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SPECTRAL ANALYSIS OF MORSE-SMALE FLOWS I: CONSTRUCTION OF THE ANISOTROPIC SPACES

NGUYEN VIET DANG AND GABRIEL RIVIÈRE

ABSTRACT. We prove the existence of a discrete correlation spectrum for Morse-Smale flows acting on smooth forms on a compact manifold. This is done by constructing spaces of currents with anisotropic Sobolev regularity on which the Lie derivative has a discrete spectrum.

1. INTRODUCTION

Given a smooth (\mathcal{C}^∞), compact, oriented, boundaryless manifold M of dimension $n \geq 1$ and a smooth flow $\varphi^t : M \rightarrow M$, a basic question from dynamical systems is to understand the long time behaviour of the flow. There are many ways to approach this problem. For instance, one can define the *correlation function* :

$$(1) \quad C_{\psi_1, \psi_2}(t) := \int_M \varphi^{-t*}(\psi_1) \wedge \psi_2,$$

with $\psi_1 \in \Omega^k(M)$ and $\psi_2 \in \Omega^{n-k}(M)$. Then, if one can describe the limit of this quantity as $t \rightarrow +\infty$, then it gives some information on the weak limit of $\varphi^{-t*}(\psi_1)$ in the sense of currents. Studying directly the limit of $C_{\psi_1, \psi_2}(t)$ as $t \rightarrow +\infty$ is not often possible and one may first introduce its Laplace transform :

$$(2) \quad \hat{C}_{\psi_1, \psi_2}(z) := \int_0^{+\infty} e^{-tz} C_{\psi_1, \psi_2}(t) dt.$$

Note that this is well defined for $\text{Re}(z) > c$ with $c > 0$ depending only on the flow φ^t . Instead of studying the long time limit, one could then try to understand if this holomorphic function has a meromorphic extension to a larger half-plane. If so, the poles and their residues also give some informations on the long time dynamics of the flow. Therefore, in the sequel, the set of poles and residues of the Laplace transformed correlation functions will be called **correlation spectrum** of the flow, elements of this spectrum will be called **Ruelle-Pollicott resonances**. These kind of questions were for instance considered by Pollicott [36] and Ruelle [38] in the framework of Axiom A flows that we shall now discuss.

In fact, this type of problem is very hard at this level of generality and some assumptions on the nature of the flow should be made to obtain some nontrivial results. A natural situation where one may expect some answer is when some hyperbolicity is involved in the system, e.g. for Axiom A flows in the sense of Smale [40]. In that framework, one can decompose the nonwandering set of the flow into finitely many invariant hyperbolic subsets

$(\Lambda_j)_{j=1}^K$, which are called the basic sets of the flow. Examples of such flows are geodesic flows on negatively curved manifolds or gradient flows associated with a Morse function. As most of the time on a given orbit is spent in some neighborhood of these basic sets, it is natural to first restrict to test forms which are supported in a small neighborhood of a given Λ_j . For a slightly different correlation function associated to a Gibbs measure of Λ_i , Pollicott [36] and Ruelle [38] proved the meromorphic extension of the Laplace transform to some half-plane $\operatorname{Re}(z) \geq -\delta$ with $\delta > 0$. Their proof relies on the symbolic coding by Markov partitions of such flows that was constructed by Bowen [5]. In the last fifteen years, many progresses have been made towards this problem by adopting a slightly different point of view. Namely, one can observe that $\varphi^{-t*}(\psi_1)$ solves the following partial differential equation:

$$\boxed{\partial_t \psi = -\mathcal{L}_V \psi, \quad \psi(t=0) = \psi_1,}$$

where $\mathcal{L}_V = (d + \iota_V)^2$ is the Lie derivative along the vector field V associated with the flow φ^t . In particular, if one can find an appropriate Banach space on which $-\mathcal{L}_V$ has a discrete spectrum (with finite multiplicity) on the half-plane $\operatorname{Re}(z) \geq -\delta$ for some positive δ , then one can verify that $\hat{C}_{\psi_1, \psi_2}(z)$ has a meromorphic extension to the same half-plane. This approach has been initiated by Liverani [29] in the context of contact Anosov flows and it was further developed in [7, 22] which show among other things the meromorphic extension of $\hat{C}_{\psi_1, \psi_2}(z)$ to the entire complex plane for Anosov flows. In these references, one of the key ingredient is the construction of Banach spaces with anisotropic Hölder regularity on which $-\mathcal{L}_V$ has good spectral properties. Alternative spaces based on microlocal tools were developed by Faure, Sjöstrand and Tsujii [42, 17, 43] – see also [13, 18]. This complementary approach allowed to bring new perspectives on the fine structure of this correlation spectrum in the Anosov case. Coming back to the case of Axiom A flows, Dyatlov and Guillarmou proved that $\hat{C}_{\psi_1, \psi_2}(z)$ admits a meromorphic extension to \mathbb{C} provided that we only consider test forms which are compactly supported in a neighborhood of a fixed basic set Λ_j [12]. We should point that progresses for flows follow earlier results for hyperbolic diffeomorphisms that we will not discuss here. We refer the reader to the book of Baladi for a recent detailed account on that case [3].

If we look at the general framework of Axiom A flows, we already noticed that these different results do not say at first sight much things on the *global dynamics* of the flow. In fact, in all the results we mentioned so far, it is important that we restrict ourselves to test forms which are compactly supported near a fixed basic set Λ_j of the flow. In the case of geodesic flows on negatively curved manifolds, this restriction is of course artificial as there is only one basic set which covers the entire manifold. However, in general, there may be several basic sets that are far from covering the entire manifold and, if we remember that Axiom A flows arised as far-reaching generalizations of gradient flows associated with a Morse function [40], then understanding the global dynamics sounds also important as it provides some informations on the topology of the manifold [39, 19]. Several difficulties appear if we want to consider this global question and let us mention at least two of them: (1) the flow is not topologically transitive on M (while it is on a fixed Λ_j), (2) hyperbolicity only holds on the basic sets.

From Morse-Smale gradient flows to Morse–Smale flows. In the case of certain Morse-Smale gradient flows, the fact that $C_{\psi_1, \psi_2}(t)$ admits a limit as $t \rightarrow +\infty$ for *any* choice of ψ_1 and ψ_2 was proved by Harvey and Lawson [24, 25] – see also [28] for earlier related results. While the proof of Harvey and Lawson was based on the theory of currents à la Federer, we recently showed how to develop an appropriate *global* spectral theory for such gradient flows [9] and to derive a complete asymptotic expansion of the correlation function – see also [21] for related results in the context of quantum field theory. In particular, this shows that \hat{C}_{ψ_1, ψ_2} has a meromorphic extension to \mathbb{C} . Moreover, as a byproduct of our spectral analysis, we obtained a new spectral interpretation of the Thom-Smale-Witten complex as the kernel of the operator $-\mathcal{L}_V$ acting on certain anisotropic spaces of currents from which one can easily deduce the finiteness of Betti numbers, the Poincaré duality and the classical Morse inequalities.

The goal of the present work is to continue to *explore the global dynamics* of Axiom A flows by focusing on the particular case of Morse-Smale flows [39, 34] for which the basic sets are either closed orbits or fixed points. They also satisfy some transversality assumptions necessary to develop proper topological applications [39] – see section 3 for more details. These flows are somewhat intermediate between gradient flows of a Morse function and Axiom A flows [40]. Recall that Peixoto proved that, in dimension 2, these flows form an open and dense family of all smooth vector fields [35] while in higher dimension, Palis showed that they form an open subset of all smooth vector fields [33].

This article is the first in a series. Here, we develop a convenient *global* functional framework for Morse-Smale flows in order to prove that $\hat{C}_{\psi_1, \psi_2}(z)$ has a meromorphic extension to the entire complex plane. In [10], we will elaborate more on Morse-Smale flows and show how to give an explicit description of the poles and of residues of $\hat{C}_{\psi_1, \psi_2}(z)$ provided certain non resonance assumptions are satisfied. Finally, in [11], we will explain how to give topological interpretations for the correlation spectrum of a class of flows which have a proper *global* spectral theory that we call **microlocally tame**. This class contains Anosov and Morse–Smale flows. We will show how to extract Morse inequalities for Pollicott-Ruelle resonant states in the kernel of \mathcal{L}_V and we will also give some new identities relating regularized products of Pollicott-Ruelle resonances on the imaginary axis with a torsion function introduced by Fried which coincides with Reidemeister torsion when V is non singular [20].

2. STATEMENT OF THE MAIN RESULTS

In all the article, M will denote a smooth (\mathcal{C}^∞), compact, oriented manifold without boundary and of dimension $n \geq 1$.

2.1. Discrete correlation spectrum. Our main result shows the existence of a meromorphic extension to \mathbb{C} of $\hat{C}_{\psi_1, \psi_2}(z)$:

Theorem 2.1 (Resonances). *Let φ^t be a Morse-Smale flow which is \mathcal{C}^1 linearizable. Denote by V the corresponding vector field and let $0 \leq k \leq n$. Then, there exists a minimal discrete subset $\mathcal{R}_k(V) \subset \mathbb{C}$ such that, given any $(\psi_1, \psi_2) \in \Omega^k(M) \times \Omega^{n-k}(M)$, the map*

$$z \mapsto \hat{C}_{\psi_1, \psi_2}(z)$$

has a meromorphic extension whose poles are contained inside $\mathcal{R}_k(V)$.

By discrete, we mean that $\mathcal{R}_k(V)$ has no accumulation points. In particular, it is at most countable. Elements inside $\mathcal{R}_k(V)$ are often referred as *Pollicott-Ruelle resonances* or as the correlation spectrum of the flow. Besides the fact that the flow is Morse-Smale, we need to make an assumption on the fact that the flow is \mathcal{C}^1 -linearizable – see paragraph 3.3 for the precise definition. Roughly speaking, it means that the flow is \mathcal{C}^1 -conjugated to a linear flow in a neighborhood of each basic set of the flow. This may sound like a big constraint. Yet, thanks to the Sternberg-Chen Theorem [8, 32, 44], it is satisfied as soon as certain number of non resonance assumptions are made on the Lyapunov exponents of the basic set. In particular, they are satisfied for a generic choice of Morse-Smale flow. We refer to appendix A.5 for a brief account on these non resonance hypothesis. This assumption of being \mathcal{C}^1 -linearizable may be artificial at this stage of our analysis but it does not look obvious to us how to remove it in an easy manner.

Note that, if ψ_1 and ψ_2 were supported in the neighborhood of a given basic set, then the existence of this discrete correlation spectrum could be deduced near critical points from [1, 2, 23] and near closed orbits from [12]. Here, the main novelty is that the result holds **globally** on the manifold, i.e. without any restriction on the supports of ψ_1 and ψ_2 . It also generalizes our previous results from [9] which were only valid for Morse-Smale **gradient flows which are not allowed to have periodic orbits**. Observe that, even if the nonwandering set is the union of finitely many basic sets, it is not obvious that the *global correlation spectrum* should be the union of the correlation spectra associated with each individual basic set. We shall see in [10] that this is indeed the case if enough nonresonance conditions are satisfied by the Lyapunov exponents. Without these assumptions, it is not completely obvious if some unexpected phenomenon may occur in the correlation spectrum.

In fact, thanks to its spectral nature, our proof will not only give the meromorphic extension of $\hat{C}_{\psi_1, \psi_2}(z)$ but also some information on its residues :

Theorem 2.2 (Resonant states). *Let φ^t be a Morse-Smale flow which is \mathcal{C}^1 linearizable. Denote by V the corresponding vector field and let $0 \leq k \leq n$.*

Then, for every $z_0 \in \mathcal{R}_k(V)$, there exists an integer $m_k(z_0) \geq 1$ and a linear map of finite rank

$$\pi_{z_0}^{(k)} : \Omega^k(M) \rightarrow \mathcal{D}^k(M)$$

such that, given any $(\psi_1, \psi_2) \in \Omega^k(M) \times \Omega^{n-k}(M)$, one has, in a small neighborhood of z_0 ,

$$\hat{C}_{\psi_1, \psi_2}(z) = \sum_{l=1}^{m_k(z_0)} (-1)^{l-1} \frac{\left\langle (\mathcal{L}_V^{(k)} + z_0)^{l-1} \pi_{z_0}^{(k)}(\psi_1), \psi_2 \right\rangle}{(z - z_0)^l} + R_{\psi_1, \psi_2}(z),$$

where $R_{\psi_1, \psi_2}(z)$ is a holomorphic function. Moreover, any element u in the range of $\pi_{z_0}^{(k)}$ satisfies the generalized eigenvalue equation

$$\boxed{(\mathcal{L}_V^{(k)} + z_0)^{m_k(z_0)}(u) = 0.}$$

Here, $\mathcal{D}^{/k}(M)$ denotes the currents of degree k , i.e. the topological dual of $\Omega^{n-k}(M)$. Elements inside the range of $\pi_{z_0}^{(k)}$ are called the *Pollicott-Ruelle resonant states*. Our proof will say more on the Sobolev regularity of these currents. In particular, their wavefront set will be contained inside the conormal of the unstable manifolds of the flow. In [10], we will show how to exploit this Sobolev regularity to give a rather precise description of these resonant states in a neighborhood of the basic sets of the flow. In [11], we will show that some of these states have a deep topological meaning related to the De Rham cohomological complex [37] and to its Reidemeister torsion [20].

2.2. About the proof of Theorems 2.1 and 2.2. We will consider a slightly more general framework than the one we described so far. Fix a complex vector bundle $\mathcal{E} \rightarrow M$ of rank N and some connection $\nabla : \Omega^0(M, \mathcal{E}) \rightarrow \Omega^1(M, \mathcal{E})$ [30] – see also paragraph 5.2 for a brief reminder. Then, one can define a covariant derivative $d^\nabla : \Omega^\bullet(M, \mathcal{E}) \rightarrow \Omega^{\bullet+1}(M, \mathcal{E})$ and introduce the operator

$$\forall 0 \leq k \leq n, \quad \mathcal{L}_{V, \nabla}^{(k)} := d^\nabla \circ \iota_V + \iota_V \circ d^\nabla : \Omega^k(M, \mathcal{E}) \rightarrow \Omega^k(M, \mathcal{E}).$$

Note that, in the present article, we will not make the assumption that ∇ is flat, i.e. that $d^\nabla \circ d^\nabla = 0$. Our goal is to introduce anisotropic Sobolev spaces of currents adapted to the dynamics of the Morse-Smale vector field V in the sense that $-\mathcal{L}_{V, \nabla}^{(k)}$ has a discrete spectrum on this space at least for $\text{Re}(z) \geq -N$. For that purpose, we will follow the microlocal approach of Faure and Sjöstrand [17]. Recall that their construction is based on the fact that the operator $-\mathcal{L}_{V, \nabla}^{(k)}$ is a differential operator whose principal symbol is $H(x; \xi) \mathbf{Id}_{\Lambda^k(T^*M) \otimes \mathcal{E}}$ where

$$(3) \quad \forall (x; \xi) \in T^*M, \quad H(x; \xi) := \xi(V(x)).$$

Then, they show that the spectrum of the operator $\mathcal{L}_{V, \nabla}^{(k)}$ (hence the correlation spectrum) can be obtained in a similar manner as in the theory of semiclassical resonances [26, 14]. In particular, this requires to understand the dynamical properties of the Hamiltonian flow induced by H , namely the topological and the dynamical properties of the set of points which are trapped by the Hamiltonian dynamics either in the future or in the past. Due to the fact that we want to deal with the global correlation spectrum, we need to proceed to a delicate analysis of the global Hamiltonian dynamics which is the content of section 4 and which is probably the main new difficulty compared with the Anosov framework treated in [17]. We emphasize that this part of the proof is implicitly related to the classical results of Smale [39] on Morse-Smale flows – see also [45] for a formulation closer to ours in the case of gradient flows. The major difference is that we are interested here in the Hamiltonian dynamics rather than on the dynamics in the configuration space M . Note that we already had to deal with similar difficulties in the context of gradient flows [9]

and we give here a more systematic approach which allows to deal with closed orbits. We should also point out that the results we obtain in that direction are in some sense related to some results of Laudenbach who gave a very precise description of the regularity of the closure of unstable manifolds for Morse–Smale gradient flows [28]. Once this is well understood, we have to construct an appropriate escape (or Lyapunov) function for the Hamiltonian dynamics – see Lemma 5.2. This construction combines our analysis of the global dynamics with some results due to Meyer on the existence of energy functions for Morse–Smale flows [31]. Given this escape function, we can follow the spectral construction of Faure and Sjöstrand whose crucial ingredient is in fact the existence of such a function – see section 5 for more details.

2.3. Organization of the article. In section 3, we review some classical facts on Morse–Smale flows and introduce some conventions that we will use all along the article. Then, in section 4 which contains the main novelty of the article, we give a precise description of the Hamiltonian dynamics induced by a Morse–Smale flow. Finally, we prove Theorems 2.1 and 2.2 by constructing anisotropic Sobolev spaces adapted to the vector field V . Finally, in appendix A, we collect some classical results on hyperbolic fixed points and closed orbits that we use in our proofs and, in appendix B, we briefly recall the proofs of some results due to Smale [39] which may be helpful to understand the proofs of our results on the Hamiltonian dynamics.

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3. REVIEW ON MORSE-SMALE FLOWS

The purpose of this preliminary section is to collect some well-known facts on the so-called Morse–Smale flows which were introduced by Smale in [39] as a generalization of gradient flows induced by a Morse function. Besides the seminal work of Smale, good references on the subject are [19, Ch. 8] and [34, Ch. 4]. All along this section, we implicitly assume that M is endowed with a Riemannian structure g .

3.1. Definition and examples. We say that $\Lambda \subset M$ is an elementary critical element if Λ is either a fixed point or a closed orbit of φ^t . Such an element is said to be hyperbolic if the fixed point or the closed orbit is hyperbolic – see appendix A for a brief reminder. Following [40, p. 798], φ^t is a **Morse–Smale flow** if the following properties hold:

- (1) the non-wandering set $NW(\varphi^t)$ is the union of finitely many elementary critical element $\Lambda_1, \dots, \Lambda_K$ which are hyperbolic,

(2) for every i, j and for every x in $W^u(\Lambda_j) \cap W^s(\Lambda_i)$, one has¹ $T_x M = T_x W^u(\Lambda_j) + T_x W^s(\Lambda_i)$.

As was already mentioned, these flows generalizes the so-called Morse-Smale gradient flows, and they are the simplest example of Axiom A flows in the sense of Smale [40, p. 803]. Before giving some remarkable properties of these flows, let us start with the following observation (which follows only from the first part of the definition):

Lemma 3.1. *For every x in M , there exists an unique couple (i, j) such that $x \in W^u(\Lambda_j) \cap W^s(\Lambda_i)$.*

In particular, *the unstable manifolds $(W^u(\Lambda_j))_{j=1, \dots, K}$ form a partition of M , i.e.*

$$M = \bigcup_{j=1}^K W^u(\Lambda_j), \quad \text{and} \quad \forall i \neq j, \quad W^u(\Lambda_i) \cap W^u(\Lambda_j) = \emptyset.$$

The same of course holds for stable manifolds.

