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► **To cite this version:**

Serban Belinschi, Hari Bercovici, Mireille Capitaine. On the outlying eigenvalues of a polynomial in large independent random matrices. 2018. hal-01494550v2

HAL Id: hal-01494550

<https://hal.science/hal-01494550v2>

Preprint submitted on 5 Nov 2018

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ON THE OUTLYING EIGENVALUES OF A POLYNOMIAL IN LARGE INDEPENDENT RANDOM MATRICES

SERBAN T. BELINSCHI, HARI BERCOVICI, AND MIREILLE CAPITAINE

ABSTRACT. Given a selfadjoint polynomial $P(X, Y)$ in two noncommuting self-adjoint indeterminates, we investigate the asymptotic eigenvalue behavior of the random matrix $P(A_N, B_N)$, where A_N and B_N are independent Hermitian random matrices and the distribution of B_N is invariant under conjugation by unitary operators. We assume that the empirical eigenvalue distributions of A_N and B_N converge almost surely to deterministic probability measures μ and ν , respectively. In addition, the eigenvalues of A_N and B_N are assumed to converge uniformly almost surely to the support of μ and ν , respectively, except for a fixed finite number of fixed eigenvalues (spikes) of A_N . It is known that almost surely the empirical distribution of the eigenvalues of $P(A_N, B_N)$ converges to a certain deterministic probability measure η (sometimes denoted $\eta = P^\square(\mu, \nu)$) and, when there are no spikes, the eigenvalues of $P(A_N, B_N)$ converge uniformly almost surely to the support of η . When spikes are present, we show that the eigenvalues of $P(A_N, B_N)$ still converge uniformly to the support of η , with the possible exception of certain isolated outliers whose location can be determined in terms of μ, ν, P , and the spikes of A_N . We establish a similar result when B_N is replaced by a Wigner matrix. The relation between outliers and spikes is described using the operator-valued subordination functions of free probability theory. These results extend known facts from the special case in which $P(X, Y) = X + Y$.

1. INTRODUCTION

Let μ and ν be two Borel probability measures with bounded support on \mathbb{R} . Suppose given, for each positive integer N , selfadjoint $N \times N$ independent random matrices A_N and B_N , with the following properties:

- (a) the distribution of B_N is invariant under conjugation by unitary $N \times N$ matrices;
- (b) the empirical eigenvalue distributions of A_N and B_N converge almost surely to μ and ν , respectively;
- (c) the eigenvalues of A_N and B_N converge uniformly almost surely to the supports of μ and ν , respectively, with the exception of a fixed number p of *spikes*, that is, fixed eigenvalues of A_N that lie outside the support of μ .

When spikes are absent, that is, when $p = 0$, it was shown in [23] that the eigenvalues of $A_N + B_N$ converge uniformly almost surely to the support of the free additive convolution $\mu \boxplus \nu$. When $p > 0$, the eigenvalues of $A_N + B_N$ also converge uniformly almost surely to a compact set $K \subset \mathbb{R}$ such that $\text{supp}(\mu \boxplus \nu) \subset K$ and $K \setminus \text{supp}(\mu \boxplus \nu)$ has no accumulation points in $\mathbb{R} \setminus \text{supp}(\mu \boxplus \nu)$. Moreover, if $t \in K \setminus \text{supp}(\mu \boxplus \nu)$, then $\omega(t)$ is one of the spikes of A_N , where ω is

HB was supported by a grant from the National Science Foundation. This work was started while HB was visiting the Institute of Mathematics of Toulouse as Professeur Invité.

a certain subordination function arising in free probability. The relative position of the eigenvectors corresponding to spikes and outliers is also given in terms of subordination functions. We refer to [11] for this result.

Our purpose is to show that analogous results hold when the sum $A_N + B_N$ is replaced by an arbitrary selfadjoint polynomial $P(A_N, B_N)$. Then, by a comparison procedure to the particular case when B_N is a G.U.E. (Gaussian unitary ensemble), we are also able to identify the outliers of an arbitrary selfadjoint polynomial $P(A_N, \frac{X_N}{\sqrt{N}})$ when X_N is a Wigner matrix independent from A_N . This extends an earlier result [22] pertaining to additive deformations of Wigner matrices. More precisely we consider a Hermitian matrix $X_N = [X_{ij}]_{i,j=1}^N$, where $[X_{ij}]_{i \geq 1, j \geq 1}$ is an infinite array of random variables such that

- (X0) X_N is independent from A_N ,
- (X1) $X_{ii}, \sqrt{2}\Re(X_{ij}), i < j, \sqrt{2}\Im(X_{ij}), i < j$, are independent, centered with variance 1,
- (X2) there exist $K, x_0 > 0, n_0 \in \mathbb{N}$, and a random variable Z with finite fourth moment such that

$$\frac{1}{n^2} \sum_{1 \leq i, j \leq n} \mathbb{P}(|X_{ij}| > x) \leq K \mathbb{P}(|Z| > x) \quad x > x_0, n > n_0.$$

- (X3) $\sup\{\mathbb{E}|X_{ij}|^3 : i, j \in \mathbb{N}, i < j\} < +\infty$.

Remark 1.1. The matrix X_N is called a G.U.E. if the variables $X_{ii}, \sqrt{2}\Re(X_{ij}), i < j$, and $\sqrt{2}\Im(X_{ij}), i < j$, are independent standard Gaussian. Assumptions (X2) and (X3) obviously hold if these variables are merely independent and identically distributed with a finite fourth moment.

Our result lies in the lineage of recent, and not so recent, works [5, 7, 8, 14, 18, 19, 21, 22, 26, 27, 31, 33, 35, 39, 40, 41] studying the influence of additive or multiplicative perturbations on the extremal eigenvalues of classical random matrix models, the seminal paper being [7], where the so-called BBP phase transition was observed.

We note that Shlyakhtenko [45] considered a framework which makes it possible to understand this kind of result as a manifestation of infinitesimal freeness. In fact, the results of [45] also allow one to detect the presence of spikes from the behaviour of the bulk of the eigenvalues of $P(A_N, B_N)$, even when $P(A_N, B_N)$ has no outlying eigenvalues. In a related result, Collins, Hasebe and Sakuma [24] study the ‘purely spike’ case in which $\mu = \nu = \delta_0$ and the eigenvalues of A_N and B_N accumulate to given sequences $(a_k)_{k=1}^\infty$ and $(b_k)_{k=1}^\infty$ of real numbers converging to zero.

2. NOTATION AND PRELIMINARIES ON STRONG ASYMPTOTIC FREENESS

We recall that a C^* -probability space is a pair (\mathcal{A}, τ) , where \mathcal{A} is a C^* -algebra and τ is a state on \mathcal{A} . We always assume that τ is faithful. The elements of \mathcal{A} are referred to as *random variables*.

If (Ω, Σ, P) is a classical probability space, then $(L^\infty(\Omega), \mathbb{E})$ is a C^* -probability space, where \mathbb{E} is the usual expected value. Given $N \in \mathbb{N}$, $(M_N(\mathbb{C}), \text{tr}_N)$ is a C^* -probability space, where $\text{tr}_N = \frac{1}{N} \text{Tr}_N$ denotes the normalized trace. More generally, if (\mathcal{A}, τ) is an arbitrary C^* -probability space and $N \in \mathbb{N}$, then $M_N(\mathcal{A}) = M_N(\mathbb{C}) \otimes \mathcal{A}$ becomes a C^* -probability space with the state $\text{tr}_N \otimes \tau$.

The *distribution* μ_a of a selfadjoint element a in a C^* -probability space (\mathcal{A}, τ) is a compactly supported probability measure on \mathbb{R} , uniquely determined by the requirement that $\int_{\mathbb{R}} t^n d\mu_a(t) = \tau(a^n)$, $n \in \mathbb{N}$. The *spectrum* of an element $a \in \mathcal{A}$ is

$$\sigma(a) = \{\lambda \in \mathbb{C}: \lambda 1 - a \text{ is not invertible in } \mathcal{A}\}.$$

For instance, if $A \in M_N(\mathbb{C})$ is a selfadjoint matrix, then the distribution of A relative to tr_N is the measure $\mu_A = \frac{1}{N} \sum_{j=1}^N \delta_{\lambda_j(A)}$, where $\{\lambda_1(A), \dots, \lambda_N(A)\}$ is the list of the eigenvalues of A , repeated according to multiplicity. As usual, the support $\text{supp}(\mu)$ of a Borel probability measure μ on \mathbb{R} is the smallest closed set $F \subset \mathbb{R}$ with the property that $\mu(F) = 1$. It is known that if $a = a^* \in \mathcal{A}$ and τ is faithful, then $\sigma(a) = \text{supp}(\mu_a)$. In the following, we assume that τ is a tracial state, that is, $\tau(ab) = \tau(ba)$, $a, b \in \mathcal{A}$.

Suppose that we are given C^* -probability spaces $\{(\mathcal{A}_N, \tau_N)\}_{N=0}^{\infty}$ and selfadjoint elements $a_N \in \mathcal{A}_N$, $N \geq 0$. We say that $\{a_N\}_{N=1}^{\infty}$ *converges in distribution* to a_0 if

$$(2.1) \quad \lim_{N \rightarrow \infty} \tau_N(a_N^k) = \tau_0(a_0^k), \quad k \in \mathbb{N}.$$

We say that $\{a_N\}_{N=1}^{\infty}$ *converges strongly in distribution* to a_0 (or to μ_{a_0}) if, in addition to (2.1), the sequence $\{\text{supp}(\mu_{a_N})\}_{N=1}^{\infty}$ converges to $\text{supp}(\mu_{a_0})$ in the Hausdorff metric. This condition simply means that for every $\varepsilon > 0$ there exists $N(\varepsilon) \in \mathbb{N}$ such that

$$\text{supp}(\mu_{a_N}) \subset \text{supp}(\mu_{a_0}) + (-\varepsilon, \varepsilon)$$

and

$$\text{supp}(\mu_{a_0}) \subset \text{supp}(\mu_{a_N}) + (-\varepsilon, \varepsilon)$$

for every $N \geq N(\varepsilon)$. If all the traces τ_N are faithful, strong convergence can be reformulated as follows:

$$\lim_{N \rightarrow \infty} \|P(a_N)\| = \|P(a_0)\|,$$

for every polynomial P with complex coefficients. This observation allows us to extend the concept of (strong) convergence in distribution to k -tuples of random variables, $k \in \mathbb{N}$. For every $k \in \mathbb{N}$, we denote by $\mathbb{C}\langle X_1, \dots, X_k \rangle$ the algebra of polynomials with complex coefficients in k noncommuting indeterminates X_1, \dots, X_k . This is a $*$ -algebra with the adjoint operation determined by

$$(\alpha X_{i_1} X_{i_2} \cdots X_{i_n})^* = \bar{\alpha} X_{i_n} \cdots X_{i_2} X_{i_1}, \quad \alpha \in \mathbb{C}, \quad i_1, i_2, \dots, i_n \in \{1, \dots, k\}.$$

Suppose that $\{(\mathcal{A}_N, \tau_N)\}_{N=0}^{\infty}$ is a sequence of C^* -probability spaces, $k \in \mathbb{N}$, and $\{a_N\}_{N=0}^{\infty}$ is a sequence of k -tuples $a_N = (a_{N,1}, \dots, a_{N,k}) \in \mathcal{A}_N^k$ of selfadjoint elements. We say that $\{a_N\}_{N=1}^{\infty}$ *converges in distribution* to a_0 if

$$(2.2) \quad \lim_{N \rightarrow \infty} \tau_N(P(a_N)) = \tau_0(P(a_0)), \quad P \in \mathbb{C}\langle X_1, \dots, X_k \rangle.$$

We say that $\{a_N\}_{N=1}^{\infty}$ *converges strongly in distribution* to a_0 if, in addition to (2.2), we have

$$\lim_{N \rightarrow \infty} \|P(a_N)\| = \|P(a_0)\|, \quad P \in \mathbb{C}\langle X_1, \dots, X_k \rangle.$$

The above concepts extend to k -tuples $a_N = (a_{N,1}, \dots, a_{N,k}) \in \mathcal{A}_N^k$ which do not necessarily consist of selfadjoint elements. The only change is that one must use polynomials in the variables $a_{N,j}$ and their adjoints $a_{N,j}^*$, $j = 1, \dots, k$.

Remark 2.1. Suppose that all the states $\tau_N, N \in \mathbb{N}$, are faithful. As seen in [23, Proposition 2.1], $\{a_N\}_{N=1}^\infty$ converges strongly in distribution to a_0 if and only if $\{P(a_N)\}_{N=1}^\infty$ converges strongly in distribution to $P(a_0)$ for every selfadjoint polynomial $P \in \mathbb{C}\langle X_1, \dots, X_k \rangle$. Moreover, strong convergence in distribution also implies strong convergence at the matricial level. The following result is [36, Proposition 7.3].

Proposition 2.2. *Let $\{(\mathcal{A}_N, \tau_N)\}_{N=0}^\infty$ be C^* -probability spaces with faithful states $\{\tau_N\}_{N=0}^\infty$, let $k \in \mathbb{N}$, and let $\{a_N\}_{N=0}^\infty$ be a sequence of k -tuples of selfadjoint elements $a_N \in \mathcal{A}_N^k$. Suppose that $\{a_N\}_{N=1}^\infty$ converges strongly in distribution to a_0 . Then $\lim_{N \rightarrow \infty} \|P(a_N)\| = \|P(a_0)\|$ for every $n \in \mathbb{N}$ and every matrix polynomial $P \in M_n(\mathbb{C}\langle X_1, \dots, X_k \rangle)$.*

A special case of strong convergence in distribution arises from the consideration of random matrices in $M_N(\mathbb{C})$. The following result follows from [23, Theorem 1.4] and [12, Theorem 1.2].

Theorem 2.3. *Let (\mathcal{A}_N, τ_N) denote the space $(M_N(\mathbb{C}), \text{tr}_N)$, $N \in \mathbb{N}$. Suppose that $k_1, k_2, k_3 \in \mathbb{N}$ are fixed, $u_N = (U_{N,1}, \dots, U_{N,k_1})$, $x_N = (X_{N,1}, \dots, X_{N,k_2})$ and $a_N = (A_{N,1}, \dots, A_{N,k_3})$ are mutually independent random tuples of matrices in some classical probability space such that:*

- (i) $U_{N,1}, \dots, U_{N,k_1}$ are independent unitaries distributed according to the Haar measure on the unitary group $U(N)$, $N \in \mathbb{N}$.
- (ii) $X_{N,1}, \dots, X_{N,k_2}$ are independent Hermitian matrices, each satisfying assumptions (X1), (X2), and (X3) in the introduction.
- (iii) a_N is a vector of $N \times N$ selfadjoint matrices such that the sequence $\{a_N\}_{N=1}^\infty$ converges strongly almost surely in distribution to some deterministic k_3 -tuple in a C^* -probability space.

Then there exist a C^* -probability space (\mathcal{A}, τ) , a free family $u = (u_1, \dots, u_{k_1}) \in \mathcal{A}^{k_1}$ of Haar unitaries, a semicircular system $x = (x_1, \dots, x_{k_2}) \in \mathcal{A}^{k_2}$ and $a = (a_1, \dots, a_{k_3}) \in \mathcal{A}^{k_3}$, such that, u, x , and a are free and $\{(u_N, x_N, a_N)\}_{N=1}^\infty$ converges strongly almost surely in distribution to (u, x, a) .

We recall that a tuple (x_1, \dots, x_k) of elements in a C^* -probability space (\mathcal{A}, τ) is called a *semicircular system* if $\{x_1, \dots, x_k\}$ is a free family of selfadjoint random variables, and for every $i = 1, \dots, k$, μ_{x_i} is the standard semicircular distribution $\nu_{0,1}$ defined by

$$(2.3) \quad d\nu_{0,1}(t) = \frac{1}{2\pi} \sqrt{4 - t^2} \mathbf{1}_{[-2,2]}(t) dt.$$

An element $u \in \mathcal{A}$ is called a *Haar unitary* if $u^* = u^{-1}$ and $\tau(u^n) = 0$ for all $n \in \mathbb{Z} \setminus \{0\}$. Note that Theorem 1.2 in [12] deals with deterministic a_N but the random case readily follows as pointed out by assertion 2 in [36, Section 3]. The point of Theorem 2.3 is, of course, that the resulting convergence is strong. Convergence in distribution was established earlier (see [49], [25], [3, Theorem 5.4.5]).

We also need a simple coupling result from [23, Lemma 5.1].

Lemma 2.4. *Suppose given selfadjoint matrices $C_N, D_N \in M_N(\mathbb{C})$, $N \in \mathbb{N}$, such that the sequences $\{C_N\}_{N \in \mathbb{N}}$ and $\{D_N\}_{N \in \mathbb{N}}$ converge strongly in distribution. Then there exist diagonal matrices $\tilde{C}_N, \tilde{D}_N \in M_N(\mathbb{C})$, $N \geq 1$, such that $\mu_{\tilde{C}_N} = \mu_{C_N}$, $\mu_{\tilde{D}_N} = \mu_{D_N}$, and the sequence $\{(\tilde{C}_N, \tilde{D}_N)\}_{N \in \mathbb{N}}$ converges strongly in distribution.*

3. DESCRIPTION OF THE MODELS

In order to describe in detail our matrix models, we need two compactly supported probability measures μ and ν on \mathbb{R} , a positive integer p , and a sequence of fixed real numbers $\theta_1 \geq \theta_2 \geq \dots \geq \theta_p$ in $\mathbb{R} \setminus \text{supp}(\mu)$. The matrix $A_N \in M_N(\mathbb{C})$ is random selfadjoint for all $N \in \mathbb{N}$, $N \geq 1$ and satisfies the following conditions:

- (A1) almost surely, the sequence $\{A_N\}_{N=1}^\infty$ converges in distribution to μ ,
- (A2) $\theta_1 \geq \theta_2 \geq \dots \geq \theta_p$ are p eigenvalues of A_N , and
- (A3) the other eigenvalues of A_N , which may be random, converge uniformly almost surely to $\text{supp}(\mu)$: almost surely, for every $\varepsilon > 0$ there exists $N(\varepsilon) \in \mathbb{N}$ such that

$$\sigma(A_N) \setminus \{\theta_1, \dots, \theta_p\} \subseteq \text{supp}(\mu) + (-\varepsilon, \varepsilon), \quad N \geq N(\varepsilon).$$

In other words, only the p eigenvalues $\theta_1, \dots, \theta_p$ prevent $\{A_N\}_{N=1}^\infty$ from converging strongly in distribution to μ .

We investigate two polynomial matricial models, both involving A_N . The first model involves a sequence $\{B_N\}_{N=1}^\infty$ of random Hermitian matrices such that

- (B0) B_N is independent from A_N ,
- (B1) B_N converges strongly in distribution to the compactly supported probability measure ν on \mathbb{R} ,
- (B2) for each N , the distribution of B_N is invariant under conjugation by arbitrary $N \times N$ unitary matrices.

The matricial model is

$$(3.1) \quad Z_N = P(A_N, B_N)$$

for an arbitrary selfadjoint polynomial $P \in \mathbb{C}\langle X_1, X_2 \rangle$.

The second model deals with $N \times N$ random Hermitian Wigner matrices $X_N = [X_{ij}]_{i,j=1}^N$, where $[X_{ij}]_{i \geq 1, j \geq 1}$ is an infinite array of random variables satisfying conditions (X0) – (X3) in the introduction. The matricial model is

$$(3.2) \quad Z_N = P\left(A_N, \frac{X_N}{\sqrt{N}}\right)$$

for an arbitrary selfadjoint polynomial $P \in \mathbb{C}\langle X_1, X_2 \rangle$.

