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CONSTRUCTIVE TENSOR FIELD THEORY: THE T_4^4 MODEL

by

V. Rivasseau & F. Vignes-Tourneret

Abstract. — We continue our constructive study of tensor field theory through the next natural model, namely the rank four tensor theory with quartic melonic interactions and propagator inverse of the Laplacian on $U(1)^4$. This superrenormalizable tensor field theory has a power counting quite similar to ordinary ϕ_3^4 . We control the model via a multiscale loop vertex expansion which has to be pushed quite beyond the one of the T_3^4 model and we establish its Borel summability in the coupling constant. This paper is also a step to prepare the constructive treatment of just renormalizable models, such as the T_5^4 model with quartic melonic interactions.

MSC. — Primary 81T08; Secondary 83C45.

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Context and outline

Perturbative quantum field theory develops the functional integrals of Lagrangian quantum field theories such as those of the standard model into a formal series of Feynman graphs and their amplitudes. The latter are the basic objects to compute in order to compare weakly coupled theories with actual particle physics experiments. However isolated Feynman amplitudes or even the full formal perturbative series cannot be considered as a complete physical theory. Indeed the non-perturbative content of Feynman functional integrals is essential to their physical interpretation, in particular when investigating stability of the vacuum and the phase structure of the model.

Axiomatic field theory, in contrast, typically does not introduce Lagrangians nor Feynman graphs but studies rigorously the general properties that any local quantum field theory ought to possess [SW64; Haa96]. Locality is indeed at the core of the mathematically rigorous formulation of quantum field theory. It is a key Wightman axiom [SW64] and in algebraic quantum field theory [Haa96] the fundamental structures are the algebras of *local observables*.

Constructive field theory is some kind of compromise between both points of view. From the start it was conceived as a model building program [VW73; GJ87; Riv91] in which specific Lagrangian field theories, typically of the superrenormalizable and renormalizable type would be studied in increasing order of complexity. Its main characteristic is the mathematical rigor with which it addresses the basic issue of divergence of the perturbative series.

The founding success of constructive field theory was the construction of the ultraviolet [Nel65] and thermodynamic [GJS73] limits of the massive ϕ_2^4 field theory [Sim74] in Euclidean space. Thanks to Osterwalder-Schrader axioms it implied the existence of a Wightman theory in real (Minkowski) space-time. Beyond this initial breakthrough, two other steps were critical for future developments. The first one was the introduction of multiscale analysis by Glimm and Jaffe to build the more complicated ϕ_3^4 model [GJ73]. It was developed as a kind of independent mathematical counterpoint to Wilson's renormalization group. All the following progress in constructive field theory and in particular the construction of just renormalizable models relied in some way on deepening this basic idea of renormalization group and multiscale analysis [GK86; Fel+86].

A bit later another key mathematical concept was introduced in constructive field theory, namely Borel summability. It is a fundamental result of the constructive quantum field theory program that the Euclidean functional integrals of many (Euclidean) quantum field theories with quartic interactions are the Borel sum of their renormalized perturbative series [EMS74; MS77; Fel+86]. This result builds a solid bridge between the Feynman amplitudes used by physicists and the Feynman-Kac functional integral which generates them. Borel summable quantum field theories have indeed a *unique* non-perturbative definition, independent of the particular cutoffs used as intermediate tools. Moreover all information contained in such theories, including the so-called "non-perturbative" issues, is *embedded* in the list of coefficients of the renormalized perturbative series. Of course to extract this information often requires an analytic continuation beyond the domains which constructive theory currently controls.

As impressive as may be the success of the standard model, it does not include gravity, the fundamental force which is the most obvious in daily life. Quantization of gravity remains a central puzzle of theoretical physics. It may require to use generalized quantum field theories with non-local interactions. Indeed near the Planck scale, space-time should fluctuate so violently that the ordinary notion of locality may no longer be the relevant concept. Among the many arguments one can list pointing into this direction are the Doplicher-Fredenhagen-Roberts remark that to distinguish two objects closer than the Planck scale would require to concentrate so much energy in such a little volume that it would create a black hole, preventing the observation [DFR94]. String theory, which (in the case of closed strings) contains a gravitational sector, is another powerful reason to abandon strict locality. Indeed strings are one-dimensional *extended* objects, whose interaction cannot be localized at any space time point. Moreover, closed strings moving in compactified background may not distinguish between small and large such backgrounds because of dualities that exchange their translational and "wrapping around" degrees of freedom. Another important remark is that in two and three dimensions pure quantum gravity is topological. In such theories, observables, being functions of the topology only, cannot be localized in a particular region of space-time.

Many approaches currently compete towards a mathematically consistent quantization of gravity, and a constructive program in this direction may seem premature. Nevertheless random tensor models have received recently increased attention, both as fundamental models for random geometry pondered by a discretized Einstein-Hilbert action [Riv13] and as efficient toy models of holography in the vicinity of a

horizon [Wit16; Gur17b; KT17; KSBS17; Fer17; Gur17a; BLT17].

Tensor models are background invariant and avoid (at least at the start) the formidable issue of fixing the gauge invariance of general relativity under diffeomorphisms (change of coordinates). Another advantage is that they remain based on functional integrals. Therefore they can be investigated with standard quantum field theory tools such as the renormalization group, and in contrast with many other approaches, with (suitably modified) constructive techniques. This paper is a step in that direction.

Random matrix and tensor models can be considered as a kind of simplification of Regge calculus [Reg61], which one could call simplicial gravity or *equilateral* Regge calculus [Amb02]. Other important discretized approaches to quantum gravity are the causal dynamical triangulations [LAJ05; Amb+13] and group field theory [Bou92; Fre05; Kra12; BGM10], in which either causality constraints or holonomy and simplicity constraints are added to bring the discretization closer to the usual formulation of general relativity in the continuum.

Random matrices are relatively well-developed and have been used successfully for discretization of two dimensional quantum gravity [Dav85; Kaz85; DFGZJ95]. They have interesting field-theoretic counterparts, such as the renormalizable Grosse-Wulkenhaar model [GW04a; GW04b; DR06; Dis+06; GW09; GW13; GW14; GW16].

Tensor models extend matrix models and were therefore introduced as promising candidates for an *ab initio* quantization of gravity in rank/dimension higher than 2 [ADJ91; Sas91; Gro92; Amb02]. However their study is much less advanced since they lacked for a long time an analog of the famous 't Hooft $1/N$ expansion for random matrix models [Hoo74] to probe their large N limit. Their modern reformulation [Gur10a; GR11b; Gur13b; BGR12b] considers *unsymmetrized* random tensors, a crucial improvement. Such tensors in fact have a larger, truly tensorial symmetry (typically in the complex case a $U(N)^{\otimes d}$ symmetry at rank d instead of the single $U(N)$ of symmetric tensors). This larger symmetry allows to probe their large N limit through $1/N$ expansions of a new type [Gur10b; GR11a; Gur11; Bon13; BDR15; Bon16].

Random tensor models can be further divided into fully invariant models, in which both propagator and interaction are invariant, and field theories in which the interaction is invariant but the propagator is not [BGR12a]. This propagator can incorporate or not a gauge invariance of the Boulatov group field theory type. In such field theories the use of tensor invariant interactions is the critical ingredient allowing in many cases for their successful renormalization [BGR12a; BGR13; OSVT13; BG14; COR14b; COR14a]. Surprisingly the simplest just renormalizable models turn out to be asymptotically free [BGR13; BG12b; BG12a; OS13; Riv15].

In all examples of random matrix and tensor models, the key issue is to understand in detail the limit in which the matrix or the tensor has many entries. Accordingly, the main constructive issue is not simply Borel summability but uniform Borel summability with the right scaling in N as $N \rightarrow \infty$. In the field theory case the corresponding key issue is to prove Borel summability of the *renormalized* perturbation expansion without cutoffs.

Recent progress has been fast on this front [Riv16]. Uniform Borel summability in the coupling constant has been proven for vector, matrix and tensor *quartic* models [Riv07; Mag+09; Gur13a; DGR14; GK15], based on the loop vertex expansion (LVE) [Riv07; MR07; RW13], which combines an intermediate field representation¹ with the use of a *forest formula* [BK87; AR95]. This relatively recent constructive technique is adapted to the study of theories without any space-time, as it works more directly at the combinatorial level and does not introduce any lattice. It was introduced precisely to make constructive sense of 't Hooft $1/N$ expansion for quartic matrix models [Riv07; GK15].

The constructive tensor field theory program started in [DR16], in which Borel summability of the renormalized series has been proved for the simplest such theory which requires some infinite renormalization, namely the $U(1)$ rank-three model with inverse Laplacian propagator and quartic interactions nicknamed T_3^4 . This model has power counting similar to the one of ϕ_2^4 . The main tool is the multiscale loop vertex expansion (MLVE) [GR14], which combines an intermediate field representation with the use of a more complicated *two-level jungle formula* [AR95]. An important additional technique is the iterated Cauchy-Schwarz bounds which allow to bound the LVE contributions. They are indeed not just standard perturbative amplitudes, but include resolvents which are delicate to bound.

¹More recently the LVE has been extended to higher order interactions by introducing another related functional integral representation called the *loop vertex representation*. It is based on the idea of forcing functional integration of a single field per vertex [Riv17]. For quartic models like the one studied in this paper, this other representation is however essentially equivalent to the intermediate field representation.

The program has been also extended recently to similar models with Boulatov-type group field theory projector [Lah15a; Lah15b].

The next natural step in this constructive tensor field theory program is to build the $U(1)$ rank-four model with inverse Laplacian propagator and quartic melonic interactions, which we nickname T_4^4 . This model is comparable in renormalization difficulty to the ordinary ϕ_3^4 theory, hence requires several additional non-trivial arguments. This is the problem we solve in the present paper.

The plan of this paper essentially extends the one of [DR16], as we follow roughly the same general strategy, but with many important additions due to the more complicated divergences of the model. As the proof of our main result, namely Theorem 2.1, is somewhat lengthy, we now outline its main steps and use this occasion to give the actual plan of this paper and to define the various classes of Feynman graphs we will encounter.

In Section 1 we provide the mathematical definition of the model. Its original or tensor representation is given in Section 1.1 as well as the full list of its perturbative counterterms. This model is a quantum field theory the fields of which are tensors namely elements of $\ell_2(\mathbb{Z})^{\otimes 4}$. As usual in quantum field theory, it is convenient to represent analytical expressions by Feynman graphs. The latter will cover many different graphical notions. As a first example, the Feynman graphs of the tensor field theory under study here (see eqs. (1.1) and (1.4)) will be called *tensor graphs*. They will be depicted as (edge-)coloured graphs like in figs. 3 to 5.

Section 1.2 then provides the intermediate field representation, at the heart of the Loop Vertex Expansion. It rewrites the partition function as a functional integral over both a main Hermitian matrix intermediate field σ and an auxiliary intermediate field τ (which is also a matrix). We will simply write *graphs* for the Feynman graphs of the intermediate field representation of the model, whereas these “graphs” are maps really, since intermediate fields are matrices.

A multiscale decomposition is introduced in Section 1.4.

Section 2 provides the multiscale loop vertex expansion (hereafter MLVE) for that model, which is surprisingly close to the one used in [DR16], with just a little bit of extra structure due to a single one of the ten divergent vacuum graphs of the theory. MLVE consists in an ordered Bosonic and Fermionic 2-jungle formula which expresses each “order” n of the partition function \mathcal{Z} (or the moments of the to-be-defined functional measure) as a sum over forests on n nodes. One of the benefits of such an expansion is that the free energy *i.e.* the logarithm of the partition function can very easily be expressed as a similar sum but over *connected* jungles namely some sort of trees.

Definition 1 (Trees, forests and jungles). — A **forest** on $[n] := \{1, 2, \dots, n\}$ is an acyclic graph the vertex-set of which is $[n]$. A **tree** is a connected acyclic graph. Connected components of forests are trees. Note that the graph with one vertex and no edges is considered a tree. A **(2-)jungle** is a forest the edges of which are marked either 0 or 1. The vertices of a jungle are called **nodes**. ♠

Jungles on $[n]$ will index the various terms composing order n of the Loop Vertex Expansion of the partition function and of the free energy of our model. More precisely, a jungle comes equipped

- with a scale attribution of its nodes (*i.e.* a function from the set of its nodes to the non-negative integers smaller than a general UV cutoff j_{\max}),
- and intermediate field derivatives at both ends of each of its edges.

