  $*$ -algebra with the adjoint operation determined by

$$(\alpha X_{i_1} X_{i_2} \cdots X_{i_n})^* = \bar{\alpha} X_{i_n} \cdots X_{i_2} X_{i_1}, \quad \alpha \in \mathbb{C}, \quad i_1, i_2, \dots, i_n \in \{1, \dots, k\}.$$

Suppose that  $\{(\mathcal{A}_N, \tau_N)\}_{N=0}^\infty$  is a sequence of  $C^*$ -probability spaces,  $k \in \mathbb{N}$ , and  $\{a_N\}_{N=0}^\infty$  is a sequence of  $k$ -tuples  $a_N = (a_{N,1}, \dots, a_{N,k}) \in \mathcal{A}_N^k$  of selfadjoint elements. We say that  $\{a_N\}_{N=1}^\infty$  *converges in distribution* to  $a_0$  if

$$(2.2) \quad \lim_{N \rightarrow \infty} \tau_N(P(a_N)) = \tau_0(P(a_0)), \quad P \in \mathbb{C}\langle X_1, \dots, X_k \rangle.$$

We say that  $\{a_N\}_{N=1}^\infty$  *converges strongly in distribution* to  $a_0$  if, in addition to (2.2), we have

$$\lim_{N \rightarrow \infty} \|P(a_N)\| = \|P(a_0)\|, \quad P \in \mathbb{C}\langle X_1, \dots, X_k \rangle.$$

The above concepts extend to  $k$ -tuples  $a_N = (a_{N,1}, \dots, a_{N,k}) \in \mathcal{A}_N^k$  which do not necessarily consist of selfadjoint elements. The only change is that one must use polynomials in the variables  $a_{N,j}$  and their adjoints  $a_{N,j}^*$ ,  $j = 1, \dots, k$ .

**Remark 2.1.** Suppose that all the states  $\tau_N, N \in \mathbb{N}$ , are faithful. As seen in [23, Proposition 2.1],  $\{a_N\}_{N=1}^\infty$  converges strongly in distribution to  $a_0$  if and only if  $\{P(a_N)\}_{N=1}^\infty$  converges strongly in distribution to  $P(a_0)$  for every selfadjoint polynomial  $P \in \mathbb{C}\langle X_1, \dots, X_k \rangle$ . Moreover, strong convergence in distribution also implies strong convergence at the matricial level. The following result is [36, Proposition 7.3].

**Proposition 2.2.** *Let  $\{(\mathcal{A}_N, \tau_N)\}_{N=0}^\infty$  be  $C^*$ -probability spaces with faithful states  $\{\tau_N\}_{N=0}^\infty$ , let  $k \in \mathbb{N}$ , and let  $\{a_N\}_{N=0}^\infty$  be a sequence of  $k$ -tuples of selfadjoint elements  $a_N \in \mathcal{A}_N^k$ . Suppose that  $\{a_N\}_{N=1}^\infty$  converges strongly in distribution to  $a_0$ . Then  $\lim_{N \rightarrow \infty} \|P(a_N)\| = \|P(a_0)\|$  for every  $n \in \mathbb{N}$  and every matrix polynomial  $P \in M_n(\mathbb{C}\langle X_1, \dots, X_k \rangle)$ .*

A special case of strong convergence in distribution arises from the consideration of random matrices in  $M_N(\mathbb{C})$ . The following result follows from [23, Theorem 1.4] and [12, Theorem 1.2].

**Theorem 2.3.** *Let  $(\mathcal{A}_N, \tau_N)$  denote the space  $(M_N(\mathbb{C}), \text{tr}_N)$ ,  $N \in \mathbb{N}$ . Suppose that  $k_1, k_2, k_3 \in \mathbb{N}$  are fixed,  $u_N = (U_{N,1}, \dots, U_{N,k_1})$ ,  $x_N = (X_{N,1}, \dots, X_{N,k_2})$  and  $a_N = (A_{N,1}, \dots, A_{N,k_3})$  are mutually independent random tuples of matrices in some classical probability space such that:*

- (i)  $U_{N,1}, \dots, U_{N,k_1}$  are independent unitaries distributed according to the Haar measure on the unitary group  $U(N)$ ,  $N \in \mathbb{N}$ .
- (ii)  $X_{N,1}, \dots, X_{N,k_2}$  are independent Hermitian matrices, each satisfying assumptions (X1), (X2), and (X3) in the introduction.
- (iii)  $a_N$  is a vector of  $N \times N$  selfadjoint matrices such that the sequence  $\{a_N\}_{N=1}^\infty$  converges strongly almost surely in distribution to some deterministic  $k_3$ -tuple in a  $C^*$ -probability space.

*Then there exist a  $C^*$ -probability space  $(\mathcal{A}, \tau)$ , a free family  $u = (u_1, \dots, u_{k_1}) \in \mathcal{A}^{k_1}$  of Haar unitaries, a semicircular system  $x = (x_1, \dots, x_{k_2}) \in \mathcal{A}^{k_2}$  and  $a = (a_1, \dots, a_{k_3}) \in \mathcal{A}^{k_3}$ , such that,  $u, x$ , and  $a$  are free and  $\{(u_N, x_N, a_N)\}_{N=1}^\infty$  converges strongly almost surely in distribution to  $(u, x, a)$ .*

We recall that a tuple  $(x_1, \dots, x_k)$  of elements in a  $C^*$ -probability space  $(\mathcal{A}, \tau)$  is called a *semicircular system* if  $\{x_1, \dots, x_k\}$  is a free family of selfadjoint random variables, and for every  $i = 1, \dots, k$ ,  $\mu_{x_i}$  is the standard semicircular distribution  $\nu_{0,1}$  defined by

$$(2.3) \quad d\nu_{0,1}(t) = \frac{1}{2\pi} \sqrt{4 - t^2} \mathbf{1}_{[-2,2]}(t) dt.$$

An element  $u \in \mathcal{A}$  is called a *Haar unitary* if  $u^* = u^{-1}$  and  $\tau(u^n) = 0$  for all  $n \in \mathbb{Z} \setminus \{0\}$ . Note that Theorem 1.2 in [12] deals with deterministic  $a_N$  but the random case readily follows as pointed out by assertion 2 in [36, Section 3]. The point of Theorem 2.3 is, of course, that the resulting convergence is strong. Convergence in distribution was established earlier (see [49], [25], [3, Theorem 5.4.5]).

We also need a simple coupling result from [23, Lemma 5.1].

**Lemma 2.4.** *Suppose given selfadjoint matrices  $C_N, D_N \in M_N(\mathbb{C})$ ,  $N \in \mathbb{N}$ , such that the sequences  $\{C_N\}_{N \in \mathbb{N}}$  and  $\{D_N\}_{N \in \mathbb{N}}$  converge strongly in distribution. Then there exist diagonal matrices  $\tilde{C}_N, \tilde{D}_N \in M_N(\mathbb{C})$ ,  $N \geq 1$ , such that  $\mu_{\tilde{C}_N} = \mu_{C_N}$ ,  $\mu_{\tilde{D}_N} = \mu_{D_N}$ , and the sequence  $\{(\tilde{C}_N, \tilde{D}_N)\}_{N \in \mathbb{N}}$  converges strongly in distribution.*

## 3. DESCRIPTION OF THE MODELS

In order to describe in detail our matrix models, we need two compactly supported probability measures  $\mu$  and  $\nu$  on  $\mathbb{R}$ , a positive integer  $p$ , and a sequence of fixed real numbers  $\theta_1 \geq \theta_2 \geq \dots \geq \theta_p$  in  $\mathbb{R} \setminus \text{supp}(\mu)$ . The matrix  $A_N \in M_N(\mathbb{C})$  is random selfadjoint for all  $N \in \mathbb{N}, N \geq 1$  and satisfies the following conditions:

- (A1) almost surely, the sequence  $\{A_N\}_{N=1}^\infty$  converges in distribution to  $\mu$ ,
- (A2)  $\theta_1 \geq \theta_2 \geq \dots \geq \theta_p$  are  $p$  eigenvalues of  $A_N$ , and
- (A3) the other eigenvalues of  $A_N$ , which may be random, converge uniformly almost surely to  $\text{supp}(\mu)$ : almost surely, for every  $\varepsilon > 0$  there exists  $N(\varepsilon) \in \mathbb{N}$  such that

$$\sigma(A_N) \setminus \{\theta_1, \dots, \theta_p\} \subseteq \text{supp}(\mu) + (-\varepsilon, \varepsilon), \quad N \geq N(\varepsilon).$$

In other words, only the  $p$  eigenvalues  $\theta_1, \dots, \theta_p$  prevent  $\{A_N\}_{N=1}^\infty$  from converging strongly in distribution to  $\mu$ .

We investigate two polynomial matricial models, both involving  $A_N$ . The first model involves a sequence  $\{B_N\}_{N=1}^\infty$  of random Hermitian matrices such that

- (B0)  $B_N$  is independent from  $A_N$ ,
- (B1)  $B_N$  converges strongly in distribution to the compactly supported probability measure  $\nu$  on  $\mathbb{R}$ ,
- (B2) for each  $N$ , the distribution of  $B_N$  is invariant under conjugation by arbitrary  $N \times N$  unitary matrices.

The matricial model is

$$(3.1) \quad Z_N = P(A_N, B_N)$$

for an arbitrary selfadjoint polynomial  $P \in \mathbb{C}\langle X_1, X_2 \rangle$ .

The second model deals with  $N \times N$  random Hermitian Wigner matrices  $X_N = [X_{ij}]_{i,j=1}^N$ , where  $[X_{ij}]_{i \geq 1, j \geq 1}$  is an infinite array of random variables satisfying conditions (X0) – (X3) in the introduction. The matricial model is

$$(3.2) \quad Z_N = P\left(A_N, \frac{X_N}{\sqrt{N}}\right)$$

for an arbitrary selfadjoint polynomial  $P \in \mathbb{C}\langle X_1, X_2 \rangle$ .

In the discussion of the first model, we use results of Voiculescu [49] (see also [54]), who showed that there exist a free pair  $(a, b)$  of selfadjoint elements in a  $\text{II}_1$ -factor  $(\mathcal{A}, \tau)$  such that, almost surely, the sequence  $\{(A_N, B_N)\}_{N=1}^\infty$  converges in distribution to  $(a, b)$ . Thus,  $\mu = \mu_a, \nu = \mu_b$ , and the sequence  $\{P(A_N, B_N)\}_{N=1}^\infty$  converges in distribution to  $P(a, b)$  (that is,

$$\lim_{N \rightarrow \infty} \mu_{P(A_N, B_N)} = \mu_{P(a, b)}$$

in the weak\* topology) for every selfadjoint polynomial  $P \in \mathbb{C}\langle X_1, X_2 \rangle$ . When  $p = 0$ , Lemma 2.4, Theorem 2.3 and Remark 2.1, show that, almost surely, this convergence is strong (see the proof of Corollary 2.2 in [23]).

For the second model we use [12, Proposition 2.2] and [3, Theorem 5.4.5], where it is seen that for every selfadjoint polynomial  $P \in \mathbb{C}\langle X_1, X_2 \rangle$  we have

$$\lim_{N \rightarrow \infty} \mu_{P(A_N, X_N/\sqrt{N})} = \mu_{P(a, b)}$$

almost surely in the weak\* topology, where  $a$  and  $b$  are freely independent self-adjoint noncommutative random variables,  $\mu_a = \mu$ , and  $\mu_b = \nu_{0,1}$ . As in the

first model, Theorem 2.3 and Remark 2.1 show that, almost surely, the sequence  $\{P(A_N, X_N/\sqrt{N})\}_{N=1}^\infty$  converges strongly in distribution to  $P(a, x)$  provided that  $p = 0$ .

Our main result applies, of course, to the case in which  $p > 0$ . Let  $Y_N$  be either  $B_N$  or  $X_N/\sqrt{N}$ . The set of outliers of  $P(A_N, Y_N)$  is calculated from the spikes  $\theta_1, \dots, \theta_p$  using Voiculescu's matrix subordination function [52]. When  $Y_N = B_N$ , we also show that the eigenvectors associated to these outlying eigenvalues have projections of computable size onto the eigenspaces of  $A_N$  corresponding to the spikes. The precise statements are Theorems 6.1 and 6.3. Sections 4 and 5 contain the necessary tools from operator-valued noncommutative probability theory while Sections 7–10 are dedicated to the proofs of the main results.

#### 4. LINEARIZATION

As in [4, 13], we use linearization to reduce a problem about a polynomial in freely independent, or asymptotically freely independent, random variables, to a problem about the addition of matrices having these random variables as entries. Suppose that  $P \in \mathbb{C}\langle X_1, \dots, X_k \rangle$ . For our purposes, a *linearization* of  $P$  is a linear polynomial of the form

$$z\alpha \otimes 1 - L,$$

where  $z$  is a complex variable, and

$$L = \gamma_0 \otimes 1 + \gamma_1 \otimes X_1 + \dots + \gamma_k \otimes X_k,$$

with  $\alpha, \gamma_0, \dots, \gamma_k \in M_n(\mathbb{C})$  for some  $n \in \mathbb{N}$ , and the following property is satisfied: *given  $z \in \mathbb{C}$  and elements  $a_1, \dots, a_k$  in a  $C^*$ -algebra  $\mathcal{A}$ ,  $z - P(a_1, \dots, a_k)$  is invertible in  $\mathcal{A}$  if and only if  $z\alpha \otimes 1 - L(a_1, \dots, a_k)$  is invertible in  $M_n(\mathcal{A})$ .* Usually, this is achieved by ensuring that  $(z\alpha \otimes 1 - L)^{-1}$  exists as an element of  $M_n(\mathbb{C}\langle X_1, \dots, X_k \rangle \langle (z - P)^{-1} \rangle)$  and  $(z - P)^{-1}$  is one of the entries of the  $(z\alpha \otimes 1 - L)^{-1}$ . It is known (see, for instance, [42]) that every polynomial has a linearization. See [29] for earlier uses of linearization in free probability.

In the following we also say, more concisely, that  $L$  is a linearization of  $P$ . We also suppress the unit of the algebra  $\mathcal{A}$  when there is no risk of confusion. For instance, we may write  $z\alpha - L$  in place of  $z\alpha \otimes 1 - L$ .

We describe in some detail a linearization procedure from [4] (see also [34]) that has several advantages. In this procedure, we always have  $\alpha = e_{1,1}$ , where  $e_{1,1}$  denotes the matrix whose only nonzero entry equals 1 and occurs in the first row and first column. Given  $P \in \mathbb{C}\langle X_1, \dots, X_k \rangle$ , we produce an integer  $n \in \mathbb{N}$  and a linear polynomial  $L \in M_n(\mathbb{C}\langle X_1, \dots, X_k \rangle)$  of the form

$$L = \begin{bmatrix} 0 & u \\ v & Q \end{bmatrix},$$

such that  $u \in M_{1 \times (n-1)}(\mathbb{C}\langle X_1, \dots, X_k \rangle)$ ,  $v \in M_{(n-1) \times 1}(\mathbb{C}\langle X_1, \dots, X_k \rangle)$ ,  $Q$  is an invertible matrix in  $M_{n-1}(\mathbb{C}\langle X_1, \dots, X_k \rangle)$  whose inverse is a polynomial of degree less than or equal to the degree of  $P$ , and  $uQ^{-1}v = -P$ . Moreover, if  $P = P^*$ , the coefficients of  $L$  can be chosen to be selfadjoint matrices in  $M_n(\mathbb{C})$ .

The construction proceeds by induction on the number of monomials in the given polynomial. If  $P$  is a monomial of degree 0 or 1, we set  $n = 1$  and  $L = P$ . If  $P = X_{i_1}X_{i_2}X_{i_3} \dots X_{i_{\ell-1}}X_{i_\ell}$ , where  $\ell \geq 2$  and  $i_1, \dots, i_\ell \in \{1, \dots, k\}$ , we set  $n = \ell$

and

$$L = - \begin{bmatrix} 0 & 0 & \cdots & 0 & X_{i_1} \\ 0 & 0 & \cdots & X_{i_2} & -1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & X_{i_{\ell-1}} & \cdots & 0 & 0 \\ X_{i_\ell} & -1 & \cdots & 0 & 0 \end{bmatrix}.$$

As noted in [34], the lower right  $(\ell-1) \times (\ell-1)$  corner of this matrix has an inverse of degree  $\ell-2$  in the algebra  $M_{\ell-1}(\mathbb{C}\langle X_1, \dots, X_k \rangle)$ . (The constant term in this inverse is a selfadjoint matrix and its spectrum is contained in  $\{-1, 1\}$ .) Suppose now that  $p = P_1 + P_2$ , where  $P_1, P_2 \in \mathbb{C}\langle X_1, \dots, X_k \rangle$ , and that linear polynomials

$$L_j = \begin{bmatrix} 0 & u_j \\ v_j & Q_j \end{bmatrix} \in M_{n_j}(\mathbb{C}\langle X_1, \dots, X_k \rangle), \quad j = 1, 2,$$

with the desired properties have been found for  $P_1$  and  $P_2$ . Then we set  $n = n_1 + n_2 - 1$  and observe that the matrix

$$L = \begin{bmatrix} 0 & u_1 & u_2 \\ v_1 & Q_1 & 0 \\ v_2 & 0 & Q_2 \end{bmatrix} = \begin{bmatrix} 0 & u \\ v & Q \end{bmatrix} \in M_{n_1+n_2-1}(\mathbb{C}\langle X_1, \dots, X_k \rangle).$$

is a linearization of  $P_1 + P_2$  with the desired properties. The construction of a linearization is now easily completed for an arbitrary polynomial. Suppose now that  $P$  is a selfadjoint polynomial, so  $P = P_0 + P_0^*$  for some other polynomial  $P_0$ . Suppose that the matrix

$$\begin{bmatrix} 0 & u_0 \\ v_0 & Q_0 \end{bmatrix}.$$

of size  $n_0$  is a linearization of  $P_0$ . Then we set  $n = 2n_0 - 1$  and observe that the selfadjoint linear polynomial

$$\begin{bmatrix} 0 & u_0 & v_0^* \\ u_0^* & 0 & Q_0^* \\ v_0 & Q_0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & u \\ u^* & Q \end{bmatrix}$$

linearizes  $P$ . It is easy to verify inductively that this construction produces a matrix  $Q$  such that the constant term of  $Q^{-1}$  has spectrum contained in  $\{1, -1\}$ . These properties of  $Q$  [34], and particularly the following observation, facilitate our analysis.

