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Boundary PI controllers for a star-shaped network of 2×2 systems governed by hyperbolic partial differential equations (long version)

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Abstract: The paper is concerned with the PI control regulation of a star-shaped network of systems governed by hyperbolic partial differential equations. The control input and measured output are on the boundary. First, each system of the network is linearized and diagonalized with Riemann invariants. Then, by using Lyapunov direct method, the PI controller is proposed for a single system. Finally, we extend the PI control design for the star-shaped network of n subsystems. The exponential stability and output regulation of closed-loop systems are all proven with the aid of a strict Lyapunov functional.

Keywords: Control of hyperbolic systems and conservation laws, infinite-dimensional multi-agent systems and networks, stability of distributed parameter systems, output regulation for distributed parameter systems, Lyapunov methods.

1. INTRODUCTION

In this paper, we consider a star-shaped network of open channels described by two Saint-Venant equations of conservation laws, studied in Halleux et al. (2003); Bastin et al. (2009); Trinh et al. (2015)

$$\begin{cases} \partial_t H(x, t) + \partial_x (H(x, t)V(x, t)) & = 0, \\ \partial_t V(x, t) + \partial_x \left(\frac{V^2(x, t)}{2} + gH(x, t) \right) & = 0 \end{cases} \quad (1)$$

where H denotes the water depth, V the water velocity, and g the gravitational acceleration. Here $x \in [0, L]$ and $t \in [0, \infty)$ are space and time coordinates, and ∂_t, ∂_x stand for the time and space partial derivative respectively.

System (1) can be linearized around a set point $H = H_e$, and $Q = Q_e$ to obtain the following equations:

$$\partial_t \begin{pmatrix} h \\ v \end{pmatrix} + \begin{pmatrix} V_e & H_e \\ g & V_e \end{pmatrix} \partial_x \begin{pmatrix} h \\ v \end{pmatrix} = 0, \quad (2)$$

where $h = H - H_e$ and $v = V - V_e$.

System (2) has been regarded as a 2×2 linear hyperbolic system. Up to now, the topic of controlling hyperbolic systems on the boundary has led to numerous studies in the literature, for example Coron et al. (2007); Greenberg and Li (1984). Furthermore, some networks of hyperbolic systems were studied, for the cascaded networks in Halleux et al. (2003); Bastin et al. (2009), and for the tree-shaped ones in Gugat et al. (2011); Leugering and Georg Schmidt (2002). Most of them apply static control laws to stabilize the closed-loop systems. Following this, dynamic control laws were developed to make the closed-loop systems

more robust confronted by constant perturbations, see for instance in Pham (2012); Perrollaz and Rosier (2014); Bastin et al. (2015); Trinh et al. (2017). However, little research has been carried out in this direction because of the complexity of systems controlled by dynamic control laws, especially for network cases. Our motivation is therefore to design a dynamic control law in the form of PI (proportional integral) controllers to stabilize the closed-loop system and regulate output measurement to desired reference, and afterward extend the control design result for a star-shaped network of hyperbolic systems.

The applying PI controller for hyperbolic systems has also been studied in the literature. Many results are based on the frequency domain approach combined with Laplace transformation or operator and semi-group theory, see in Bastin et al. (2015); Xu and Sallet (2014). In other works of Dos Santos et al. (2008); Trinh et al. (2016), the Lyapunov direct method is employed to prove the stability of the closed-loop system in L^2 norm. In this paper, our contribution is making use of Lyapunov techniques to prove the exponential stability of the closed-loop system controlled by PI controllers in H^1 norm, and then obtain the regulation of the output. In addition, our approach is also extended for a star-shaped network of hyperbolic systems, and it is novel to the best of our knowledge.

This paper is the long version of the paper published in Trinh et al. (2017) in which some proofs have been removed due to space constraints.

The paper is organized as follows. Section II is devoted to present the characteristic form and PI controller structure used for the next sections. In Section 3, our PI controller design for a single system and the mathematical proof based on Lyapunov method are given. The main contribution is presented in Section 4, in which we analyze how to generalize the PI control design from the single system to the star-shaped network. Finally, the conclusions are addressed in Section 5.

2. CHARACTERISTIC FORM AND PI CONTROLLER STRUCTURE

We consider the linearised system (2) with the output measurement $y(t)$ which is chosen as the water level at $x = 0$, i.e, $y(t) = h(0, t) + y_r$ where $y_r = H_e$. By using the Riemann coordinates, the equations in (2) can be transformed to the characteristic form as follows:

$$\partial_t \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} + \begin{pmatrix} \lambda_1 & 0 \\ 0 & -\lambda_2 \end{pmatrix} \partial_x \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} = 0, \quad (3)$$

$$y(t) - y_r = a\phi_1(0, t) + b\phi_2(0, t)$$

where

$$\phi_1 = h + v\sqrt{\frac{H_e}{g}}, \quad \phi_2 = h - v\sqrt{\frac{H_e}{g}} \quad (4)$$

and $\lambda_1 = V_e + \sqrt{gH_e}$, $\lambda_2 = -V_e + \sqrt{gH_e}$, $a = \frac{1}{2}$, $b = \frac{1}{2}$. Here, we consider only the subcritical water flow, i.e $\lambda_1 > 0$ and $\lambda_2 > 0$.