In the discussion of the first model, we use results of Voiculescu [49] (see also [54]), who showed that there exist a free pair (a, b) of selfadjoint elements in a II_1 -factor (\mathcal{A}, τ) such that, almost surely, the sequence $\{(A_N, B_N)\}_{N=1}^\infty$ converges in distribution to (a, b) . Thus, $\mu = \mu_a, \nu = \mu_b$, and the sequence $\{P(A_N, B_N)\}_{N=1}^\infty$ converges in distribution to $P(a, b)$ (that is,

$$\lim_{N \rightarrow \infty} \mu_{P(A_N, B_N)} = \mu_{P(a, b)}$$

in the weak* topology) for every selfadjoint polynomial $P \in \mathbb{C}\langle X_1, X_2 \rangle$. When $p = 0$, Lemma 2.4, Theorem 2.3 and Remark 2.1, show that, almost surely, this convergence is strong (see the proof of Corollary 2.2 in [23]).

For the second model we use [12, Proposition 2.2] and [3, Theorem 5.4.5], where it is seen that for every selfadjoint polynomial $P \in \mathbb{C}\langle X_1, X_2 \rangle$ we have

$$\lim_{N \rightarrow \infty} \mu_{P(A_N, X_N/\sqrt{N})} = \mu_{P(a, b)}$$

almost surely in the weak* topology, where a and b are freely independent selfadjoint noncommutative random variables, $\mu_a = \mu$, and $\mu_b = \nu_{0,1}$. As in the

first model, Theorem 2.3 and Remark 2.1 show that, almost surely, the sequence $\{P(A_N, X_N/\sqrt{N})\}_{N=1}^\infty$ converges strongly in distribution to $P(a, x)$ provided that $p = 0$.

Our main result applies, of course, to the case in which $p > 0$. Let Y_N be either B_N or X_N/\sqrt{N} . The set of outliers of $P(A_N, Y_N)$ is calculated from the spikes $\theta_1, \dots, \theta_p$ using Voiculescu's matrix subordination function [52]. When $Y_N = B_N$, we also show that the eigenvectors associated to these outlying eigenvalues have projections of computable size onto the eigenspaces of A_N corresponding to the spikes. The precise statements are Theorems 6.1 and 6.3. Sections 4 and 5 contain the necessary tools from operator-valued noncommutative probability theory while Sections 7–10 are dedicated to the proofs of the main results.

4. LINEARIZATION

As in [4, 13], we use linearization to reduce a problem about a polynomial in freely independent, or asymptotically freely independent, random variables, to a problem about the addition of matrices having these random variables as entries. Suppose that $P \in \mathbb{C}\langle X_1, \dots, X_k \rangle$. For our purposes, a *linearization* of P is a linear polynomial of the form

$$z\alpha \otimes 1 - L,$$

where z is a complex variable, and

$$L = \gamma_0 \otimes 1 + \gamma_1 \otimes X_1 + \dots + \gamma_k \otimes X_k,$$

with $\alpha, \gamma_0, \dots, \gamma_k \in M_n(\mathbb{C})$ for some $n \in \mathbb{N}$, and the following property is satisfied: *given $z \in \mathbb{C}$ and elements a_1, \dots, a_k in a C^* -algebra \mathcal{A} , $z - P(a_1, \dots, a_k)$ is invertible in \mathcal{A} if and only if $z\alpha \otimes 1 - L(a_1, \dots, a_k)$ is invertible in $M_n(\mathcal{A})$.* Usually, this is achieved by ensuring that $(z\alpha \otimes 1 - L)^{-1}$ exists as an element of $M_n(\mathbb{C}\langle X_1, \dots, X_k \rangle \langle (z - P)^{-1} \rangle)$ and $(z - P)^{-1}$ is one of the entries of the $(z\alpha \otimes 1 - L)^{-1}$. It is known (see, for instance, [42]) that every polynomial has a linearization. See [29] for earlier uses of linearization in free probability.

In the following we also say, more concisely, that L is a linearization of P . We also suppress the unit of the algebra \mathcal{A} when there is no risk of confusion. For instance, we may write $z\alpha - L$ in place of $z\alpha \otimes 1 - L$.

We describe in some detail a linearization procedure from [4] (see also [34]) that has several advantages. In this procedure, we always have $\alpha = e_{1,1}$, where $e_{1,1}$ denotes the matrix whose only nonzero entry equals 1 and occurs in the first row and first column. Given $P \in \mathbb{C}\langle X_1, \dots, X_k \rangle$, we produce an integer $n \in \mathbb{N}$ and a linear polynomial $L \in M_n(\mathbb{C}\langle X_1, \dots, X_k \rangle)$ of the form

$$L = \begin{bmatrix} 0 & u \\ v & Q \end{bmatrix},$$

such that $u \in M_{1 \times (n-1)}(\mathbb{C}\langle X_1, \dots, X_k \rangle)$, $v \in M_{(n-1) \times 1}(\mathbb{C}\langle X_1, \dots, X_k \rangle)$, Q is an invertible matrix in $M_{n-1}(\mathbb{C}\langle X_1, \dots, X_k \rangle)$ whose inverse is a polynomial of degree less than or equal to the degree of P , and $uQ^{-1}v = -P$. Moreover, if $P = P^*$, the coefficients of L can be chosen to be selfadjoint matrices in $M_n(\mathbb{C})$.

The construction proceeds by induction on the number of monomials in the given polynomial. If P is a monomial of degree 0 or 1, we set $n = 1$ and $L = P$. If $P = X_{i_1} X_{i_2} X_{i_3} \dots X_{i_{\ell-1}} X_{i_\ell}$, where $\ell \geq 2$ and $i_1, \dots, i_\ell \in \{1, \dots, k\}$, we set $n = \ell$

and

$$L = - \begin{bmatrix} 0 & 0 & \cdots & 0 & X_{i_1} \\ 0 & 0 & \cdots & X_{i_2} & -1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & X_{i_{\ell-1}} & \cdots & 0 & 0 \\ X_{i_\ell} & -1 & \cdots & 0 & 0 \end{bmatrix}.$$

As noted in [34], the lower right $(\ell-1) \times (\ell-1)$ corner of this matrix has an inverse of degree $\ell-2$ in the algebra $M_{\ell-1}(\mathbb{C}\langle X_1, \dots, X_k \rangle)$. (The constant term in this inverse is a selfadjoint matrix and its spectrum is contained in $\{-1, 1\}$.) Suppose now that $p = P_1 + P_2$, where $P_1, P_2 \in \mathbb{C}\langle X_1, \dots, X_k \rangle$, and that linear polynomials

$$L_j = \begin{bmatrix} 0 & u_j \\ v_j & Q_j \end{bmatrix} \in M_{n_j}(\mathbb{C}\langle X_1, \dots, X_k \rangle), \quad j = 1, 2,$$

with the desired properties have been found for P_1 and P_2 . Then we set $n = n_1 + n_2 - 1$ and observe that the matrix

$$L = \begin{bmatrix} 0 & u_1 & u_2 \\ v_1 & Q_1 & 0 \\ v_2 & 0 & Q_2 \end{bmatrix} = \begin{bmatrix} 0 & u \\ v & Q \end{bmatrix} \in M_{n_1+n_2-1}(\mathbb{C}\langle X_1, \dots, X_k \rangle).$$

is a linearization of $P_1 + P_2$ with the desired properties. The construction of a linearization is now easily completed for an arbitrary polynomial. Suppose now that P is a selfadjoint polynomial, so $P = P_0 + P_0^*$ for some other polynomial P_0 . Suppose that the matrix

$$\begin{bmatrix} 0 & u_0 \\ v_0 & Q_0 \end{bmatrix}.$$

of size n_0 is a linearization of P_0 . Then we set $n = 2n_0 - 1$ and observe that the selfadjoint linear polynomial

$$\begin{bmatrix} 0 & u_0 & v_0^* \\ u_0^* & 0 & Q_0^* \\ v_0 & Q_0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & u \\ u^* & Q \end{bmatrix}$$

linearizes P . It is easy to verify inductively that this construction produces a matrix Q such that the constant term of Q^{-1} has spectrum contained in $\{1, -1\}$. These properties of Q [34], and particularly the following observation, facilitate our analysis.

Lemma 4.1. *Let $P \in \mathbb{C}\langle X_1, \dots, X_k \rangle$, and let*

$$L = \begin{bmatrix} 0 & u \\ v & Q \end{bmatrix} \in M_n(\mathbb{C}\langle X_1, \dots, X_k \rangle)$$

be a linearization of P as constructed above. There exist a permutation matrix $T \in M_{n-1}$ and a strictly lower triangular matrix $N \in M_{n-1}(\mathbb{C}\langle X_1, \dots, X_k \rangle)$ such that $Q^{-1} = T(1_{n-1} + N)$.

Proof. We show that there exist a permutation matrix $T_0 \in M_{n-1}$ and a strictly lower triangular matrix permutation matrices $N_0 \in M_{n-1}(\mathbb{C}\langle X_1, \dots, X_k \rangle)$ such that $Q = (1_{n-1} - N_0)T_0$. Then we can define $T = T_0^{-1}$ and $N = \sum_{j=1}^{n-2} N_0^j$. The

existence of T_0 and N_0 is proved by following inductively the construction of L . If $P = X_{i_1} \cdots X_{i_\ell}$, $\ell \geq 2$, we define

$$T_0 = \begin{bmatrix} 0 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 0 \end{bmatrix},$$

and let the only nonzero entries of N_0 be $X_{i_2}, \dots, X_{i_\ell}$ just below the main diagonal. If $P = P_1 + P_2$, and linearizations for P_1 and P_2 have been found, then the desired matrices are obtained simply by taking direct sums of the matrices corresponding to P_1 and P_2 . The case in which $P = P_0 + P_0^*$ is treated similarly (different factorizations must be used for Q_0 and Q_0^*). \square

Lemma 4.2. *Suppose that $P \in \mathbb{C}\langle X_1, \dots, X_k \rangle$, and let*

$$L = \begin{bmatrix} 0 & u \\ v & Q \end{bmatrix} \in M_n(\mathbb{C}\langle X_1, \dots, X_k \rangle)$$

be a linearization of P with the properties outlined above. Then for every $N \in \mathbb{N}$, and for every $S_1, \dots, S_k \in M_N(\mathbb{C})$, we have

$$\det(ze_{1,1} \otimes I_N - L(S_1, \dots, S_k)) = \pm \det(zI_n - P(S_1, \dots, S_k)),$$

where the sign is $\det(Q(S_1, \dots, S_k))$. Moreover,

$$\dim \ker(zI_n - P(S_1, \dots, S_k)) = \dim \ker(ze_{1,1} \otimes I_N - L(S_1, \dots, S_k)) \quad z \in \mathbb{C}.$$

Proof. Suppressing the variables S_1, \dots, S_k , we have

$$\begin{bmatrix} 1 & -uQ^{-1} \\ 0 & 1_{n-1} \end{bmatrix} \begin{bmatrix} z & -u \\ -v & -Q \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -Q^{-1}v & 1_{n-1} \end{bmatrix} = \begin{bmatrix} z - P & 0 \\ 0 & -Q \end{bmatrix}, \quad z \in \mathbb{C}.$$

Lemma 4.1 implies that $\det Q(S_1, \dots, S_k)$ is ± 1 and the determinant identity follows immediately. The dimension of the kernel of a square matrix does not change if the matrix is multiplied by some other invertible matrices. Also, since Q is invertible, the kernel of the matrix on the right hand side of the last equality is easily identified with $\ker(z - P)$. The last assertion follows from these observations. \square

In the case of selfadjoint polynomials, applied to selfadjoint matrices, we can estimate how far $ze_{1,1} - L$ is from not being invertible.

Lemma 4.3. *Suppose that $P = P^* \in \mathbb{C}\langle X_1, \dots, X_k \rangle$, and let*

$$L = \begin{bmatrix} 0 & u^* \\ u & Q \end{bmatrix} \in M_n(\mathbb{C}\langle X_1, \dots, X_k \rangle)$$

be a linearization of P with the properties outlined above. There exist polynomials $T_1, T_2 \in \mathbb{C}[X_1, \dots, X_k]$ with nonnegative coefficients with the following property: given arbitrary selfadjoint elements S_1, \dots, S_k in a unital C^ -algebra \mathcal{A} , and given $z_0 \in \mathbb{C}$ such that $z_0 - P(S)$ is invertible, we have*

$$\|(z_0 e_{1,1} - L(S))^{-1}\| \leq T_1(\|S_1\|, \dots, \|S_k\|) \|(z_0 - P(S))^{-1}\| + T_2(\|S_1\|, \dots, \|S_k\|).$$

In particular, given two real constants $C, \delta > 0$, there exists $\varepsilon > 0$ such that $\text{dist}(z_0, \sigma(P(S))) \geq \delta$ and $\|S_1\| + \dots + \|S_k\| \leq C$ imply $\text{dist}(0, \sigma(z_0 e_{1,1} - L(S))) \geq \varepsilon$.

Proof. For every element a of a C^* -algebra, we have $\text{dist}(0, \sigma(a)) \geq 1/\|a^{-1}\|$. Equality is achieved, for instance, if $a = a^*$. A matrix calculation (in which we suppress the variables S) shows that

$$(z_0 e_{1,1} - L)^{-1} = \begin{bmatrix} 1 & 0 \\ -Q^{-1}u & 1_{n-1} \end{bmatrix} \begin{bmatrix} (z_0 - P)^{-1} & 0 \\ 0 & -Q^{-1} \end{bmatrix} \begin{bmatrix} 1 & -u^*Q^{-1} \\ 0 & 1_{n-1} \end{bmatrix}.$$

The lemma follows now because the entries of $u(S)$, $u^*(S)$, and $Q(S)^{-1}$ are polynomials in S , and

$$\|(z_0 - P(S))^{-1}\| = 1/\text{dist}(z_0, \sigma(P(S)))$$

because $P(S)$ is selfadjoint. \square

The dependence on L in the above lemma is given via the norms of Q^{-1} and of u . Since $\lim_{z \rightarrow \infty} \|(z e_{1,1} - L(S))^{-1}\| \neq 0$, we see that $T_2 \neq 0$.

5. SUBORDINATION

Consider a von Neumann algebra \mathcal{M} endowed with a normal faithful tracial state τ , let $\mathcal{B} \subset \mathcal{N} \subset \mathcal{M}$ be unital von Neumann subalgebras, and denote by $E_{\mathcal{N}} : \mathcal{M} \rightarrow \mathcal{N}$ the unique trace-preserving conditional expectation of \mathcal{M} onto \mathcal{N} (see [46, Proposition V.2.36]). Denote by $\mathbb{H}^+(\mathcal{M})$ the operator upper-half plane of \mathcal{M} : $\mathbb{H}^+(\mathcal{M}) = \{x \in \mathcal{M} : \Im x := (x - x^*)/2i > 0\}$. Given two arbitrary selfadjoint elements $c, d \in \mathcal{M}$, we define the open set $\mathcal{G}_{c,d,\mathcal{B},\mathcal{N}}$ to consist of those elements $\beta \in \mathcal{B}$ such that $\beta - (c + d)$ is invertible and $E_{\mathcal{N}}((\beta - (c + d))^{-1})$ is invertible as well. Then the function

$$\omega_{c,d,\mathcal{B},\mathcal{N}} : \mathcal{G}_{c,d,\mathcal{B},\mathcal{N}} \rightarrow \mathcal{M}$$

defined by

$$(5.1) \quad \omega_{c,d,\mathcal{B},\mathcal{N}}(\beta) = c + [E_{\mathcal{N}}((\beta - (c + d))^{-1})]^{-1}, \quad \beta \in \mathcal{G}_{c,d,\mathcal{B},\mathcal{N}},$$

is analytic. This equation can also be written as

$$(5.2) \quad E_{\mathcal{N}}((\beta - (c + d))^{-1}) = (\omega_{c,d,\mathcal{B},\mathcal{N}}(\beta) - c)^{-1} \quad \beta \in \mathcal{G}_{c,d,\mathcal{B},\mathcal{N}}.$$

Properties (1), (2), and (3) in the following lemma are easy observations, while (4) follows as in [10, Remark 2.5].

Lemma 5.1. *Fix $\mathcal{B} \subset \mathcal{N} \subset \mathcal{M}$ and $c, d \in \mathcal{M}$ as above. Then:*

- (1) *The set $\mathcal{G}_{c,d,\mathcal{B},\mathcal{N}}$ is selfadjoint.*
- (2) *$\omega_{c,d,\mathcal{B},\mathcal{N}}(\beta^*) = \omega_{c,d,\mathcal{B},\mathcal{N}}(\beta)^*$, $\beta \in \mathcal{G}_{c,d,\mathcal{B},\mathcal{N}}$.*
- (3) *$\mathbb{H}^+(\mathcal{B}) \subset \mathcal{G}_{c,d,\mathcal{B},\mathcal{N}}$ and $\omega_{c,d,\mathcal{B},\mathcal{N}}(\mathbb{H}^+(\mathcal{B})) \subset \mathbb{H}^+(\mathcal{N})$.*
- (4) *$\Im(\omega_{c,d,\mathcal{B},\mathcal{N}}(\beta)) \geq \Im(\beta)$, $\beta \in \mathbb{H}^+(\mathcal{B})$.*

There is one important case in which $\omega_{c,d,\mathcal{B},\mathcal{N}}$ takes values in \mathcal{B} , and thus (5.2) allows us to view $\omega_{c,d,\mathcal{B},\mathcal{N}}|_{\mathbb{H}^+(\mathcal{B})}$ as a subordination function in the sense of Littlewood. Denote by $\mathcal{G}_{c,d,\mathcal{B},\mathcal{N}}^0$ the connected component of $\mathcal{G}_{c,d,\mathcal{B},\mathcal{N}}$ that contains $\mathbb{H}^+(\mathcal{B})$. The following basic result is from [48].

Theorem 5.2. *With the above notation, suppose that c and d are free over \mathcal{B} and $\mathcal{N} = \mathcal{B}\langle c \rangle$ is the unital von Neumann generated by \mathcal{B} and c . Then*

$$\omega_{c,d,\mathcal{B},\mathcal{N}}(\mathcal{G}_{c,d,\mathcal{B},\mathcal{N}}^0) \subset \mathcal{B}.$$

In our applications, the algebra \mathcal{B} is (isomorphic to) $M_n(\mathbb{C})$ for some $n \in \mathbb{N}$. More precisely, let \mathcal{M} be a von Neumann algebra endowed with a normal faithful tracial state τ , and let $n \in \mathbb{N}$. Then $M_n(\mathbb{C})$ can be identified with the subalgebra $M_n(\mathbb{C}) \otimes 1$ of $M_n(\mathcal{M}) = M_n(\mathbb{C}) \otimes \mathcal{M}$. Moreover, $M_n(\mathcal{M})$ is endowed with the faithful normal tracial state $\text{tr}_n \otimes \tau = (1/n)\text{Tr}_n \otimes \tau$, and $\text{Id}_{M_n(\mathbb{C})} \otimes \tau$ is the trace-preserving conditional expectation from $M_n(\mathcal{M})$ to $M_n(\mathbb{C})$. The following result is from [37].