Each node a of a jungle represents a functional expression, namely $W_{j_a} = e^{-V_{j_a}} - 1$ where V_{j_a} is the quartic interaction of the model at scale j_a . The MLVE expresses $\log \mathcal{Z}$ as follows:

$$(2.3) \quad \mathcal{W}_{\leq j_{\max}}(g) := \log \mathcal{Z}_{\leq j_{\max}}(g) = \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{\mathcal{J} \text{ tree}} \sum_{j_1=1}^{j_{\max}} \cdots \sum_{j_n=1}^{j_{\max}} \int dw_{\mathcal{J}} \int d\nu_{\mathcal{J}} \partial_{\mathcal{J}} \left[\prod_{\mathcal{B}} \prod_{a \in \mathcal{B}} (-\bar{\chi}_{j_a}^{\mathcal{B}} W_{j_a}(\vec{\sigma}^a, \vec{\tau}^a) \chi_{j_a}^{\mathcal{B}}) \right]$$

where \mathcal{B} represents a connected component of the Bosonic part of the jungle \mathcal{J} . Each Bosonic block \mathcal{B} is thus a subtree of \mathcal{J} . Our main result, Theorem 2.1, consists in the analyticity of $\lim_{j_{\max} \rightarrow \infty} \mathcal{W}_{\leq j_{\max}}(g)$

in a non empty cardioid domain of the complex plane as well as the Borel summability of its perturbative *renormalised* series. The rest of the paper is entirely devoted to its proof.

The jungles of the MLVE are considered hereafter *abstract graphs*. Each edge of an abstract forest comes equipped with intermediate field derivatives at both of its ends (represented by the $\partial_{\mathcal{J}}$ operator in the preceding equation). The result of these derivatives (with respect to the σ - and χ -fields) on the W_{j_a} 's is a sum, the terms of which can be indexed by still another type of graphs that we name *skeletons*, see Section 3.

Definition 2 (Skeleton graphs). — *Skeleton graphs* are plane forests possibly with external edges, marked subgraphs, marked external edges and marked corners. External edges are unpaired half-edges. We will denote skeleton graphs with sans serif characters such as \mathbf{G} . The possibly marked subgraphs are \mathbb{I} and \mathbb{I}^\bullet . The marked ones will be depicted in gray and basically represent renormalised amplitudes of 2-point subgraphs noted respectively D_1 and D_2 . Unmarked external edges will be pictured \mathbb{I} and marked ones by dotted lines \mathbb{I}^\bullet . The latter represent resolvent insertions. Each vertex of a skeleton graph has a unique marked corner (*i.e.* an angular sector between two consecutive half-edges, marked or not, adjacent to a same vertex). Each such marked corner bears an integer between 1 and $\lfloor \frac{m+1}{2} \rfloor + 1$ if the graph has m vertices. \spadesuit

Let us consider a skeleton graph $\mathbf{G}(\mathcal{J})$ derived from a jungle \mathcal{J} on $[n]$. Thanks to the Faà di Bruno formula, eq. (3.1), each node a of \mathcal{J} might be split into several (in fact up to the degree of a) vertices of \mathbf{G} . For $a \in [n]$, let $V_a(\mathbf{G})$ be the subset of vertices of \mathbf{G} originating from node a of \mathcal{J} . The set $\{V_a(\mathbf{G}), a \in [n]\}$ forms a partition of $V(\mathbf{G})$. For all $a \in [n]$, the marked corners of the vertices in $V_a(\mathbf{G})$ bear integer a .

To reach analyticity of $\mathcal{W}_{\leq j_{\max}}$ we prove that it converges normally. We must then compute an upper bound on the module of its order n . The Fermionic integrals is standard and can be performed exactly, see Section 5.1. It leads to the following bound

$$|\mathcal{W}_{\leq j_{\max}}(g)| \leq \sum_{n=1}^{\infty} \frac{2^n}{n!} \sum_{\mathcal{J} \text{ tree}} \sum_{\{j_a\}} \left(\prod_{\mathcal{B}} \prod_{\substack{a,b \in \mathcal{B} \\ a \neq b}} (1 - \delta_{j_a j_b}) \right) \left(\prod_{\substack{\ell_F \in \mathcal{F}_F \\ \ell_F = (a,b)}} \delta_{j_a j_b} \right) \prod_{\mathcal{B}} |I_{\mathcal{B}}|,$$

$$I_{\mathcal{B}} = \int d\mathbf{w}_{\mathcal{B}} \int d\nu_{\mathcal{B}} \partial_{\tau_{\mathcal{B}}} \prod_{a \in \mathcal{B}} (e^{-V_{j_a}} - 1)(\vec{\sigma}^a, \vec{\tau}^a).$$

The main difficulty resides in the estimation of the Bosonic contributions $I_{\mathcal{B}}$. A Hölder inequality rewrites it as (see eq. (3.9))

$$|I_{\mathcal{B}}| \leq I_{\mathcal{B}}^{NP} \sum_{\mathbf{G}} \underbrace{\left(\int d\nu_{\mathcal{B}} | :A_{\mathbf{G}}(\vec{\sigma}) : |^4 \right)^{1/4}}_{\text{perturbative}}.$$

This bound consists in two parts: a perturbative one, the terms of which are indexed by skeleton graphs \mathbf{G} and a non perturbative one, $I_{\mathcal{B}}^{NP}$, made of exponentials of interaction terms and counterterms. Sections 4 and 5 are devoted to the non perturbative terms and lead in particular to Theorem 5.5. Section 4 is a technical preparation for the next section and consists in proving two very different but essential bounds, one of which is *quadratic*, see Lemma 4.1, and the other *quartic*, Corollary 4.6, on the main part of V_j . In Section 5 we find some echo of the main Glimm and Jaffe idea of expanding more and more the functional integral at higher and higher energy scale [GJ73]. Indeed to compensate for linearly divergent vacuum graphs we need to push quite far a Taylor expansion of the non-perturbative factor. However of course a key difference is that there are no geometrical objects such as the scaled ‘‘Russian dolls’’ lattices of cubes so central to traditional multiscale constructive analysis.

In Section 6 we bound the perturbative terms in $I_{\mathcal{B}}$ using an improved version of the Iterated Cauchy-Schwarz bounds. Indeed the trees of the LVE and MLVE are not perturbative; they still resum infinite power series through resolvents, which are however uniformly bounded in norm, see Lemma 1.4. The ICS bound is a technique which allows to bound such ‘‘quasi-perturbative’’ LVE contributions by truly perturbative contributions, but with no longer any resolvent included. More precisely, remember that

skeleton graphs \mathbf{G} are intermediate field graphs (thus maps) both with unmarked external edges (corresponding to σ -fields still to be integrated out) and marked ones representing resolvents. We first get rid of those external σ -fields by integrating by parts (with respect to the Gaussian measure $d\nu_{\mathcal{B}}$), see eq. (6.1), in what we call the contraction process (see Section 6.1). Note that unmarked external edges will then be paired both with marked and unmarked external edges. When an unmarked external edge contracts to another unmarked external edge, it simply creates a new edge. But when it contracts to a marked external edge, it actually creates a new corner, as depicted in fig. 1 and according to eq. (6.2). The result

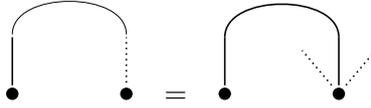


Figure 1. Contraction of half-edges in skeleton graphs.

of all the possible contractions of all the unmarked external edges of a skeleton graph \mathbf{G} consists in a set of *resolvent graphs*.

Definition 3 (Resolvent graphs). — A *resolvent graph* is a map with external edges, marked subgraphs and marked corners. External edges, pictured , represent resolvents. Possible marked subgraphs are the same than for skeleton graphs. Marked corners bear an integer between 1 and $\lfloor \frac{m+1}{2} \rfloor + 1$ if the graph has m vertices. Resolvent graphs will be denoted with calligraphic fonts such as \mathcal{G} for example. We also let $s(\mathcal{G})$ be the set of marked corners of \mathcal{G} and for any corner c in $s(\mathcal{G})$, we let i_c be the corresponding integer. ♠

Let \mathbf{G} be a skeleton graph and $\mathcal{G}(\mathbf{G})$ one of the resolvent graphs created from \mathbf{G} by the contraction process. As the latter does not create nor destroy vertices, the sets of vertices of \mathbf{G} and \mathcal{G} have the same cardinality. Nevertheless the contraction process may create new corners. In fact it creates two new corners each time an unmarked external edge is paired to a marked external one. Thus there is a natural injection ι from the corners of \mathbf{G} to the ones of \mathcal{G} . Moreover it is such that the marked corners of \mathcal{G} are the images of the marked corners of \mathbf{G} via ι .

Amplitudes of resolvent graphs still contain σ -fields in the resolvents. In Section 6.2 we will apply iterated Cauchy-Schwarz estimates to such amplitudes in order to bound them by the geometric mean of resolvent-free amplitudes, using that the norm of the resolvent is bounded in a cardioid domain of the complex plane. To this aim, it will be convenient to represent resolvent graph amplitudes by the partial duals of resolvent graphs with respect to a spanning subtree, see Section 6.2. It results in one-vertex maps that we will actually represent as chord diagrams. Resolvents in such maps will not be pictured anymore as dotted external edges but as encircled \mathcal{R} 's. See fig. 2 for an example.

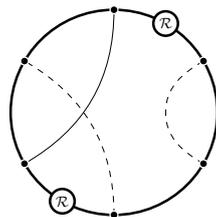


Figure 2. Example of the partial dual with respect to a spanning subtree of a resolvent graph, represented as a chord diagram. Edges of the tree correspond to plain lines whereas edges in the complement are dashed lines. Resolvent insertions are explicitly represented.

In Section 7 we prove that the good power counting of convergent amplitudes is sufficient to both compensate the large combinatorial factors inherent in the perturbative sector of the theory and sum over the scales j_a of the jungle \mathcal{J} .

Finally appendices contain some of the proofs and details.

1. The model

1.1. Laplacian, bare and renormalized action. — Consider a pair of conjugate rank-4 tensor fields

$$T_{\mathbf{n}}, \bar{T}_{\bar{\mathbf{n}}}, \text{ with } \mathbf{n} = (n_1, n_2, n_3, n_4) \in \mathbb{Z}^4, \bar{\mathbf{n}} = (\bar{n}_1, \bar{n}_2, \bar{n}_3, \bar{n}_4) \in \mathbb{Z}^4.$$

They belong respectively to the tensor product $\mathcal{H}^{\otimes} := \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathcal{H}_3 \otimes \mathcal{H}_4$ and to its dual, where each \mathcal{H}_i is an independent copy of $\ell_2(\mathbb{Z}) = L_2(U(1))$, and the colour or strand index i takes values in $\{1, 2, 3, 4\}$. Indeed by Fourier transform these fields can be considered also as ordinary scalar fields $T(\theta_1, \theta_2, \theta_3, \theta_4)$ and $\bar{T}(\bar{\theta}_1, \bar{\theta}_2, \bar{\theta}_3, \bar{\theta}_4)$ on the four torus $\mathbb{T}_4 = U(1)^4$ [BGR12a; DR16].

If we restrict the indices \mathbf{n} to lie in $[-N, N]^4$ rather than in \mathbb{Z}^4 we have a proper (finite dimensional) tensor model. We can consider N as the ultraviolet cutoff, and we are interested in performing the ultraviolet limit $N \rightarrow \infty$.

Unless specified explicitly, short notations such as $\sum_{\mathbf{n}}$, $\prod_{\mathbf{n}}$ mean either cutoff sums $\sum_{\mathbf{n} \in [-N, N]^4}$, $\prod_{\mathbf{n} \in [-N, N]^4}$ in the initial sections of this paper, before renormalization has been performed, or simply $\sum_{\mathbf{n} \in \mathbb{Z}^4}$ and $\prod_{\mathbf{n} \in \mathbb{Z}^4}$ in the later sections when renormalization has been performed.