**Lemma 4.1.** *Let  $P \in \mathbb{C}\langle X_1, \dots, X_k \rangle$ , and let*

$$L = \begin{bmatrix} 0 & u \\ v & Q \end{bmatrix} \in M_n(\mathbb{C}\langle X_1, \dots, X_k \rangle)$$

*be a linearization of  $P$  as constructed above. There exist a permutation matrix  $T \in M_{n-1}$  and a strictly lower triangular matrix  $N \in M_{n-1}(\mathbb{C}\langle X_1, \dots, X_k \rangle)$  such that  $Q^{-1} = T(1_{n-1} + N)$ .*

*Proof.* We show that there exist a permutation matrix  $T_0 \in M_{n-1}$  and a strictly lower triangular matrix permutation matrices  $N_0 \in M_{n-1}(\mathbb{C}\langle X_1, \dots, X_k \rangle)$  such that  $Q = (1_{n-1} - N_0)T_0$ . Then we can define  $T = T_0^{-1}$  and  $N = \sum_{j=1}^{n-2} N_0^j$ . The



existence of  $T_0$  and  $N_0$  is proved by following inductively the construction of  $L$ . If  $P = X_{i_1} \cdots X_{i_\ell}$ ,  $\ell \geq 2$ , we define

$$T_0 = \begin{bmatrix} 0 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 0 \end{bmatrix},$$

and let the only nonzero entries of  $N_0$  be  $X_{i_2}, \dots, X_{i_\ell}$  just below the main diagonal. If  $P = P_1 + P_2$ , and linearizations for  $P_1$  and  $P_2$  have been found, then the desired matrices are obtained simply by taking direct sums of the matrices corresponding to  $P_1$  and  $P_2$ . The case in which  $P = P_0 + P_0^*$  is treated similarly (different factorizations must be used for  $Q_0$  and  $Q_0^*$ ).  $\square$

**Lemma 4.2.** *Suppose that  $P \in \mathbb{C}\langle X_1, \dots, X_k \rangle$ , and let*

$$L = \begin{bmatrix} 0 & u \\ v & Q \end{bmatrix} \in M_n(\mathbb{C}\langle X_1, \dots, X_k \rangle)$$

*be a linearization of  $P$  with the properties outlined above. Then for every  $N \in \mathbb{N}$ , and for every  $S_1, \dots, S_k \in M_N(\mathbb{C})$ , we have*

$$\det(ze_{1,1} \otimes I_N - L(S_1, \dots, S_k)) = \pm \det(zI_n - P(S_1, \dots, S_k)),$$

*where the sign is  $\det(Q(S_1, \dots, S_k))$ . Moreover,*

$$\dim \ker(zI_n - P(S_1, \dots, S_k)) = \dim \ker(ze_{1,1} \otimes I_N - L(S_1, \dots, S_k)) \quad z \in \mathbb{C}.$$

*Proof.* Suppressing the variables  $S_1, \dots, S_k$ , we have

$$\begin{bmatrix} 1 & -uQ^{-1} \\ 0 & 1_{n-1} \end{bmatrix} \begin{bmatrix} z & -u \\ -v & -Q \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -Q^{-1}v & 1_{n-1} \end{bmatrix} = \begin{bmatrix} z - P & 0 \\ 0 & -Q \end{bmatrix}, \quad z \in \mathbb{C}.$$

Lemma 4.1 implies that  $\det Q(S_1, \dots, S_k)$  is  $\pm 1$  and the determinant identity follows immediately. The dimension of the kernel of a square matrix does not change if the matrix is multiplied by some other invertible matrices. Also, since  $Q$  is invertible, the kernel of the matrix on the right hand side of the last equality is easily identified with  $\ker(z - P)$ . The last assertion follows from these observations.  $\square$

In the case of selfadjoint polynomials, applied to selfadjoint matrices, we can estimate how far  $ze_{1,1} - L$  is from not being invertible.

**Lemma 4.3.** *Suppose that  $P = P^* \in \mathbb{C}\langle X_1, \dots, X_k \rangle$ , and let*

$$L = \begin{bmatrix} 0 & u^* \\ u & Q \end{bmatrix} \in M_n(\mathbb{C}\langle X_1, \dots, X_k \rangle)$$

*be a linearization of  $P$  with the properties outlined above. There exist polynomials  $T_1, T_2 \in \mathbb{C}[X_1, \dots, X_k]$  with nonnegative coefficients with the following property: given arbitrary selfadjoint elements  $S_1, \dots, S_k$  in a unital  $C^*$ -algebra  $\mathcal{A}$ , and given  $z_0 \in \mathbb{C}$  such that  $z_0 - P(S)$  is invertible, we have*

$$\|(z_0 e_{1,1} - L(S))^{-1}\| \leq T_1(\|S_1\|, \dots, \|S_k\|) \|(z_0 - P(S))^{-1}\| + T_2(\|S_1\|, \dots, \|S_k\|).$$

*In particular, given two real constants  $C, \delta > 0$ , there exists  $\varepsilon > 0$  such that  $\text{dist}(z_0, \sigma(P(S))) \geq \delta$  and  $\|S_1\| + \dots + \|S_k\| \leq C$  imply  $\text{dist}(0, \sigma(z_0 e_{1,1} - L(S))) \geq \varepsilon$ .*

*Proof.* For every element  $a$  of a  $C^*$ -algebra, we have  $\text{dist}(0, \sigma(a)) \geq 1/\|a^{-1}\|$ . Equality is achieved, for instance, if  $a = a^*$ . A matrix calculation (in which we suppress the variables  $S$ ) shows that

$$(z_0 e_{1,1} - L)^{-1} = \begin{bmatrix} 1 & 0 \\ -Q^{-1}u & 1_{n-1} \end{bmatrix} \begin{bmatrix} (z_0 - P)^{-1} & 0 \\ 0 & -Q^{-1} \end{bmatrix} \begin{bmatrix} 1 & -u^*Q^{-1} \\ 0 & 1_{n-1} \end{bmatrix}.$$

The lemma follows now because the entries of  $u(S)$ ,  $u^*(S)$ , and  $Q(S)^{-1}$  are polynomials in  $S$ , and

$$\|(z_0 - P(S))^{-1}\| = 1/\text{dist}(z_0, \sigma(P(S)))$$

because  $P(S)$  is selfadjoint.  $\square$

The dependence on  $L$  in the above lemma is given via the norms of  $Q^{-1}$  and of  $u$ . Since  $\lim_{z \rightarrow \infty} \|(ze_{1,1} - L(S))^{-1}\| \neq 0$ , we see that  $T_2 \neq 0$ .

## 5. SUBORDINATION

Consider a von Neumann algebra  $\mathcal{M}$  endowed with a normal faithful tracial state  $\tau$ , let  $\mathcal{B} \subset \mathcal{N} \subset \mathcal{M}$  be unital von Neumann subalgebras, and denote by  $E_{\mathcal{N}} : \mathcal{M} \rightarrow \mathcal{N}$  the unique trace-preserving conditional expectation of  $\mathcal{M}$  onto  $\mathcal{N}$  (see [46, Proposition V.2.36]). Denote by  $\mathbb{H}^+(\mathcal{M})$  the operator upper-half plane of  $\mathcal{M}$ :  $\mathbb{H}^+(\mathcal{M}) = \{x \in \mathcal{M} : \Im x := (x - x^*)/2i > 0\}$ . Given two arbitrary selfadjoint elements  $c, d \in \mathcal{M}$ , we define the open set  $\mathcal{G}_{c,d,\mathcal{B},\mathcal{N}}$  to consist of those elements  $\beta \in \mathcal{B}$  such that  $\beta - (c + d)$  is invertible and  $E_{\mathcal{N}}((\beta - (c + d))^{-1})$  is invertible as well. Then the function

$$\omega_{c,d,\mathcal{B},\mathcal{N}} : \mathcal{G}_{c,d,\mathcal{B},\mathcal{N}} \rightarrow \mathcal{M}$$

defined by

$$(5.1) \quad \omega_{c,d,\mathcal{B},\mathcal{N}}(\beta) = c + [E_{\mathcal{N}}((\beta - (c + d))^{-1})]^{-1}, \quad \beta \in \mathcal{G}_{c,d,\mathcal{B},\mathcal{N}},$$

is analytic. This equation can also be written as

$$(5.2) \quad E_{\mathcal{N}}((\beta - (c + d))^{-1}) = (\omega_{c,d,\mathcal{B},\mathcal{N}}(\beta) - c)^{-1} \quad \beta \in \mathcal{G}_{c,d,\mathcal{B},\mathcal{N}}.$$

Properties (1), (2), and (3) in the following lemma are easy observations, while (4) follows as in [10, Remark 2.5].

**Lemma 5.1.** *Fix  $\mathcal{B} \subset \mathcal{N} \subset \mathcal{M}$  and  $c, d \in \mathcal{M}$  as above. Then:*

- (1) *The set  $\mathcal{G}_{c,d,\mathcal{B},\mathcal{N}}$  is selfadjoint.*
- (2)  *$\omega_{c,d,\mathcal{B},\mathcal{N}}(\beta^*) = \omega_{c,d,\mathcal{B},\mathcal{N}}(\beta)^*$ ,  $\beta \in \mathcal{G}_{c,d,\mathcal{B},\mathcal{N}}$ .*
- (3)  *$\mathbb{H}^+(\mathcal{B}) \subset \mathcal{G}_{c,d,\mathcal{B},\mathcal{N}}$  and  $\omega_{c,d,\mathcal{B},\mathcal{N}}(\mathbb{H}^+(\mathcal{B})) \subset \mathbb{H}^+(\mathcal{N})$ .*
- (4)  *$\Im(\omega_{c,d,\mathcal{B},\mathcal{N}}(\beta)) \geq \Im(\beta)$ ,  $\beta \in \mathbb{H}^+(\mathcal{B})$ .*

There is one important case in which  $\omega_{c,d,\mathcal{B},\mathcal{N}}$  takes values in  $\mathcal{B}$ , and thus (5.2) allows us to view  $\omega_{c,d,\mathcal{B},\mathcal{N}}|_{\mathbb{H}^+(\mathcal{B})}$  as a subordination function in the sense of Littlewood. Denote by  $\mathcal{G}_{c,d,\mathcal{B},\mathcal{N}}^0$  the connected component of  $\mathcal{G}_{c,d,\mathcal{B},\mathcal{N}}$  that contains  $\mathbb{H}^+(\mathcal{B})$ . The following basic result is from [48].

**Theorem 5.2.** *With the above notation, suppose that  $c$  and  $d$  are free over  $\mathcal{B}$  and  $\mathcal{N} = \mathcal{B}\langle c \rangle$  is the unital von Neumann generated by  $\mathcal{B}$  and  $c$ . Then*

$$\omega_{c,d,\mathcal{B},\mathcal{N}}(\mathcal{G}_{c,d,\mathcal{B},\mathcal{N}}^0) \subset \mathcal{B}.$$

In our applications, the algebra  $\mathcal{B}$  is (isomorphic to)  $M_n(\mathbb{C})$  for some  $n \in \mathbb{N}$ . More precisely, let  $\mathcal{M}$  be a von Neumann algebra endowed with a normal faithful tracial state  $\tau$ , and let  $n \in \mathbb{N}$ . Then  $M_n(\mathbb{C})$  can be identified with the subalgebra  $M_n(\mathbb{C}) \otimes 1$  of  $M_n(\mathcal{M}) = M_n(\mathbb{C}) \otimes \mathcal{M}$ . Moreover,  $M_n(\mathcal{M})$  is endowed with the faithful normal tracial state  $\text{tr}_n \otimes \tau = (1/n)\text{Tr}_n \otimes \tau$ , and  $\text{Id}_{M_n(\mathbb{C})} \otimes \tau$  is the trace-preserving conditional expectation from  $M_n(\mathcal{M})$  to  $M_n(\mathbb{C})$ . The following result is from [37].

**Proposition 5.3.** *Let  $\mathcal{M}$  be a von Neumann algebra endowed with a normal faithful tracial state  $\tau$ , let  $c, d \in \mathcal{M}$  be freely independent, let  $n$  be a positive integer, and let  $\gamma_1, \gamma_2 \in M_n(\mathbb{C})$ . Then  $\gamma_1 \otimes c$  and  $\gamma_2 \otimes d$  are free over  $M_n(\mathbb{C})$ .*

We show next how the spectrum of  $P(c, d)$  relates with the functions  $\omega$  defined above. Thus, we fix  $P = P^* \in \mathbb{C}\langle X_1, X_2 \rangle$  and a linearization  $L = \gamma_0 \otimes 1 + \gamma_1 \otimes X_1 + \gamma_2 \otimes X_2$  of  $P$  as constructed in Section 4. Thus,  $\gamma_0, \gamma_1, \gamma_2 \in M_n(\mathbb{C})$  are selfadjoint matrices for some  $n \in \mathbb{N}$ . (Clearly  $\gamma_1 \neq 0$  unless  $P \in \mathbb{C}\langle X_2 \rangle$ .) Then we consider the random variables  $\gamma_1 \otimes c$  and  $\gamma_2 \otimes d$  in  $M_n(\mathcal{M})$ , the algebra  $\mathcal{B} = M_n(\mathbb{C}) \subset M_n(\mathcal{M})$ , and  $\mathcal{N} = M_n(\mathbb{C}\langle c \rangle)$ ; clearly  $\mathcal{B} \subset \mathcal{N} \subset M_n(\mathcal{M})$ . We set

$$\mathcal{G} = \mathcal{G}_{\gamma_1 \otimes c, \gamma_2 \otimes d, \mathcal{B}, \mathcal{N}}^0$$

and

$$\omega = \omega_{\gamma_1 \otimes c, \gamma_2 \otimes d, \mathcal{B}, \mathcal{N}} : \mathcal{G} \rightarrow \mathcal{N}.$$

Thus,

$$E_{\mathcal{N}} [(\beta \otimes 1 - \gamma_1 \otimes c - \gamma_2 \otimes d)^{-1}] = (\omega(\beta) \otimes 1 - \gamma_1 \otimes c)^{-1}, \quad \beta \in \mathcal{G}.$$

The left hand side of this equation is defined if  $\beta = ze_{11} - \gamma_0$  for some  $z \in \mathbb{C} \setminus \sigma(P(c, d))$ , so it would be desirable that  $ze_{11} - \gamma_0 \in \mathcal{G}$  for such values of  $z$ . This is not true except for special cases. (One such case applies to  $P = X_1 + X_2$  if  $d$  is a semicircular variable free from  $c$  [16, 9].) The following lemma offers a partial result.

**Lemma 5.4.** *With the notation above, there exists  $k > 0$  depending only on  $L$ ,  $\|c\|$ , and  $\|d\|$  such that  $ze_{11} - \gamma_0 \in \mathcal{G}$  if  $|z| > k$ . The analytic function  $u(z) = \omega(ze_{11} - \gamma_0)$  satisfies the equation  $u(z) = u(z)^*$  for  $|z| > k$ .*

*Proof.* Define an analytic function  $F : \mathbb{C} \setminus \sigma(P(c, d)) \rightarrow \mathcal{N}$  by

$$(5.3) \quad F(z) = E_{\mathcal{N}} [((ze_{11} - \gamma_0) \otimes 1 - \gamma_1 \otimes c - \gamma_2 \otimes d)^{-1}], \quad z \in \mathbb{C} \setminus \sigma(P(c, d)).$$

We show that  $F(z)$  is invertible if  $|z|$  is sufficiently large. Suppressing the variables  $c$  and  $d$  from the notation, it follows from the factorization used in the proof of Lemma 4.2 that

$$F(z) = E_{\mathcal{N}} \begin{bmatrix} (z - P)^{-1} & -(z - P)^{-1}u^*Q^{-1} \\ -Q^{-1}u(z - P)^{-1} & Q^{-1}u(z - P)^{-1}u^*Q^{-1} - Q^{-1} \end{bmatrix}.$$

Moreover, because of the matrix structure of  $\mathcal{N}$ , this matrix can be obtained by applying  $E_{\mathbb{C}\langle c \rangle}$  entrywise. According to the Schur complement formula, a matrix  $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$  is invertible if both  $A$  and  $D - BA^{-1}C$  are invertible. For our matrix, we have  $A = A(z) = E_{\mathbb{C}\langle c \rangle}((z - P)^{-1})$ . The fact that  $\|zA(z) - 1\| < 1$  for  $|z| > 2\|P\|$  implies that  $A(z)$  is invertible. Next, we see that  $\|(z - P)^{-1}\|$  and  $\|A(z)^{-1}\|$  are comparable to  $1/|z|$  and  $|z|$ , respectively. Using these estimates, one sees also that

$\lim_{|z| \rightarrow \infty} \|B(z)A(z)^{-1}C(z)\| = 0$ , so the invertibility of  $F(z)$  would follow from the invertibility of  $D(z)$  for large  $|z|$ . Since

$$\lim_{|z| \rightarrow \infty} \|D(z) + E_{M_{n-1}(\mathbb{C}\langle c \rangle)}(Q^{-1})\| = 0,$$

we only need to verify that  $E_{M_{n-1}(\mathbb{C}\langle c \rangle)}(Q^{-1})$  is invertible. Write  $Q^{-1} = T(1_{n-1} + N)$  as in Lemma 4.1. We have

$$E_{M_{n-1}(\mathbb{C}\langle c \rangle)}(Q^{-1}) = T(1_{n-1} + E_{M_{n-1}(\mathbb{C}\langle c \rangle)}(N)),$$

and  $E_{M_{n-1}(\mathbb{C}\langle c \rangle)}(N)$  is strictly lower triangular. The invertibility of  $E_{M_{n-1}(\mathbb{C}\langle c \rangle)}(Q^{-1})$  follows. The quantities  $\|D(z) + E_{M_{n-1}(\mathbb{C}\langle c \rangle)}(Q^{-1})\|$  and  $\|Q^{-1}\|$  can be estimated using only  $L$ ,  $\|c\|$ , and  $\|d\|$ , and this shows that  $k$  can be chosen as a function of these objects. The last assertion of the lemma is immediate.  $\square$

The estimates in the preceding proof apply, by virtue of continuity, to nearby points in  $M_n(\mathbb{C})$ . We record the result for later use.

**Corollary 5.5.** *Let  $c, d$  and  $k$  be as in Lemma 5.4 and let  $z \in \mathbb{C} \setminus [-k, k]$ . Then there exist a constant  $k' > 0$  and a neighborhood  $W$  of  $ze_{1,1} - \gamma_0$ , depending only on  $\|c\|$ ,  $\|d\|$ , and  $L$ , such that  $V \subset \mathcal{G}$  and  $\|\omega(\beta)\| \leq k'$  for  $\beta \in W$ .*

In some cases of interest, the analytic function  $u$  extends to the entire upper and lower half-planes. We recall that a function  $v$  defined in a domain  $G \subset \mathbb{C}$  with values in a Banach space  $\mathcal{X}$  is said to be meromorphic if, for every  $z_0 \in G$ , the function  $(z - z_0)^n v(z)$  is analytic in a neighborhood of  $z_0$  for sufficiently large  $n$ . For instance, if  $\mathcal{X}$  is a finite dimensional Banach algebra and  $h : G \rightarrow \mathcal{X}$  is an analytic function such that  $h(z)$  is invertible for some  $z \in G$ , then the function  $v(z) = h(z)^{-1}$  is meromorphic in  $G$ . This fact follows easily once we identify  $\mathcal{X}$  with an algebra of matrices, so the inverse can be calculated using determinants.

**Lemma 5.6.** *The function  $u$  defined in Lemma 5.4 is meromorphic in  $\mathbb{C} \setminus \sigma(P(c, d))$ .*

*Proof.* The lemma follows immediately from the observation preceding the statement applied to the function  $F$  defined in (5.3) which is analytic in  $\mathbb{C} \setminus \sigma(P(c, d))$  since, by hypothesis and by Theorem 5.2,  $u$  takes values in a finite dimensional algebra.  $\square$

The conclusion of the preceding lemma applies, for instance, when  $\mathcal{M} = M_N \otimes L^\infty(\Omega)$  with the usual trace  $\text{tr}_N \otimes \mathbb{E}$ . This situation arises in the study of random matrices. The function  $u$  is also meromorphic provided that  $c$  and  $d$  are free random variables and  $\mathcal{N} = M_n(\mathbb{C})\langle \gamma_1 \otimes c \rangle$ . Since  $\gamma_1 \neq 0$ , we have  $\mathcal{N} = M_n(\mathbb{C}\langle c \rangle)$ .