Proof. Note that the hyperbolicity assumption ensures that the sets $(\Lambda_j)_j$ are disjoint. Hence, we can assume that $x \notin \cup_{j=1}^K \Lambda_j$. Set J to be the subset of $\{1, \dots, K\}$ such that $\forall j \in J, x \in W^s(\Lambda_j)$ and $\forall j \notin J, x \notin W^s(\Lambda_j)$. Assume that J contains more than two elements then there is some open neighborhood O of $\cup_{j \in J} \Lambda_j$ such that $(\varphi^t(x))_{t \geq 0}$ stays infinitely often in O , since O contains at least two components which are bounded away and by continuity of $\varphi^t(x)$ in t , we can extract a subsequence $(t_n)_n$ of times such that $\varphi^{t_n}(x) \in M \setminus \tilde{O}$ where \tilde{O} is some open neighborhood of the compact set $NW(\varphi^t)$. Therefore, by extracting again we get a sequence which converges to some point $y \in M$. Moreover, $y \notin O \implies y \notin NW(\varphi^t)$ which contradicts the fact that y belongs to $NW(\varphi^t)$ by construction. \square

3.2. Classical results on Morse-Smale flows. We shall now collect some other useful facts on Morse-Smale flows following the seminal article of Smale [39]. As the proofs are not very long and as they are instructive for the arguments of the upcoming sections, we briefly recall how to prove most of them in appendix B.

3.2.1. First properties. We start with the following direct consequence from the definition – see appendix B for details.

Lemma 3.2. *Let x be an element in $W^u(\Lambda_j) \cap W^s(\Lambda_i)$. Then, one has*

$$\dim W^u(\Lambda_j) \geq \dim W^u(\Lambda_i).$$

Moreover, if $x \notin \Lambda_i$, equality can occur only if Λ_i is a closed orbit.

It roughly means that the dimension of the unstable manifolds must decrease along the limit sets of the flow. We continue our description of the properties of Morse-Smale flow with the no-cycle property.

Lemma 3.3 (No-cycle). *If x belongs to $W^u(\Lambda_j) \cap W^s(\Lambda_j)$, then $x \in \Lambda_j$.*

¹See appendix A for the definition of the stable/unstable manifolds $W^{s/u}(\Lambda)$.

Remark 3.4. Note that, in its original article [39], Smale took this property as one of the axioms satisfied by his flows. Yet, in [40], this assumption was removed as it can be deduced from the other axioms combined with the so-called λ -Lemma – see appendix B.

3.2.2. *Ordering unstable manifolds.* Let us now turn to the main feature of these flows which was proved by Smale in [39] – see also [40, p.52]:

Theorem 3.5 (Smale). *Suppose that φ^t is a Morse-Smale flow. Then, for every $1 \leq j \leq K$, the closure of $W^u(\Lambda_j)$ is the union of certain $W^u(\Lambda_{j'})$. Moreover, if we say that $W^u(\Lambda_{j'}) \leq W^u(\Lambda_j)$ whenever $W^u(\Lambda_{j'})$ is contained in the closure of $W^u(\Lambda_j)$, then \leq is a partial ordering. Finally if $W^u(\Lambda_{j'}) \leq W^u(\Lambda_j)$, then $\dim W^u(\Lambda_{j'}) \leq \dim W^u(\Lambda_j)$.*

We refer to appendix B for a reminder on the proof of this result. This partial ordering relation, which was crucial for Smale to construct a filtration of the manifold in order to prove his Morse inequalities [39], is related to the concept of topological stratification. A stratum is less than a bigger stratum if it lies in the closure of the bigger stratum. Following [40, p. 753] – see also [34, Ch. 4], we can set

Definition 3.6 (Smale quiver.). The partial order relation on the collection of subsets $W^u(\Lambda_j)_{j=1}^K$ defined above is called **Smale causality relation**. Following Smale, we define an oriented graph² D , called **Smale quiver**, whose K vertices are given by $W^u(\Lambda_j)_{j=1}^K$. Two vertices $W^u(\Lambda_j), W^u(\Lambda_i)$ are connected by an oriented path starting at $W^u(\Lambda_j)$ and ending at $W^u(\Lambda_i)$ iff $W^u(\Lambda_j) \geq W^u(\Lambda_i)$.

3.3. **\mathcal{C}^1 Linearizable flows.** In order to construct anisotropic Sobolev spaces adapted to a Morse-Smale flow, we will need to make some extra assumption that will roughly say that the flow is linearizable in a \mathcal{C}^1 chart near every critical element Λ_j . Such an assumption may sound quite restrictive. Yet, thanks to Sternberg-Chen's Theorem (see appendix A.5 for a brief reminder), it is automatically satisfied as soon as certain (generic) non resonance assumptions on the Lyapunov exponents are satisfied.

For a hyperbolic fixed point Λ , we say that φ^t is \mathcal{C}^1 -linearizable near Λ if there exists a \mathcal{C}^1 diffeomorphism $h : B_n(0, r) \rightarrow O$ (where O is a small open neighborhood of Λ and $B_n(0, r)$ is a small ball of radius r centered at 0 in \mathbb{R}^n) s.t. $V \circ h = dh \circ L$ with

$$L(x) := Ax\partial_x,$$

for some $A \in M_n(\mathbb{R})$.

For a hyperbolic closed orbit Λ of minimal period \mathcal{P}_Λ , we say that φ^t is \mathcal{C}^1 -linearizable near Λ if there exists a \mathcal{C}^1 diffeomorphism $h : B_{n-1}(0, r) \times \mathbb{R} / (\mathcal{P}_\Lambda \mathbb{Z}) \rightarrow O$ (where O is a small open neighborhood of Λ and $r > 0$ is small) and a smooth map $A : \mathbb{R} / (\mathcal{P}_\Lambda \mathbb{Z}) \rightarrow M_{n-1}(\mathbb{R})$ such that $V \circ h = dh \circ L$ with

$$L(x, \theta) = A(\theta)x.\partial_x + \partial_\theta.$$

Finally, we say that a Morse-Smale flow is \mathcal{C}^1 -linearizable if it is \mathcal{C}^1 -linearizable near each of its critical element $(\Lambda_j)_{j=1, \dots, K}$. More generally, for any $k \geq 1$, we will say that it

²This diagram is the Hasse diagram associated to the poset $(W^u(\Lambda_j)_{j=1}^K, \leq)$.

is \mathcal{C}^k -linearizable if the “linearizing” diffeomorphism can be chosen of class \mathcal{C}^k for every critical element.

3.4. Existence of an energy function. To conclude our short review on Morse-Smale flows, we mention the following result due to Meyer [31, p. 1034] that will be central in our construction of anisotropic Sobolev spaces:

Theorem 3.7 (Meyer). *Let V be a vector field generating some smooth Morse-Smale flow. Then, there exists a smooth function $E : M \rightarrow \mathbb{R}$ such that*

$$\mathcal{L}_V.E \geq 0 \text{ on } M, \text{ and } \mathcal{L}_V.E > 0 \text{ on } M - (\cup_{i=1}^K \Lambda_i).$$

Moreover, E is constant on every connected component of $\cup_{j=1}^K \Lambda_j$ and one can choose E such that $E|_{\Lambda_j} > E|_{\Lambda_i}$ whenever $W^u(\Lambda_j) \not\subseteq W^u(\Lambda_i)$ with $i \neq j$.

4. DYNAMICS ON THE COTANGENT SPACE

The spectral analysis of flows via microlocal technics as in the works of Faure-Sjöstrand is intimately related to the induced dynamics on the cotangent bundle [17]. The purpose of this section is to consider the Hamiltonian lift of a Morse-Smale flow and to describe some of its dynamical features. The main results of this section are Theorems 4.2 and 4.4.

4.1. Hamiltonian lift. A flow on M can be lifted to the cotangent space T^*M as follows. We associate to the vector field V an Hamiltonian function,

$$\forall (x; \xi) \in T^*M, \quad H(x; \xi) := \xi(V(x)).$$

This Hamiltonian function also induces an Hamiltonian flow that we denote by $\Phi^t : T^*M \rightarrow T^*M$ whose vector field will be denoted by X_H . We note that, by construction,

$$\Phi^t(x; \xi) := \left(\varphi^t(x), (d\varphi^t(x)^T)^{-1} \xi \right),$$

and that *this flow induces a diffeomorphism between $T^*M - \{0\}$ and $T^*M - \{0\}$* . This lifted flow also induces a smooth flow on the unit cotangent bundle S^*M , i.e.

$$\forall t \in \mathbb{R}, \forall (x; \xi) \in S^*M, \quad \tilde{\Phi}^t(x; \xi) = \left(\varphi^t(x), \frac{(d\varphi^t(x)^T)^{-1} \xi}{\|(d\varphi^t(x)^T)^{-1} \xi\|} \right).$$

We denote by \tilde{X}_H the induced smooth vector field on S^*M .

4.2. Writing the flows in local coordinates. Let us rewrite the Morse-Smale flow and its Hamiltonian lift near the critical elements of φ^t in some well-chosen local coordinates. For that purpose, we now make the assumption that φ^t is a Morse-Smale flow which is \mathcal{C}^1 -linearizable. If Λ is a *fixed point*, then we can choose local \mathcal{C}^1 coordinates $(x_1, \dots, x_r, y_{r+1}, \dots, y_n)$ (with $0 \leq r(=r_\Lambda) \leq n$) such that

$$(4) \quad \varphi^t(x_1, \dots, x_r, y_{r+1}, \dots, y_n) = (e^{-t\Omega_s} x, e^{t\Omega_u} y),$$

with Ω_s (resp. Ω_u) an element in $M_r(\mathbb{R})$ (resp. $M_{n-r}(\mathbb{R})$) all of whose eigenvalues have positive real part from the hyperbolicity assumption. In particular, there exist some positive constants $0 < C_1 < C_2$ and $0 < \chi_- < \chi_+$ such that, for every $t \geq 0$ and for every (x, y) ,

$$(5) \quad C_1 e^{-t\chi_+} \|(x, y)\| \leq \|(e^{-t\Omega_s} x, e^{-t\Omega_u} y)\| \leq C_2 e^{-t\chi_-} \|(x, y)\|.$$

As the chart is of class \mathcal{C}^1 , we can also write the Hamiltonian flow (in the induced local coordinates $(x, y; \xi, \eta)$ on T^*M) under the form:

$$(6) \quad \Phi^t(x, y; \xi, \eta) = (e^{-t\Omega_s} x, e^{t\Omega_u} y; e^{t\Omega_s^T} \xi, e^{-t\Omega_u^T} \eta).$$

We now turn to the case where Λ is a *closed orbit* of minimal period $\mathcal{P}_\Lambda > 0$. For that purpose, we shall use the notations of paragraph A.6 on Floquet theory. Under the same assumptions, we can fix a system of \mathcal{C}^1 local coordinates $(z, \theta) = (x_1, \dots, x_r, y_{r+1}, \dots, y_{n-1}, \theta) \in \mathbb{R}^{n-1} \times \mathbb{R}/(\mathcal{P}_\Lambda \mathbb{Z})$ with $0 \leq r (= r_\Lambda) \leq n-1$ such that

$$(7) \quad \varphi^t(z, \theta) = (P(\theta + t, 0) e^{t\Omega} P(\theta, 0)^{-1} z, \theta + t \bmod \mathcal{P}_\Lambda),$$

where $t \mapsto P(\theta + t, 0)$ is $2\mathcal{P}_\Lambda$ -periodic (see appendix A.6) and $\Omega = \text{diag}(-\Omega_s, \Omega_u)$ with Ω_s and Ω_u having the same eigenvalues property as in the case of fixed points (see appendix A.6). Equivalently, one has

$$(8) \quad \varphi^t(z, \theta) = (U(t + \theta, \theta) z, t + \theta \bmod \mathcal{P}_\Lambda),$$

where $U(t + \theta, \theta)$ is defined in paragraph A.5 as the fundamental solution for the Floquet problem associated with $A(t)$. We can now write the corresponding Hamiltonian flow:

$$(9) \quad \Phi^t(z, \theta, \zeta, \Theta) = (\varphi^t(z, \theta), (U(t + \theta, \theta)^T)^{-1} \zeta, \Theta + R(t, z, \theta, \zeta)),$$

with $R(t, z; \theta, \zeta)$ which is defined in paragraph A.6. Equivalently, one has

$$(10) \quad \Phi^t(z, \theta, \zeta, \Theta) = \left(\varphi^t(z, \theta); (P(\theta + t, 0)^T)^{-1} e^{-t\Omega^T} P(\theta, 0)^T(\zeta), \Theta + R(t, z, \theta, \zeta) \right).$$

Note that we have a good control (as t becomes large) on the value of the term $R(t, z; \theta, \zeta)$ thanks to (37).

4.3. Conormals of the unstable/stable manifolds. Given a smooth submanifold S inside M , one can define its *conormal* as follows:

$$N^*S := \{(x; \xi) \in T^*M : x \in S, \xi \neq 0, \text{ and } \forall v \in T_x S, \xi(v) = 0\}.$$

Here, the relevant sets for the lifted dynamics will be

$$\Sigma_{uu} := \bigcup_{j=1}^K N^*(W^s(\Lambda_j)) \cap S^*M \text{ and } \Sigma_{ss} := \bigcup_{j=1}^K N^*(W^u(\Lambda_j)) \cap S^*M.$$

In forthcoming Lemma 4.1, we will see that the set $\bigcup_{j=1}^K N^*(W^u(\Lambda_j)) \cap S^*M$ is in fact an attractor for the lifted dynamics $(\tilde{\Phi}^t)_t$ this is why it is denoted Σ_{ss} to emphasize it is

the **stable set** for the contact flow $(\tilde{\Phi}^t)_t$ in S^*M . From appendix A, we know that the unstable (resp. stable) manifolds are invariantly fibered by smooth submanifolds, i.e.

$$\forall 1 \leq j \leq K, W^u(\Lambda_j) := \bigcup_{x \in \Lambda_j} W^{uu}(x), \text{ and } W^s(\Lambda_j) := \bigcup_{x \in \Lambda_j} W^{ss}(x).$$

We then define

$$\Sigma_u := \bigcup_{j=1}^K \bigcup_{x \in \Lambda_j} N^*(W^{ss}(x)) \cap S^*M \text{ and } \Sigma_s := \bigcup_{j=1}^K \bigcup_{x \in \Lambda_j} N^*(W^{uu}(x)) \cap S^*M.$$

Note that, in the case of a fixed point $\Lambda_j = \{x\}$, $W^{ss/uu}(x)$ coincide with $W^{s/u}(\Lambda_j)$. We also observe that we have the inclusions $\Sigma_{uu} \subset \Sigma_u$ and $\Sigma_{ss} \subset \Sigma_s$. Finally, we can translate the transversality assumption in this setting:

$$(11) \quad \Sigma_{uu} \cap \Sigma_{ss} = \Sigma_{uu} \cap \Sigma_s = \Sigma_{ss} \cap \Sigma_u = \emptyset.$$

4.4. First properties. Let us start our description of the dynamics in the cotangent bundle with the following Lemma:

Lemma 4.1. *Let φ^t be a Morse-Smale flow which is C^1 -linearizable. Then, one has*

$$\forall (x; \xi) \in S^*M - \Sigma_{ss}, \lim_{t \rightarrow -\infty} d_{S^*M}(\tilde{\Phi}^t(x; \xi), \Sigma_u) = 0,$$

and

$$\forall (x; \xi) \in S^*M - \Sigma_u, \lim_{t \rightarrow +\infty} d_{S^*M}(\tilde{\Phi}^t(x; \xi), \Sigma_{ss}) = 0.$$

By reversing the time, one can verify that the same properties hold if we intertwine the roles of s and u .

Proof. Suppose that $\rho = (q; p)$ belongs to $S^*M - \Sigma_{ss}$. From Lemma 3.1, there exists a unique $1 \leq i \leq K$ such that the corresponding point q on M belongs to $W^u(\Lambda_i)$. Up to applying the flow in backward times, we can suppose that $\rho \in S^*M - \Sigma_{ss}$ belongs to the linearizing chart near Λ_i which was defined in paragraph 4.2. If Λ_i is a critical point, then the coordinates of ρ are of the form $(x = 0, y, \xi, \eta)$ with $\eta \neq 0$ as $\rho \notin N^*(W^u(\Lambda_i))$ by definition. We can then use the linearized expression (6) of the Hamiltonian flow :

$$\Phi^t(\rho) = (0, e^{t\Omega_u}y; e^{t\Omega_s^T}\xi, e^{-t\Omega_u^T}\eta)$$

where $\eta \neq 0$ and $e^{t\Omega_s^T}\xi \rightarrow 0$ when $t \rightarrow -\infty$. Combined with (5), this implies that $\tilde{\Phi}^t(\rho)$ accumulates on Σ_u as $t \rightarrow -\infty$. In the case where Λ_i is a closed orbit, the coordinates of ρ are of the form $(P(\theta, 0)(x = 0, y), \theta; (P(\theta, 0)^T)^{-1}(\xi, \eta), \Theta)$ with $(\eta, \Theta) \neq 0$ as $\rho \notin \Sigma_{ss}$. We can now combine the expression of the Hamiltonian in local coordinates (10) with the estimates (37) and (35) from appendix A.6. This exactly shows that $\tilde{\Phi}^t(\rho)$ accumulates on Σ_u as $t \rightarrow -\infty$.

Suppose now that ρ belongs to $S^*M - \Sigma_u$. Again from Lemma 3.1, there exists a unique $1 \leq j \leq K$ such that the base point associated with ρ belongs to $W^s(\Lambda_j)$. As ρ does not belong to Σ_u , we know that ρ does not belong to $\bigcup_{x \in \Lambda_j} N^*(W^s(x)) \cap S^*M$ since $\Sigma_u \subset \Sigma_{uu}$ by definition. Then, we have to distinguish again the case where Λ_j is a fixed point and

the one where Λ_j is a closed orbit. In both cases, the proof is exactly the same and it is even simpler in the case of closed orbits as our assumption implies that $\eta \neq 0$ and not only $(\eta, \Theta) \neq 0$. \square

4.5. Compactness. The first main result of this section is a topological statement on the sets we have just defined:

Theorem 4.2. *Let φ^t be a Morse-Smale flow which is \mathcal{C}^1 -linearizable. Then, Σ_u , Σ_{uu} , Σ_s and Σ_{ss} are compact subsets of S^*M .*

The proof of this result was already given in [9, Lemma 3.6] in the particular case of Morse-Smale gradient flows satisfying a certain (generic) linearization property. We shall give here a proof which is valid for *any Morse-Smale flow* (including of course the case of gradient flows) satisfying also certain (generic) linearization property. The property of being \mathcal{C}^1 -linearizable seems crucial in our proof as it allows us to control the asymptotic behaviour of cotangent vectors under the flow near a critical element Λ_j . In fact, having a \mathcal{C}^1 -chart enables us to use the local expressions of paragraph 4.2 for the Hamiltonian flow. Using only hyperbolicity³ at Λ_j does not seem to be enough to control the asymptotic behaviour of cotangent vectors outside Λ_j in our proof of compactness.