We introduce the normalized Gaussian measure

$$d\mu_C(T, \bar{T}) := \left(\prod_{\mathbf{n}, \bar{\mathbf{n}}} \frac{dT_{\mathbf{n}} d\bar{T}_{\bar{\mathbf{n}}}}{2i\pi} \right) \det C^{-1} e^{-\sum_{\mathbf{n}, \bar{\mathbf{n}}} T_{\mathbf{n}} C_{\mathbf{n}\bar{\mathbf{n}}}^{-1} \bar{T}_{\bar{\mathbf{n}}}}$$

where the covariance C is the inverse of the Laplacian on \mathbb{T}_4 plus a unit mass term

$$C_{\mathbf{n}, \bar{\mathbf{n}}} = \frac{\delta_{\mathbf{n}, \bar{\mathbf{n}}}}{\mathbf{n}^2 + 1}, \quad \mathbf{n}^2 := n_1^2 + n_2^2 + n_3^2 + n_4^2.$$

The *formal*² generating function for the moments of the model is then

$$(1.1) \quad \mathcal{Z}_0(g, J, \bar{J}) = \mathcal{N} \int e^{T \cdot \bar{J} + J \cdot \bar{T}} e^{-\frac{g}{2} \sum_c V_c(T, \bar{T})} d\mu_C(T, \bar{T}),$$

where the scalar product of two tensors $A \cdot B$ means $\sum_{\mathbf{n}} A_{\mathbf{n}} B_{\mathbf{n}}$, g is the coupling constant, the source tensors J and \bar{J} are dual respectively to \bar{T} and T and \mathcal{N} is a normalization. To compute correlation functions it is common to choose $\mathcal{N}^{-1} = \int e^{-\frac{g}{2} \sum_c V_c(T, \bar{T})} d\mu_C(T, \bar{T})$ which is the sum of all vacuum amplitudes. However following the constructive tradition for such superrenormalizable models, we shall limit \mathcal{N} to be the exponential of the finite sum of the *divergent* connected vacuum amplitudes. The interaction is $\sum_c V_c(T, \bar{T})$ with

$$(1.2) \quad V_c(T, \bar{T}) := \text{Tr}_c(T \mathbb{I}_{\hat{c}} \bar{T})^2 = \sum_{\substack{\mathbf{n}_c, \bar{\mathbf{n}}_c, \\ \mathbf{m}_c, \bar{\mathbf{m}}_c}} \left(\sum_{\mathbf{n}_{\hat{c}}, \bar{\mathbf{n}}_{\hat{c}}} T_{\mathbf{n}} \bar{T}_{\bar{\mathbf{n}}} \delta_{\mathbf{n}_{\hat{c}} \bar{\mathbf{n}}_{\hat{c}}} \right) \delta_{\mathbf{n}_c \bar{\mathbf{m}}_c} \delta_{\mathbf{m}_c \bar{\mathbf{n}}_c} \left(\sum_{\mathbf{m}_{\hat{c}}, \bar{\mathbf{m}}_{\hat{c}}} T_{\mathbf{m}} \bar{T}_{\bar{\mathbf{m}}} \delta_{\mathbf{m}_{\hat{c}} \bar{\mathbf{m}}_{\hat{c}}} \right),$$

and where Tr_c means the trace over \mathcal{H}_c , $\mathbf{n}_{\hat{c}} := \{n_{c'}, c' \neq c\}$ (and similarly for $\bar{\mathbf{n}}_{\hat{c}}$, $\mathbf{m}_{\hat{c}}$, $\bar{\mathbf{m}}_{\hat{c}}$) and $(\mathbb{I}_{\hat{c}})_{\mathbf{n}_{\hat{c}} \bar{\mathbf{n}}_{\hat{c}}} = \delta_{\mathbf{n}_{\hat{c}} \bar{\mathbf{n}}_{\hat{c}}}$. Hence it is the symmetric sum of the four quartic melonic interactions of random tensors at rank four [DGR14] with equal couplings.

This model is globally symmetric under colour permutations and has a power counting almost similar to the one of ordinary ϕ_3^4 [GJ73; FO76; MS76]. It has eleven divergent graphs (regardless of their colours) including two (melonic) two-point graphs: the tadpole \mathcal{M}_1 , linearly divergent, and the graph \mathcal{M}_2 log divergent (see fig. 3). Note that each of these eleven graphs has several coloured versions. For example, there are four different coloured graphs corresponding to \mathcal{M}_1 , sixteen to \mathcal{M}_2 , and ten to the unique melonic divergent vacuum graph of order two (see fig. 4).

²Here *formal* simply means that \mathcal{Z}_0 is ill-defined in the limit $N \rightarrow \infty$.

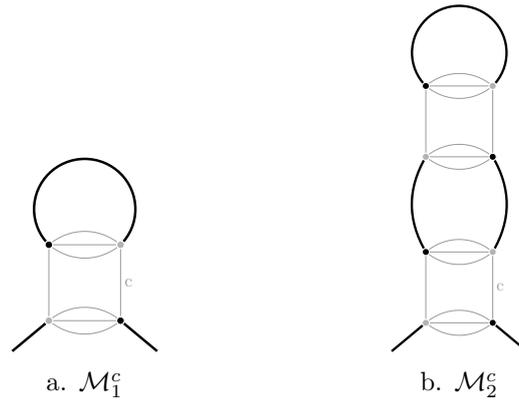


Figure 3. The two divergent (melonic) two-point graphs. The melonic quartic vertex is shown with gray edges, and the bold edges correspond to Wick contractions of T with \overline{T} , hence bear an inverse Laplacian.

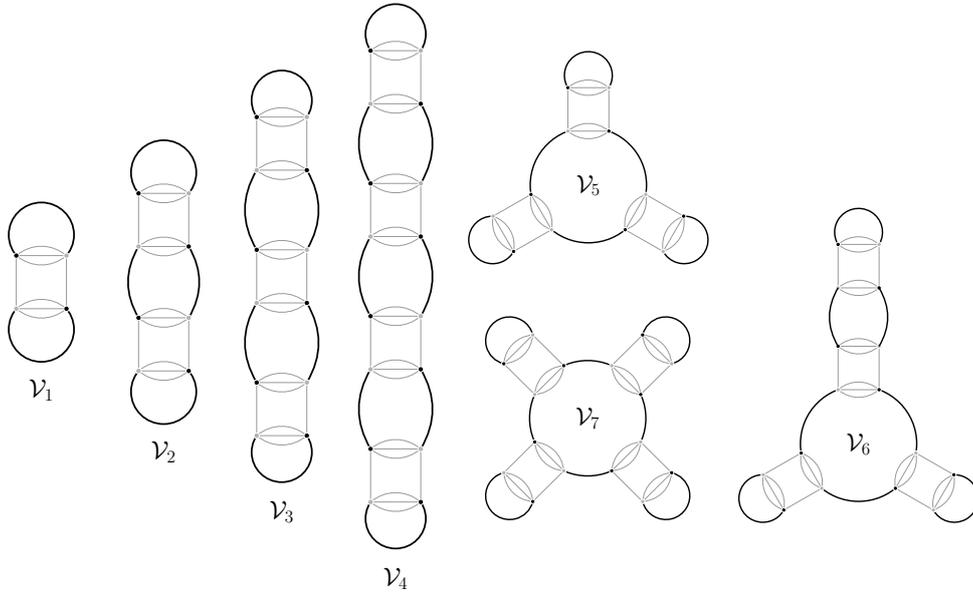


Figure 4. The seven divergent melonic vacuum connected graphs.

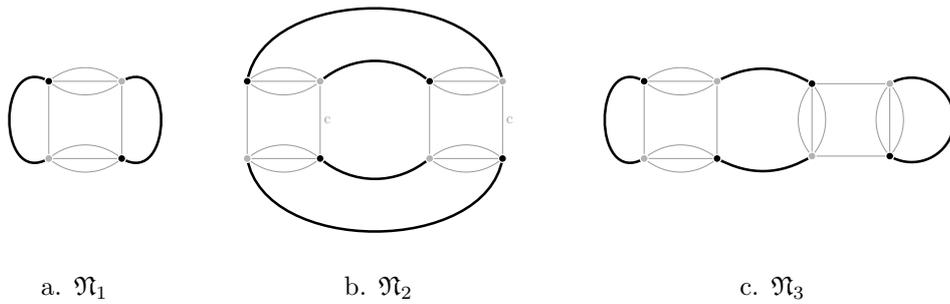


Figure 5. The three divergent non-melonic vacuum connected graphs.

The main problem in quantum field theory is to compute $\mathcal{W}(g, J, \bar{J}) = \log \mathcal{Z}(g, J, \bar{J})$ which is the generating function for the *connected* Schwinger functions

$$S_{2k}(\bar{n}_1, \dots, \bar{n}_k; n_1, \dots, n_k) = \frac{\partial}{\partial J_{\bar{n}_1}} \cdots \frac{\partial}{\partial J_{\bar{n}_k}} \frac{\partial}{\partial \bar{J}_{n_1}} \cdots \frac{\partial}{\partial \bar{J}_{n_k}} \mathcal{W}(g, J, \bar{J})|_{J=\bar{J}=0}.$$

Thus our main concern in this work will be to prove the analyticity (in g) of \mathcal{W} in some (non empty) domain of the complex plane, in the limit $N \rightarrow \infty$. Of course there is no chance for \mathcal{Z}_0 to be well defined in this limit and some (well-known) modifications of the action have to be done, namely it has to be supplemented with the counterterms of all its divergent subgraphs.

Let G be a tensor Feynman graph and τ_G be the operator which sets to 0 the external indices of its Feynman amplitude A_G . The counterterm associated to G is given by

$$\delta_G = -\tau_G \left(\sum_{\mathcal{F} \not\supset G} \prod_{g \in \mathcal{F}} -\tau_g \right) A_G$$

where the sum runs over all the forests of divergent subgraphs of G which do not contain G itself (including the empty one). The renormalized amplitude of G is

$$A_G^r = (1 - \tau_G) \left(\sum_{\mathcal{F} \not\supset G} \prod_{g \in \mathcal{F}} -\tau_g \right) A_G.$$

The behaviour of the renormalized amplitudes at large external momenta is a remainder of the initial power counting of the graph. In particular, let \mathcal{M} be the set of the two divergent 2-point graphs, namely $\mathcal{M} = \{\mathcal{M}_1, \mathcal{M}_2\}$. Their renormalized amplitudes are (neither including the coupling constants nor the symmetry factors, and seen as linear operators on \mathcal{H}^{\otimes})

$$(1.3) \quad \left\{ \begin{array}{l} A_{\mathcal{M}_1}^r(\mathbf{n}, \bar{\mathbf{n}}) = \sum_c a_1(n_c) \delta_{\mathbf{n}, \bar{\mathbf{n}}}, \\ A_{\mathcal{M}_2}^r(\mathbf{n}, \bar{\mathbf{n}}) = \sum_c (a_2(n_c) + \sum_{c' \neq c} a_2^{c'}(n_c)) \delta_{\mathbf{n}, \bar{\mathbf{n}}}, \\ a_1(n_c) = \sum_{p \in \mathbb{Z}^4} \frac{\delta(p_c - n_c) - \delta(p_c)}{p^2 + 1} = \sum_{p \in \mathbb{Z}^3} \frac{n_c^2}{(n_c^2 + p^2 + 1)(p^2 + 1)}, \\ a_2(n_c) = \sum_{p, q \in \mathbb{Z}^4} \frac{\delta(p_c - n_c) - \delta(p_c)}{p^2 + 1} \frac{\delta(q_c - n_c) - \delta(q_c)}{(q^2 + 1)^2}, \\ a_2^{c'}(n_c) = \sum_{p, q \in \mathbb{Z}^4} \frac{\delta(p_{c'} - q_{c'}) - \delta(p_{c'})}{p^2 + 1} \frac{\delta(q_c - n_c) - \delta(q_c)}{(q^2 + 1)^2}. \end{array} \right.$$

Remark that $a_2^{c'}$ is in fact independent of c' .

From now on we shall use the time-honored constructive practice of noting $O(1)$ any inessential constant. The large \mathbf{n} behaviour of the renormalized graphs \mathcal{M}_1 and \mathcal{M}_2 is controlled by the following

Lemma 1.1. — *Let $\mathbf{n} \in \mathbb{Z}^4$ and $\|\mathbf{n}\|$ be $\sqrt{\sum_{i=1}^4 n_i^2}$. Then*

$$|A_{\mathcal{M}_1}^r(\mathbf{n}, \bar{\mathbf{n}})| \leq O(1) \|\mathbf{n}\| \delta_{\mathbf{n}, \bar{\mathbf{n}}}, \quad |A_{\mathcal{M}_2}^r(\mathbf{n}, \bar{\mathbf{n}})| \leq O(1) \log(1 + \|\mathbf{n}\|) \delta_{\mathbf{n}, \bar{\mathbf{n}}}.$$

Proof. — Elementary from eq. (1.3).

Let \mathcal{V} be the set of divergent vacuum graphs of the model (1.1). For any Feynman graph G , let $|G|$ be its order (a.k.a. number of vertices). Then the regularized generating function \mathcal{Z} of the renormalized Schwinger functions is defined by

$$(1.4) \quad \mathcal{Z}_N(g, J, \bar{J}) := \mathcal{N} \int e^{T \cdot \bar{J} + J \cdot \bar{T}} e^{-\frac{g}{2} \sum_c V_c(T, \bar{T}) + T \cdot \bar{T}} \left(\sum_{G \in \mathcal{M}} \frac{(-g)^{|G|}}{s_G} \delta_G \right) d\mu_C(T, \bar{T}).$$

where S_G is the usual symmetry factor of the Feynman graph G , and the normalization \mathcal{N} is, as announced, the exponential of the finite sum of the counterterms of the divergent vacuum connected graphs, computed with cutoff N :

$$\mathcal{N} := \exp \left(\sum_{G \in \mathcal{V}} \frac{(-g)^{|G|}}{S_G} \delta_G \right).$$

As a final step of this section, let us rewrite eq. (1.4) a bit differently. We want to absorb the mass counterterms in a translation of the quartic interaction. So let us define $g\delta_m := \sum_{G \in \mathcal{M}} \frac{(-g)^{|G|}}{S_G} \delta_G$ and $\delta_m := \sum_c \delta_m^c$. Then the integrand of \mathcal{Z}_N contains $e^{-g} \sum_c I_c$ with

$$I_c = \frac{1}{2} V_c(T, \bar{T}) - \delta_m^c T \cdot \bar{T} = \frac{1}{2} \text{Tr}_c(T \mathbb{I}_c \bar{T})^2 - \delta_m^c T \cdot \bar{T}.$$

By simply noting that for all c , $T \cdot \bar{T} = \text{Tr}_c(T \mathbb{I}_c \bar{T})$, we get

$$I_c = \frac{1}{2} \text{Tr}_c(T \mathbb{I}_c \bar{T} - \delta_m^c \mathbb{I}_c)^2 - \frac{1}{2} (2N + 1) (\delta_m^c)^2.$$

Thus \mathcal{Z}_N rewrites as

$$\mathcal{Z}_N(g, J, \bar{J}) = \mathcal{N} e^{\delta_t} \int e^{T \cdot \bar{J} + J \cdot \bar{T}} e^{-\frac{g}{2} \sum_c V_c^r(T, \bar{T})} d\mu_C(T, \bar{T}),$$

where $V_c^r(T, \bar{T}) := \text{Tr}_c(T \mathbb{I}_c \bar{T} - \delta_m^c \mathbb{I}_c)^2$ and

$$\delta_t := \frac{g}{2} \sum_c \text{Tr}_c \mathbb{I}_c (\delta_m^c)^2 = \frac{g}{2} (2N + 1) \sum_c (\delta_m^c)^2,$$

where the last equality uses the particular form of the cutoff $[-N, N]$.

