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Hochschild cohomology algebra of radical square zero algebras

Claude Cibils

Abstract

We compute the dimension at each degree of the Hochschild cohomology of a radical square zero finite dimensional algebra, in terms of the combinatorics of its quiver. The multiplicative structure is obtained as well, those algebras are finitely generated only for quivers without oriented cycles and for crowns.

1991 Mathematics Subject Classification : 16E40

1 Introduction

The Hochschild cohomology algebra of a finite dimensional algebra is not easy to compute in general. It gives inside to the homological global dimension (see [13]) and is closely related to the deformations of the algebra *via* its value in degree 2 and suitable obstructions (see [12]). In [4, 5, 8, 7, 9] a classification of rigid families is performed and cohomology groups are described. Recently Holm obtained in [14] an explicit computation for finite abelian group algebras; then A. Solotar and the author showed that for these algebras the Hochschild cohomology algebra is isomorphic to the tensor product of the group algebra and the usual cohomology algebra of the group ([10]). Moreover a close relation seems to exist between the structure of the Hochschild cohomology algebra and the monoidal structure of the category of Hopf bimodules over a Hopf algebra, see [3].

In view of these results, it became interesting to write observations obtained some years ago: I lectured at Séminaire P. Dubreuil and M.-P. Mallavin in Paris (1990) on the additive structure of the Hochschild cohomology algebra of a radical square zero algebra; in Salt Lake City meeting on Homological methods in representation theory (1991) I lectured on the multiplicative aspects.

A radical square zero algebra is a finite dimensional algebra over a field k such that the square of its Jacobson radical is already zero. Since Hochschild cohomology is Morita invariant, we consider the corresponding basic algebra. More precisely, let Q be a connected and finite quiver and let $(kQ)_2$ be the quotient of the path algebra of Q by the two-sided ideal generated by paths of length 2. The case when the quiver is a crown is exceptional; out of this case, the dimension of each Hochschild cohomology vector space $H^n((kQ)_2, (kQ)_2)$ is the difference between the number of couples composed by a path of length n and an arrow sharing the same starting and ending vertices and the number of oriented cycles of length $n - 1$. This computation can be performed using a reduced projective resolution presented in [4, 5] of an algebra having a maximal separable subalgebra. Then we are able to produce suitable models for the cochains vector spaces and coboundaries maps.

The cup-product (or Yoneda product) can also be described. We obtain that the Hochschild cohomology algebra is not finitely generated in general, except of course if Q has no oriented cycles – the Hochschild cohomology vanish in this case at high enough degrees –, and in the exceptional crown case. This result has to be compared with Venkov's Theorem ([16] and also [2]): for finite group algebras the cohomology algebra is always finitely generated.

We will also relate the results with previous work on the deformation theory of square radical zero algebras from [8]; cyclic and Hochschild homology of this class of algebras has been obtained in [6].

The work presented in this paper has the advantage of being mostly self-contained: once the correct framework is established, there is no difficulty for carrying on the observations. Partial aspects of the present computations have been written in a manuscript by a former Diploma student M. Blater at Geneva in 1994. E. Marcos told me at the Conference on representations of algebras Geiranger (1996) that he has made computations for crowns, obtaining that the Hochschild cohomology algebra is finitely generated in that case. Notice also recent results on Hochschild homology of truncated and quadratic monomial algebras obtained by Sköldberg in [15] using the minimal projective resolution of Annick and Green ([1]).

2 Resolution and models

Let Λ be a finite dimensional k -algebra, where k is a field, affording a maximal semisimple algebra E which is separable; the Wedderburn-Malcev Theorem (see for instance [11]) holds and we have a decomposition $\Lambda = E \oplus r$ where r is the Jacobson radical of Λ . Then the Hochschild cohomology of a Λ -bimodule M is the cohomology of the following complex of cochains (see [4, 5]):

$$0 \rightarrow M^E \rightarrow \text{Hom}_{E-E}(r, M) \rightarrow \cdots \rightarrow \text{Hom}_{E-E}\left(r^{\otimes n}, M\right) \rightarrow \cdots$$

where $M^E = \{m \in M \mid em = me \text{ for all } e \text{ in } M\}$ and $\text{Hom}_{E-E}\left(r^{\otimes n}, M\right)$ is the vector space of E -bimodule morphisms from the tensor product over E of n copies of r to the E -bimodule M obtained by restricting the Λ actions. The coboundaries δ are given by

$$(\delta m)(x) = xm - mx \text{ for } m \in M^E \text{ and } x \in r,$$

$$\delta f(x_1, \dots, x_{n+1}) = x_1 f(x_2, \dots, x_{n+1}) +$$

$$\sum (-1)^i f(x_1, \dots, x_i x_{i+1}, \dots, x_{n+1}) + (-1)^{n+1} f(x_1, \dots, x_n) x_{n+1}$$

for $f \in \text{Hom}_{E-E}\left(r^{\otimes n}, M\right)$ and where we have replaced the tensor product symbols by commas.

