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A stable cut finite element method for multiple unilateral contact

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ABSTRACT

This paper presents a novel CutFEM-LaTIn algorithm to solve multiple unilateral contact problems over geometries that do not conform with the finite element mesh. We show that our method is (i) stable, independently of the interface locations (ii) optimally convergent with mesh refinement and (iii) efficient from an algorithmic point of view.

Key Words: cut finite elements, contact, multiple materials, LaTIn

1. Introduction

Since the pioneer work presented in [5], several research teams have successfully developed stable XFEM schemes for unilateral contact (see *e.g.* [10, 6, 1]). However, it is extremely challenging to make sure that these methods are optimally convergent with mesh refinement. As far as we are aware, this has only been achieved recently in [4, 3]. In this paper, we propose an alternative to the latter contributions that is based on an extension of the CutFEM [7, 2] technology for embedded interface problems, coupled with the LaTIn mixed formulation of contact [9, 5, 8, 6]. We show that the formulation is always stable, and that our high-performance computing implementation allows us to perform 3D computations involving arbitrarily complex intersections of solids, potentially leading to field singularities. Moreover, we show optimal convergence with mesh refinement of the background grid. Incidentally, our work is also a stabilisation of the LaTIn hybrid-mixed formulation. In this respect, our method is an alternative to the two previous non-locking LaTIn formulations based on (i) a local coarsening of trace meshes (see *e.g.* [9, 8]) and (ii) a relaxation of interface kinematic conditions together with the addition of penalty terms, as presented in [6]. Our approach is algorithmically quite advanced. We therefore choose, in this short contribution, to focus on the most novel aspects of it, which is the discrete non-conforming and non-locking mixed LaTIn formulation of contact, enriched and further stabilised with tools of the CutFEM technology. This scheme is developed in section 2, whilst section 3 contains some representative examples and short conclusions.

2. Finite Element Formulation

2.1. Domain Discretisation

We consider a domain Ω divided into n_d non-overlapping subdomains Ω^i . We partition the boundary of domain Ω , $\partial\Omega$, into a Dirichlet $\partial\Omega_D$ (body is clamped) and a Neumann part $\partial\Omega_N$ (tractions are imposed). Furthermore, each subdomain Ω^i is considered to be covered by a linear elastic body (*i*). We assume that if two elastic bodies share an interface $\Gamma^{i,j} = \partial\Omega^i \cap \partial\Omega^j$ these two bodies are expected to come into contact. For example, in Figure 1a, we employ contact conditions on the interfaces $\Gamma^{1,2}$, $\Gamma^{1,3}$ and $\Gamma^{2,3}$. We introduce a non-conforming finite element discretisation of the domains as follows. Firstly, let \mathcal{T}_h denote a tessellation of domain Ω independent of the interface locations $\Gamma^{i,j}$. Secondly, we introduce the following mesh subsets of our background mesh \mathcal{T}_h . For each subdomain Ω^i , we introduce a so-called fictitious domain $\tilde{\Omega}^{*,i} := \{K \in \mathcal{T}_h : K \cap \Omega^i \neq \emptyset\}$. In a similar fashion, we define the set of elements intersected by an interface $\Gamma^i = \bigcup_{(k,l) \in \mathcal{I}_i, (k=i \text{ or } l=i)} \Gamma^{k,l}$ as $\mathcal{G}^i := \{K \in \mathcal{T}_h : K \cap \Gamma^i \neq \emptyset\}$. We also define the following two subsets of faces between two elements $\mathcal{F}(K, K') = K \cap K'$: the set of element faces associated with \mathcal{G}^i , namely $\mathcal{F}_G^i := \{F \in \mathcal{F}(K, K') : K \in \mathcal{G}^i \text{ or } K' \in \mathcal{G}^i\}$, which we call ghost penalty faces, and the set of intersected faces $\mathcal{F}_I^i := \{F \in \mathcal{F}(K, K') : K \in \mathcal{G}^i \text{ and } K' \in \mathcal{G}^i\}$ (see Figure 1b).