1.2. Intermediate field representation. — The main message of the Loop Vertex Expansion (a.k.a. LVE) is that it is easier (and to a certain extent better) to perform constructive renormalization within the intermediate field setting. Initially designed for matrix models [Riv07] LVE has proven to be very efficient for tensor models in general [Gur13a].

1.2.1. Integrating out the tensors. — So we now decompose the four interactions V_c^r by introducing four intermediate Hermitian $N \times N$ matrix fields σ_c^T acting on \mathcal{H}_c (here the superscript T refers to transposition). To simplify the formulas we put $g =: \lambda^2$ and write

$$e^{-\frac{\lambda^2}{2} V_c^r(T, \bar{T})} = \int e^{i\lambda \text{Tr}_c [(T \mathbb{I}_c \bar{T} - \delta_m^c \mathbb{I}_c) \sigma_c^T]} d\nu(\sigma_c^T)$$

where $d\nu(\sigma_c^T) = d\nu(\sigma_c)$ is the GUE law of covariance 1. $\mathcal{Z}_N(g, J, \bar{J})$ is now a Gaussian integral over (T, \bar{T}) , hence can be evaluated:

$$\begin{aligned} \mathcal{Z}_N(g, J, \bar{J}) &= \mathcal{N} e^{\delta_t} \int \left(\prod_c d\nu(\sigma_c) \right) d\mu_C(T, \bar{T}) e^{T \cdot \bar{J} + J \cdot \bar{T}} e^{i\lambda (T \sigma \bar{T} - \sum_c \delta_m^c \text{Tr}_c \sigma_c)} \\ (1.5) \quad &= \mathcal{N} e^{\delta_t} \int \left(\prod_c d\nu(\sigma_c) \right) e^{J C^{1/2} R(\sigma) C^{1/2} \bar{J} - \text{Tr} \log(\mathbb{I} - \Sigma) - i\lambda \sum_c \delta_m^c \text{Tr}_c \sigma_c} \end{aligned}$$

where $\sigma := \sigma_1 \otimes \mathbb{I}_2 \otimes \mathbb{I}_3 \otimes \mathbb{I}_4 + \mathbb{I}_1 \otimes \sigma_2 \otimes \mathbb{I}_3 \otimes \mathbb{I}_4 + \mathbb{I}_1 \otimes \mathbb{I}_2 \otimes \sigma_3 \otimes \mathbb{I}_4 + \mathbb{I}_1 \otimes \mathbb{I}_2 \otimes \mathbb{I}_3 \otimes \sigma_4$, \mathbb{I} is the identity operator on \mathcal{H}^\otimes , Tr denotes the trace over \mathcal{H}^\otimes ,

$$\Sigma(\sigma) := i\lambda C^{1/2} \sigma C^{1/2} =: i\lambda H$$

is the σ operator sandwiched³ with appropriate square roots $C^{1/2}$ of propagators and includes the $i\lambda$ factor, hence H is always Hermitian and Σ is *anti-Hermitian* for g real positive. The *symmetrized resolvent* operator is

$$R(\sigma) := (\mathbb{I} - i\lambda C^{1/2} \sigma C^{1/2})^{-1} = (\mathbb{I} - \Sigma)^{-1}.$$

³Using cyclicity of the trace, it is possible to work either with $C\sigma$ operators or with *symmetrized* “sandwiched” operators but the latter are more convenient for the future constructive bounds of Section 5.

In the sequel it will also be convenient to consider the inner product space $L(\mathcal{H})^\times := L(\mathcal{H}_1) \times L(\mathcal{H}_2) \times L(\mathcal{H}_3) \times L(\mathcal{H}_4)$ where each $L(\mathcal{H}_i)$ is the space of linear operators on \mathcal{H}_i . Let \vec{a} and \vec{b} be elements of $L(\mathcal{H})^\times$. Their inner product, denoted $\vec{a} \cdot \vec{b}$, is defined as $\sum_c \text{Tr}_c(a_c^\dagger b_c)$. For any $\vec{a} \in L(\mathcal{H})^\times$, to simplify notations, we will write its c -component $(\vec{a})_c$ as a_c . Similarly we define $\vec{\sigma}$ as the element of $L(\mathcal{H})^\times$ the c -component of which is σ_c . Finally let \mathbb{I} be the multiplicative identity element ($\mathbb{I}_{cc'} = \delta_{cc'} \mathbb{I}_c$) of the linear operators $L(L(\mathcal{H})^\times)$ on $L(\mathcal{H})^\times$.

The Gaussian measure $\prod_c d\nu(\sigma_c)$ is now interpreted as the normalized Gaussian measure on $L(\mathcal{H})^\times$ of covariance \mathbb{I} and denoted $d\nu_{\mathbb{I}}(\vec{\sigma})$.

1.2.2. Renormalized action. — It is well known that each order of the Taylor expansion around $g = 0$ of \mathcal{Z}_N (see eq. (1.4)) is finite in the limit $N \rightarrow \infty$. The counterterms added to the action precisely compensate the divergences of the Feynman graphs created by the bare action. Proving such a result is by now very classical but still somewhat combinatorially involved. We exhibit here one of the advantages of the intermediate field representation. We are indeed going to rewrite eq. (1.5) in such a way that the compensations between terms and counterterms are more explicit. Such a new form of an action will be called **renormalized σ -action**. The idea is to Taylor expand $\log(\mathbb{I} - i\lambda C\sigma)$ “carefully”, *i.e.* order by order in a way somewhat similar in spirit to the way multiscale analysis teaches us how to renormalize a quantum field theory.

Order 1. So let us start with the first order of the log:

$$\log(\mathbb{I} - i\lambda C\sigma) =: -i\lambda C\sigma + \log_2(\mathbb{I} - i\lambda C\sigma),$$

where $\log_p(1 - x) = \sum_{k=1}^{p-1} x^k/k + \log(1 - x)$. The integrand now includes the exponential of a linear term in σ , namely $i\lambda(\text{Tr}(C\sigma) - \sum_c \delta_m^c \text{Tr}_c \sigma_c)$. Recall that $\delta_m^c = -\delta_{\mathcal{M}_1^c} + \lambda^2 \delta_{\mathcal{M}_2^c}$ (see Appendix A.1 for the explicit expressions). Let us rewrite (part of) this linear term as follows:

$$i\lambda(\text{Tr}(C\sigma) + \sum_c \delta_{\mathcal{M}_1^c} \text{Tr}_c \sigma_c) =: i\lambda \vec{A}_{\mathcal{M}_1} \cdot \vec{\sigma}, \quad (\vec{A}_{\mathcal{M}_1})_c = \text{Tr}_c C + \delta_{\mathcal{M}_1^c} \mathbb{I}_c.$$

Note that $(\vec{A}_{\mathcal{M}_1})_c$ is, up to a factor \mathbb{I}_c , the truncated renormalized amplitude of \mathcal{M}_1^c , considered here as a linear operator on \mathcal{H}_c . Therefore

$$\mathcal{Z}_N(g, J, \bar{J}) = \mathcal{N} e^{\delta_t} \int d\nu_{\mathbb{I}}(\vec{\sigma}) e^{JC^{1/2}R(\sigma)C^{1/2}\bar{J} - \text{Tr} \log_2(\mathbb{I} - \Sigma) + i\lambda \vec{A}_{\mathcal{M}_1} \cdot \vec{\sigma} - i\lambda^3 \sum_c \delta_{\mathcal{M}_2^c} \text{Tr}_c \sigma_c}.$$

The next step consists in translating the $\vec{\sigma}$ field in order to absorb the $i\lambda \vec{A}_{\mathcal{M}_1} \cdot \vec{\sigma}$ term in the preceding equation through a translation of integration contour for the diagonal part of $\vec{\sigma}$: $\vec{\sigma} \rightarrow \vec{\sigma} + \vec{B}_1$, where $\vec{B}_1 := i\lambda \vec{A}_{\mathcal{M}_1}$:

$$\mathcal{Z}_N(g, J, \bar{J}) = \mathcal{N} e^{\delta_t} \int d\nu_{\mathbb{I}}(\vec{\sigma} - \vec{B}_1) e^{JC^{1/2}R(\sigma)C^{1/2}\bar{J} - \text{Tr} \log_2(\mathbb{I} - \Sigma) - i\lambda^3 \sum_c \delta_{\mathcal{M}_2^c} \text{Tr}_c \sigma_c + \frac{1}{2}(\vec{B}_1)^2}.$$

To simplify the writing of the result of the translation, we introduce the following notations:

$$\begin{aligned} A_{\mathcal{M}_1}^r &= \sum_c (\vec{A}_{\mathcal{M}_1})_c \otimes \mathbb{I}_c \in L(\mathcal{H}^{\otimes}), & B_1 &:= i\lambda A_{\mathcal{M}_1}^r, \\ D_1 &:= i\lambda C^{1/2} B_1 C^{1/2}, & U_1 &:= \Sigma + D_1, \\ R_1(\sigma) &:= (\mathbb{I} - U_1)^{-1}. \end{aligned}$$

Remark that as $(\vec{A}_{\mathcal{M}_1})_c$ is diagonal (*i.e.* proportional to \mathbb{I}_c), the operator B_1 is diagonal too:

$$(1.6) \quad (B_1)_{mn} =: \sum_c b_1(m_c) \delta_{mn}.$$

The partition function thus rewrites as

$$\begin{aligned} \mathcal{Z}_N(g, J, \bar{J}) &= \mathcal{N}_1 \int d\nu_{\mathbb{I}}(\vec{\sigma}) e^{JC^{1/2}R_1(\sigma)C^{1/2}\bar{J} - \text{Tr} \log_2(\mathbb{I} - U_1) - i\lambda^3 \sum_c \delta_{\mathcal{M}_2^c} \text{Tr}_c \sigma_c} \\ \mathcal{N}_1 &:= \mathcal{N} e^{\delta_t} e^{\frac{1}{2}(\vec{B}_1)^2 - i\lambda^3 \sum_c \delta_{\mathcal{M}_2^c} \text{Tr}_c (\vec{B}_1)_c}, \end{aligned}$$

provided the contour translation does not cross any singularity of the integrand (which is proven in Lemma 1.5).

Order 2. We go on by pushing the Taylor expansion of the log to the next order:

$$\log_2(\mathbb{I} - U_1) = -\frac{1}{2}U_1^2 + \log_3(\mathbb{I} - U_1).$$

Using $\text{Tr}[D_1\Sigma] - i\lambda^3 \sum_c \delta_{\mathcal{M}_2^c} \text{Tr}_c \sigma_c = -i\lambda^3 \overline{A}_{\mathcal{M}_2}^{\dagger} \cdot \vec{\sigma}$, and adding and subtracting a term $\text{Tr}[D_1\Sigma^2]$ to prepare for the cancellation of the vacuum non-melonic graph in fig. 5c, we obtain

$$(1.7) \quad \mathcal{Z}_N(g, J, \bar{J}) = \mathcal{N}_1 e^{\frac{1}{2} \text{Tr} D_1^2} \int d\nu_{\mathbb{I}}(\vec{\sigma}) e^{JC^{1/2}R_1(\sigma)C^{1/2}\bar{J} - \text{Tr} \log_3(\mathbb{I} - U_1)} \\ \times e^{\frac{1}{2} \text{Tr}[\Sigma^2(\mathbb{I} + 2D_1)] - \text{Tr}[D_1\Sigma^2] - i\lambda^3 \overline{A}_{\mathcal{M}_2}^{\dagger} \cdot \vec{\sigma}}$$

where, as for \mathcal{M}_1 , $(\overline{A}_{\mathcal{M}_2}^{\dagger})_c$ is the truncated renormalized amplitude of \mathcal{M}_2^c . We now define the operator $Q \in L(L(\mathcal{H})^{\times})$ as the real symmetric operator such that

$$\lambda^2 \vec{\sigma} \cdot Q \vec{\sigma} = -\text{Tr}[\Sigma^2(\mathbb{I} + 2D_1)].$$