We are concerned in the paper the system (3) under the initial conditions

$$\phi_1(x, 0) = \phi_1^0(x), \quad \phi_2(x, 0) = \phi_2^0(x), \quad \forall x \in [0, L], \quad (5)$$

and suppose two following boundary conditions

$$\begin{aligned} \phi_1(0, t) &= R_1\phi_2(0, t) + u(t), \\ \phi_2(L, t) &= R_2\phi_1(L, t) \end{aligned} \quad (6)$$

where $u(t)$ is control input and R_1, R_2 are real constant parameters, computed from the adaptive gate openings controlled on the boundary (see for example in Section 4). Our control target is to propose a dynamic feedback control law $u(t)$ by the PI controller form such that the closed-loop system is asymptotically stable and the output $y(t)$ is regulated to the desired reference y_r . Precisely, we need to design the PI controller on the boundary with the real gain parameters K_P and K_I ,

$$u(t) = K_P(y(t) - y_r) + K_I \int_0^t (y(s) - y_r) ds \quad (7)$$

to guarantee the exponential stability of the closed-loop system (3)-(7) and the output regulation.

The results of control design is presented in the following: in Section 3 for a single channel and in Section 4 for network of multichannels.

3. PI CONTROLLER DESIGN FOR A SINGLE CHANNEL

In this section, we consider a single channel with the linearised characteristic model analyzed in (3) with boundary conditions in (6) and the PI control feedback in (7). The direct Lyapunov method is used to prove the exponential stability and the output regulation of the closed-loop system controlled by PI controller.

Let denote the new state variable $\xi(t)$ where $\partial_t \xi = y(t) - y_r$, the closed-loop system (3), (6), and (7) is governed by

$$\begin{cases} \partial_t \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} = \begin{pmatrix} -\lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \partial_x \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \\ \partial_t \xi = a\phi_1(0, t) + b\phi_2(0, t) \\ \phi_1(0, t) = R_1\phi_2(0, t) + K_P(y(t) - y_r) + K_I \xi \\ \phi_2(L, t) = R_2\phi_1(L, t) \\ y(t) - y_r = a\phi_1(0, t) + b\phi_2(0, t). \end{cases} \quad (8)$$

with the following initial condition:

$$(\phi_1^0(x), \phi_2^0(x), \xi^0) \in (H^1(0, L))^2 \times \mathbb{R}, \quad \forall x \in [0, L]$$

For each initial condition $(\phi_1^0(x), \phi_2^0(x), \xi^0)$ in $(H^1(0, L))^2 \times \mathbb{R}$ and satisfying C^0 and C^1 compatibility conditions, then there exists a unique solution $(\phi_1(\cdot, t), \phi_2(\cdot, t), \xi(t))$ of (8) in $(C^0([0, +\infty), H^1(0, L)))^2 \times \mathbb{R}$ for all t . In the paper, we study the stability of the closed-loop system and the regulation of the output to the desired reference.

Let $\mathbb{X} = (H^1(0, L))^2 \times \mathbb{R}$ be the state space of the closed-loop system (8) equipped with the following norm :

$$\|Z\|_{\mathbb{X}}^2 = \|Z_1(\cdot, t)\|_{H^1(0, L)}^2 + \|Z_2(\cdot, t)\|_{H^1(0, L)}^2 + |Z_3(t)|^2$$

where $Z = (Z_1, Z_2, Z_3) \in \mathbb{X}$.

The PI controller is designed on the boundary at first for a single system as following

Theorem 1. There exists $\mu^* > 0$ such that for each $\mu \in (0, \mu^*)$, and for each PI controller with the following proportional gain K_P and the integral gain K_I :

$$K_P = \frac{-R_1}{b}, \quad K_I = -\mu \frac{(a + bR_2e^{\mu L})(b + aR_1)}{b} \quad (9)$$

then the two following properties hold true :

- The closed-loop system (8) with the PI controller design in (9) is exponentially stable toward the origin in \mathbb{X} .
- The output $y(t)$ is regulated to the desired reference y_r , i.e:

$$\lim_{t \rightarrow \infty} |y(t) - y_r| = 0$$

Proof : Applying the PI controller design in (9), the closed-loop system (8) is described as follows

$$\begin{cases} \partial_t \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} = \begin{pmatrix} -\lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \partial_x \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \\ \partial_t \xi = a\phi_1(0, t) + b\phi_2(0, t) \\ \phi_1(0, t) = -k_i \xi \\ \phi_2(L, t) = R_2\phi_1(L, t) \\ y(t) - y_r = a\phi_1(0, t) + b\phi_2(0, t). \end{cases} \quad (10)$$

where

$$k_i = \mu(a + bR_2e^{\mu L}) \quad (11)$$

Let us put $\phi_{1x} = \partial_x \phi_1$ and $\phi_{2x} = \partial_x \phi_2$. The dynamics of $\phi_{1x}(x, t)$ and $\phi_{2x}(x, t)$ are given by

$$\begin{aligned}
& \partial_t \begin{pmatrix} \phi_{1x} \\ \phi_{2x} \end{pmatrix} + \begin{pmatrix} \lambda_1 & 0 \\ 0 & -\lambda_2 \end{pmatrix} \partial_x \begin{pmatrix} \phi_{1x} \\ \phi_{2x} \end{pmatrix} = 0, \\
& (\phi_{1x}(x, 0), \phi_{2x}(x, 0)) \in (L^2(0, L))^2, \\
& \phi_{1x}(0, t) = \frac{k_i}{\lambda_1} (a\phi_1(0, t) + b\phi_2(0, t)) \\
& \phi_{2x}(L, t) = -\frac{R_2\lambda_1}{\lambda_2} \phi_{1x}(L, t)
\end{aligned} \tag{12}$$