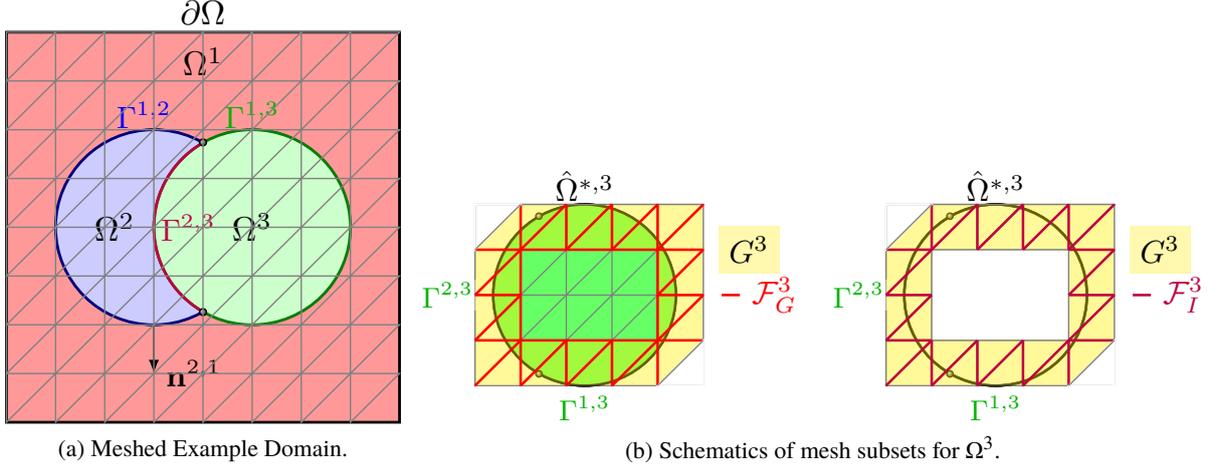


Figure 1: Example of rectangular domain Ω with two circular inclusions Ω^2 and Ω^3 and the definition of important mesh subsets for domain Ω^3 .

2.2. Discrete hybrid-mixed formulation of the contact problem

Let \mathcal{U}_h^i denote the vector valued space of continuous piecewise linear polynomials defined on the fictitious domain $\widehat{\Omega}^{*,i}$. Our weak formulation of linear elasticity reads: For all $\widehat{\Omega}^{*,i}$, find $\underline{u}_h^i \in \mathcal{U}_h^i$ such that for all $\delta \underline{u}_h^i \in \mathcal{U}_h^i$

$$a(\underline{u}_h^i, \delta \underline{u}_h^i) + a_b(\underline{u}_h^i, \delta \underline{u}_h^i) + j(\underline{u}_h^i, \delta \underline{u}_h^i) = l(\delta \underline{u}_h^i) + l_b(\delta \underline{u}_h^i). \quad (1)$$

The terms of $a(\underline{u}_h^i, \delta \underline{u}_h^i)$ and $l(\delta \underline{u}_h^i)$ are given by

$$\begin{aligned} a(\underline{u}_h^i, \delta \underline{u}_h^i) &= \int_{\Omega_h^i} \underline{\sigma}(\underline{u}_h^i) : \underline{\epsilon}(\delta \underline{u}_h^i) d\Omega - \int_{\Gamma_h^{i,j}} (\underline{\sigma}(\underline{u}_h^i) \cdot \underline{n}^{i,j}) \delta \underline{u}_h^i d\Gamma, \\ l(\delta \underline{u}_h^i) &= \int_{\Omega_h^i} \mathbf{f} \cdot \delta \underline{u}_h^i d\Omega, \end{aligned} \quad (2)$$