Using eq. (1.6),

$$(1.8) \quad (Q)_{cc'; m_c n_c, p_{c'} q_{c'}} := \delta_{cc'} \delta_{m_c p_c} \delta_{n_c q_c} \sum_{\mathbf{m}_{\hat{c}}} \frac{1}{(m_c^2 + \mathbf{m}_{\hat{c}}^2 + 1)(n_c^2 + \mathbf{m}_{\hat{c}}^2 + 1)} \\ \times \left(1 + 2i\lambda \sum_{c''} \frac{b_1(m_{c''})}{m_c^2 + \mathbf{m}_{\hat{c}}^2 + 1}\right) \\ + (1 - \delta_{cc'}) \delta_{m_c n_c} \delta_{p_{c'} q_{c'}} \sum_{\mathbf{r} \in [-N, N]^2} \frac{1}{(m_c^2 + p_{c'}^2 + \mathbf{r}^2 + 1)^2} \\ \times \left(1 + \frac{2i\lambda}{m_c^2 + p_{c'}^2 + \mathbf{r}^2 + 1} (b_1(m_c) + b_1(p_{c'}) + \sum_{c'' \neq c, c'} b_1(r_{c''}))\right).$$

It is also convenient to give a special name, Q_0 , to the leading part of Q . More precisely Q_0 is a diagonal operator, both in colour and in index space, defined by:

$$(1.9) \quad (Q_0)_{cc'; m_c n_c, p_{c'} q_{c'}} = \delta_{cc'} \sum_{\mathbf{m}_{\hat{c}}, \mathbf{p}_{\hat{c}}} C_{\mathbf{p}_{\hat{c}} \mathbf{m}_{\hat{c}}}^{q_c n_c} C_{\mathbf{m}_{\hat{c}} \mathbf{p}_{\hat{c}}}^{m_c p_c} \\ = \delta_{cc'} \delta_{m_c p_c} \delta_{n_c q_c} \sum_{\mathbf{m}_{\hat{c}}} \frac{1}{(m_c^2 + \mathbf{m}_{\hat{c}}^2 + 1)(n_c^2 + \mathbf{m}_{\hat{c}}^2 + 1)},$$

so that minus half of its trace, which is linearly divergent, is precisely canceled by the $\delta_{\mathfrak{N}_1}$ counterterm

$$-\frac{\lambda^2}{2} \text{Tr} Q_0 = -\delta_{\mathfrak{N}_1} = -\frac{\lambda^2}{2} \sum_{m_c, n_c, \mathbf{m}_{\hat{c}}} \frac{1}{(m_c^2 + \mathbf{m}_{\hat{c}}^2 + 1)(n_c^2 + \mathbf{m}_{\hat{c}}^2 + 1)}.$$

We also define $Q_1 := Q - Q_0$. Remark that in $\text{Tr} Q_1$, only the diagonal part of Q contributes, hence $\text{Tr} Q_1$ is exactly canceled by the counterterm for the graph \mathfrak{N}_3 : $-\frac{\lambda^2}{2} \text{Tr} Q_1 = -\delta_{\mathfrak{N}_3}$. Consequently

$$(1.10) \quad -\frac{\lambda^2}{2} \vec{\sigma} \cdot Q \vec{\sigma} + \delta_{\mathfrak{N}_1} + \delta_{\mathfrak{N}_3} = -\frac{\lambda^2}{2} (\vec{\sigma} \cdot Q \vec{\sigma} - \text{Tr} Q) = -\frac{\lambda^2}{2} \vec{\sigma} \cdot Q \vec{\sigma}$$

is nothing but a *Wick-ordered* quadratic interaction with respect to the Gaussian measure $d\nu_{\mathbb{I}}(\vec{\sigma})$. Therefore we can rewrite eq. (1.7) as

$$\mathcal{Z}_N(g, J, \bar{J}) = \mathcal{N}_2 \int d\nu_{\mathbb{I}}(\vec{\sigma}) e^{JC^{1/2}R_1(\sigma)C^{1/2}\bar{J} - \text{Tr} \log_3(\mathbb{I} - U_1) - \frac{\lambda^2}{2} \vec{\sigma} \cdot Q \vec{\sigma} - \text{Tr}[D_1\Sigma^2] - i\lambda^3 \overline{A}_{\mathcal{M}_2}^{\dagger} \cdot \vec{\sigma}}$$

with $\mathcal{N}_2 := \mathcal{N}_1 e^{\frac{1}{2} \text{Tr} D_1^2 - \delta_{\mathfrak{N}_1} - \delta_{\mathfrak{N}_3}}$.

The counterterm $\delta_{\mathfrak{N}_2}$ for \mathfrak{N}_2 is a bit more difficult to express in this language since it corresponds to the Wick ordering of $\frac{\lambda^4}{4}\vec{\sigma}\cdot Q_0^2\vec{\sigma}$. It is in fact a square: $\delta_{\mathfrak{N}_2} = -\frac{\lambda^4}{4}\text{Tr} Q_0^2$. We first represent it as an integral over an auxiliary tensor $\vec{\tau}$ which is also a collection of four random matrices τ_{mn}^c :

$$e^{\delta_{\mathfrak{N}_2}} = \int d\nu_{\mathbb{I}}(\vec{\tau}) e^{i\frac{\lambda^2}{\sqrt{2}}Q_0\cdot\vec{\tau}}$$

where the scalar product is taken over both colour and m, n indices *i.e.*

$$Q_0\cdot\vec{\tau} := \sum_{c,m,n} (Q_0)_{cc;mn,mn}\tau_{mn}^c.$$

Then

$$\begin{aligned} \mathcal{Z}_N(g, J, \bar{J}) &= \mathcal{N}_3 \int d\nu_{\mathbb{I}}(\vec{\sigma}, \vec{\tau}) e^{JC^{1/2}R_1(\sigma)C^{1/2}\bar{J} - \text{Tr} \log_3(\mathbb{I} - U_1)} \\ &\quad \times e^{-\frac{\lambda^2}{2}:\vec{\sigma}\cdot Q\vec{\sigma}: + i\frac{\lambda^2}{\sqrt{2}}Q_0\cdot\vec{\tau} - \text{Tr}[D_1\Sigma^2] - i\lambda^3\bar{A}_{\mathcal{M}_2}^r\cdot\vec{\sigma}} \end{aligned}$$

with $\mathcal{N}_3 := \mathcal{N}_2 e^{-\delta_{\mathfrak{N}_2}}$ and $d\nu_{\mathbb{I}}(\vec{\sigma}, \vec{\tau}) := d\nu_{\mathbb{I}}(\vec{\sigma}) \otimes d\nu_{\mathbb{I}}(\vec{\tau})$. The next step of the rewriting of the σ -action consists in one more translation of the $\vec{\sigma}$ field: $\bar{B}_2 := -i\lambda^3\bar{A}_{\mathcal{M}_2}^r$,

$$\begin{aligned} \mathcal{Z}_N(g, J, \bar{J}) &= \mathcal{N}_3 e^{\frac{1}{2}\bar{B}_2\cdot\bar{B}_2} \int d\nu_{\mathbb{I}}(\vec{\sigma} - \bar{B}_2, \vec{\tau}) e^{JC^{1/2}R_1(\sigma)C^{1/2}\bar{J} - \text{Tr} \log_3(\mathbb{I} - U_1)} \\ &\quad \times e^{-\frac{\lambda^2}{2}:\vec{\sigma}\cdot Q\vec{\sigma}: + i\frac{\lambda^2}{\sqrt{2}}Q_0\cdot\vec{\tau} - \text{Tr}[D_1\Sigma^2]}. \end{aligned}$$

Finally we introduce the following notations:

$$(1.11) \quad \begin{aligned} A_{\mathcal{M}_2}^r &:= \sum_c (\bar{A}_{\mathcal{M}_2}^r)_c \otimes \mathbb{I}_c, & B_2 &:= -i\lambda^3 A_{\mathcal{M}_2}^r, \\ D_2 &:= i\lambda C^{1/2} B_2 C^{1/2}, & U &:= \Sigma + D_1 + D_2 =: \Sigma + D, \\ \mathcal{R}(\sigma) &:= (\mathbb{I} - U)^{-1}, & \tilde{V}^{\geq 3}(\sigma) &:= \text{Tr} \log_3(\mathbb{I} - U). \end{aligned}$$

Remark indeed that $-\text{Tr} \log_3(\mathbb{I} - U)$ expands as $\sum_{q \geq 3} \text{Tr} \frac{U^q}{q}$, which can be interpreted as a sum over cycles (also called *loop vertices*) of length at least three with σ or D insertions. We get

$$\begin{aligned} \mathcal{Z}_N(g, J, \bar{J}) &= \mathcal{N}_4 \int d\nu_{\mathbb{I}}(\vec{\sigma}, \vec{\tau}) e^{JC^{1/2}\mathcal{R}(\sigma)C^{1/2}\bar{J} - \tilde{V}^{\geq 3}(\sigma) - V^{\leq 2}(\sigma, \tau) - \text{Tr}[D_1\Sigma^2]} \\ V^{\leq 2}(\sigma, \tau) &:= \frac{\lambda^2}{2}:\vec{\sigma}\cdot Q\vec{\sigma}: - i\frac{\lambda^2}{\sqrt{2}}Q_0\cdot\vec{\tau} - \text{Tr}[D_2\Sigma] \\ \mathcal{N}_4 &:= \mathcal{N}_3 e^{\frac{1}{2}\bar{B}_2\cdot\bar{B}_2 + \frac{1}{2}\text{Tr}[D_2^2]} \end{aligned}$$

provided the contour translation does not cross any singularity of the integrand, see again Lemma 1.5.

Returning to figs. 3 to 5, we see that Feynman graphs made out solely of loop vertices of length at least three are all convergent at the perturbative level, except the last three of the seven divergent vacuum melonic graphs in fig. 4, which correspond respectively to a loop vertex of length three with three \mathcal{M}_1 insertions, a loop vertex of length three with two \mathcal{M}_1 insertions and one \mathcal{M}_2 insertion, and a loop vertex of length 4 with four \mathcal{M}_1 insertions. The three missing terms corresponding to the three remaining divergent vacuum graphs are $\mathcal{E} := \text{Tr}(\frac{1}{3}D_1^3 + D_1^2 D_2 + \frac{1}{4}D_1^4)$. Once again, we add and subtract those missing terms from the action. Thus, defining $V^{\geq 3}(\sigma) := \tilde{V}^{\geq 3}(\sigma) + \text{Tr}[D_1\Sigma^2] + \mathcal{E}$, we get

$$\begin{aligned} \mathcal{Z}_N(g, J, \bar{J}) &= \mathcal{N}_5 \int d\nu_{\mathbb{I}}(\vec{\sigma}, \vec{\tau}) e^{JC^{1/2}\mathcal{R}(\sigma)C^{1/2}\bar{J} - V^{\geq 3}(\sigma) - V^{\leq 2}(\sigma, \tau)}, \\ \mathcal{N}_5 &:= \mathcal{N}_4 e^{\mathcal{E}}. \end{aligned}$$

Lemma 1.2. — $\mathcal{Z}^{(0)}(g) := \log \mathcal{N}_5 = 0$.

The proof is given in Appendix A.3.

The goal is therefore from now on to build the $N \rightarrow \infty$ limit of

$$\mathcal{W}_N(g, J, \bar{J}) = \log \mathcal{Z}_N(g, J, \bar{J})$$

and to prove that it is the Borel sum of its (well-defined and ultraviolet finite) perturbative expansion in $g = \lambda^2$. In fact, like in [DR16], we shall only prove the convergence theorem for the pressure

$$(1.12) \quad \mathcal{W}_N(g) := \mathcal{W}_N(g, J, \bar{J})|_{J=\bar{J}=0} = \log \mathcal{Z}_N(g), \quad \mathcal{Z}_N(g) := \int d\nu_{\mathbb{1}}(\vec{\sigma}, \vec{\tau}) e^{-V},$$

where the intermediate field interaction V is

$$V := V^{\geq 3}(\sigma) + V^{\leq 2}(\sigma, \tau),$$

since adding the external sources leads to inessential technicalities that may obscure the essential constructive argument, namely the perturbative and non perturbative bounds of Sections 5 and 6.

1.3. Justifying contour translations. — In this subsection we prove that the successive translations performed in the previous subsection did not cross singularities of the integrand. This will lead us to introduce some basic uniform bounds on D and \mathcal{R} when g varies in the small open cardioid domain Card_ρ defined by $|g| < \rho \cos^2(\frac{1}{2} \arg g)$ (see fig. 6).

