Synchronization in abstract mean field models

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Abstract

We give a sufficient conditions for the existence of solutions, both synchronized and periodically locked states, in abstract mean field models.

Keywords: Coupled oscillators, mean field, synchronization, differential equation, periodic orbit.

1 Introduction

This article generalizes the results obtained in [6]. The class of abstract mean field systems that we study is defined by the following system:

$$\dot{x}_i = F(X, x_i) + H_i(X), \quad i = 1, .., N, \quad t \geq t_0, \tag{P}$$

where $N \geq 2$ and $X = (x_1, \ldots, x_N)$ is the state of the system. $F : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ and $H = (H_1, \ldots, H_N) : \mathbb{R}^N \to \mathbb{R}^N$ are a continuous functions. We note Φ^t the flow (if exists) of the system (P). The results do not appear to be trivial for the periodic, non-perturbed system.

1.1 Notations and definitions

In this section, we introduce some notations and definitions. For $q, p \in \mathbb{N}^*$ let G be a function from $\mathbb{R}^q$ to $\mathbb{R}^p$. Put $G = (G_1, \ldots, G_p)$ we consider the quasi-norm on the space of continues functions from $\mathbb{R}^q$ to $\mathbb{R}^p$ defined by the next quantity

$$||G||_B = \max_{1 \leq i \leq p} \sup_{Y \in B} |G_i(Y)|,$$

where

$$B = \{Y = (y_1, \dots, y_q) \in \mathbb{R}^q : \ \max |y_i - y_j| \leq 1\}.$$

This quasi-norm is a norm on the space of continues functions from B to $\mathbb{R}^p$. We note d^iG, $i = 1, 2, \dots$, the i^{th} differential of G. We define

$$||dG||_B = \max_{\substack{1\leq i\leq p\\ 1\leq j\leq q}} ||\partial_j G_i(Y)||_B, \quad ||d^2G||_B = \max_{\substack{1\leq i\leq p\\ 1\leq j,k\leq q}} ||\partial_k\partial_j G_i(Y)||_B.$$

Let $G : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$, $Y = (y_1, \dots, y_N) \in \mathbb{R}^N$ and $z \in \mathbb{R}$, We note

$$\partial_i G(Y, z) := \begin{cases} \dfrac{\partial}{\partial z} G(Y, z) \quad \text{if} \quad i = N + 1, \\ \dfrac{\partial}{\partial y_i} G(Y, z) \quad \text{if} \quad i = 1, \dots, N. \end{cases}$$

A function $G : \mathbb{R}^q \to \mathbb{R}^p$ is called $\mathbb{1}$-*periodic* in the following sense

Definition 1. [$\mathbb{1}$-periodic function] Let $G : \mathbb{R}^q \to \mathbb{R}^p$ be a function and note $\mathbb{1} := (1, \dots, 1) \in \mathbb{R}^q$. The function G is called $\mathbb{1}$-*periodic* if

$$G(Y + \mathbb{1}) = G(Y), \quad \forall Y \in \mathbb{R}^q.$$

Remark that the previous definition do not imply that the function G is periodic relative to each variable. Now we define a positive Φ^t-invariant set,

Definition 2. Suppose that the flow Φ^t of system (P) exists for every $t \geq t_0$. We say that a open set $C \subset \mathbb{R}^N$ is a positive Φ^t-invariant if $\Phi^t(C) \subset C$ for all $t \geq t_0$.

Synchronization and locking may have several meanings or definitions depending on the authors. We choose the following definitions.

Definition 3 (Dynamical oscillator)**.** The oscillator $x_i(t)$ of a solution $X(t) = (x_1(t), .., x_N(t))$ of system (P) is called *dynamical* if

$$\inf_{t\geq t_0} \dot{x}_i(t) > 0.$$

Definition 4 (Synchronisation)**.** We say that the oscillators $\{x_i(t)\}_{i=1}^N$ are synchronized if they are dynamical and if

$$\sup_{t\geq t_0} \sup_{1\leq i,j\leq N} |x_i(t) - x_j(t)| < +\infty.$$

Definition 5 (Periodic locked solution)**.** We say that the oscillators $\{x_i(t)\}_{i=1}^N$ are periodically locked to the frequency $\rho > 0$ if they are synchronized and if there exist a periodic functions $(\Psi_i(t) : \mathbb{R} \to \mathbb{R})_{i=1}^N$ such that

$$x_i(t) = \rho t + \Psi_i(t), \quad \forall i = 1, \dots, N, \ \forall t \geq t_0.$$

1.2 Synchronization Hypothesis (H) and (H_*)

The goal is to prove the existence of the synchronization state of the system (P) when $||H||_B \approx 0$. Consider the following hypotheses:

- (H): F is class C^2 *on* B, $\mathbb{1}$-periodic and $\min_{s\in[0,1]} F(s\mathbb{1}, s) > 0$,
- (H_*): $\int_0^1 \frac{\partial_{n+1}F(s\mathbb{1},s)}{F(s\mathbb{1},s)}ds < 0$.