We consider the following Lyapunov function candidate

$$V(\phi_1, \phi_2, \xi) = \int_0^L \phi^T P \phi dx \tag{13}$$

$$\text{where } \phi = \begin{pmatrix} \phi_1 e^{-\frac{\mu x}{2}} \\ \phi_2 e^{\frac{\mu x}{2}} \\ \xi \\ \phi_{1x} e^{-\frac{\mu x}{2}} \\ \phi_{2x} e^{\frac{\mu x}{2}} \end{pmatrix}, P = \begin{pmatrix} 1 & 0 & q_3 & 0 & 0 \\ 0 & q_1 & q_4 & 0 & 0 \\ q_3 & q_4 & q_2 & 0 & 0 \\ 0 & 0 & 0 & q_5 & 0 \\ 0 & 0 & 0 & 0 & q_6 \end{pmatrix}, \mu >$$

0, and $q_i \in \mathbb{R}$ with $i = \overline{1, 6}$ are parameters that have to be designed. The following lemma shows our design for q_i .

Lemma 1. Let k_i be defined in (11) and q_1, q_2, \dots, q_6 be defined by

$$\begin{aligned}
q_1 &= \frac{\lambda_1 e^{-2\mu L}}{2\lambda_2 R_2^2}, \quad q_2 = \mu\lambda_1, \\
q_3 &= \mu b R_2 e^{\mu L}, \quad q_4 = \mu \frac{b\lambda_1}{\lambda_2}, \\
q_5 &= \gamma 2e^{2\mu L} R_2^2 \lambda_1 \lambda_2, \quad q_6 = \gamma.
\end{aligned} \tag{14}$$

Then there exists $\gamma > 0$ and $\mu^* > 0$ such that for every $\mu \in (0, \mu^*)$, we have :

- (1) There exists $M > 0$ such that $\forall (\phi_1, \phi_2, \xi)$ in \mathbb{X} :

$$\frac{1}{M} V(\phi_1, \phi_2, \xi) \leq \|(\phi_1, \phi_2, \xi)\|_{\mathbb{X}}^2 \leq M V(\phi_1, \phi_2, \xi). \tag{15}$$

- (2) There exists $\beta > 0$ such that along the solution of (10), for all t at which the solution is well defined

$$\dot{V}(t) \leq -\beta V(t) - \phi_1^2(L, t) \frac{\lambda_1 e^{-\mu L}}{2}. \tag{16}$$

The proof of Lemma 1 is given in Appendix.

Now applying the Lemma 1 and the Lyapunov candidate to prove the Theorem 1.

At first, we prove the exponential stability of the closed-loop system (10) toward the origin in \mathbb{X} . According to Lemma 1, there exists $\beta > 0$ such that:

$$V(\phi_1, \phi_2, \xi) \leq V(\phi_1^0(x), \phi_2^0(x), \xi^0) e^{-\beta t}.$$

Moreover, using (15), one can find $K > 0$ such that for all initial conditions $(\phi_1^0(x), \phi_2^0(x), \xi^0) \in (H^1(0, L))^2 \times \mathbb{R}$ and satisfies C^0 and C^1 compatibility conditions, the solution is defined for all positive time in \mathbb{X} and satisfies:

$$\|(\phi_1, \phi_2, \xi)\|_{\mathbb{X}}^2 \leq K e^{-\beta t} \|(\phi_1^0, \phi_2^0, \xi^0)\|_{\mathbb{X}}^2. \tag{17}$$

This implies that the origin of the closed-loop system (10) is exponentially stable in \mathbb{X} .

Secondly, we prove the regulation part. Employing (17), it is easy to show that

$$\lim_{t \rightarrow \infty} \|\phi_1(x, t)\|_{H^1(0, L)} = 0, \quad \lim_{t \rightarrow \infty} \|\phi_2(x, t)\|_{H^1(0, L)} = 0. \tag{18}$$

Now employing the Sobolev embedding theorem, see in Brezis (1983), one can find that for $\forall x \in [0, L]$

$$\lim_{t \rightarrow \infty} \phi_1(x, t) = 0, \quad \lim_{t \rightarrow \infty} \phi_2(x, t) = 0.$$

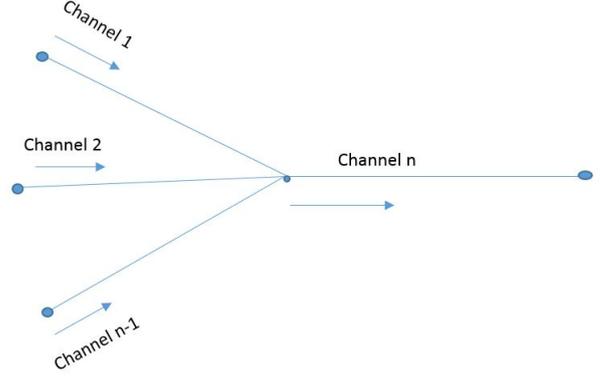


Fig. 1. Star-shaped networks of n channels

As a result, $\lim_{t \rightarrow \infty} \|y(t) - y_r\|_{\mathbb{R}^2}^2 = 0$. This completes the proof of the Theorem 1. \square