where $\underline{\sigma}(\underline{u}) = \lambda^i \text{tr}(\underline{\epsilon}(\underline{u})) \mathbf{I} + 2\mu_i \underline{\epsilon}(\underline{u})$ is the Cauchy-stress tensor, $\underline{\epsilon}(\underline{u}) = \frac{1}{2} (\nabla \underline{u} + \nabla \underline{u}^T)$ is the strain tensor, $\underline{n}^{i,j}$ is the normal on $\Gamma_h^{i,j}$ pointing from Ω_h^i to Ω_h^j , λ^i and μ^i are the two Lamé coefficients, *i.e.* $\lambda^i = \frac{E^i \nu}{(1+\nu)(1-2\nu)}$, $\mu^i = \frac{E^i}{2(1+\nu)}$, with E^i the Young's modulus and $\nu = 0.3$ the Poisson's ratio. Ω_h^i and $\Gamma_h^{i,j}$ denote piecewise polynomial approximations of exact geometrical sets. The terms $a_b(\underline{u}_h^i, \delta \underline{u}_h^i)$ and $l_b(\delta \underline{u}_h^i)$ weakly enforce Dirichlet conditions using Nitsche's method and Neumann boundary conditions

$$\begin{aligned} a_b(\underline{u}_h^i, \delta \underline{u}_h^i) &= - \int_{\partial\Omega_D} (\underline{\sigma}(\underline{u}_h^i) \cdot \underline{n}^{i,j}) \delta \underline{u}_h^i d\Gamma - \int_{\partial\Omega_D} (\underline{\sigma}(\delta \underline{u}_h^i) \cdot \underline{n}^{i,j}) \underline{u}_h^i d\Gamma + \int_{\partial\Omega_D} \frac{\alpha E^i}{h} \underline{u}_h^i \cdot \delta \underline{u}_h^i d\Gamma, \\ l_b(\delta \underline{u}_h^i) &= - \int_{\partial\Omega_D} (\underline{\sigma}(\delta \underline{u}_h^i) \cdot \underline{n}^{i,j}) \mathbf{g} d\Gamma + \int_{\partial\Omega_D} \frac{\alpha E^i}{h} \mathbf{g} \cdot \delta \underline{u}_h^i + \int_{\partial\Omega_N} \mathbf{T} \delta \underline{u}_h^i d\Gamma d\Gamma, \end{aligned} \quad (3)$$

where \mathbf{T} is a prescribed surface load, \mathbf{g} is a prescribed displacement and $\alpha > 0$ is the Nitsche penalty parameter. The term $j(\underline{u}_h^i, \delta \underline{u}_h^i)$ regularises the solution in the interface region and is given by

$$j(\underline{u}_h^i, \delta \underline{u}_h^i) = \sum_{F \in \mathcal{F}_G^i} \int_F \frac{\gamma_g h}{E^i} \llbracket \underline{\sigma}(\underline{u}_h^i) \cdot \underline{n}_F \rrbracket \cdot \llbracket \underline{\sigma}(\delta \underline{u}_h^i) \cdot \underline{n}_F \rrbracket ds. \quad (4)$$

We call $j(\underline{u}_h^i, \delta \underline{u}_h^i)$ ghost penalty stabilisation ([2]). Here, $\llbracket x \cdot \underline{n}_F \rrbracket$ denotes the normal jump of the quantity x over the face, F , defined as $\llbracket x \cdot \underline{n}_F \rrbracket = x|_K \cdot \underline{n}_F - x|_{K'} \cdot \underline{n}_F$, where \underline{n}_F denotes a unit normal to

the facet F with fixed but arbitrary orientation and $\gamma_g > 0$ is the ghost penalty parameter.

At contact interfaces $\Gamma_h^{i,j}$, we introduce interface force \underline{F}_h^i and interface displacement \underline{W}_h^i satisfying

$$\int_{\Gamma_h^{i,j}} \underline{F}_h^i \cdot \delta \underline{u}_h^i d\Gamma = \int_{\Gamma_h^{i,j}} \left(\underline{\underline{\sigma}}(\underline{u}_h^i) \cdot \underline{n}^{i,j} \right) \cdot \delta \underline{u}_h^i d\Gamma \quad , \text{ and } \quad \underline{W}_h^i = \underline{u}_h^i \quad \text{at } \Gamma_h^{i,j}. \quad (5)$$