We call the hypothesis (H_*) the *synchronization hypothesis*. The particularity of the hypothesis (H_*) is the fact : $H \approx 0$ and $x_i \approx x_j (\approx x)$ implies that the system (P) is equivalent to

$$\frac{d}{dt}x_i \approx F(x\mathbb{1}, x) \text{ and } \frac{d}{dt}(x_i - x_j) \approx \partial_{N+1}F(x\mathbb{1}, x)(x_i - x_j).$$

The condition $\min_{s\in[0,1]} F(s\mathbb{1}, s) > 0$ is a sufficient condition to get a dynamical oscillators as defined in definition 3.

1.3 Main Results

The following main result, I, states the existence of a solution and a synchronized state as defined in the Definition. 4

Main Result (I)**.** *Consider the system* (P)*. Suppose that F satisfies the hypotheses (H) and (H_*). Then there exists $r_* > 0$ such that for all $r \in (0, r_*]$ and every C^1 function H on B satisfying $||H||_B < r$ there exists an open set $C_r \subset \mathbb{R}^N$ such that*

1. Existence of solution. *The flow Φ^t of the system* (P) *exists for all initial condition $X \in C_r$ and for all $t \geq t_0$.*
2. Synchronization. *The open set C_r is positive Φ^t-invariant . Further, for every $X \in C_r$ we have*

$$\min_{1\leq i\leq N} \inf_{t\geq t_0} \frac{d}{dt}\Phi_i^t(X) > 0 \text{ and } \sup_{t\geq t_0} |\Phi_i^t(X) - \Phi_j^t(X)| < 2r, \forall 1 \leq i, j \leq N.$$

The next main result II shows the existence of a periodic locked solution as defined in definition 5

Main Result (II)**.** *Under assumptions of the Main result (I), suppose in addition that H is $\mathbb{1}$-periodic. Then there exists $\rho > 0$ and an initial condition $X_* \in C_r$ such that*

$$\Phi_i^t(X_*) = \rho t + \Psi_{i,X_*}(t), \quad \forall i = 1, .., N, \ \forall t \geq t_0,$$

where $\Psi_{i,X_*} : \mathbb{R} \to \mathbb{R}$ are a C^1 and $\frac{1}{\rho}$-periodic functions.

Remark. *Main result I can be generalized to a function* $(t, X) \in \mathbb{R} \times \mathbb{R}^N \mapsto H(t, X)$ *that depends on the variable time* t.

1.4 Remarks and motivation

The results can be applied to models of coupled oscillators such as the Winfree model [10] and the Kuramoto model [9].

Example 6. [Winfree and Kuramoto Models] Winfree [10] proposed a model describing the synchronization of a population of organisms or oscillators that interact simultaneously. The Winfree model is also studied in [4, 5, 7, 8]. The Kuramoto model is a refined version of the Winfree model. The Kuramoto model is applied, for example, in neuroscience to study the synchronization of neurons in the brain [2, 3]. We call the frequency of each oscillator its natural frequency, as if it were isolated from the others.

The explicit Winfree [1] and Kuramoto models are defined by the following equations, respectively.

$$\dot{x}_i = \omega_i + \mathrm{Win}(X, x_i), \quad i = 1 \dots N \quad , t \geq t_0, \tag{W}$$

$$\dot{x}_i = \omega_i + \mathrm{Kur}(X, x_i), \quad i = 1 \dots N, \quad t \geq t_0, \tag{K}$$

where for $(\omega, \kappa) \in \mathbb{R}^2_+$, $\mathrm{Win}(Y, z) = \omega - \kappa \frac{1}{N} \sum_{j=1}^{N} [1 + \cos(y_j)] \sin(z)$ and $\mathrm{Kur}(Y, z) = \omega - \kappa \frac{1}{N} \sum_{j=1}^{N} \sin(y_j - z)$ for all $Y = (y_1, \dots, y_N) \in \mathbb{R}^N$ and $z \in \mathbb{R}$. $X(t) = (x_1(t), \dots, x_N(t))$ is the state of the systems, and $x_i(t)$ is the phase of the i$^{\text{th}}$-oscillator. The parameter $\kappa \geq 0$ is the strong coupling; the vector $(\omega_1 + \omega, \dots, \omega_N + \omega) \in \mathbb{R}^N$ is the vector of the natural frequencies. In order to apply the main result I and II we need only shows that the functions (W) and (K) satisfies the synchronization hypothesis (H_*) and the hypothesis (H) as proved in the following proposition.

Proposition 7. *There exists an open set of parameters* $(\kappa, \omega) \in \mathbb{R}^2_+$, *such that the functions Win and Kur of the systems* (W) *and* (K) *respectively, satisfies both hypotheses* (H) *and* (H^*).