Remark : Our result for a single system with the dynamic feedback PI control represented by terms of ξ in Lyapunov candidate is an extension of the one in Coron et al. (2007); Bastin et al. (2009). Furthermore, by adding the first order derivative terms of ϕ_{1x} and ϕ_{2x} in Lyapunov function, it leads to the proof of the stability with H^1 norm, and compared to the one with L^2 norm in results of Dos Santos et al. (2008); Trinh et al. (2016), our stability is useful to directly imply the output regulation. \blacktriangleleft

4. PI CONTROLLER DESIGN FOR A STAR-SHAPED NETWORK OF CHANNELS

4.1 Statement of star-shaped network control problem

In this section, we extend the PI control results acquired from the previous section to a star-shaped network of n channels ($n \geq 3$). The connection between channels is depicted in Fig. 1, with $n - 1$ inlet channels (1th to $(n - 1)$ th) and a outlet channel (n th channel).

Modeling Each channel of the network is modeled by two PDE hyperbolic equations (1). Without loss of generality, the lengths of channels are assumed identical and equal L . The following notations are used for the next : $i = \overline{1, n}$ and $j = \overline{1, n-1}$; $H_i(x, t)$ is water level in the i th channel, $H_{i0}(t) = H_i(0, t)$ and $H_{iL}(t) = H_i(L, t)$; $V_i(x, t)$ is water velocity in the i th channel, $V_{i0}(t) = V_i(0, t)$ and $V_{iL}(t) = V_i(L, t)$. The network is thus governed by $2n$ equations :

$$\begin{aligned}
\partial_t H_i(x, t) + \partial_x (H_i(x, t) V_i(x, t)) &= 0, \\
\partial_t V_i(x, t) + \partial_x \left(\frac{V_i^2(x, t)}{2} + g H_i(x, t) \right) &= 0.
\end{aligned} \tag{19}$$

The $2n - 1$ online measured outputs at each time t are

$$y_{j0}(t) = H_{j0}(t), \quad y_{iL}(t) = H_{iL}(t) \tag{20}$$

We assume that the network is controlled by $2n - 1$ input controls, in which each channel is controlled by an independent gate at $x = L$, i.e

$$H_{iL}^2(t) V_{iL}^2(t) - U_{iL}(t) (H_{iL}(t) - H_{di}) = 0, \tag{21}$$

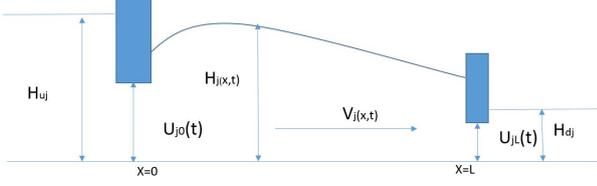


Fig. 2. Channel j th

and $n - 1$ inlet channels is actively controlled at $x = 0$ by

$$H_{j0}^2(t)V_{j0}^2(t) - U_{j0}(t)(H_{uj} - H_{j0}(t)) = 0. \quad (22)$$

where control inputs U_{iL} and U_{j0} are gate openings at the boundary, the constants H_{di} and H_{uj} are water levels outside channels (see in Fig 2).

Note however that at the junction, the constraint of flow-rate conservation is expressed by

$$H_{n0}(t)V_{n0}(t) = \sum_{j=1}^{n-1} H_{jL}(t)V_{jL}(t). \quad (23)$$

The equation (23) implies that, at the boundary $x = 0$ the outlet channel (n th channel) cannot be controlled, but its dynamic can be deduced from the conservation of the flow.

Equilibrium states Each $2n$ constant values H_i^* , V_i^* satisfying the following conditions

- $gH_i^* > (V_i^*)^2$ (subcritical conditions),
- $H_{uj} > H_j^* > H_{dj}$, $H_n^* > H_{dn}$ and $V_i^* > 0$,
- $H_n^*V_n^* = \sum_{j=1}^{n-1} H_j^*V_j^*$,

can be chosen as an equilibrium state with the appropriate constant controls U_{j0}^* , U_{iL}^* .

Control and regulation problem The objective is to stabilize the linearised model of the network (19)-(23) to the set-points H_i^* , V_i^* and regulate the output measurements of each inlet channel $y_j(t)$ to the desired references $y_{jr} = H_j^*$.

Apply the similar Riemann coordinates in (4) for the linearized models around the set-point H_i^* , V_i^* , one obtains the characteristic forms as follows:

$$\begin{aligned} \partial_t \phi_{i1}(x, t) + \lambda_{i1} \partial_x \phi_{i1} &= 0, \\ \partial_t \phi_{i2}(x, t) - \lambda_{i2} \partial_x \phi_{i2} &= 0 \end{aligned} \quad (24)$$

and $2n - 1$ measured outputs

$$\begin{aligned} y_{j0}(t) - y_{jr} &= a\phi_{j1}(0, t) + b\phi_{j2}(0, t), \\ y_{iL}(t) - y_{ir} &= a\phi_{n1}(L, t) + b\phi_{n2}(L, t) \end{aligned} \quad (25)$$

where the characteristic variables are

$$\begin{aligned} \phi_{i1} &= (H_i - H_i^*) + (V - V_i^*) \sqrt{\frac{H_i^*}{g}}, \\ \phi_{i2} &= (H_i - H_i^*) - (V_i - V_i^*) \sqrt{\frac{H_i^*}{g}} \end{aligned} \quad (26)$$

and $\lambda_{i1} = V_i^* + \sqrt{gH_i^*} > 0$, $\lambda_{i2} = -V_i^* + \sqrt{gH_i^*} > 0$, $a = \frac{1}{2}$, $b = \frac{1}{2}$.