Of course, the interesting feature of this cochain complex in case of a square radical zero algebra is that the middle-sum terms of the coboundary vanish. We have

$$\delta f(x_1, \dots, x_{n+1}) = x_1 f(x_2, \dots, x_{n+1}) + (-1)^{n+1} f(x_1, \dots, x_n) x_{n+1}.$$

Let now Q be a quiver, that is a finite oriented graph given by two sets, the set of vertices Q_0 and the set of arrows Q_1 , provided with two maps s and $t : Q_1 \rightarrow Q_0$ corresponding to the source and the terminus vertices of each arrow. A path of length n is a sequence of arrows $\gamma = a_n \cdots a_1$ such that $t(a_i) = s(a_{i+1})$ for each possible i . We put $s(\gamma) = s(a_1)$ and $t(\gamma) = t(a_n)$. We agree that a vertex e is a path of length zero with source and terminus vertices e itself.

The quotient of the path algebra of Q by the two-sided ideal generated by paths of length 2 can be described directly as follows: let kQ_0 be the

semisimple commutative algebra which has Q_0 as a complete set of primitive orthogonal idempotents. Let also kQ_1 be the vector space which has Q_1 as a basis, equipped with its natural bimodule structure:

$$ea = \delta_{e,t(a)}a \text{ and } ae = \delta_{e,s(a)}a$$

where e is a vertex, a is an arrow and $\delta_{x,y}$ the Kronecker symbol having value 1 if $x = y$ and 0 otherwise.

We denote $(kQ)_2$ the vector space $kQ_0 \oplus kQ_1$ with the algebra structure given by

$$(e, x)(e', x') = (ee', ex' + xe').$$

This finite dimensional algebra has square zero Jacobson radical $r = kQ_1$, and $E = kQ_0$ is a maximal semisimple subalgebra already separable. Actually any finite dimensional algebra having radical square zero over an algebraically closed field is Morita equivalent to an algebra $(kQ)_2$, where Q has vertices given by the set of isomorphism classes of simple modules, and the number of arrows between two vertices is the dimension of Ext^1 between the corresponding simple modules.

We consider now more in detail the cochain space $\text{Hom}_{E-E}(r^{\otimes n}, \Lambda)$, where the bimodule M is given by the algebra itself $\Lambda = (kQ)_2$, where Q is an arbitrary quiver and k any field. First we have the following immediate result:

Lemma 2.1 *Let $r = kQ_1$ be the Jacobson radical of $(kQ)_2$ and let $E = kQ_0$. Then $r^{\otimes n}$ has a basis given by Q_n , the set of paths of length n .*

We introduce now the following notation: let X and Y be sets of paths. The set $X//Y$ of parallel paths is the set of couples (γ, γ') from $X \times Y$ such that $s(\gamma) = s(\gamma')$ and $t(\gamma) = t(\gamma')$. For example $Q_n//Q_0$ is the set of oriented cycles of length n . In the following, we will denote kB the vector space having a given set B as a basis.

Lemma 2.2 *For the algebra $\Lambda = (kQ)_2$ the vector space $\text{Hom}_{E-E}(r^{\otimes n}, \Lambda)$ is isomorphic to $k(Q_n//Q_0) \oplus k(Q_n//Q_1)$.*

Proof: Since $\Lambda = E \oplus r$ is a decomposition of Λ as a direct sum of E -bimodules, we have

$$\text{Hom}_{E-E}(r^{\otimes n}, \Lambda) = \text{Hom}_{E-E}(r^{\otimes n}, E) \oplus \text{Hom}_{E-E}(r^{\otimes n}, r).$$