Proof. The function Win is C^2 and $2\pi\mathbb{1}$-periodic. Further

$$\min_{s \in [0, 2\pi]} \mathrm{Win}(s\mathbb{1}, s) > 0 \iff \forall \omega > (1 + \cos(\frac{\pi}{3})) \sin(\frac{\pi}{3}) \kappa, \quad \forall s \in [0, 2\pi].$$

For every $\omega > (1+\cos(\frac{\pi}{3}))\sin(\frac{\pi}{3})\kappa$ we have

$$\begin{aligned}\int_0^{2\pi} \frac{\partial_{N+1}\mathrm{Win}(s\mathbb{1},s)}{\mathrm{Win}(s\mathbb{1},s)}ds &= -\int_0^{2\pi} \frac{\kappa[1+\cos(s)]\cos(s)}{\omega-\kappa(1+\cos(s))\sin(s)}ds \\ &= -\int_0^{2\pi} \frac{\kappa\sin^2(s)}{\omega-\kappa(1+\cos(s))\sin(s)}ds < 0.\end{aligned}$$

Same for the Kuramoto model, we have Kur is $2\pi\mathbb{1}$-periodic, and

$$\min_{s\in[0,2\pi]} \mathrm{Kur}(s\mathbb{1},s) > 0, \quad \forall\omega > 0, \quad \forall s \in [0,2\pi],$$

For every $\omega > 0$ and $\kappa > 0$ we have

$$\int_0^{2\pi} \frac{\partial_{N+1}\mathrm{Kur}(s\mathbb{1},s)}{\mathrm{Kur}(s\mathbb{1},s)}ds = -\int_0^{2\pi} \frac{\kappa}{\omega}ds = -\frac{2\pi\kappa}{\omega} < 0.$$

□

2 Dispersion curve

The strategy to prove the main results is to utilize the comparison theorem for differential equations. We assume a priori that the distance between the oscillators is small and derive a differential equation estimation to conclude that the distance between oscillators remains uniformly bounded over time. We refer to the upper-solution as the *dispersion curve*. We present the following lemma:

Lemma 8. *Let $\eta = (\eta_1,\eta_2,\eta_3) \in \mathbb{R}^3_+/\{(0,0,0)\}$. Let $P_1(a,b) = \eta_1 a + \eta_2 b^2$ a polynomial defined for all $(a,b) \in \mathbb{R}\times\mathbb{R}$ and let $\Lambda : \mathbb{R} \to \mathbb{R}$ a C^1 and 1-periodic function satisfying*

$$\int_0^1 \Lambda(s)ds < 0.$$

Then for all $(a,b) \in \mathbb{R}^*_+ \times (0,\eta_3)$ the following differential equation

$$\frac{d}{ds}z(s) = \frac{P_1(a,b)}{\eta_3 - b} + \Lambda(s)z(s), \tag{1}$$

admits a positive solution C^1 and 1-periodic solution that we note $\Delta_{a,b}(s)$. Further, there exists $D_{\eta,\Delta} \in (0,\eta_3)$ such that for all $D \in (0, D_{\eta,\Delta}]$ there exists $r > 0$ such that the solution $\Delta_r := \Delta_{r,D}$ satisfies

$$\max_{s\in[0,1]} \Delta_r(s) \le D.$$

Proof. Remark that for all $(a,b) \in \mathbb{R}_+^* \times (0,\eta_3)$, the differential equation (1) admit a positive C^2 and 1-periodic solution $\Delta_{a,b}(s)$ of the form

$$\Delta_{a,b}(s) = \frac{P_1(a,b)}{\eta_3 - b} \frac{\int_s^{1+s} \exp\Big(\int_t^{1+s} \Lambda(v)dv\Big)dt}{1 - \exp\Big(\int_0^1 \Lambda(v)dv\Big)}$$

Put

$$\lambda_1 = -\int_0^1 \Lambda(s)ds \quad \text{and} \quad \lambda_2 = \max_{0\le s,t\le 1} \int_t^{1+s} \Lambda(v)dv.$$

$$\max_{s\in[0,1]} \Delta_{a,b}(s) \le \frac{P_1(a,b)}{\eta_3 - b} \frac{\exp(\lambda_2)}{1-\exp(-\lambda_1)}.$$

To get $\max_{s\in[0,1]} \Delta_r(s) \le D$ it sufficient to choose r and D such that

$$\frac{P_1(r,D)}{\eta_3 - D} \frac{\exp(\lambda_2)}{1-\exp(-\lambda_1)} = D, \tag{2}$$

which is satisfied for all $D \in (0, D_{\eta,\Lambda}]$ such that

$$D_{\eta,\Lambda} = \frac{\eta_3}{2} \frac{1-\exp(-\lambda_1)}{1-\exp(-\lambda_1) + \eta_2 \exp(\lambda_2)}.$$

where $r > 0$ is given by the following formula

$$r = \frac{D}{\eta_1}\Big[\eta_3 \frac{1-\exp(-\lambda_1)}{\exp(\lambda_2)} - [\frac{1-\exp(-\lambda_1)}{\exp(\lambda_2)} + \eta_2]D\Big].$$