Inspired by the stability and regulation analysis in Section 3, we want to receive the following boundary conditions :

$$\begin{aligned} \phi_{j1}(0, t) &= R_{j1}\phi_{j2}(0, t) + u_j(t), \\ \phi_{i2}(L, t) &= R_{i2}\phi_{i1}(L, t), \end{aligned} \quad (27)$$

where R_{j1} and R_{i2} are arbitrary constants, $u_j(t)$ is the dynamic feedback control law which we want to design by PI controller for j th channel.

Hence, using (25), (26), (21) and (22), we can compute the adaptive control gate openings on the boundary to satisfy conditions (27) as follows

$$\begin{aligned} U_{j0} &= \frac{y_{j0}^2}{H_{uj} - y_{j0}} \left(V_j^* + \sqrt{\frac{g}{H_j^*}} \frac{(R_{j1} - 1)(y_{j0} - H_j^*) + u_j}{R_{j1} + 1} \right)^2, \\ U_{iL} &= \frac{y_{iL}^2}{y_{iL} - H_{dn}} \left(V_i^* + \sqrt{\frac{g}{H_i^*}} \frac{(1 - R_{i2})(y_{iL} - H_i^*)}{R_{i2} + 1} \right)^2. \end{aligned} \quad (28)$$

Remark : Here, to implement control inputs U_{iL} and U_{j0} , we only need $2n - 1$ online measurements y_{j0} and y_{iL} as water levels H_{j0} and H_{iL} . It is practical because measure flow discharge in reality is difficult. ◀

The condition at the junction (23) therefore becomes

$$\phi_{n1}(0, t) = R_{n1}\phi_{n2}(0, t) + \sum_{j=1}^{n-1} \alpha_j \phi_{j1}(L, t). \quad (29)$$

where

$$R_{n1} = \frac{\lambda_{n2}}{\lambda_{n1}}, \quad \alpha_j = \frac{\lambda_{j1} + R_{j2}\lambda_{j2}}{\lambda_{n1}}.$$

To summarize, the control target is to design the dynamic feedback PI controllers $u_j(t)$,

$$u_j(t) = K_{jP}(y_j(t) - y_{jr}) + K_{jI} \int_0^L (y_j(s) - y_{jr}) ds \quad (30)$$

such that the network of closed-loop systems (24), (25), (27), (29) and (30) is stabilized toward the origin and the outputs $y_j(t)$ of inlet channels are regulated to the reference y_{jr} ($\forall j = \overline{1, n-1}$).

4.2 PI controller design

The ideal of control design is to extend from the results in Section 3, we propose for each inlet channel a PI controller motivated by Theorem 1 and guarantee the exponential stability and the output regulation for the network of closed-loop systems controlled by PI controllers.

Denoting the new state variables $\xi_j(t)$ where $\partial_t \xi_j = y_j(t) - y_{jr}$, the network of closed-loop systems (24), (25), (27), (29) and (30) is governed by :

$$\begin{cases} \partial_t \phi_{i1}(x, t) = -\lambda_{i1} \partial_x \phi_{i1}, \\ \partial_t \phi_{i2}(x, t) = \lambda_{i2} \partial_x \phi_{i2}, \\ \partial_t \xi_j = a\phi_{j1}(0, t) + b\phi_{j2}(0, t) \\ \phi_{i2}(L, t) = R_{i2}\phi_{i1}(L, t), \\ \phi_{j1}(0, t) = R_{j1}\phi_{j2}(0, t) + K_{jP}(y_j(t) - y_{jr}) + K_{jI}\xi_j(t), \\ \phi_{n1}(0, t) = R_{n1}\phi_{n2}(0, t) + \sum_{j=1}^{n-1} \alpha_j \phi_{j1}(L, t). \\ y_j(t) - y_{jr} = a\phi_{j1}(0, t) + b\phi_{j2}(0, t). \end{cases} \quad (31)$$

The closed-loop system (31) is completed by the following initial conditions

$$\begin{aligned} & (\phi_{11}^0(x), \phi_{12}^0(x), \dots, \phi_{n1}^0(x), \phi_{n2}^0(x), \xi_1^0, \dots, \xi_{n-1}^0) \\ & \in (H^1(0, L))^{2n-2} \times (L^2(0, L))^2 \times \mathbb{R}^{n-1}. \end{aligned} \quad (32)$$

Let $\mathbb{E} = (H^1(0, L))^{2n-2} \times (L^2(0, L))^2 \times \mathbb{R}^{n-1}$ be the state space of the closed-loop system (31) equipped with the following norm :

$$\begin{aligned} \|Y\|_{\mathbb{E}}^2 &= \sum_{i=1}^{2n-2} \|Y_i(\cdot, t)\|_{H^1(0, L)}^2 + \|Y_{2n-1}(\cdot, t)\|_{L^2(0, L)}^2 \\ &+ \|Y_{2n}(\cdot, t)\|_{L^2(0, L)}^2 + \sum_{j=2n+1}^{3n-1} |Y_j(t)|^2, \end{aligned}$$

where $Y = (Y_1, Y_2, \dots, Y_{3n-1}) \in \mathbb{E}$.