**Lemma 5.7.** *If  $c$  and  $d$  are free and  $\mathcal{N} = M_n(\mathbb{C})\langle \gamma_1 \otimes c \rangle$ , then the function  $u$  defined in Lemma 5.4 is meromorphic in  $\mathbb{C} \setminus \sigma(P(c, d))$  with values in  $M_n(\mathbb{C})$ . Moreover, given an arbitrary  $\lambda \in \sigma(c)$ , the function  $(u - \lambda\gamma_1)^{-1}$  extends analytically to  $\mathbb{C} \setminus \sigma(P(c, d))$ .*

*Proof.* Theorem 5.2 shows that  $\omega$  takes values in  $M_n(\mathbb{C})$ . We have established that the domain  $\mathcal{G}$  of  $\omega$  contains  $z \otimes e_{11} - \gamma_0$  for sufficiently large  $|z|$ . Fix a character  $\chi$  of the commutative  $C^*$ -algebra  $\mathbb{C}\langle c \rangle$  and denote by  $\chi_n : M_n(\mathbb{C})\langle \gamma_1 \otimes c \rangle \rightarrow M_n(\mathbb{C})$  the algebra homomorphism obtained by applying  $\chi$  to each entry. Using the notation (5.3), we have

$$(\omega(z \otimes e_{11} - \gamma_0) - \gamma_1 \otimes \chi(c))\chi_n(F(z)) = I_n$$

for sufficiently large  $|z|$ . It follows immediately that the function

$$u_1(z) = \gamma_1 \otimes \chi(c) + [\chi_n(F(z))]^{-1}$$

is a meromorphic continuation of  $u$  to  $\mathbb{C} \setminus \sigma(P(c, d))$ . Moreover, the equation

$$(5.4) \quad (u_1(z) - \gamma_1 \otimes c)F(z) = I_n$$

holds for large values  $|z|$  and hence it holds on the entire domain of analyticity of  $u_1$ . It follows that  $\mathcal{G}$  contains  $z \otimes e_{11} - \gamma_0$  whenever  $z \in \mathbb{C} \setminus \sigma(P(c, d))$  is not a pole of  $u_1$ , and  $\omega(z \otimes e_{11} - \gamma_0) = u_1(z)$  for such values of  $z$ . To verify the last assertion, choose  $\chi$  such that  $\chi(c) = \lambda$  and apply  $\chi_n$  to (5.4) to obtain

$$(u_1(z) - \lambda \gamma_1) \chi_n(F(z)) = I_n.$$

Thus  $\chi_n \circ F$  is an analytic extension of  $(u - \lambda \gamma_1)^{-1}$  to  $\mathbb{C} \setminus \sigma(P(c, d))$ .  $\square$

## 6. MAIN RESULTS AND EXAMPLE

Fix a polynomial  $P = P^* \in \mathbb{C}\langle X_1, X_2 \rangle$  and choose, as in Section 4, a linearization of  $P$  of the form  $ze_{1,1} - L$ , where  $L = \gamma_0 \otimes 1 + \gamma_1 \otimes X_1 + \gamma_2 \otimes X_2 \in M_n(\mathbb{C}\langle X_1, X_2 \rangle)$ . In particular,  $\gamma_0, \gamma_1, \gamma_2 \in M_n(\mathbb{C})$  are selfadjoint matrices.

Suppose that  $\{A_N\}_{N \in \mathbb{N}}$  and  $\{B_N\}_{N \in \mathbb{N}}$  are two sequences of selfadjoint random matrices satisfying the hypotheses (A1)–(A3) and (B0)–(B2) of Section 3. As noted earlier, the pairs  $(A_N, B_N)$  in  $M_N(\mathbb{C})$  converge almost surely in distribution to a pair  $(a, b)$  of freely independent selfadjoint random variables in a  $C^*$ -probability space  $(\mathcal{A}, \tau)$  such that  $\mu_a = \mu$  and  $\mu_b = \nu$ . By Theorem 5.2, there exists a selfadjoint open set  $\mathcal{G} \subset M_n(\mathbb{C})$ , and an analytic function  $\omega: \mathcal{G} \rightarrow M_n(\mathbb{C})$  such that

$$(\omega(\beta) \otimes 1 - \gamma_1 \otimes a)^{-1} = E_{M_n(\mathbb{C}\langle a \rangle)} [(\beta \otimes 1 - (\gamma_1 \otimes a + \gamma_2 \otimes b))^{-1}], \quad \beta \in \mathcal{G}.$$

As shown in Lemma 5.7, the map

$$u(z) = \omega(z \otimes e_{1,1} - \gamma_0)$$

is meromorphic on  $\mathbb{C} \setminus \sigma(P(a, b))$ . Define a new function

$$u_0(z) = (u(ze_{1,1} - \gamma_0) + iI_n)^{-1}.$$

It follows from Lemma 5.1 that  $u_0$  continues analytically to a neighbourhood of  $\mathbb{R} \setminus \sigma(P(a, b))$ . (Indeed,  $u_0$  is bounded near every pole of  $u$ .) Define

$$H_j(z) = \det[(\theta_j \gamma_1 + i)u_0(z) - I_n], \quad j = 1, \dots, p$$

and denote by  $m_j(t)$  the order of  $t$  as a zero of  $H_j(z)$  at  $z = t$ . Also set  $m(t) = m_1(t) + \dots + m_p(t)$  for  $t \in \mathbb{R} \setminus \sigma(P(a, b))$ , and note that  $\{t : m(t) \neq 0\}$  is an isolated set in  $\mathbb{R} \setminus \sigma(P(a, b))$ . With this notation, we are ready to state our first main result. The notation  $E_{A_N}$  indicates the spectral measure of the matrix  $A_N$ , that is,  $E_{A_N}(S)$  is the orthogonal projection onto the linear span of the eigenvectors of  $A$  corresponding to eigenvalues in the Borel set  $S$ .

**Theorem 6.1.** (1) *Suppose that  $t \in \mathbb{R} \setminus \sigma(P(a, b))$ . Then there exists  $\delta_0 > 0$  such that for every  $\delta \in (0, \delta_0)$ , almost surely for large  $N$ , the random matrix  $P(A_N, B_N)$  has exactly  $m(t)$  eigenvalues in the interval  $(t - \delta, t + \delta)$ , counting multiplicity.*

(2) *Suppose in addition that the spikes of  $A_N$  are distinct and  $\det H_{i_0}(t) = 0$ . Then, for  $\varepsilon$  small enough, almost surely*

$$(6.1) \quad \lim_{N \rightarrow \infty} \|E_{A_N}(\{\theta_i\}) [E_{P(A_N, B_N)}((t - \varepsilon, t + \varepsilon)) - \delta_{i, i_0} \mathcal{C}_i(t) I_N] E_{A_N}(\{\theta_i\})\| = 0,$$

where  $\mathcal{C}_i(t) = \lim_{z \rightarrow t} (z - t) [(u(z) - \theta_i \gamma_1)^{-1}]_{1,1}$  is the residue of the meromorphic function  $[(u(z) - \theta_i \gamma_1)^{-1}]_{1,1}$  at  $z = t$ .

**Remark 6.2.** If we know in addition that  $\omega$  is analytic at the point  $\beta = te_{1,1} - \gamma_0$ , then the function  $H_j(z)$  can be replaced by  $z \mapsto \det[\theta_j \gamma_1 - u(z)]$ . In that case,  $m(t)$  is equal to the multiplicity of  $t$  as a zero of

$$z \mapsto \prod_{j=1}^p \det[\theta_j \gamma_1 - u(z)].$$

This situation arises, for instance, when  $b$  is a semicircular variable and it is relevant when  $B_N$  is replaced by a Wigner matrix  $X_N/\sqrt{N}$ . Under the hypotheses (X0) – (X3) of Section 3, we obtain the following result. Note that the subordination function  $\omega$  has the more explicit form

$$\omega(\beta) = \beta - \gamma_2(\text{Id}_{M_n(\mathbb{C})} \otimes \tau) [(\beta \otimes 1_{\mathcal{A}} - \gamma_1 \otimes a - \gamma_2 \otimes b)^{-1}] \gamma_2, \quad \beta \in \mathcal{G}.$$

**Theorem 6.3.** Let  $a$  and  $b$  be free selfadjoint elements in a  $C^*$ -probability space  $(\mathcal{A}, \tau)$  with distribution  $\mu$  and  $\nu_{0,1}$  respectively (see (2.3)),  $t \in \mathbb{R} \setminus \sigma(P(a, b))$ , and let  $m(t)$  be defined as in Remark 6.2. Then, for sufficiently small  $\varepsilon$ , almost surely for large  $N$ , there are exactly  $m(t)$  eigenvalues of  $P(A_N, X_N/\sqrt{N})$  in an  $\varepsilon$ -neighborhood of  $t$ .

**Remark 6.4.** The subordination function can be calculated more explicitly if  $\mu = \delta_0$  (and hence  $a = 0$ ). In this case,

$$\omega(\beta) = \{(\text{Id}_{M_n(\mathbb{C})} \otimes \tau) [(w \otimes 1 - \gamma_2 \otimes b)^{-1}]\}^{-1}.$$

As an illustration, consider the random matrix

$$M = A_N \frac{X_N}{\sqrt{N}} + \frac{X_N}{\sqrt{N}} A_N + \frac{X_N^2}{N},$$

where  $X_N$  is a standard G.U.E. matrix of size  $N$  (thus, each entry of  $X_N$  has unit norm in  $L^2(\Omega)$ ) and

$$A_N = \text{Diag}(\theta, 0, \dots, 0), \quad \theta \in \mathbb{R} \setminus \{0\}.$$

In this case,  $A_N$  has rank one, and thus  $\mu = \delta_0$ . It follows that the limit spectral measure  $\rho$  of  $M$  is the same as the limit spectral measure of  $X_N^2/N$ . Thus,  $\eta$  is the Marchenko-Pastur distribution  $\rho$  with parameter 1:

$$d\eta(x) = \frac{\sqrt{(4-x)x}}{2\pi x} 1_{(0,4)}(x) dx.$$

The polynomial  $P$  is  $P(X_1, X_2) = X_1 X_2 + X_2 X_1 + X_2^2$ ,  $\mu = \delta_0$ , and  $\nu$  is the standard semi-circular distribution. An economical linearization of  $P$  is provided by  $L = \gamma_0 \otimes 1 + \gamma_1 \otimes X_1 + \gamma_2 \otimes X_2$ , where

$$\gamma_0 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix}, \quad \gamma_1 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad \gamma_2 = \begin{bmatrix} 0 & 1 & \frac{1}{2} \\ 1 & 0 & 0 \\ \frac{1}{2} & 0 & 0 \end{bmatrix}.$$

Denote by

$$G_\eta(z) = \int_0^4 \frac{1}{z-t} d\eta(t) = \frac{z - \sqrt{z^2 - 4z}}{2z}, \quad z \in \mathbb{C} \setminus [0, 4]$$

the Cauchy transform of the measure  $\eta$ . (The branch of the square root is chosen so  $\sqrt{z^2 - 4z} > 0$  for  $z > 4$ .) This function satisfies the quadratic equation  $zG_\eta(z)^2 - zG_\eta(z) + 1 = 0$ . Suppose now that  $x \notin [0, 4]$ . Denoting by  $E = \text{Id}_{M_3(\mathbb{C})} \otimes \tau: M_3(\mathcal{A}) \rightarrow M_3(\mathbb{C})$  the usual expectation and using Remark 6.4, we have

$$\omega(xe_{1,1} - \gamma_0) = E[(xe_{1,1} - \gamma_0 - \gamma_2 \otimes b)^{-1}]^{-1}, \quad x \in \mathbb{R} \setminus [0, 4].$$

The inverse of  $(xe_{1,1} - \gamma_0) \otimes 1 - \gamma_2 \otimes b$  is then calculated explicitly and application of the expected value to its entries yields eventually

$$\omega(xe_{11} - \gamma_0) = \begin{bmatrix} \frac{1}{G_\eta(x)} & 0 & 0 \\ 0 & \frac{1}{xG_\eta(x)} - 1 & \frac{1}{2xG_\eta(x)} + \frac{1}{2} \\ 0 & \frac{1}{2xG_\eta(x)} + \frac{1}{2} & \frac{1}{4xG_\eta(x)} - \frac{1}{4} \end{bmatrix}.$$

After calculation, the equation  $\det[\gamma_1\theta - \omega(xe_{11} - \gamma_0)] = 0$  reduces to

$$(6.2) \quad \theta^2 G_\eta(x)^2 - (1 - G_\eta(x)) = 0.$$

This equation has two solutions, namely

$$\frac{2\theta^4}{-(3\theta^2 + 1) \pm \sqrt{4\theta^2 + 1}(\theta^2 + 1)},$$

one of which is negative. The positive solution belongs to  $[4, +\infty)$  precisely when  $|\theta| > \sqrt{2}$ . Thus, the matrix  $M_N$  exhibits one (negative) outlier when  $0 < |\theta| \leq \sqrt{2}$  and two outliers (one negative and one  $> 4$ ) when  $|\theta| > \sqrt{2}$ . The second situation is illustrated by the simulation presented in Figure 1.

## 7. OUTLINE OF THE PROOFS

We consider first the matricial model (3.1), that is,  $Z_N = P(A_N, B_N)$ , where  $A_N$  and  $B_N$  are independent and the distribution of  $B_N$  is invariant under unitary conjugation. As seen in [23, Proposition 6.1],  $B_N$  can be written as  $B_N = U_N D_N U_N^*$  almost surely, where  $U_N$  is distributed according to the Haar measure on the unitary group  $U(N)$ ,  $D_N$  is a diagonal random matrix, and  $U_N$  is independent from  $D_N$ . As pointed out in [36, Section 3, Assertion 2], it suffices to prove Theorem 6.1 under the assumption that  $A_N$  and  $D_N$  are constant selfadjoint matrices that can be taken to be diagonal in the standard basis. Thus, we work with

$$A_N = \text{Diag}(\lambda_1(A_N), \dots, \lambda_N(A_N))$$

and

$$B_N = U_N D_N U_N^*, \quad D_N = \text{Diag}(\lambda_1(D_N), \dots, \lambda_N(D_N))$$

where  $\lambda_j(A_N) = \theta_j$ ,  $1 \leq j \leq p$ , and  $U_N$  is uniformly distributed in  $U(N)$ .

Similarly, the proof for the second model  $Z_N = P(A_N, X_N/\sqrt{N})$  reduces to the special case in which  $A_N$  is a constant matrix.

Choose a linearization  $L$  of  $P$  as in Section 4. In the spirit of [14], the first step in the proofs of Theorems 6.1 and 6.3 consists of reducing the problem to the convergence of random matrix function  $F_N$  of fixed size  $np$ , involving the generalized resolvent of the linearization applied to  $Z_N$ . For the first model, this convergence is established in Section 8 by extending the arguments of [11] and making use of the properties of the operator-valued subordination function described in Section 5. For the second model, the convergence of  $F_N$  is obtained in Section 10 via a

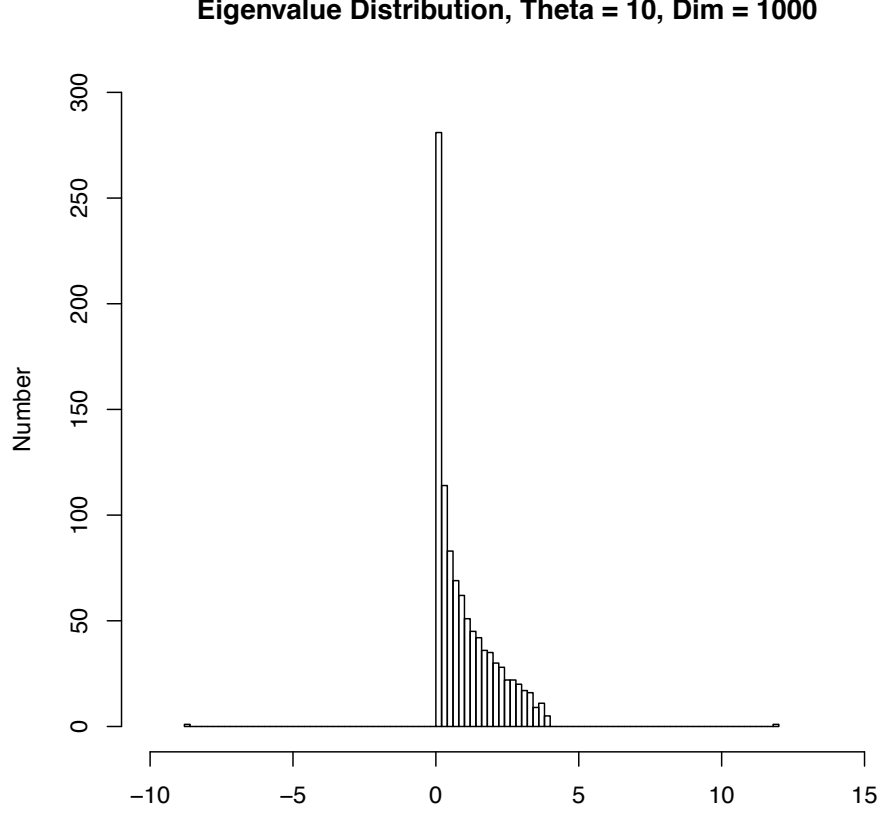


FIGURE 1. One sample from the model described in remark 6.4 corresponding to  $\theta = 10$ , with matrix size  $N = 1000$ .

comparison with the G.U.E. case. The case in which  $X_N$  is a G.U.E. is, of course, a particular case of the unitarily invariant model.

#### 8. EXPECTATIONS OF MATRIX-VALUED RANDOM ANALYTIC MAPS

As seen earlier in this paper, it suffices to prove Theorem 6.1 in the special case in which the matrix  $A_N$  is constant and  $B_N$  is a random unitary conjugate of another constant matrix. In this section, we establish some useful ingredients specific to this situation. We fix sequences  $\{C_N\}_{N \in \mathbb{N}}$  and  $\{D_N\}_{N \in \mathbb{N}}$ , where  $C_N, D_N \in M_N(\mathbb{C})$ , and a sequence  $\{U_N\}_{N \in \mathbb{N}}$  of random matrices such that  $U_N$  is uniformly distributed in the unitary group  $U(N)$ . We also fix a selfadjoint polynomial  $P \in \mathbb{C}\langle X_1, X_2 \rangle$  and a selfadjoint linearization

$$L = \gamma_0 \otimes 1 + \gamma_1 \otimes X_1 + \gamma_2 \otimes X_2 \in M_n(\mathbb{C}\langle X_1, X_2 \rangle)$$

of  $P$  as in Section 4. The random variables  $c_N = C_N \otimes 1_\Omega$  and  $d_N = U_N D_N U_N^*$  are viewed as elements of the noncommutative probability space  $(\mathcal{M}_N, \tau_N)$ , where



$\mathcal{M}_N = M_N(\mathbb{C}) \otimes L^\infty(\Omega)$  and  $\tau_N = \text{tr}_N \otimes \mathbb{E}$ . For every  $N \in \mathbb{N}$  we consider the elements  $\gamma_1 \otimes c_N, \gamma_2 \otimes d_N \in M_n(\mathbb{C}) \otimes \mathcal{M}_N$ , and the algebras  $\mathcal{B}_N = M_n(\mathbb{C}) \otimes I_N$  and  $\mathcal{N}_N = M_n(\mathbb{C}) \otimes M_N(\mathbb{C})$ , both identified with subalgebras of  $M_n(\mathbb{C}) \otimes \mathcal{M}_N$ . The conditional expectation  $E_{\mathcal{N}_N} : M_n(\mathbb{C}) \otimes \mathcal{M}_N \rightarrow \mathcal{N}_N$  is simply the expected value and it is accordingly denoted  $\mathbb{E}$ . We use the notation

$$\mathcal{G}_N = \mathcal{G}_{\gamma_1 \otimes c_N, \gamma_2 \otimes d_N, \mathcal{B}_N, \mathcal{N}_N}, \quad \omega_N = \omega_{\gamma_1 \otimes c_N, \gamma_2 \otimes d_N, \mathcal{B}_N, \mathcal{N}_N}.$$

Recall that  $\mathcal{G}_N$  consists of those matrices  $\beta \in M_n(\mathbb{C})$  with the property that  $\beta - \gamma_1 \otimes c_N - \gamma_2 \otimes d_N$  is invertible in  $M_n(\mathbb{C}) \otimes \mathcal{M}_N$  and  $\mathbb{E}((\beta - \gamma_1 \otimes c_N - \gamma_2 \otimes d_N)^{-1})$  is invertible in  $\mathcal{N}_N$ . In particular, if  $\beta \in \mathcal{G}_N$  then the matrix  $\beta - \gamma_1 \otimes C_N - \gamma_2 \otimes V D_N V^*$  is invertible for every  $V \in U(N)$ . The set  $\mathcal{G}_N$  is open and it contains  $\mathbb{H}^+(M_n(\mathbb{C}))$ . According to Lemma 5.6 and the remarks following it, the function  $u_N(z) = \omega_N(ze_{1,1} - \gamma_0)$  is meromorphic in  $\mathbb{C} \setminus \sigma(P(c_N, d_N))$ .