Proposition 5.3. *Let \mathcal{M} be a von Neumann algebra endowed with a normal faithful tracial state τ , let $c, d \in \mathcal{M}$ be freely independent, let n be a positive integer, and let $\gamma_1, \gamma_2 \in M_n(\mathbb{C})$. Then $\gamma_1 \otimes c$ and $\gamma_2 \otimes d$ are free over $M_n(\mathbb{C})$.*

We show next how the spectrum of $P(c, d)$ relates with the functions ω defined above. Thus, we fix $P = P^* \in \mathbb{C}\langle X_1, X_2 \rangle$ and a linearization $L = \gamma_0 \otimes 1 + \gamma_1 \otimes X_1 + \gamma_2 \otimes X_2$ of P as constructed in Section 4. Thus, $\gamma_0, \gamma_1, \gamma_2 \in M_n(\mathbb{C})$ are selfadjoint matrices for some $n \in \mathbb{N}$. (Clearly $\gamma_1 \neq 0$ unless $P \in \mathbb{C}\langle X_2 \rangle$.) Then we consider the random variables $\gamma_1 \otimes c$ and $\gamma_2 \otimes d$ in $M_n(\mathcal{M})$, the algebra $\mathcal{B} = M_n(\mathbb{C}) \subset M_n(\mathcal{M})$, and $\mathcal{N} = M_n(\mathbb{C}\langle c \rangle)$; clearly $\mathcal{B} \subset \mathcal{N} \subset M_n(\mathcal{M})$. We set

$$\mathcal{G} = \mathcal{G}_{\gamma_1 \otimes c, \gamma_2 \otimes d, \mathcal{B}, \mathcal{N}}^0$$

and

$$\omega = \omega_{\gamma_1 \otimes c, \gamma_2 \otimes d, \mathcal{B}, \mathcal{N}}: \mathcal{G} \rightarrow \mathcal{N}.$$

Thus,

$$E_{\mathcal{N}} [(\beta \otimes 1 - \gamma_1 \otimes c - \gamma_2 \otimes d)^{-1}] = (\omega(\beta) \otimes 1 - \gamma_1 \otimes c)^{-1}, \quad \beta \in \mathcal{G}.$$

The left hand side of this equation is defined if $\beta = ze_{11} - \gamma_0$ for some $z \in \mathbb{C} \setminus \sigma(P(c, d))$, so it would be desirable that $ze_{11} - \gamma_0 \in \mathcal{G}$ for such values of z . This is not true except for special cases. (One such case applies to $P = X_1 + X_2$ if d is a semicircular variable free from c [16, 9].) The following lemma offers a partial result.

Lemma 5.4. *With the notation above, there exists $k > 0$ depending only on L , $\|c\|$, and $\|d\|$ such that $ze_{11} - \gamma_0 \in \mathcal{G}$ if $|z| > k$. The analytic function $u(z) = \omega(ze_{11} - \gamma_0)$ satisfies the equation $u(\bar{z}) = u(z)^*$ for $|z| > k$.*

Proof. Define an analytic function $F: \mathbb{C} \setminus \sigma(P(c, d)) \rightarrow \mathcal{N}$ by

$$(5.3) \quad F(z) = E_{\mathcal{N}} [((ze_{11} - \gamma_0) \otimes 1 - \gamma_1 \otimes c - \gamma_2 \otimes d)^{-1}], \quad z \in \mathbb{C} \setminus \sigma(P(c, d)).$$

We show that $F(z)$ is invertible if $|z|$ is sufficiently large. Suppressing the variables c and d from the notation, it follows from the factorization used in the proof of Lemma 4.2 that

$$F(z) = E_{\mathcal{N}} \begin{bmatrix} (z - P)^{-1} & -(z - P)^{-1}u^*Q^{-1} \\ -Q^{-1}u(z - P)^{-1} & Q^{-1}u(z - P)^{-1}u^*Q^{-1} - Q^{-1} \end{bmatrix}.$$

Moreover, because of the matrix structure of \mathcal{N} , this matrix can be obtained by applying $E_{\mathbb{C}\langle c \rangle}$ entrywise. According to the Schur complement formula, a matrix $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$ is invertible if both A and $D - BA^{-1}C$ are invertible. For our matrix, we have $A = A(z) = E_{\mathbb{C}\langle c \rangle}((z - P)^{-1})$. The fact that $\|zA(z) - 1\| < 1$ for $|z| > 2\|P\|$ implies that $A(z)$ is invertible. Next, we see that $\|(z - P)^{-1}\|$ and $\|A(z)^{-1}\|$ are comparable to $1/|z|$ and $|z|$, respectively. Using these estimates, one sees also that

$\lim_{|z| \rightarrow \infty} \|B(z)A(z)^{-1}C(z)\| = 0$, so the invertibility of $F(z)$ would follow from the invertibility of $D(z)$ for large $|z|$. Since

$$\lim_{|z| \rightarrow \infty} \|D(z) + E_{M_{n-1}(\mathbb{C}\langle c \rangle)}(Q^{-1})\| = 0,$$

we only need to verify that $E_{M_{n-1}(\mathbb{C}\langle c \rangle)}(Q^{-1})$ is invertible. Write $Q^{-1} = T(1_{n-1} + N)$ as in Lemma 4.1. We have

$$E_{M_{n-1}(\mathbb{C}\langle c \rangle)}(Q^{-1}) = T(1_{n-1} + E_{M_{n-1}(\mathbb{C}\langle c \rangle)}(N)),$$

and $E_{M_{n-1}(\mathbb{C}\langle c \rangle)}(N)$ is strictly lower triangular. The invertibility of $E_{M_{n-1}(\mathbb{C}\langle c \rangle)}(Q^{-1})$ follows. The quantities $\|D(z) + E_{M_{n-1}(\mathbb{C}\langle c \rangle)}(Q^{-1})\|$ and $\|Q^{-1}\|$ can be estimated using only L , $\|c\|$, and $\|d\|$, and this shows that k can be chosen as a function of these objects. The last assertion of the lemma is immediate. \square

The estimates in the preceding proof apply, by virtue of continuity, to nearby points in $M_n(\mathbb{C})$. We record the result for later use.

Corollary 5.5. *Let c, d and k be as in Lemma 5.4 and let $z \in \mathbb{C} \setminus [-k, k]$. Then there exist a constant $k' > 0$ and a neighborhood W of $ze_{1,1} - \gamma_0$, depending only on $\|c\|$, $\|d\|$, and L , such that $V \subset \mathcal{G}$ and $\|\omega(\beta)\| \leq k'$ for $\beta \in W$.*

In some cases of interest, the analytic function u extends to the entire upper and lower half-planes. We recall that a function v defined in a domain $G \subset \mathbb{C}$ with values in a Banach space \mathcal{X} is said to be meromorphic if, for every $z_0 \in G$, the function $(z - z_0)^n v(z)$ is analytic in a neighborhood of z_0 for sufficiently large n . For instance, if \mathcal{X} is a finite dimensional Banach algebra and $h : G \rightarrow \mathcal{X}$ is an analytic function such that $h(z)$ is invertible for some $z \in G$, then the function $v(z) = h(z)^{-1}$ is meromorphic in G . This fact follows easily once we identify \mathcal{X} with an algebra of matrices, so the inverse can be calculated using determinants.

Lemma 5.6. *The function u defined in Lemma 5.4 is meromorphic in $\mathbb{C} \setminus \sigma(P(c, d))$.*

Proof. The lemma follows immediately from the observation preceding the statement applied to the function F defined in (5.3) which is analytic in $\mathbb{C} \setminus \sigma(P(c, d))$ since, by hypothesis and by Theorem 5.2, u takes values in a finite dimensional algebra. \square

The conclusion of the preceding lemma applies, for instance, when $\mathcal{M} = M_N \otimes L^\infty(\Omega)$ with the usual trace $\text{tr}_N \otimes \mathbb{E}$. This situation arises in the study of random matrices. The function u is also meromorphic provided that c and d are free random variables and $\mathcal{N} = M_n(\mathbb{C})\langle \gamma_1 \otimes c \rangle$. Since $\gamma_1 \neq 0$, we have $\mathcal{N} = M_n(\mathbb{C}\langle c \rangle)$.

Lemma 5.7. *If c and d are free and $\mathcal{N} = M_n(\mathbb{C})\langle \gamma_1 \otimes c \rangle$, then the function u defined in Lemma 5.4 is meromorphic in $\mathbb{C} \setminus \sigma(P(c, d))$ with values in $M_n(\mathbb{C})$. Moreover, given an arbitrary $\lambda \in \sigma(c)$, the function $(u - \lambda\gamma_1)^{-1}$ extends analytically to $\mathbb{C} \setminus \sigma(P(c, d))$.*

Proof. Theorem 5.2 shows that ω takes values in $M_n(\mathbb{C})$. We have established that the domain \mathcal{G} of ω contains $z \otimes e_{11} - \gamma_0$ for sufficiently large $|z|$. Fix a character χ of the commutative C^* -algebra $\mathbb{C}\langle c \rangle$ and denote by $\chi_n : M_n(\mathbb{C})\langle \gamma_1 \otimes c \rangle \rightarrow M_n(\mathbb{C})$ the algebra homomorphism obtained by applying χ to each entry. Using the notation (5.3), we have

$$(\omega(z \otimes e_{11} - \gamma_0) - \gamma_1 \otimes \chi(c))\chi_n(F(z)) = I_n$$

for sufficiently large $|z|$. It follows immediately that the function

$$u_1(z) = \gamma_1 \otimes \chi(c) + [\chi_n(F(z))]^{-1}$$

is a meromorphic continuation of u to $\mathbb{C} \setminus \sigma(P(c, d))$. Moreover, the equation

$$(5.4) \quad (u_1(z) - \gamma_1 \otimes c)F(z) = I_n$$

holds for large values $|z|$ and hence it holds on the entire domain of analyticity of u_1 . It follows that \mathcal{G} contains $z \otimes e_{11} - \gamma_0$ whenever $z \in \mathbb{C} \setminus \sigma(P(c, d))$ is not a pole of u_1 , and $\omega(z \otimes e_{11} - \gamma_0) = u_1(z)$ for such values of z . To verify the last assertion, choose χ such that $\chi(c) = \lambda$ and apply χ_n to (5.4) to obtain

$$(u_1(z) - \lambda\gamma_1)\chi_n(F(z)) = I_n.$$

Thus $\chi_n \circ F$ is an analytic extension of $(u - \lambda\gamma_1)^{-1}$ to $\mathbb{C} \setminus \sigma(P(c, d))$. \square

6. MAIN RESULTS AND EXAMPLE

Fix a polynomial $P = P^* \in \mathbb{C}\langle X_1, X_2 \rangle$ and choose, as in Section 4, a linearization of P of the form $ze_{1,1} - L$, where $L = \gamma_0 \otimes 1 + \gamma_1 \otimes X_1 + \gamma_2 \otimes X_2 \in M_n(\mathbb{C}\langle X_1, X_2 \rangle)$. In particular, $\gamma_0, \gamma_1, \gamma_2 \in M_n(\mathbb{C})$ are selfadjoint matrices.

Suppose that $\{A_N\}_{N \in \mathbb{N}}$ and $\{B_N\}_{N \in \mathbb{N}}$ are two sequences of selfadjoint random matrices satisfying the hypotheses (A1)–(A3) and (B0)–(B2) of Section 3. As noted earlier, the pairs (A_N, B_N) in $M_N(\mathbb{C})$ converge almost surely in distribution to a pair (a, b) of freely independent selfadjoint random variables in a C^* -probability space (\mathcal{A}, τ) such that $\mu_a = \mu$ and $\mu_b = \nu$. By Theorem 5.2, there exists a selfadjoint open set $\mathcal{G} \subset M_n(\mathbb{C})$, and an analytic function $\omega: \mathcal{G} \rightarrow M_n(\mathbb{C})$ such that

$$(\omega(\beta) \otimes 1 - \gamma_1 \otimes a)^{-1} = E_{M_n(\mathbb{C}(a))} [(\beta \otimes 1 - (\gamma_1 \otimes a + \gamma_2 \otimes b))^{-1}], \quad \beta \in \mathcal{G}.$$

As shown in Lemma 5.7, the map

$$u(z) = \omega(z \otimes e_{1,1} - \gamma_0)$$

is meromorphic on $\mathbb{C} \setminus \sigma(P(a, b))$. Define a new function

$$u_0(z) = (u(ze_{1,1} - \gamma_0) + iI_n)^{-1}.$$

It follows from Lemma 5.1 that u_0 continues analytically to a neighbourhood of $\mathbb{R} \setminus \sigma(P(a, b))$. (Indeed, u_0 is bounded near every pole of u .) Define

$$H_j(z) = \det[(\theta_j \gamma_1 + i)u_0(z) - I_n], \quad j = 1, \dots, p$$

and denote by $m_j(t)$ the order of t as a zero of $H_j(z)$ at $z = t$. Also set $m(t) = m_1(t) + \dots + m_p(t)$ for $t \in \mathbb{R} \setminus \sigma(P(a, b))$, and note that $\{t : m(t) \neq 0\}$ is an isolated set in $\mathbb{R} \setminus \sigma(P(a, b))$. With this notation, we are ready to state our first main result. The notation E_{A_N} indicates the spectral measure of the matrix A_N , that is, $E_{A_N}(S)$ is the orthogonal projection onto the linear span of the eigenvectors of A corresponding to eigenvalues in the Borel set S .

Theorem 6.1. (1) *Suppose that $t \in \mathbb{R} \setminus \sigma(P(a, b))$. Then there exists $\delta_0 > 0$ such that for every $\delta \in (0, \delta_0)$, almost surely for large N , the random matrix $P(A_N, B_N)$ has exactly $m(t)$ eigenvalues in the interval $(t - \delta, t + \delta)$, counting multiplicity.*

(2) *Suppose in addition that the spikes of A_N are distinct and $\det H_{i_0}(t) = 0$. Then, for ε small enough, almost surely*

$$(6.1) \quad \lim_{N \rightarrow \infty} \|E_{A_N}(\{\theta_i\}) [E_{P(A_N, B_N)}((t - \varepsilon, t + \varepsilon)) - \delta_{i, i_0} \mathcal{C}_i(t) I_N] E_{A_N}(\{\theta_i\})\| = 0,$$

where $\mathcal{C}_i(t) = \lim_{z \rightarrow t} (z - t) [(u(z) - \theta_i \gamma_1)^{-1}]_{1,1}$ is the residue of the meromorphic function $[(u(z) - \theta_i \gamma_1)^{-1}]_{1,1}$ at $z = t$.

Remark 6.2. If we know in addition that ω is analytic at the point $\beta = te_{1,1} - \gamma_0$, then the function $H_j(z)$ can be replaced by $z \mapsto \det[\theta_j \gamma_1 - u(z)]$. In that case, $m(t)$ is equal to the multiplicity of t as a zero of

$$z \mapsto \prod_{j=1}^p \det[\theta_j \gamma_1 - u(z)].$$

This situation arises, for instance, when b is a semicircular variable and it is relevant when B_N is replaced by a Wigner matrix X_N/\sqrt{N} . Under the hypotheses (X0) – (X3) of Section 3, we obtain the following result. Note that the subordination function ω has the more explicit form

$$\omega(\beta) = \beta - \gamma_2 (\text{Id}_{M_n(\mathbb{C})} \otimes \tau) [(\beta \otimes 1_{\mathcal{A}} - \gamma_1 \otimes a - \gamma_2 \otimes b)^{-1}] \gamma_2, \quad \beta \in \mathcal{G}.$$

Theorem 6.3. *Let a and b be free selfadjoint elements in a C^* -probability space (\mathcal{A}, τ) with distribution μ and $\nu_{0,1}$ respectively (see (2.3)), $t \in \mathbb{R} \setminus \sigma(P(a, b))$, and let $m(t)$ be defined as in Remark 6.2. Then, for sufficiently small ε , almost surely for large N , there are exactly $m(t)$ eigenvalues of $P(A_N, X_N/\sqrt{N})$ in an ε -neighborhood of t .*

Remark 6.4. The subordination function can be calculated more explicitly if $\mu = \delta_0$ (and hence $a = 0$). In this case,

$$\omega(\beta) = \{(\text{Id}_{M_n(\mathbb{C})} \otimes \tau) [(w \otimes 1 - \gamma_2 \otimes b)^{-1}]\}^{-1}.$$

As an illustration, consider the random matrix

$$M = A_N \frac{X_N}{\sqrt{N}} + \frac{X_N}{\sqrt{N}} A_N + \frac{X_N^2}{N},$$

where X_N is a standard G.U.E. matrix of size N (thus, each entry of X_N has unit norm in $L^2(\Omega)$) and

$$A_N = \text{Diag}(\theta, 0, \dots, 0), \quad \theta \in \mathbb{R} \setminus \{0\}.$$

In this case, A_N has rank one, and thus $\mu = \delta_0$. It follows that the limit spectral measure ρ of M is the same as the limit spectral measure of X_N^2/N . Thus, η is the Marchenko-Pastur distribution ρ with parameter 1:

$$d\eta(x) = \frac{\sqrt{(4-x)x}}{2\pi x} 1_{(0,4)}(x) dx.$$

The polynomial P is $P(X_1, X_2) = X_1 X_2 + X_2 X_1 + X_2^2$, $\mu = \delta_0$, and ν is the standard semi-circular distribution. An economical linearization of P is provided by $L = \gamma_0 \otimes 1 + \gamma_1 \otimes X_1 + \gamma_2 \otimes X_2$, where

$$\gamma_0 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix}, \quad \gamma_1 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad \gamma_2 = \begin{bmatrix} 0 & 1 & \frac{1}{2} \\ 1 & 0 & 0 \\ \frac{1}{2} & 0 & 0 \end{bmatrix}.$$

Denote by

$$G_\eta(z) = \int_0^4 \frac{1}{z-t} d\eta(t) = \frac{z - \sqrt{z^2 - 4z}}{2z}, \quad z \in \mathbb{C} \setminus [0, 4]$$

the Cauchy transform of the measure η . (The branch of the square root is chosen so $\sqrt{z^2 - 4z} > 0$ for $z > 4$.) This function satisfies the quadratic equation $zG_\eta(z)^2 - zG_\eta(z) + 1 = 0$. Suppose now that $x \notin [0, 4]$. Denoting by $E = \text{Id}_{M_3(\mathbb{C})} \otimes \tau: M_3(\mathcal{A}) \rightarrow M_3(\mathbb{C})$ the usual expectation and using Remark 6.4, we have

$$\omega(xe_{1,1} - \gamma_0) = E [(xe_{1,1} - \gamma_0 - \gamma_2 \otimes b)^{-1}]^{-1}, \quad x \in \mathbb{R} \setminus [0, 4].$$

The inverse of $(xe_{1,1} - \gamma_0) \otimes 1 - \gamma_2 \otimes b$ is then calculated explicitly and application of the expected value to its entries yields eventually

$$\omega(xe_{11} - \gamma_0) = \begin{bmatrix} \frac{1}{G_\eta(x)} & 0 & 0 \\ 0 & \frac{1}{xG_\eta(x)} - 1 & \frac{1}{2xG_\eta(x)} + \frac{1}{2} \\ 0 & \frac{1}{2xG_\eta(x)} + \frac{1}{2} & \frac{1}{4xG_\eta(x)} - \frac{1}{4} \end{bmatrix}.$$

After calculation, the equation $\det[\gamma_1\theta - \omega(xe_{11} - \gamma_0)] = 0$ reduces to

$$(6.2) \quad \theta^2 G_\eta(x)^2 - (1 - G_\eta(x)) = 0.$$

This equation has two solutions, namely

$$\frac{2\theta^4}{-(3\theta^2 + 1) \pm \sqrt{4\theta^2 + 1}(\theta^2 + 1)},$$

one of which is negative. The positive solution belongs to $[4, +\infty)$ precisely when $|\theta| > \sqrt{2}$. Thus, the matrix M_N exhibits one (negative) outlier when $0 < |\theta| \leq \sqrt{2}$ and two outliers (one negative and one > 4) when $|\theta| > \sqrt{2}$. The second situation is illustrated by the simulation presented in Figure 1.

7. OUTLINE OF THE PROOFS

We consider first the matricial model (3.1), that is, $Z_N = P(A_N, B_N)$, where A_N and B_N are independent and the distribution of B_N is invariant under unitary conjugation. As seen in [23, Proposition 6.1], B_N can be written as $B_N = U_N D_N U_N^*$ almost surely, where U_N is distributed according to the Haar measure on the unitary group $U(N)$, D_N is a diagonal random matrix, and U_N is independent from D_N . As pointed out in [36, Section 3, Assertion 2], it suffices to prove Theorem 6.1 under the assumption that A_N and D_N are constant selfadjoint matrices that can be taken to be diagonal in the standard basis. Thus, we work with

$$A_N = \text{Diag}(\lambda_1(A_N), \dots, \lambda_N(A_N))$$

and

$$B_N = U_N D_N U_N^*, \quad D_N = \text{Diag}(\lambda_1(D_N), \dots, \lambda_N(D_N))$$

where $\lambda_j(A_N) = \theta_j$, $1 \leq j \leq p$, and U_N is uniformly distributed in $U(N)$.