The kQ_0 -bimodule decomposition of $r^{\otimes n} = kQ_n$ is among Q_n , more precisely $k\gamma$ is a one-dimensional sub-bimodule of $r^{\otimes n}$ for each $\gamma \in Q_n$. Moreover the action of the vertices are zero on γ except for $t(\gamma)$ on the left and $s(\gamma)$ on the right, this vertices acts as the identity: we say that $k\gamma$ is of type $(t(\gamma), s(\gamma))$. The kQ_0 -bimodule decomposition of kQ_0 is among Q_0 , each vertex e provides a direct summand of type (e, e) . Now we have that $\text{Hom}_{E-E}(U, V) = 0$ if U and V are simple E-bimodules not of the same type (this is Schur's Lemma which is immediate in this context). Finally we infer the following linear isomorphism

$$k(Q_n//Q_0) \rightarrow \text{Hom}_{kQ_0-kQ_0}(kQ_n, kQ_0)$$

where $(\gamma, e) \in Q_n//Q_0$ is sent to the elementary map which associates e to γ and 0 to any other path of Q_n . Similarly we have that

$$(\gamma, a) \mapsto (\gamma' \mapsto \delta_{\gamma, \gamma'} a)$$

is a linear isomorphism from $k(Q_n//Q_1)$ to $\text{Hom}_{kQ_0-kQ_0}(kQ_n, kQ_1)$.

Remark 2.3 In degree 0, the vector space Λ^E is identified with $k(Q_0//Q_0) \oplus k(Q_0//Q_1)$. Notice that $Q_0//Q_0$ is simply Q_0 , and that $Q_0//Q_1$ is the set of loops of Q .

We wish now to translate the coboundaries through these linear isomorphisms. Let

$$D : k(Q_n//Q_0) \longrightarrow k(Q_{n+1}//Q_1) \text{ given by}$$

$$D(\gamma, e) = \sum_{a \in Q_1 e} (a\gamma, a) + (-1)^{n+1} \sum_{a \in e Q_1} (\gamma a, a)$$

where $Q_1 e$ (resp. $e Q_1$) is the subset of Q_1 of arrows with prescribed source (resp. terminus) vertex e .

Proposition 2.4 *The diagram where the vertical maps are given by the linear isomorphisms of Lemma 2.2, the coboundary map δ of the reduced complex of cochains at the top horizontal, and at the bottom the map*

$$k(Q_n//Q_0) \oplus k(Q_n//Q_1) \begin{pmatrix} 0 & 0 \\ D & 0 \end{pmatrix} \longrightarrow k(Q_{n+1}//Q_0) \oplus k(Q_{n+1}//Q_1)$$

is commutative.

Verifications are straightforward, using that the algebra has zero radical square.

Remark 2.5 If Q has no oriented cycles, there is a path of maximum length m . From the above result we infer that $H^n((kQ)_2, (kQ)_2)$ is zero for $n > m$

3 Hochschild cohomology

Since Hochschild cohomology is additive on a product of algebras, we will assume that the considered quivers are connected.

A c -crown is a quiver with c vertices cyclically labeled by the cyclic group of order c , and c arrows a_0, \dots, a_{c-1} such that $s(a_i) = i$ and $t(a_i) = i + 1$. A 1-crown is a loop, and a 2-crown is the two-way quiver.

Theorem 3.1 *Let Q be a connected quiver which is not a crown. Then for $n > 0$ we have*

$$\dim_k H^n((kQ)_2, (kQ)_2) = |Q_n//Q_1| - |Q_{n-1}//Q_0|.$$

We also have $\dim_k H^0((kQ)_2, (kQ)_2) = |Q_1//Q_0| + 1$.

Proof: We will show the probably most interesting observation of this work, namely that D is injective for a positive n and for quivers which are not crowns. We fix a notation: if γ is an oriented path of positive length, $f(\gamma)$ and $l(\gamma)$ denotes the first and last arrows of γ respectively. Moreover the support of an element x expressed in a basis B of a vector space is the subset of B determined by the non zero coefficients of x .

Let $x = \sum_{(\gamma, e) \in k(Q_n//Q_0)} x_{(\gamma, e)}(\gamma, e)$ be an element of the kernel of D . We fix an oriented cycle (γ, e) in order to show that its field coefficient $x_{(\gamma, e)}$ is zero. The contribution of $x_{(\gamma, e)}(\gamma, e)$ in $D(x)$ is

$$x_{(\gamma, e)} \left[\sum_{a \in Q_1 e} (a\gamma, a) + (-1)^{n+1} \sum_{a \in e Q_1} (\gamma a, a) \right].$$