For simplicity of notation, we write

$$(8.1) \quad R_N(\beta) = (\beta - \gamma_1 \otimes c_N - \gamma_2 \otimes d_N)^{-1}, \quad \beta \in \mathcal{G}_N,$$

and observe that  $R_N(\beta)$  is an element of  $M_n(\mathbb{C}) \otimes M_N(\mathbb{C}) \otimes L^\infty(\Omega)$ , that is, a random matrix of size  $nN$ . We also write

$$(8.2) \quad R_N(V, \beta) = (\beta - \gamma_1 \otimes C_N - \gamma_2 \otimes V D_N V^*)^{-1}, \quad \beta \in \mathcal{G}_N, V \in U(N),$$

for sample values of this random variable. The function  $\omega_N$  is given by

$$(8.3) \quad \omega_N(\beta) = \gamma_1 \otimes C_N + (\mathbb{E}(R_N(\beta)))^{-1}, \quad \beta \in \mathcal{G}_N.$$

We start by showing that the matrix  $\omega_N(\beta)$  has a block diagonal form, thus extending [11, Lemma 4.7]. We recall that the commutant and double commutant of a set  $S \subset M_m(\mathbb{C})$  are denoted by  $S'$  and  $S''$ , respectively. We use the fact that  $M_n(\mathbb{C}) \otimes S'' = (I_n \otimes S')'$ . If  $S = \{C_N\}$  then  $\{C_N\}''$  is the linear span of the matrices  $\{I_N, C_N, \dots, C_N^{N-1}\}$ . In particular, every eigenvector of  $C_N$  is a common eigenvector for the elements of  $\{C_N\}''$ .

For each  $N \in \mathbb{N}$ , we select an eigenbasis  $\{f_N^{(1)}, \dots, f_N^{(N)}\}$  for the operator  $C_N$  and denote by  $\lambda_N^{(j)}$  the corresponding eigenvalues, that is,  $C_N f_N^{(j)} = \lambda_N^{(j)} f_N^{(j)}$ . We write  $P_N^{(j)} = f_N^{(j)} \otimes f_N^{(j)*} \in M_N(\mathbb{C})$  for the orthogonal projection onto the space generated by  $f_N^{(j)}$ ,  $j = 1, \dots, N$ . Thus, the double commutant  $\{C_N\}''$  is contained in the linear span of  $\{P_N^{(j)} : j = 1, \dots, N\}$ .

We write  $[x, y] = xy - yx$  for the commutator of two elements  $x, y$  in an algebra.

**Lemma 8.1.** *For every  $\beta \in \mathcal{G}_N$  we have:*

- (1)  $\mathbb{E}(R_N(\beta)) \in M_n(\mathbb{C}) \otimes \{C_N\}''$ . In particular, there exist analytic functions  $\omega_N^{(j)} : \mathcal{G}_N \rightarrow M_n(\mathbb{C})$ ,  $j = 1, \dots, N$ , such that

$$\omega_N(\beta) = \sum_{j=1}^N \omega_N^{(j)}(\beta) \otimes P_N^{(j)}, \quad \beta \in \mathcal{G}_N.$$

- (2) For every  $Z \in M_N(\mathbb{C})$ ,

$$[\mathbb{E}(R_N(\beta)), I_n \otimes Z] = \mathbb{E}(R_N(\beta)[\gamma_1 \otimes C_N, I_n \otimes Z]R_N(\beta)).$$

*Proof.* The first assertion in (1) follows from an application of (2) to an arbitrary matrix  $Z \in \{C_N\}'$  and from the fact that  $M_n(\mathbb{C}) \otimes \{C_N\}'' = (I_n \otimes \{C_N\}')'$ . The

second assertion follows because we also have  $\gamma_1 \otimes C_N \in M_n(\mathbb{C}) \otimes \{C_N\}''$ . To prove (2), observe that the analytic map

$$H(W) = \mathbb{E} \left( (\beta \otimes I_N - (\gamma_1 \otimes c_N + \gamma_2 \otimes e^{iW} d_N e^{-iW}))^{-1} \right)$$

is defined for  $W$  in an open neighbourhood of the set of selfadjoint matrices in  $M_N(\mathbb{C})$ . The unitary invariance of  $d_N$  implies that  $H(W) = \mathbb{E}(R(\beta))$  if  $W$  is self-adjoint. Since the selfadjoint matrices form a uniqueness set for analytic functions, we conclude that  $H$  is constant on an open subset of  $M_N(\mathbb{C})$  containing the self-adjoint matrices. Given an arbitrary  $Z \in M_N(\mathbb{C})$ , we conclude that the function  $\varepsilon \mapsto H(\varepsilon Z)$  is defined and constant for  $\varepsilon \in \mathbb{C}$  with  $|\varepsilon|$  sufficiently small. Differentiate with respect to  $\varepsilon$  and set  $\varepsilon = 0$ , we obtain

$$\mathbb{E}(R_N(\beta)[I_n \otimes Z, (\gamma_2 \otimes d_N)]R_N(\beta)) = 0.$$

The equality

$$R_N(\beta)(\gamma_2 \otimes d_N) = R_N(\beta)(\beta \otimes I_N - \gamma_1 \otimes C_N) - I_n \otimes I_N,$$

applied in the relation above, yields (2).  $\square$

The following result is simply a reformulation of Lemma 8.1 that emphasizes the fact that the functions  $\beta \mapsto (\omega_N^{(j)}(\beta) - \lambda_N^{(j)} \gamma_1)^{-1}$  extend holomorphically to the open set  $\{\beta \in \mathbb{M}_N(\mathbb{C}) : \beta - \gamma_1 \otimes c_N - \gamma_2 \otimes d_N \text{ is invertible}\}$ .

**Corollary 8.2.** *We have*

$$(I_n \otimes P_N^{(j)}) \mathbb{E}(R_N(\beta))(I_n \otimes P_N^{(j)}) = (\omega_N^{(j)}(\beta) - \lambda_N^{(j)} \gamma_1)^{-1} \otimes P_N^{(j)}, \quad j = 1, \dots, N,$$

for every  $\beta \in \mathcal{G}_N$  such that  $\omega_N(j)(\beta) - \lambda_N^{(j)} \gamma_1$  is invertible.

It is useful to rewrite assertion (2) of Lemma 8.1 as follows:

$$(8.4) \quad \begin{aligned} [\omega_N(\beta), I_n \otimes Z] &= (\mathbb{E}(R_N(\beta)))^{-1} \mathbb{E} \left( (R_N(\beta) - \mathbb{E}(R_N(\beta))) \right. \\ &\quad \left. \times (\gamma_1 \otimes [Z, C_N])(R_N(\beta) - \mathbb{E}(R_N(\beta))) \right) (\mathbb{E}(R_N(\beta)))^{-1}. \end{aligned}$$

This is analogous to [11, (4.10)] and the derivation is practically identical. Relation (8.4) allows us to estimate the differences between the matrices  $\omega_N^{(j)}(\beta)$  once we control the differences  $R_N(\beta) - \mathbb{E}(R_N(\beta))$ . For this purpose, we use the concentration of measure result in [3, Corollary 4.4.28]. This requires estimating the Lipschitz constant of the map  $V \mapsto R_N(V, \beta)$  (see (8.2)) in the Hilbert-Schmidt norm. We use the notation  $\|T\|_2 = \text{Tr}_m(T^*T)$  for the Hilbert-Schmidt norm of an arbitrary matrix  $T \in M_m(\mathbb{C})$ .

**Lemma 8.3.** *Suppose that  $N \in \mathbb{N}$  and  $\beta \in \mathcal{G}_N$ . Then*

$$\|R_N(V, \beta) - R_N(W, \beta)\|_2 \leq 2r^2 \|\gamma_2\|_2 \|D_N\| \|V - W\|_2, \quad V, W \in \text{U}(N),$$

where  $r = \|R_N(\beta)\|_{M_n(\mathbb{C}) \otimes \mathcal{M}_N}$ .

*Proof.* A simple calculation shows that

$$R_N(V, \beta) - R_N(W, \beta) = R_N(V, \beta)[\gamma_2 \otimes (W D_N W^* - V D_N V^*)] R_N(W, \beta).$$

Next we see that

$$W D_N W^* - V D_N V^* = (W - V) D_N W^* + V D_N (W^* - V^*),$$

and thus

$$\|WD_NW^* - VD_NV^*\|_2 \leq 2\|D_N\|\|W - V\|_2.$$

Use now the equality  $\|T \otimes S\|_2 = \|T\|_2\|S\|_2$  to deduce that

$$\begin{aligned} & \|R_N(V, \beta) - R_N(W, \beta)\|_2 \\ & \leq \|R_N(V, \beta)\|\|\gamma_2\|_2\|WD_NW^* - VD_NV^*\|_2\|R_N(W, \beta)\| \\ & \leq 2\|\gamma_2\|_2\|D_N\|\|R_N(V, \beta)\|\|R_N(W, \beta)\|\|V - W\|_2. \end{aligned}$$

The lemma follows from this estimate.  $\square$

**Proposition 8.4.** *Suppose that  $\sup\{\|D_N\| : N \in \mathbb{N}\} < +\infty$ ,  $\beta \in \bigcap_{N \in \mathbb{N}} \mathcal{G}_N$ , and moreover,*

$$r = \sup\{\|R_N(\beta)\|_{M_n(\mathbb{C}) \otimes \mathcal{M}_N} : N \in \mathbb{N}\} < +\infty.$$

*Let  $X_N, Y_N \in M_n \otimes M_N$  be matrices of norm one and rank uniformly bounded by  $m \in \mathbb{N}$ . Then:*

(1) *Almost surely,*

$$(8.5) \quad \lim_{N \rightarrow \infty} \|X_N(R_N(\beta) - \mathbb{E}(R_N(\beta)))Y_N\| = 0.$$

(2) *There exists  $k > 0$  such that*

$$(8.6) \quad \mathbb{E}(\|X_N(R_N(\beta) - \mathbb{E}(R_N(\beta)))Y_N\|^2) \leq \frac{k}{N}r^4, \quad N \in \mathbb{N}.$$

*In particular, there exists a dense countable subset  $\Lambda \subset \bigcap_{N \in \mathbb{N}} \mathcal{G}_N$  such that almost surely, (8.5) holds for any  $\beta \in \Lambda$ .*

*Proof.* An arbitrary operator of rank  $m$  can be written as a sum of  $m$  operators of rank one (with the same or smaller norm). Thus we may, and do, restrict ourselves to the case in which the operators  $X_N$  and  $Y_N$  are projections of rank 1. In this case,  $X_N R_N(V, \beta) Y_N$  is a scalar multiple of a fixed operator of rank one and Lemma 8.3 shows that this function satisfies a Lipschitz estimate. This estimate, combined with [3, Corollary 4.4.28], shows that

$$\mathbb{P}(\|X_N(R_N(\beta) - \mathbb{E}(R_N(\beta)))Y_N\| > \varepsilon) \leq 2 \exp\left(\frac{-N\varepsilon^2}{8r^4\|\gamma_2\|^2\|D_N\|^2}\right)$$

for every  $\varepsilon > 0$  and every  $\alpha \in (0, 1/2)$ . The hypothesis implies that the last denominator has a bound independent of  $N$ . Part (1) of the lemma follows from this inequality, while (2) follows from the formula  $\mathbb{E}(|Z|) = \int_0^{+\infty} \mathbb{P}(|Z| > t) dt$ , valid for arbitrary random variables  $Z$ .  $\square$

**Remark 8.5.** While Proposition 8.4 was formulated for  $\beta \in \bigcap_{N \in \mathbb{N}} \mathcal{G}_N$ , the hypothesis  $r < +\infty$  (and therefore the conclusion of the proposition) is also satisfied in the following cases:

- (1)  $\Im\beta > 0$  with  $r \leq \|(\Im\beta)^{-1}\|$ ;
- (2)  $\beta = ze_{1,1} - \gamma_0$  with  $z \in \mathbb{C}^+ \cup \mathbb{C}^-$ , by an estimate provided by Lemma 4.3;
- (3)  $\beta = xe_{1,1} - \gamma_0$  with  $x \in \mathbb{R}$ , by the same estimate provided that

$$|x| > \sup\{\|P(c_N, d_N)\|_{M_n(\mathbb{C}) \otimes \mathcal{M}_N} : N \in \mathbb{N}\}.$$

**Corollary 8.6.** *Under the assumptions of Proposition 8.4 suppose that we also have  $\sup\{\|C_N\| : N \in \mathbb{N}\} < +\infty$ , and set*

$$t = \sup\{\|P(c_N, d_N)\|_{M_n(\mathbb{C}) \otimes \mathcal{M}_N} : N \in \mathbb{N}\}.$$

*Let  $K \subset \mathbb{C} \setminus [-t, t]$  be a compact set. Then, almost surely, the functions*

$$z \mapsto \|X_N(R_N(ze_{1,1} - \gamma_0) - \mathbb{E}(R_N(ze_{1,1} - \gamma_0)))Y_N\|$$

*converge to zero uniformly on  $K$  as  $N \rightarrow +\infty$ .*

*Proof.* According to Proposition 8.4 and Remark 8.5(2) and (3), almost surely, for every  $z \in \mathbb{C} \setminus [-t, t]$  such that  $\Im z \in \mathbb{Q}$  and  $\Re z \in \mathbb{Q}$ , we have pointwise convergence to zero. A second application of Remark 8.5 yields uniform bounds on  $K$  for all of these functions, and this implies uniform convergence because the resolvents involved are analytic in  $z$ .  $\square$

We apply the concentration results just proved to operators  $X_N$  and  $Y_N$ , of the form  $I_n \otimes P_N^{(j)}$ , where  $\{P_N^{(j)} : j = 1, \dots, N\}$  are the projections used in Lemma 8.1. The rank of  $I_n \otimes P_N^{(j)}$  is equal to  $n$ .

**Proposition 8.7.** *Suppose that  $\sup\{\|C_N\| + \|D_N\| : N \in \mathbb{N}\} < +\infty$ , and let  $\beta \in \bigcap_{N \in \mathbb{N}} \mathcal{G}_N$  be such that*

$$r = \sup\{\|R_N(\beta)\|_{M_n(\mathbb{C}) \otimes \mathcal{M}_N} : N \in \mathbb{N}\} < +\infty$$

*and*

$$r' = \sup\{\|(\mathbb{E}(R_N(\beta)))^{-1}\| : N \in \mathbb{N}\} < +\infty.$$

*Then  $\sup_{N \in \mathbb{N}} N\|\omega_N(\beta) - \omega_N^{(1)}(\beta) \otimes I_n\| < +\infty$ .*

*Proof.* The conclusion of the proposition is equivalent to

$$\max_{j \in \{2, \dots, N\}} \|\omega_N^{(j)}(\beta) - \omega_N^{(1)}(\beta)\| = O(1/N)$$

as  $N \rightarrow +\infty$ . Fix  $j \in \{2, \dots, N\}$  and set  $Z_N = f_j \otimes f_1^* \in M_N(\mathbb{C}^N)$ . Thus,  $Z_N$  is an operator of rank one such that  $Z_N h = \langle h, f_1 \rangle f_j$  for every  $h \in \mathbb{C}^N$ . We have

$$\begin{aligned} [\omega_N(\beta), I_n \otimes Z_N] &= (\omega_N^{(j)}(\beta) - \omega_N^{(1)}(\beta)) \otimes Z_N \\ &= (I_n \otimes P_N^{(j)})((\omega_N^{(j)}(\beta) - \omega_N^{(1)}(\beta)) \otimes Z_N)(I_n \otimes P_N^{(1)}) \end{aligned}$$

and, similarly,

$$[C_N, Z_N] = (\lambda_N^{(j)} - \lambda_N^{(1)})Z_N = P_N^{(j)}(\lambda_N^{(j)} - \lambda_N^{(1)})Z_N P_N^{(1)}.$$

Next, we apply (8.4) and use the fact that  $I \otimes P_N^{(j)}$  commutes with  $\mathbb{E}(R_N(U_N, \beta))$ . Setting  $X_N = I_n \otimes P_N^{(k)}$ ,  $Y_N = I_n \otimes P_N^{(1)}$ , we obtain

$$\begin{aligned} (\omega_N^{(j)}(\beta) - \omega_N^{(1)}(\beta)) \otimes Z_N &= (\lambda_N^{(j)} - \lambda_N^{(1)})(\mathbb{E}(R_N(\beta)))^{-1} \\ &\quad \times \mathbb{E}(X_N(R_N(\beta) - \mathbb{E}(R_N(\beta)))X_N(\gamma_1 \otimes Z_N)Y_N(R_N(\beta) - \mathbb{E}(R_N(\beta)))Y_N) \\ &\quad \times (\mathbb{E}(R_N(\beta)))^{-1}. \end{aligned}$$

Since  $\|Z_N\| = 1$ , an application of the Cauchy-Schwarz inequality leads to the following estimate:

$$\begin{aligned} \|\omega_N^{(j)}(\beta) - \omega_N^{(1)}(\beta)\| &\leq |\lambda_N^{(j)} - \lambda_N^{(1)}| \|(\mathbb{E}(R_N(\beta)))^{-1}\|^2 \|\gamma_1\| \\ &\quad \times \mathbb{E}(\|X_N(R_N - \mathbb{E}(R_N(\beta)))X_N\|^2)^{1/2} \mathbb{E}(\|Y_N(R_N - \mathbb{E}(R_N(\beta)))Y_N\|^2)^{1/2}. \end{aligned}$$

We have  $|\lambda_N^{(j)} - \lambda_N^{(1)}| \leq 2\|C_N\|$  and the product of the last two factors above is estimated via (8.6) by  $kr^4/N$  with  $k$  independent of  $N$ . Thus,

$$\|\omega_N^{(j)}(\beta) - \omega_N^{(1)}(\beta)\| \leq 2k\|C_N\|r'^2\|\gamma_1\|r^4/N \leq k'/N,$$

with  $k'$  independent of  $j$  and  $N$ . The lemma follows.  $\square$

In the probability model we consider, the sequences  $\{C_N\}_{N \in \mathbb{N}}$  and  $\{D_N\}_{N \in \mathbb{N}}$  are uniformly bounded in norm and, in addition, the sequences  $\{\mu_{C_N}\}_{N \in \mathbb{N}}$  and  $\{\mu_{D_N}\}_{N \in \mathbb{N}}$  converge weakly to  $\mu$  and  $\nu$ , respectively. We denote by  $(a, b)$  a pair of free random variables in some  $C^*$ -probability space  $(\mathcal{M}, \tau)$  such that the sequence  $\{(c_N, d_N)\}_{N \in \mathbb{N}}$  converges in distribution to  $(a, b)$ . We also set

$$\mathcal{G} = \mathcal{G}_{\gamma_1 \otimes a, \gamma_2 \otimes b, M_n(\mathbb{C}), M_n(\mathbb{C}(a))}, \quad \omega = \omega_{\gamma_1 \otimes a, \gamma_2 \otimes b, M_n(\mathbb{C}), M_n(\mathbb{C}(a))}.$$

In other words,  $\omega$  is the usual matrix subordination function associated to the pair  $(\gamma_1 \otimes a, \gamma_2 \otimes b)$ .