Proof of Theorem 4.2. It is sufficient to prove that Σ_s and Σ_{ss} are compact. The other cases follow by reversing the time. We start with the compactness of Σ_{ss} and, for that purpose, we consider a sequence $(x_m, \xi_m)_{m \geq 1}$ inside Σ_{ss} which converges to a certain point (x_∞, ξ_∞) . We aim at proving that the limit point belongs to Σ_{ss} . For that purpose, we proceed by contradiction and suppose that $(x_\infty, \xi_\infty) \notin \Sigma_{ss}$.

Without loss of generality, we can suppose that there exists $1 \leq i \leq K$ such that, for every $m \geq 1$, x_m belongs to $W^u(\Lambda_i)$. From Lemma 3.1, we also know that x_∞ belongs to $W^u(\Lambda_j)$ for a certain $1 \leq j \leq K$. Then, either $i = j$ or $i \neq j$. As a first step of the proof, let us show that we can assume $i \neq j$. If $i = j$, we have two distinguish two cases:

- **Λ_j is a critical point.** Fix $0 < \delta < \delta_0$ small enough. As x_∞ belongs to $W^u(\Lambda_j)$, there exists some $T_\delta > 0$ such that, for every $T \geq T_\delta$, $\varphi^{-T}(x_\infty)$ belongs to the ball of radius $\delta > 0$ centered at Λ_j . Moreover, thanks to the local expression (6) for the Hamiltonian flow near Λ_j and thanks to (5), we can also assume that, for every $T \geq T_\delta$, the cotangent component of $\tilde{\Phi}^{-T}(x_\infty, \xi_\infty)$ is δ -close to the stable direction by Lemma 4.1 since (x_∞, ξ_∞) does not belong to $N^*(W^u(\Lambda_j))$, $\lim_{t \rightarrow +\infty} \tilde{\Phi}^{-t}(x_\infty, \xi_\infty)$ must tend to the conormal $N^*(W^s(\Lambda_j))$. In particular, for every $\delta > 0$, there exists m_δ such that this property remains true for every point $\tilde{\Phi}^{-T_\delta}(x_m, \xi_m)$ with $m \geq m_\delta$ by **continuity** of the map $(x; \xi) \mapsto \tilde{\Phi}^{-T_\delta}(x; \xi)$. Recall now that x_m belongs to $W^u(\Lambda_i)$ with $i \neq j$. Therefore, there exists some $T'_\delta > 0$ such that, for every $0 \leq T < T'_\delta$, $\varphi^{-(T+T_\delta)}(x_{m_\delta})$ belongs to the ball of radius δ_0 centered at Λ_j while $\varphi^{-(T'_\delta+T_\delta)}(x_{m_\delta})$ will not belong to this ball since for every m , $\lim_{t \rightarrow +\infty} \varphi^{-t}(x_m) = \Lambda_i$ hence we must escape any neighborhood of Λ_j in finite time. Take now p large

³For instance, hyperbolicity allows us to use the Grobman-Hartman Theorem but it only provides a \mathcal{C}^0 -chart.

enough and define the sequence $(x_p^{(1)}, \xi_p^{(1)}) = \varphi^{-(T'_{1/p} + T_{1/p})}(x_{m_{1/p}})$ inside Σ_{ss} . Up to extraction, one can verify that this sequence converges to a certain limit $(x_\infty^{(1)}, \xi_\infty^{(1)})$. By construction, one can verify that this limit point verifies $x_\infty^{(1)} \in W^s(\Lambda_j) - \Lambda_j$ and $(x_\infty^{(1)}, \xi_\infty^{(1)})$ belongs to $N^*(W^s(\Lambda_j)) \cap S^*M$ hence to Σ_{uu} . The fact that the cotangent vectors $\xi_p^{(1)}$ converge to some element in the stable direction comes from the fact that the cotangent component is $1/p$ -close to this direction at time $-T_{1/p}$ and that, thanks to (6) and to (5), it remains $1/p$ -close after iterating the flow at time $-T'_{1/p}$ times. In particular, from our transversality assumption (11), the limit point is not in Σ_{ss} even if all the points of the sequence are inside Σ_{ss} .

- **Λ_j is a closed orbit.** Fix again $0 < \delta < \delta_0$. We use the \mathcal{C}^1 linearizing property to work (near Λ_j) in a \mathcal{C}^1 system of coordinates $(z, \theta) \in B_{n-1}(0, 2\delta_0) \times \mathbb{R}/(\mathcal{P}_{\Lambda_j}\mathbb{Z})$ which allows us to use the formulas of paragraph 4.2 in order to represent the flow near Λ_j . As x_∞ belongs to $W^u(\Lambda_j)$, we can find some time $T_\delta > 0$ such that $\varphi^{-T_\delta}(x_\infty)$ can be written in local coordinates under the form $(z, 0 \bmod \mathcal{P}_{\Lambda_j})$ with z belonging to $B_{n-1}(0, \delta)$. As (x_∞, ξ_∞) does not belong to $N^*(W^u(\Lambda_j))$ and according to Lemma 4.1, we can find some large integer $N_\delta > 0$ so that the cotangent component of $\tilde{\Phi}^{-(T_\delta + N_\delta \mathcal{P}_{\Lambda_j})}(x_\infty, \xi_\infty)$ is in a δ -neighborhood of $\cup_{x \in \Lambda_j} N^*(W^{ss}(x)) \cap S^*M$. By continuity of the map $\tilde{\Phi}^{-(T_\delta + N_\delta \mathcal{P}_{\Lambda_j})}$, we can find $m_\delta > 0$ and $T_{\delta, m_\delta} \in (-\mathcal{P}_{\Lambda_j} + T_\delta, \mathcal{P}_{\Lambda_j} + T_\delta)$ such that the previous properties remain true for $\tilde{\Phi}^{-(T_{\delta, m_\delta} + N_\delta \mathcal{P}_{\Lambda_j})}(x_{m_\delta}, \xi_{m_\delta})$ except that the base point belongs to $W^u(\Lambda_i)$ instead of $W^u(\Lambda_j)$. Then, we proceed as in the case of critical points, i.e. we iterate the flow $\tilde{\Phi}^{-N'_\delta \mathcal{P}_{\Lambda_j}}$ so that $\phi^{-(T_{\delta, m_\delta} + (N_\delta + N'_\delta) \mathcal{P}_{\Lambda_j})}(x_{m_\delta})$ escapes the tubular neighborhood $B_{n-1}(0, \delta) \times \mathbb{R}/(\mathcal{P}_{\Lambda_j}\mathbb{Z})$ of Λ_j . We have to verify that, during this step, the cotangent component of the evolved point remains 2δ -close to $N^*(W^{ss}(\Lambda_j))$. For that purpose, we shall use the local coordinates introduced in paragraph 4.2. This allows us to express explicitly the evolved cotangent vector. Indeed, knowing that this cotangent vector is δ -close to the weakly stable direction at time $T_{\delta, m_\delta} + N_\delta \mathcal{P}_{\Lambda_j}$, we can use (10) and the estimates (35) and (37) from the appendix to verify that this remains true after applying the flow $\tilde{\Phi}^t$ at time $-N'_\delta \mathcal{P}_{\Lambda_j}$. Then, by letting $\delta = 1/p$ with $p \geq 1$ large enough and up to an extraction, we can one more time construct a sequence $(x_p^{(1)}, \xi_p^{(1)})$ inside $N^*(W^u(\Lambda_i)) \cap S^*M \subset \Sigma_{ss}$ which converges to a point $(x_\infty^{(1)}, \xi_\infty^{(1)}) \in \cup_{x \in \Lambda_j} N^*(W^{ss}(x)) \cap S^*M$ with $x_\infty^{(1)} \notin \Lambda_j$. In particular, from the transversality assumption (11), it does not belong to Σ_{ss} .

To summarize, we have verified that, if $i \neq j$, then we can construct a sequence $(x_p^{(1)}, \xi_p^{(1)})$ inside $N^*(W^u(\Lambda_i)) \cap S^*M \subset \Sigma_{ss}$ which converges to a point $(x_\infty^{(1)}, \xi_\infty^{(1)}) \notin \Sigma_{ss}$ with $x_\infty^{(1)} \in W^s(\Lambda_j) - \Lambda_j$. As there are only a finite number of critical elements, we can iterate this procedure a finite number of times and assume without loss of generality that $i = j$. In other words, we can suppose that $(x_m, \xi_m)_{m \geq 1}$ is a sequence in Σ_{ss} which converges to (x_∞, ξ_∞) and that, for every $m \geq 1$, $x_m \in W^u(\Lambda_i)$ and $x_\infty \in W^u(\Lambda_i)$. Up to iterating the flow in backward times and up to considering some large enough m , we can suppose that

all the elements of the sequence $(x_m, \xi_m)_{m \geq 1}$ are in the linearizing chart near Λ_j defined in paragraph 4.2. In this system of coordinates, one can verify that any accumulation point of the sequence $(x_m, \xi_m)_m$ must belong to $N^*(W^u(\Lambda_i)) \cap S^*M$ which gives the expected contradiction. The proof of compactness of Σ_s follows exactly the same lines except that the arguments for the convergence of cotangent vectors near the closed orbits have to be slightly modified. Again, we get the appropriate convergence property in these cases by using our linearizing expressions for the Hamiltonian flow from paragraph 4.2 and the estimates from paragraph A.6. \square

Remark 4.3. Reproducing the same argument allows in fact to prove something slightly stronger. Fix $J \subset \{1, \dots, K\}$. Then,

$$\Sigma_{ss}^J := \bigcup_{j \in J} \bigcup_{W^u(\Lambda_i) \leq W^u(\Lambda_j)} N^*(W^u(\Lambda_i)) \cap S^*M,$$

and

$$\Sigma_s^J := \bigcup_{j \in J} \bigcup_{W^u(\Lambda_i) \leq W^u(\Lambda_j)} \bigcup_{x \in \Lambda_j} N^*(W^{uu}(x)) \cap S^*M$$

are compact subsets of S^*M . By considering negative times of the flow, the same of course holds for the stable manifolds with the associated partial order relation.

4.6. Attractor and repeller. Now that we have described the topological properties of the subsets Σ_* , we will turn to more dynamical properties. More precisely, our main result is that the sets Σ_* are attractors or repeller of the flow $\tilde{\Phi}^t$:

Theorem 4.4. *Let φ^t be a Morse-Smale flow which is \mathcal{C}^1 -linearizable. Let $\epsilon > 0$. Then, there exists an open neighborhood V^{ss} (resp. V^s) of Σ_{ss} (resp. Σ_s) which is of size $\leq \epsilon$ and such that*

$$\forall t \geq 0, \tilde{\Phi}^t(V^{ss}) \subset V^{ss},$$

resp.

$$\forall t \geq 0, \tilde{\Phi}^t(V^s) \subset V^s,$$

If we replace s by u , the same conclusion holds except that we have to replace positive times by negative ones. Like the property of compactness, this result was already proved in [9] in the particular case of a Morse-Smale *gradient* flow satisfying certain linearizing properties given for instance by the Sternberg-Chen's Theorem. We will prove that the extension to more general Morse-Smale flows is still true.

Proof. In order to prove this result, we proceed in two stages. First, we construct a small neighborhood of the projection of Σ_{ss} (resp. Σ_s) on M . Then, we lift this neighborhood to S^*M into an ϵ -neighborhood of Σ_{ss} (resp. Σ_s) satisfying the expected properties. In the case of Σ_{ss} , the projection on M is exactly given by

$$\tilde{\Sigma}_{ss} := \bigcup_{j: 0 \leq \dim W^u(\Lambda_j) < n} W^u(\Lambda_j),$$

while the projection of Σ_s is

$$\tilde{\Sigma}_s := \tilde{\Sigma}_{ss} \cup \bigcup_{j: \dim W^u(\Lambda_j)=n, \Lambda_j \text{ closed orbit}} W^u(\Lambda_j).$$

As in [9], we will construct these neighborhoods by induction on the Smale diagrams, starting from the “smallest” critical elements. Indeed, even if it was not explicitly mentioned, this induction procedure was in some sense ordered by the Smale causality relation defined by Theorem 3.5. Fix $\epsilon > 0$. We start with the case of $\tilde{\Sigma}_{ss}$ and we will then extend the obtained neighborhood into a neighborhood of $\tilde{\Sigma}_s$ by making one more step in the induction.

Before starting this construction by induction, let us fix some a priori neighborhoods of the critical elements $(\Lambda_j)_{1 \leq j \leq K}$. Again, the cases of fixed points and closed orbits have to be treated in a slightly different manner. Suppose first that Λ_j is a *critical point* and fix two small parameters $\epsilon_j, \epsilon'_j > 0$. Near this critical point, we use the coordinates (x, y) defined in paragraph 4.2, and we set

$$W(\Lambda_j, \epsilon_j, \epsilon'_j) := \{(x, y) : \|(x, 0)\| < \epsilon_j, \|(0, y)\| < \epsilon'_j\}.$$

Recall that all norms on \mathbb{R}^n are equivalent. Thus, proceeding as in [6, Prop. 5.2.2], we can suppose that the norm in (5) is adapted in the sense that the constants C_1 and C_2 appearing in these inequalities are in fact equal to 1. From the expression of the flow (4), we know that every point in this small neighborhood will in forward times either stay in this neighborhood (and converges to Λ_j) or escape this neighborhood in finite time through $\{(x, y) : \|(x, 0)\| < \epsilon_j, \|(0, y)\| = \epsilon'_j\}$. Let us now discuss the future of the trajectory in the second case. We note that every point in $\{(0, y) : \|(0, y)\| = \epsilon'_j\} \subset W^u(\Lambda_j)$ will reach in the future a critical element Λ_i . In particular, this imposes that $W^u(\Lambda_i) \subseteq W^u(\Lambda_j)$. Fix now a family of small neighborhoods $\tilde{V}_i \subset \subset \tilde{W}_i$ around every critical element Λ_i satisfying $W^u(\Lambda_i) \subseteq W^u(\Lambda_j)$. One can verify that there exists a uniform time T_j depending only on the size of the neighborhoods \tilde{V}_i and on ϵ'_j such that every point in $\{(0, y) : \|(0, y)\| = \epsilon'_j\}$ will cross one of the \tilde{V}_i (with $i \neq j$) during the time interval $[0, T_j]$. According to [46, Lemma 11.11], the distance between points can only grow at most with an exponential rate under the flow. Hence, one knows that, for $\epsilon_j > 0$ small enough (depending only on T_j , on ϵ'_j and on the neighborhoods \tilde{W}_i with $i \neq j$), every point in $\{(x, y) : \|(x, 0)\| < \epsilon_j, \|(0, y)\| = \epsilon'_j\}$ will cross one of the neighborhood \tilde{W}_i during the time interval $[0, T_j]$. Applying [46, Lemma 11.11] one more time, we can also suppose without loss of generality that $\epsilon'_j > 0$ is small enough to ensure that the trajectory of these points is ϵ -close to $W^u(\Lambda_j)$ during all the time interval $[0, T_j]$.

We now construct an a priori neighborhood when Λ_j is a *closed orbit*. The main difference is that we have to take into account the direction of the flow. Again, we fix two small parameters $\epsilon_j, \epsilon'_j > 0$ and we use the coordinates of paragraph 4.2. Then, we set

$$W(\Lambda_j, \epsilon_j, \epsilon'_j) := \{(P(\theta, 0)(x, y), \theta) : \theta \in \mathbb{R}/\mathcal{P}_{\Lambda_j}\mathbb{Z}, \|(x, 0)\| < \epsilon_j, \|(0, y)\| < \epsilon'_j\}.$$

As for the case of critical points, we can without loss of generality suppose that the constants C_1 and C_2 appearing in (35) are both equal to 1. As before, every point in this tubular neighborhood of the closed orbit will either stay in this neighborhood for every $t \geq 0$ or it will escape it in finite time by crossing the subset

$$\{(P(\theta, 0)(x, y), \theta) : \theta \in \mathbb{R}/\mathcal{P}_{\Lambda_j}\mathbb{Z}, \|(x, 0)\| < \epsilon_j, \|(0, y)\| = \epsilon'_j\}.$$

Then, every point which lies in the intersection of $W^u(\Lambda_j)$ with the former subset will accumulate on some Λ_i as $t \rightarrow +\infty$. Once again, this implies that $W^u(\Lambda_i) \subseteq W^u(\Lambda_j)$ with $i \neq j$. As before, we can fix two families of neighborhoods $\tilde{V}_i \subset \subset \tilde{W}_i$ around every critical element Λ_i satisfying $W^u(\Lambda_i) \subseteq W^u(\Lambda_j)$ with $i \neq j$. Arguing as above, we can find $T_j > 0$ depending only on the size of the neighborhoods \tilde{V}_i and on ϵ'_j such that every point in

$$\{(P(\theta, 0)(0, y), \theta) : \theta \in \mathbb{R}/\mathcal{P}_{\Lambda_j}\mathbb{Z}, \|(0, y)\| = \epsilon'_j\} \subset W^u(\Lambda_j)$$

will cross one the \tilde{V}_i (with $i \neq j$) during the time interval $[0, T_j]$. We can also verify by similar arguments that, for $\epsilon_j > 0$ small enough (depending only on T_j , on ϵ'_j and on the neighborhoods \tilde{W}_i with $i \neq j$), every point in

$$\{(P(\theta, 0)(x, y), \theta) : \theta \in \mathbb{R}/\mathcal{P}_{\Lambda_j}\mathbb{Z}, \|(x, 0)\| < \epsilon_j, \|(0, y)\| = \epsilon'_j\}$$

will cross one of the neighborhood \tilde{W}_i during the time interval $[0, T_j]$ and will remain in an ϵ -neighborhood of $W^u(\Lambda_j)$ on the interval $[0, T_j]$.

To summarize, we have constructed some small neighborhoods near every critical element “adapted” to the forward dynamics. Let us now explain how to paste these neighborhoods together in order to construct a neighborhood of $\tilde{\Sigma}_{ss}$ and $\tilde{\Sigma}_s$. We denote by K_0 the number of levels in Smale’s diagram and we proceed by induction from $k = 1$ to $k = K_0$. We start our induction by considering all the critical elements which are minimal for Smale’s partial order relation on the unstable manifolds, i.e. the ones which are at the bottom of Smale’s diagram. All these elements correspond to unstable manifolds which are either of dimension 0 or of dimension 1. Around these critical elements, we fix the neighborhood $W(\Lambda_j, \epsilon_j, \epsilon'_j = 0)$ defined above. From the expression of the flow in local chart, one can verify that all the points in this neighborhood will remain in it in forward times.