Lemma 1.1 easily implies

Lemma 1.3 (D, D_1, D_2 estimates). — D, D_1 and D_2 are compact operators on \mathcal{H}^\otimes , diagonal in the momentum basis, with

$$\sup(|(D)_{mn}|, |(D_1)_{mn}|) \leq \frac{O(1)|g|}{1 + \|\mathbf{n}\|} \delta_{mn}, \quad |(D_2)_{mn}| \leq \frac{O(1)|g|^2[1 + \log(1 + \|\mathbf{n}\|)]}{1 + \|\mathbf{n}\|^2} \delta_{mn}.$$

Lemma 1.4 (Resolvent bound). — For g in the small open cardioid domain Card_ρ , the translated resolvent $\mathcal{R} = (\mathbb{I} - U)^{-1}$ is well defined and uniformly bounded:

$$\|\mathcal{R}\| \leq 2 \cos^{-1}(\frac{1}{2} \arg g).$$

Proof. — In the cardioid domain we have $|\arg g| < \pi$. For any self-adjoint operator L , by the spectral mapping theorem [RS80, Theorem VII.1], we have

$$(1.13) \quad \|(\mathbb{I} - i\sqrt{g}L)^{-1}\| \leq \cos^{-1}(\frac{1}{2} \arg g).$$

Applying to $L = H$, remembering that $\lambda = \sqrt{g}$, the lemma follows from the power series expansion

$$\|(\mathbb{I} - U)^{-1}\| = \|(\mathbb{I} - i\lambda H - D)^{-1}\| \leq \|J^{-1}\| \sum_{q=0}^{\infty} \|DJ^{-1}\|^q,$$

with $J := \mathbb{I} - i\lambda H$. Indeed by eq. (1.13), $\|J^{-1}\| \leq \cos^{-1}(\frac{1}{2} \arg g)$, and, by Lemma 1.3,

$$\|DJ^{-1}\| \leq O(1)|g| \cos^{-1}(\frac{1}{2} \arg g) \leq O(1)\rho.$$

Taking ρ small enough, we can ensure $\|DJ^{-1}\| < 1/2$, hence $\sum_{q=0}^{\infty} \|DJ^{-1}\|^q < 2$. \square

Lemma 1.5 (Contour translation). — For g in the cardioid domain Card_ρ , the contour translation from $(\sigma_c)_{n_c n_c}$ to $(\sigma_c)_{n_c n_c} + B_1$ does not cross any singularity of $\text{Tr} \log_2(\mathbb{I} - i\lambda C^{1/2} \sigma C^{1/2})$, and the translation $(\sigma_c)_{n_c n_c} + B_2$ does not cross any singularity of $\text{Tr} \log_3(\mathbb{I} - i\lambda C^{1/2} \sigma C^{1/2} + D_1)$.

Proof. — To prove that $\text{Tr} \log_2(\mathbb{I} - i\lambda C^{1/2} \sigma C^{1/2})$ is analytic in the band corresponding to $(\sigma_c)_{n_c n_c} + B_1$ for the $(\sigma_c)_{n_c n_c}$ variables, one can write

$$\log_2(1-x) = x - \int_0^1 \frac{x}{1-tx} dt = - \int_0^1 \frac{tx^2}{1-tx} dt$$

and then use the previous Lemma to prove that, for g in the small open cardioid domain Card_ρ , the resolvent $R(t) := (\mathbb{I} - it\lambda C^{1/2} \sigma C^{1/2})^{-1}$, is also well-defined for any $t \in [0, 1]$ by a power series uniformly convergent in the band considered.

For the second translation, we use a similar argument, writing □

$$\log_3(1-x) = x + \frac{x^2}{2} - \int_0^1 \frac{x}{1-tx} dt = \int_0^1 \frac{x^2(1-2t-tx)}{2(1-tx)} dt.$$

1.4. Multiscale analysis. — The cutoff $[-N, N]^4$ of the previous section is not well adapted to the rotation invariant \mathbf{n}^2 term in the propagator, nor very convenient for multi-slice analysis as in [GR14]. From now on we introduce other cutoffs, which are still sharp in the “momentum space” $\ell_2(\mathbb{Z})^4$, hence equivalent⁴ to the previous ones, but do not longer factorize over colours⁵.

We fix an integer $M > 1$ as ratio of a geometric progression M^j , where $j \in \mathbb{N}^*$ is the slice index and define the ultraviolet cutoff as a maximal slice index j_{\max} so that the previous N roughly corresponds to $M^{j_{\max}}$. More precisely, our notation convention is that $\mathbf{1}_x$ is the characteristic function of the event x , and we define the following diagonal operators on \mathcal{H}^{\otimes} :

$$\begin{aligned} (\mathbf{1}_{\leq 1})_{mn} &= (\mathbf{1}_1)_{mn} := \mathbf{1}_{1+\|n\|^2 \leq M^2} \delta_{mn}, \\ (\mathbf{1}_{\leq j})_{mn} &:= \mathbf{1}_{1+\|n\|^2 \leq M^{2j}} \delta_{mn} && \text{for } j \geq 2, \\ \mathbf{1}_j &:= \mathbf{1}_{\leq j} - \mathbf{1}_{\leq j-1} && \text{for } j \geq 2. \end{aligned}$$

(Beware we choose the convention of *lower* indices for slices, as in [GR14], not upper indices as in [Riv91].) We also write $C_{\leq j}^{1/2}$ for $\mathbf{1}_{\leq j} C^{1/2}$ and $C_j^{1/2}$ for $\mathbf{1}_j C^{1/2}$. Since our cutoffs are sharp (projectors) we still have the natural relations

$$(C_{\leq j}^{1/2})^2 = C_{\leq j}, \quad (C_j^{1/2})^2 = C_j.$$

We start with the (σ, τ) functional integral (1.12) which we have reached in the previous section, and organize it according to the new cutoffs, so that the previous limit $N \rightarrow \infty$ becomes a limit $j_{\max} \rightarrow \infty$. The interaction with cutoff j is obtained by cutting the propagators in the loop vertices. Remark that we do not need to introduce cutoffs on the propagators hidden in $A_{\mathcal{M}_1}^x$ or $A_{\mathcal{M}_2}^x$, as these are convergent integrals anyway. It means we define the cutoff version of the quantities introduced in the previous subsection as

$$(1.14a) \quad V_{\leq j}(\sigma, \tau) := V_{\leq j}^{\geq 3}(\sigma) + V_{\leq j}^{\leq 2}(\sigma, \tau),$$

$$(1.14b) \quad V_{\leq j}^{\geq 3}(\sigma) := \text{Tr} \log_3(\mathbb{I} - U_{\leq j}) + \text{Tr}[D_{1, \leq j} \Sigma_{\leq j}^2] + \mathcal{E}_{\leq j},$$

$$(1.14c) \quad \mathcal{E}_{\leq j} := \text{Tr}(\frac{1}{3} D_{1, \leq j}^3 + D_{1, \leq j}^2 D_{2, \leq j} + \frac{1}{4} D_{1, \leq j}^4),$$

$$(1.14d) \quad V_{\leq j}^{\leq 2} := \frac{\lambda^2}{2} : \vec{\sigma} \cdot Q_{\leq j} \vec{\sigma} : - i \frac{\lambda^2}{\sqrt{2}} Q_{0, \leq j} \cdot \vec{\tau} - \text{Tr}[D_{2, \leq j} \Sigma_{\leq j}],$$

$$(1.14e) \quad Q_{\leq j} = Q_{0, \leq j} + Q_{1, \leq j},$$

$$(1.14f) \quad \mathcal{R}_{\leq j} := \frac{1}{\mathbb{I} - U_{\leq j}}, \quad U_{\leq j} := \Sigma_{\leq j} + D_{\leq j}, \quad \Sigma_{\leq j} := i\lambda C_{\leq j}^{1/2} \sigma C_{\leq j}^{1/2},$$

$$(1.14g) \quad D_{1, \leq j} := i\lambda C_{\leq j}^{1/2} B_1 C_{\leq j}^{1/2}, \quad D_{2, \leq j} := i\lambda C_{\leq j}^{1/2} B_2 C_{\leq j}^{1/2}, \quad D_{\leq j} := D_{1, \leq j} + D_{2, \leq j}.$$

The functional integral (1.12) with cutoff j_{\max} is then defined as

$$\mathcal{W}_{\leq j_{\max}}(g) := \log \mathcal{Z}_{\leq j_{\max}}(g), \quad \mathcal{Z}_{\leq j_{\max}}(g) := \int d\nu_{\mathbb{I}}(\vec{\sigma}, \tau) e^{-V_{\leq j_{\max}}}.$$

⁴The sup and square norm in our finite dimension four are equivalent.

⁵We could also use parametric cutoffs as in [Riv91; BGR12a], but sharp cutoffs are simpler.

Defining $V_{\leq 0} := 0$ and, for all $1 \leq j \leq j_{\max}$, $V_j := V_{\leq j} - V_{\leq j-1}$, we note that $V_{\leq j_{\max}} = \sum_{j=1}^{j_{\max}} V_j$ so that

$$(1.15) \quad \mathcal{Z}_{\leq j_{\max}}(g) = \int d\nu_{\mathbb{I}}(\vec{\sigma}, \vec{\tau}) \prod_{j=1}^{j_{\max}} e^{-V_j}.$$

To define the specific part of the interaction which should be attributed to scale j we introduce

$$\mathbf{1}_{\leq j}(t_j) = \mathbf{1}_{\leq j-1} + t_j \mathbf{1}_j$$

where $t_j \in [0, 1]$ is an interpolation parameter for the j -th scale. Remark that

$$\mathbf{1}_{\leq j}^2(t_j) = \mathbf{1}_{\leq j-1} + t_j^2 \mathbf{1}_j.$$

The interpolated interaction and resolvents are defined as $V_{\leq j}(t_j)$, $\Sigma_{\leq j}(t_j)$, $D_{\leq j}(t_j)$, $\mathcal{R}_{\leq j}(t_j)$ and so on by eqs. (1.14a) to (1.14g) in which we substitute $\mathbf{1}_{\leq j}(t_j)$ for $\mathbf{1}_{\leq j}$. When the context is clear, we write simply $V_{\leq j}$ for $V_{\leq j}(t_j)$, $U_{\leq j}$ for $U_{\leq j}(t_j)$, U'_j for $\frac{d}{dt_j} U_{\leq j}$ and so on. In these notations we have

$$(1.16) \quad \begin{cases} V_j = V_j^{\geq 3} + V_j^{\leq 2}, \\ V_j^{\geq 3} = \mathcal{E}_j + \int_0^1 dt_j \operatorname{Tr}[U'_j(\mathbb{I} + U_{\leq j} - \mathcal{R}_{\leq j}) + D'_{1,j} \Sigma^2 + D_{1,\leq j} \Sigma'_j \Sigma + D_{1,\leq j} \Sigma \Sigma'_j], \\ V_j^{\leq 2} = \frac{\lambda^2}{2} : \vec{\sigma} \cdot (Q_{0,j} + Q_{1,j}) \vec{\sigma} : - i \frac{\lambda^2}{\sqrt{2}} Q_{0,j} \cdot \vec{\tau} - 3 \int_0^1 dt_j \operatorname{Tr}[D'_{2,j} \Sigma_{\leq j}], \\ \mathcal{E}_j = \mathcal{E}_{\leq j} - \mathcal{E}_{\leq j-1}, \quad Q_{1,j} = Q_{1,\leq j} - Q_{1,\leq j-1}, \quad Q_{0,j} = Q_{0,\leq j} - Q_{0,\leq j-1}. \end{cases}$$

Finally, as in [GR14], we define

$$W_j(\sigma, \tau) := e^{-V_j} - 1$$

and encode the factorization of the interaction in (1.15) through Grassmann numbers as

$$\mathcal{Z}_{\leq j_{\max}}(g) = \int d\nu_{\mathbb{I}}(\vec{\sigma}, \vec{\tau}) \left(\prod_{j=1}^{j_{\max}} d\mu(\bar{\chi}_j, \chi_j) \right) e^{-\sum_{j=1}^{j_{\max}} \bar{\chi}_j W_j(\sigma, \tau) \chi_j},$$

where $d\mu(\bar{\chi}, \chi) = d\bar{\chi} d\chi e^{-\bar{\chi}\chi}$ is the standard normalized Grassmann Gaussian measure with covariance 1.

2. The Multiscale Loop Vertex Expansion

We perform now the two-level jungle expansion of [AR95; GR14; DR16]. This section is almost identical to those of [GR14; DR16], as it was precisely the goal of [GR14] to create a combinatorial constructive “black box” to automatically compute and control the logarithm of a functional integral of the type of \mathcal{Z}_N . Nevertheless we reproduce the section here, in abridged form, since the MLVE technique is still relatively recent and since there is a slight change compared to the standard version. Indeed we have now two sets of Bosonic fields, the main σ field and the auxiliary τ field, and the τ field requires slightly different interpolation parameters, namely w^2 instead of w parameters.