Similarly, the proof for the second model $Z_N = P(A_N, X_N/\sqrt{N})$ reduces to the special case in which A_N is a constant matrix.

Choose a linearization L of P as in Section 4. In the spirit of [14], the first step in the proofs of Theorems 6.1 and 6.3 consists of reducing the problem to the convergence of random matrix function F_N of fixed size np , involving the generalized resolvent of the linearization applied to Z_N . For the first model, this convergence is established in Section 8 by extending the arguments of [11] and making use of the properties of the operator-valued subordination function described in Section 5. For the second model, the convergence of F_N is obtained in Section 10 via a

Eigenvalue Distribution, Theta = 10, Dim = 1000

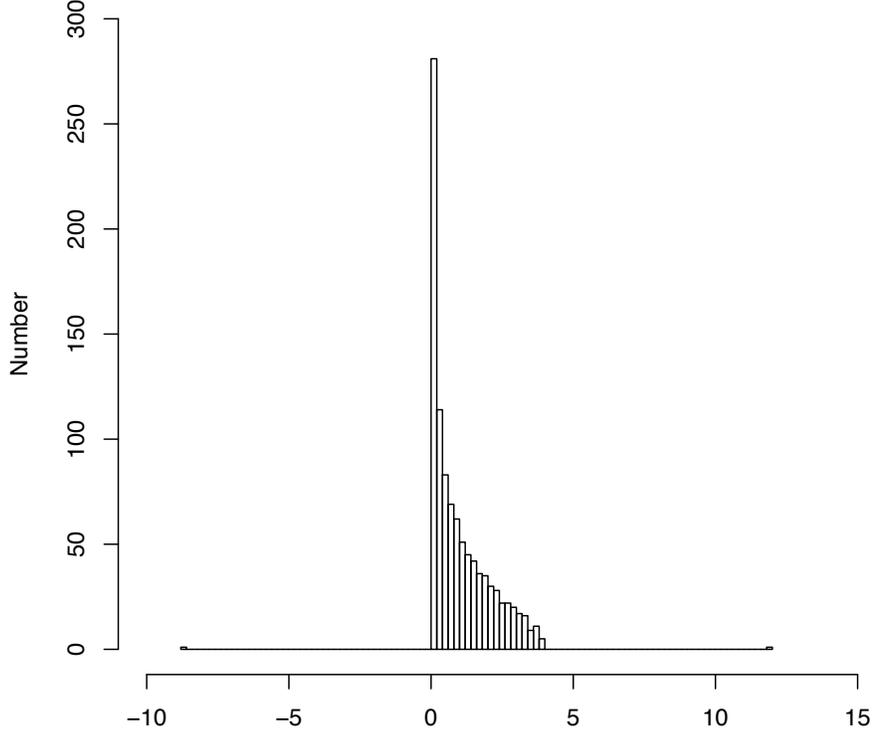


FIGURE 1. One sample from the model described in remark 6.4 corresponding to $\theta = 10$, with matrix size $N = 1000$.

comparison with the G.U.E. case. The case in which X_N is a G.U.E. is, of course, a particular case of the unitarily invariant model.

8. EXPECTATIONS OF MATRIX-VALUED RANDOM ANALYTIC MAPS

As seen earlier in this paper, it suffices to prove Theorem 6.1 in the special case in which the matrix A_N is constant and B_N is a random unitary conjugate of another constant matrix. In this section, we establish some useful ingredients specific to this situation. We fix sequences $\{C_N\}_{N \in \mathbb{N}}$ and $\{D_N\}_{N \in \mathbb{N}}$, where $C_N, D_N \in M_N(\mathbb{C})$, and a sequence $\{U_N\}_{N \in \mathbb{N}}$ of random matrices such that U_N is uniformly distributed in the unitary group $U(N)$. We also fix a selfadjoint polynomial $P \in \mathbb{C}\langle X_1, X_2 \rangle$ and a selfadjoint linearization

$$L = \gamma_0 \otimes 1 + \gamma_1 \otimes X_1 + \gamma_2 \otimes X_2 \in M_n(\mathbb{C}\langle X_1, X_2 \rangle)$$

of P as in Section 4. The random variables $c_N = C_N \otimes 1_\Omega$ and $d_N = U_N D_N U_N^*$ are viewed as elements of the noncommutative probability space (\mathcal{M}_N, τ_N) , where

$\mathcal{M}_N = M_N(\mathbb{C}) \otimes L^\infty(\Omega)$ and $\tau_N = \text{tr}_N \otimes \mathbb{E}$. For every $N \in \mathbb{N}$ we consider the elements $\gamma_1 \otimes c_N, \gamma_2 \otimes d_N \in M_n(\mathbb{C}) \otimes \mathcal{M}_N$, and the algebras $\mathcal{B}_N = M_n(\mathbb{C}) \otimes I_N$ and $\mathcal{N}_N = M_n(\mathbb{C}) \otimes M_N(\mathbb{C})$, both identified with subalgebras of $M_n(\mathbb{C}) \otimes \mathcal{M}_N$. The conditional expectation $E_{\mathcal{N}_N} : M_n(\mathbb{C}) \otimes \mathcal{M}_N \rightarrow \mathcal{N}_N$ is simply the expected value and it is accordingly denoted \mathbb{E} . We use the notation

$$\mathcal{G}_N = \mathcal{G}_{\gamma_1 \otimes c_N, \gamma_2 \otimes d_N, \mathcal{B}_N, \mathcal{N}_N}, \quad \omega_N = \omega_{\gamma_1 \otimes c_N, \gamma_2 \otimes d_N, \mathcal{B}_N, \mathcal{N}_N}.$$

Recall that \mathcal{G}_N consists of those matrices $\beta \in M_n(\mathbb{C})$ with the property that $\beta - \gamma_1 \otimes c_N - \gamma_2 \otimes d_N$ is invertible in $M_n(\mathbb{C}) \otimes \mathcal{M}_N$ and $\mathbb{E}((\beta - \gamma_1 \otimes c_N - \gamma_2 \otimes d_N)^{-1})$ is invertible in \mathcal{N}_N . In particular, if $\beta \in \mathcal{G}_N$ then the matrix $\beta - \gamma_1 \otimes C_N - \gamma_2 \otimes VD_NV^*$ is invertible for every $V \in U(N)$. The set \mathcal{G}_N is open and it contains $\mathbb{H}^+(M_n(\mathbb{C}))$. According to Lemma 5.6 and the remarks following it, the function $u_N(z) = \omega_N(ze_{1,1} - \gamma_0)$ is meromorphic in $\mathbb{C} \setminus \sigma(P(c_N, d_N))$.

For simplicity of notation, we write

$$(8.1) \quad R_N(\beta) = (\beta - \gamma_1 \otimes c_N - \gamma_2 \otimes d_N)^{-1}, \quad \beta \in \mathcal{G}_N,$$

and observe that $R_N(\beta)$ is an element of $M_n(\mathbb{C}) \otimes M_N(\mathbb{C}) \otimes L^\infty(\Omega)$, that is, a random matrix of size nN . We also write

$$(8.2) \quad R_N(V, \beta) = (\beta - \gamma_1 \otimes C_N - \gamma_2 \otimes VD_NV^*)^{-1}, \quad \beta \in \mathcal{G}_N, V \in U(N),$$

for sample values of this random variable. The function ω_N is given by

$$(8.3) \quad \omega_N(\beta) = \gamma_1 \otimes C_N + (\mathbb{E}(R_N(\beta)))^{-1}, \quad \beta \in \mathcal{G}_N.$$

We start by showing that the matrix $\omega_N(\beta)$ has a block diagonal form, thus extending [11, Lemma 4.7]. We recall that the commutant and double commutant of a set $S \subset M_m(\mathbb{C})$ are denoted by S' and S'' , respectively. We use the fact that $M_n(\mathbb{C}) \otimes S'' = (I_n \otimes S')'$. If $S = \{C_N\}$ then $\{C_N\}''$ is the linear span of the matrices $\{I_N, C_N, \dots, C_N^{N-1}\}$. In particular, every eigenvector of C_N is a common eigenvector for the elements of $\{C_N\}''$.

For each $N \in \mathbb{N}$, we select an eigenbasis $\{f_N^{(1)}, \dots, f_N^{(N)}\}$ for the operator C_N and denote by $\lambda_N^{(j)}$ the corresponding eigenvalues, that is, $C_N f_N^{(j)} = \lambda_N^{(j)} f_N^{(j)}$. We write $P_N^{(j)} = f_N^{(j)} \otimes f_N^{(j)*} \in M_N(\mathbb{C})$ for the orthogonal projection onto the space generated by $f_N^{(j)}$, $j = 1, \dots, N$. Thus, the double commutant $\{C_N\}''$ is contained in the linear span of $\{P_N^{(j)} : j = 1, \dots, n\}$.

We write $[x, y] = xy - yx$ for the commutator of two elements x, y in an algebra.

Lemma 8.1. *For every $\beta \in \mathcal{G}_N$ we have:*

- (1) $\mathbb{E}(R_N(\beta)) \in M_n(\mathbb{C}) \otimes \{C_N\}''$. In particular, there exist analytic functions $\omega_N^{(j)} : \mathcal{G}_N \rightarrow M_n(\mathbb{C})$, $j = 1, \dots, N$, such that

$$\omega_N(\beta) = \sum_{j=1}^N \omega_N^{(j)}(\beta) \otimes P_N^{(j)}, \quad \beta \in \mathcal{G}_N.$$

- (2) For every $Z \in M_N(\mathbb{C})$,

$$[\mathbb{E}(R_N(\beta)), I_n \otimes Z] = \mathbb{E}(R_N(\beta)[\gamma_1 \otimes C_N, I_n \otimes Z]R_N(\beta)).$$

Proof. The first assertion in (1) follows from an application of (2) to an arbitrary matrix $Z \in \{C_N\}'$ and from the fact that $M_n(\mathbb{C}) \otimes \{C_N\}'' = (I_n \otimes \{C_N\}')'$. The

second assertion follows because we also have $\gamma_1 \otimes C_N \in M_n(\mathbb{C}) \otimes \{C_N\}''$. To prove (2), observe that the analytic map

$$H(W) = \mathbb{E} \left((\beta \otimes I_N - (\gamma_1 \otimes c_N + \gamma_2 \otimes e^{iW} d_N e^{-iW}))^{-1} \right)$$

is defined for W in an open neighbourhood of the set of selfadjoint matrices in $M_N(\mathbb{C})$. The unitary invariance of d_N implies that $H(W) = \mathbb{E}(R(\beta))$ if W is selfadjoint. Since the selfadjoint matrices form a uniqueness set for analytic functions, we conclude that H is constant on an open subset of $M_N(\mathbb{C})$ containing the selfadjoint matrices. Given an arbitrary $Z \in M_N(\mathbb{C})$, we conclude that the function $\varepsilon \mapsto H(\varepsilon Z)$ is defined and constant for $\varepsilon \in \mathbb{C}$ with $|\varepsilon|$ sufficiently small. Differentiate with respect to ε and set $\varepsilon = 0$, we obtain

$$\mathbb{E}(R_N(\beta)[I_n \otimes Z, (\gamma_2 \otimes d_N)]R_N(\beta)) = 0.$$

The equality

$$R_N(\beta)(\gamma_2 \otimes d_N) = R_N(\beta)(\beta \otimes I_N - \gamma_1 \otimes C_N) - I_n \otimes I_N,$$

applied in the relation above, yields (2). \square

The following result is simply a reformulation of Lemma 8.1 that emphasizes the fact that the functions $\beta \mapsto (\omega_N^{(j)}(\beta) - \lambda_N^{(j)} \gamma_1)^{-1}$ extend holomorphically to the open set $\{\beta \in \mathbb{M}_N(\mathbb{C}) : \beta - \gamma_1 \otimes c_N - \gamma_2 \otimes d_N \text{ is invertible}\}$.

Corollary 8.2. *We have*

$$(I_n \otimes P_N^{(j)}) \mathbb{E}(R_N(\beta))(I_n \otimes P_N^{(j)}) = (\omega_N^{(j)}(\beta) - \lambda_N^{(j)} \gamma_1)^{-1} \otimes P_N^{(j)}, \quad j = 1, \dots, N,$$

for every $\beta \in \mathcal{G}_N$ such that $\omega_N^{(j)}(\beta) - \lambda_N^{(j)} \gamma_1$ is invertible.

It is useful to rewrite assertion (2) of Lemma 8.1 as follows:

$$(8.4) \quad \begin{aligned} [\omega_N(\beta), I_n \otimes Z] &= (\mathbb{E}(R_N(\beta)))^{-1} \mathbb{E} \left((R_N(\beta) - \mathbb{E}(R_N(\beta))) \right. \\ &\quad \left. \times (\gamma_1 \otimes [Z, C_N])(R_N(\beta) - \mathbb{E}(R_N(\beta))) \right) (\mathbb{E}(R_N(\beta)))^{-1}. \end{aligned}$$

This is analogous to [11, (4.10)] and the derivation is practically identical. Relation (8.4) allows us to estimate the differences between the matrices $\omega_N^{(j)}(\beta)$ once we control the differences $R_N(\beta) - \mathbb{E}(R_N(\beta))$. For this purpose, we use the concentration of measure result in [3, Corollary 4.4.28]. This requires estimating the Lipschitz constant of the map $V \mapsto R_N(V, \beta)$ (see (8.2)) in the Hilbert-Schmidt norm. We use the notation $\|T\|_2 = \text{Tr}_m(T^*T)$ for the Hilbert-Schmidt norm of an arbitrary matrix $T \in M_m(\mathbb{C})$.

Lemma 8.3. *Suppose that $N \in \mathbb{N}$ and $\beta \in \mathcal{G}_N$. Then*

$$\|R_N(V, \beta) - R_N(W, \beta)\|_2 \leq 2r^2 \|\gamma_2\|_2 \|D_N\| \|V - W\|_2, \quad V, W \in \text{U}(N),$$

where $r = \|R_N(\beta)\|_{M_n(\mathbb{C}) \otimes \mathcal{M}_N}$.

Proof. A simple calculation shows that

$$R_N(V, \beta) - R_N(W, \beta) = R_N(V, \beta)[\gamma_2 \otimes (W D_N W^* - V D_N V^*)]R_N(W, \beta).$$

Next we see that

$$W D_N W^* - V D_N V^* = (W - V)D_N W^* + V D_N (W^* - V^*),$$

and thus

$$\|WD_NW^* - VD_NV^*\|_2 \leq 2\|D_N\|\|W - V\|_2.$$

Use now the equality $\|T \otimes S\|_2 = \|T\|_2\|S\|_2$ to deduce that

$$\begin{aligned} & \|R_N(V, \beta) - R_N(W, \beta)\|_2 \\ & \leq \|R_N(V, \beta)\|\|\gamma_2\|_2\|WD_NW^* - VD_NV^*\|_2\|R_N(W, \beta)\| \\ & \leq 2\|\gamma_2\|_2\|D_N\|\|R_N(V, \beta)\|\|R_N(W, \beta)\|\|V - W\|_2. \end{aligned}$$

The lemma follows from this estimate. \square

Proposition 8.4. *Suppose that $\sup\{\|D_N\| : N \in \mathbb{N}\} < +\infty$, $\beta \in \bigcap_{N \in \mathbb{N}} \mathcal{G}_N$, and moreover,*

$$r = \sup\{\|R_N(\beta)\|_{M_n(\mathbb{C}) \otimes \mathcal{M}_N} : N \in \mathbb{N}\} < +\infty.$$

Let $X_N, Y_N \in M_n \otimes M_N$ be matrices of norm one and rank uniformly bounded by $m \in \mathbb{N}$. Then:

(1) *Almost surely,*

$$(8.5) \quad \lim_{N \rightarrow \infty} \|X_N(R_N(\beta) - \mathbb{E}(R_N(\beta)))Y_N\| = 0.$$

(2) *There exists $k > 0$ such that*

$$(8.6) \quad \mathbb{E}(\|X_N(R_N(\beta) - \mathbb{E}(R_N(\beta)))Y_N\|^2) \leq \frac{k}{N}r^4, \quad N \in \mathbb{N}.$$

In particular, there exists a dense countable subset $\Lambda \subset \bigcap_{N \in \mathbb{N}} \mathcal{G}_N$ such that almost surely, (8.5) holds for any $\beta \in \Lambda$.

Proof. An arbitrary operator of rank m can be written as a sum of m operators of rank one (with the same or smaller norm). Thus we may, and do, restrict ourselves to the case in which the operators X_N and Y_N are projections of rank 1. In this case, $X_N R_N(V, \beta) Y_N$ is a scalar multiple of a fixed operator of rank one and Lemma 8.3 shows that this function satisfies a Lipschitz estimate. This estimate, combined with [3, Corollary 4.4.28], shows that

$$\mathbb{P}(\|X_N(R_N(\beta) - \mathbb{E}(R_N(\beta)))Y_N\| > \varepsilon) \leq 2 \exp\left(\frac{-N\varepsilon^2}{8r^4\|\gamma_2\|_2^2\|D_N\|_2^2}\right)$$

for every $\varepsilon > 0$ and every $\alpha \in (0, 1/2)$. The hypothesis implies that the last denominator has a bound independent of N . Part (1) of the lemma follows from this inequality, while (2) follows from the formula $\mathbb{E}(|Z|) = \int_0^{+\infty} \mathbb{P}(|Z| > t) dt$, valid for arbitrary random variables Z . \square

Remark 8.5. While Proposition 8.4 was formulated for $\beta \in \bigcap_{N \in \mathbb{N}} \mathcal{G}_N$, the hypothesis $r < +\infty$ (and therefore the conclusion of the proposition) is also satisfied in the following cases:

- (1) $\Im\beta > 0$ with $r \leq \|(\Im\beta)^{-1}\|$;
- (2) $\beta = ze_{1,1} - \gamma_0$ with $z \in \mathbb{C}^+ \cup \mathbb{C}^-$, by an estimate provided by Lemma 4.3;
- (3) $\beta = xe_{1,1} - \gamma_0$ with $x \in \mathbb{R}$, by the same estimate provided that

$$|x| > \sup\{\|P(c_N, d_N)\|_{M_n(\mathbb{C}) \otimes \mathcal{M}_N} : N \in \mathbb{N}\}.$$

Corollary 8.6. *Under the assumptions of Proposition 8.4 suppose that we also have $\sup\{\|C_N\| : N \in \mathbb{N}\} < +\infty$, and set*

$$t = \sup\{\|P(c_N, d_N)\|_{M_n(\mathbb{C}) \otimes \mathcal{M}_N} : N \in \mathbb{N}\}.$$

Let $K \subset \mathbb{C} \setminus [-t, t]$ be a compact set. Then, almost surely, the functions

$$z \mapsto \|X_N(R_N(ze_{1,1} - \gamma_0) - \mathbb{E}(R_N(ze_{1,1} - \gamma_0)))Y_N\|$$

converge to zero uniformly on K as $N \rightarrow +\infty$.

Proof. According to Proposition 8.4 and Remark 8.5(2) and (3), almost surely, for every $z \in \mathbb{C} \setminus [-t, t]$ such that $\Im z \in \mathbb{Q}$ and $\Re z \in \mathbb{Q}$, we have pointwise convergence to zero. A second application of Remark 8.5 yields uniform bounds on K for all of these functions, and this implies uniform convergence because the resolvents involved are analytic in z . \square

We apply the concentration results just proved to operators X_N and Y_N , of the form $I_n \otimes P_N^{(j)}$, where $\{P_N^{(j)} : j = 1, \dots, N\}$ are the projections used in Lemma 8.1. The rank of $I_n \otimes P_N^{(j)}$ is equal to n .