Let us say that a vertex is concerned by an arrow a if it coincides with either the source or the terminus vertex of a . We claim first that if the vertex e is concerned by other arrows than $f(\gamma)$ or $l(\gamma)$, then $x_{(\gamma, e)}$ is already zero. Indeed, let a be such an arrow with for instance $s(a) = e$ and $a \neq f(\gamma)$; then the basis element $(a\gamma, a)$ is not in the support of $D(\gamma', e')$ for any (γ', e')

other than (γ, e) . To see this, suppose first that $(a\gamma, a) = (a'\gamma', a')$; then $a = a'$ and $\gamma = \gamma'$. Secondly suppose $(a\gamma, a) = (\gamma'a', a')$; then $a = a'$ and a is the first arrow of γ : we have supposed this is not the case. When a is an arrow with $t(a) = e$ and $a \neq l(\gamma)$, the analogous proof applies.

We assume now we are in the leaving situation, that is $Q_1e = \{f(\gamma)\}$ and $eQ_1 = \{l(\gamma)\}$. Consider the “rotated” $(\bar{\gamma}, \bar{e})$ of (γ, e) , namely remove $f(\gamma)$ from the beginning of γ and add it at its end: if $\gamma = l(\gamma) \cdots a_2 f(\gamma)$ then $\bar{\gamma} = f(\gamma)l(\gamma) \cdots a_2$ and $\bar{e} = s(a_2) = t(f(\gamma))$. Our second claim is that $x_{(\gamma, e)} = (-1)^n x_{(\bar{\gamma}, \bar{e})}$. Indeed we have that $(f(\gamma)\gamma, f(\gamma))$ is of course in the support of $D(\gamma, e)$ and also in the support of $D(\bar{\gamma}, \bar{e})$ since $(\bar{\gamma}f(\gamma), f(\gamma)) = (f(\gamma)\gamma, f(\gamma))$. But $(f(\gamma)\gamma, f(\gamma))$ is not in any other support of the image by D of a basis element. To prove this, let first (γ', e') be a cycle such that there exist $a \in Q_1e'$ verifying $(a\gamma', a) = (f(\gamma)\gamma, f(\gamma))$. Then $f(\gamma) = a$ and $\gamma = \gamma'$. Let secondly (γ', e') be a cycle such that there exist $a \in e'Q_1$ with $(\gamma'a, a) = (f(\gamma)\gamma, f(\gamma))$. Then $a = f(\gamma)$, $\gamma'f(\gamma) = f(\gamma)\gamma$ and γ' is the rotated $\bar{\gamma}$ of γ . We infer that the coefficient of $(f(\gamma)\gamma, f(\gamma))$ in $D(x)$ is $x_{(\gamma, e)} + (-1)^{n+1}x_{(\bar{\gamma}, \bar{e})}$. This has to be zero since we are assuming that $D(x) = 0$, and since Q_{n+1}/Q_1 is a basis.

The end of the proof is an induction process: if \bar{e} is concerned by some other arrow than $f(\bar{\gamma})$ or $l(\bar{\gamma})$, we infer $x_{(\bar{\gamma}, \bar{e})} = 0$ by the first claim. The second claim provides our aim $x_{(\gamma, e)} = 0$. If \bar{e} is not concerned by other arrows than $f(\bar{\gamma})$ or $l(\bar{\gamma})$, we rotate $\bar{\gamma}$ and we analyze arrows at \bar{e} . The process has a successful end: rotating n times (γ, e) returns to (γ, e) and at some rotation stage we have to meet an “exotic” arrow – not the first, not the last –, otherwise the connected quiver is a crown and γ some power of the minimal oriented cycle at one of its vertices.

The linear map D at zero degree is certainly not injective, since we know that $H^0(\Lambda, \Lambda)$ is always the center of Λ ; for $\Lambda = (kQ)_2$, a basis of the center is provided by the set of loops Q_1/Q_0 together with the unit element of the algebra, namely the sum of all the vertices. Furthermore it is immediate to verify that the kernel of

$$D : k(Q_0/Q_0) \rightarrow k(Q_1/Q_1)$$

given by $D(e, e) = \sum_{a \in Q_1e} (a, a) - \sum_{a \in eQ_1} (a, a)$ is indeed one-dimensional, generated by $\sum_{e \in Q_0} (e, e)$ (recall that Q is connected).

Before considering the exceptional crown case, we point out a fact from the previous dimensional computation.