**Proposition 8.8.** *Suppose that  $\sup\{\|C_N\| + \|D_N\| : N \in \mathbb{N}\} < +\infty$  and that the sequence  $\{(c_N, d_N)\}_{N \in \mathbb{N}}$  converges to  $(a, b)$  in distribution. Let  $\mathcal{D} \subset \mathcal{G} \cap \bigcap_{n \in \mathbb{N}} \mathcal{G}_N$  be a connected open set containing  $\mathbb{H}^+(M_n(\mathbb{C}))$  and such that the sequence of functions  $\{\|\omega_N\|\}_{N \in \mathbb{N}}$  is locally uniformly bounded on  $\mathcal{D}$ . Then*

$$\lim_{N \rightarrow \infty} \|\omega_N(\beta) - \omega(\beta) \otimes I_N\| = 0, \quad \beta \in \mathcal{D}.$$

*Proof.* By hypothesis, the analytic functions  $\{\omega_N^{(1)}\}_{N \in \mathbb{N}}$  form a normal family on  $\mathcal{D}$ . By Proposition 8.7, it suffices to prove that every subsequential limit of this sequence equals  $\omega$ . Suppose that  $\{\omega_{N_k}^{(1)}\}_{k \in \mathbb{N}}$  converges on  $\mathcal{D}$  to a function  $\tilde{\omega}$ . Fix  $\beta \in H^+(M_n(\mathbb{C}))$  such that  $\Im \beta > \sup\{\|C_N\| + \|D_N\| : N \in \mathbb{N}\}$ . Then

$$\begin{aligned} \mathbb{E}(R_N(\beta)) &= (\omega_N^{(1)}(\beta) \otimes I_N - \gamma_1 \otimes C_N + O(1/N))^{-1} \\ &= (\omega_N^{(1)}(\beta) \otimes I_N - \gamma_1 \otimes C_N)^{-1} + O(1/N) \\ &= \sum_{m=0}^{\infty} (\omega_N^{(1)}(\beta)^{-1})(\gamma_1 \omega_N^{(1)}(\beta)^{-1})^m \otimes C_N^m + O(1/N), \end{aligned}$$

and thus

$$(\text{Id}_{M_n(\mathbb{C})} \otimes \text{tr}_N)(\mathbb{E}(R_N(\beta))) = \sum_{m=0}^{\infty} (\omega_N^{(1)}(\beta)^{-1})(\gamma_1 \omega_N^{(1)}(\beta)^{-1})^m \cdot \text{tr}_N(C_N^m) + O(1/N).$$

Setting  $N = N_k$ , letting  $k \rightarrow +\infty$ , and observing that the series on the right is uniformly dominated, we conclude that

$$\begin{aligned} \lim_{k \rightarrow \infty} (\text{Id}_{M_n(\mathbb{C})} \otimes \text{tr}_{N_k})(\mathbb{E}(R_{N_k}(\beta))) &= \sum_{m=0}^{\infty} (\tilde{\omega}(\beta)^{-1})(\gamma_1 \tilde{\omega}(\beta)^{-1})^m \cdot \tau(a^m) \\ &= (\text{Id}_{M_n(\mathbb{C})} \otimes \tau)((\tilde{\omega}(\beta) - \gamma_1 \otimes a)^{-1}). \end{aligned}$$

The fact that the pairs  $(C_N, D_N)$  converge to  $(a, b)$  implies that

$$\begin{aligned} \lim_{k \rightarrow \infty} (\text{Id}_{M_n(\mathbb{C})} \otimes \text{tr}_{N_k})(\mathbb{E}(R_{N_k}(\beta))) &= (\text{Id}_{M_n(\mathbb{C})} \otimes \tau)((\beta - \gamma_1 \otimes a - \gamma_2 \otimes b)^{-1}) \\ &= (\text{Id}_{M_n(\mathbb{C})} \otimes \tau)((\omega(\beta) - \gamma_1 \otimes a)^{-1}) \end{aligned}$$

Thus, we obtain the equality

$$(\text{Id}_{M_n(\mathbb{C})} \otimes \tau)((\omega(\beta) - \gamma_1 \otimes a)^{-1}) = (\text{Id}_{M_n(\mathbb{C})} \otimes \tau)((\tilde{\omega}(\beta) - \gamma_1 \otimes a)^{-1}),$$

and this easily yields  $\tilde{\omega}(\beta) = \omega(\beta)$ . Since  $\mathcal{D}$  is connected, we must have  $\tilde{\omega} = \omega$ , thus concluding the proof.  $\square$

Corollary 5.5 implies the following result.

**Corollary 8.9.** *Suppose that  $\mathfrak{s} := \sup\{\|C_N\| + \|D_N\| : N \in \mathbb{N}\} < +\infty$  and the sequence  $\{(c_N, d_N)\}_{N \in \mathbb{N}}$  converges to  $(a, b)$  in distribution. Let  $\mathfrak{k} = \max\{k, \mathfrak{s}\}$ , where  $k$  is the constant provided by Lemma 5.4. Then, for every  $z \in \mathbb{C} \setminus [-\mathfrak{k}, \mathfrak{k}]$ , we have  $ze_{1,1} - \gamma_0 \in \mathcal{G} \cap \bigcap_{N \in \mathbb{N}} \mathcal{G}_N$  and*

$$\lim_{N \rightarrow \infty} \|\omega_N(ze_{1,1} - \gamma_0) - \omega(ze_{1,1} - \gamma_0) \otimes I_N\| = 0.$$

The preceding results combine to yield convergence results for sample resolvents. For the following statement, it is useful to identify  $\mathbb{C}^m$  with a subspace of  $\mathbb{C}^N$  if  $m < N$  and to denote by  $\{f_1, \dots, f_N\}$  the standard basis in  $\mathbb{C}^N$ . Thus, we have  $f_j \in \mathbb{C}^m$  provided that  $j \leq m$ .

**Proposition 8.10.** *Suppose that  $\sup\{\|C_N\| + \|D_N\| : N \in \mathbb{N}\} < +\infty$  and that the sequence  $\{(c_N, d_N)\}_{N \in \mathbb{N}}$  converges to  $(a, b)$  in distribution. Suppose also that  $C_N$  is diagonal in the standard basis, that is,  $C_N f_j = \lambda_N^{(j)} f_j$ ,  $j = 1, \dots, N$ , and that  $p \in \mathbb{N}$  is such that the limits  $\lambda^{(j)} = \lim_{N \rightarrow \infty} \lambda_N^{(j)}$  exist for  $j = 1, \dots, p$ . Denote by  $P_N : \mathbb{C}^N \rightarrow \mathbb{C}^p$  the orthogonal projection onto  $\mathbb{C}^p$ ,  $N \geq p$ . Let  $\mathcal{D} \subset \mathcal{G} \cap \bigcap_{N \in \mathbb{N}} \mathcal{G}_N$  be a connected open set containing  $\mathbb{H}^+(M_n(\mathbb{C}))$  and such that the sequence of functions  $\{\|\omega_N\|\}_{N \in \mathbb{N}}$  is locally uniformly bounded on  $\mathcal{D}$ . Then almost surely*

$$\lim_{N \rightarrow \infty} (I_n \otimes P_N) R_N(\beta) (I_n \otimes P_N) = (\omega(\beta) \otimes I_p - \gamma_1 \otimes \text{diag}(\lambda^{(1)}, \dots, \lambda^{(p)}))^{-1}$$

in the norm topology for every  $\beta \in \mathcal{D}$ .

*Proof.* By Proposition 8.4, it suffices to show that the conclusion holds with  $\mathbb{E}(R_N(\beta))$  in place of  $R_N(\beta)$ . We observe that

$$(I_n \otimes P_N) \mathbb{E}(R_N(\beta)) (I_n \otimes P_N) = \sum_{j=1}^p (\omega_N^{(j)}(\beta) - \lambda_N^{(j)} \gamma_1)^{-1} \otimes P_N^{(j)},$$

so the desired conclusion follows from Proposition 8.8.  $\square$

We observe for use in the following result that there exists a domain  $\mathcal{D}$  as in the above statement such that  $ze_{1,1} - \gamma_0 \in \mathcal{D}$  for every  $z \in \mathbb{C}^+$ .

When the convergence of  $(C_N, U_N D_N U_N^*)$  to  $(a, b)$  is strong, the preceding result extends beyond  $\mathbb{H}^+(M_n(\mathbb{C}))$ . In this case,  $\sigma(P(C_N, U_N D_N U_N^*))$  converges almost surely to  $\sigma(P(a, b))$  and thus the sample resolvent  $R_N(U_N, ze_{1,1} - \gamma_0)$  is defined almost surely for large  $N$  even for  $z \in \mathbb{R} \setminus \sigma(P(a, b))$ . We also recall that the function

$$(\omega(ze_{1,1} - \gamma_0) - \lambda \gamma_1)^{-1}$$

extends analytically to  $\mathbb{C} \setminus \sigma(P(a, b))$  if  $\lambda \in \sigma(a)$ . These analytic extensions are used in the following statement.

**Proposition 8.11.** *Under the hypothesis of Proposition 8.10, suppose that the pairs  $\{(C_N, U_N D_N U_N^*)\}_{N \in \mathbb{N}}$  converge strongly to  $(a, b)$ . Then almost surely*

$$\lim_{N \rightarrow \infty} (I_n \otimes P_N) R_N(\beta) (I_n \otimes P_N) = (\omega(\beta) \otimes I_p - \gamma_1 \otimes \text{diag}(\lambda^{(1)}, \dots, \lambda^{(p)}))^{-1}$$

for  $\beta = ze_{1,1} - \gamma_0$ ,  $z \in \mathbb{C} \setminus \sigma(a, b)$ . The convergence is uniform on compact subsets of  $\mathbb{C} \setminus \sigma(P(a, b))$ .

*Proof.* Strong convergence implies that  $\lambda^{(j)} \in \sigma(a)$  for  $j = 1, \dots, p$ , so the functions  $(\omega(ze_{1,1} - \gamma_0) - \lambda^{(j)}\gamma_1)^{-1}$  extend analytically to  $\mathbb{R} \setminus \sigma(P(a, b))$ . Let  $\mathcal{O} \subset \mathbb{C} \setminus \sigma(P(a, b))$  be a connected open set containing  $\{z\} \cup \mathbb{C}^+$  which is at a strictly positive distance from  $\mathbb{C} \setminus \sigma(P(a, b))$ . We prove that the conclusion of the proposition holds for  $U_N(\xi)$  provided that  $\xi \in \Omega$  is such that the conclusion of Corollary 8.6 holds and  $\sigma(P(C_N, VD_NV^*)) \subset \mathbb{R} \setminus \mathcal{O}$  for  $V = U_N(\xi)$  and sufficiently large  $N$ . By hypothesis, the collection of such points  $\xi \in \Omega$  has probability 1. Lemma 4.3 shows that the family of functions  $\|R_N(U_N(\xi), ze_{1,1} - \gamma_0)\|$  is locally uniformly bounded on  $\mathcal{O}$  for large  $N$ . By Montel's theorem, we can conclude the proof by verifying the conclusions of the proposition for  $\beta = ze_{1,1} - \gamma_0$  with  $z \in \mathbb{C}^+$ . For such values of  $z$ , the result follows from Proposition 8.10.  $\square$

## 9. THE UNITARILY INVARIANT MODEL

In this section we prove Theorem 6.1 under the additional condition that  $A_N$  is a constant matrix and  $B_N$  is a random unitary conjugate of another constant matrix. We may, and do, assume that for  $N \geq p$ ,  $A_N$  is diagonal in the standard basis  $\{f_1, \dots, f_N\}$  with eigenvalues  $\theta_1, \dots, \theta_p, \lambda_N^{(p+1)}, \dots, \lambda_N^{(N)}$  and, as before,  $B_N = U_N D_N U_N^*$ . Then the random matrices  $a_N = A_N \otimes 1_\Omega$  and  $d_N = U_N D_N U_N^*$  are viewed as elements of the noncommutative probability space  $M_N(\mathbb{C}) \otimes L^\infty(\Omega)$ . We fix free selfadjoint random variables  $(a, b)$  in some tracial  $W^*$ -probability space such that the pairs  $(a_N, d_N)$  converge in distribution to  $(a, b)$  as  $N \rightarrow \infty$ . This convergence is not strong because of the spikes  $\theta_1, \dots, \theta_p$ . As in [11], we consider closely related pairs  $(c_N, d_N)$  that do converge strongly to  $(a, b)$ . Namely, we set  $c_N = C_N \otimes 1_\Omega$ , where  $C_N$  is diagonal in the standard basis of  $\mathbb{C}^N$  with eigenvalues  $\lambda_N^{(1)}, \dots, \lambda_N^{(N)}$  that coincide with those of  $A_N$  except that  $\lambda_N^{(1)} = \dots = \lambda_N^{(p)} = s$  is an arbitrary (but fixed for the remainder of this section) element of  $\text{supp}(\mu)$ . For  $N \geq p$ , the difference  $\Delta_N = A_N - C_N$  can then be written as  $\Delta_N = P_N^* T P_N$ , where  $T \in M_p(\mathbb{C})$  is the diagonal matrix with eigenvalues  $\theta_1 - s, \dots, \theta_p - s$  and  $P_N: \mathbb{C}^N \rightarrow \mathbb{C}^p$  is the orthogonal projection.

According to Lemma 4.2,  $\ker(tI_N - P(A_N, B_N))$  and  $\ker(te_{1,1} \otimes I_N - L(A_N, B_N))$  have the same dimension for every  $t \in \mathbb{R}$ . Setting  $\beta = ze_{1,1} - \gamma_0$ , strong convergence of the pairs  $c_N, d_N$  implies that almost surely the sample resolvent  $R_N(U_N, \beta)$  is defined for sufficiently large  $N$  if  $z \in \mathbb{C} \setminus \sigma(P(c, d))$ . (We continue using the notation introduced in (8.1), (8.2), and (8.3).) We need to consider the matrix

$$\begin{aligned} \beta \otimes I_N - \gamma_1 \otimes A_N - \gamma_2 \otimes B_N &= \beta \otimes I_N - \gamma_1 \otimes C_N - \gamma_2 \otimes B_N - \gamma_1 \otimes \Delta_N \\ &= (I_n \otimes I_N - (\gamma_1 \otimes \Delta_N) R_N(U_N, \beta)) (\beta \otimes I_N - \gamma_1 \otimes C_N - \gamma_2 \otimes B_N), \end{aligned}$$

since the order of  $t \in \mathbb{R}$  as a zero of its determinant equals  $\dim \ker(tI_N - P(A_N, B_N))$ , and hence the number of eigenvalues of  $P(A_N, B_N)$  in a neighborhood  $V$  of a given  $t \in \mathbb{R} \setminus \sigma(P(a, b))$  is the number of zeros of this determinant in  $V$ . Thus, we need to consider the zeros in  $V$  of

$$\det(I_n \otimes I_N - (\gamma_1 \otimes \Delta_N) R_N(U_N, \beta)).$$

Using Sylvester's identity ( $\det(I_r - XY) = \det(I_p - YX)$  if  $X$  is an  $r \times p$  matrix and  $Y$  is a  $p \times r$  matrix) and the fact that  $\Delta_N = P_N^* T P_N$ , this determinant can be rewritten as

$$F_N(U_N, \beta) = \det(I_n \otimes I_p - (\gamma_1 \otimes T)(I_n \otimes P_N) R_N(U_N, \beta)(I_n \otimes P_N^*)).$$

At this point, we observe that the hypothesis of Proposition 8.11 are satisfied with  $\lambda^{(1)} = \dots = \lambda^{(p)} = s$ . We conclude that almost surely

$$\lim_{n \rightarrow \infty} F_N(U_N, \beta) = F(\beta), \quad \beta = ze_{1,1} - \gamma_0, z \notin \sigma(P(a, b)),$$

where

$$F(\beta) = \det(I_n \otimes I_p - (\gamma_1(\omega(\beta) - s\gamma_1)^{-1}) \otimes T),$$

and the convergence is uniform on compact sets. The limit  $F(ze_{1,1} - \gamma_0)$  is a (deterministic) analytic function on  $\mathbb{C} \setminus \sigma(P(a, b))$ . An application of Hurwitz's theorem on zeros of analytic functions (see [1, Theorem 5.2]) yields the following result.

**Proposition 9.1.** *Suppose that  $t_1, t_2 \in \mathbb{R}$ ,  $t_1 < t_2$ ,  $[t_1, t_2] \subset \mathbb{R} \setminus \sigma(P(a, b))$ ,  $F(t_j e_{1,1} - \gamma_0) \neq 0$  for  $j = 1, 2$ , and the function  $F(ze_{1,1} - \gamma_0)$  has at most one zero  $t$  in the interval  $(t_1, t_2)$ . Then, almost surely for large  $N$ , the matrix  $P(A_N, B_N)$  has exactly  $m$  eigenvalues in the interval  $[t_1, t_2]$ , where  $m$  is the order of  $t$  as a zero of  $F(ze_{1,1} - \gamma_0)$  and  $m = 0$  if this function does not vanish on  $[t_1, t_2]$ .*

Part(1) of Theorem 6.1 is a reformulation of Proposition 9.1. To see that this is the case, we observe that  $T$  is a diagonal matrix and thus the matrix

$$I_n \otimes I_p - (\gamma_1(\omega(ze_{1,1} - \gamma_0) - s\gamma_1)^{-1}) \otimes T$$

is block diagonal with diagonal blocks

$$G_{j,s}(z) = I_n - (\theta_j \gamma_1 - s\gamma_1)(\omega(ze_{1,1} - \gamma_0) - s\gamma_1)^{-1}, \quad j = 1, \dots, p.$$

If  $\det(G_{j,s}(z))$  has a zero of order  $m_j$  at  $t$  then the number  $m$  in the statement is  $m_1 + \dots + m_p$ . We recall that  $G_{j,s}$  is analytic on  $\mathbb{C} \setminus \sigma(P(a, b))$  but  $\omega(ze_{1,1} - \gamma_0)$  is only meromorphic. It is not immediately apparent that the number  $m_j$  does not depend on  $s$  but this is a consequence of the following result.