Proposition 8.7. *Suppose that $\sup\{\|C_N\| + \|D_N\| : N \in \mathbb{N}\} < +\infty$, and let $\beta \in \bigcap_{N \in \mathbb{N}} \mathcal{G}_N$ be such that*

$$r = \sup\{\|R_N(\beta)\|_{M_n(\mathbb{C}) \otimes \mathcal{M}_N} : N \in \mathbb{N}\} < +\infty$$

and

$$r' = \sup\{\|(\mathbb{E}(R_N(\beta)))^{-1}\| : N \in \mathbb{N}\} < +\infty.$$

Then $\sup_{N \in \mathbb{N}} N\|\omega_N(\beta) - \omega_N^{(1)}(\beta) \otimes I_n\| < +\infty$.

Proof. The conclusion of the proposition is equivalent to

$$\max_{j \in \{2, \dots, N\}} \|\omega_N^{(j)}(\beta) - \omega_N^{(1)}(\beta)\| = O(1/N)$$

as $N \rightarrow +\infty$. Fix $j \in \{2, \dots, N\}$ and set $Z_N = f_j \otimes f_1^* \in M_N(\mathbb{C}^N)$. Thus, Z_N is an operator of rank one such that $Z_N h = \langle h, f_1 \rangle f_j$ for every $h \in \mathbb{C}^N$. We have

$$\begin{aligned} [\omega_N(\beta), I_n \otimes Z_N] &= (\omega_N^{(j)}(\beta) - \omega_N^{(1)}(\beta)) \otimes Z_N \\ &= (I_n \otimes P_N^{(j)})((\omega_N^{(j)}(\beta) - \omega_N^{(1)}(\beta)) \otimes Z_N)(I_n \otimes P_N^{(1)}) \end{aligned}$$

and, similarly,

$$[C_N, Z_N] = (\lambda_N^{(j)} - \lambda_N^{(1)})Z_N = P_N^{(j)}(\lambda_N^{(j)} - \lambda_N^{(1)})Z_N P_N^{(1)}.$$

Next, we apply (8.4) and use the fact that $I \otimes P_N^{(j)}$ commutes with $\mathbb{E}(R_N(U_N, \beta))$. Setting $X_N = I_n \otimes P_N^{(k)}$, $Y_N = I_n \otimes P_N^{(1)}$, we obtain

$$\begin{aligned} (\omega_N^{(j)}(\beta) - \omega_N^{(1)}(\beta)) \otimes Z_N &= (\lambda_N^{(j)} - \lambda_N^{(1)})(\mathbb{E}(R_N(\beta)))^{-1} \\ &\quad \times \mathbb{E}(X_N(R_N(\beta) - \mathbb{E}(R_N(\beta)))X_N(\gamma_1 \otimes Z_N)Y_N(R_N(\beta) - \mathbb{E}(R_N(\beta)))Y_N) \\ &\quad \times (\mathbb{E}(R_N(\beta)))^{-1}. \end{aligned}$$

Since $\|Z_N\| = 1$, an application of the Cauchy-Schwarz inequality leads to the following estimate:

$$\begin{aligned} \|\omega_N^{(j)}(\beta) - \omega_N^{(1)}(\beta)\| &\leq |\lambda_N^{(j)} - \lambda_N^{(1)}| \|(\mathbb{E}(R_N(\beta)))^{-1}\|^2 \|\gamma_1\| \\ &\quad \times \mathbb{E}(\|X_N(R_N - \mathbb{E}(R_N(\beta)))X_N\|^2)^{1/2} \mathbb{E}(\|Y_N(R_N - \mathbb{E}(R_N(\beta)))Y_N\|^2)^{1/2}. \end{aligned}$$

We have $|\lambda_N^{(j)} - \lambda_N^{(1)}| \leq 2\|C_N\|$ and the product of the last two factors above is estimated via (8.6) by kr^4/N with k independent of N . Thus,

$$\|\omega_N^{(j)}(\beta) - \omega_N^{(1)}(\beta)\| \leq 2k\|C_N\|r'^2\|\gamma_1\|r^4/N \leq k'/N,$$

with k' independent of j and N . The lemma follows. \square

In the probability model we consider, the sequences $\{C_N\}_{N \in \mathbb{N}}$ and $\{D_N\}_{N \in \mathbb{N}}$ are uniformly bounded in norm and, in addition, the sequences $\{\mu_{C_N}\}_{N \in \mathbb{N}}$ and $\{\mu_{D_N}\}_{N \in \mathbb{N}}$ converge weakly to μ and ν , respectively. We denote by (a, b) a pair of free random variables in some C^* -probability space (\mathcal{M}, τ) such that the sequence $\{(c_N, d_N)\}_{N \in \mathbb{N}}$ converges in distribution to (a, b) . We also set

$$\mathcal{G} = \mathcal{G}_{\gamma_1 \otimes a, \gamma_2 \otimes b, M_n(\mathbb{C}), M_n(\mathbb{C}(a))}, \quad \omega = \omega_{\gamma_1 \otimes a, \gamma_2 \otimes b, M_n(\mathbb{C}), M_n(\mathbb{C}(a))}.$$

In other words, ω is the usual matrix subordination function associated to the pair $(\gamma_1 \otimes a, \gamma_2 \otimes b)$.

Proposition 8.8. *Suppose that $\sup\{\|C_N\| + \|D_N\| : N \in \mathbb{N}\} < +\infty$ and that the sequence $\{(c_N, d_N)\}_{N \in \mathbb{N}}$ converges to (a, b) in distribution. Let $\mathcal{D} \subset \mathcal{G} \cap \bigcap_{n \in \mathbb{N}} \mathcal{G}_N$ be a connected open set containing $\mathbb{H}^+(M_n(\mathbb{C}))$ and such that the sequence of functions $\{\|\omega_N\|\}_{N \in \mathbb{N}}$ is locally uniformly bounded on \mathcal{D} . Then*

$$\lim_{N \rightarrow \infty} \|\omega_N(\beta) - \omega(\beta) \otimes I_N\| = 0, \quad \beta \in \mathcal{D}.$$

Proof. By hypothesis, the analytic functions $\{\omega_N^{(1)}\}_{N \in \mathbb{N}}$ form a normal family on \mathcal{D} . By Proposition 8.7, it suffices to prove that every subsequential limit of this sequence equals ω . Suppose that $\{\omega_{N_k}^{(1)}\}_{k \in \mathbb{N}}$ converges on \mathcal{D} to a function $\tilde{\omega}$. Fix $\beta \in H^+(M_n(\mathbb{C}))$ such that $\Im\beta > \sup\{\|C_N\| + \|D_N\| : N \in \mathbb{N}\}$. Then

$$\begin{aligned} \mathbb{E}(R_N(\beta)) &= (\omega_N^{(1)}(\beta) \otimes I_N - \gamma_1 \otimes C_N + O(1/N))^{-1} \\ &= (\omega_N^{(1)}(\beta) \otimes I_N - \gamma_1 \otimes C_N)^{-1} + O(1/N) \\ &= \sum_{m=0}^{\infty} (\omega_N^{(1)}(\beta)^{-1}) (\gamma_1 \omega_N^{(1)}(\beta)^{-1})^m \otimes C_N^m + O(1/N), \end{aligned}$$

and thus

$$(\text{Id}_{M_n(\mathbb{C})} \otimes \text{tr}_N)(\mathbb{E}(R_N(\beta))) = \sum_{m=0}^{\infty} (\omega_N^{(1)}(\beta)^{-1}) (\gamma_1 \omega_N^{(1)}(\beta)^{-1})^m \cdot \text{tr}_N(C_N^m) + O(1/N).$$

Setting $N = N_k$, letting $k \rightarrow +\infty$, and observing that the series on the right is uniformly dominated, we conclude that

$$\begin{aligned} \lim_{k \rightarrow \infty} (\text{Id}_{M_n(\mathbb{C})} \otimes \text{tr}_{N_k})(\mathbb{E}(R_{N_k}(\beta))) &= \sum_{m=0}^{\infty} (\tilde{\omega}(\beta)^{-1}) (\gamma_1 \tilde{\omega}(\beta)^{-1})^m \cdot \tau(a^m) \\ &= (\text{Id}_{M_n(\mathbb{C})} \otimes \tau)((\tilde{\omega}(\beta) - \gamma_1 \otimes a)^{-1}). \end{aligned}$$

The fact that the pairs (C_N, D_N) converge to (a, b) implies that

$$\begin{aligned} \lim_{k \rightarrow \infty} (\text{Id}_{M_n(\mathbb{C})} \otimes \text{tr}_{N_k})(\mathbb{E}(R_{N_k}(\beta))) &= (\text{Id}_{M_n(\mathbb{C})} \otimes \tau)((\beta - \gamma_1 \otimes a - \gamma_2 \otimes b)^{-1}) \\ &= (\text{Id}_{M_n(\mathbb{C})} \otimes \tau)((\omega(\beta) - \gamma_1 \otimes a)^{-1}) \end{aligned}$$

Thus, we obtain the equality

$$(\text{Id}_{M_n(\mathbb{C})} \otimes \tau)((\omega(\beta) - \gamma_1 \otimes a)^{-1}) = (\text{Id}_{M_n(\mathbb{C})} \otimes \tau)((\tilde{\omega}(\beta) - \gamma_1 \otimes a)^{-1}),$$

and this easily yields $\tilde{\omega}(\beta) = \omega(\beta)$. Since \mathcal{D} is connected, we must have $\tilde{\omega} = \omega$, thus concluding the proof. \square

Corollary 5.5 implies the following result.

Corollary 8.9. *Suppose that $\mathfrak{s} := \sup\{\|C_N\| + \|D_N\| : N \in \mathbb{N}\} < +\infty$ and the sequence $\{(c_N, d_N)\}_{N \in \mathbb{N}}$ converges to (a, b) in distribution. Let $\mathfrak{k} = \max\{k, \mathfrak{s}\}$, where k is the constant provided by Lemma 5.4. Then, for every $z \in \mathbb{C} \setminus [-\mathfrak{k}, \mathfrak{k}]$, we have $ze_{1,1} - \gamma_0 \in \mathcal{G} \cap \bigcap_{N \in \mathbb{N}} \mathcal{G}_N$ and*

$$\lim_{N \rightarrow \infty} \|\omega_N(ze_{1,1} - \gamma_0) - \omega(ze_{1,1} - \gamma_0) \otimes I_N\| = 0.$$

The preceding results combine to yield convergence results for sample resolvents. For the following statement, it is useful to identify \mathbb{C}^m with a subspace of \mathbb{C}^N if $m < N$ and to denote by $\{f_1, \dots, f_N\}$ the standard basis in \mathbb{C}^N . Thus, we have $f_j \in \mathbb{C}^m$ provided that $j \leq m$.

Proposition 8.10. *Suppose that $\sup\{\|C_N\| + \|D_N\| : N \in \mathbb{N}\} < +\infty$ and that the sequence $\{(c_N, d_N)\}_{N \in \mathbb{N}}$ converges to (a, b) in distribution. Suppose also that C_N is diagonal in the standard basis, that is, $C_N f_j = \lambda_N^{(j)} f_j$, $j = 1, \dots, N$, and that $p \in \mathbb{N}$ is such that the limits $\lambda^{(j)} = \lim_{N \rightarrow \infty} \lambda_N^{(j)}$ exist for $j = 1, \dots, p$. Denote by $P_N : \mathbb{C}^N \rightarrow \mathbb{C}^p$ the orthogonal projection onto \mathbb{C}^p , $N \geq p$. Let $\mathcal{D} \subset \mathcal{G} \cap \bigcap_{N \in \mathbb{N}} \mathcal{G}_N$ be a connected open set containing $\mathbb{H}^+(M_n(\mathbb{C}))$ and such that the sequence of functions $\{\|\omega_N\|\}_{N \in \mathbb{N}}$ is locally uniformly bounded on \mathcal{D} . Then almost surely*

$$\lim_{N \rightarrow \infty} (I_n \otimes P_N) R_N(\beta) (I_n \otimes P_N) = (\omega(\beta) \otimes I_p - \gamma_1 \otimes \text{diag}(\lambda^{(1)}, \dots, \lambda^{(p)}))^{-1}$$

in the norm topology for every $\beta \in \mathcal{D}$.

Proof. By Proposition 8.4, it suffices to show that the conclusion holds with $\mathbb{E}(R_N(\beta))$ in place of $R_N(\beta)$. We observe that

$$(I_n \otimes P_N) \mathbb{E}(R_N(\beta)) (I_n \otimes P_N) = \sum_{j=1}^p (\omega_N^{(j)}(\beta) - \lambda_N^{(j)} \gamma_1)^{-1} \otimes P_N^{(j)},$$

so the desired conclusion follows from Proposition 8.8. \square

We observe for use in the following result that there exists a domain \mathcal{D} as in the above statement such that $ze_{1,1} - \gamma_0 \in \mathcal{D}$ for every $z \in \mathbb{C}^+$.

When the convergence of $(C_N, U_N D_N U_N^*)$ to (a, b) is strong, the preceding result extends beyond $\mathbb{H}^+(M_n(\mathbb{C}))$. In this case, $\sigma(P(C_N, U_N D_N U_N^*))$ converges almost surely to $\sigma(P(a, b))$ and thus the sample resolvent $R_N(U_N, ze_{1,1} - \gamma_0)$ is defined almost surely for large N even for $z \in \mathbb{R} \setminus \sigma(P(a, b))$. We also recall that the function

$$(\omega(ze_{1,1} - \gamma_0) - \lambda \gamma_1)^{-1}$$

extends analytically to $\mathbb{C} \setminus \sigma(P(a, b))$ if $\lambda \in \sigma(a)$. These analytic extensions are used in the following statement.

Proposition 8.11. *Under the hypothesis of Proposition 8.10, suppose that the pairs $\{(C_N, U_N D_N U_N^*)\}_{N \in \mathbb{N}}$ converge strongly to (a, b) . Then almost surely*

$$\lim_{N \rightarrow \infty} (I_n \otimes P_N) R_N(\beta) (I_n \otimes P_N) = (\omega(\beta) \otimes I_p - \gamma_1 \otimes \text{diag}(\lambda^{(1)}, \dots, \lambda^{(p)}))^{-1}$$

for $\beta = ze_{1,1} - \gamma_0$, $z \in \mathbb{C} \setminus \sigma(a, b)$. The convergence is uniform on compact subsets of $\mathbb{C} \setminus \sigma(P(a, b))$.

Proof. Strong convergence implies that $\lambda^{(j)} \in \sigma(a)$ for $j = 1, \dots, p$, so the functions $(\omega(ze_{1,1} - \gamma_0) - \lambda^{(j)}\gamma_1)^{-1}$ extend analytically to $\mathbb{R} \setminus \sigma(P(a, b))$. Let $\mathcal{O} \subset \mathbb{C} \setminus \sigma(P(a, b))$ be a connected open set containing $\{z\} \cup \mathbb{C}^+$ which is at a strictly positive distance from $\mathbb{C} \setminus \sigma(P(a, b))$. We prove that the conclusion of the proposition holds for $U_N(\xi)$ provided that $\xi \in \Omega$ is such that the conclusion of Corollary 8.6 holds and $\sigma(P(C_N, VD_NV^*)) \subset \mathbb{R} \setminus \mathcal{O}$ for $V = U_N(\xi)$ and sufficiently large N . By hypothesis, the collection of such points $\xi \in \Omega$ has probability 1. Lemma 4.3 shows that the family of functions $\|R_N(U_N(\xi), ze_{1,1} - \gamma_0)\|$ is locally uniformly bounded on \mathcal{O} for large N . By Montel's theorem, we can conclude the proof by verifying the conclusions of the proposition for $\beta = ze_{1,1} - \gamma_0$ with $z \in \mathbb{C}^+$. For such values of z , the result follows from Proposition 8.10. \square

9. THE UNITARILY INVARIANT MODEL

In this section we prove Theorem 6.1 under the additional condition that A_N is a constant matrix and B_N is a random unitary conjugate of another constant matrix. We may, and do, assume that for $N \geq p$, A_N is diagonal in the standard basis $\{f_1, \dots, f_N\}$ with eigenvalues $\theta_1, \dots, \theta_p, \lambda_N^{(p+1)}, \dots, \lambda_N^{(N)}$ and, as before, $B_N = U_N D_N U_N^*$. Then the random matrices $a_N = A_N \otimes 1_\Omega$ and $d_N = U_N D_N U_N^*$ are viewed as elements of the noncommutative probability space $M_N(\mathbb{C}) \otimes L^\infty(\Omega)$. We fix free selfadjoint random variables (a, b) in some tracial W^* -probability space such that the pairs (a_N, d_N) converge in distribution to (a, b) as $N \rightarrow \infty$. This convergence is not strong because of the spikes $\theta_1, \dots, \theta_p$. As in [11], we consider closely related pairs (c_N, d_N) that do converge strongly to (a, b) . Namely, we set $c_N = C_N \otimes 1_\Omega$, where C_N is diagonal in the standard basis of \mathbb{C}^N with eigenvalues $\lambda_N^{(1)}, \dots, \lambda_N^{(N)}$ that coincide with those of A_N except that $\lambda_N^{(1)} = \dots = \lambda_N^{(p)} = s$ is an arbitrary (but fixed for the remainder of this section) element of $\text{supp}(\mu)$. For $N \geq p$, the difference $\Delta_N = A_N - C_N$ can then be written as $\Delta_N = P_N^* T P_N$, where $T \in M_p(\mathbb{C})$ is the diagonal matrix with eigenvalues $\theta_1 - s, \dots, \theta_p - s$ and $P_N: \mathbb{C}^N \rightarrow \mathbb{C}^p$ is the orthogonal projection.

According to Lemma 4.2, $\ker(tI_N - P(A_N, B_N))$ and $\ker(te_{1,1} \otimes I_N - L(A_N, B_N))$ have the same dimension for every $t \in \mathbb{R}$. Setting $\beta = ze_{1,1} - \gamma_0$, strong convergence of the pairs c_N, d_N implies that almost surely the sample resolvent $R_N(U_N, \beta)$ is defined for sufficiently large N if $z \in \mathbb{C} \setminus \sigma(P(c, d))$. (We continue using the notation introduced in (8.1), (8.2), and (8.3).) We need to consider the matrix

$$\begin{aligned} \beta \otimes I_N - \gamma_1 \otimes A_N - \gamma_2 \otimes B_N &= \beta \otimes I_N - \gamma_1 \otimes C_N - \gamma_2 \otimes B_N - \gamma_1 \otimes \Delta_N \\ &= (I_n \otimes I_N - (\gamma_1 \otimes \Delta_N) R_N(U_N, \beta)) (\beta \otimes I_N - \gamma_1 \otimes C_N - \gamma_2 \otimes B_N), \end{aligned}$$

since the order of $t \in \mathbb{R}$ as a zero of its determinant equals $\dim \ker(tI_N - P(A_N, B_N))$, and hence the number of eigenvalues of $P(A_N, B_N)$ in a neighborhood V of a given $t \in \mathbb{R} \setminus \sigma(P(a, b))$ is the number of zeros of this determinant in V . Thus, we need to consider the zeros in V of

$$\det(I_n \otimes I_N - (\gamma_1 \otimes \Delta_N) R_N(U_N, \beta)).$$

Using Sylvester's identity ($\det(I_r - XY) = \det(I_p - YX)$ if X is an $r \times p$ matrix and Y is a $p \times r$ matrix) and the fact that $\Delta_N = P_N^* T P_N$, this determinant can be rewritten as

$$F_N(U_N, \beta) = \det(I_n \otimes I_p - (\gamma_1 \otimes T)(I_n \otimes P_N) R_N(U_N, \beta)(I_n \otimes P_N^*)).$$

At this point, we observe that the hypothesis of Proposition 8.11 are satisfied with $\lambda^{(1)} = \dots = \lambda^{(p)} = s$. We conclude that almost surely

$$\lim_{n \rightarrow \infty} F_N(U_N, \beta) = F(\beta), \quad \beta = ze_{1,1} - \gamma_0, z \notin \sigma(P(a, b)),$$

where

$$F(\beta) = \det(I_n \otimes I_p - (\gamma_1(\omega(\beta) - s\gamma_1)^{-1}) \otimes T),$$

and the convergence is uniform on compact sets. The limit $F(ze_{1,1} - \gamma_0)$ is a (deterministic) analytic function on $\mathbb{C} \setminus \sigma(P(a, b))$. An application of Hurwitz's theorem on zeros of analytic functions (see [1, Theorem 5.2]) yields the following result.