Suppose now that we have fixed some values for ϵ_j and ϵ'_j for every Λ_j at height $k \leq k_0$ in Smale’s diagram and satisfying $\dim W^u(\Lambda_j) < n$. We can then determine the values of ϵ_j and ϵ'_j at the level $k_0 + 1$ for elements satisfying $\dim W^u(\Lambda_j) < n$. This can be done just by taking $\epsilon_j > 0$ and $\epsilon'_j > 0$ to ensure that the above construction is valid with \tilde{W}_i being equal to $W(\Lambda_i, \epsilon_i, \epsilon'_i)$ for every Λ_i below $k_0 + 1$ in Smale’s diagram. Once we have constructed all these neighborhoods, we can verify that

$$\bigcup_{j: \dim W^u(\Lambda_j) < n} \bigcup_{t \geq 0} \varphi^t(V(\Lambda_j, \epsilon_j, \epsilon'_j))$$

is an ϵ -neighborhood of $\tilde{\Sigma}_{ss}$ which we recall is the projection on M of Σ_{ss} . By construction, this neighborhood is invariant under forward times of the Morse-Smale flow. In order to get

a neighborhood of $\tilde{\Sigma}_s$, we just need to add in the induction the neighborhoods associated with the closed orbits Λ_j whose unstable manifold has full dimension.

Remark 4.5. Before lifting these neighborhoods to the unit cotangent bundle, let us make some observation. Our proof did not really use the fact that we were considering $\tilde{\Sigma}_s$ or $\tilde{\Sigma}_{ss}$. Indeed, if we fix a subfamily $(\Lambda_j)_{j \in J}$ of the critical elements, then our procedure allows to construct, for every $\epsilon > 0$, an ϵ -neighborhood of

$$\bigcup_{j \in J} \bigcup_{W^u(\Lambda_i) \leq W^u(\Lambda_j)} W^u(\Lambda_i)$$

which is invariant under application of the flow in forward times.

Let us now explain how these neighborhoods can be lifted into neighborhoods of Σ_s and Σ_{ss} . Again, we start with defining a priori neighborhoods close to every critical element. First, we consider the case of a fixed point Λ_j . In local coordinates, the conormal of the unstable manifold is given by points of the form $(0, y, \xi, 0)$. Hence, for $\epsilon_j'' > 0$, we lift the neighborhood $W(\Lambda_j, \epsilon_j, \epsilon_j')$ into a neighborhood $W(\Lambda_j, \epsilon_j, \epsilon_j', \epsilon_j'')$ inside S^*M by adding cotangent vectors which are ϵ_j'' -close to the direction $\eta = 0$. By compactness of Σ_{ss} , we can find some small enough $\epsilon_{j,0}'' > 0$ such that, for every $0 < \epsilon_j'' \leq \epsilon_{j,0}''$, there exists $\epsilon_{j,0} > 0$ such that, for any $0 < \epsilon_j < \epsilon_{j,0}$, $W(\Lambda_j, \epsilon_j, \epsilon_j', \epsilon_j'')$ contains $\Sigma_{ss} \cap S^*W(\Lambda_j, \epsilon_j, \epsilon_j')$. Moreover, using again the fact that the distance between points can grow at most exponentially fast in time [46, Lemma 11.11], we can ensure that, for $\epsilon_j > 0$ and $\epsilon_j'' > 0$ small enough, every point in this neighborhood will remain ϵ -close to $N^*(W^u(\Lambda_j))$ under the evolution by $\tilde{\Phi}^t$ for $t \in [0, T_j]$ and will reach during the interval $[0, T_j]$ (up to the fact that we may need to increase T_j a little bit) one of the neighborhoods associated to $W^u(\Lambda_i) \leq W^u(\Lambda_j)$. In the case of closed orbit, we proceed similarly, i.e. we lift the a priori neighborhood near $\Lambda_j \subset M$ into a neighborhood of S^*M by adding the cotangent vectors which are $\epsilon_j'' > 0$ -close to $(\eta, \Theta) = 0$ (in the case of Σ_{ss}) or to $\eta = 0$ (in the case of Σ_s). This defines a neighborhood $W(\Lambda_j, \epsilon_j, \epsilon_j', \epsilon_j'')$ in the case of Σ_{ss} and a neighborhood $W_0(\Lambda_j, \epsilon_j, \epsilon_j', \epsilon_j'')$ in the case of Σ_s . Then, by similar arguments, we can ensure that points inside this neighborhood have the same properties under the forward evolution by $\tilde{\Phi}^t$. Having settled these neighborhoods, we can then define an ϵ -neighborhood of Σ_{ss} by setting

$$V_{ss} := \bigcup_{j: \dim W^u(\Lambda_j) < n} \bigcup_{t \geq 0} \tilde{\Phi}^t (W(\Lambda_j, \epsilon_j, \epsilon_j', \epsilon_j'')).$$

Obviously, this neighborhood is invariant under $\tilde{\Phi}^t$ for $t \geq 0$. Similarly, for Σ_s , we set

$$V_s := \bigcup_{j: \dim \Lambda_j = 1} \bigcup_{t \geq 0} \tilde{\Phi}^t (W_0(\Lambda_j, \epsilon_j, \epsilon_j', \epsilon_j'')) \bigcup_{j: \dim \Lambda_j = 0, \dim W^u(\Lambda_j) < n} \bigcup_{t \geq 0} \tilde{\Phi}^t (W(\Lambda_j, \epsilon_j, \epsilon_j', \epsilon_j'')).$$

Again, this defines by construction an ϵ -neighborhood of Σ_s which is invariant under the forward flow. \square

Remark 4.6. We emphasized the case of Σ_{ss} and Σ_s as they will appear naturally in our construction of anisotropic Sobolev spaces. Yet, if we fix $J \subset \{1, \dots, K\}$, our proof also

shows how to construct forward invariant small neighborhoods of the following subsets of S^*M :

$$\bigcup_{j \in J} \bigcup_{W^u(\Lambda_i) \leq W^u(\Lambda_j)} N^*(W^u(\Lambda_i)) \cap S^*M,$$

and

$$\bigcup_{j \in J} \bigcup_{W^u(\Lambda_i) \leq W^u(\Lambda_j)} \bigcup_{x \in \Lambda_i} N^*(W^{uu}(x)) \cap S^*M.$$

5. ANISOTROPIC SOBOLEV SPACES

In this section, we aim at proving Theorems 2.1 and 2.2 or more precisely their generalization to vector bundles (\mathcal{E}, ∇) . For that purpose, we shall construct anisotropic Sobolev spaces adapted to the operator $-\mathcal{L}_{V, \nabla}^{(k)}$ for every $0 \leq k \leq n$. More precisely, for every $N > 0$ and for every $0 \leq k \leq n$, we will construct Hilbert spaces of currents $\mathcal{H}_k^{mN}(M, \mathcal{E})$ such that

$$-\mathcal{L}_{V, \nabla}^{(k)} : \mathcal{H}_k^{mN}(M, \mathcal{E}) \longrightarrow \mathcal{H}_k^{mN}(M, \mathcal{E})$$

has discrete spectrum (with finite multiplicity) on $\text{Re}(z) > -N$ – see Proposition 5.5. Once this spectral framework is properly settled, one can easily deduce our main Theorems on the existence of a discrete set of Pollicott-Ruelle resonances – see paragraph 5.5. Hence, everything boils down to constructing appropriate Hilbert spaces. For that purpose, we follow the (microlocal) strategy initiated by Faure and Sjöstrand in [17]. This can be essentially decomposed in two steps. First, we construct an escape function (or Lyapunov function) adapted to the Morse-Smale dynamics – see Lemma 5.2. This can be achieved by combining the dynamical results of section 4 and Meyer’s Theorem 3.7 with the strategy used by Faure and Sjöstrand in the Anosov framework. This escape function allows to define a Sobolev space of currents. Following exactly the proof of Faure and Sjöstrand, we can finally show that $-\mathcal{L}_{V, \nabla}^{(k)}$ has nice spectral properties on that space via microlocal arguments and analytic Fredholm theory. This kind of strategy is natural in the context of the study of semiclassical resonances of Schrödinger operators [26] and we refer to [17] for a more detailed discussion on that aspect.

Remark 5.1. The proof in [17] only deals with the case $k = 0$ and $\mathcal{E} = M \times \mathbb{C}$. Yet, the argument can be adapted to any $0 \leq k \leq n$ as it only relies on the construction of a proper escape function for the flow as in the case $k = 0$. We emphasize that the case of vector bundles in the Anosov flows framework was considered via microlocal methods by Dyatlov and Zworski in order to prove the meromorphic continuation of the zeta function [13]. We chose to use the microlocal technics from these references but it is most likely that the methods developed in [7, 22] could also be developed to define a proper spectral framework.

5.1. Escape function. In order to state the first result, we introduce the following subset inside T^*M :

$$\Gamma_0 := \{(x; \xi) \in T^*M \setminus 0 : \exists 1 \leq j \leq K, x \in \Lambda_j \text{ and } (x; \xi) \in N^*(W^{uu}(x)) \cap N^*(W^{ss}(x))\}.$$

Then, the main dynamical ingredient to construct appropriate Sobolev spaces for $-\mathcal{L}_{V,\nabla}$ is the following Lemma:

Lemma 5.2 (Escape function). *Let V be a smooth vector field inducing a \mathcal{C}^1 linearizable Morse-Smale flow. Let (u, n_0, s) be elements in \mathbb{R} with $u < -2\|E\|_\infty < 2\|E\|_\infty < n_0 < s$ where E is the energy function of Theorem 3.7. Let N_0 be an arbitrarily small conic neighborhood of Γ_0 inside $T^*M \setminus 0$.*

*Then, there exists a smooth function $m(x; \xi) \in \mathcal{C}^\infty(T^*M)$ called an order function, taking values in $[u, s]$, and an escape function on T^*M defined by*

$$G_m(x; \xi) := m(x; \xi) \log(1 + f(x; \xi)^2)^{\frac{1}{2}},$$

*where $f(x; \xi) \in \mathcal{C}^\infty(T^*M)$. Moreover, $f(x; \xi)$ is positive and positively homogeneous of degree 1 for $\|\xi\| \geq 1$, $f(x; \xi) = \|\xi\|_x$ outside a small neighborhood of N_0 and $f(x; \xi) = |\xi(V(x))| = |H(x; \xi)|$ in a small neighborhood of N_0 . Finally, one has*

- (1) *For $|\xi| \geq 1$, $m(x; \xi)$ depends only on $\xi/\|\xi\|$ and it takes values $\leq \frac{u}{4}$ (resp. $\geq \frac{2n_0}{4}$ and $\geq \frac{s}{4}$) in a small neighborhood of $\cup_{j=1}^K N^*(W^u(\Lambda_j))$ (resp. Γ_0 and $\cup_{j=1}^K N^*(W^s(\Lambda_j))$).*
- (2) *There exists $R > 0$ such that, for every $(x; \xi)$ in T^*M satisfying $\|\xi\| \geq R$, one has*

$$X_H \cdot (G_m)(x; \xi) \leq 0.$$

- (3) *If in addition $(x; \xi) \notin N_0$, then*

$$X_H \cdot (G_m)(x; \xi) \leq -C_m < 0,$$

with

$$C_m := c \min(|u|, s),$$

for some constant $c > 0$ independent of u , n_0 and s .

The same result was proved in the Anosov framework in [17, Lemma 1.2]. The main input compared with that case are Meyer's Theorem 3.7 on the one hand and Theorems 4.2 and 4.4 on the other. We postpone the proof of this Lemma to Paragraph 5.6 and we will first draw some spectral consequences of it following [17].

5.2. Vector bundles and connection. Before defining the Sobolev spaces, we collect a few definitions and properties and we refer to [30, Ch.6-8-12] for more details. We fix a smooth *complex* vector bundle $\pi : \mathcal{E} \rightarrow M$ [30, Def. 6.21] of rank N . We suppose without loss of generality that this vector bundle is equipped with a smooth Hermitian structure $\langle \cdot, \cdot \rangle_{\mathcal{E}}$. We now consider the space $\Omega^k(M, \mathcal{E})$ of smooth differential k -forms with coefficients in \mathcal{E} . If we are given a metric g on M , we can fix the inner product $\langle \cdot, \cdot \rangle_{g^*}^{(k)}$ on $\Lambda^k(T^*M)$ which is induced by the metric g on M . Then, the Hodge star operator is the unique isomorphism $\star_k : \Lambda^k(T^*M) \rightarrow \Lambda^{n-k}(T^*M)$ such that, for every ψ_1 in $\Omega^k(M) = \Omega^k(M, \mathbb{C})$ and ψ_2 in $\Omega^{n-k}(M) = \Omega^{n-k}(M, \mathbb{C})$,

$$\int_M \psi_1 \wedge \psi_2 = \int_M \langle \psi_1, \star_k^{-1} \psi_2 \rangle_{g^*(x)}^{(k)} \omega_g(x),$$

where ω_g is the volume form induced by the Riemannian metric on M . This induces a map $\star_k : \Omega^k(M, \mathcal{E}) \rightarrow \Omega^{n-k}(M, \mathcal{E})$ which acts trivially on the \mathcal{E} -coefficients. Using the metric $g_{\mathcal{E}}$ on \mathcal{E} , we can also introduce the following pairing, for every $0 \leq k, l \leq n$,

$$\langle \cdot \wedge \cdot \rangle_{\mathcal{E}} : \Omega^k(M, \mathcal{E}) \times \Omega^l(M, \mathcal{E}) \rightarrow \Omega^{k+l}(M).$$

Then we can define the positive definite Hodge inner product on $\Omega^k(M, \mathcal{E})$ as

$$(\psi_1, \psi_2) \in \Omega^k(M, \mathcal{E}) \times \Omega^k(M, \mathcal{E}) \mapsto \int_M \langle \psi_1 \wedge \star_k(\psi_2) \rangle_{\mathcal{E}}.$$

In particular, we can define $L^2(M, \Lambda^k(T^*M) \otimes \mathcal{E})$ as the completion of $\Omega^k(M, \mathcal{E})$ for this scalar product.

We would now like to define an analogue of the Lie derivative along a vector field V in the context of differential forms with values in a vector bundle \mathcal{E} . For that purpose, we observe that the contraction operator ι_V is well-defined and we have to introduce a proper definition for the coboundary operator. For that purpose, we fix a smooth (Koszul) connection [30, Def. 12.1 and p. 505], i.e. a linear map

$$\nabla : \Gamma(M, \mathcal{E}) \simeq \Omega^0(M, \mathcal{E}) \rightarrow \Gamma(M, T^*M \otimes \mathcal{E}) \simeq \Omega^1(M, \mathcal{E}),$$

satisfying, for every $\psi \in C^\infty(M)$ and for every u in $\Omega^0(M, \mathcal{E})$,

$$\nabla(\psi u) = \psi \nabla u + (d\psi) u,$$

where d is the usual coboundary operator on M . According to [30, Th. 12.57], this map uniquely extends to a map d^∇ such that, for every $0 \leq k, l \leq n$,

- $d^\nabla : \Omega^k(M, \mathcal{E}) \rightarrow \Omega^{k+1}(M, \mathcal{E})$,
 - for every (ψ, u) in $\Omega^k(M) \times \Omega^l(M, \mathcal{E})$,
- $$d^\nabla(\psi \wedge u) = d\psi \wedge u + (-1)^k \psi \wedge d^\nabla u \text{ and } d^\nabla(u \wedge \psi) = d^\nabla u \wedge \psi + (-1)^l u \wedge d\psi,$$
- for every u in $\Omega^0(M, \mathcal{E})$, $d^\nabla u = \nabla u$.

The operator d^∇ is the coboundary operator associated with (\mathcal{E}, ∇) , and we define the corresponding *Lie derivative* along the vector field V as

$$(12) \quad \boxed{\mathcal{L}_{V, \nabla} = d^\nabla \circ \iota_V + \iota_V \circ d^\nabla.}$$

Remark 5.3. We define the *curvature* of the vector bundle (\mathcal{E}, ∇) as the unique map F_∇ in $\Omega^2(M, \text{End}(\mathcal{E}))$ such that, for every $0 \leq k \leq n$ and for every u in $\Omega^k(M, \mathcal{E})$,

$$d^\nabla \circ d^\nabla u = F_\nabla \wedge u.$$

We say (\mathcal{E}, ∇) is a *flat vector bundle* if $F_\nabla = 0$ (equivalently, $d^\nabla \circ d^\nabla = 0$).

Finally, one can define a connection

$$\nabla^\dagger : \Omega^0(M, \mathcal{E}') \rightarrow \Omega^1(M, \mathcal{E}'),$$

which is completely determined by requiring the following property, for any (ψ_1, ψ_2) in $\Omega^0(M, \mathcal{E}) \times \Omega^0(M, \mathcal{E}')$,

$$d(\psi_2(\psi_1)) = \psi_2(\nabla \psi_1) + (\nabla^\dagger \psi_2)(\psi_1).$$

This connection induces a coboundary operator $d^{\nabla^\dagger} : \Omega^k(M, \mathcal{E}) \rightarrow \Omega^{k+1}(M, \mathcal{E})$. By construction, one has, for any ψ_1 in $\Omega^{k-1}(M, \mathcal{E})$ and for any ψ_2 in $\Omega^{n-k}(M, \mathcal{E}')$,

$$(13) \quad \int_M \psi_2(d^\nabla \psi_1) = (-1)^k \int_M (d^{\nabla^\dagger} \psi_2)(\psi_1).$$

In the following, $\mathcal{E} \rightarrow M$ is always a smooth complex vector bundle endowed with an Hermitian structure $\langle \cdot, \cdot \rangle_{\mathcal{E}}$ and a connection ∇ (not necessarily flat).

5.3. Definition of the anisotropic Sobolev spaces. First, we set

$$(14) \quad A_m(x; \xi) := \exp G_m(x; \xi),$$

where $G_m(x; \xi)$ is given by Lemma 5.2. Let now $0 \leq k \leq n$ and $\mathcal{E} \rightarrow M$ be a smooth vector bundle equipped with a Riemannian structure. We consider the vector bundle $\Lambda^k(T^*M) \otimes \mathcal{E}$. We define $\mathbf{A}_m^{(k)}(x; \xi) := A_m(x; \xi) \mathbf{Id}$ belonging to $\text{Hom}(\Lambda^k(T^*M) \otimes \mathcal{E})$. We define an anisotropic Sobolev space of currents by setting

$$\boxed{\mathcal{H}_k^m(M, \mathcal{E}) = \text{Op}(\mathbf{A}_m^{(k)})^{-1} L^2(M, \Lambda^k(T^*M) \otimes \mathcal{E}),}$$

where $\text{Op}(\mathbf{A}_m^{(k)})$ is a (essentially selfadjoint) pseudodifferential operator with principal symbol $\mathbf{A}_m^{(k)}$.

Remark 5.4. Note that this requires to deal with symbols of variable orders whose symbolic calculus was described in Appendix A of [16]. This can be done as the symbol $m(x; \xi)$ belongs to the standard class of symbols $S^0(T^*M)$. We also refer to [13, App. C.1] or to [4, Part I] for a brief reminder of pseudodifferential operators with values in vector bundles. In particular, adapting the proof of [16, Cor. 4] to the vector bundle valued framework, one can verify that $\mathbf{A}_m^{(k)}$ is an elliptic symbol, and thus $\text{Op}(\mathbf{A}_m^{(k)})$ can be chosen to be invertible.