**Lemma 9.2.** *Suppose that  $\alpha_1, \alpha_2 \in M_n(\mathbb{C})$  are such that  $(\omega(ze_{1,1} - \gamma_0) - \alpha_k)^{-1}$  extends analytically to  $t$  for  $k = 1, 2$ . Then the order of  $t$  as a zero of*

$$\det(I_n - (\theta_j \gamma_1 - \alpha_k)(\omega(ze_{1,1} - \gamma_0) - \alpha_k)^{-1})$$

*does not depend on  $k$ .*

*Proof.* An easy calculation shows that

$$I_n - (\theta_j \gamma_1 - \alpha_k)(\omega(ze_{1,1} - \gamma_0) - \alpha_k)^{-1} = (\omega(ze_{1,1} - \gamma_0) - s\gamma_1)(\omega(ze_{1,1} - \gamma_0) - \alpha_k)^{-1}.$$

The desired conclusion follows if we prove that the function

$$H(z) = (\omega(ze_{1,1} - \gamma_0) - \alpha_1)(\omega(ze_{1,1} - \gamma_0) - \alpha_2)^{-1}$$

is analytic and invertible at  $z = t$ . We have

$$H(z) = I_n + (\alpha_2 - \alpha_1)(\omega(ze_{1,1} - \gamma_0) - \alpha_2)^{-1}$$

and

$$H(z)^{-1} = I_n + (\alpha_1 - \alpha_2)(\omega(ze_{1,1} - \gamma_0) - \alpha_1)^{-1},$$

so the analyticity and invertibility of  $H$  follow from the hypothesis.  $\square$



We proceed now to Part(2) of Theorem 6.1. Thus, assumptions (A1–A3) and (B0–B2) are in force and, in addition, the spikes  $\theta_1, \dots, \theta_p$  are distinct. In particular,  $\sup\{\|C_N\| + \|D_N\| : N \in \mathbb{N}\} < +\infty$ .

If  $S \subset \mathbb{R}$  is a Borel set and  $A \in M_N(\mathbb{C})$  is a selfadjoint operator, then  $E_A(S)$  denotes the orthogonal projection onto the linear span of the eigenvectors of  $A$  corresponding to eigenvalues in  $S$ . For instance, under the hypotheses of Part(2) of Theorem 6.1,  $E_{A_N}(\{\theta_j\})$  is a projection of rank one for  $j = 1, \dots, p$ . If  $h : \mathbb{R} \rightarrow \mathbb{C}$  is a continuous function, then  $h(A)$  denotes the usual functional calculus for selfadjoint matrices. Thus, if  $Ax = tx$  for some  $t \in \mathbb{R}$  and  $x \in \mathbb{C}^N$ , then  $h(A)x = h(t)x$ .

Fix  $t \in \mathbb{R} \setminus \sigma(P(a, b))$  and  $\varepsilon > 0$  small enough. We need to show that, almost surely,

$$(9.1) \quad \lim_{N \rightarrow \infty} \text{Tr}_N [E_{A_N}(\{\theta_i\})E_{P(A_N, B_N)}((t - \varepsilon, t + \varepsilon))] = \delta_{i, i_0} C_i(t).$$

Choose  $\delta > 0$  and  $N_0 \in \mathbb{N}$  such that  $[\theta_j - \delta, \theta_j + \delta] \cap \sigma(A_N) = \{\theta_j\}$ , for  $N \geq N_0$  and  $j = 1, \dots, p$ . Pick infinitely differentiable functions  $f_j : \mathbb{R} \rightarrow [0, 1]$  supported in  $[\theta_j - \delta, \theta_j + \delta]$  such that  $f_j(\theta_j) = 1$ ,  $j = 1, \dots, p$ . Also pick an infinitely differentiable function  $h : \mathbb{R} \rightarrow [0, 1]$  supported in  $(t - \varepsilon, t + \varepsilon)$  such that  $h$  is identically 1 on  $[t - \varepsilon/2, t + \varepsilon/2]$ . Then part (1) of Theorem 6.1 implies that, almost surely for large  $N$ , we have

$$E_{A_N}(\{\theta_j\})E_{P(A_N, B_N)}((t - \varepsilon, t + \varepsilon)) = f_j(A_N)h(P(A_N, B_N)).$$

In anticipation of a concentration inequality, we prove a Lipschitz estimate for the functions  $g_{N,j} : \text{U}(N) \rightarrow M_N(\mathbb{C})$  defined by

$$g_{N,j}(V) = \text{Tr}_N(f_j(A_N)h(P(A_N, VD_NV^*))), \quad V \in \text{U}(N), j = 1, \dots, p.$$

**Lemma 9.3.** *There exists  $k > 0$ , independent of  $N$ , such that*

$$|g_{N,j}(V) - g_{N,j}(W)| \leq k\|V - W\|_2, \quad V, W \in \text{U}(N), j = 1, \dots, p.$$

*Proof.* Given a Lipschitz function  $u : \text{U}(N) \rightarrow M_N(\mathbb{C})$ , we denote by  $\text{Lip}(u)$  the smallest constant  $c$  such that

$$\|u(V) - u(W)\|_2 \leq c\|V - W\|_2, \quad V, W \in \text{U}(N),$$

and we set  $\|u\|_\infty = \sup\{\|u(V)\| : V \in \text{U}(N)\}$ . If  $u_1, u_2 : \text{U}(N) \rightarrow M_N(\mathbb{C})$  are two Lipschitz functions, then  $\text{Lip}(u_1 + u_2) \leq \text{Lip}(u_2) + \text{Lip}(u_1)$  and

$$\text{Lip}(u_1 u_2) \leq \|u_1\|_\infty \text{Lip}(u_2) + \text{Lip}(u_1) \|u_2\|_\infty.$$

Since the functions  $V \mapsto V$  and  $V \mapsto V^*$  are Lipschitz with constant 1, we deduce immediately that the map  $V \mapsto P(A_N, VD_NV^*)$  is Lipschitz with constant bounded independently of  $N$ . It is well-known that a Lipschitz function  $f : \mathbb{R} \rightarrow \mathbb{R}$  is also Lipschitz, with the same constant, when viewed as a map on the selfadjoint matrices with the Hilbert-Schmidt norm (see for instance, [18, Lemma A.2]). The function  $h$  is infinitely differentiable with compact support, hence Lipschitz. We deduce that the map  $V \mapsto h(P(A_N, VD_NV^*))$  is Lipschitz with constant bounded independently of  $N$ . Finally, we have

$$|g_{N,j}(V) - g_{N,j}(W)| \leq \|f_j(A_N)\|_2 \|h(P(A_N, VD_NV^*)) - h(P(A_N, WD_NW^*))\|_2,$$

and the lemma follows because  $\|f_j(A_N)\|_2 = 1$ .  $\square$

An application of [3, Corollary 4.4.28] yields

$$\mathbb{P}(|g_{N,j}(U_N) - \mathbb{E}(g_{N,j}(U_N))| > \eta) \leq 2 \exp\left(-\frac{\eta^2 N}{4k^2}\right), \quad \eta > 0, j = 1, \dots, p,$$

and the Borel-Cantelli lemma shows that, almost surely,

$$(9.2) \quad \lim_{N \rightarrow \infty} (g_{N,j}(U_N) - \mathbb{E}(g_{N,j}(U_N))) = 0, \quad j = 1, \dots, p.$$

The expected value in (9.2) is estimated using [18, Lemma 6.3] and the fact that  $f_j(A_N)$  is the projection onto the  $j$ th coordinate. If we set  $r_N(z) = (zI_N - P(A_N, B_N))^{-1}$  and  $\tilde{R}_N(ze_{1,1} - \gamma_0) = ((ze_{1,1} - \gamma_0) \otimes I_N - L(A_N, B_N))^{-1}$ ,  $z \in \mathbb{C}^+$ , then

$$(9.3) \quad \begin{aligned} \mathbb{E}(g_{N,j}(U_N)) &= -\lim_{y \downarrow 0} \frac{1}{\pi} \Im \int_{\mathbb{R}} \mathbb{E}(\text{Tr}_N(f_j(A_N)r_N(\xi + iy))h(\xi)) d\xi \\ &= -\lim_{y \downarrow 0} \frac{1}{\pi} \Im \int_{\mathbb{R}} \mathbb{E}(r_N(\xi + iy)_{j,j})h(\xi) d\xi. \end{aligned}$$

The construction of the linearization  $L$  (Section 4) is such that the matrix  $\tilde{R}_N(ze_{1,1} - \gamma_0)$ , viewed as an  $n \times n$  block matrix, has  $r_N(z)$  as its  $(1, 1)$  entry. By Propositions 8.4(1) and 8.10 (with  $A_N$  in place of  $C_N$ ) and the unitary invariance of the distribution of  $B_N$ , for  $z \in \mathbb{C}^+$  the matrices

$$(I_n \otimes P_N) \mathbb{E}(\tilde{R}_N(ze_{1,1} - \gamma_0))(I_n \otimes P_N)^*$$

converge as  $N \rightarrow \infty$  to the block diagonal matrix with diagonal entries

$$(\omega(ze_{1,1} - \gamma_0) - \theta_j \gamma_1)^{-1}.$$

It follows that

$$(9.4) \quad \lim_{N \rightarrow \infty} \mathbb{E}((r_N(z))_{j,j}) = ((\omega(ze_{1,1} - \gamma_0) - \theta_j \gamma_1)^{-1})_{1,1}.$$

We intend to let  $N \rightarrow \infty$  in (9.3) using (9.4), so we consider the differences

$$\Delta_{j,N}(z) = ((\omega(ze_{1,1} - \gamma_0) - \theta_j \gamma_1)^{-1})_{1,1} - \mathbb{E}(r_N(z)_{j,j}).$$

By Corollary 5.5, these functions are defined on  $\mathbb{C} \setminus [-k, k]$  for some  $k > 0$  and satisfy  $\Delta_{j,N}(\bar{z}) = \overline{\Delta_{j,N}(z)}$ . We claim that there exists a sequence  $\{v_N\}_{N \in \mathbb{N}} \subset (0, +\infty)$  such that  $\lim_{N \rightarrow \infty} v_N = 0$  and

$$(9.5) \quad |\Delta_{j,N}(z)| \leq v_N \left(1 + \frac{1}{(\Im z)^2}\right), \quad z \in \mathbb{C}^+.$$

To verify this claim, we observe first [2] that the function  $\mathbb{E}(r_N(z)_{j,j})$  is the Cauchy-Stieltjes transform of a Borel probability measure  $\sigma_{N,j}$  on  $\mathbb{R}$ . Since

$$\sup\{\|P(A_N, B_N)\| : N \in \mathbb{N}\} < \infty,$$

the measures  $\{\sigma_{N,j}\}_{N \in \mathbb{N}}$  have uniformly bounded supports. Now, (9.4) shows that the Cauchy-Stieltjes transform of any accumulation point of this sequence of measures is equal to  $((\omega(ze_{1,1} - \beta_0) - \theta_j \beta_1)^{-1})_{1,1}$ . It follows that this sequence has a weak limit  $\sigma_j$  that is a Borel probability measure with compact support. The existence of the sequence  $\{v_N\}_{N \in \mathbb{N}}$  follows from [11, Lemma 4.1] applied to the signed measures  $\rho_N = \sigma_{N,i} - \sigma_i$ .

We use (9.5), (9.3), and the Lemma from [20, Appendix] to obtain

$$\begin{aligned}
\lim_{N \rightarrow \infty} \mathbb{E}(g_{N,j}(U_N)) &= - \lim_{N \rightarrow \infty} \lim_{y \downarrow 0} \frac{1}{\pi} \Im \int_{\mathbb{R}} \mathbb{E}(r_N(\xi + iy)_{j,j}) h(\xi) d\xi \\
&= \lim_{N \rightarrow \infty} \lim_{y \downarrow 0} \frac{1}{\pi} \Im \int_{\mathbb{R}} \Delta_{j,N}(\xi + iy) h(\xi) d\xi \\
&\quad - \lim_{y \downarrow 0} \frac{1}{\pi} \Im \int_{\mathbb{R}} ((\omega((\xi + iy)e_{1,1} - \gamma_0) - \theta_j \gamma_1)^{-1})_{1,1} h(\xi) d\xi \\
(9.6) \quad &= - \lim_{y \downarrow 0} \frac{1}{\pi} \Im \int_{\mathbb{R}} ((\omega((\xi + iy)e_{1,1} - \gamma_0) - \theta_j \gamma_1)^{-1})_{1,1} h(\xi) d\xi.
\end{aligned}$$

The choice of  $h$ , and the fact that

$$u(z) = ((\omega(ze_{1,1} - \gamma_0) - \theta_j \gamma_1)^{-1})_{1,1}$$

is analytic and real-valued on the intervals  $[t - \varepsilon, t - \varepsilon/2]$  and  $[t + \varepsilon/2, t + \varepsilon]$ , imply that the last line in (9.6) can be rewritten as

$$\begin{aligned}
&\lim_{y \downarrow 0} \frac{1}{\pi} \int_{t-\varepsilon}^{t-\frac{\varepsilon}{2}} \Im(u(\xi + iy)) h(\xi) d\xi + \lim_{y \downarrow 0} \frac{1}{\pi} \int_{t+\frac{\varepsilon}{2}}^{t+\varepsilon} \Im(u(\xi + iy)) h(\xi) d\xi \\
&\quad - \lim_{y \downarrow 0} \frac{1}{\pi} \int_{t-\frac{\varepsilon}{2}}^{t+\frac{\varepsilon}{2}} \Im(u(\xi + iy)) d\xi \\
(9.7) \quad &= \lim_{y \downarrow 0} \frac{1}{2\pi i} \int_{t-\frac{\varepsilon}{2}}^{t+\frac{\varepsilon}{2}} u(\xi + iy) d\xi - \lim_{y \downarrow 0} \frac{1}{2\pi i} \int_{t-\frac{\varepsilon}{2}}^{t+\frac{\varepsilon}{2}} u(\xi - iy) d\xi.
\end{aligned}$$

Recall (see, for instance, [1, Chapter 4]) that if  $f$  is an analytic function on a simply connected domain  $D$ , except for an isolated singularity  $a$ , then  $\frac{1}{2\pi i} \int_{\gamma} f(z) dz = n(\gamma, a) \text{Res}_{z=a} f(z)$ . Here  $\gamma$  is a closed Jordan path in  $D$  not containing  $a$ ,  $n(\gamma, a)$  is the winding number of  $\gamma$  with respect to  $a$ , and  $\text{Res}_{z=a} f(z)$  is that number  $R$  which satisfies the condition that  $f(z) - \frac{R}{z-a}$  has vanishing period (called the residue of  $f$  at  $a$ ). Denote by  $\Gamma_y$  the rectangle with corners  $t \pm (\varepsilon/2) \pm iy$  and let  $\gamma_y$  be the boundary of  $\Gamma_y$  oriented counterclockwise. The expression in (9.7) represents the integral of  $u$  on the horizontal segments in  $\gamma_y$ . It is clear that the integral of  $u$  on the vertical segments is  $O(y)$ , and thus (9.6) implies the equality

$$\lim_{N \rightarrow \infty} \mathbb{E}(g_{N,j}(U_N)) = \lim_{y \downarrow 0} \frac{1}{2\pi i} \int_{\gamma_y} u(z) dz = \text{Res}_{z=t} u(z).$$

The alternative formula in Theorem 6.1 follows from the fact that  $u$  has a simple pole at  $t$  because  $u$  maps  $\mathbb{C}^+$  to  $\mathbb{C}^-$ .

## 10. THE WIGNER MODEL

We proceed now to the proof of Theorem 6.3. The matrices  $A_N$  are subject to the hypotheses (A1–A3), while  $X_N/\sqrt{N}$  and  $X_N$  satisfies conditions (X0–X3). By [36, Section 3, Assertion 2], it suffices to proceed under the additional hypothesis that each  $A_N$  is a constant matrix. The free variables  $a$  and  $b$  are such that  $b$  has standard semicircular distribution  $\nu_{0,1}$ .

One consequence of the fact that  $b$  is a semicircular variable is that the function  $\omega$  is analytic on the entire set

$$\{\beta \in M_n(\mathbb{C}) : \beta \otimes 1 - \gamma_1 \otimes a - \gamma_2 \otimes b \text{ is invertible.}\}$$

This justifies the comment from Remark 6.2. We recall that in this special case the subordination function is given by

$$\omega(\beta) = \beta - \gamma_2(\text{Id}_{M_n(\mathbb{C})} \otimes \tau)(\beta \otimes 1 - \gamma_1 \otimes a - \gamma_2 \otimes b)^{-1} \gamma_2.$$

Since the distribution of the random matrix  $X_N$  is not usually invariant under unitary conjugation, we can no longer assume that  $A_N$  is diagonal in the standard basis  $\{f_1, \dots, f_N\}$ . There is however a (constant) unitary matrix  $V_N \in M_N(\mathbb{C})$  such that  $A_N$  is diagonal in the basis  $\{V_N f_1, \dots, V_N f_N\}$  with eigenvalues  $\theta_1, \dots, \theta_p, \lambda_N^{(p+1)}, \dots, \lambda_N^{(N)}$ ,  $N \geq p$ . Viewing each realization of the random matrices  $A_N$  and  $X_N/\sqrt{N}$  as elements of the noncommutative probability space  $(M_N(\mathbb{C}), \text{tr}_N)$ , almost surely the pairs  $(A_N, X_N/\sqrt{N})$  converge in distribution, but not strongly, to  $(a, b)$  as  $N \rightarrow \infty$ . A modification of  $A_N$  provides almost surely strongly convergent pairs  $(C_N, X_N/\sqrt{N})$ . Thus, let  $C_N$  be diagonal in the basis  $\{V_N f_1, \dots, V_N f_N\}$  with eigenvalues  $\lambda_N^{(1)}, \dots, \lambda_N^{(N)}$  that coincide with those of  $A_N$  except that  $\lambda_N^{(1)} = \dots = \lambda_N^{(p)} = s$  is an arbitrary (but fixed for the remainder of this section) element of  $\text{supp}(\mu)$ . For  $N \geq p$ , the difference  $\Delta_N = A_N - C_N$  can then be written as  $\Delta_N = V_N P_N^* T P_N V_N^*$ , where  $T \in M_p(\mathbb{C})$  is the diagonal matrix with eigenvalues  $\theta_1 - s, \dots, \theta_p - s$  and  $P_N : \mathbb{C}^N \rightarrow \mathbb{C}^p$  is the orthogonal projection. Almost surely, the pairs  $(C_N, X_N/\sqrt{N})$  converge strongly to  $(a, b)$  as shown in [12, Theorem 1.2] and [23, Proposition 2.1]. We continue using the notation introduced in (8.1) and (8.3) with  $c_N = C_N$  and  $d_N = \frac{X_N}{\sqrt{N}}$ . The calculation in Section 9 show that, almost surely, for  $N$  large enough, the number of eigenvalues of  $P(A_N, X_N/\sqrt{N})$  in a small enough neighborhood of  $t \in \mathbb{R} \setminus \sigma(P(a, b))$  is equal to the number of zeros of  $F_N(ze_{1,1} - \gamma_0)$  in this neighborhood, where

$F_N(ze_{1,1} - \gamma_0) = \det(I_n \otimes I_p - (\gamma_1 \otimes T)(I_n \otimes P_N V_N^*) R_N(ze_{1,1} - \gamma_0)(I_n \otimes P_N^* V_N))$  is a random analytic function. We focus on the study of the large  $N$  behavior of the matrix function

$$(10.1) \quad \mathcal{F}_N(\beta) = (I_n \otimes P_N V_N^*) R_N(\beta) (I_n \otimes V_N P_N^*).$$

We start with the special case in which  $X_N$  is replaced by a standard G.U.E.. The following proposition is a consequence of the results in Section 9. Thus, suppose that  $(X_N^g)_{N \in \mathbb{N}}$  is a sequence of standard G.U.E. ensembles and we set  $R_N^g(\beta) = (\beta \otimes I_N - L(C_N, X_N^g/\sqrt{N}))^{-1}$ , and

$$(10.2) \quad \mathcal{F}_N^g(\beta) = (I_n \otimes P_N V_N^*) R_N^g(\beta) (I_n \otimes V_N P_N^*).$$

**Proposition 10.1.** *We have*

$$\lim_{N \rightarrow \infty} \mathbb{E}(\mathcal{F}_N^g(\beta)) = (\omega(\beta) - s\gamma_1)^{-1} \otimes I_p$$

for every  $\beta \in \mathbb{H}^+(M_n(\mathbb{C}))$ .