Now, let $\mathcal{Q}_h(\mathcal{G}^i) := \{ \underline{v}_h \in C^0(\mathcal{G}^i) : \underline{v}_h|_K \in \mathcal{P}_1(K)^d, \forall K \in \mathcal{G}^i \}$ denote the space of continuous piecewise linear polynomials on the band of intersected elements. Then, we determine the interface fields as follows. Find $\underline{F}_h^i \in \mathcal{Q}_h(\mathcal{G}^i)$ and $\underline{W}_h^i \in \mathcal{Q}_h(\mathcal{G}^i)$, such that for all $\delta \underline{F}_h^i \in \mathcal{Q}_h(\mathcal{G}^i)$ and for all $\delta \underline{W}_h^i \in \mathcal{Q}_h(\mathcal{G}^i)$

$$\begin{aligned} \int_{\Gamma_h^{i,j}} (\underline{F}_h^i - \underline{F}_\heartsuit^i) \delta \underline{F}_h^i d\Gamma + j_F(\underline{F}_h^i, \delta \underline{F}_h^i) &= \underline{0}, \\ \int_{\Gamma_h^{i,j}} (\underline{W}_h^i - \underline{W}_\heartsuit^i) \delta \underline{W}_h^i d\Gamma + j_F(\underline{W}_h^i, \delta \underline{W}_h^i) &= \underline{0}, \end{aligned} \quad (6)$$

where

$$j_F(\underline{F}_h^i, \delta \underline{F}_h^i) = \sum_{F \in \mathcal{F}_h^i} \gamma_\Pi h^2 \int_F \llbracket \nabla \underline{F}_h^i \cdot \underline{n}_F \rrbracket \cdot \llbracket \nabla \delta \underline{F}_h^i \cdot \underline{n}_F \rrbracket ds \quad (7)$$

regularises the interface fields with a penalty parameter $\gamma_\Pi > 0$. Heart quantities $\underline{F}_\heartsuit^i, \underline{W}_\heartsuit^i$ are required to fulfill contact at a set of quadrature points along the interface $\Gamma_h^{i,j}$

$$\left(\underline{W}_\heartsuit^j - \underline{W}_\heartsuit^i \right) \cdot \underline{n}^{i,j} \geq 0; \quad \underline{F}_\heartsuit^i \cdot \underline{n}^{i,j} \leq 0; \quad \left(\left(\underline{W}_\heartsuit^j - \underline{W}_\heartsuit^i \right) \cdot \underline{n}^{i,j} \right) \cdot \left(\underline{F}_\heartsuit^i \cdot \underline{n}^{i,j} \right) = 0. \quad (8)$$

and the system is closed by requiring that the fluctuation of the heart quantities around \underline{F}_h^i and \underline{W}_h^i satisfy a Robin condition, which reads as

$$\left(\underline{F}_h^i - \underline{F}_\heartsuit^i \right) - \beta \left(\underline{W}_h^i - \underline{W}_\heartsuit^i \right) = \underline{0}, \quad (9)$$

where β is an algorithmic parameter that is homogeneous to a stiffness (we set it equal to the LaTIn search direction parameter for reasons not detailed here). We can finally solve the nonlinear coupled system of equations (1,5,6,8) for $\underline{u}_h^i, \underline{F}_h^i, \underline{W}_h^i, \underline{F}_\heartsuit^i, \underline{W}_\heartsuit^i$, using an extension of the LaTIn iterative algorithm [9] whereby the local stage is "non-localised" by two-scale continuity condition (6).