□

Definition 9. Let $D \in (0, D_{\eta,\Lambda}]$. We call *the dispersion curve associated to* D the solution

$$\Delta_r := \Delta_{r,D}(s),$$

of the differential equation (1) where r is defined by

$$r = \frac{D}{\eta_1}\Big[\eta_3 \frac{1-\exp(-\lambda_1)}{\exp(\lambda_2)} - [\frac{1-\exp(-\lambda_1)}{\exp(\lambda_2)} + \eta_2]D\Big], \tag{3}$$

and where

$$\lambda_1 = -\int_0^1 \Lambda(s)ds \quad \text{et} \quad \lambda_2 = \max_{0\le s,t\le 1} \int_t^{1+s} \Lambda(v)dv.$$

Definition 10. Let $D \in (0, D_{\eta,\Lambda}]$. We call *the synchronization open set associated to* D and we note C_r the open set on $\mathbb{R}^N$ defined by

$$C_r := \Big\{ X = (x_i)_{i=1}^N \in \mathbb{R}^N \,:\, \exists \nu_X \in \mathbb{R}, \quad \max_i |x_i - \nu_X| < \Delta_r(\nu_X) \Big\}, \quad (4)$$

where Δ_r is the dispersion curve associated to D.

Remark 11. Remark that

$$D < D_{\eta,\Lambda} < \frac{\eta_3}{2\eta_2} \exp(-\lambda_2) \quad \text{and} \quad r < D\frac{\eta_3}{\eta_1} \exp(-\lambda_2).$$

3 Reduction of the system (P)

The goal of this Section is to prove that the perturbed system (P) can be studied by using a scalar periodic differential equation such as equation (1) of lemma 8. Define the following new system

Definition 12. Let $X \in \mathbb{R}^N$ and let $\mu_0 \in \mathbb{R}$, we call the (NPS) *system associated to* $\Phi^t(X)$ the following system

$$\dot{\mu}_X = F(\Phi^t(X), \mu_X), \quad t \in I_X, \qquad \text{(NPS)}$$

where $I_X = [t_0, T_X)$ is the maximal interval of the solution $X(t) := \Phi^t(X)$ of the system (P) of initial condition $\phi^{t_0}(X) = X$. We say that $\mu_X(t)$ is the solution of the system (NPS) associated to $\Phi^t(X)$ of initial condition $\mu_X(t_0) \in \mathbb{R}$.

We note

$$L := ||F||_B + ||dF||_B + ||d^2F||_B, \quad \text{and} \quad \alpha := \min_{s\in[0,1]} F(s\mathbb{1}, s). \quad (5)$$

Let $X \in \mathbb{R}^N$ and let $\mu_X(t)$ be the solution of the system (NPS) associated to $\Phi^t(X)$ of initial condition $\mu_0 \in \mathbb{R}$. We also note $X := \Phi^t(X)$ and $\mu_X := \mu_X(t)$ without loss of generality. We consider the following quantities

$$\begin{aligned} &\delta_{i,1}(X) := x_i - \mu_X, \quad \delta_{i,2}(X) := \mu_X - x_i, \\ \text{and} \quad &\delta(X) := \max_{1\le i\le N} |\delta_{i,1}(X)| = \max_{1\le i\le N} |\delta_{i,2}(X)|. \end{aligned}$$

We have the next lemma

Proposition 13. *We consider the system* (P). *Suppose that the function F satisfies the hypothesis (H) and suppose that $\Phi^t(X)$ is defined for all $t \in [t_1, t_2]$. Let $D \in (0, \frac{\alpha}{L})$ and suppose that $\delta(X) < D$ for all $t \in [t_1, t_2]$, then*

$$\dot{\mu}_X > -LD + \alpha > 0, \quad \forall t \in [t_1, t_2].$$

In particular, $t \to \mu_X(t)$ is a diffeomorphism from $[t_1, t_2]$ to $[\mu_X(t_1), \mu_X(t_2)]$.

Proof. The strategy is to use the Mean value theorem . Since $\delta(X) < D$ we get, $|F(X, \mu_X) - F(\mu_X \mathbb{1}, \mu_X)| \leq ||dF||_B D < LD$. Hence

$$\dot{\mu}_X = F(X, \mu_X) = [F(X, \mu_X) - F(\mu_X \mathbb{1}, \mu_X)] + F(\mu_X \mathbb{1}, \mu_X) > -LD + \alpha.$$

Thanks to hypothesis $0 < D < \frac{\alpha}{L}$ to get $\dot{\mu}_X(t) > -LD + \alpha > 0$ for all $t \in [t_1, t_2]$. □

Proposition 14. *We consider the system* (P). *Suuppose that F satisfies the hypothesis (H) and suppose that $\Phi^t(X)$ is defined for all $t \in [t_1, t_2]$. Let $D \in (0, \frac{\alpha}{L})$ and $r > 0$. Suppose that*

$$||H||_B < r, \quad \text{and} \quad \delta(X) < D, \quad \forall t \in [t_1, t_2].$$

Then for all $1 \leq i \leq N$, $k \in \{1, 2\}$ and $s \in [\mu_X(t_1), \mu_X(t_2)]$ we have

$$\frac{d}{ds}\delta^*_{i,k}(s) < \frac{1}{\alpha}\frac{\alpha r + LD^2(L + 2\alpha)}{\alpha - LD} + \frac{\partial F_{N+1}(s\mathbb{1}, s)}{F(s\mathbb{1}, s)}\delta^*_{i,k}(s), \tag{6}$$

where
$\delta^*_{i,k}(s) := \delta_{i,k}(X(\mu_X^{-1}(s)))$ *and* $X(\mu_X^{-1}(s)) = (x_1(\mu_X^{-1}(s)), \ldots, x_N(\mu_X^{-1}(s)))$.