The main result of this Section is given in the following theorem

Theorem 2. There exists $\mu^* > 0$ such that for each $\mu \in (0, \mu^*)$, each $R_{n2} \in \left(-\frac{\lambda_{n1}}{\lambda_{n2}}, \frac{\lambda_{n1}}{\lambda_{n2}}\right)$ and for each PI controller with the following proportional gain K_{jP} and the integral gain K_{jI}

$$K_{jP} = \frac{-R_{j1}}{b}, \quad K_{jI} = -\mu \frac{(a + bR_{j2}e^{\mu L})(b + aR_{j1})}{b} \quad (33)$$

then the two following properties hold true :

- The network of closed-loop systems (31) with the initial condition in (32) is exponentially stable toward the origin in \mathbb{E} .
- The measured outputs $y_j(t)$ of each inlet channel are regulated to the desired set-points y_{jr} , i.e :

$$\lim_{t \rightarrow \infty} |y_j(t) - y_{jr}| = 0$$

4.3 Lyapunov method for the proof

Inspired from the proof of the Theorem 1, we construct the following Lyapunov candidate function for the closed-loop system (31):

$$\mathbb{S} = \sum_{j=1}^{n-1} V_j + V_n \quad (34)$$

where V_j has the same form in (13), i.e

$$V_j = \int_0^L F_j^T P_j F dx$$

$$\text{with } F_j = \begin{pmatrix} \phi_{j1} e^{-\frac{\mu x}{2}} \\ \phi_{j2} e^{\frac{\mu x}{2}} \\ \xi_j \\ \phi_{j1x} e^{-\frac{\mu x}{2}} \\ \phi_{j2x} e^{\frac{\mu x}{2}} \end{pmatrix}, \quad P_j = \begin{pmatrix} 1 & 0 & q_{j3} & 0 & 0 \\ 0 & q_{j1} & q_{j4} & 0 & 0 \\ q_{j3} & q_{j4} & q_{j2} & 0 & 0 \\ 0 & 0 & 0 & q_{j5} & 0 \\ 0 & 0 & 0 & 0 & q_{j6} \end{pmatrix},$$

$$V_n = q_n \int_0^L (q \phi_{n1}^2 e^{-\mu x} + \phi_{n2}^2 e^{\mu x}) dx$$

Employing the Lemma 1, we can design the parameters q_{j1}, \dots, q_{j6} for every $j = \overline{1, n-1}$ satisfying that

- There exists $M_j > 0$ such that

$$\frac{1}{M_j} V_j \leq \|(\phi_{j1}, \phi_{j2}, \xi_j)\|_{\mathbb{X}}^2 \leq M_j V_j. \quad (35)$$

- There exists $\beta_j > 0$ such that

$$\dot{V}_j(t) \leq -\beta_j V_j(t) - \phi_{j1}^2(L, t) \frac{\lambda_{j1} e^{-\mu L}}{2}. \quad (36)$$

Now analysis the time derivative of V_n along the solution of the system (24),

$$\begin{aligned} \dot{V}_n(t) &= -\mu q_n \int_0^L (\lambda_{n1} q \phi_{n1}^2 e^{-\mu x} + \lambda_{n2} \phi_{n2}^2 e^{\mu x}) dx \\ &- \phi_{n1}^2(L, t) e^{-\mu L} q_n (\lambda_{n1} q - \lambda_{n2} R_{n2}^2) - \phi_{n2}^2(0, t) \lambda_{2n} q_n \\ &+ \lambda_{n1} q_n q \left(R_{n1} \phi_{n2}(0, t) + \sum_{j=1}^{n-1} \alpha_j \phi_{j1}(L, t) \right)^2. \end{aligned} \quad (37)$$

Since $|R_{n2}| < \frac{\lambda_{n1}}{\lambda_{n2}}$, one can choose q such that $\frac{R_{n2}^2 \lambda_{n2}}{\lambda_{n1}} < q < \frac{\lambda_{n1}}{\lambda_{n2}}$, and take $q_n > 0$ small enough, we have

$$\begin{aligned} & - \sum_{j=1}^{n-1} \phi_{j1}^2(L, t) \frac{\lambda_{j1} e^{-\mu L}}{2} - \phi_{n2}^2(0, t) \lambda_{n2} q_n \\ & + \lambda_{n1} q_n q \left(R_{n1} \phi_{n2}(0, t) + \sum_{j=1}^{n-1} \alpha_j \phi_{j1}(L, t) \right)^2 \leq 0, \\ & - \phi_{n1}^2(L, t) e^{-\mu L} (\lambda_{n1} q - \lambda_{n2} R_{n2}^2) q_n \leq 0. \end{aligned} \quad (38)$$

From (36), (37) and (38), there exists $\delta > 0$ such that

$$\sum_{j=1}^{n-1} \dot{V}_j + \dot{V}_n \leq -\delta \left(\sum_{j=1}^{n-1} V_j + V_n \right)$$

This implies that

$$\dot{\mathbb{S}}(t) \leq -\delta \mathbb{S}(t) \quad (39)$$

Moreover, one can find $A > 0$ such that

$$\frac{1}{A} \mathbb{S} \leq \|(\phi_{11}, \phi_{12}, \dots, \phi_{n1}, \phi_{n2}, \xi_1, \dots, \xi_{n-1})\|_{\mathbb{E}} \leq A \mathbb{S} \quad (40)$$

From (39) and (40), it leads to the exponential stability toward the origin of network closed-loop system (31) in \mathbb{E} . Finally, the proof of the output regulation for $n - 1$ inlet channels is similar to the one in Theorem 1.