Mimicking the proofs of [16], we can deduce some properties of these spaces of currents. First of all, they are endowed with a Hilbert structure inherited from the L^2 -structure on M . The space

$$\mathcal{H}_k^m(M, \mathcal{E})' = \text{Op}(\mathbf{A}_m^{(k)}) L^2(M, \Lambda^k(T^*M) \otimes \mathcal{E})$$

is the topological dual of $\mathcal{H}_k^m(M, \mathcal{E})$ which is in fact reflexive. We also note that the space $\mathcal{H}_k^m(M, \mathcal{E})$ can be identified with $\mathcal{H}_0^m(M, \mathbb{R}) \otimes_{C^\infty(M)} \Omega^k(M, \mathcal{E})$. Finally, one has

$$\Omega^k(M, \mathcal{E}) \subset \mathcal{H}_k^m(M, \mathcal{E}) \subset \mathcal{D}'^k(M, \mathcal{E}),$$

where the injections are continuous.

The Hermitian structure on \mathcal{E} allows to define a canonical isomorphism $\tau_{\mathcal{E}} : \mathcal{E} \rightarrow \mathcal{E}'$ by setting, for every ψ in \mathcal{E} , $\tau_{\mathcal{E}}(\psi) = \langle \psi, \cdot \rangle_{\mathcal{E}}$. Then, combined with the Hodge star map, it induces an isomorphism from $\mathcal{H}_k^m(M, \mathcal{E})'$ to $\mathcal{H}_{n-k}^{-m}(M, \mathcal{E}')$, whose Hilbert structure is given by the scalar product

$$(\psi_1, \psi_2) \in \mathcal{H}_{n-k}^{-m}(M, \mathcal{E}')^2 \mapsto \langle \star_k^{-1} \tau_{\mathcal{E}}^{-1}(\psi_1), \star_k^{-1} \tau_{\mathcal{E}}^{-1}(\psi_2) \rangle_{\mathcal{H}_k^m(M, \mathcal{E})}.$$

Thus, the topological dual of $\mathcal{H}_k^m(M, \mathcal{E})$ can be identified with $\mathcal{H}_{n-k}^{-m}(M, \mathcal{E}')$, where, for every ψ_1 in $\Omega^k(M, \mathcal{E})$ and ψ_2 in $\Omega^{n-k}(M, \mathcal{E}')$, one has the following duality relation:

$$\begin{aligned} \langle \psi_2, \psi_1 \rangle_{\mathcal{H}_{n-k}^{-m}(M, \mathcal{E}') \times \mathcal{H}_k^m(M, \mathcal{E})} &= \int_M \psi_2 \wedge \psi_1 \\ &= \langle \text{Op}(\mathbf{A}_m^{(k)})^{-1} \star_k^{-1} \tau_{\mathcal{E}}^{-1}(\overline{\psi_2}), \text{Op}(\mathbf{A}_m^{(k)})\psi_1 \rangle_{L^2(M, \Lambda^k(T^*M) \otimes \mathcal{E})} \\ &= \langle \star_k^{-1} \tau_{\mathcal{E}}^{-1}(\psi_2), \psi_1 \rangle_{\mathcal{H}_k^m(M, \mathcal{E}) \times \mathcal{H}_k^m(M, \mathcal{E})'}. \end{aligned}$$

5.4. Pollicott–Ruelle resonances and their resonant states. Now that we have defined our Sobolev in a similar fashion as for the Anosov setting of [17], we can follow almost verbatim the argument of Faure and Sjöstrand in [17] in order to show the existence of a discrete dynamical spectrum on these spaces. Indeed, this part of their arguments only made use of the dynamical properties of m combined with microlocal tools and analytic Fredholm theory. More precisely, the main result on the spectral properties of $-\mathcal{L}_{V, \nabla}^{(k)}$ acting on these anisotropic spaces is the following Proposition:

Proposition 5.5 (Discrete spectrum). *The operator $-\mathcal{L}_{V, \nabla}^{(k)}$ defines a maximal closed unbounded operator on $\mathcal{H}_k^m(M, \mathcal{E})$,*

$$-\mathcal{L}_{V, \nabla}^{(k)} : \mathcal{H}_k^m(M, \mathcal{E}) \rightarrow \mathcal{H}_k^m(M, \mathcal{E}),$$

with domain given by $\mathcal{D}(-\mathcal{L}_{V, \nabla}^{(k)}) := \{\psi \in \mathcal{H}_k^m(M, \mathcal{E}) : -\mathcal{L}_{V, \nabla}^{(k)}\psi \in \mathcal{H}_k^m(M, \mathcal{E})\}$. It coincides with the closure of $-\mathcal{L}_{V, \nabla}^{(k)} : \Omega^k(M, \mathcal{E}) \rightarrow \Omega^k(M, \mathcal{E})$ in the graph norm for operators. Moreover, there exists a constant C_0 in \mathbb{R} (that depends on the choice of the order function $m(x; \xi)$) such that $-\mathcal{L}_{V, \nabla}^{(k)}$ has empty spectrum for $\text{Re}(z) > C_0$. Finally, the operator

$$-\mathcal{L}_{V, \nabla}^{(k)} : \mathcal{H}_k^m(M, \mathcal{E}) \rightarrow \mathcal{H}_k^m(M, \mathcal{E}),$$

has a discrete spectrum with finite multiplicity in the domain

$$\text{Re}(z) > -C_m + C,$$

where $C > 0$ depends only the choice of the metric g , $\langle, \rangle_{\mathcal{E}}$ and $C_m > 0$ is the constant from Lemma 5.2.

The second part on the discrete spectrum is obtained by showing that the operator $(-\mathcal{L}_{V, \nabla}^{(k)} - z)$ is a Fredholm operator of index 0 depending analytically on z in the corresponding half plane [26, 46]. In the case of Anosov flows, the proof of this result was given by Faure-Sjöstrand in [17, Sect. 3] for $k = 0$ while the extension to the case of currents was done by Dyatlov-Zworski in [13, Sect. 3]. Note that the proofs in both references are of slightly different nature but they both crucially rely on the properties of the escape function used to define the anisotropic space $\mathcal{H}_k^m(M, \mathcal{E})$. The proof of this Proposition was given in great details in [17, Th. 1.4] in the case $k = 0$ and $\mathcal{E} = M \times \mathbb{C}$. As was already mentionned, the extension to the case where $0 \leq k \leq n$ and where \mathcal{E} is an arbitrary vector

bundle can be adapted almost verbatim except that we have to deal with pseudodifferential operators with values in $\Lambda^k(T^*M) \otimes \mathcal{E}$. As was already observed in [13, 9], the main point to adapt to the vector bundle framework is that the (pseudodifferential) operators under consideration have a *diagonal symbol*. In fact, given any local basis $(e_j)_{j=1, \dots, J_k}$ of $\Lambda^k(T^*M) \otimes \mathcal{E}$ and any family $(u_j)_{j=1, \dots, J_k}$ of smooth functions $\mathcal{C}^\infty(M)$, one has

$$\mathcal{L}_{V, \nabla}^{(k)} \left(\sum_{j=1}^{J_k} u_j e_j \right) = \sum_{j=1}^{J_k} \mathcal{L}_V(u_j) e_j + \sum_{j=1}^{J_k} \mathcal{L}_{V, \nabla}^{(k)}(e_j) u_j,$$

where the second part of the sum in the right-hand side is a lower order term (of order 0). In other words, the principal symbol of $\mathcal{L}_{V, \nabla}^{(k)}$ is $\xi(V(x)) \mathbf{Id}_{\Lambda^k(T^*M) \otimes \mathcal{E}}$. This diagonal form allows to adapt the proofs of [17] to this vector bundle framework and we refer to [17] for a detailed proof. Note that, in this reference, the author use the convention $-i\mathcal{L}_V$ instead of $-\mathcal{L}_V$.

Remark 5.6. We also note that they implicitly show [17, Lemma 3.3] that, for every z in \mathbb{C} satisfying $\text{Im} z > C_0$, one has

$$(15) \quad \left\| \left(\mathcal{L}_{V, \nabla}^{(k)} + z \right)^{-1} \right\|_{\mathcal{H}_k^m(M, \mathcal{E}) \rightarrow \mathcal{H}_k^m(M, \mathcal{E})} \leq \frac{1}{\text{Re}(z) - C_0}.$$

In particular, combining Proposition 5.5 to the Hille-Yosida Theorem [15, Cor. 3.6, p. 76], one knows that

$$(16) \quad (\varphi^{-t})^* : \mathcal{H}_k^m(M, \mathcal{E}) \rightarrow \mathcal{H}_k^m(M, \mathcal{E}),$$

generates a strongly continuous semigroup which is defined for every $t \geq 0$ and whose norm is bounded by e^{tC_0} .

We now list some properties of this spectrum:

- As in [17, Th. 1.5], we can show that the eigenvalues (counted with their algebraic multiplicity) and the eigenspaces of $-\mathcal{L}_{V, \nabla}^{(k)} : \mathcal{H}_k^m(M, \mathcal{E}) \rightarrow \mathcal{H}_k^m(M, \mathcal{E})$ are in fact independent of the choice of escape function. For every $0 \leq k \leq n$, we call the eigenvalues the **Pollicott-Ruelle resonances of index k** and we denote by $\mathcal{R}_k(V, \nabla)$ this set.
- By duality, the same spectral properties holds for the dual operator

$$(17) \quad (-\mathcal{L}_{V, \nabla}^{(k)})^\dagger = -\mathcal{L}_{-V, \nabla^\dagger}^{(n-k)} : \mathcal{H}_{n-k}^{-m}(M, \mathcal{E}') \rightarrow \mathcal{H}_{n-k}^{-m}(M, \mathcal{E}').$$

- Given any z_0 in $\mathcal{R}_k(V, \nabla)$, the corresponding spectral projector $\pi_{z_0}^{(k)}$ is given by [26, Appendix]:

$$(18) \quad \pi_{z_0}^{(k)} := \frac{1}{2i\pi} \int_{\gamma_{z_0}} (z + \mathcal{L}_{V, \nabla}^{(k)})^{-1} dz : \mathcal{H}_k^m(M, \mathcal{E}) \rightarrow \mathcal{H}_k^m(M, \mathcal{E}),$$

where γ_{z_0} is a small contour around z_0 which only contains the eigenvalue z_0 in its interior.

- Given any z_0 in \mathbb{C} with $\operatorname{Re}(z_0) > -C_m + C$, there exists $m_k(z_0) \geq 1$ such that, in a small neighborhood of z_0 , one has

$$(19) \quad (z + \mathcal{L}_{V,\nabla}^{(k)})^{-1} = \sum_{l=1}^{m_k(z_0)} (-1)^{l-1} \frac{(\mathcal{L}_{V,\nabla}^{(k)} + z_0)^{l-1} \pi_{z_0}^{(k)}}{(z - z_0)^j} + R_{\lambda,k}(z) : \mathcal{H}_k^m(M, \mathcal{E}) \rightarrow \mathcal{H}_k^m(M, \mathcal{E}),$$

with $R_{z_0,k}(z)$ an holomorphic function and with $\pi_{z_0}^{(k)} = 0$ whenever $z_0 \notin \mathcal{R}_k(V, \nabla)$.

5.5. Proof of Theorems 2.1 and 2.2. We will now briefly deduce the proofs of our main Theorems (in that case $\mathcal{E} = M \times \mathbb{R}$). We will in fact prove something slightly stronger as we will verify that these statements hold for any $\psi_1 \in \mathcal{H}_k^m(M)$ and any $\psi_2 \in \mathcal{H}_{n-k}^{-m}(M)$. Let ψ_1 be an element in $\mathcal{H}_k^m(M)$. Thanks to Remark 5.6, we know that

$$\int_0^{+\infty} e^{-tz} \varphi^{-t*}(\psi_1) dt = (\mathcal{L}_V^{(k)} + z)^{-1}(\psi_1)$$

holds in $\mathcal{H}_k^m(M)$ for $\operatorname{Re}(z) > C_0$. From Proposition 5.5, we have a meromorphic extension of the right-hand side on $\operatorname{Re}(z) > -C_m + C$ whose poles are included in the set of Pollicott-Ruelle resonances $\mathcal{R}_k(V)$. Moreover, from (19), we also know that the following holds in $\mathcal{H}_k^m(M)$:

$$(z + \mathcal{L}_V^{(k)})^{-1}(\psi_1) = \sum_{l=1}^{m_k(z_0)} (-1)^{l-1} \frac{(\mathcal{L}_V^{(k)} + z_0)^{l-1} \pi_{z_0}^{(k)}(\psi_1)}{(z - z_0)^l} + R_{\lambda,k}(z)(\psi_1)$$

in a neighborhood of any z in $\mathcal{R}_k(V)$ which satisfies $\operatorname{Re}(\lambda) > -C_m + C$. Finally, as all the results holds in $\mathcal{H}_k^m(M)$, we can always pair these equalities with some ψ_2 in $\mathcal{H}_{d-k}^{-m}(M)$.

5.6. Proof of Lemma 5.2. It remains to construct the escape function of Lemma 5.2. The main lines of the argument are very close to the one given for Anosov vector fields by Faure and Sjöstrand in [17]. Yet, several steps need to be adapted in order to fit into our dynamical framework. To begin with, we recall a dynamical Lemma from [17, Lemma 2.1]:

Lemma 5.7. *Let V^{uu} and V^s be small open neighborhoods of Σ_{uu} and Σ_s respectively, and let $\epsilon > 0$. Then, there exist smaller open neighborhoods $\mathcal{W}^{uu} \subset V^{uu}$ and $\mathcal{W}^s \subset V^s$ of Σ_{uu} and Σ_s respectively, \tilde{m}_1 in $\mathcal{C}^\infty(S^*M, [0, 1])$, $\eta_1 > 0$ such that $\tilde{X}_{H_f} \cdot \tilde{m}_1 \geq 0$ on S^*M , $\tilde{X}_{H_f} \cdot \tilde{m}_1 \geq \eta_1 > 0$ on $S^*M - (\mathcal{W}^{uu} \cup \mathcal{W}^s)$, $\tilde{m}_1(x; \xi) > 1 - \epsilon$ for $(x; \xi) \in \mathcal{W}^s$ and $\tilde{m}_1(x; \xi) < \epsilon$ for $(x; \xi) \in \mathcal{W}^{uu}$.*

We will not give the proof of this Lemma and we refer to [17, Lemma 8] for a proof of it. We just emphasize that the main ingredients to make the proof of Faure and Sjöstrand works are the following facts

- the **disjoint** sets Σ_{uu} and Σ_s are the respective repellers and attractors of the lifted Hamiltonian flow generated by \tilde{X}_H on the cosphere S^*M by Lemma 4.1,
- both sets Σ_{uu} and Σ_s are **compact** by Theorem 4.2,
- there exists **disjoint** neighborhoods V_{uu} and V_s of Σ_{uu} and Σ_s respectively which are stable by the flow taken in the negative and positive time directions by Theorem 4.4.

In [17, paragraph 3.3.1], these dynamical statements were the only elements needed to prove the properties of $\tilde{m} \in C^\infty(S^*M)$ and were deduced from the Anosov structure and we showed how they can be proved in the Morse-Smale framework in section 4.

From the above properties, one deduces as Faure–Sjöstrand that the *travel time* in the complement $S^*M \setminus (V_{uu} \cup V_s)$ of $(V_{uu} \cup V_s)$ of any integral curve for the lifted Hamiltonian flow $\exp(t\tilde{X}_H)$ is **uniformly bounded** by a constant $\tau > 0$. Therefore starting from a function $m_0 \in C^\infty(S^*M, [0, 1])$ s.t. $m_0|_{V_{uu}} = 0$ and $m_0|_{V_s} = 1$, for T large enough, the continuous Birkhoff mean :

$$m_1 = \frac{1}{2T} \int_{-T}^T m_0 \circ \exp(t\tilde{X}_H)$$

is a Lyapunov function which satisfies the assumptions of the Lemma.

Remark 5.8. We remark that, by inverting the sense of time, we obtain a similar result for Σ_{ss} and Σ_u . More precisely, we let V^{ss} and V^u be small open neighborhoods of Σ_{ss} and Σ_u respectively, and we let $\epsilon > 0$. Then, there exist open neighborhoods $\mathcal{W}^{ss} \subset V^{ss}$ and $\mathcal{W}^u \subset V^u$, \tilde{m}_2 in $C^\infty(S^*M, [0, 1])$, $\eta_2 > 0$ such that $\tilde{X}_{H_f}.\tilde{m}_2 \geq 0$ on S^*M , $\tilde{X}_{H_f}.\tilde{m}_2 \geq \eta_2 > 0$ on $S^*M - (\mathcal{W}^{ss} \cup \mathcal{W}^u)$, $\tilde{m}_2(x; \xi) > 1 - \epsilon$ for $(x; \xi) \in \mathcal{W}^{ss}$ and $\tilde{m}_2(x; \xi) < \epsilon$ for $(x; \xi) \in \mathcal{W}^u$.

Proof of property (2) from Lemma 5.2.

Let now $u < 0 < n_0 < s$ and N_0 a conic neighborhood of Γ_0 be as in the statement of Lemma 5.2. Let us also consider the constant $\epsilon > 0$ and the collection $W^{uu} \subset V^u, W^s \subset V^s, W^{ss} \subset V^{ss}, W^u \subset V^u$ of neighborhoods from the statements of Lemma 5.7 and Remark 5.8. Following [17, paragraph 3.3.1] and [9, appendix A], we set

$$(20) \quad \tilde{m}(x; \xi) := -E(x) + s + (n_0 - s)\tilde{m}_1 + (u - n_0)\tilde{m}_2,$$

where E is the energy function of Theorem 3.7. We already observe that

$$(21) \quad \forall (x; \xi) \in S^*M, \tilde{X}_H.\tilde{m}(x; \xi) \leq 0$$

since the function $-E$ decreases along the flow, both $u - n_0 < 0, n_0 - s < 0$ and finally $\tilde{X}_H\tilde{m}_i \geq 0$, for $i = (1, 2)$.

Proof of property (1) from Lemma 5.2.