Considering the set of scales $S := \llbracket 1, j_{\max} \rrbracket$, we denote \mathbb{I}_S the $|S|$ by $|S|$ identity matrix. The product Gaussian measure on the χ_i 's and $\bar{\chi}_i$'s can then be recast into the following form:

$$\prod_{j=1}^{j_{\max}} d\mu(\bar{\chi}_j, \chi_j) = d\mu_{\mathbb{I}_S}(\bar{\boldsymbol{\chi}}, \boldsymbol{\chi}), \quad \boldsymbol{\chi} := (\chi_i)_{1 \leq i \leq j_{\max}}, \quad \bar{\boldsymbol{\chi}} := (\bar{\chi}_i)_{1 \leq i \leq j_{\max}}$$

so that the partition function rewrites as

$$\mathcal{Z}_{\leq j_{\max}}(g) = \int d\nu_S e^{-W}, \quad d\nu_S := d\nu_{\mathbb{I}}(\vec{\sigma}, \vec{\tau}) d\mu_{\mathbb{I}_S}(\bar{\boldsymbol{\chi}}, \boldsymbol{\chi}), \quad W = \sum_{j=1}^{j_{\max}} \bar{\chi}_j W_j(\vec{\sigma}, \vec{\tau}) \chi_j.$$

The first step expands to infinity the exponential of the interaction:

$$\mathcal{Z}_{\leq j_{\max}}(g) = \sum_{n=0}^{\infty} \frac{1}{n!} \int d\nu_S (-W)^n.$$

The second step introduces Bosonic replicas for all the *nodes*⁶ in $[n] := \llbracket 1, n \rrbracket$:

$$\mathcal{Z}_{\leq j_{\max}}(g) = \sum_{n=0}^{\infty} \frac{1}{n!} \int d\nu_{S,[n]} \prod_{a=1}^n (-W_a),$$

so that each node $W_a = \sum_{j=0}^{j_{\max}} \bar{\chi}_j^a W_j(\bar{\sigma}^a, \bar{\tau}^a) \chi_j^a$ has now its own set of Bosonic matrix fields $\bar{\sigma}^a = ((\sigma^1)^a, (\sigma^2)^a, (\sigma^3)^a, (\sigma^4)^a)$ and $\bar{\tau}^a = ((\tau^1)^a, (\tau^2)^a, (\tau^3)^a, (\tau^4)^a)$, and its own Fermionic replicas $(\bar{\chi}_j^a, \chi_j^a)$. The sequence of Bosonic replicas $(\bar{\sigma}^a, \bar{\tau}^a)_{a \in [n]}$ will be denoted by $(\bar{\sigma}; \bar{\tau})$ and belongs to the product space for the σ and τ fields (which is also a direct sum)

$$\tilde{V}_{[n]} := [L(\mathcal{H})^\times \otimes \mathbb{R}^n] \times [L(\mathcal{H})^\times \otimes \mathbb{R}^n] = [L(\mathcal{H})^\times \oplus L(\mathcal{H})^\times] \otimes \mathbb{R}^n.$$

The replicated *normalised* measure is completely degenerate between replicas (each of the four colours remaining independent of the others):

$$d\nu_{S,[n]} := d\nu_{\mathbb{1} \otimes \mathbb{1}_{[n]}}(\bar{\sigma}, \bar{\tau}) d\mu_{\mathbb{1}_S \otimes \mathbb{1}_{[n]}}(\bar{\chi}, \chi)$$

where $\mathbb{1}$ means the “full” matrix with all entries equal to 1.

The obstacle to factorize the functional integral \mathcal{Z} over nodes and to compute $\log \mathcal{Z}$ lies in the degenerate blocks $\mathbb{1}_{[n]}$ of both the Bosonic and Fermionic covariances. In order to remove this obstacle we simply apply the 2-level jungle Taylor formula of [AR95] with priority to Bosonic links over Fermionic links. However beware that since the τ field counts for two σ fields, we have to introduce the parameters w differently in σ and τ namely we interpolate off-diagonal covariances between vertices a and $b \neq a$ with ordinary parameters w for the σ covariance but with parameters w^2 for the τ covariance. Indeed with this precise prescription a sigma tree link (a, b) of type $\bar{\sigma} \cdot Q_{0,j_a} Q_{0,j_b} \bar{\sigma}$ term will be exactly Wick-ordered with respect to the interpolated $d\nu(\bar{\sigma})$ measure by the associated tau link $\ell = (a, b)$, see Section 3. In other words the \mathfrak{N}_2 graph when it occurs as such a link, is exactly renormalized.

It means that a first Taylor forest formula is applied to $\mathbb{1}_{[n]}$ in $d\nu_{\mathbb{1} \otimes \mathbb{1}_{[n]}}(\bar{\sigma}, \bar{\tau})$, with weakening parameters w for the σ covariance and parameters w^2 for the τ covariance. The forest formula simply interpolates iteratively off-diagonal covariances between 0 and 1. The prescription described is legitimate since when w monotonically parametrizes the $[0, 1]$ interval, w^2 also parametrizes the $[0, 1]$ interval monotonically; hence a Taylor formula can be written just as well as $F(1) = F(0) + \int_0^1 F'(x) dx$ or as $F(1) = F(0) + \int_0^1 2xF'(x^2) dx$.

It is then followed by a second Taylor forest formula of $\mathbb{1}_{[n]}$ in $d\mu_{\mathbb{1}_S \otimes \mathbb{1}_{[n]}}(\bar{\chi}, \chi)$, decoupling the connected components \mathcal{B} of the first forest.

The definition of m -level jungle formulas and their equivalence to m successive forests formulas is given in [AR95]; the application (with $m = 2$) to the current context is described in detail in [GR14; DR16], so we shall not repeat it here.

The 2-jungle Taylor formula rewrites our partition function as:

$$(2.1) \quad \mathcal{Z}_{\leq j_{\max}}(g) = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{\mathcal{J}} \sum_{j_1=1}^{j_{\max}} \cdots \sum_{j_n=1}^{j_{\max}} \int d\mathbf{w}_{\mathcal{J}} \int d\nu_{\mathcal{J}} \partial_{\mathcal{J}} \left[\prod_{\mathcal{B}} \prod_{a \in \mathcal{B}} (-\bar{\chi}_{j_a}^{\mathcal{B}} W_{j_a}(\bar{\sigma}^a, \bar{\tau}^a) \chi_{j_a}^{\mathcal{B}}) \right],$$

where

- the sum over \mathcal{J} runs over all 2-level jungles, hence over all ordered pairs $\mathcal{J} = (\mathcal{F}_B, \mathcal{F}_F)$ of two (each possibly empty) disjoint forests on $[n]$, such that \mathcal{F}_B is a (Bosonic) forest, \mathcal{F}_F is a (Fermionic) forest

⁶We use the new word “node” rather than “vertex” for the W factors, in order not to confuse them with the ordinary vertices of the initial perturbative expansion, nor with the loop vertices of the intermediate field expansion, which are not equipped with Fermionic fields.

and $\overline{\mathcal{J}} = \mathcal{F}_B \cup \mathcal{F}_F$ is still a forest on $[n]$. The forests \mathcal{F}_B and \mathcal{F}_F are the Bosonic and Fermionic components of \mathcal{J} . Fermionic edges $\ell_F \in E(\mathcal{F}_F)$ carry a scale data j .

- $\int d\mathbf{w}_{\mathcal{J}}$ means integration from 0 to 1 over parameters w_{ℓ} , one for each edge $\ell \in E(\overline{\mathcal{J}})$, namely $\int d\mathbf{w}_{\mathcal{J}} = \prod_{\ell \in E(\overline{\mathcal{J}})} \int_0^1 dw_{\ell}$. There is no integration for the empty forest since by convention an empty product is 1. A generic integration point $\mathbf{w}_{\mathcal{J}}$ is therefore made of $m(\overline{\mathcal{J}})$ parameters $w_{\ell} \in [0, 1]$, one for each $\ell \in E(\overline{\mathcal{J}})$.
- In any $\mathcal{J} = (\mathcal{F}_B, \mathcal{F}_F)$, each block \mathcal{B} corresponds to a tree $\mathcal{T}_{\mathcal{B}}$ of \mathcal{F}_B .

$$(2.2a) \quad \partial_{\mathcal{J}} := \partial_F \partial_B, \quad \partial_B := \prod_{\mathcal{B} \in \mathcal{F}_B} \partial_{\mathcal{T}_{\mathcal{B}}},$$

$$(2.2b) \quad \partial_F := \prod_{\substack{\ell_F \in E(\mathcal{F}_F), \\ \ell_F = (d,e)}} \delta_{j_d j_e} \left(\frac{\partial}{\partial \chi_{j_d}^{\mathcal{B}(d)}} \frac{\partial}{\partial \chi_{j_e}^{\mathcal{B}(e)}} + \frac{\partial}{\partial \chi_{j_e}^{\mathcal{B}(e)}} \frac{\partial}{\partial \chi_{j_d}^{\mathcal{B}(d)}} \right),$$

$$(2.2c) \quad \partial_{\mathcal{T}_{\mathcal{B}}} := \prod_{\substack{\ell_B \in E(\mathcal{T}_{\mathcal{B}}), \\ \ell_B = (a,b)}} \left[\sum_{c=1}^4 \sum_{m,n} \left(\frac{\partial}{\partial (\sigma_{mn}^c)^a} \frac{\partial}{\partial (\sigma_{mn}^c)^b} + 2w_{\ell} \frac{\partial}{\partial (\tau_{mn}^c)^a} \frac{\partial}{\partial (\tau_{mn}^c)^b} \right) \right]$$

where $\mathcal{B}(d)$ denotes the Bosonic block to which the node d belongs. Remark the factor $2w_{\ell}$ in (2.2c) corresponding to the use of w^2 parameters for τ .

- The measure $d\nu_{\mathcal{J}}$ has covariance $\mathbb{I} \otimes X(\mathbf{w}_B)$ on Bosonic variables σ , covariance $\mathbb{I} \otimes X^{\circ 2}(\mathbf{w}_B)$ on Bosonic variables τ and $\mathbb{I}_S \otimes Y(\mathbf{w}_F)$ on Fermionic variables, hence

$$\int d\nu_{\mathcal{J}} F = \left[e^{\frac{1}{2} \sum_{a,b=1}^n \sum_{c=1}^4 \sum_{m,n} (X_{ab}(\mathbf{w}_B) \frac{\partial}{\partial (\sigma_{mn}^c)^a} \frac{\partial}{\partial (\sigma_{mn}^c)^b} + X_{ab}^{\circ 2}(\mathbf{w}_B) \frac{\partial}{\partial (\tau_{mn}^c)^a} \frac{\partial}{\partial (\tau_{mn}^c)^b})} \sum_{\mathcal{B}, \mathcal{B}'} Y_{\mathcal{B}\mathcal{B}'}(\mathbf{w}_F) \sum_{a \in \mathcal{B}, b \in \mathcal{B}'} \delta_{j_a j_b} \frac{\partial}{\partial \chi_{j_a}^{\mathcal{B}}} \frac{\partial}{\partial \chi_{j_b}^{\mathcal{B}'}} F \right]_{\sigma=\tau=\overline{\chi}=\chi=0}$$

where $X^{\circ 2}$ means the Hadamard square of the matrix, hence the matrix whose elements are the squares of the matrix elements of X , *not* the square in the ordinary matrix product sense.

- $X_{ab}(\mathbf{w}_B)$ is the infimum of the w_{ℓ_B} parameters for all the Bosonic edges ℓ_B in the unique path $P_{a \rightarrow b}^{\mathcal{F}_B}$ from node a to node b in \mathcal{F}_B . The infimum is set to zero if such a path does not exist and to 1 if $a = b$.
- $Y_{\mathcal{B}\mathcal{B}'}(\mathbf{w}_F)$ is the infimum of the w_{ℓ_F} parameters for all the Fermionic edges ℓ_F in any of the paths $P_{a \rightarrow b}^{\mathcal{F}_B \cup \mathcal{F}_F}$ from some node $a \in \mathcal{B}$ to some node $b \in \mathcal{B}'$. The infimum is set to 0 if there are no such paths, and to 1 if $\mathcal{B} = \mathcal{B}'$ (i.e. if such paths exist but do not contain any Fermionic edges).

Remember that a main property of the forest formula is that the symmetric n by n matrices $X_{ab}(\mathbf{w}_B)$ or $X_{ab}^{\circ 2}(\mathbf{w}_B)$ are positive for any value of $\mathbf{w}_{\mathcal{J}}$, hence the Gaussian measure $d\nu_{\mathcal{J}}$ is well-defined. The matrix $Y_{\mathcal{B}\mathcal{B}'}(\mathbf{w}_F)$ is also positive.

Since the slice assignments, the fields, the measure and the integrand are now factorized over the connected components of $\overline{\mathcal{J}}$, the logarithm of \mathcal{Z} is easily computed as exactly the same sum but restricted to 2-level spanning trees:

$$(2.3) \quad \mathcal{W}_{\leq j_{\max}}(g) = \log \mathcal{Z}_{\leq j_{\max}}(g) = \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{\mathcal{J} \text{ tree}} \sum_{j_1=1}^{j_{\max}} \cdots \sum_{j_n=1}^{j_{\max}} \int d\mathbf{w}_{\mathcal{J}} \int d\nu_{\mathcal{J}} \partial_{\mathcal{J}} \left[\prod_{\mathcal{B}} \prod_{a \in \mathcal{B}} (-\overline{\chi}_{j_a}^{\mathcal{B}} W_{j_a}(\vec{\sigma}^a, \vec{\tau}^a) \chi_{j_a}^{\mathcal{B}}) \right]$$

where the sum is the same but conditioned on $\overline{\mathcal{J}} = \mathcal{F}_B \cup \mathcal{F}_F$ being a *spanning tree* on $[n]$.