Proof. The strategy is to use several times the Taylor formula. Let $D \in (0, \frac{\alpha}{L})$ and suppose that $\delta(X) < D$ for all $t \in [t_1, t_2]$. Use the Taylor formula, there exists $c_i \in [x_i, \mu_X]$ such that for all $1 \leq i \leq N$

$$\begin{aligned} F(X, x_i) - F(X, \mu_X) &= \partial_{N+1}F(X, \mu_X)\delta_{i,1} + \frac{1}{2}\partial_{N+1}[\partial_{N+1}F(X, c_i)]\delta^2_{i,1} \\ &< \partial_{N+1}F(X, \mu_X)\delta_{i,1} + \frac{1}{2}||d\partial_{N+1}F||_B D^2 \\ &< \partial_{N+1}F(X, \mu_X)\delta_{i,1} + LD^2. \end{aligned}$$

For $k = 2$ we also obtain

$$\begin{aligned} F(X, \mu_X) - F(X, x_i) &= -\partial_{N+1}F(X, \mu_X)\delta_{i,1} - \frac{1}{2}\partial_{N+1}[\partial_{N+1}F(X, c_i)]\delta^2_{i,1} \\ &= \partial_{N+1}F(X, \mu_X)\delta_{i,2} - \frac{1}{2}\partial_{N+1}[\partial_{N+1}F(X, c_i)]\delta^2_{i,2} \\ &< \partial_{N+1}F(X, \mu_X)\delta_{i,2} + LD^2. \end{aligned}$$

We have $||H||_B < r$, use equations (P) and (NPS) we obtain for all $1 \leq i \leq N$ and $k \in \{1, 2\}$

$$\frac{d}{dt}\delta_{i,k} = H_i(X) + [F(X, x_i) - F(X, \mu_X)] < r + LD^2 + \partial_{N+1}F(X, \mu_X)\delta_{i,k}. \tag{7}$$

Use again the Taylor formula to get

$$\begin{aligned}\partial_{N+1}F(X, \mu_X)\delta_{i,k} &= [\partial_{N+1}F(X, \mu_X) - \partial_{N+1}F(\mu_X\mathbb{1}, \mu_X) + \partial_{N+1}F(\mu_X\mathbb{1}, \mu_X)]\delta_{i,k} \\ &< ||d\partial_{N+1}F||_B D|\delta_{i,j}| + \partial_{N+1}F(\mu_X\mathbb{1}, \mu_X)\delta_{i,k} \\ &< LD^2 + \partial_{N+1}F(\mu_X\mathbb{1}, \mu_X)\delta_{i,k}.\end{aligned}$$

Equation (7) implies that for all $1 \leq i \leq N$ and $k \in \{1, 2\}$

$$\frac{d}{dt}\delta_{i,k} < r + 2LD^2 + \partial_{N+1}F(\mu_X\mathbb{1}, \mu_X)\delta_{i,k}.$$

Thanks to proposition 13, $\dot{\mu}_X > \alpha - LD$. We consider the change of variable : $t \to s := \mu_X(t)$ for $t \in [t_1, t_2]$. Put $\delta^*_{i,k}(s) := \delta_{i,k}(X(\mu_X^{-1}(s)))$ and $X(\mu_X^{-1}(s)) = (x_1(\mu_X^{-1}(s)), \ldots, x_N(\mu_X^{-1}(s)))$. We deduce that for all $s \in [\mu_X(t_1), \mu_X(t_2)]$

$$\begin{aligned}\frac{d}{dt}\delta_{i,k}(X) &= \frac{d}{ds}\delta^*_{i,k}(s)\frac{d}{dt}\mu_X(t) < r + 2LD^2 + \partial_{N+1}F(s\mathbb{1}, s)\delta^*_{i,k}(s) \\ \frac{d}{ds}\delta^*_{i,k}(s) &= \frac{r + 2LD^2}{\dot{\mu}_X} + \frac{\partial_{N+1}F(s\mathbb{1}, s)}{\dot{\mu}_X}\delta^*_{i,k}(s) < \frac{r + 2LD^2}{\alpha - LD} + \frac{\partial_{N+1}F(s\mathbb{1}, s)}{\dot{\mu}_X}\delta^*_{i,k}(s) \\ &= \frac{r + 2LD^2}{\alpha - LD} + \frac{\partial_{N+1}F(s\mathbb{1}, s)}{F(s\mathbb{1}, s)}\frac{F(s\mathbb{1}, s)}{\dot{\mu}_X}\delta^*_{i,k}(s).\end{aligned}$$