The nice properties of the energy function allow us to control the decay of \tilde{m} for x outside some neighborhood of the nonwandering set $NW(\varphi^t)$. Indeed from Meyer's Theorem 3.7, if we fix $\delta > 0$ and if x does not belong to a δ -neighborhood \mathcal{V}_δ of $NW(\varphi^t)$, then, there exists some constant $c(\delta) > 0$ such that

$$(22) \quad \forall (x; \xi) \notin S^*\mathcal{V}_\delta, \tilde{X}_H.\tilde{m}(x; \xi) \leq -c(\delta) < 0.$$

We now analyze more precisely the properties near the critical elements. For that purpose, we set $\eta := \min(\eta_1, \eta_2)$ and we define several open sets as follows :

$$\tilde{N}_s := \mathcal{W}^{uu} \cap \mathcal{W}^u, \tilde{N}_0 := \mathcal{W}^u \cap \mathcal{W}^s, \tilde{N}_u := \mathcal{W}^{ss} \cap \mathcal{W}^s.$$

We already observe that $(\tilde{N}_s, \tilde{N}_0, \tilde{N}_u)$ define disjoint open neighborhoods inside S^*M of $(\Sigma_{uu}, \Gamma_0 \cap S^*M, \Sigma_{ss})$ respectively. Then, as in [17, p. 338] we can verify the following properties:

- On $S^*M - (\tilde{N}_s \cup \tilde{N}_0 \cup \tilde{N}_u) = (S^*M - (\mathcal{W}^{uu} \cap \mathcal{W}^s)) \cup (S^*M - (\mathcal{W}^{ss} \cap \mathcal{W}^u))$, one has

$$(23) \quad \tilde{X}_H \tilde{m}(x; \xi) \leq -\eta \min(|n_0 - s|, |u - n_0|),$$

with $\eta = \min\{\eta_1, \eta_2\}$ since for $(x; \xi)$ in this region, either $\tilde{X}_H \tilde{m}_1(x; \xi) \geq \eta_1 \geq \eta$ or $\tilde{X}_H \tilde{m}_2(x; \xi) \geq \eta_2 \geq \eta$.

- On $\tilde{N}_s = \mathcal{W}^{uu} \cap \mathcal{W}^u$, one has both $\tilde{m}_1 \leq \varepsilon, \tilde{m}_2 \leq \varepsilon$ therefore

$$\tilde{m}(x; \xi) = \underbrace{-E(x)}_{\geq -\|E\|_\infty} + s + \underbrace{(n_0 - s)\tilde{m}_1 + (u - n_0)\tilde{m}_2}_{\geq \varepsilon(u-s)}$$

hence by $\|E\|_\infty < \frac{s}{2}$, we find that

$$(24) \quad \tilde{m}(x; \xi) \geq -\|E\|_\infty + u\varepsilon + (1 - \varepsilon)s \geq \left(\frac{1}{2} - \varepsilon\right)s + u\varepsilon \geq \frac{s}{4},$$

where the last inequality holds provided we take $\varepsilon < \frac{s}{4(s-u)}$.

- Similarly, on $\tilde{N}_u = \mathcal{W}^{ss} \cap \mathcal{W}^s$, one has both $\tilde{m}_1 \geq 1 - \varepsilon, \tilde{m}_2 \geq 1 - \varepsilon$ therefore

$$\tilde{m}(x; \xi) = \underbrace{-E(x)}_{\leq -\min E} + s + \underbrace{(n_0 - s)\tilde{m}_1 + (u - n_0)\tilde{m}_2}_{\leq (1-\varepsilon)(u-s)}$$

hence by $\frac{u}{2} < -\|E\|_\infty$, we find that

$$(25) \quad \tilde{m}(x; \xi) \leq -\min E + u(1 - \varepsilon) + s\varepsilon \leq \left(\frac{1}{2} - \varepsilon\right)u + s\varepsilon \leq \frac{u}{4},$$

where the last inequality holds provided we take $\varepsilon < \frac{u}{4(u-s)}$.

- Finally, on $\tilde{N}_0 = \mathcal{W}^u \cap \mathcal{W}^s$, one has both $1 - \varepsilon < \tilde{m}_1 \leq 1$ and $\tilde{m}_2 \leq \varepsilon$ hence

$$\tilde{m}(x; \xi) = \underbrace{-E(x)}_{\geq -\|E\|_\infty} + s + \underbrace{(n_0 - s)\tilde{m}_1 + (u - n_0)\tilde{m}_2}_{\geq (n_0-s) + (u-n_0)\varepsilon}$$

$$(26) \quad \tilde{m}(x; \xi) \geq -\max E + (1 - \varepsilon)n_0 + (u - n_0)\varepsilon \geq \left(\frac{1}{2} - 2\varepsilon\right)n_0 + u\varepsilon \geq \frac{n_0}{4},$$

where the last inequality holds provided $\varepsilon < \frac{n_0}{4(2n_0-u)}$.

We extend \tilde{m} into a smooth function m defined on T^*M by setting $m(x; \xi) = 0$ for $\|\xi\| \leq 1/2$ and

$$\forall (x; \xi) \in T^*M \text{ s.t. } \|\xi\|_x \geq 1, \quad m(x; \xi) = \tilde{m}\left(x, \frac{\xi}{\|\xi\|}\right).$$

From (24), (25) and (26), we remark that for $|\xi| \geq 1$, $m(x; \xi)$ depends only on $\xi/\|\xi\|$ and it takes values $\leq \frac{u}{4}$ (resp. $\geq \frac{n_0}{4}$ and $\geq \frac{s}{4}$) in a small neighborhood of $\cup_{j=1}^K N^*(W^u(\Lambda_j))$ (resp. Γ_0 and $\cup_{j=1}^K N^*(W^s(\Lambda_j))$). Hence point (1) of Lemma 5.2 is proved.

Property 3 of Lemma 5.2 on the decay of G_m along the flow.

We start by defining $f(x; \xi)$ according to the neighborhoods that have already been introduced. Recall that \mathcal{V}_δ was some δ -neighborhood of the nonwandering set. We set f to be a smooth function which is positive and positively homogeneous of degree 1 for

$\|\xi\| \geq 1$. Moreover, we suppose that, for $\xi \in N_0 \cap T^*\mathcal{V}_\delta$, one has $f(x; \xi) = |H(x; \xi)|$ and $f(x; \xi) = \|\xi\|_x$ when ξ does not belong to a small open neighborhood of N_0 . Without loss of generality, we can assume that $f(x; \xi) \geq C_0\|\xi\|$ for $\|\xi\| \geq 1$ and for some positive constant $C_0 > 0$ depending on N_0 .

Remark 5.9. We observe that, up to choosing a smaller value of δ and smaller neighborhoods in the statements of Lemma 5.7 and Remark 5.8, we can always suppose that $T^*\mathcal{V}_\delta \cap \tilde{N}_0 \subset S^*M \cap N_0$.

Let us now discuss the decay properties of $G_m = m(x; \xi) \log(1 + f^2(x; \xi))^{\frac{1}{2}}$ to prove property 3 of Lemma 5.2. It suffices to show that in some conic neighborhood of N_0 , we have the bound $X_H G_m \leq 0$ and outside some conic neighborhood of N_0 , we have a **uniform bound** on $X_H G_m$. We decompose $X_H G_m$ as the sum of two terms :

$$(27) \quad X_H \cdot G_m(x; \xi) = (X_H m) (\log(1 + f(x; \xi)^2)^{\frac{1}{2}}) + m(x; \xi) X_H (\log(1 + f(x; \xi)^2)^{\frac{1}{2}}).$$

We start by bounding the term $m(x; \xi) X_H \cdot (\log(1 + f(x; \xi)^2)^{\frac{1}{2}})$ uniformly in $(x; \xi) \in T^*M$ and $|\xi|$ larger than some $R > 0$. Note that there exists a constant $C_1 \geq 0$ such that $\sup_{(x; \xi) \notin T^*\mathcal{V}_\delta} |X_H \cdot (\log(1 + f(x; \xi)^2)^{\frac{1}{2}})| \leq C_1$ since we remark that

$$(28) \quad X_H \cdot (\log(1 + f(x; \xi)^2)^{\frac{1}{2}}) = \frac{X_H \cdot (f(x; \xi)^2)}{2(1 + f(x; \xi)^2)},$$

defines a **bounded function** on T^*M thanks to the homogeneity properties of f .

Then from the bounds $u < -2\|E\|_\infty < 2\|E\|_\infty < n_0 < s, 0 \leq \tilde{m}_1, \tilde{m}_2 \leq 1$ and the definition $\tilde{m} = -E + s + (n_0 - s)\tilde{m}_1 + (u - n_0)\tilde{m}_2$, we find that

$$u - \|E\|_\infty = -\|E\|_\infty + s + (n_0 - s) + (u - n_0) \leq \tilde{m} \leq s + \|E\|_\infty \implies |\tilde{m}| \leq \|E\|_\infty + |u| + s$$

which is a kind of rough bound.

To bound the first term $(X_H m) (\log(1 + f(x; \xi)^2)^{\frac{1}{2}})$ in the decomposition from equation 27, we split the problems between points $x \in M$ which are close to the nonwandering set and those which are far from it.

The element x does not belong to the δ neighborhood \mathcal{V}_δ of the nonwandering set.

In fact, for every $(x; \xi) \notin T^*\mathcal{V}_\delta$ satisfying $\|\xi\| \geq 1$, we have the estimates $X_H m \leq -c(\delta)$ and $(\log(1 + f(x; \xi)^2)^{\frac{1}{2}}) \geq (\log(1 + C_0|\xi|^2)^{\frac{1}{2}})$, which yields the bound $(X_H m) (\log(1 + f(x; \xi)^2)^{\frac{1}{2}}) \leq -c(\delta)(\log(1 + f(x; \xi)^2)^{\frac{1}{2}})$.

Hence combining both bounds yields :

$$(29) \quad X_H \cdot G_m(x; \xi) \leq -c(\delta) \log(1 + C_0^2\|\xi\|^2)^{\frac{1}{2}} + C_1(\|E\|_\infty + s + |u|),$$

for some positive constant $C_1 > 0$ depending on N_0 . Hence, provided R is large enough (in terms of $(C_i)_{i=0,1}$, δ , u and s) in the statement of Lemma 5.2, part (3) of the Lemma will always hold for these points of phase space.

The element $x \in \mathcal{V}_\delta$ and $(x; \xi) \in N_0$.

It now remains to analyze the situation near the nonwandering set (and thus fix a small enough value of δ). Here, the situation follows closely what was done in [17, p. 339]. First,

we note that, as $X_H.H = \{H, H\} = 0$, one can show that, for $(x; \xi) \in N_0$ satisfying $\|\xi\| \geq 1$, one has

$$(30) \quad X_H.G_m(x; \xi) = X_H.m(x; \xi) \log(1 + H(x; \xi)^2)^{\frac{1}{2}} + m(x; \xi) X_H(\log(1 + H(x; \xi)^2)^{\frac{1}{2}}) \leq 0.$$

The element $x \in \mathcal{V}_\delta$ and $(x; \xi) \notin N_0$.

Thanks to remark 5.9, we will cover $S^*\mathcal{V}_\delta \setminus N_0$ by three regions where we will prove uniform decay of G_m along the Hamiltonian flow :

$$S^*\mathcal{V}_\delta \setminus N_0 \subset \tilde{N}_u \cap S^*\mathcal{V}_\delta \cup \tilde{N}_s \cap S^*\mathcal{V}_\delta \cup S^*\mathcal{V}_\delta - (\tilde{N}_0 \cup \tilde{N}_u \cup \tilde{N}_s).$$

- We begin with the case where $(x; \tilde{\xi}) \in S^*M - (\tilde{N}_0 \cup \tilde{N}_u \cup \tilde{N}_s)$. In that case, $f(x; \xi) = H(x; \xi)$ by definition and it follows from (23) that

$$\begin{aligned} X_H.G_m(x; \xi) &= (X_H.m)(x; \xi) \log(1 + H(x; \xi)^2)^{\frac{1}{2}} + m(x; \xi) X_H(\log(1 + H(x; \xi)^2)^{\frac{1}{2}}) \\ &\leq -\eta \min(|n_0 - s|, |u - n_0|) \log(1 + C_0^2 \|\xi\|^2) + C_1(\|E\|_{C^0} + s + |u|). \end{aligned}$$

Hence, as for the case $x \notin \mathcal{V}_\delta$, we can ensure that part (3) of the Lemma is satisfied by picking $R > 0$ large enough (in a way that depends on η, u, s, δ and N_0) so that the negative term $-\eta \min(|n_0 - s|, |u - n_0|) \log(1 + C_0^2 \|\xi\|^2)$ predominates over the positive term $C_1(\|E\|_{C^0} + s + |u|)$ and makes sure the r.h.s. is uniformly negative as long as $|\xi| \geq R$.

- Assume that $(x; \tilde{\xi}) \in \tilde{N}_s \cap S^*\mathcal{V}_\delta$. We show that we can pick $\delta > 0$ small enough to ensure that the bound holds for $(x; \tilde{\xi}) \in \tilde{N}_s \cap S^*\mathcal{V}_\delta$. In this case $f(x; \xi) = |\xi|_x^2$, hence

$$\begin{aligned} X_H.G_m(x; \xi) &= \underbrace{(X_H m)}_{\leq 0} \log(1 + |\xi|_x^2)^{\frac{1}{2}} + m X_H \log(1 + |\xi|_x^2)^{\frac{1}{2}} \\ &\leq m(x; \xi) \frac{X_H \cdot (\|\xi\|_x^2)}{2(1 + \|\xi\|_x^2)}. \end{aligned}$$

We will now make use of the hyperbolicity of the flow one more time in order to control the term $X_H \cdot (\|\xi\|_x^2)$ ⁴. We assume w.l.o.g (periodic orbits are treated similarly) that we are near a critical point Λ_j . From the linearized expression of the Hamiltonian flow in paragraph 4.2, we know that, for every $1 \leq j \leq K$ and for every $(x; \xi) \in T_{\Lambda_j}^*M \cap N^*(W^u(\Lambda_j))$, hence $\xi = (\xi_x, 0)$ in coordinates and

$$\begin{aligned} X_H \cdot (\|(\xi_x, 0)\|_x^2) &= \frac{d}{dt} \Big|_{t=0} \left(\| (e^{t\Omega_s^T} \xi_x, 0) \|_x^2 \right) \\ &= \langle \xi_x, \Omega_s^T \xi_x \rangle \geq c_0 \|\xi\|^2, \end{aligned}$$

for some positive constant c_0 depending only on (M, g) and φ^t . By compactness of Σ_{ss} , we can ensure that this inequality remains true in $\tilde{N}_u \cap S^*\mathcal{V}_\delta$, where \tilde{N}_u is a conical neighborhood of $N^*(W^u(\Lambda_j))$, provided that we replace c_0 by $c_0/2$ and that

⁴Observe that hyperbolicity was already used in Meyer's Theorem and in the proof of Lemma 5.7 which relied on Theorems 4.2 and 4.4.

$\delta > 0$ and all neighborhoods $(V^{ss}, V^{uu}, V^s, V^u)$ from Lemma 5.7 are chosen small enough. Thus, thanks to inequality (25) that yields $\tilde{m}(x; \xi) \leq \frac{u}{4} < 0$, one finds that

$$\tilde{\xi} \in \tilde{N}_u \cap S^* \mathcal{V}_\delta \implies X_H \cdot G_m(x; \xi) \leq \frac{c_0 u}{8},$$

where $c_0 > 0$ is a geometric constant.

- The case where $(x; \tilde{\xi}) \in \tilde{N}_u \cap S^* \mathcal{V}_\delta$ is treated similarly thanks to (24). Then we can derive similar bounds of the form

$$\tilde{\xi} \in \tilde{N}_s \cap S^* \mathcal{V}_\delta \implies X_H \cdot G_m(x; \xi) \leq -\frac{c_0 s}{8},$$

which concludes the construction of the escape function.

APPENDIX A. HYPERBOLIC CRITICAL ELEMENTS

In the definition of Morse-Smale flows, we implicitly assumed some results on hyperbolic fixed points and hyperbolic closed orbits that we will briefly review in this appendix. For more details, we invite the reader to look at the classical textbook of Palis and de Melo [34, Ch. 2,3] – see also [27] for general results on (partially) hyperbolic invariant subsets.

A.1. Limit sets. We start with some terminology from the theory of dynamical systems. We say that a point x in M is wandering if there exist some open neighborhood U of x and some $t_0 > 0$ such that

$$U \cap (\cup_{|t| > t_0} \varphi^t(U)) = \emptyset.$$

The *nonwandering set* of the flow is given by the points which are not wandering. The set of nonwandering points is denoted by $\text{NW}(\varphi^t)$. Given any $x \in M$, we define

$$\alpha(x) := \overline{\cap_{T \leq 0} \{\varphi^t(x) : t \leq T\}},$$

and

$$\omega(x) := \overline{\cap_{T \geq 0} \{\varphi^t(x) : t \geq T\}}.$$

We note that, for every x in M , $\alpha(x)$ and $\omega(x)$ are contained in $\text{NW}(\varphi^t)$. For any invariant closed subset Λ of M , we define the *unstable and stable manifolds* of Λ :

$$W^u(\Lambda) := \{x \in M : \alpha(x) \subset \Lambda\},$$

and

$$W^s(\Lambda) := \{x \in M : \omega(x) \subset \Lambda\}.$$

A.2. Hyperbolic fixed points. We say that a point x_0 in M is an *hyperbolic fixed point* of $(\varphi^t)_{t \in \mathbb{R}}$ if $V(x_0) = 0$ and if $d_{x_0} V : T_{x_0} M \rightarrow T_{x_0} M$ has no eigenvalue on the imaginary axis. Equivalently, it means that, for every $t \neq 0$, x_0 is a fixed point of the smooth diffeomorphism φ^t and that $d_{x_0} \varphi^t : T_{x_0} M \rightarrow T_{x_0} M$ has no eigenvalue of modulus one.

Consider now an hyperbolic fixed point x_0 and some small enough $\delta > 0$. We can define the local unstable and stable manifolds as follows:

$$W_\delta^u(x_0) := \{x \in B(x_0, \delta) : \forall t \leq 0, \varphi^t(x) \in B(x_0, \delta)\},$$

and

$$W_\delta^s(x_0) := \{x \in B(x_0, \delta) : \forall t \geq 0, \varphi^t(x) \in B(x_0, \delta)\}.$$

Then, one has [34, Ch. 2, Prop. 6.1 and Th. 6.2]

$$W^u(x_0) = \bigcap_{t \geq 0} \varphi^t(W_\delta^u(x_0)), \text{ and } W^s(x_0) = \bigcap_{t \geq 0} \varphi^{-t}(W_\delta^s(x_0)).$$

Moreover, $W_\delta^u(x_0)$ (resp. $W_\delta^s(x_0)$) is a smooth embedded disk whose dimension is that of the unstable (resp. stable) space of $d_{x_0}V$ while $W^u(x_0)$ (resp. $W^s(x_0)$) is a smooth injectively immersed manifold in M whose tangent space at x_0 is the unstable (resp. stable) space of $d_{x_0}V : T_{x_0}M \rightarrow T_{x_0}M$.

A.3. Hyperbolic closed orbits. We say that a point x_0 in M is an hyperbolic periodic point if $V(x_0) \neq 0$, if there exists $T_0 > 0$ such that $\varphi^{T_0}(x_0) = x_0$ and if $d_{x_0}\varphi^{T_0} : T_{x_0}M \rightarrow T_{x_0}M$ has 1 as a simple eigenvalue and no other eigenvalue of modulus 1. Equivalently, we will say that $\{\varphi^t(x_0) : 0 \leq t \leq T_0\}$ is an hyperbolic closed orbit.