*Proof.* Since G.U.E. ensembles are invariant under unitary conjugation we may, and do, assume that  $V_N = I_N$  for every  $N \in \mathbb{N}$ . For every  $\beta \in \mathbb{H}^+(M_N(\mathbb{C}))$  we have

$$\mathbb{E}(\mathcal{F}_N^g(\beta)) = \mathbb{E}(\mathcal{F}_N^g(\beta) \mathbf{1}_{\|X_N^g/\sqrt{N}\| \leq 3}) + \mathbb{E}(\mathcal{F}_N^g(\beta) \mathbf{1}_{\|X_N^g/\sqrt{N}\| > 3}),$$

and the second term is at most  $\|(\Im \beta)^{-1}\| \mathbb{P}(\|X_N^g/\sqrt{N}\| > 3)$ . As shown by Bai and Yin [6], this number tends to zero as  $N \rightarrow \infty$ . To estimate the first term, we recall that  $X_N^g/\sqrt{N} = U_N D_N U_N^*$ , where  $U_N$  is a random matrix uniformly distributed in

$U(N)$  and  $D_N$  is a random diagonal matrix, independent from  $U_N$ , whose empirical spectral measure converges almost surely to  $\nu_{0,1}$  as  $N \rightarrow \infty$ . Thus, we can write

$$\begin{aligned} & \mathbb{E}(\mathcal{F}_N^g(\beta) \mathbf{1}_{\|X_N^g/\sqrt{N}\| \leq 3}) \\ &= \int_{\Omega} \int_{\Omega} (I_n \otimes P_N) R_N((\xi_1, \xi_2), \beta) (I_n \otimes P_N^*) d\mathbb{P}(\xi_1) \mathbf{1}_{\|D_N(\xi_2)\| \leq 3} d\mathbb{P}(\xi_2), \\ & R_N((\xi_1, \xi_2), \beta) := (\beta \otimes I_N - L(C_N, U_N(\xi_1) D_N(\xi_2) U_N(\xi_1)^*))^{-1}. \end{aligned}$$

Proposition 8.10 can be applied for almost every  $\xi_2$  and it shows that

$$\int_{\Omega} (I_n \otimes P_N) R_N((\xi_1, \xi_2), \beta) (I_n \otimes P_N^*) d\mathbb{P}(\xi_1) \mathbf{1}_{\|D_N(\xi_2)\| \leq 3}$$

converges to  $(\omega(\beta) - s\gamma_1)^{-1} \otimes I_p$ . The proposition follows now from an application of the dominated convergence theorem.  $\square$

Passing to arbitrary Wigner matrices requires an approximation procedure from [12, Section 2]. For every  $\varepsilon > 0$ , there exist random selfadjoint matrices  $X_N(\varepsilon) = [(X(\varepsilon))_{ij}]_{1 \leq i, j \leq N}$  such that

- (H1) the variables  $\sqrt{2}\Re X_{ij}(\varepsilon)$ ,  $\sqrt{2}\Im X_{ij}(\varepsilon)$ ,  $X_{ii}(\varepsilon)$ ,  $i, j \in \mathbb{N}$ ,  $i < j$ , are independent, centered with variance 1 and satisfy a Poincaré inequality with common constant  $C_{PI}(\varepsilon)$ ,
- (H2) for every  $m \in \mathbb{N}$ ,

$$(10.3) \quad \sup_{(i,j) \in \mathbb{N}^2} \mathbb{E}(|X_{ij}(\varepsilon)|^m) < +\infty,$$

and almost surely for large  $N$ ,

$$\left\| \frac{X_N - X_N(\varepsilon)}{\sqrt{N}} \right\| \leq \varepsilon.$$

Set

$$(10.4) \quad R_N^\varepsilon(\beta) = (\beta \otimes I_N - L(C_N, X_N(\varepsilon)/\sqrt{N}))^{-1}$$

and

$$(10.5) \quad \mathcal{F}_N^\varepsilon(\beta) = (I_n \otimes P_N V_N^*) R_N^\varepsilon(\beta) (I_n \otimes V_N P_N^*)$$

for  $\beta \in \mathbb{H}^+(M_n(\mathbb{C}^N))$ . It readily follows that, almost surely for large  $N$ ,

$$(10.6) \quad \|\mathcal{F}_N^\varepsilon(\beta) - \mathcal{F}_N(\beta)\| \leq \varepsilon \|\gamma_2\| \|(\Im \beta)^{-1}\|^2.$$

Properties (H1) and (H2) imply that, for every  $\varepsilon > 0$ ,

$$\forall (i, j) \in \mathbb{N}^2, \kappa_1^{i,j,\varepsilon} = 0, \kappa_2^{i,j,\varepsilon} = 1,$$

$$\forall (i, j) \in \mathbb{N}^2, i \neq j, \tilde{\kappa}_1^{i,j,\varepsilon} = 0, \tilde{\kappa}_2^{i,j,\varepsilon} = 1,$$

and for any  $m \in \mathbb{N} \setminus \{0\}$ ,

$$(10.7) \quad \sup_{(i,j) \in \mathbb{N}^2} |\kappa_m^{i,j,\varepsilon}| < +\infty, \quad \sup_{(i,j) \in \mathbb{N}^2} |\tilde{\kappa}_m^{i,j,\varepsilon}| < +\infty,$$

where for  $i \neq j$ ,  $(\kappa_m^{i,j,\varepsilon})_{m \geq 1}$  and  $(\tilde{\kappa}_m^{i,j,\varepsilon})_{m \geq 1}$  denote the classical cumulants of  $\sqrt{2}\Re X_{ij}(\varepsilon)$  and  $\sqrt{2}\Im X_{ij}(\varepsilon)$  respectively, and  $(\kappa_m^{i,i,\varepsilon})_{m \geq 1}$  denote the classical cumulants of  $X_{ii}(\varepsilon)$  (we set  $(\tilde{\kappa}_m^{i,i,\varepsilon})_{m \geq 1} \equiv 0$ ).

We use the following notation for an arbitrary matrix  $M \in M_n(\mathbb{C}) \otimes M_N(\mathbb{C})$ :

$$(10.8) \quad M^{ij} = (\text{Id}_{M_n(\mathbb{C})} \otimes \text{Tr}_N)(M(I_n \otimes \hat{e}_{j,i})) \in M_n(\mathbb{C}),$$

and

$$M_{ij} = (\text{Tr}_n \otimes \text{Id}_{M_N(\mathbb{C})}) (M(e_{j,i} \otimes I_N)) \in M_N(\mathbb{C}),$$

where  $e_{j,i}$  (resp.  $\hat{e}_{j,i}$ ) denotes the  $n \times n$  (resp.  $N \times N$ ) matrix whose unique nonzero entry equals 1 and occurs in row  $j$  and column  $i$ .

**Proposition 10.2.** *There exists a polynomial  $P_\varepsilon$  in one variable with nonnegative coefficients such that for all large  $N$ , for every  $v, u \in \{1, \dots, N\}$ , for every  $\beta \in \mathbb{H}^+(M_n(\mathbb{C}))$ , and for every deterministic  $B_N^{(1)}, B_N^{(2)} \in M_n(\mathbb{C}) \otimes M_N(\mathbb{C})$  such that  $\|B_N^{(1)}\| \leq 1$  and  $\|B_N^{(2)}\| \leq 1$ , we have*

$$(10.9) \quad \left\| \mathbb{E}(B_N^{(1)} R_N^g(\beta) B_N^{(2)})^{vu} - \mathbb{E}(B_N^{(1)} R_N^\varepsilon(\beta) B_N^{(2)})^{vu} \right\| \leq \frac{1}{\sqrt{N}} P_\varepsilon(\|(\Im \beta)^{-1}\|),$$

and

$$(10.10) \quad \left\| \mathbb{E}(\mathcal{F}_N^g(\beta)) - \mathbb{E}(\mathcal{F}_N^\varepsilon(\beta)) \right\| \leq \frac{1}{\sqrt{N}} P_\varepsilon(\|(\Im w)^{-1}\|).$$

The proof uses a well-known lemma.

**Lemma 10.3.** *Let  $Z$  be a real-valued random variable such that  $\mathbb{E}(|Z|^{p+2}) < \infty$ . Let  $\phi: \mathbb{R} \rightarrow \mathbb{C}$  be a function whose first  $p+1$  derivatives are continuous and bounded. Then,*

$$(10.11) \quad \mathbb{E}(Z\phi(Z)) = \sum_{a=0}^p \frac{\kappa_{a+1}}{a!} \mathbb{E}(\phi^{(a)}(Z)) + \eta,$$

where  $\kappa_a$  are the cumulants of  $Z$ ,  $|\eta| \leq C \sup_t |\phi^{(p+1)}(t)| \mathbb{E}(|Z|^{p+2})$ , and  $C$  only depends on  $p$ .

*Proof of Proposition 10.2.* Following the approach of [38, Ch. 18 and 19] we introduce the interpolation matrix  $X_\varepsilon(\alpha) = \cos \alpha X_N(\varepsilon) + \sin \alpha Y_N$ ,  $\alpha \in [0, \pi/2]$ , and the corresponding resolvent

$$R_N^{\varepsilon, \alpha}(\beta) = (\beta \otimes I_N - L(C_N, X_\varepsilon(\alpha)/\sqrt{N}))^{-1}, \quad \beta \in \mathbb{H}^+(M_n(\mathbb{C})).$$

We have

$$B_N^{(1)} (\mathbb{E} R_N^g(\beta) - \mathbb{E} R_N^\varepsilon(\beta)) B_N^{(2)} = \int_0^{\pi/2} \mathbb{E} \left( B_N^{(1)} \frac{\partial}{\partial \alpha} R_N^{\varepsilon, \alpha}(\beta) B_N^{(2)} \right) d\alpha,$$

$$\frac{\partial}{\partial \alpha} R_N^{\varepsilon, \alpha}(\beta) = R_N^{\varepsilon, \alpha}(\beta) \gamma_2 \otimes \left( \cos \alpha \frac{X_N^g}{\sqrt{N}} - \sin \alpha \frac{X_N(\varepsilon)}{\sqrt{N}} \right) R_N^{\varepsilon, \alpha}(\beta).$$

Define a basis of the real vector space of selfadjoint matrices in  $M_N(\mathbb{C})$  as follows:

$$\tilde{e}_{j,j} = \hat{e}_{j,j}, 1 \leq j \leq N,$$

$$\tilde{e}_{j,k} =: \frac{1}{\sqrt{2}} (\hat{e}_{j,k} + \hat{e}_{k,j}), 1 \leq j < k \leq N,$$

$$\tilde{f}_{j,k} =: \frac{i}{\sqrt{2}} (\hat{e}_{j,k} - \hat{e}_{k,j}), 1 \leq j < k \leq N.$$

In the following calculation, we write simply  $R_N^{\varepsilon, R}$  in place of  $R_N^{\varepsilon, \alpha}(\beta)$  and  $X^g$  in place of  $X_N^g$ :

$$\begin{aligned} \frac{\partial}{\partial \alpha} R_N^{\varepsilon, \alpha} &= \frac{1}{\sqrt{N}} \sum_{k=1}^N (-\sin \alpha X_{kk}(\varepsilon) + \cos \alpha X_{kk}^g) R_N^{\varepsilon, \alpha} \beta_2 \otimes \tilde{e}_{k,k} R_N^{\varepsilon, \alpha} \\ &\quad + \frac{1}{\sqrt{N}} \sum_{1 \leq j < k \leq N} (-\sin \alpha \sqrt{2} \Re X_{jk}(\varepsilon) + \cos \alpha \sqrt{2} \Re X_{jk}^g) R_N^{\varepsilon, \alpha} \gamma_2 \otimes \tilde{e}_{j,k} R_N^{\varepsilon, \alpha} \\ &\quad + \frac{1}{\sqrt{N}} \sum_{1 \leq j < k \leq N} (-\sin \alpha \sqrt{2} \Im X_{jk}(\varepsilon) + \cos \alpha \sqrt{2} \Im X_{jk}^g) R_N^{\varepsilon, \alpha} \gamma_2 \otimes \tilde{f}_{j,k} R_N^{\varepsilon, \alpha}. \end{aligned}$$

Next, we apply Lemma 10.3 with  $p = 3$  for  $1 \leq k \leq N$ ,  $j < k$ , to each random variable  $Z$  in the set

$$\{\sqrt{2} \Re X_{jk}(\varepsilon), \sqrt{2} \Re X_{jk}^g, \sqrt{2} \Im X_{jk}(\varepsilon), \sqrt{2} \Im X_{jk}^g, X_{kk}(\varepsilon), X_{kk}^g, j < k\}$$

and to each  $\phi$  in the set

$$\begin{aligned} &\{\text{Tr}(B_N^{(1)} R_N^{\varepsilon, \alpha} \gamma_2 \otimes \tilde{e}_{k,k} R_N^{\varepsilon, \alpha} B_N^{(2)} e_{q,l} \otimes \hat{e}_{u,v}), \text{Tr}(B_N^{(1)} R_N^{\varepsilon, \alpha} \gamma_2 \otimes \tilde{e}_{j,k} R_N^{\varepsilon, \alpha} B_N^{(2)} e_{q,l} \otimes \hat{e}_{u,v}), \\ &\quad \text{Tr}(B_N^{(1)} R_N^{\varepsilon, \alpha} \gamma_2 \otimes \tilde{f}_{j,k} R_N^{\varepsilon, \alpha} B_N^{(2)} e_{q,l} \otimes \hat{e}_{u,v}) : 1 \leq u, v \leq N, 1 \leq q, l \leq m\}. \end{aligned}$$

Setting now  $B = B_N^{(2)} e_{q,l} \otimes \hat{e}_{u,v} B_N^{(1)}$ , we have:

$$\begin{aligned} \text{Tr}\left(\frac{\partial}{\partial \alpha} R_N^{\varepsilon, \alpha} B\right) &= \frac{C(\alpha)}{N^{\frac{3}{2}}} \sum_{k=1}^N \kappa_3^{k,k,\varepsilon} \text{Tr}(R_N^{\varepsilon, \alpha} \gamma_2 \otimes \tilde{e}_{k,k} R_N^{\varepsilon, \alpha} \gamma_2 \otimes \tilde{e}_{k,k} R_N^{\varepsilon, \alpha} \gamma_2 \otimes \tilde{e}_{k,k} R_N^{\varepsilon, \alpha} B) \\ &\quad + \frac{C(\alpha)}{N^{\frac{3}{2}}} \sum_{1 \leq j < k < N} \kappa_3^{j,k,\varepsilon} \text{Tr}(R_N^{\varepsilon, \alpha} \gamma_2 \otimes \tilde{e}_{j,k} R_N^{\varepsilon, \alpha} \gamma_2 \otimes \tilde{e}_{j,k} R_N^{\varepsilon, \alpha} \gamma_2 \otimes \tilde{e}_{j,k} R_N^{\varepsilon, \alpha} B) \\ &\quad + \frac{C(\alpha)}{N^{\frac{3}{2}}} \sum_{1 \leq j < k < N} \tilde{\kappa}_3^{j,k,\varepsilon} \text{Tr}(R_N^{\varepsilon, \alpha} \gamma_2 \otimes \tilde{f}_{j,k} R_N^{\varepsilon, \alpha} \gamma_2 \otimes \tilde{f}_{j,k} R_N^{\varepsilon, \alpha} \gamma_2 \otimes \tilde{f}_{j,k} R_N^{\varepsilon, \alpha} B) \\ &\quad + \frac{\tilde{C}(\alpha)}{N^2} \sum_{k=1}^N \kappa_4^{k,k,\varepsilon} \text{Tr}(R_N^{\varepsilon, \alpha} \gamma_2 \otimes \tilde{e}_{k,k} R_N^{\varepsilon, \alpha} \gamma_2 \otimes \tilde{e}_{k,k} R_N^{\varepsilon, \alpha} \gamma_2 \otimes \tilde{e}_{k,k} R_N^{\varepsilon, \alpha} \gamma_2 \otimes \tilde{e}_{k,k} R_N^{\varepsilon, \alpha} B) \\ &\quad + \frac{\tilde{C}(\alpha)}{N^2} \sum_{1 \leq j < k < N} \kappa_4^{j,k,\varepsilon} \text{Tr}(R_N^{\varepsilon, \alpha} \gamma_2 \otimes \tilde{e}_{j,k} R_N^{\varepsilon, \alpha} \gamma_2 \otimes \tilde{e}_{j,k} R_N^{\varepsilon, \alpha} \gamma_2 \otimes \tilde{e}_{j,k} R_N^{\varepsilon, \alpha} \gamma_2 \otimes \tilde{e}_{j,k} R_N^{\varepsilon, \alpha} B) \\ &\quad + \frac{\tilde{C}(\alpha)}{N^2} \sum_{1 \leq j < k < N} \tilde{\kappa}_4^{j,k,\varepsilon} \text{Tr}(R_N^{\varepsilon, \alpha} \gamma_2 \otimes \tilde{f}_{j,k} R_N^{\varepsilon, \alpha} \gamma_2 \otimes \tilde{f}_{j,k} R_N^{\varepsilon, \alpha} \gamma_2 \otimes \tilde{f}_{j,k} R_N^{\varepsilon, \alpha} \gamma_2 \otimes \tilde{f}_{j,k} R_N^{\varepsilon, \alpha} B) + \delta \\ &= I_1 + I_2 + I_3 + I_4 + I_5 + I_6 + \delta, \end{aligned}$$

where

$$|\delta| \leq C_\varepsilon \frac{\|(\Im w)^{-1}\|^6}{\sqrt{N}},$$

for some  $C_\varepsilon \geq 0$ , while  $C(\alpha)$  and  $\tilde{C}(\alpha)$  are polynomials in  $\cos \alpha$  and  $\sin \alpha$ . In the following,  $C_\varepsilon$  may vary from line to line. It is clear that

$$|I_1| \leq C_\varepsilon \frac{\|(\Im \beta)^{-1}\|^4}{\sqrt{N}}, \text{ and } |I_4| \leq C_\varepsilon \frac{\|(\Im w)^{-1}\|^5}{N}.$$