Use the Mean value theorem and the change of variable $t \to s := \mu_X(t)$ we get

$$|F(\mu_X\mathbb{1}, \mu_X) - \dot{\mu}_X| = |F(\mu_X\mathbb{1}, \mu_X) - F(X, \mu_X)| < ||dF||_B D < LD,$$

which is equivalent to

$$\frac{F(s\mathbb{1}, s)}{\dot{\mu}_X} = 1 + \theta(s), \quad |\theta(s)| < \frac{LD}{\alpha - LD}, \quad \forall s \in [\mu_X(t_1), \mu_X(t_2)].$$

Finlay, since $|\frac{\partial_{N+1}F(s\mathbb{1}, s)}{F(s\mathbb{1}, s)}| < \frac{L}{\alpha}$ and since $|\delta_{i,k}(t)| \leq \delta(X) < D$ for all

$t \in [t_1, t_2]$ we obtain for all $s \in [\mu_X(t_1), \mu_X(t_2)]$

$$\begin{aligned}\frac{d}{ds}\delta^*_{i,k}(s) &< \frac{r + 2LD^2}{\alpha - LD} + \frac{\partial_{N+1}F(s\mathbb{1}, s)}{F(s\mathbb{1}, s)}[1 + \theta(s)]\delta^*_{i,k}(s) \\ &< \frac{r + 2LD^2}{\alpha - LD} + \frac{L^2}{\alpha}\frac{D^2}{\alpha - LD} + \frac{\partial_{N+1}F(s\mathbb{1}, s)}{F(s\mathbb{1}, s)}\delta^*_{i,k}(s) \\ &= \frac{1}{\alpha}\frac{\alpha(r + 2LD^2) + L^2D^2}{\alpha - LD} + \frac{\partial_{N+1}F(s\mathbb{1}, s)}{F(s\mathbb{1}, s)}\delta^*_{i,k}(s).\end{aligned}$$

□

We have the following proposition

Proposition 15. *Let F be a function satisfying hypotheses (H) and (H_*). Then there exists $D_* \in (0, 1)$ such that for all $D \in (0, D_*]$, there exists $r > 0$ and an open set C_r (as in definition 10), such that for any function H satisfying $||H||_B < r$ we have*

$$\forall X \in C_r \ : \ \Phi^t(X) \in C_r, \quad \forall t \in I_X.$$

Proof. Use equation (5) and hypothesis (H)

$$\max\{\int_{t_1}^{t_2} \frac{\partial_{N+1}F(s\mathbb{1}, s)}{F(s\mathbb{1}, s)}ds : \quad 0 \le t_1 \le t_2 \le 1\} \le \frac{L}{\alpha}.$$

Let $D_{\eta,\Lambda}$ the constant defined by lemma 8 such that η et Λ are defined by

$$\eta = (\frac{1}{L}, 2 + \frac{L}{\alpha}, \frac{\alpha}{L}), \quad \text{and} \quad \Lambda(s) = \frac{\partial_{N+1}F(s\mathbb{1}, s)}{F(s\mathbb{1}, s)}.$$

Put $D_* := D_{\eta,\Lambda}$. Let $D \in (0, D_*]$ and the dispersion function Δ_r associated to D (See definition 9). The dispersion curve Δ_r is solution of the periodic scalar differential equation

$$\frac{d}{ds}\Delta_r(s) = \frac{1}{\alpha}\frac{\alpha r + LD^2(L + 2\alpha)}{\alpha - LD} + \frac{F_{N+1}(s\mathbb{1}, s)}{F(s\mathbb{1}, s)}\Delta_r(s),$$

and satisfies the folioing estimation

$$\max_{s\in[0,1]} \Delta_r(s) \le D.$$

Let C_r be the synchronization open set associated to D, as defined in definition 10.

For any function H satisfying $||H||_B < r$, where r is given by formula (3), let $X(t) = (x_1(t), \dots, x_N(t)) := \Phi^t(X)$ be the solution of the system (P) of initial condition $X = (x_1, \dots, x_N) \in C_r$. There exists $\nu_X \in \mathbb{R}$ such that $\max_{1\leq i\leq N} |x_i - \nu_X| < \Delta_r(\nu_X) \leq D$. Let $\mu_X(t)$ be the solution of the system (NPS) associated to $X(t)$ of initial condition $\mu_X(t_0) = \nu_X$, then $\delta(X) < \Delta_r(\mu_X(t_0))$. Let

$$T^* := \sup\{t \in I_X : \forall\ t_0 < s < t,\ \max_i |x_i(s) - \mu_X(s)| < \Delta_r(\mu_X(s))\}.$$

By continuity we have $t_0 \neq T^*$. The proposition is proved if we shows that $T^* = \sup\{t \in I_X\}$. By contradiction, suppose that $T^* \in I_X$. Using the change of variable $s = \mu_X(t)$ the proposition 14 implies that for all $s \in [\nu_X, \mu_X^* := \mu_X(T^*)]$

$$\frac{d}{ds}\delta^*_{i,k}(s) < \frac{1}{\alpha}\frac{\alpha r + LD^2(L+2\alpha)}{\alpha - LD} + \frac{F_{N+1}(s\mathbb{1}, s)}{F(s\mathbb{1}, s)}\delta^*_{i,k}(s), \quad \forall s \in [\mu_X, \mu_X^*].$$