This can also be defined in terms of Poincaré sections which allows to make the connection with the case of hyperbolic fixed points. Let Σ be a smooth transversal to the vector field V containing x_0 . We denote by $P_\Sigma : O \subset \Sigma \rightarrow \Sigma$ the corresponding Poincaré. Then, the point x_0 is said to be an hyperbolic periodic point of the flow if x_0 is an hyperbolic point for the Poincaré map. Note that this definition does not depend on the choice of the Poincaré section Σ . Fix now a neighborhood \tilde{O} of the closed orbit Λ generated by the point x_0 . We can define one more time the (local) unstable and stable manifolds:

$$W_{\tilde{O}}^u(x_0) := \{x \in \tilde{O} : \forall t \leq 0, \varphi^t(x) \in \tilde{O}\},$$

and

$$W_{\tilde{O}}^s(x_0) := \{x \in \tilde{O} : \forall t \geq 0, \varphi^t(x) \in \tilde{O}\}.$$

As in the case of fixed points, the following holds:

$$W^u(\Lambda) = \bigcap_{t \geq 0} \varphi^t(W_{\tilde{O}}^u(x_0)), \text{ and } W^s(\Lambda) = \bigcap_{t \leq 0} \varphi^t(W_{\tilde{O}}^s(x_0)).$$

Moreover, $W_{\tilde{O}}^u(x_0)$ and $W_{\tilde{O}}^s(x_0)$ are smooth submanifolds of M which are transversal to each other [34, Ch. 3, Prop. 1.5], and $W_{\tilde{O}}^u(x_0) \cap W_{\tilde{O}}^s(x_0) = \Lambda$. In fact, for a given Poincaré section Σ , one can define $W_\delta^u(x_0)$ and $W_\delta^s(x_0)$ for the induced Poincaré map P_Σ and $W_{\tilde{O}}^u(x_0)$ (resp. $W_{\tilde{O}}^s(x_0)$) is an open neighborhood of Λ inside $\cup_{t \in (0, 2T_0)} \varphi^{-t}(W_\delta^u(x_0))$ (resp. $\cup_{t \in (0, 2T_0)} \varphi^t(W_\delta^s(x_0))$). Also, $W^u(\Lambda)$ and $W^s(\Lambda)$ are smooth immersed submanifolds of M [34, Ch. 3, Coro. 1.6].

Finally, $W^u(\Lambda)$ and $W^s(\Lambda)$ are invariantly fibered by smooth submanifolds $(W^{uu}(x_0))_{x_0 \in \Lambda}$ (resp. $(W^{ss}(x_0))_{x_0 \in \Lambda}$) tangent to the unstable (resp. stable) space of $d_{x_0}\varphi^{T_0}$ [27, Th. 4.1]. Points of these submanifolds are characterized as follows, for every x_0 in Λ ,

$$W^{uu}(x_0) := \left\{ x \in M : \lim_{n \rightarrow +\infty} \varphi^{-nT_0}(x) = x_0 \right\},$$

and

$$W^{ss}(x_0) := \left\{ x \in M : \lim_{n \rightarrow +\infty} \varphi^{nT_0}(x) = x_0 \right\}.$$

A.4. λ -Lemma. In the previous paragraphs, we saw that understanding the dynamics near a critical element is related to understanding the dynamics of a diffeomorphism f near an hyperbolic point $\Lambda := \{x_0\}$. This reduction can be done either by considering the time one map of the flow in the case of fixed points, or by looking at the Poincaré map associated with a certain transversal to the orbit. In this paragraph, we would like to give some quantitative features of the dynamics for a diffeomorphism $f : M \rightarrow M$ near an hyperbolic point x_0 . In particular, we would like to recall the λ -Lemma [39, 33] (sometimes called the inclination lemma). We follow closely the presentation of [34, Ch. 2] – see also [6, Ch. 5] in the case of more general hyperbolic subsets. Before that, recall that two smooth submanifolds S and S' of M are ϵ - \mathcal{C}^1 close if there exists a \mathcal{C}^1 diffeomorphism $h : S \rightarrow S'$ such that $i' \circ h$ is ϵ -close to i in the \mathcal{C}^1 topology⁵.

We fix some local coordinates $(x, y) \in \mathbb{R}^{n'} = E_s \oplus E_u$ around the hyperbolic point x_0 (with $n' = n$ or $n - 1$). The local stable (resp. unstable) manifold is then the graph of a smooth function $\kappa_s : B_s(0, r_1) \rightarrow E_u$ (resp. $\kappa_u : B_u(0, r_1) \rightarrow E_s$) where $B_s(0, r_1)$ (resp. $B_u(0, r_1)$) is the ball of radius r_1 centered at 0 inside E_s (resp. E_u). Moreover, we have that $\kappa_u(0) = \kappa_s(0) = 0$ and $d_0\kappa_s = d_0\kappa_u = 0$. We then introduce the following change of coordinates:

$$\kappa : B_s(0, r_1) \oplus B_u(0, r_1) \rightarrow E_s \oplus E_u, (x, y) \mapsto (x - \kappa_u(y), y - \kappa_s(x)).$$

From our construction, one can verify that, for $r_1 > 0$ small enough, κ induces a diffeomorphism near the origin. Hence, one can assume without loss of generality that we are working in a local chart where the local stable (resp. unstable) manifold is represented by the stable (resp. unstable) linear space. The λ -Lemma can then be formulated as follows [34, Ch. 2, Lemma 7.1] (see also [6, Th. 5.7.2]):

Theorem A.1 (λ -Lemma). *We use the above conventions. Let $O = B_s(x_0, r) \times B_u(x_0, r)$, let x be an element in $W^s(x_0)$ and let D^u be a small disk of dimension $\dim(E^u(x_0))$ and which is transversal to $W^s(x_0)$ at x .*

If we denote by D_N^u the connected component of $f^N(D^u) \cap O$ containing $f^N(x)$, then, for every $\epsilon > 0$, there exists N_0 such that, for every $N \geq N_0$, D_N^u is ϵ \mathcal{C}^1 -close to $W^u(x_0)$.

A.5. Sternberg-Chen's Theorem. In this paragraph, we collect a few results on the linearization of vector fields near hyperbolic critical elements in order to illustrate that our assumption of being \mathcal{C}^1 linearizable is in some sense generic.

A.5.1. The case of a fixed point. Recall Sternberg-Chen's Theorem on the linearization of vector fields near hyperbolic critical points [8] (see also [32, Th. 9, p.50]):

Theorem A.2 (Sternberg-Chen). *Let $V(x) = \sum_j a_j(x) \partial_{x_j}$ be a smooth vector field defined in a neighborhood of 0 in \mathbb{R}^n . Suppose that $V(0) = 0$ and that 0 is an hyperbolic fixed point.*

⁵Here $i : S \rightarrow M$ and $i' : S' \rightarrow M$ denote the inclusion maps.

Denote by (μ_j) the eigenvalues of $A := (\partial_{x_k} a_j(0))_{k,j}$. Suppose that the eigenvalues satisfy the **non resonant assumption**,

$$\forall \alpha_1, \dots, \alpha_n \in \mathbb{N} \text{ s.t. } \alpha_1 + \dots + \alpha_n \geq 2, \forall 1 \leq j \leq n, \mu_j \neq \sum_{i=1}^d \alpha_i \mu_i.$$

Then, there exists a smooth diffeomorphism h which is defined in a neighborhood of 0 such that

$$V \circ h = dh \circ L,$$

where $L(x) = Ax \partial_x$.

Here \mathbb{N} denotes the set of *nonnegative* integers. The classical Grobman-Hartman Theorem ensures the existence of a conjugating homeomorphism. The crucial observation for us is that the conjugating map is smooth provided some non resonance assumption is made.

Remark A.3. We stated here a version of the Theorem which give conditions to have a smooth diffeomorphism. Yet, if we only search a \mathcal{C}^k conjugating diffeomorphism (with $k \geq 1$), we can restrict our assumptions by imposing only a *finite* number of conditions on the eigenvalues – see [32, Th. 10, p. 52] for the precise statement.

A.5.2. The case of a closed orbit. Sternberg-Chen's Theorem can be generalized in the case of a closed hyperbolic periodic orbit [44]. Consider a smooth vector field $V(x, \theta)$ defined on $B_{n-1}(0, r) \times \mathbb{R}/(\mathcal{P}\mathbb{Z})$ where $B_{n-1}(0, r)$ is a small ball of radius $r > 0$ centered at 0 in \mathbb{R}^{n-1} . We make the assumption that

$$(31) \quad V(x, \theta) = (1 + g(x, \theta)) \partial_\theta + (A(\theta)x + f(x, \theta)) \cdot \partial_x,$$

with $f(x, \theta) = \mathcal{O}(\|x\|^2)$, $A(\theta)$ smooth and $g(x, \theta) = \mathcal{O}(\|x\|)$.

Remark A.4. Let us explain how one can reduce the problem to the above situation. Recall that the tubular neighborhood Theorem states that some neighborhood of $\Lambda \subset M$ is C^∞ diffeomorphic to some neighborhood of the zero section of the normal bundle $N(\Lambda \subset M)$ induced by some Riemannian metric on M . But Λ is a circle (with global coordinates θ in $\mathbb{S}^1 := \mathbb{R}/(2\pi\mathbb{Z})$) hence $N(\Lambda \subset M)$ is an oriented real vector bundle over \mathbb{S}^1 . Hence, $N(\Lambda)$ is trivial and diffeomorphic to a cartesian product $\mathbb{S}^1 \times \mathbb{R}^{d-1}$. Equivalently, triviality and the tubular neighborhood Theorem guarantee that we have some germ of coordinate system (x, θ) near Λ where Λ is defined by the global equation $\{x = 0\}$. In these coordinates, the vector field V associated with φ^t can be written

$$(32) \quad V(x, \theta) = \tilde{g}(x, \theta) \partial_\theta + \tilde{f}(x, \theta) \partial_x,$$

where (x, θ) belongs to $O_\Lambda \times \mathbb{S}^1$ for some small neighborhood $O_\Lambda \subset \mathbb{R}^{n-1}$ of 0. As Λ is a closed orbit, we can suppose that $\tilde{f}(0, \theta) = 0$ for all $\theta \in \mathbb{S}^1$ and $\tilde{g}(0, \theta) > 0$ for all $\theta \in \mathbb{S}^1$. Before explaining the analogue of the Sternberg-Chen's Theorem for closed hyperbolic orbit, let us first reparametrize the θ variable. Set $\tilde{\theta}(\theta) = \int_0^\theta \tilde{g}^{-1}(0, s) ds$ and $\mathcal{P}_\Lambda := \int_0^{2\pi} \tilde{g}^{-1}(0, s) ds$. In these new coordinates, one has

$$d\tilde{\theta} = d \int_0^\theta \tilde{g}^{-1}(0, s) ds = \tilde{g}^{-1}(0, \theta) d\theta \implies d\tilde{\theta}(\tilde{g}(0, \theta) \partial_\theta) = 1.$$

The fact that this is a diffeomorphism relies on the fact that $\tilde{g}(0, s) > 0$ for all $s \in [0, 2\pi]$. Therefore up to doing this reparametrization, we may assume that

$$(33) \quad V(x, \theta) = \tilde{f}(x, \theta)\partial_x + \tilde{g}(x, \theta)\partial_\theta$$

with $\theta \in \mathbb{R}/(\mathcal{P}_\Lambda\mathbb{Z})$,

$$\tilde{f}(x, \theta) = A(\theta)x + f(x, \theta), \text{ and } \tilde{g}(x, \theta) = 1 + g(x, \theta),$$

Moreover, $f(x, \theta) = \mathcal{O}(\|x\|^2)$ and $g(x, \theta) = \mathcal{O}(\|x\|)$ uniformly in a small neighborhood of $\{x = 0\}$.

Before stating the analogue of Sternberg-Chen's Theorem in that case, let us review a few facts from Floquet theory [41, p. 91] – see also the next paragraph. For every $\theta_0 \in \mathbb{R}/(\mathcal{P}_\Lambda\mathbb{Z})$, we consider the matrix valued ordinary differential equation:

$$(34) \quad \frac{dU(\theta, \theta_0)}{d\theta} = A(\theta)U(\theta, \theta_0), \quad U(\theta_0, \theta_0) = \text{Id}.$$

The solution satisfies $U(\theta + \mathcal{P}, \theta_0 + \mathcal{P}) = U(\theta, \theta_0)$. The matrix $M(\theta_0) = U(\theta_0 + \mathcal{P}, \theta_0)$ is called the **monodromy matrix** of the system. Recall that $M(\theta_1) = U(\theta_1, \theta_0)M(\theta_0)U(\theta_1, \theta_0)^{-1}$. In particular, its eigenvalues are independent of θ_0 . We denote by $(\lambda_1, \dots, \lambda_n)$ the eigenvalues of $M = M(0)$. We write these eigenvalues under an exponential form, i.e. $\lambda_j := e^{\mathcal{P}_\Lambda\mu_j(\Lambda)}$. The $\mu_j(\Lambda)$ are called the **Floquet exponents of the closed orbit** $\{0\} \times \mathbb{R}/(\mathcal{P}\mathbb{Z})$ while their real parts are the so-called **Lyapunov exponents**. Following [44], we introduce the following nonresonance conditions:

- $(\mu_j(\Lambda))_{j=1, \dots, n-1}$ are **nonresonant in space** if, for every $\alpha \in \mathbb{N}^{n-1}$ with $\sum_{k=1}^{n-1} \alpha_k \geq 2$ and for every $1 \leq j \leq n-1$,

$$\mu_j(\Lambda) - \sum_{k=1}^{n-1} \alpha_k \mu_k(\Lambda) \notin \frac{2i\pi}{\mathcal{P}}\mathbb{Z},$$

- $(\mu_j(\Lambda))_{j=1, \dots, n-1}$ are **nonresonant in time** if, for every $\alpha \in \mathbb{N}^{n-1}$ with $\sum_{k=1}^{n-1} \alpha_k \geq 1$,

$$\sum_{k=1}^{n-1} \alpha_k \mu_k(\Lambda) \notin \frac{2i\pi}{\mathcal{P}}\mathbb{Z},$$

Then, we can state the analogue of Sternberg-Chen's Theorem in the case of an hyperbolic closed orbit [44, Th. 3]:

Theorem A.5. *Let $V(x, \theta)$ be a smooth vector field of the form (31). Suppose that $\{0\} \times \mathbb{R}/(\mathcal{P}_\Lambda\mathbb{Z})$ is an hyperbolic closed orbit for the flow generated by V and that $(\mu_j(\Lambda))_{j=1, \dots, n-1}$ are both nonresonant in time and in space. Then, there exists a smooth diffeomorphism h which is defined in a neighborhood of $\{0\} \times \mathbb{R}/(\mathcal{P}_\Lambda\mathbb{Z})$ such that*

$$V \circ h = dh \circ L,$$

with

$$L(x, \theta) = \partial_\theta + A(\theta)x.\partial_x.$$

Remark A.6. Fix $k \geq 1$. As in [32], the proof of [44] could be adapted to ensure that, under a *finite* number of non resonant conditions in space and times, h can be chosen to be \mathcal{C}^k . However, we are not aware of a place in the literature where this condition is explicitly written.

A.6. More on Floquet theory. According to [41, Lemma 3.34], we can put the matrix $M(\theta_0)^2$ under a real exponential form, i.e. $M(\theta_0)^2 = e^{2\mathcal{P}_\Lambda \Omega(\theta_0)}$ with $\Omega(\theta_0)$ real valued. Note that we have to take the square of $M(\theta_0)$ in order to take into account the fact that $M(\theta_0)$ may have negative eigenvalues. Then, thanks to [41, Cor. 3.16], one knows that there exists a *real valued* matrix $P(\theta_0 + \theta, \theta_0)$ which is $2\mathcal{P}$ -periodic in θ and such that, for every θ in $[0, 2\mathcal{P}]$,

$$U(\theta, \theta_0) = P(\theta, \theta_0)e^{(\theta - \theta_0)\Omega(\theta_0)}.$$

Suppose now that the eigenvalues of the matrix $A_\Lambda = \Omega(0)$ have a nonzero real part (which corresponds to the case of an hyperbolic closed orbit Λ) and assume for the sake of simplicity that the vector field is defined on $\mathbb{R}^{n-1} \times \mathbb{R}/\mathcal{P}_\Lambda\mathbb{Z}$. We can split $\mathbb{R}^{n-1} = E_{ss} \oplus E_{uu}$ where E_{uu} (resp. E_{ss}) corresponds to the eigenvalues with positive (resp. negative) real part. This allows to rewrite the matrix A_Λ under a block diagonal form, i.e. $A_\Lambda = \text{diag}(-\Omega_s, \Omega_u)$ with $\Omega_{u/s}$ having eigenvalues with positive real parts. In particular, there exists some positive constants $0 < C_1 < C_2$ and $0 < \chi_- < \chi_+$ such that, for every $t \geq 0$,

$$(35) \quad C_1 e^{-t\chi_+} \|(x, y)\| \leq \|(e^{-t\Omega_s} x, e^{-t\Omega_u} y)\| \leq C_2 e^{t\chi_-} \|(x, y)\|.$$

We set $z = (x, y)$ for this new system of coordinates and (ξ, η) for the dual coordinates. The flow can then be written as

$$\varphi^t(z, \theta) = (U(\theta + t, \theta)z, \theta + t) = (P(t + \theta, 0)e^{t\Omega}P(\theta, 0)^{-1}z, \theta + t),$$

with $t \mapsto P(t, 0)$ which is $2\mathcal{P}_\Lambda$ -periodic. From this expression, one can verify that the unstable and stable manifolds are given by

$$W^u(\Lambda) := \{(P(\theta, 0)(0, y), \theta) : \theta \in \mathbb{R}/\mathcal{P}_\Lambda\mathbb{Z}, (0, y) \in \mathbb{R}^{n-1}\}$$

and

$$W^s(\Lambda) := \{(P(\theta, 0)(x, 0), \theta) : \theta \in \mathbb{R}/\mathcal{P}_\Lambda\mathbb{Z}, (x, 0) \in \mathbb{R}^{n-1}\}.$$

Similarly, we can verify that the strong unstable and stable manifolds at a given point $(0, 0, \theta)$ of the closed orbit Λ is

$$W^{uu}(0, 0, \theta) := \{(P(\theta, 0)(0, y), \theta) : (0, y) \in \mathbb{R}^{n-1}\}$$

and

$$W^{ss}(0, 0, \theta) := \{(P(\theta, 0)(x, 0), \theta) : (x, 0) \in \mathbb{R}^{n-1}\}.$$

Our construction requires to understand the topological and dynamical properties of the conormal of these submanifolds – see paragraph 4.3 for the precise definition. Let us give the expression of the conormal of the (strong) unstable manifolds⁶:

$$N^*(W^u(\Lambda)) := \{(P(\theta, 0)(0, y), \theta; (P(\theta, 0)^T)^{-1}(\xi, 0), 0) : (0, y, \theta; \xi, 0, 0) \in T^*(\mathbb{R}^{n-1} \times \mathbb{R}/\mathcal{P}_\Lambda\mathbb{Z})\},$$

⁶The case of stable manifolds is similar.