Proposition 9.1. *Suppose that $t_1, t_2 \in \mathbb{R}$, $t_1 < t_2$, $[t_1, t_2] \subset \mathbb{R} \setminus \sigma(P(a, b))$, $F(t_j e_{1,1} - \gamma_0) \neq 0$ for $j = 1, 2$, and the function $F(ze_{1,1} - \gamma_0)$ has at most one zero t in the interval (t_1, t_2) . Then, almost surely for large N , the matrix $P(A_N, B_N)$ has exactly m eigenvalues in the interval $[t_1, t_2]$, where m is the order of t as a zero of $F(ze_{1,1} - \gamma_0)$ and $m = 0$ if this function does not vanish on $[t_1, t_2]$.*

Part(1) of Theorem 6.1 is a reformulation of Proposition 9.1. To see that this is the case, we observe that T is a diagonal matrix and thus the matrix

$$I_n \otimes I_p - (\gamma_1(\omega(ze_{1,1} - \gamma_0) - s\gamma_1)^{-1}) \otimes T$$

is block diagonal with diagonal blocks

$$G_{j,s}(z) = I_n - (\theta_j \gamma_1 - s\gamma_1)(\omega(ze_{1,1} - \gamma_0) - s\gamma_1)^{-1}, \quad j = 1, \dots, p.$$

If $\det(G_{j,s}(z))$ has a zero of order m_j at t then the number m in the statement is $m_1 + \dots + m_p$. We recall that $G_{j,s}$ is analytic on $\mathbb{C} \setminus \sigma(P(a, b))$ but $\omega(ze_{1,1} - \gamma_0)$ is only meromorphic. It is not immediately apparent that the number m_j does not depend on s but this is a consequence of the following result.

Lemma 9.2. *Suppose that $\alpha_1, \alpha_2 \in M_n(\mathbb{C})$ are such that $(\omega(ze_{1,1} - \gamma_0) - \alpha_k)^{-1}$ extends analytically to t for $k = 1, 2$. Then the order of t as a zero of*

$$\det(I_n - (\theta_j \gamma_1 - \alpha_k)(\omega(ze_{1,1} - \gamma_0) - \alpha_k)^{-1})$$

does not depend on k .

Proof. An easy calculation shows that

$$I_n - (\theta_j \gamma_1 - \alpha_k)(\omega(ze_{1,1} - \gamma_0) - \alpha_k)^{-1} = (\omega(ze_{1,1} - \gamma_0) - s\gamma_1)(\omega(ze_{1,1} - \gamma_0) - \alpha_k)^{-1}.$$

The desired conclusion follows if we prove that the function

$$H(z) = (\omega(ze_{1,1} - \gamma_0) - \alpha_1)(\omega(ze_{1,1} - \gamma_0) - \alpha_2)^{-1}$$

is analytic and invertible at $z = t$. We have

$$H(z) = I_n + (\alpha_2 - \alpha_1)(\omega(ze_{1,1} - \gamma_0) - \alpha_2)^{-1}$$

and

$$H(z)^{-1} = I_n + (\alpha_1 - \alpha_2)(\omega(ze_{1,1} - \gamma_0) - \alpha_1)^{-1},$$

so the analyticity and invertibility of H follow from the hypothesis. \square

We proceed now to Part(2) of Theorem 6.1. Thus, assumptions (A1–A3) and (B0–B2) are in force and, in addition, the spikes $\theta_1, \dots, \theta_p$ are distinct. In particular, $\sup\{\|C_N\| + \|D_N\| : N \in \mathbb{N}\} < +\infty$.

If $S \subset \mathbb{R}$ is a Borel set and $A \in M_N(\mathbb{C})$ is a selfadjoint operator, then $E_A(S)$ denotes the orthogonal projection onto the linear span of the eigenvectors of A corresponding to eigenvalues in S . For instance, under the hypotheses of Part(2) of Theorem 6.1, $E_{A_N}(\{\theta_j\})$ is a projection of rank one for $j = 1, \dots, p$. If $h: \mathbb{R} \rightarrow \mathbb{C}$ is a continuous function, then $h(A)$ denotes the usual functional calculus for selfadjoint matrices. Thus, if $Ax = tx$ for some $t \in \mathbb{R}$ and $x \in \mathbb{C}^N$, then $h(A)x = h(t)x$.

Fix $t \in \mathbb{R} \setminus \sigma(P(a, b))$ and $\varepsilon > 0$ small enough. We need to show that, almost surely,

$$(9.1) \quad \lim_{N \rightarrow \infty} \text{Tr}_N [E_{A_N}(\{\theta_i\})E_{P(A_N, B_N)}((t - \varepsilon, t + \varepsilon))] = \delta_{i, i_0} \mathcal{C}_i(t).$$

Choose $\delta > 0$ and $N_0 \in \mathbb{N}$ such that $[\theta_j - \delta, \theta_j + \delta] \cap \sigma(A_N) = \{\theta_j\}$, for $N \geq N_0$ and $j = 1, \dots, p$. Pick infinitely differentiable functions $f_j: \mathbb{R} \rightarrow [0, 1]$ supported in $[\theta_j - \delta, \theta_j + \delta]$ such that $f_j(\theta_j) = 1$, $j = 1, \dots, p$. Also pick an infinitely differentiable function $h: \mathbb{R} \rightarrow [0, 1]$ supported in $(t - \varepsilon, t + \varepsilon)$ such that h is identically 1 on $[t - \varepsilon/2, t + \varepsilon/2]$. Then part (1) of Theorem 6.1 implies that, almost surely for large N , we have

$$E_{A_N}(\{\theta_j\})E_{P(A_N, B_N)}((t - \varepsilon, t + \varepsilon)) = f_j(A_N)h(P(A_N, B_N)).$$

In anticipation of a concentration inequality, we prove a Lipschitz estimate for the functions $g_{N,j}: \text{U}(N) \rightarrow M_N(\mathbb{C})$ defined by

$$g_{N,j}(V) = \text{Tr}_N(f_j(A_N)h(P(A_N, VD_NV^*))), \quad V \in \text{U}(N), j = 1, \dots, p.$$

Lemma 9.3. *There exists $k > 0$, independent of N , such that*

$$|g_{N,j}(V) - g_{N,j}(W)| \leq k\|V - W\|_2, \quad V, W \in \text{U}(N), j = 1, \dots, p.$$

Proof. Given a Lipschitz function $u: \text{U}(N) \rightarrow M_N(\mathbb{C})$, we denote by $\text{Lip}(u)$ the smallest constant c such that

$$\|u(V) - u(W)\|_2 \leq c\|V - W\|_2, \quad V, W \in \text{U}(N),$$

and we set $\|u\|_\infty = \sup\{\|u(V)\| : v \in \text{U}(N)\}$. If $u_1, u_2: \text{U}(N) \rightarrow M_N(\mathbb{C})$ are two Lipschitz functions, then $\text{Lip}(u_1 + u_2) \leq \text{Lip}(u_2) + \text{Lip}(u_1)$ and

$$\text{Lip}(u_1 u_2) \leq \|u_1\|_\infty \text{Lip}(u_2) + \text{Lip}(u_1) \|u_2\|_\infty.$$

Since the functions $V \mapsto V$ and $V \mapsto V^*$ are Lipschitz with constant 1, we deduce immediately that the map $V \mapsto P(A_N, VD_NV^*)$ is Lipschitz with constant bounded independently of N . It is well-known that a Lipschitz function $f: \mathbb{R} \rightarrow \mathbb{R}$ is also Lipschitz, with the same constant, when viewed as a map on the selfadjoint matrices with the Hilbert-Schmidt norm (see for instance, [18, Lemma A.2]). The function h is infinitely differentiable with compact support, hence Lipschitz. We deduce that the map $V \mapsto h(P(A_N, VD_NV^*))$ is Lipschitz with constant bounded independently of N . Finally, we have

$$|g_{N,j}(V) - g_{N,j}(W)| \leq \|f_j(A_N)\|_2 \|h(P(A_N, VD_NV^*)) - h(P(A_N, WD_NW^*))\|_2,$$

and the lemma follows because $\|f_j(A_N)\|_2 = 1$. \square

An application of [3, Corollary 4.4.28] yields

$$\mathbb{P}(|g_{N,j}(U_N) - \mathbb{E}(g_{N,j}(U_N))| > \eta) \leq 2 \exp\left(-\frac{\eta^2 N}{4k^2}\right), \quad \eta > 0, j = 1, \dots, p,$$

and the Borel-Cantelli lemma shows that, almost surely,

$$(9.2) \quad \lim_{N \rightarrow \infty} (g_{N,j}(U_N) - \mathbb{E}(g_{N,j}(U_N))) = 0, \quad j = 1, \dots, p.$$

The expected value in (9.2) is estimated using [18, Lemma 6.3] and the fact that $f_j(A_N)$ is the projection onto the j th coordinate. If we set $r_N(z) = (zI_N - P(A_N, B_N))^{-1}$ and $\tilde{R}_N(ze_{1,1} - \gamma_0) = ((ze_{1,1} - \gamma_0) \otimes I_N - L(A_N, B_N))^{-1}$, $z \in \mathbb{C}^+$, then

$$(9.3) \quad \begin{aligned} \mathbb{E}(g_{N,j}(U_N)) &= -\lim_{y \downarrow 0} \frac{1}{\pi} \Im \int_{\mathbb{R}} \mathbb{E}(\text{Tr}_N(f_j(A_N)r_N(\xi + iy))h(\xi)) d\xi \\ &= -\lim_{y \downarrow 0} \frac{1}{\pi} \Im \int_{\mathbb{R}} \mathbb{E}(r_N(\xi + iy)_{j,j})h(\xi) d\xi. \end{aligned}$$

The construction of the linearization L (Section 4) is such that the matrix $\tilde{R}_N(ze_{1,1} - \gamma_0)$, viewed as an $n \times n$ block matrix, has $r_N(z)$ as its $(1, 1)$ entry. By Propositions 8.4(1) and 8.10 (with A_N in place of C_N) and the unitary invariance of the distribution of B_N , for $z \in \mathbb{C}^+$ the matrices

$$(I_n \otimes P_N)\mathbb{E}(\tilde{R}_N(ze_{1,1} - \gamma_0))(I_n \otimes P_N)^*$$

converge as $N \rightarrow \infty$ to the block diagonal matrix with diagonal entries

$$(\omega(ze_{1,1} - \gamma_0) - \theta_j \gamma_1)^{-1}.$$

It follows that

$$(9.4) \quad \lim_{N \rightarrow \infty} \mathbb{E}((r_N(z))_{j,j}) = ((\omega(ze_{1,1} - \gamma_0) - \theta_j \gamma_1)^{-1})_{1,1}.$$

We intend to let $N \rightarrow \infty$ in (9.3) using (9.4), so we consider the differences

$$\Delta_{j,N}(z) = ((\omega(ze_{1,1} - \gamma_0) - \theta_j \gamma_1)^{-1})_{1,1} - \mathbb{E}(r_N(z)_{j,j}).$$

By Corollary 5.5, these functions are defined on $\mathbb{C} \setminus [-k, k]$ for some $k > 0$ and satisfy $\Delta_{j,N}(\bar{z}) = \overline{\Delta_{j,N}(z)}$. We claim that there exists a sequence $\{v_N\}_{N \in \mathbb{N}} \subset (0, +\infty)$ such that $\lim_{N \rightarrow \infty} v_N = 0$ and

$$(9.5) \quad |\Delta_{j,N}(z)| \leq v_N \left(1 + \frac{1}{(\Im z)^2}\right), \quad z \in \mathbb{C}^+.$$

To verify this claim, we observe first [2] that the function $\mathbb{E}(r_N(z)_{j,j})$ is the Cauchy-Stieltjes transform of a Borel probability measure $\sigma_{N,j}$ on \mathbb{R} . Since

$$\sup\{\|P(A_N, B_N)\| : N \in \mathbb{N}\} < \infty,$$

the measures $\{\sigma_{N,j}\}_{N \in \mathbb{N}}$ have uniformly bounded supports. Now, (9.4) shows that the Cauchy-Stieltjes transform of any accumulation point of this sequence of measures is equal to $((\omega(ze_{1,1} - \beta_0) - \theta_j \beta_1)^{-1})_{1,1}$. It follows that this sequence has a weak limit σ_j that is a Borel probability measure with compact support. The existence of the sequence $\{v_N\}_{N \in \mathbb{N}}$ follows from [11, Lemma 4.1] applied to the signed measures $\rho_N = \sigma_{N,i} - \sigma_i$.

We use (9.5), (9.3), and the Lemma from [20, Appendix] to obtain

$$\begin{aligned}
\lim_{N \rightarrow \infty} \mathbb{E}(g_{N,j}(U_N)) &= - \lim_{N \rightarrow \infty} \lim_{y \downarrow 0} \frac{1}{\pi} \Im \int_{\mathbb{R}} \mathbb{E}(r_N(\xi + iy)_{j,j}) h(\xi) d\xi \\
&= \lim_{N \rightarrow \infty} \lim_{y \downarrow 0} \frac{1}{\pi} \Im \int_{\mathbb{R}} \Delta_{j,N}(\xi + iy) h(\xi) d\xi \\
&\quad - \lim_{y \downarrow 0} \frac{1}{\pi} \Im \int_{\mathbb{R}} ((\omega((\xi + iy)e_{1,1} - \gamma_0) - \theta_j \gamma_1)^{-1})_{1,1} h(\xi) d\xi \\
(9.6) \quad &= - \lim_{y \downarrow 0} \frac{1}{\pi} \Im \int_{\mathbb{R}} ((\omega((\xi + iy)e_{1,1} - \gamma_0) - \theta_j \gamma_1)^{-1})_{1,1} h(\xi) d\xi.
\end{aligned}$$

The choice of h , and the fact that

$$u(z) = ((\omega(z e_{1,1} - \gamma_0) - \theta_j \gamma_1)^{-1})_{1,1}$$

is analytic and real-valued on the intervals $[t - \varepsilon, t - \varepsilon/2]$ and $[t + \varepsilon/2, t + \varepsilon]$, imply that the last line in (9.6) can be rewritten as

$$\begin{aligned}
&\lim_{y \downarrow 0} \frac{1}{\pi} \int_{t-\varepsilon}^{t-\frac{\varepsilon}{2}} \Im(u(\xi + iy)) h(\xi) d\xi + \lim_{y \downarrow 0} \frac{1}{\pi} \int_{t+\frac{\varepsilon}{2}}^{t+\varepsilon} \Im(u(\xi + iy)) h(\xi) d\xi \\
&\quad - \lim_{y \downarrow 0} \frac{1}{\pi} \int_{t-\frac{\varepsilon}{2}}^{t+\frac{\varepsilon}{2}} \Im(u(\xi + iy)) d\xi \\
(9.7) \quad &= \lim_{y \downarrow 0} \frac{1}{2\pi i} \int_{t-\frac{\varepsilon}{2}}^{t+\frac{\varepsilon}{2}} u(\xi + iy) d\xi - \lim_{y \downarrow 0} \frac{1}{2\pi i} \int_{t-\frac{\varepsilon}{2}}^{t+\frac{\varepsilon}{2}} u(\xi - iy) d\xi.
\end{aligned}$$

Recall (see, for instance, [1, Chapter 4]) that if f is an analytic function on a simply connected domain D , except for an isolated singularity a , then $\frac{1}{2\pi i} \int_{\gamma} f(z) dz = n(\gamma, a) \text{Res}_{z=a} f(z)$. Here γ is a closed Jordan path in D not containing a , $n(\gamma, a)$ is the winding number of γ with respect to a , and $\text{Res}_{z=a} f(z)$ is that number R which satisfies the condition that $f(z) - \frac{R}{z-a}$ has vanishing period (called the residue of f at a). Denote by Γ_y the rectangle with corners $t \pm (\varepsilon/2) \pm iy$ and let γ_y be the boundary of Γ_y oriented counterclockwise. The expression in (9.7) represents the integral of u on the horizontal segments in γ_y . It is clear that the integral of u on the vertical segments is $O(y)$, and thus (9.6) implies the equality

$$\lim_{N \rightarrow \infty} \mathbb{E}(g_{N,j}(U_N)) = \lim_{y \downarrow 0} \frac{1}{2\pi i} \int_{\gamma_y} u(z) dz = \text{Res}_{z=t} u(z).$$

The alternative formula in Theorem 6.1 follows from the fact that u has a simple pole at t because u maps \mathbb{C}^+ to \mathbb{C}^- .

10. THE WIGNER MODEL

We proceed now to the proof of Theorem 6.3. The matrices A_N are subject to the hypotheses (A1–A3), while X_N/\sqrt{N} and X_N satisfies conditions (X0–X3). By [36, Section 3, Assertion 2], it suffices to proceed under the additional hypothesis that each A_N is a constant matrix. The free variables a and b are such that b has standard semicircular distribution $\nu_{0,1}$.