4.4 Numerical simulations

Because of the lack of space, in this Section, we give only the numerical simulations for a single channel with the following data:

- Length of channel $L = 50 \text{ m}$ with the outside level:
 $H_{d1} = 3, \quad H_{u1} = 1 \text{ m}.$
- The state equilibrium:
 $H^* = 2 \text{ m}, \quad V^* = 0.5 \text{ m/s}.$

Clearly, the above data satisfy the subcritical conditions and physical conditions. The tuning parameter μ is chosen 0.02 in the simulations.

The simulations are based on Preissmann schema, with weighting coefficient $\theta = 0.55$, space discretization step $\Delta x = 1 \text{ m}$.

Figure 3 and Figure 4 show the stability of the closed-loop system controlled by PI controller. One can see that the equilibrium state ($H^* = 2 \text{ m}$ and $V^* = 0.5 \text{ m/s}$) is stable

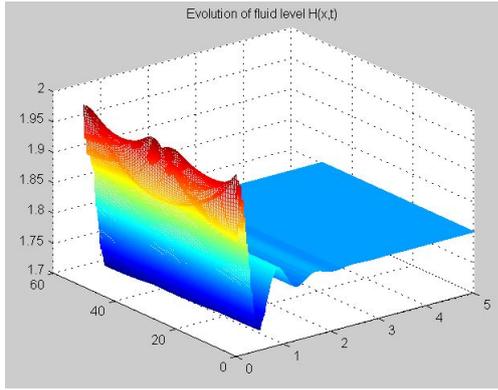


Fig. 3. Water level $H(x, t)$

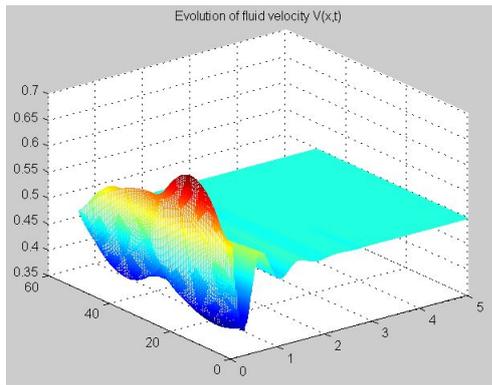


Fig. 4. Water velocity $V(x, t)$

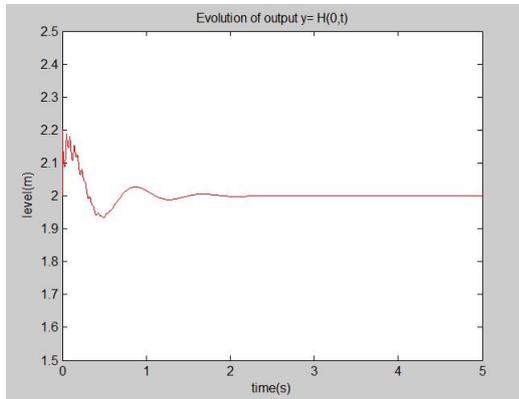


Fig. 5. Output $H(0, t)$

for the closed-loop system

In Figure 5, we see that the output measurement $H(0, t)$ converges to the desired reference $H^* = 2$ m.

Note also that with the big value of tuning parameter μ (bigger than 0.002) in simulations, the controlled system becomes unstable.

5. CONCLUSIONS

In the paper, we have studied PI control design for a star-shaped network of hydraulic systems governed by two hyperbolic partial differential equations. The boundary PI controller is proposed by using Lyapunov direct method not only for a single system, but also for a star-shaped network in which each inlet system is applied a PI controller at the free extremity. Here the stability analysis is

in H^1 norm and used to prove the output regulation effect for the closed-loop system. In the future, the objective is to extend the control design for a tree-shaped network, and to consider the PI control design at the junctions.

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Appendix A. PROOF OF LEMMA 1

A.1 Proof of the first part

To begin with, we prove that matrix P is definite positive. Applying the Sylvester criterion, matrix P is positive definite if and only if :

$$q_1 > 0, q_2 > q_3^2 + \frac{q_4^2}{q_1}, q_5 > 0, q_6 > 0.$$

Employing (14), one can find that :

$$q_1 > 0, q_5 > 0, q_6 > 0,$$

$$q_2 - q_3^2 - \frac{q_4^2}{q_1} = \mu \left(\lambda_1 - \mu b^2 R_2^2 e^{2\mu L} - \mu b^2 R_2^2 e^{2\mu L} \frac{2\lambda_1}{\lambda_2} \right).$$

It is clearly that if μ is small enough, $q_2 - q_3^2 - \frac{q_4^2}{q_1} > 0$.

It therefore yields that the matrix P is symmetric positive definite. Hence, there exists $\sigma_1, \sigma_2 > 0$ such that

$$\sigma_1 \int_0^L \phi^T \phi \leq V(\phi_1, \phi_2, \xi) \leq \sigma_2 \int_0^L \phi^T \phi$$

with $\phi = \left[\phi_1 e^{-\frac{\mu x}{2}}, \phi_2 e^{\frac{\mu x}{2}}, \xi, \phi_{1x} e^{-\frac{\mu x}{2}}, \phi_{2x} e^{\frac{\mu x}{2}} \right]^T$

As a result, there exists $M > 0$ such that (15) holds.