3. Results

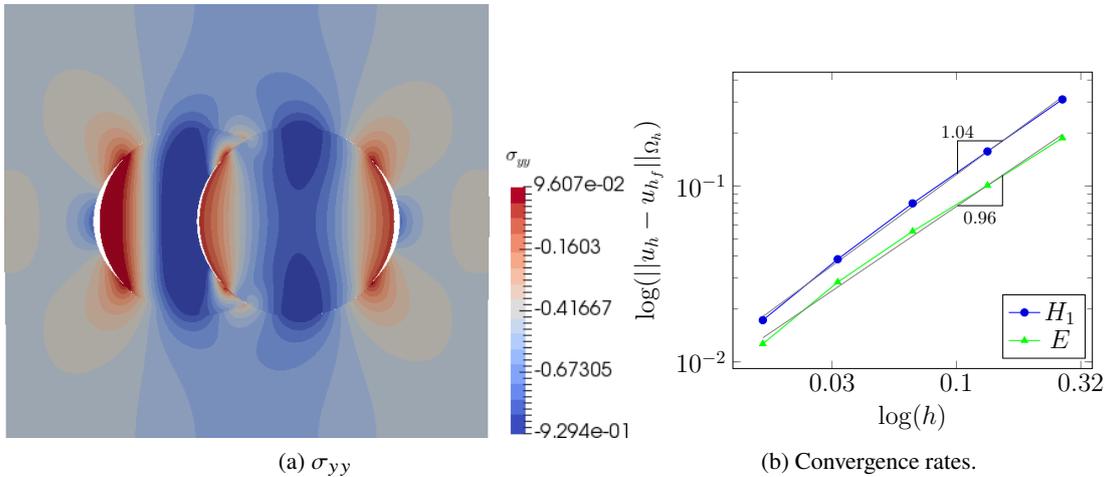


Figure 2: Convergence rates and energy norm error for two inclusions with $E_1 = E_2 = E_3 = 1$.

The structure illustrated in Figure 2 is made of two interacting inclusions within a square matrix. The square matrix is meshed using a regular grid of triangles (Figure 1a). The matrix is compressed in

the vertical direction. We plot the vertical compressive component of the stress. We also show the convergence rate together with the element size h . The convergence rate is optimum in both H^1 and energy semi-norm. Figure 3 shows the versatility of the method and the stability of the implementation in 3D.

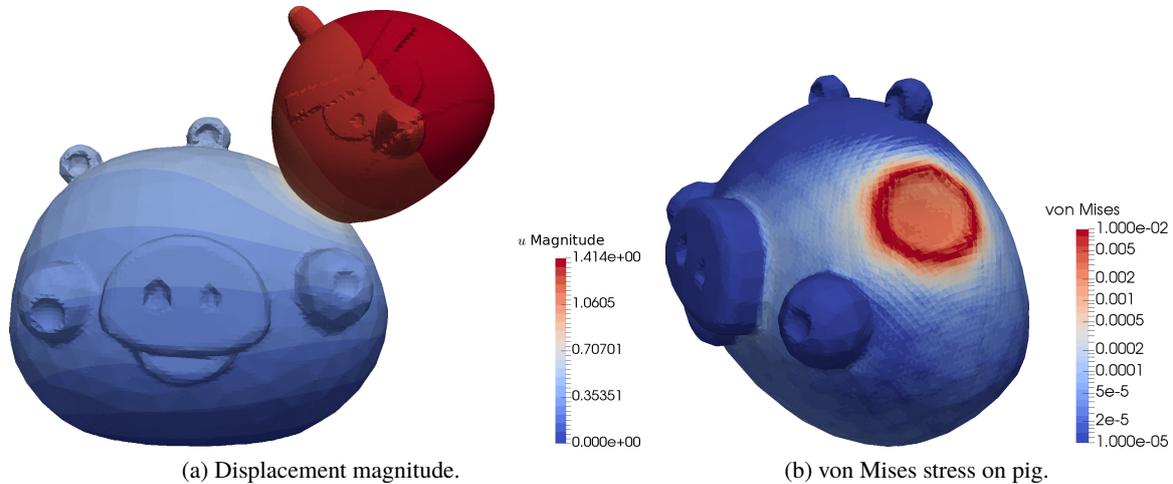


Figure 3: Angry bird in contact with a pig. The bird is being pushed into the pig, which results in the development of a stress concentration around the contact zone.

4. Conclusions

We have presented a novel non-conforming finite element formulation for multiple linear elastic bodies in contact. We have demonstrated that our discretisation scheme is stable independent of how the contact interfaces intersect our fixed regular background mesh and that the contact problem solution converges optimally with mesh refinement. The next stage of our investigations is to extend the approach to the context of large displacements, whereby the contact region is an unknown of the problem.

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