One consequence of the fact that b is a semicircular variable is that the function ω is analytic on the entire set

$$\{\beta \in M_n(\mathbb{C}) : \beta \otimes 1 - \gamma_1 \otimes a - \gamma_2 \otimes b \text{ is invertible.}\}$$

This justifies the comment from Remark 6.2. We recall that in this special case the subordination function is given by

$$\omega(\beta) = \beta - \gamma_2(\text{Id}_{M_n(\mathbb{C})} \otimes \tau)(\beta \otimes 1 - \gamma_1 \otimes a - \gamma_2 \otimes b)^{-1} \gamma_2.$$

Since the distribution of the random matrix X_N is not usually invariant under unitary conjugation, we can no longer assume that A_N is diagonal in the standard basis $\{f_1, \dots, f_N\}$. There is however a (constant) unitary matrix $V_N \in M_N(\mathbb{C})$ such that A_N is diagonal in the basis $\{V_N f_1, \dots, V_N f_N\}$ with eigenvalues $\theta_1, \dots, \theta_p, \lambda_N^{(p+1)}, \dots, \lambda_N^{(N)}$, $N \geq p$. Viewing each realization of the random matrices A_N and X_N/\sqrt{N} as elements of the noncommutative probability space $(M_N(\mathbb{C}), \text{tr}_N)$, almost surely the pairs $(A_N, X_N/\sqrt{N})$ converge in distribution, but not strongly, to (a, b) as $N \rightarrow \infty$. A modification of A_N provides almost surely strongly convergent pairs $(C_N, X_N/\sqrt{N})$. Thus, let C_N be diagonal in the basis $\{V_N f_1, \dots, V_N f_N\}$ with eigenvalues $\lambda_N^{(1)}, \dots, \lambda_N^{(N)}$ that coincide with those of A_N except that $\lambda_N^{(1)} = \dots = \lambda_N^{(p)} = s$ is an arbitrary (but fixed for the remainder of this section) element of $\text{supp}(\mu)$. For $N \geq p$, the difference $\Delta_N = A_N - C_N$ can then be written as $\Delta_N = V_N P_N^* T P_N V_N^*$, where $T \in M_p(\mathbb{C})$ is the diagonal matrix with eigenvalues $\theta_1 - s, \dots, \theta_p - s$ and $P_N : \mathbb{C}^N \rightarrow \mathbb{C}^p$ is the orthogonal projection. Almost surely, the pairs $(C_N, X_N/\sqrt{N})$ converge strongly to (a, b) as shown in [12, Theorem 1.2] and [23, Proposition 2.1]. We continue using the notation introduced in (8.1) and (8.3) with $c_N = C_N$ and $d_N = \frac{X_N}{\sqrt{N}}$. The calculation in Section 9 show that, almost surely, for N large enough, the number of eigenvalues of $P(A_N, X_N/\sqrt{N})$ in a small enough neighborhood of $t \in \mathbb{R} \setminus \sigma(P(a, b))$ is equal to the number of zeros of $F_N(ze_{1,1} - \gamma_0)$ in this neighborhood, where

$F_N(ze_{1,1} - \gamma_0) = \det(I_n \otimes I_p - (\gamma_1 \otimes T)(I_n \otimes P_N V_N^*) R_N(ze_{1,1} - \gamma_0)(I_n \otimes P_N^* V_N))$ is a random analytic function. We focus on the study of the large N behavior of the matrix function

$$(10.1) \quad \mathcal{F}_N(\beta) = (I_n \otimes P_N V_N^*) R_N(\beta) (I_n \otimes V_N P_N^*).$$

We start with the special case in which X_N is replaced by a standard G.U.E.. The following proposition is a consequence of the results in Section 9. Thus, suppose that $(X_N^g)_{N \in \mathbb{N}}$ is a sequence of standard G.U.E. ensembles and we set $R_N^g(\beta) = (\beta \otimes I_n - L(C_N, X_N^g/\sqrt{N}))^{-1}$, and

$$(10.2) \quad \mathcal{F}_N^g(\beta) = (I_n \otimes P_N V_N^*) R_N^g(\beta) (I_n \otimes V_N P_N^*).$$

Proposition 10.1. *We have*

$$\lim_{N \rightarrow \infty} \mathbb{E}(\mathcal{F}_N^g(\beta)) = (\omega(\beta) - s\gamma_1)^{-1} \otimes I_p$$

for every $\beta \in \mathbb{H}^+(M_n(\mathbb{C}))$.

Proof. Since G.U.E. ensembles are invariant under unitary conjugation we may, and do, assume that $V_N = I_N$ for every $N \in \mathbb{N}$. For every $\beta \in \mathbb{H}^+(M_N(\mathbb{C}))$ we have

$$\mathbb{E}(\mathcal{F}_N^g(\beta)) = \mathbb{E}(\mathcal{F}_N^g(\beta) \mathbf{1}_{\|X_N^g/\sqrt{N}\| \leq 3}) + \mathbb{E}(\mathcal{F}_N^g(\beta) \mathbf{1}_{\|X_N^g/\sqrt{N}\| > 3}),$$

and the second term is at most $\|(\Im \beta)^{-1}\| \mathbb{P}(\|X_N^g/\sqrt{N}\| > 3)$. As shown by Bai and Yin [6], this number tends to zero as $N \rightarrow \infty$. To estimate the first term, we recall that $X_N^g/\sqrt{N} = U_N D_N U_N^*$, where U_N is a random matrix uniformly distributed in

$U(N)$ and D_N is a random diagonal matrix, independent from U_N , whose empirical spectral measure converges almost surely to $\nu_{0,1}$ as $N \rightarrow \infty$. Thus, we can write

$$\begin{aligned} & \mathbb{E}(\mathcal{F}_N^g(\beta) \mathbf{1}_{\|X_N^g/\sqrt{N}\| \leq 3}) \\ &= \int_{\Omega} \int_{\Omega} (I_n \otimes P_N) R_N((\xi_1, \xi_2), \beta) (I_n \otimes P_N^*) d\mathbb{P}(\xi_1) \mathbf{1}_{\|D_N(\xi_2)\| \leq 3} d\mathbb{P}(\xi_2), \\ & R_N((\xi_1, \xi_2), \beta) := (\beta \otimes I_N - L(C_N, U_N(\xi_1) D_N(\xi_2) U_n(\xi_1)^*))^{-1}. \end{aligned}$$

Proposition 8.10 can be applied for almost every ξ_2 and it shows that

$$\int_{\Omega} (I_n \otimes P_N) R_N((\xi_1, \xi_2), \beta) (I_n \otimes P_N^*) d\mathbb{P}(\xi_1) \mathbf{1}_{\|D_N(\xi_2)\| \leq 3}$$

converges to $(\omega(\beta) - s\gamma_1)^{-1} \otimes I_p$. The proposition follows now from an application of the dominated convergence theorem. \square

Passing to arbitrary Wigner matrices requires an approximation procedure from [12, Section 2]. For every $\varepsilon > 0$, there exist random selfadjoint matrices $X_N(\varepsilon) = [(X(\varepsilon))_{ij}]_{1 \leq i, j \leq N}$ such that

- (H1) the variables $\sqrt{2}\Re X_{ij}(\varepsilon)$, $\sqrt{2}\Im X_{ij}(\varepsilon)$, $X_{ii}(\varepsilon)$, $i, j \in \mathbb{N}$, $i < j$, are independent, centered with variance 1 and satisfy a Poincaré inequality with common constant $C_{PI}(\varepsilon)$,
- (H2) for every $m \in \mathbb{N}$,

$$(10.3) \quad \sup_{(i,j) \in \mathbb{N}^2} \mathbb{E}(|X_{ij}(\varepsilon)|^m) < +\infty,$$

and almost surely for large N ,

$$\left\| \frac{X_N - X_N(\varepsilon)}{\sqrt{N}} \right\| \leq \varepsilon.$$

Set

$$(10.4) \quad R_N^\varepsilon(\beta) = (\beta \otimes I_N - L(C_N, X_N(\varepsilon)/\sqrt{N}))^{-1}$$

and

$$(10.5) \quad \mathcal{F}_N^\varepsilon(\beta) = (I_n \otimes P_N V_N^*) R_N^\varepsilon(\beta) (I_n \otimes V_N P_N^*)$$

for $\beta \in \mathbb{H}^+(M_n(\mathbb{C}^N))$. It readily follows that, almost surely for large N ,

$$(10.6) \quad \|\mathcal{F}_N^\varepsilon(\beta) - \mathcal{F}_N(\beta)\| \leq \varepsilon \|\gamma_2\| \|(\Im \beta)^{-1}\|^2.$$

Properties (H1) and (H2) imply that, for every $\varepsilon > 0$,

$$\forall (i, j) \in \mathbb{N}^2, \kappa_1^{i,j,\varepsilon} = 0, \kappa_2^{i,j,\varepsilon} = 1,$$

$$\forall (i, j) \in \mathbb{N}^2, i \neq j, \tilde{\kappa}_1^{i,j,\varepsilon} = 0, \tilde{\kappa}_2^{i,j,\varepsilon} = 1,$$

and for any $m \in \mathbb{N} \setminus \{0\}$,

$$(10.7) \quad \sup_{(i,j) \in \mathbb{N}^2} |\kappa_m^{i,j,\varepsilon}| < +\infty, \quad \sup_{(i,j) \in \mathbb{N}^2} |\tilde{\kappa}_m^{i,j,\varepsilon}| < +\infty,$$

where for $i \neq j$, $(\kappa_m^{i,j,\varepsilon})_{m \geq 1}$ and $(\tilde{\kappa}_m^{i,j,\varepsilon})_{m \geq 1}$ denote the classical cumulants of $\sqrt{2}\Re X_{ij}(\varepsilon)$ and $\sqrt{2}\Im X_{ij}(\varepsilon)$ respectively, and $(\kappa_m^{i,i,\varepsilon})_{m \geq 1}$ denote the classical cumulants of $X_{ii}(\varepsilon)$ (we set $(\tilde{\kappa}_m^{i,i,\varepsilon})_{m \geq 1} \equiv 0$).

We use the following notation for an arbitrary matrix $M \in M_n(\mathbb{C}) \otimes M_N(\mathbb{C})$:

$$(10.8) \quad M^{ij} = (\text{Id}_{M_n(\mathbb{C})} \otimes \text{Tr}_N) (M (I_n \otimes \hat{e}_{j,i})) \in M_n(\mathbb{C}),$$

and

$$M_{ij} = (\text{Tr}_n \otimes \text{Id}_{M_N(\mathbb{C})})(M(e_{j,i} \otimes I_N)) \in M_N(\mathbb{C}),$$

where $e_{j,i}$ (resp. $\hat{e}_{j,i}$) denotes the $n \times n$ (resp. $N \times N$) matrix whose unique nonzero entry equals 1 and occurs in row j and column i .

Proposition 10.2. *There exists a polynomial P_ε in one variable with nonnegative coefficients such that for all large N , for every $v, u \in \{1, \dots, N\}$, for every $\beta \in \mathbb{H}^+(M_n(\mathbb{C}))$, and for every deterministic $B_N^{(1)}, B_N^{(2)} \in M_n(\mathbb{C}) \otimes M_N(\mathbb{C})$ such that $\|B_N^{(1)}\| \leq 1$ and $\|B_N^{(2)}\| \leq 1$, we have*

$$(10.9) \quad \left\| \mathbb{E}(B_N^{(1)} R_N^g(\beta) B_N^{(2)})^{vu} - \mathbb{E}(B_N^{(1)} R_N^\varepsilon(\beta) B_N^{(2)})^{vu} \right\| \leq \frac{1}{\sqrt{N}} P_\varepsilon(\|(\Im \beta)^{-1}\|),$$

and

$$(10.10) \quad \left\| \mathbb{E}(\mathcal{F}_N^g(\beta)) - \mathbb{E}(\mathcal{F}_N^\varepsilon(\beta)) \right\| \leq \frac{1}{\sqrt{N}} P_\varepsilon(\|(\Im w)^{-1}\|).$$

The proof uses a well-known lemma.

Lemma 10.3. *Let Z be a real-valued random variable such that $\mathbb{E}(|Z|^{p+2}) < \infty$. Let $\phi: \mathbb{R} \rightarrow \mathbb{C}$ be a function whose first $p+1$ derivatives are continuous and bounded. Then,*

$$(10.11) \quad \mathbb{E}(Z\phi(Z)) = \sum_{a=0}^p \frac{\kappa_{a+1}}{a!} \mathbb{E}(\phi^{(a)}(Z)) + \eta,$$

where κ_a are the cumulants of Z , $|\eta| \leq C \sup_t |\phi^{(p+1)}(t)| \mathbb{E}(|Z|^{p+2})$, and C only depends on p .

Proof of Proposition 10.2. Following the approach of [38, Ch. 18 and 19] we introduce the interpolation matrix $X_\varepsilon(\alpha) = \cos \alpha X_N(\varepsilon) + \sin \alpha Y_N$, $\alpha \in [0, \pi/2]$, and the corresponding resolvent

$$R_N^{\varepsilon, \alpha}(\beta) = (\beta \otimes I_N - L(C_N, X_\varepsilon(\alpha)/\sqrt{N}))^{-1}, \quad \beta \in \mathbb{H}^+(M_n(\mathbb{C})).$$

We have

$$B_N^{(1)}(\mathbb{E}R_N^g(\beta) - \mathbb{E}R_N^\varepsilon(\beta))B_N^{(2)} = \int_0^{\pi/2} \mathbb{E} \left(B_N^{(1)} \frac{\partial}{\partial \alpha} R_N^{\varepsilon, \alpha}(\beta) B_N^{(2)} \right) d\alpha,$$

$$\frac{\partial}{\partial \alpha} R_N^{\varepsilon, \alpha}(\beta) = R_N^{\varepsilon, \alpha}(\beta) \gamma_2 \otimes \left(\cos \alpha \frac{X_N^g}{\sqrt{N}} - \sin \alpha \frac{X_N(\varepsilon)}{\sqrt{N}} \right) R_N^{\varepsilon, \alpha}(\beta).$$

Define a basis of the real vector space of selfadjoint matrices in $M_N(\mathbb{C})$ as follows:

$$\tilde{e}_{j,j} = \hat{e}_{j,j}, 1 \leq j \leq N,$$

$$\tilde{e}_{j,k} =: \frac{1}{\sqrt{2}}(\hat{e}_{j,k} + \hat{e}_{k,j}), 1 \leq j < k \leq N,$$

$$\tilde{f}_{j,k} =: \frac{i}{\sqrt{2}}(\hat{e}_{j,k} - \hat{e}_{k,j}), 1 \leq j < k \leq N.$$

In the following calculation, we write simply $R_N^{\varepsilon;R}$ in place of $R_N^{\varepsilon,\alpha}(\beta)$ and X^g in place of X_N^g :

$$\begin{aligned} \frac{\partial}{\partial \alpha} R_N^{\varepsilon,\alpha} &= \frac{1}{\sqrt{N}} \sum_{k=1}^N (-\sin \alpha X_{kk}(\varepsilon) + \cos \alpha X_{kk}^g) R_N^{\varepsilon,\alpha} \beta_2 \otimes \tilde{e}_{k,k} R_N^{\varepsilon,\alpha} \\ &\quad + \frac{1}{\sqrt{N}} \sum_{1 \leq j < k \leq N} (-\sin \alpha \sqrt{2} \Re X_{jk}(\varepsilon) + \cos \alpha \sqrt{2} \Re X_{jk}^g) R_N^{\varepsilon,\alpha} \gamma_2 \otimes \tilde{e}_{j,k} R_N^{\varepsilon,\alpha} \\ &\quad + \frac{1}{\sqrt{N}} \sum_{1 \leq j < k \leq N} (-\sin \alpha \sqrt{2} \Im X_{jk}(\varepsilon) + \cos \alpha \sqrt{2} \Im X_{jk}^g) R_N^{\varepsilon,\alpha} \gamma_2 \otimes \tilde{f}_{j,k} R_N^{\varepsilon,\alpha}. \end{aligned}$$

Next, we apply Lemma 10.3 with $p = 3$ for $1 \leq k \leq N$, $j < k$, to each random variable Z in the set

$$\{\sqrt{2} \Re X_{jk}(\varepsilon), \sqrt{2} \Re X_{jk}^g, \sqrt{2} \Im X_{jk}(\varepsilon), \sqrt{2} \Im X_{jk}^g, X_{kk}(\varepsilon), X_{kk}^g, j < k\}$$

and to each ϕ in the set

$$\begin{aligned} &\{\text{Tr}(B_N^{(1)} R_N^{\varepsilon,\alpha} \gamma_2 \otimes \tilde{e}_{k,k} R_N^{\varepsilon,\alpha} B_N^{(2)} e_{q,l} \otimes \hat{e}_{u,v}), \text{Tr}(B_N^{(1)} R_N^{\varepsilon,\alpha} \gamma_2 \otimes \tilde{e}_{j,k} R_N^{\varepsilon,\alpha} B_N^{(2)} e_{q,l} \otimes \hat{e}_{u,v}), \\ &\quad \text{Tr}(B_N^{(1)} R_N^{\varepsilon,\alpha} \gamma_2 \otimes \tilde{f}_{j,k} R_N^{\varepsilon,\alpha} B_N^{(2)} e_{q,l} \otimes \hat{e}_{u,v}): 1 \leq u, v \leq N, 1 \leq q, l \leq m\}. \end{aligned}$$

Setting now $B = B_N^{(2)} e_{q,l} \otimes \hat{e}_{u,v} B_N^{(1)}$, we have:

$$\begin{aligned} \text{Tr}\left(\frac{\partial}{\partial \alpha} R_N^{\varepsilon,\alpha} B\right) &= \frac{C(\alpha)}{N^{\frac{3}{2}}} \sum_{k=1}^N \kappa_3^{k,k,\varepsilon} \text{Tr}(R_N^{\varepsilon,\alpha} \gamma_2 \otimes \tilde{e}_{k,k} R_N^{\varepsilon,\alpha} \gamma_2 \otimes \tilde{e}_{k,k} R_N^{\varepsilon,\alpha} \gamma_2 \otimes \tilde{e}_{k,k} R_N^{\varepsilon,\alpha} B) \\ &\quad + \frac{C(\alpha)}{N^{\frac{3}{2}}} \sum_{1 \leq j < k < N} \kappa_3^{j,k,\varepsilon} \text{Tr}(R_N^{\varepsilon,\alpha} \gamma_2 \otimes \tilde{e}_{j,k} R_N^{\varepsilon,\alpha} \gamma_2 \otimes \tilde{e}_{j,k} R_N^{\varepsilon,\alpha} \gamma_2 \otimes \tilde{e}_{j,k} R_N^{\varepsilon,\alpha} B) \\ &\quad + \frac{C(\alpha)}{N^{\frac{3}{2}}} \sum_{1 \leq j < k < N} \tilde{\kappa}_3^{j,k,\varepsilon} \text{Tr}(R_N^{\varepsilon,\alpha} \gamma_2 \otimes \tilde{f}_{j,k} R_N^{\varepsilon,\alpha} \gamma_2 \otimes \tilde{f}_{j,k} R_N^{\varepsilon,\alpha} \gamma_2 \otimes \tilde{f}_{j,k} R_N^{\varepsilon,\alpha} B) \\ &\quad + \frac{\tilde{C}(\alpha)}{N^2} \sum_{k=1}^N \kappa_4^{k,k,\varepsilon} \text{Tr}(R_N^{\varepsilon,\alpha} \gamma_2 \otimes \tilde{e}_{k,k} R_N^{\varepsilon,\alpha} \gamma_2 \otimes \tilde{e}_{k,k} R_N^{\varepsilon,\alpha} \gamma_2 \otimes \tilde{e}_{k,k} R_N^{\varepsilon,\alpha} \gamma_2 \otimes \tilde{e}_{k,k} R_N^{\varepsilon,\alpha} B) \\ &\quad + \frac{\tilde{C}(\alpha)}{N^2} \sum_{1 \leq j < k < N} \kappa_4^{j,k,\varepsilon} \text{Tr}(R_N^{\varepsilon,\alpha} \gamma_2 \otimes \tilde{e}_{j,k} R_N^{\varepsilon,\alpha} \gamma_2 \otimes \tilde{e}_{j,k} R_N^{\varepsilon,\alpha} \gamma_2 \otimes \tilde{e}_{j,k} R_N^{\varepsilon,\alpha} \gamma_2 \otimes \tilde{e}_{j,k} R_N^{\varepsilon,\alpha} B) \\ &\quad + \frac{\tilde{C}(\alpha)}{N^2} \sum_{1 \leq j < k < N} \tilde{\kappa}_4^{j,k,\varepsilon} \text{Tr}(R_N^{\varepsilon,\alpha} \gamma_2 \otimes \tilde{f}_{j,k} R_N^{\varepsilon,\alpha} \gamma_2 \otimes \tilde{f}_{j,k} R_N^{\varepsilon,\alpha} \gamma_2 \otimes \tilde{f}_{j,k} R_N^{\varepsilon,\alpha} \gamma_2 \otimes \tilde{f}_{j,k} R_N^{\varepsilon,\alpha} B) + \delta \\ &= I_1 + I_2 + I_3 + I_4 + I_5 + I_6 + \delta, \end{aligned}$$

where

$$|\delta| \leq C_\varepsilon \frac{\|(\Im w)^{-1}\|^6}{\sqrt{N}},$$

for some $C_\varepsilon \geq 0$, while $C(\alpha)$ and $\tilde{C}(\alpha)$ are polynomials in $\cos \alpha$ and $\sin \alpha$. In the following, C_ε may vary from line to line. It is clear that

$$|I_1| \leq C_\varepsilon \frac{\|(\Im \beta)^{-1}\|^4}{\sqrt{N}}, \quad \text{and} \quad |I_4| \leq C_\varepsilon \frac{\|(\Im w)^{-1}\|^5}{N}.$$