and, for every θ in $\mathbb{R}/\mathcal{P}_\Lambda\mathbb{Z}$

$$N^*(W^{uu}(0, 0, \theta)) := \left\{ (P(\theta, 0)(0, y), \theta; (P(\theta, 0)^T)^{-1}(\xi, 0), \Theta) : (0, y, \theta; \xi, 0, \Theta) \in T^*(\mathbb{R}^{n-1} \times \mathbb{R}/\mathcal{P}_\Lambda\mathbb{Z}) \right\}.$$

In our construction of anisotropic Sobolev spaces, we need to understand the induced dynamics on the cotangent space, i.e. the Hamiltonian flow Φ^t which was defined in paragraph 4.1. A straightforward calculation tells us that

$$((d_{x,y,\theta}\varphi^t)^T)^{-1} = \begin{pmatrix} (U(\theta+t, \theta)^{-1})^T & 0 \\ -\left(U(\theta+t, \theta)^{-1} \partial_\theta \left(U(\theta+t, \theta) \begin{pmatrix} x \\ y \end{pmatrix} \right) \right)^T & 1 \end{pmatrix}.$$

Fix now a point $(P(\theta, 0)(x, y), \theta; (P(\theta, 0)^T)^{-1}(\xi, \eta), \Theta)$ in $T^*(\mathbb{R}^{n-1} \times \mathbb{R}/\mathcal{P}_\Lambda\mathbb{Z})$. Applying the previous formula, we find

(36)

$$\left((d_{P(\theta, 0)(x, y), \theta}\varphi^t)^T \right)^{-1} \left((P(\theta, 0)^T)^{-1}(\xi, \eta), \Theta \right) = \left((P(\theta+t, 0)^T)^{-1}(e^{t\Omega_s^T}\xi, e^{-t\Omega_u^T}\eta), \Theta + R(t, x, y, \theta, \xi, \eta) \right),$$

where

$$R(t, x, y, \theta, \xi, \eta) = \left\langle \partial_\theta(U(\theta+t, \theta))P(\theta, 0)(x, y), (P(\theta+t, 0)^T)^{-1}(e^{t\Omega_s^T}\xi, e^{-t\Omega_u^T}\eta) \right\rangle.$$

Remark A.7. The expression of the remainder term can be controlled in terms of t . Indeed, write first

$$\partial_\theta(U(\theta+t, \theta))P(\theta, 0)(x, y) = \partial_\theta(P(\theta+t, 0))e^{t\Omega}(x, y) - U(\theta+t, \theta)\partial_\theta(P(\theta, 0))(x, y).$$

Hence, after simplification, we get

$$\begin{aligned} R(t, x, y, \theta, \xi, \eta) &= -\left\langle \partial_\theta(P(\theta, 0))(x, 0), (P(\theta, 0)^T)^{-1}(\xi, 0) \right\rangle - \left\langle \partial_\theta(P(\theta, 0))(0, y), (P(\theta, 0)^T)^{-1}(0, \eta) \right\rangle \\ &\quad + \left\langle \partial_\theta(P(\theta+t, 0))(e^{-t\Omega_s}x, 0), (P(\theta+t, 0)^T)^{-1}(e^{t\Omega_s^T}\xi, 0) \right\rangle \\ &\quad + \left\langle \partial_\theta(P(\theta+t, 0))(0, e^{t\Omega_u}y), (P(\theta+t, 0)^T)^{-1}(0, e^{-t\Omega_u^T}\eta) \right\rangle. \end{aligned}$$

In particular, for $t \leq 0$, one has

$$(37) \quad R(t, x, y, \theta, \xi, \eta) = \tilde{R}(x, y, \theta, \xi, \eta) + o(\|e^{-t\Omega_s}x\|) + o(\|e^{-t\Omega_u^T}\eta\|),$$

with

$$\tilde{R}(x, y, \theta, \xi, \eta) = -\left\langle \partial_\theta(P(\theta, 0))(x, 0), (P(\theta, 0)^T)^{-1}(\xi, 0) \right\rangle - \left\langle \partial_\theta(P(\theta, 0))(0, y), (P(\theta, 0)^T)^{-1}(0, \eta) \right\rangle.$$

APPENDIX B. PROOF OF THE DYNAMICAL STATEMENTS FROM SECTION 3

In this appendix, for the sake of completeness, we collect the instructive proofs of some dynamical results due to Smale.

B.1. Proof of Lemma 3.2. Let x be an element in $W^u(\Lambda_j) \cap W^s(\Lambda_i)$ which does not belong to Λ_i (otherwise $i = j$ and the conclusion is trivial). Then, the flow line $t \mapsto \varphi^t(x)$ is contained in the intersection $W^u(\Lambda_j) \cap W^s(\Lambda_i)$ which should have dimension at least 1. Also, one knows

$$\dim T_x W^u(\Lambda_j) + \dim T_x W^s(\Lambda_i) = \dim(T_x W^u(\Lambda_j) + T_x W^s(\Lambda_i)) + \dim(T_x W^u(\Lambda_j) \cap T_x W^s(\Lambda_i)).$$

From the transversality assumption $T_x W^u(\Lambda_j) + T_x W^s(\Lambda_i) = T_x M$ which, combined to the previous observation, implies

$$\dim W^u(\Lambda_j) + \dim W^s(\Lambda_i) \geq n + 1.$$

We now distinguish two cases. On the one hand, if Λ_i is a critical point, then $\dim W^s(\Lambda_i) = n - \dim W^u(\Lambda_i)$ (by transversality at Λ_i). Hence, $\dim W^u(\Lambda_j) \geq 1 + \dim W^u(\Lambda_i)$ as expected. On the other hand, if Λ_i is a closed orbit, one has, by transversality at Λ_i , $\dim W^s(\Lambda_i) = n - \dim W^u(\Lambda_i) + 1$ from which one can conclude that $\dim W^u(\Lambda_j) \geq \dim W^u(\Lambda_i)$.

B.2. Proof of Lemma 3.3. From Lemma 3.2, we note that, if $W^u(\Lambda_j) \cap W^s(\Lambda_j) \neq \Lambda_j$, then Λ_j is necessarily a closed orbit. In fact, it means that there exists x that belongs to $W^u(\Lambda_j) \cap W^s(\Lambda_j)$ but not to Λ_j . From the case of equality in Lemma 3.2, it follows that Λ_j is a closed orbit.

Suppose now that there exists such a point x_0 , i.e. x_0 belongs to $W^u(\Lambda_j) \cap W^s(\Lambda_j)$ but not to Λ_j . Note that x_0 does not belong to Λ_i for every $i \neq j$ – see Lemma 3.1. Let U be a small open set containing x_0 . We would like to prove that, for every $t_0 > 0$, there exists $t \geq t_0$ such that $\varphi^t(U) \cap U \neq \emptyset$, which would contradict the fact that $x_0 \notin NW(\varphi^t)$.

For that purpose, fix Σ a small Poincaré section associated with the closed orbit Λ_j and centered at the point $y_0 \in \Lambda_j$ – see appendix A. We also set D^u to be a small open disk containing x_0 inside $W^u(\Lambda_j) \cap U$ and D^s to be a small open disk containing x_0 inside $W^s(\Lambda_j) \cap U$. Fix now $t_0 > 0$ and let us show the expected contradiction. As x_0 belongs to $W^s(\Lambda_j)$, we know that there exists $t \geq t_0$ such that $\varphi^t(x_0)$ belongs to Σ . We denote by $D^u(t, \Sigma)$ the connected component of $\varphi^t(D^u) \cap \Sigma$ containing $\varphi^t(x_0)$. From the λ -Lemma A.1, we know that, for t large enough, $D^u(t, \Sigma)$ is ϵ \mathcal{C}^1 -close to the unstable manifold of the Poincaré map near the origin (for some small ϵ). Similarly, we can work in negative times and construct $D^s(-t', \Sigma)$ which is ϵ \mathcal{C}^1 -close to the stable manifold of the Poincaré map near the origin. As $\dim W^u(\Lambda_j) + \dim W^s(\Lambda_j) = n + 1$, $D^u(t, \Sigma)$ (resp. $D^s(-t', \Sigma)$) have the same dimension as the unstable (resp. stable) manifolds of the induced Poincaré map on Σ . Therefore, there exists a point y_1 which lies in the intersection of $D^u(t, \Sigma)$ and $D^s(-t', \Sigma)$. Then, we set $y_2 = \varphi^{-t}(y_1) = \varphi^{-(t+t')}(y_3)$ which belongs to $U \cap \varphi^{-(t+t')}(U)$. This gives the contradiction

B.3. Proof of Theorem 3.5. The proof of Theorem 3.5 was given by Smale in [39]. The proof of this classical result contains important ideas that enlightens our general strategy. Hence, we it seems useful to us to recall Smale's argument. The proof starts with the following Lemma [39, Lemma 3.3]:

Lemma B.1. *Suppose $W^u(\Lambda_i) \cap W^s(\Lambda_j) \neq \emptyset$. Then, $W^u(\Lambda_j) \subset \overline{W^u(\Lambda_i)}$.*

This first step means the following. If $\varphi^t(x)$ is repelled by Λ_j and attracted by Λ_i , then the unstable manifold $W^u(\Lambda_i)$ of the set Λ_i at the arrival is contained in the closure of the bigger stratum $W^u(\Lambda_j)$.

Proof. Let x_0 be a point in $W^u(\Lambda_j)$ and let $\epsilon > 0$. We start with the case where Λ_j is an hyperbolic fixed point.

We first note that there exists $T_0 > 0$ such that $\varphi^{-T_0}(x_0)$ belongs to an ϵ -neighborhood O of Λ_j inside $W^u(\Lambda_j)$. We shall use the λ -Lemma A.1. Let y_0 be an element in $W^u(\Lambda_i) \cap W^s(\Lambda_j) \neq \emptyset$. Let $D^u(y_0)$ be a small disk contained in $W^u(\Lambda_i)$ which is of dimension $\dim W^u(\Lambda_j)$, which contains y_0 in its interior and which is transversal to $W^s(\Lambda_j)$. The existence of such a disk is provided by the dimension bound $\dim W^u(\Lambda_j) \leq \dim W^u(\Lambda_i)$ from Lemma 3.2. Using the map $f = \varphi^1$ in the λ -Lemma, we can find n large enough such that the connected component of $\varphi^n(D^u(y_0)) \cap W$ containing $\varphi^n(y_0)$ is ϵe^{-CT_0} \mathcal{C}^1 -close to $W^u(\Lambda_j)$ near Λ_j (for some C large enough to be determined). In particular, we can find a point $y_1 \in W^u(\Lambda_i)$ which is at a distance ϵe^{-CT_0} of $\varphi^{-T_0}(x_0)$. Recall that the flow φ^t can only expand the distance exponentially with time – see for instance [46, Lemma 11.11]. Hence, if we choose $C > 0$ large enough, we find that $\varphi^{T_0}(y_1) \in W^u(\Lambda_i)$ is ϵ -close to x_0 .

In the case where Λ_j is a closed orbit, we shall fix Σ to be a Poincaré section transversal to Λ_j . For $T_0 > 0$ large enough, we know that $\varphi^{-T_0}(x_0)$ belongs to Σ . As in the proof of Lemma 3.3, we have to use the λ -lemma for the induced Poincaré map and we deduce the results following the same lines as for a fixed point. \square

We continue with the following Lemma [39, Lemma 3.5]:

Lemma B.2. *Suppose $W^u(\Lambda_{i_1}) \cap W^s(\Lambda_{i_2}) \neq \emptyset$ and $W^u(\Lambda_{i_2}) \cap W^s(\Lambda_{i_3}) \neq \emptyset$. Then, one has*

$$W^u(\Lambda_{i_1}) \cap W^s(\Lambda_{i_3}) \neq \emptyset.$$

Proof. The argument looks very much like the proof of the no-cycle Lemma 3.3. Let x_2 be an element in $W^u(\Lambda_{i_2}) \cap W^s(\Lambda_{i_3})$. Let \tilde{D}^s be a small disk inside $W^s(\Lambda_{i_3})$ which is of dimension $\dim(W^s(\Lambda_{i_2}))$ and which contains x_2 . Again, this is possible according to Lemma 3.2. By applying φ^{-t} with $t > 0$ large enough, we have two options which follows from the λ -Lemma:

- Λ_{i_2} is an hyperbolic fixed point. If we set $D^s(-t)$ to be connected component of $\varphi^{-t}(\tilde{D}^s) \cap O$ (where O is a neighborhood of Λ_{i_2}) containing $\varphi^{-t}(x_2)$ in its interior, then $D^s(-t)$ is ϵ \mathcal{C}^1 -close to the local stable manifold of Λ_{i_2} containing Λ_{i_2} . Note that any element in $D^s(-t)$ belongs to $W^s(\Lambda_{i_3})$ and that this small piece of disk has dimension $\dim W^s(\Lambda_{i_2})$.
- Λ_{i_2} is an hyperbolic closed orbit. As in the proof of Lemma 3.3, we fix a Poincaré section Σ and we define a small disk $D^s(-t, \Sigma)$ inside $W^s(\Lambda_{i_3})$ which is ϵ close to the stable manifold of the induced Poincaré map on Σ . Again, any point in $D^s(-t, \Sigma)$ is contained in the stable manifold of Λ_{i_3} . Note that this small disk has dimension $\dim W^s(\Lambda_{i_2}) - 1$.

Then, we fix x_1 be an element in $W^u(\Lambda_{i_1}) \cap W^s(\Lambda_{i_2})$. Working with positive times, we can similarly define either $D^u(t')$ (which is of dimension $\dim W^u(\Lambda_{i_2})$) or $D^u(t', \Sigma)$ (which is of

dimension $\dim W^u(\Lambda_{i_2}) - 1$) inside $W^u(\Lambda_{i_1})$. Using the relations on the dimension as in the proof of Lemma 3.3, we know that these small pieces of disks have at least one point of intersection y_0 . Hence, the point y_0 belongs to $W^u(\Lambda_{i_1}) \cap W^s(\Lambda_{i_3})$ which concludes the proof. \square

Finally, we also have [39, Lemmas 3.6 and 3.7] whose proof has inspired Theorem 4.2 (see also [45]):

Lemma B.3. *Suppose that $W^u(\Lambda_j) \cap \overline{W^u(\Lambda_i)} \neq \emptyset$. Then, there exists a sequence $i = i_1, \dots, i_m = j$ such that $W^u(\Lambda_{i_p}) \cap W^s(\Lambda_{i_{p+1}}) \neq \emptyset$ for every $1 \leq p \leq m - 1$. In particular, from Lemmas B.1 and B.2, $W^u(\Lambda_j) \subset \overline{W^u(\Lambda_i)}$.*

Proof. We fix a point x_0 in $W^u(\Lambda_j) \cap \overline{W^u(\Lambda_i)}$ with $i \neq j$ (otherwise the conclusion is clear). For the sake of simplicity, let us consider the case where Λ_j is a critical point (the proof works similarly for closed orbits except that we have to introduce some Poincaré section as in the above proofs). We fix $\delta_0 > 0$ and we pick a small disk D_0 of dimension $\dim W^s(\Lambda_j)$ containing x_0 in its interior and which is transversal to $W^u(\Lambda_j)$. For every $\epsilon > 0$, we can apply the flow in backward times to this small piece of disk and we obtain a small disk inside $\varphi^{-T}(D_0)$ which is ϵ -close to the stable manifold $W^s(\Lambda_j)$ of Λ_j from the λ -Lemma. Inside this small disk, we consider the points which are at a distance $\geq \delta_0$ and $\leq 2\delta_0$ of Λ_j . We would like to verify that we can find such a point y_1 which would also belong to $W^u(\Lambda_i)$. For that purpose, let us use our assumption that x_0 belongs to $W^u(\Lambda_j) \cap \overline{W^u(\Lambda_i)}$. From this hypothesis and from the fact that D_0 is transversal to $W^u(\Lambda_j)$, we can always find a point $y_0 \in D_0$ whose image under φ^{-T} will fall inside this subset. For this, we may need to increase $T > 0$. Hence, we set $y_1 = \varphi^{-T}(y_0)$ which belongs to $W^u(\Lambda_i)$, which is ϵ -close to $W^s(\Lambda_j)$ but at a distance $\geq \delta_0$ of Λ_j . This is valid for every $\epsilon > 0$. In particular, thanks to Lemma 3.3, this shows that there exists some $j' \neq j$ such that $W^u(\Lambda_{j'}) \cap \overline{W^u(\Lambda_i)} \neq \emptyset$ with $W^s(\Lambda_j) \cap W^u(\Lambda_{j'}) \neq \emptyset$. Either $j' = i$ (and we are done), or $j' \neq i$. In that case, thanks to Lemmas B.2 and 3.3, we can reproduce the same argument a finite number of times to get the expected conclusion. \square

The combination of Lemmas B.1, B.2 and B.3 gives the proof of Smale's Theorem 3.5 as we shall now explain. First, we note that, according to Lemma 3.1, one can find, for every $x \in M$, an unique $1 \leq j(x) \leq K$ such that $x \in W^u(\Lambda_{j(x)})$. To prove the first part of Smale's Theorem, we shall verify that, for every $1 \leq i \leq K$,

$$\overline{W^u(\Lambda_i)} = \bigcup_{x \in \overline{W^u(\Lambda_i)}} W^u(\Lambda_{j(x)}).$$

Note that that the first inclusion \subset is obvious by definition while the other one follows from Lemma B.3. It now remains to show the partial order relation on the unstable manifolds inside M . We start with the transitivity property. Suppose that $\overline{W^u(\Lambda_{i_1})} \leq W^u(\Lambda_{i_2})$ and that $W^u(\Lambda_{i_2}) \leq W^u(\Lambda_{i_3})$. This exactly means that $W^u(\Lambda_{i_1}) \subset \overline{W^u(\Lambda_{i_2})}$ and $W^u(\Lambda_{i_2}) \subset \overline{W^u(\Lambda_{i_3})}$. Applying Lemma B.3 to (i_1, i_2) and to (i_2, i_3) , we find a sequence $i_3 = j_1, \dots, j_m = i_1$ such that, for every $1 \leq p \leq m - 1$, $W^u(\Lambda_{j_p}) \cap W^s(\Lambda_{j_{p+1}}) \neq \emptyset$. Then,

a combination of Lemmas B.1 and B.2 shows that $W^u(\Lambda_{i_1}) \subset \overline{W^u(\Lambda_{i_3})}$ which implies that $W^u(\Lambda_{i_1}) \subseteq W^u(\Lambda_{i_3})$. For the reflexivity, we suppose that both $W^u(\Lambda_i) \subseteq W^u(\Lambda_j)$ and $W^u(\Lambda_j) \subseteq W^u(\Lambda_i)$ hold true for some couple (i, j) with $i \neq j$. Then, applying Lemmas B.3 and B.2 in this order, and as $i \neq j$, it provides the existence of a point x accumulating to Λ_i in the past and to Λ_j in the future. Similarly, we can find a point y such that the roles of i and j are reversed. Proceeding as in the proof of Lemma B.2, we can find a point z outside Λ_i which accumulate to Λ_i in the past and in the future. This contradicts Lemma 3.3 and concludes the proof of the reflexivity.

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