Corollary 3.2 *Let Q be a connected quiver which is not a crown, k be a field and $(kQ)_2$ the corresponding radical square zero algebra. The graded cohomology $H^*((kQ)_2, (kQ)_2) = \bigoplus_{n \geq 0} H^n((kQ)_2, (kQ)_2)$ is finite dimensional if and only if Q has no oriented cycles. More precisely, if there is an oriented cycle of length c , then $H^{cn+1}((kQ)_2, (kQ)_2) \neq 0$ for all positive n .*

Proof: We already noticed that if Q has no oriented cycles, the degree of the non-zero cohomology vector spaces is bounded by the maximal length of a path. We need to prove that the inequality $|Q_{n-1} // Q_0| < |Q_n // Q_1|$ holds whenever $Q_{n-1} // Q_0$ is non empty. Consider the following set monomorphism

$$Q_{n-1} // Q_0 \rightarrow Q_n // Q_1$$

given by $(\gamma, e) \mapsto (l(\gamma)\gamma, l(\gamma))$ where $l(\gamma)$ denotes as before the last arrow of γ . Since Q is not a crown, this map is not surjective.

Proposition 3.3 *Let Q be a c -crown with $c \geq 2$. The center of $(kQ)_2$ is one-dimensional. If the characteristic of k is not two, let n be an even multiple of c . Then*

$$\dim_k H^n((kQ)_2, (kQ)_2) = \dim_k H^{n+1}((kQ)_2, (kQ)_2) = 1.$$

The cohomology vanishes in all other degrees.

Moreover, a non-trivial n -cochain is provided by the sum of all the cycles of length n . A non-trivial $n+1$ -cochain is given by $(a\gamma, a)$ for any chosen cycle γ of length n with first arrow denoted a .

If the characteristic of k is two, the above dimensions are valid for any multiple of c , the non-trivial cochains are given as before.

Proof: We have to consider the kernel of

$$D : k(Q_n // Q_0) \rightarrow k(Q_{n+1} // Q_1).$$

Notice first that $Q_n // Q_0$ and $Q_{n+1} // Q_1$ are empty if n is not a multiple of c . Now assume that c divides n and consider the cyclic group $G = \langle t \mid t^c = 1 \rangle$ labeling the vertices of the crown; each cycle of length n is determined by its source and terminus vertex in G , hence we obtain a bijection between $Q_n // Q_0$ and G . Similarly, each couple of $Q_{n+1} // Q_1$ is determined by their common source vertex, we also have a bijection with G . Through this bijections the map D becomes a linear endomorphism D' of kG given by $D'(t^i) = t^i + (-1)^{n+1}t^{i-1}$. Now it is straightforward to verify

that D' has one-dimensional kernel given by the sum of elements of G if n is even, or for any n in case of characteristic two. If n is odd, D' is injective for a field of characteristic not two.

The result follows, since the assumption $c \geq 2$ insures enough distance between non-zero cochain vector spaces.

The case of a loop ($c = 1$) is special since there is no longer vanishing cochain vector spaces, and the coboundaries maps interact:

Proposition 3.4 *Let Q be a loop a , the corresponding algebra $(kQ)_2$ is $k[a]/a^2$. Assume k is not of characteristic two. For every $n > 0$, we have*

$$\dim_k H^n((kQ)_2, (kQ)_2) = 1.$$

If n is even (a^n, a) generates linearly the cohomology at degree n ; for n odd, $(a^n, 1)$ is a linear generator at degree $n + 1$.

If the characteristic of k is two, the coboundaries are zero and each Hochschild cohomology vector space is two-dimensional.

4 Cup-product

In the context of the reduced cochain complex of the beginning of Section 1, the cup-product is the following: let f and g be cochains of degree n and m respectively. Then $f \smile g$ is the cochain of degree $n + m$ given by

$$f \smile g(x_1, \dots, x_n, y_1, \dots, y_m) = f(x_1, \dots, x_n)g(y_1, \dots, y_m).$$

As usual we have: $\delta(f \smile g) = \delta f \smile g + (-1)^n g \smile \delta f$. This implies the existence of a well defined product in Hochschild cohomology. In case of a radical square zero algebra this can be easily described. Recall that if Q is connected and is not a crown, any cocycle can be represented as a linear combination of elements of $Q_n//Q_1$. Relations are provided by the image of D .

Lemma 4.1 *If Q is a connected quiver which is not a crown, the cup-product of elements of positive degree is always zero for the Hochschild cohomology of $(kQ)_2$.*

Proof: Consider elements of $k(Q_n//Q_1)$ and $k(Q_m//Q_1)$ as cochains representing cocycles, and recall that the corresponding E -bimodule morphisms have values in the radical. Since $r^2 = 0$, the cup product is already zero at the cochain level.