Hence there exists $1 \leq i_0 \leq N$ and $k \in \{1, 2\}$ such that $|\delta^*_{i_0,k_0}(\mu_X^*)| = \Delta_r(\mu_X^*)$. Suppose that $\delta^*_{i_0,k_0}(\mu_X^*) = \Delta_r(\mu_X^*)$ without loss of generality. We get

$$\begin{aligned}\frac{d}{ds}\delta^*_{i_0,k_0}(\mu_X^*) &< \frac{1}{\alpha}\frac{\alpha r + LD^2(L+2\alpha)}{\alpha - LD} + \frac{F_{N+1}(\mu_X^*\mathbb{1}, \mu_X^*)}{F(\mu_X^*\mathbb{1}, \mu_X^*)}\delta^*_{i_0,k_0}(\mu_X^*) \\ &= \frac{1}{\alpha}\frac{\alpha r + LD^2(L+2\alpha)}{\alpha - LD} + \frac{F_{N+1}(\mu_X^*\mathbb{1}, \mu_X^*)}{F(\mu_X^*\mathbb{1}, \mu_X^*)}\Delta_r(\mu_X^*) = \frac{d}{ds}\Delta_r(\mu_X^*).\end{aligned}$$

There exists $s < \mu_X^*$ close enough to μ_X^* such that $\delta^*_{i_0,k_0}(s) > \Delta_r(s)$ or in other words there exists $t < T^*$ close enough to T^* such that $\delta_{i_0,k_0}(t) > \Delta_r(\mu_X(t))$. We have obtained a contradiction. □

4 Proof of main result I : Existence of solution and the synchronized state

Theorem 16. *Let F be a function satisfying the hypotheses (H) and (H_*). Then there exists $D_* \in (0, 1)$ such that for all $r \in (0, D_*]$, there exists a synchronization open set C_r (as defined in the definition 10), such that for any function H satisfying $||H||_B < r$, and for all $X \in C_r$ we have $I_X = [t_0, +\infty[$. Further C_r is positive Φ^t-invariant and*

$$\forall X \in C_r,\ \exists \nu_X \in \mathbb{R}\ :\ |\Phi_i^t(X) - \mu_X(t)| < r,\ \forall i = 1, .., N,\ \forall t \geq t_0,$$

where $\mu_X(t)$ is the solution of the system (NPS) *associated to $\Phi^t(X)$ of initial condition $\mu_X(t_0) = \nu_X$.*

Proof. Thanks to proposition 15, it is sufficient to prove that $I_X = [t_0, +\infty[$. By contradiction suppose that there exists $t_0 < t_X < +\infty$ such that the solution $X(t)$ is defined only on $I_X = [t_0, t_X[$. Then $\lim_{t\to t_X} ||\Phi^t(t)|| = +\infty$. Proposition 15 implies that

$$|\Phi_i^t(X) - \Phi_j^t(X)| < D, \forall 1 \leq i, j \leq N, \ \forall t_X > t \geq t_0,$$

For all $i = 1, .., N$,

$$\alpha - LD - r < \frac{d}{dt}x_i < \max_{s\in[0,1]} F(s\mathbb{1}, s) + LD + r, \ \forall t_X > t \geq t_0.$$

Hence $||\Phi^t(X)|| < +\infty$ for all $t \in [t_0, t_X]$, in particular $\lim_{t\to t_X} ||\Phi^t(t)|| < +\infty$. We have obtained a contradiction. □

5 Proof of main result II : Periodic locked solution

We use the fixed point Theorem to prove the existence of periodic locked state as follow

Lemma 17. *Let F be a function satisfying the hypotheses $(H)$and (H_*). For any $\mathbb{1}$-periodic C^1 function H satisfying $||H||_B < r$ let C_r the synchronization Φ^t-invariant open set given by theorem 16. Let the set Σ defined by*

$$\Sigma = \{X \in \mathbb{R}^N, \ \max_i |x_i| < \Delta_r(0)\} \subset C_r.$$

Then there exists a C^1 function $P : \Sigma \to \Sigma$ (the Poincaré map) and a C^1 function $\theta : \Sigma \to \mathbb{R}^+$ (the return time map) such that

$$\Phi^{t_0+\theta(X)}(X) = P(X) + \mathbb{1}, \quad \mathbb{1} = (1, \cdots, 1) \in \mathbb{R}^N,$$
$$\frac{1}{L} < \theta(X) < \frac{2}{\alpha}.$$