A.2 Proof of the second part

The time derivative of V along the solution of system (10) has the following form:

$$\dot{V} = - \int_0^L \mathbf{\Lambda}^T Q \mathbf{\Lambda} - L(t),$$

where $\mathbf{\Lambda} = \left[\phi_1 e^{-\frac{\mu x}{2}}, \phi_2 e^{\frac{\mu x}{2}}, \xi, \phi_2(0, t), \phi_{1x} e^{-\frac{\mu x}{2}}, \phi_{2x} e^{\frac{\mu x}{2}} \right]$,

$$\begin{aligned} L(t) = & \phi_1^2(L, t) \frac{\lambda_1 e^{-\mu L}}{2} + \phi_{1x}^2(L, t) \gamma e^{\mu L} R_2^2 \frac{\lambda_1^2}{\lambda_2} \\ & + \phi_{2x}^2(0, t) \lambda_2 \gamma + \phi_2^2(0, t) \frac{\lambda_1 e^{-2\mu L}}{4R_2^2} \\ & + \xi^2 \frac{\lambda_1 k_i^2}{2} - \frac{\gamma k_i^2}{\lambda_1} (b\phi_2(0, t) - ak_i \xi(t))^2, \end{aligned}$$

$$Q = \frac{1}{L} \begin{pmatrix} \mu\lambda_1 L & 0 & T_{\phi_1, \xi} & T_{\phi_1, \phi_{20}} & 0 & 0 \\ 0 & \mu q_1 \lambda_2 L & T_{\phi_2, \xi} & T_{\phi_2, \phi_{20}} & 0 & 0 \\ T_{\phi_1, \xi} & T_{\phi_2, \xi} & T_\xi & 0 & 0 & 0 \\ T_{\phi_1, \phi_{20}} & T_{\phi_2, \phi_{20}} & 0 & \frac{\lambda_2 q_1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \lambda_1 q_5 & 0 \\ 0 & 0 & 0 & 0 & 0 & \lambda_2 q_6 \end{pmatrix}$$

with

$$\begin{aligned} T_\xi &= \mu^2 \frac{\lambda_1 c^2}{2}, T_{\phi_1, \phi_{20}} = -\mu L b^2 R_2 e^{\mu L}, \\ T_{\phi_2, \phi_{20}} &= -\mu L \frac{b^2 \lambda_1}{\lambda_2}, T_{\phi_1, \xi} = \mu^2 L b R_2 e^{\mu L} \frac{2ac + \lambda_1}{2}, \\ T_{\phi_2, \xi} &= \mu^2 L b \lambda_1 \frac{2ac + \lambda_2}{2\lambda_2}, c = a + b R_2 e^{\mu L}. \end{aligned}$$

At first, one can easily see that if taking γ small enough, we have $L(t) \geq \phi_1^2(L, t) \frac{\lambda_1 e^{-\mu L}}{2}$. In the following, we prove that by picking μ small enough, matrix Q is symmetric positive definite. From the Sylvester criterion, Q is symmetric positive definite if and only if

$$D_1 = \det \begin{pmatrix} \mu\lambda_1 L & 0 & T_{\phi_1, \xi} \\ 0 & \mu q_1 \lambda_2 L & T_{\phi_2, \xi} \\ T_{\phi_1, \xi} & T_{\phi_2, \xi} & T_\xi \end{pmatrix} > 0,$$

$$D_2 = \det \begin{pmatrix} \mu\lambda_1 L & 0 & T_{\phi_1, \xi} & T_{\phi_1, \phi_{20}} \\ 0 & \mu q_1 \lambda_2 L & T_{\phi_2, \xi} & T_{\phi_2, \phi_{20}} \\ T_{\phi_1, \xi} & T_{\phi_2, \xi} & T_\xi & 0 \\ T_{\phi_1, \phi_{20}} & T_{\phi_2, \phi_{20}} & 0 & \frac{\lambda_2 q_1}{2} \end{pmatrix} > 0.$$

By the direct computing, we have that

$$D_1 = \mu^4 f_1(\mu), D_2 = \mu^4 f_2(\mu)$$

with $\lim_{\mu \rightarrow 0} f_1(\mu) = \frac{1}{2} L^2 \lambda_1^2 \lambda_2 c^2 q_1 > 0$ and $\lim_{\mu \rightarrow 0} f_2(\mu) = \frac{1}{4} L^2 \lambda_1^2 \lambda_2^2 c^2 q_1^2 > 0$.

Taking μ small enough, it yields that the two terms D_1, D_2 are both positive. Consequently, the matrix Q is symmetric positive definite.

Therefore, with the adaptive choice of γ and μ , there exists a positive real number $K > 0$ such that for all t at which the solution is well defined we have:

$$\dot{V}(t) \leq -K \|(\phi_1, \phi_2, \xi)\|_{\mathbb{X}}^2 - \phi_1^2(L, t) \frac{\lambda_1 e^{-\mu L}}{2}.$$

With (15) the former inequality implies that we can find $\beta > 0$ such that (16) holds. This completes the proof of Lemma 1.