Lemma 4.2 *The center of $(kQ)_2$ is isomorphic to the algebra $k[l_1, \dots, l_r]$ where $\{l_1, \dots, l_r\}$ is the set of loops; each loop has square zero and is orthogonal to the others. Moreover, loops and elements of positive degree are orthogonal in Hochschild cohomology.*

Let Q be a connected quiver which is not a crown. Let $Z = k[l_1, \dots, l_r]$ be the center of $(kQ)_2$ as above, and let X be a vector space of dimension $\sum_{i>0} \dim_k H^i((kQ)_2, (kQ)_2)$ – this sum is finite if and only if Q has no oriented cycles, see Corollary 3.2 –. Consider on X the trivial Z -bimodule structure, i.e. $l_i x = x l_i = 0$ for all i and for all $x \in X$, the unit element acts of course as the identity.

Proposition 4.3 *Let Q be a connected quiver which is not a crown. Then the Hochschild cohomology algebra $H^*((kQ)_2, (kQ)_2)$ is isomorphic to $Z \oplus X$ where the product is given by*

$$(z, x)(z', x') = (zz', zx' + xz').$$

Remark 4.4 Any finite subset of the above algebra generates a finite dimensional sub-algebra. Hence as soon as the quiver has an oriented cycle (but is not a crown), the Hochschild cohomology is not finitely generated.

We observe now that we retrieve the conditions obtained in [8] in order to insure the rigidity of a radical square zero algebra:

Proposition 4.5 *Let Q be a connected quiver. Then $H^2((kQ)_2, (kQ)_2) = 0$ if and only if*

- 1) Q has no loops,
- 2) Q do not contain unoriented triangles, i.e. 3 vertices and 3 arrows not performing an oriented cycle.
- 3) Q is not the 2-crown.

Proof: If Q is a 2-crown or a loop, $H^2((kQ)_2, (kQ)_2) \neq 0$ regardless the characteristic of the field; other crowns has zero 2-cohomology. Then assume Q is not a crown. If Q has a loop, we know from Corollary 3.2 that $H^2((kQ)_2, (kQ)_2) \neq 0$. Now assume Q has no loops; then

$$\dim_k H^2((kQ)_2, (kQ)_2) = |Q_2 // Q_1|$$

which is exactly the number of unoriented triangles for such quivers.

Theorem 4.6 *Let k be a field and let Q be a c -crown with $c \geq 2$. Then $H^*((kQ)_2, (kQ)_2)$ is isomorphic to $k[f, g]/g^2$, where f and g are commuting variables of degrees c' and $c' + 1$ respectively (if k has characteristic two or if c is even, we put $c' = c$; otherwise $c' = 2c$).*

Proof: We have proved that the Hochschild cohomology of a c -crown ($c \geq 2$) is non-zero only for degrees n and $n + 1$, where n is a multiple of c' . At degree n the cohomology is linearly generated by the E -bimodule morphism

$$f_n : r^{\otimes n} \rightarrow E$$

which sends each oriented cycle of length n to its source-terminus vertex. At degree $n + 1$, the cohomology is given by

$$g_n : r^{\otimes n+1} \rightarrow r$$

obtained through the choice of a particular oriented cycle γ of length n with first arrow called a : we have $g_n(a\gamma) = a$ and g_n vanish on any other basis element. The element in cohomology does not depend on the choice of γ .

From this description, we obtain the following: since we know that $f_{c'} \neq 0$ in cohomology, we have $(f_{c'})^m = f_{c'm}$. We know also that $g_{c'} \neq 0$, and we have $g_{c'}^2 = 0$. Moreover

$$g_{c'} (f_{c'})^m = (f_{c'})^m g_{c'} = g_{c'm}.$$

The presentation follows, with $f_{c'} = f$ and $g_{c'} = g$.

There is no difficulty in order to perform the analogous computations for a crown and we leave them to the reader.

Discussion: The methods developed in this work can perhaps be extended in order to describe the Hochschild cohomology algebra of a truncated quiver algebra – every path of a given length is zero –. More generally, the monomial algebras – some chosen paths are zero – can perhaps also be reached. An interesting application will be to compute for a non-trivial 2-cocycle the obstructions in order to realize it as a non-trivial deformation of the algebra (see [12]).

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Université de Montpellier 2, Département de Mathématiques, Pl. E. Bataillon, F-34095 Montpellier cedex 5

`cibils@math.univ-montp2.fr`

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