Next, I_2 and I_3 are a finite linear combinations of terms of the form

$$(10.12) \quad \frac{C(\alpha)}{N^{\frac{3}{2}}} \sum_{j,k \in \mathcal{E}} C^{j,k,\varepsilon} \text{Tr}_n(\gamma_2(R_N^{\varepsilon,\alpha})^{p_1 p_2} \gamma_2(R_N^{\varepsilon,\alpha})^{p_3 p_4} \gamma_2(R_N^{\varepsilon,\alpha} B_N^{(2)})^{p_5 u} e_{q,l}(B_N^{(1)} R_N^{\varepsilon,\alpha})^{v p_6}),$$

where \mathcal{E} is some subset of $\{1, \dots, N\}^2$, $C^{j,k,\varepsilon} \in \{\kappa_3^{j,k,\varepsilon}, \tilde{\kappa}_3^{j,k,\varepsilon}\}$, and (p_1, \dots, p_6) is a permutation of (k, k, k, j, j, j) . The two following cases hold:

- $p_5 = p_6$, in which case Lemma 11.1 yields

$$\begin{aligned} & \left\| \sum_{j,k \in \mathcal{E}} C^{j,k,\varepsilon} \text{Tr}_n(\gamma_2(R_N^{\varepsilon,\alpha})^{p_1 p_2} \gamma_2(R_N^{\varepsilon,\alpha})^{p_3 p_4} \gamma_2(R_N^{\varepsilon,\alpha} B_N^{(2)})^{p_5 u} e_{q,l}(B_N^{(1)} R_N^{\varepsilon,\alpha})^{v p_6}) \right\| \\ & \leq C_\varepsilon \|\gamma_2\|^3 \|(\Im\beta)^{-1}\|^2 N \sum_{j=1}^N \|(B_N^{(1)} R_N^{\varepsilon,\alpha})^{v j}\| \|(R_N^{\varepsilon,\alpha} B_N^{(2)})^{j u}\| \\ & \leq C_\varepsilon \|\gamma_2\|^3 \|(\Im\beta)^{-1}\|^2 N \left(\sum_{j=1}^N \|(B_N^{(1)} R_N^{\varepsilon,\alpha})^{v j}\|^2 \right)^{1/2} \left(\sum_{j=1}^N \|(R_N^{\varepsilon,\alpha} B_N^{(2)})^{j u}\|^2 \right)^{1/2} \\ & \leq C_\varepsilon \|\gamma_2\|^3 \|(\Im\beta)^{-1}\|^4 n N. \end{aligned}$$

- $p_5 \neq p_6$, in which case Lemma 11.1 yields

$$\begin{aligned} & \left\| \sum_{j,k \in \mathcal{E}} C^{j,k,\varepsilon} \text{Tr}_n(\gamma_2(R_N^{\varepsilon,\alpha})^{p_1 p_2} \gamma_2(R_N^{\varepsilon,\alpha})^{p_3 p_4} \gamma_2(R_N^{\varepsilon,\alpha} B_N^{(2)})^{p_5 u} e_{q,l}(B_N^{(1)} R_N^{\varepsilon,\alpha})^{v p_6}) \right\| \\ & \leq C_\varepsilon \|\gamma_2\|^3 \|(\Im\beta)^{-1}\|^2 \left(\sum_{k=1}^N \|(R_N^{\varepsilon,\alpha} B_N^{(2)})^{k u}\| \right) \left(\sum_{j=1}^N \|(B_N^{(1)} R_N^{\varepsilon,\alpha})^{v j}\| \right) \\ & \leq C_\varepsilon N \|\gamma_2\|^3 \|(\Im\beta)^{-1}\|^2 \left(\sum_{k=1}^N \|(R_N^{\varepsilon,\alpha} B_N^{(2)})^{k u}\|^2 \right)^{1/2} \left(\sum_{j=1}^N \|(B_N^{(1)} R_N^{\varepsilon,\alpha})^{v j}\|^2 \right)^{1/2} \\ & \leq C_\varepsilon n N \|\gamma_2\|^3 \|(\Im\beta)^{-1}\|^4. \end{aligned}$$

We see now that $|I_j| \leq C_\varepsilon \|(\Im\beta)^{-1}\|^4 / \sqrt{N}$ for $j = 2, 3$. Finally, I_5 and I_6 are finite linear combinations of terms of the form

$$(10.13) \quad \frac{\tilde{C}(\alpha)}{N^2} \sum_{j,k \in \mathcal{E}} C^{j,k,\varepsilon} \text{Tr}_n(\gamma_2(R_N^{\varepsilon,\alpha})^{p_1 p_2} \gamma_2(R_N^{\varepsilon,\alpha})^{p_3 p_4} \gamma_2(R_N^{\varepsilon,\alpha})^{p_5 p_6} \gamma_2(R_N^{\varepsilon,\alpha} B_N^{(2)})^{p_7 u} e_{q,l}(B_N^{(1)} R_N^{\varepsilon,\alpha})^{v p_8})$$

where \mathcal{E} is some subset of $\{1, \dots, N\}^2$, $C^{j,k,\varepsilon} \in \{\kappa_4^{j,k,\varepsilon}, \tilde{\kappa}_4^{j,k,\varepsilon}\}$ and (p_1, \dots, p_6) is a permutation of (k, k, k, k, j, j, j, j) . Lemma 11.1 shows that the norm of such a term can be estimated by

$$\begin{aligned} & \frac{1}{N^2} C_\varepsilon N \|\gamma_2\|^4 \|(\Im\beta)^{-1}\|^4 \left(\sum_{k=1}^N \|(R_N^{\varepsilon,\alpha} B_N^{(2)})^{k u}\| \right) \\ & \leq \frac{1}{N^2} C_\varepsilon N^{\frac{3}{2}} \|\gamma_2\|^4 \|(\Im\beta)^{-1}\|^4 \left(\sum_{k=1}^N \|(R_N^{\varepsilon,\alpha} B_N^{(2)})^{k u}\|^2 \right)^{1/2} \leq \frac{1}{N^2} C_\varepsilon \sqrt{n} N^{\frac{3}{2}} \|\gamma_2\|^4 \|(\Im\beta)^{-1}\|^5. \end{aligned}$$

It follows that $|I_j| \leq C_\varepsilon \|(\Im w)^{-1}\|^5 / \sqrt{N}$ for $j = 5, 6$. The proposition follows. \square

We show next that $\mathcal{F}_N^\varepsilon(\beta)$ is close to its expected value. This result uses concentration inequalities in the presence of a Poincaré inequality. We recall that if the law of a random variable X satisfies the Poincaré inequality with constant C and $\alpha \in \mathbb{R} \setminus \{0\}$, then the law of αX satisfies the Poincaré inequality with constant $\alpha^2 C$. Moreover, suppose that the probability measures μ_1, \dots, μ_r on \mathbb{R} satisfy the Poincaré inequality with constants C_1, \dots, C_r respectively. Then the product measure $\mu_1 \otimes \dots \otimes \mu_r$ on \mathbb{R}^r satisfies the Poincaré inequality with constant $C = \max\{C_1, \dots, C_r\}$. That is, if $f : \mathbb{C}^r \rightarrow \mathbb{R}$ is an arbitrary differentiable function such that f and its gradient $\text{grad} f$ are square integrable relative to $\mu_1 \otimes \dots \otimes \mu_r$, then

$$\mathbb{V}(f) \leq C \int_{\mathbb{R}^r} \|\text{grad} f\|_2^2 d\mu_1 \otimes \dots \otimes \mu_r.$$

Here $\mathbb{V}(f) = \int |f - \int f d\mu_1 \otimes \dots \otimes \mu_r|^2 d\mu_1 \otimes \dots \otimes \mu_r$ (see [28, Theorem 2.5]).

We use the following concentration result (see [3, Lemma 4.4.3 and Exercise 4.4.5] or [32, Chapter 3]).

Lemma 10.4. *Let \mathbb{P} be a probability measure on \mathbb{R}^r which satisfies a Poincaré inequality with constant C . Then there exist $K_1 > 0$ and $K_2 > 0$ such that, for every Lipschitz function F on \mathbb{R}^r with Lipschitz constant $|F|_{\text{Lip}}$, and for every $\varepsilon > 0$, we have*

$$\mathbb{P}(|F - \mathbb{E}_{\mathbb{P}}(F)| > \varepsilon) \leq K_1 \exp\left(-\frac{\varepsilon}{K_2 \sqrt{C} |F|_{\text{Lip}}}\right).$$

The following result is similar to Proposition 8.4(1).

Proposition 10.5. *Suppose that $T_N, S_N \in M_N(\mathbb{C})$ are contractions of uniformly bounded rank. Given $\varepsilon > 0$ and $\beta \in \mathbb{H}^+(M_n(\mathbb{C}))$, almost surely*

$$\lim_{N \rightarrow \infty} \|(I_n \otimes T_N)(R_N^\varepsilon(\beta) - \mathbb{E}R_N^\varepsilon(\beta))(I_n \otimes S_N)\| = 0.$$

Proof. As in the proof of Proposition 8.4, it suffices to consider the case in which T_N and S_N are contractions of rank 1. Write I_n as a sum $Q_1 + \dots + Q_n$ of rank 1 projections. Then the norm in the statement is at most equal to

$$\sum_{j,k=1}^n \|(Q_j \otimes T_N)(R_N^\varepsilon(\beta) - \mathbb{E}R_N^\varepsilon(\beta))(Q_k \otimes S_N)\| = \sum_{j,k=1}^n |\text{Tr}(R_N^\varepsilon(\beta) - \mathbb{E}R_N^\varepsilon(\beta))Q_{j,k}|,$$

where each $Q_{j,k}$ is a contraction of rank 1. Given a selfadjoint matrix $Z_N \in M_N(\mathbb{C})$, we set $R(Z, \beta) = (\beta \otimes I_N - L(C_N, Z))^{-1}$ and $f_{N,j,k}(Z) = \text{Tr}R(Z, \beta)Q_{j,k}$. We have

$$f_N(Z_1) - f_N(Z_2) = \text{Tr}(R(Z_1, \beta)(\gamma_2 \otimes (Z_1 - Z_2))R(Z_2, \beta)Q_{j,k}),$$

and thus

$$\begin{aligned} |f_N(Z_1) - f_N(Z_2)| &\leq \|\gamma_2 \otimes (Z_1 - Z_2)\|_2 \|R(Z_1, \beta)Q_{j,k}R(Z_2, \beta)\|_2 \\ &\leq \|\gamma_2\|_2 \|(Z_1 - Z_2)\|_2 \|(\mathfrak{S}\beta)^{-1}\|^2. \end{aligned}$$

An application of Lemma 10.4 and of the comment preceding it yield

$$\mathbb{P}(|\text{Tr}(R_N^\varepsilon(\beta) - \mathbb{E}R_N^\varepsilon(\beta))Q_{j,k}| > \delta) \leq 2 \exp(-CN^{1/2} \|(\mathfrak{S}\beta)^{-1}\|^{-2} \delta)$$

for every $\delta > 0$, with a constant C that does not depend in N, j , or k . The proposition follows by an application of the Borel-Cantelli lemma. \square

Corollary 10.6. *For every $\beta \in \mathbb{H}^+(M_n(\mathbb{C}))$ and every $\varepsilon > 0$ we have, almost surely,*

$$(10.14) \quad \lim_{N \rightarrow \infty} \mathcal{F}_N^\varepsilon(\beta) - \mathbb{E}(\mathcal{F}_N^\varepsilon(\beta)) = 0.$$

Observe now that

$$\begin{aligned} \mathcal{F}_N(\beta) - (\omega(\beta) - s\gamma_1)^{-1} \otimes I_p &= \mathcal{F}_N(\beta) - \mathcal{F}_N^\varepsilon(\beta) + \mathcal{F}_N^\varepsilon(\beta) - \mathbb{E}(\mathcal{F}_N^\varepsilon(\beta)) \\ &\quad + \mathbb{E}(\mathcal{F}_N^\varepsilon(\beta)) - \mathbb{E}(\mathcal{F}_N^g(\beta)) + \mathbb{E}(\mathcal{F}_N^g(\beta)) - (\omega(\beta) - s\gamma_1)^{-1} \otimes I_p. \end{aligned}$$

We let $N \rightarrow \infty$ and then $\varepsilon \rightarrow 0$ and apply (10.6), (10.14), Proposition 10.2, and Proposition 10.1 to obtain the following result.

Theorem 10.7. *For every $\beta \in \mathbb{H}^+(M_n(\mathbb{C}))$ we have, almost surely, when N goes to infinity, $\lim_{N \rightarrow \infty} \mathcal{F}_N(\beta) = (\omega(\beta) - s\gamma_1)^{-1} \otimes I_p$.*

Everything is now in place for completing the argument.

Proof of Theorem 6.3. We noted earlier that $\omega(ze_{1,1} - \gamma_0)$ is analytic in $\mathbb{C} \setminus \sigma(P(a, b))$. For fixed $t \in \mathbb{R} \setminus \sigma(P(a, b))$, set $\Psi(\beta) = \beta \otimes 1 - L(a, b)$ and $\Psi_N(\beta) = \beta \otimes 1 - L(C_N, X_N/\sqrt{N})$. According to Lemma 4.2, $\Psi(te_{1,1} - \gamma_0)$ is invertible, and thus there exists $\delta > 0$ such that

$$(10.15) \quad d(0, \sigma(\Psi(te_{1,1} - \gamma_0))) \geq \delta > 0.$$

Theorem 2.3 and Proposition 2.2 imply that, almost surely, for every complex polynomial Q in one variable we have

$$\lim_{N \rightarrow \infty} \|Q(\Psi_N(te_{1,1} - \gamma_0))\| = \|Q(\Psi(te_{1,1} - \gamma_0))\|.$$

Asymptotic freeness implies that almost surely, for every complex polynomial Q in one variable,

$$\lim_{N \rightarrow \infty} (\text{tr}_n \otimes \text{tr}_N)(Q(\Psi_N(te_{1,1} - \gamma_0))) = (\text{tr}_n \otimes \tau)(Q(\Psi(te_{1,1} - \gamma_0))).$$

Thus, denoting the Hausdorff distance by d_H , we deduce that almost surely for N large enough,

$$d_H(\sigma(\Psi_N(te_{1,1} - \gamma_0)), \sigma(\Psi(te_{1,1} - \gamma_0))) \leq \delta/4.$$

Note that $\Psi_N(te_{1,1} - \gamma_0)$ is selfadjoint. For an arbitrary $\beta \in M_n(\mathbb{C})$, we have

$$\|\Psi_N(te_{1,1} - \gamma_0) - \Psi_N(\beta)\| \leq \|(te_{1,1} - \gamma_0) - \beta\|.$$

It follows from (10.15) that, almost surely for all large N , if $\|\beta - (te_{1,1} - \gamma_0)\| < \delta/4$, then

$$d(0, \sigma(\Psi_N(\beta))) \geq \delta/2.$$

Moreover, denoting by $s_1(M)$ the smallest singular value of an arbitrary matrix M ,

$$s_1(\Psi_N(\beta)) \geq s_1(\Psi_N(te_{1,1} - \gamma_0)) - \|\beta - (te_{1,1} - \gamma_0)\|.$$

Thus, almost surely for all large N , provided that $\|\beta - (te_{1,1} - \gamma_0)\| < \delta/4$, we have $\|\mathcal{F}_N(\beta)\| = \|(\Psi_N(\beta))^{-1}\| \leq 4/\delta$. In other words, almost surely, the family $\{\mathcal{F}_N\}_{N \in \mathbb{N}}$ is normal in a neighborhood of $te_{1,1} - \gamma_0$. According to Theorem 10.7, for any $\beta \in \mathbb{H}^+(M_n(\mathbb{C}))$, almost surely \mathcal{F}_N converges towards $(\omega(\beta) - s\gamma_1)^{-1} \otimes I_p$. Set

$$\Lambda = \{w \in M_n(\mathbb{C}), \|w - (te_{1,1} - \gamma_0)\| < \delta/4, \Im w > 0\}.$$

Almost surely for any $w \in \Lambda$ such that $\Im w \in M_n(\mathbb{Q})$ and $\Re w \in M_n(\mathbb{Q})$, $\mathcal{F}_N(w)$ converges towards $(\omega(w) - s\gamma_1)^{-1} \otimes I_p$. The Vitali-Montel convergence theorem

implies that that almost surely \mathcal{F}_N converges towards a holomorphic function on $\{w \in M_n(\mathbb{C}), \|w - (te_{1,1} - \gamma_0)\| < \delta/4\}$, and, in particular, $\mathcal{F}_N(ze_{1,1} - \beta_0)$ converges for any $z \in \mathbb{C}$ such that $|z - t| < \delta/4$ towards $(\omega((ze_{1,1} - \gamma_0) - s\gamma_1)^{-1} \otimes I_p)$.

Now, the Hurwitz theorem on zeros of analytic functions implies that, almost surely for large N , the function $F_N(ze_{1,1} - \gamma_0) = \det(I_n \otimes I_p - \mathcal{F}_N(ze_{1,1} - \gamma_0))$ has as many zeros in a neighborhood of t as the function

$$\det(I_n \otimes I_p - (\gamma_1 \otimes T)((\omega((ze_{1,1} - \gamma_0) - s\gamma_1)^{-1} \otimes I_p)).$$

Now, note that

$$\begin{aligned} \gamma_1((\omega(ze_{11} - \gamma_0) - s\gamma_1)^{-1} \otimes T - I_n \otimes I_p) \\ = (\gamma_1 \otimes T - s\gamma_1 \otimes I_p) (\omega(ze_{11} - \gamma_0) - s\gamma_1)^{-1} \otimes I_p - I_n \otimes I_p \\ = (\gamma_1 \otimes T - \omega(ze_{11} - \gamma_0) \otimes I_p) (\omega(ze_{11} - \gamma_0) - s\gamma_1)^{-1} \otimes I_p \end{aligned}$$

and

$$\omega(ze_{1,1} - \gamma_0) \otimes I_p - \gamma_1 \otimes T = \sum_{i=1}^p (\omega(ze_{11} - \gamma_0) - \gamma_1 \theta_i) \otimes e_{i,i}.$$

Therefore

$$\det(\omega(ze_{11} - \gamma_0) \otimes I_p - \gamma_1 \otimes T) = \prod_{i=1}^p \det(\omega(ze_{11} - \gamma_0) - \gamma_1 \theta_i).$$

The theorem follows. \square

11. APPENDIX

The following result is [12, Lemma 8.1].

Lemma 11.1. *For any matrix $M \in M_n(\mathbb{C}) \otimes M_N(\mathbb{C})$,*

$$(11.1) \quad \frac{1}{N} \sum_{k,l=1}^N \|M^{kl}\|^2 \leq n \|M\|^2$$

and for any fixed k ,

$$(11.2) \quad \sum_{l=1}^N \|M^{lk}\|^2 \leq n \|M\|^2$$

and

$$(11.3) \quad \sum_{l=1}^N \|M^{kl}\|^2 \leq n \|M\|^2,$$

where M^{kl} is defined by (10.8).

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INSTITUT DE MATHÉMATIQUES DE TOULOUSE; UMR5219; UNIVERSITÉ DE TOULOUSE; CNRS;
UPS, F-31062 TOULOUSE, FRANCE

E-mail address: `serban.belinschi@math.univ-toulouse.fr`

DEPARTMENT OF MATHEMATICS AND STATISTICS, INDIANA UNIVERSITY, BLOOMINGTON, IN
47405 U.S.A.

E-mail address: `bercovic@indiana.edu`

INSTITUT DE MATHÉMATIQUES DE TOULOUSE; UMR5219; UNIVERSITÉ DE TOULOUSE; CNRS;
UPS, F-31062 TOULOUSE, FRANCE

E-mail address: `mireille.capitaine@math.univ-toulouse.fr`