Next,  $I_2$  and  $I_3$  are a finite linear combinations of terms of the form

$$(10.12) \quad \frac{C(\alpha)}{N^{\frac{3}{2}}} \sum_{j,k \in \mathcal{E}} C^{j,k,\varepsilon} \text{Tr}_n(\gamma_2(R_N^{\varepsilon,\alpha})^{p_1 p_2} \gamma_2(R_N^{\varepsilon,\alpha})^{p_3 p_4} \gamma_2(R_N^{\varepsilon,\alpha} B_N^{(2)})^{p_5 u} e_{q,l}(B_N^{(1)} R_N^{\varepsilon,\alpha})^{v p_6}),$$

where  $\mathcal{E}$  is some subset of  $\{1, \dots, N\}^2$ ,  $C^{j,k,\varepsilon} \in \{\kappa_3^{j,k,\varepsilon}, \tilde{\kappa}_3^{j,k,\varepsilon}\}$ , and  $(p_1, \dots, p_6)$  is a permutation of  $(k, k, k, j, j, j)$ . The two following cases hold:

- $p_5 = p_6$ , in which case Lemma 11.1 yields

$$\begin{aligned} & \left\| \sum_{j,k \in \mathcal{E}} C^{j,k,\varepsilon} \text{Tr}_n(\gamma_2(R_N^{\varepsilon,\alpha})^{p_1 p_2} \gamma_2(R_N^{\varepsilon,\alpha})^{p_3 p_4} \gamma_2(R_N^{\varepsilon,\alpha} B_N^{(2)})^{p_5 u} e_{q,l}(B_N^{(1)} R_N^{\varepsilon,\alpha})^{v p_6}) \right\| \\ & \leq C_\varepsilon \|\gamma_2\|^3 \|(\Im \beta)^{-1}\|^2 N \sum_{j=1}^N \|(B_N^{(1)} R_N^{\varepsilon,\alpha})^{v j}\| \|(R_N^{\varepsilon,\alpha} B_N^{(2)})^{j u}\| \\ & \leq C_\varepsilon \|\gamma_2\|^3 \|(\Im \beta)^{-1}\|^2 N \left( \sum_{j=1}^N \|(B_N^{(1)} R_N^{\varepsilon,\alpha})^{v j}\|^2 \right)^{1/2} \left( \sum_{j=1}^N \|(R_N^{\varepsilon,\alpha} B_N^{(2)})^{j u}\|^2 \right)^{1/2} \\ & \leq C_\varepsilon \|\gamma_2\|^3 \|(\Im \beta)^{-1}\|^4 n N. \end{aligned}$$

- $p_5 \neq p_6$ , in which case Lemma 11.1 yields

$$\begin{aligned} & \left\| \sum_{j,k \in \mathcal{E}} C^{j,k,\varepsilon} \text{Tr}_n(\gamma_2(R_N^{\varepsilon,\alpha})^{p_1 p_2} \gamma_2(R_N^{\varepsilon,\alpha})^{p_3 p_4} \gamma_2(R_N^{\varepsilon,\alpha} B_N^{(2)})^{p_5 u} e_{q,l}(B_N^{(1)} R_N^{\varepsilon,\alpha})^{v p_6}) \right\| \\ & \leq C_\varepsilon \|\gamma_2\|^3 \|(\Im \beta)^{-1}\|^2 \left( \sum_{k=1}^N \|(R_N^{\varepsilon,\alpha} B_N^{(2)})^{k u}\| \right) \left( \sum_{j=1}^N \|(B_N^{(1)} R_N^{\varepsilon,\alpha})^{v j}\| \right) \\ & \leq C_\varepsilon N \|\gamma_2\|^3 \|(\Im \beta)^{-1}\|^2 \left( \sum_{k=1}^N \|(R_N^{\varepsilon,\alpha} B_N^{(2)})^{k u}\|^2 \right)^{1/2} \left( \sum_{j=1}^N \|(B_N^{(1)} R_N^{\varepsilon,\alpha})^{v j}\|^2 \right)^{1/2} \\ & \leq C_\varepsilon n N \|\gamma_2\|^3 \|(\Im \beta)^{-1}\|^4. \end{aligned}$$

We see now that  $|I_j| \leq C_\varepsilon \|(\Im \beta)^{-1}\|^4 / \sqrt{N}$  for  $j = 2, 3$ . Finally,  $I_5$  and  $I_6$  are finite linear combinations of terms of the form

$$(10.13) \quad \frac{\tilde{C}(\alpha)}{N^2} \sum_{j,k \in \mathcal{E}} C^{j,k,\varepsilon} \text{Tr}_n(\gamma_2(R_N^{\varepsilon,\alpha})^{p_1 p_2} \gamma_2(R_N^{\varepsilon,\alpha})^{p_3 p_4} \gamma_2(R_N^{\varepsilon,\alpha})^{p_5 p_6} \gamma_2(R_N^{\varepsilon,\alpha} B_N^{(2)})^{p_7 u} e_{q,l}(B_N^{(1)} R_N^{\varepsilon,\alpha})^{v p_8})$$

where  $\mathcal{E}$  is some subset of  $\{1, \dots, N\}^2$ ,  $C^{j,k,\varepsilon} \in \{\kappa_4^{j,k,\varepsilon}, \tilde{\kappa}_4^{j,k,\varepsilon}\}$  and  $(p_1, \dots, p_6)$  is a permutation of  $(k, k, k, k, j, j, j, j)$ . Lemma 11.1 shows that the norm of such a term can be estimated by

$$\begin{aligned} & \frac{1}{N^2} C_\varepsilon N \|\gamma_2\|^4 \|(\Im \beta)^{-1}\|^4 \left( \sum_{k=1}^N \|(R_N^{\varepsilon,\alpha} B_N^{(2)})^{k u}\| \right) \\ & \leq \frac{1}{N^2} C_\varepsilon N^{\frac{3}{2}} \|\gamma_2\|^4 \|(\Im \beta)^{-1}\|^4 \left( \sum_{k=1}^N \|(R_N^{\varepsilon,\alpha} B_N^{(2)})^{k u}\|^2 \right)^{1/2} \leq \frac{1}{N^2} C_\varepsilon \sqrt{n} N^{\frac{3}{2}} \|\gamma_2\|^4 \|(\Im \beta)^{-1}\|^5. \end{aligned}$$

It follows that  $|I_j| \leq C_\varepsilon \|(\Im w)^{-1}\|^5 / \sqrt{N}$  for  $j = 5, 6$ . The proposition follows.  $\square$



We show next that  $\mathcal{F}_N^\varepsilon(\beta)$  is close to its expected value. This result uses concentration inequalities in the presence of a Poincaré inequality. We recall that if the law of a random variable  $X$  satisfies the Poincaré inequality with constant  $C$  and  $\alpha \in \mathbb{R} \setminus \{0\}$ , then the law of  $\alpha X$  satisfies the Poincaré inequality with constant  $\alpha^2 C$ . Moreover, suppose that the probability measures  $\mu_1, \dots, \mu_r$  on  $\mathbb{R}$  satisfy the Poincaré inequality with constants  $C_1, \dots, C_r$  respectively. Then the product measure  $\mu_1 \otimes \dots \otimes \mu_r$  on  $\mathbb{R}^r$  satisfies the Poincaré inequality with constant  $C = \max\{C_1, \dots, C_r\}$ . That is, if  $f : \mathbb{C}^r \rightarrow \mathbb{R}$  is an arbitrary differentiable function such that  $f$  and its gradient  $\text{grad} f$  are square integrable relative to  $\mu_1 \otimes \dots \otimes \mu_r$ , then

$$\mathbb{V}(f) \leq C \int_{\mathbb{R}^r} \|\text{grad} f\|_2^2 d\mu_1 \otimes \dots \otimes \mu_r.$$

Here  $\mathbb{V}(f) = \int |f - \int f d\mu_1 \otimes \dots \otimes \mu_r|^2 d\mu_1 \otimes \dots \otimes \mu_r$  (see [28, Theorem 2.5]).

We use the following concentration result (see [3, Lemma 4.4.3 and Exercise 4.4.5] or [32, Chapter 3]).

**Lemma 10.4.** *Let  $\mathbb{P}$  be a probability measure on  $\mathbb{R}^r$  which satisfies a Poincaré inequality with constant  $C$ . Then there exist  $K_1 > 0$  and  $K_2 > 0$  such that, for every Lipschitz function  $F$  on  $\mathbb{R}^r$  with Lipschitz constant  $|F|_{\text{Lip}}$ , and for every  $\varepsilon > 0$ , we have*

$$\mathbb{P}(|F - \mathbb{E}_{\mathbb{P}}(F)| > \varepsilon) \leq K_1 \exp\left(-\frac{\varepsilon}{K_2 \sqrt{C} |F|_{\text{Lip}}}\right).$$

The following result is similar to Proposition 8.4(1).

**Proposition 10.5.** *Suppose that  $T_N, S_N \in M_N(\mathbb{C})$  are contractions of uniformly bounded rank. Given  $\varepsilon > 0$  and  $\beta \in \mathbb{H}^+(M_n(\mathbb{C}))$ , almost surely*

$$\lim_{N \rightarrow \infty} \|(I_n \otimes T_N)(R_N^\varepsilon(\beta) - \mathbb{E}R_N^\varepsilon(\beta))(I_n \otimes S_N)\| = 0.$$

*Proof.* As in the proof of Proposition 8.4, it suffices to consider the case in which  $T_N$  and  $S_N$  are contractions of rank 1. Write  $I_n$  as a sum  $Q_1 + \dots + Q_n$  of rank 1 projections. Then the norm in the statement is at most equal to

$$\sum_{j,k=1}^n \|(Q_j \otimes T_N)(R_N^\varepsilon(\beta) - \mathbb{E}R_N^\varepsilon(\beta))(Q_k \otimes S_N)\| = \sum_{j,k=1}^n |\text{Tr}(R_N^\varepsilon(\beta) - \mathbb{E}R_N^\varepsilon(\beta))Q_{j,k}|,$$

where each  $Q_{j,k}$  is a contraction of rank 1. Given a selfadjoint matrix  $Z_N \in M_N(\mathbb{C})$ , we set  $R(Z, \beta) = (\beta \otimes I_N - L(C_N, Z))^{-1}$  and  $f_{N,j,k}(Z) = \text{Tr}R(Z, \beta)Q_{j,k}$ . We have

$$f_N(Z_1) - f_N(Z_2) = \text{Tr}(R(Z_1, \beta)(\gamma_2 \otimes (Z_1 - Z_2))R(Z_2, \beta)Q_{j,k}),$$

and thus

$$\begin{aligned} |f_N(Z_1) - f_N(Z_2)| &\leq \|\gamma_2 \otimes (Z_1 - Z_2)\|_2 \|R(Z_1, \beta)Q_{j,k}R(Z_2, \beta)\|_2 \\ &\leq \|\gamma_2\|_2 \|(Z_1 - Z_2)\|_2 \|(\Im \beta)^{-1}\|^2. \end{aligned}$$

An application of Lemma 10.4 and of the comment preceding it yield

$$\mathbb{P}(|\text{Tr}(R_N^\varepsilon(\beta) - \mathbb{E}R_N^\varepsilon(\beta))Q_{j,k}| > \delta) \leq 2 \exp(-CN^{1/2} \|(\Im \beta)^{-1}\|^{-2} \delta)$$

for every  $\delta > 0$ , with a constant  $C$  that does not depend in  $N, j$ , or  $k$ . The proposition follows by an application of the Borel-Cantelli lemma.  $\square$

**Corollary 10.6.** *For every  $\beta \in \mathbb{H}^+(M_n(\mathbb{C}))$  and every  $\varepsilon > 0$  we have, almost surely,*

$$(10.14) \quad \lim_{N \rightarrow \infty} \mathcal{F}_N^\varepsilon(\beta) - \mathbb{E}(\mathcal{F}_N^\varepsilon(\beta)) = 0.$$

Observe now that

$$\begin{aligned} \mathcal{F}_N(\beta) - (\omega(\beta) - s\gamma_1)^{-1} \otimes I_p &= \mathcal{F}_N(\beta) - \mathcal{F}_N^\varepsilon(\beta) + \mathcal{F}_N^\varepsilon(\beta) - \mathbb{E}(\mathcal{F}_N^\varepsilon(\beta)) \\ &\quad + \mathbb{E}(\mathcal{F}_N^\varepsilon(\beta)) - \mathbb{E}(\mathcal{F}_N^g(\beta)) + \mathbb{E}(\mathcal{F}_N^g(\beta)) - (\omega(\beta) - s\gamma_1)^{-1} \otimes I_p. \end{aligned}$$

We let  $N \rightarrow \infty$  and then  $\varepsilon \rightarrow 0$  and apply (10.6), (10.14), Proposition 10.2, and Proposition 10.1 to obtain the following result.

**Theorem 10.7.** *For every  $\beta \in \mathbb{H}^+(M_n(\mathbb{C}))$  we have, almost surely, when  $N$  goes to infinity,  $\lim_{N \rightarrow \infty} \mathcal{F}_N(\beta) = (\omega(\beta) - s\gamma_1)^{-1} \otimes I_p$ .*

Everything is now in place for completing the argument.

*Proof of Theorem 6.3.* We noted earlier that  $\omega(ze_{1,1} - \gamma_0)$  is analytic in  $\mathbb{C} \setminus \sigma(P(a, b))$ . For fixed  $t \in \mathbb{R} \setminus \sigma(P(a, b))$ , set  $\Psi(\beta) = \beta \otimes 1 - L(a, b)$  and  $\Psi_N(\beta) = \beta \otimes 1 - L(C_N, X_N/\sqrt{N})$ . According to Lemma 4.2,  $\Psi(te_{1,1} - \gamma_0)$  is invertible, and thus there exists  $\delta > 0$  such that

$$(10.15) \quad d(0, \sigma(\Psi(te_{1,1} - \gamma_0))) \geq \delta > 0.$$

Theorem 2.3 and Proposition 2.2 imply that, almost surely, for every complex polynomial  $Q$  in one variable we have

$$\lim_{N \rightarrow \infty} \|Q(\Psi_N(te_{1,1} - \gamma_0))\| = \|Q(\Psi(te_{1,1} - \gamma_0))\|.$$

Asymptotic freeness implies that almost surely, for every complex polynomial  $Q$  in one variable,

$$\lim_{N \rightarrow \infty} (\text{tr}_n \otimes \text{tr}_N)(Q(\Psi_N(te_{1,1} - \gamma_0))) = (\text{tr}_n \otimes \tau)(Q(\Psi(te_{1,1} - \gamma_0))).$$

Thus, denoting the Hausdorff distance by  $d_H$ , we deduce that almost surely for  $N$  large enough,

$$d_H(\sigma(\Psi_N(te_{1,1} - \gamma_0)), \sigma(\Psi(te_{1,1} - \gamma_0))) \leq \delta/4.$$

Note that  $\Psi_N(te_{1,1} - \gamma_0)$  is selfadjoint. For an arbitrary  $\beta \in M_n(\mathbb{C})$ , we have

$$\|\Psi_N(te_{1,1} - \gamma_0) - \Psi_N(\beta)\| \leq \|(te_{1,1} - \gamma_0) - \beta\|.$$

It follows from (10.15) that, almost surely for all large  $N$ , if  $\|\beta - (te_{1,1} - \gamma_0)\| < \delta/4$ , then

$$d(0, \sigma(\Psi_N(\beta))) \geq \delta/2.$$

Moreover, denoting by  $s_1(M)$  the smallest singular value of an arbitrary matrix  $M$ ,

$$s_1(\Psi_N(\beta)) \geq s_1(\Psi_N(te_{1,1} - \gamma_0)) - \|\beta - (te_{1,1} - \gamma_0)\|.$$

Thus, almost surely for all large  $N$ , provided that  $\|\beta - (te_{1,1} - \gamma_0)\| < \delta/4$ , we have  $\|\mathcal{F}_N(\beta)\| = \|(\Psi_N(\beta))^{-1}\| \leq 4/\delta$ . In other words, almost surely, the family  $\{\mathcal{F}_N\}_{N \in \mathbb{N}}$  is normal in a neighborhood of  $te_{1,1} - \gamma_0$ . According to Theorem 10.7, for any  $\beta \in \mathbb{H}^+(M_n(\mathbb{C}))$ , almost surely  $\mathcal{F}_N$  converges towards  $(\omega(\beta) - s\gamma_1)^{-1} \otimes I_p$ . Set

$$\Lambda = \{w \in M_n(\mathbb{C}), \|w - (te_{1,1} - \gamma_0)\| < \delta/4, \Im w > 0\}.$$

Almost surely for any  $w \in \Lambda$  such that  $\Im w \in M_n(\mathbb{Q})$  and  $\Re w \in M_n(\mathbb{Q})$ ,  $\mathcal{F}_N(w)$  converges towards  $(\omega(w) - s\gamma_1)^{-1} \otimes I_p$ . The Vitali-Montel convergence theorem

implies that that almost surely  $\mathcal{F}_N$  converges towards a holomorphic function on  $\{w \in M_n(\mathbb{C}), \|w - (te_{1,1} - \gamma_0)\| < \delta/4\}$ , and, in particular,  $\mathcal{F}_N(ze_{1,1} - \beta_0)$  converges for any  $z \in \mathbb{C}$  such that  $|z - t| < \delta/4$  towards  $(\omega((ze_{1,1} - \gamma_0) - s\gamma_1)^{-1} \otimes I_p)$ .

Now, the Hurwitz theorem on zeros of analytic functions implies that, almost surely for large  $N$ , the function  $F_N(ze_{1,1} - \gamma_0) = \det(I_n \otimes I_p - \mathcal{F}_N(ze_{1,1} - \gamma_0))$  has as many zeros in a neighborhood of  $t$  as the function

$$\det(I_n \otimes I_p - (\gamma_1 \otimes T)((\omega((ze_{1,1} - \gamma_0) - s\gamma_1)^{-1} \otimes I_p).$$

Now, note that

$$\begin{aligned} \gamma_1((\omega(ze_{11} - \gamma_0) - s\gamma_1)^{-1} \otimes T - I_n \otimes I_p) \\ = (\gamma_1 \otimes T - s\gamma_1 \otimes I_p)(\omega(ze_{11} - \gamma_0) - s\gamma_1)^{-1} \otimes I_p - I_n \otimes I_p \\ = (\gamma_1 \otimes T - \omega(ze_{11} - \gamma_0) \otimes I_p)(\omega(ze_{11} - \gamma_0) - s\gamma_1)^{-1} \otimes I_p \end{aligned}$$

and

$$\omega(ze_{1,1} - \gamma_0) \otimes I_p - \gamma_1 \otimes T = \sum_{i=1}^p (\omega(ze_{11} - \gamma_0) - \gamma_1 \theta_i) \otimes e_{i,i}.$$

Therefore

$$\det(\omega(ze_{11} - \gamma_0) \otimes I_p - \gamma_1 \otimes T) = \prod_{i=1}^p \det(\omega(ze_{11} - \gamma_0) - \gamma_1 \theta_i).$$

The theorem follows.  $\square$

## 11. APPENDIX

The following result is [12, Lemma 8.1].

**Lemma 11.1.** *For any matrix  $M \in M_n(\mathbb{C}) \otimes M_N(\mathbb{C})$ ,*

$$(11.1) \quad \frac{1}{N} \sum_{k,l=1}^N \|M^{kl}\|^2 \leq n \|M\|^2$$

and for any fixed  $k$ ,

$$(11.2) \quad \sum_{l=1}^N \|M^{lk}\|^2 \leq n \|M\|^2$$

and

$$(11.3) \quad \sum_{l=1}^N \|M^{kl}\|^2 \leq n \|M\|^2,$$

where  $M^{kl}$  is defined by (10.8).

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