Proof. Let $X \in \Sigma \subset C_r$. Let $\mu_X(t)$ be the solution of the system (NPS) associated to $X(t)$ of initial condition $\mu_X(t_0) = 0$. Let τ_X be the inverse function of the function $\mu_X := \mu_X(t)$. By the proposition 13 and the theorem 16 we obtain

$$\alpha - LD < \dot{\mu}_X(t) < L.$$

Remark (11) in the Section 2 shows that for all $D < \frac{\alpha}{2L}$ we have

$$\frac{\alpha}{2} < \dot{\mu}_X(t) < L.$$

Let $\theta(X) := \tau_X(1) - t_0$. Then $\int_{t_0}^{\tau_X(1)} \dot{\mu}_X(t)dt = 1$ which implies the second estimation of lemma. Recall that $\max_{1\leq i\leq N} |\Phi_i^t(X) - \mu_X(t)| < \Delta(\mu_X(t)) < D$ for all $t \geq t_0$. Put $P(X) := \Phi^{t_0+\theta(X)}(X) - \mathbb{1}$, $P = (P_1, \ldots, P_N)$. Since $\mu_X(t_0 + \theta(X)) = 1$ then

$$\max_{1\leq i\leq N} |P_i(X)| = \max_{1\leq i\leq N} |\Phi_i^{t_0+\theta(X)}(X) - 1| < \Delta(1) = \Delta(0).$$

We have shown that P is a map from Σ into itself. □

Corollary 18. *The Poincaré map P defined in lemma 17 admits a fixed point $X_* \in \Sigma$.*

Proof. $\bar{\Sigma}$ is compact and convex; $P : \bar{\Sigma} \to \bar{\Sigma}$ is continuous. By Brouwer fixed point theorem, P admits a fixed point in $X_* \in \bar{\Sigma}$. We claim that $X_* \notin \partial\bar{\Sigma}$. Suppose by contradiction $X_* \in \partial\bar{\Sigma}$. There exists $1 \leq i_0 \leq N$, such that $|x_{i_0,*}| = \Delta_r(0)$; Put

$$X_*(t) = \Phi^t(X_*), \quad X_*(t) = (x_{1,*}(t), \ldots, x_{N,*}(t)).$$

Let $\mu_{X_*}(t)$ the solution of the system (NPS) associated to the solution $X_*(t)$ of initial condition $\mu_{X_*}(t_0) = 0$. We note

$$\delta_{i,1}(X_*(t)) = x_{i,*}(t) - \mu_{X_*}(t), \quad \text{and} \quad \delta_{i,2}(X_*(t)) = \mu_{X_*}(t) - x_{i,*}(t).$$

There exists $1 \leq i \leq N$, $k \in \{1, 2\}$, and $t' > t_0$ close to t_0 such that $\delta_{i_0,k}(X_*(t)) < \Delta_r(\mu_{X_*}(t))$ for all $t' > t > t_0$. By repeating this argument for every $1 \leq i_1 \leq N$ and $k \in \{1, 2\}$ satisfying the equality $\delta_{i_1,k}(X_*(t)) = \Delta_r(\mu_{X_*}(t))$ we obtain for some $t^* > t_0$, $\delta(X_*(t^*)) < \Delta_r(\mu_{X_*}(t^*))$. But 16 implies that $\max_{1\leq i\leq N} |x_{i,*}(t) - \mu_{X_*}(t)| < \Delta_r(\mu_{X_*}(t))$ for all $t > t^*$. We have obtained a contradiction with the fact that

$$\begin{aligned}\max_{1\leq i\leq N} |\Phi_i^{t_0+\theta(X_*)}(X_*) - \mu_{X_*}(t_0 + \theta(X_*))| &= \max_{1\leq i\leq N} |x_{i,*} + 1 - 1| \\ &= \Delta_r(0) = \Delta_r(\mu_{X_*}(t_0 + \theta(X_*))),\end{aligned}$$

knowing that $\mu_{X_*}(t_0 + \theta(X_*)) = 1$ and $\Phi^{t_0+\theta(X_*)}(X_*) = X_* + \mathbb{1}$. □

The main result II is a consequence of the previous corollary.

Proof of the main result II. Corollary 18 implies the existence of a fixed point $X_* \in C_r$ and return time $\theta_* > 0$ such that

$$\Phi^{t_0+\theta_*}(X_*) = X_* + \mathbb{1}.$$

Thanks to periodicity and uniqueness of solution of differential equation, we obtain

$$\Phi^{\theta_*+t}(X_*) = \Phi^t(X_*) + \mathbb{1}, \quad \forall t \geq t_0.$$

Let $\Psi : \mathbb{R}^N \to \mathbb{R}^N$ the function defined by

$$\Psi(s) := \Phi^s(X_*) - \frac{s}{\theta_*}\mathbb{1} = (\Psi_1(s), \cdots, \Psi_N(s)), \quad \forall s \geq t_0.$$

The theorem is proved if we show that Ψ_i are θ_*-periodic. We have

$$\Psi(s+\theta_*) = \Phi^{s+\theta_*}(X_*) - \frac{s+\theta_*}{\theta_*}\mathbb{1} = \Phi^s(X_*) + \mathbb{1} - \frac{s+\theta_*}{\theta_*}\mathbb{1} = \Psi(s).$$

Lemma 17 implies that θ_* is uniformly bounded. □

6 Conclusion

We have extended the results obtained in [6] to a class of abstract mean field models. We have demonstrated the existence of solutions and synchronized solutions under small perturbations. Additionally, we have established the existence of periodic locked states for periodic systems.

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