Our main result is similar to the one of [DR16] in the more convergent three dimensional case:

Theorem 2.1. — Fix $\rho > 0$ small enough. The series (2.3) is absolutely and uniformly in j_{max} convergent for g in the small open cardioid domain $Card_\rho$ (defined by $|\arg g| < \pi$ and $|g| < \rho \cos^2(\frac{1}{2} \arg g)$, see fig. 6). Its ultraviolet limit $\mathcal{W}_\infty(g) := \lim_{j_{max} \rightarrow \infty} \log \mathcal{Z}_{\leq j_{max}}(g)$ is therefore well-defined and analytic in that cardioid domain; furthermore it is the Borel sum of its perturbative series in powers of g .

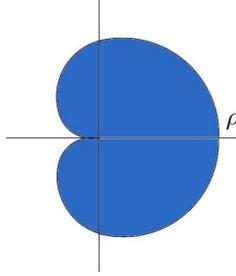


Figure 6. A Cardioid Domain

The rest of the paper is devoted to the proof of this Theorem.

3. Block Bosonic integrals

Since the Bosonic functional integral factorizes over the Bosonic blocks, it is sufficient to compute and bound the Bosonic functional integrals over a fixed block \mathcal{B} .

3.1. The single node case. — Let us consider first the simple case in which the Bosonic block \mathcal{B} is reduced to a single node a . We have then a relatively simple contribution

$$\int d\nu_{\mathbb{I}}(\vec{\sigma}^a, \vec{\tau}^a) W_{j_a} = \int d\nu_{\mathbb{I}}(e^{-V_{j_a}} - 1) = \int_0^1 dt \int d\nu_{\mathbb{I}} e^{-tV_{j_a}}(-V_{j_a}).$$

We consider in the term $-V_{j_a}$ down from the exponential two particular pieces of $V_{j_a}^{\leq 2}$, namely the terms $-\frac{\lambda^2}{2} \vec{\sigma} \cdot Q_{0,j_a} \vec{\sigma}$: and $i \frac{\lambda^2}{\sqrt{2}} Q_{0,j_a} \cdot \vec{\tau}$. In the first one, we integrate by parts one of its two σ fields, obtaining $t \frac{\lambda^4}{2} \vec{\sigma} \cdot Q_{0,j_a}^2 \vec{\sigma}$ plus (perturbatively convergent) terms

$$PC_{j_a}(\vec{\sigma}) = t \frac{\lambda^2}{2} \vec{\sigma} \cdot Q_{0,j_a} \frac{\partial}{\partial \vec{\sigma}} \left(\frac{\lambda^2}{2} \vec{\sigma} \cdot Q_{1,j_a} \vec{\sigma} + 3 \int_0^1 dt_j \text{Tr}[D'_{2,j} \Sigma_{\leq j}] + V_{j_a}^{\geq 3}(\sigma) \right).$$

We also integrate by parts the τ term in $V_{j_a}^{\leq 2}$ and remark that it gives $-t \frac{\lambda^4}{2} \text{Tr}[Q_{0,j_a}^2]$, hence exactly Wick-orders the previous $\vec{\sigma} \cdot Q_{0,j_a}^2 \vec{\sigma}$ term. Finally we integrate out the τ field, which gives back the $t^2 \delta_{\mathfrak{N}_{2,a}}$ counterterm. Hence altogether we have proven:

Lemma 3.1. — The result of this computation is

$$\int d\nu_{\mathbb{I}}(\vec{\sigma}, \vec{\tau}) W_{j_a}(\vec{\sigma}, \vec{\tau}) = - \int_0^1 dt e^{t^2 \delta_{\mathfrak{N}_{2,a}}} \int d\nu_{\mathbb{I}}(\vec{\sigma}) e^{-tV_{j_a}(\vec{\sigma})} ::V_{j_a}(\vec{\sigma})::$$

where

$$::V_{j_a}(\vec{\sigma}):: := V_{j_a}^{\geq 3}(\sigma) - \frac{t\lambda^4}{2} \vec{\sigma} \cdot Q_{0,j_a}^2 \vec{\sigma} - PC_{j_a}(\sigma) + \frac{\lambda^2}{2} \vec{\sigma} \cdot Q_{1,j} \vec{\sigma} - 3 \int_0^1 dt_j \text{Tr}[D'_{2,j} \Sigma_{\leq j}].$$

This Lemma will be sufficient to bound the single node contribution by $O(1)M^{-O(1)j_a}$, see next sections.

In order to treat the single node case and the cases of Bosonic blocks with more than one node in a unified manner, it is convenient to regard $::V_{j_a}(\vec{\sigma})::$ as a sum of (Wick-ordered) skeleton graph amplitudes, see Definition 2. These Feynman graphs are one-vertex maps except those which correspond to the terms in $PC_{j_a}(\sigma)$ which are trees with only one edge. Therefore we will write

$$::V_{j_a}(\vec{\sigma}):: =: \sum_{\mathbf{G}} A_{\mathbf{G}}(\vec{\sigma})::$$

3.2. Blocks with more than one node. — In a Bosonic block with two or more nodes, the Bosonic forest $\mathcal{F}_{\mathcal{B}}$ is a non-empty Bosonic tree $\mathcal{T}_{\mathcal{B}}$. Consider a fixed such block \mathcal{B} , a fixed tree $\mathcal{T}_{\mathcal{B}}$ and the fixed set of frequencies $\{j_a\}$, $a \in \mathcal{B}$, all distinct. We shall write simply $d\nu_{\mathcal{B}}$ for $d\nu_{\mathcal{T}_{\mathcal{B}}}(\vec{\sigma}, \vec{\tau})$. The corresponding covariance of the Gaussian measure $d\nu_{\mathcal{B}}$ is also a symmetric matrix on the vector space $\tilde{V}_{\mathcal{B}}$, whose vectors, in addition to the colour and double momentum components and their type σ or τ have also a node index $a \in \mathcal{B}$; hence $\tilde{V}_{\mathcal{B}} = \mathbb{R}^{|\mathcal{B}|} \otimes [L(\mathcal{H})^{\times} \oplus L(\mathcal{H})^{\times}]$. It can be written as $\mathbf{X}_{\mathcal{B}} := \mathbb{I} \otimes [X(\mathbf{w}_{\mathcal{B}}) + X^{\circ 2}(\mathbf{w}_{\mathcal{B}})]$ where X acts on the σ part hence on the first factor in $[L(\mathcal{H})^{\times} \oplus L(\mathcal{H})^{\times}]$ and $X^{\circ 2}$ on the τ part hence on the second factor in $[L(\mathcal{H})^{\times} \oplus L(\mathcal{H})^{\times}]$.

3.2.1. *From trees to forests.* — We want to compute

$$I_{\mathcal{B}} := \int d\nu_{\mathcal{B}} \partial_{\mathcal{T}_{\mathcal{B}}} \prod_{a \in \mathcal{B}} (e^{-V_{j_a}} - 1)(\vec{\sigma}^a, \vec{\tau}^a).$$

When \mathcal{B} has more than one node, since $\mathcal{T}_{\mathcal{B}}$ is a tree, each node $a \in \mathcal{B}$ is touched by at least one derivative and we can replace $W_{j_a} = e^{-V_{j_a}} - 1$ by $e^{-V_{j_a}}$ (the derivative of 1 giving 0). The partial derivative $\partial_{\mathcal{T}_{\mathcal{B}}}$ can be rewritten as follows:

$$\partial_{\mathcal{T}_{\mathcal{B}}} = \left(\prod_{\substack{\ell \in E(\mathcal{T}_{\mathcal{B}}), \\ \ell = (a,b)}} \sum_{c_{\ell}=1}^4 \sum_{m_{\ell}, n_{\ell}} \right) \prod_{a \in \mathcal{B}} \prod_{s \in S_{\mathcal{B}}^a} (\partial_{\sigma_s} + \partial_{\tau_s})$$

where $S_{\mathcal{B}}^a$ is the set of edges of $\mathcal{T}_{\mathcal{B}}$ which ends at a , and

$$\partial_{\sigma_s} := \frac{\partial}{\partial (\sigma_{m_s n_s})^a}, \quad \partial_{\tau_s} := \frac{\partial}{\partial (\tau_{m_s n_s})^a}.$$

We thus have to compute

$$I_{\mathcal{B}} = \int d\nu_{\mathcal{B}} \prod_{\substack{\ell \in E(\mathcal{T}_{\mathcal{B}}), \\ \ell = (a,b)}} \sum_{c_{\ell}=1}^4 \sum_{m_{\ell}, n_{\ell}} F_{\mathcal{B}}, \quad F_{\mathcal{B}} := \prod_{a \in \mathcal{B}} \left[\prod_{s \in S_{\mathcal{B}}^a} (\partial_{\sigma_s} + \partial_{\tau_s}) e^{-V_{j_a}} \right].$$

We can evaluate the derivatives in the preceding equation through the Faà di Bruno formula:

$$\prod_{s \in S} [\partial_{\sigma_s} + \partial_{\tau_s}] f(g(\sigma, \tau)) = \sum_{\pi} f^{|\pi|}(g(\sigma, \tau)) \prod_{b \in \pi} \left(\prod_{s \in b} [\partial_{\sigma_s} + \partial_{\tau_s}] g(\sigma, \tau) \right),$$

where π runs over the partitions of the set S , b runs through the blocks of the partition π , and $|\pi|$ denotes the number of blocks of π . In our case f , the exponential function, is its own derivative, hence the formula simplifies to

$$(3.1) \quad F_{\mathcal{B}} = \prod_{a \in \mathcal{B}} e^{-V_{j_a}} \left(\sum_{\pi^a} \prod_{b^a \in \pi^a} \left[\left[\prod_{s \in b^a} (\partial_{\sigma_s} + \partial_{\tau_s}) \right] (-V_{j_a}) \right] \right),$$

where π^a runs over partitions of $S_{\mathcal{B}}^a$ into blocks b^a . The Bosonic integral in a block \mathcal{B} can be written therefore in a simplified manner as:

$$(3.2) \quad I_{\mathcal{B}} = \sum_{\mathbf{G}} \int d\nu_{\mathcal{B}} \left(\prod_{a \in \mathcal{B}} e^{-V_{j_a}(\vec{\sigma}^a, \vec{\tau}^a)} \right) A_{\mathbf{G}}(\vec{\sigma}),$$

where we gather the result of the derivatives as a sum over graphs \mathbf{G} of corresponding amplitudes $A_{\mathbf{G}}(\vec{\sigma})$. Indeed, the dependence of V_j being linear in τ , the corresponding τ derivatives are constant, hence amplitudes $A_{\mathbf{G}}(\vec{\sigma})$ do not depend on $\vec{\tau}$. The graphs \mathbf{G} will be called **skeleton graphs**, see Definition 2. They are still forests, with loop vertices⁷, one for each $b^a \in \pi^a$, $a \in \mathcal{B}$. We now detail the different types of those four-stranded loop vertices.

To this aim, let us actually compute $\partial_{b^a} := [\prod_{s \in b^a} (\partial_{\sigma_s} + \partial_{\tau_s})](-V_{j_a})$, part of eq. (3.1). First of all, remark that as V_j is linear in τ and $\partial_{\tau_s} V_j$ is independent of σ , if $|b^a| \geq 2$, $\partial_{b^a} = [\prod_{s \in b^a} \partial_{\sigma_s}](-V_{j_a})$. Then, we rewrite eq. (1.16) using $\mathbb{I} + U - \mathcal{R} = -U^2 \mathcal{R}$:

$$(3.3) \quad V_j = \mathcal{E}_j + \frac{\lambda^2}{2} : \vec{\sigma} \cdot Q_j \vec{\sigma} : - i \frac{\lambda^2}{\sqrt{2}} Q_{0,j} \cdot \vec{\tau} + \int_0^1 dt_j \operatorname{Tr}[-U'_j U_{\leq j}^2 \mathcal{R}_{\leq j} + D'_{1,j} \Sigma_{\leq j}^2 + D_{1,\leq j} (\Sigma'_j \Sigma_{\leq j} + \Sigma_{\leq j} \Sigma'_j) - 3D'_{2,j} \Sigma_{\leq j}].$$

Remembering that ∂_{σ_s} and ∂_{τ_s} stand for derivatives with well defined colour and matrix elements, we introduce the notations

$$\Delta_{\leq j}^s := \frac{\partial U_{\leq j}}{\partial \sigma_{m_s n_s}^{c_s}} = \frac{\partial \Sigma_{\leq j}}{\partial \sigma_{m_s n_s}^{c_s}} = i\lambda C_{\leq j}^{1/2} \delta^s C_{\leq j}^{1/2},$$

$$\Delta_j^s := \frac{\partial U'_j}{\partial \sigma_{m_s n_s}^{c_s}} = \frac{\partial \Sigma'_j}{\partial \sigma_{m_s n_s}^{c_s}} = i\lambda (C_j^{1/2} \delta^s C_j^{1/2} + C_{\leq j}^{1/2} \delta^s C_j^{1/2})$$

where δ^s , defined as $(\delta^s)_{mn} := \frac{\partial \sigma}{\partial \sigma_{m_s n_s}^{c_s}} = \frac{\partial \tau}{\partial \tau_{m_s n_s}^{c_s}}$, equals $\mathbf{e}_{m_s n_s} \otimes \mathbb{I}_{\hat{c}_s}$ where $\mathbf{e}_{m_s n_s}$ has zero entries everywhere except at position $m_s n_s$ where it has entry one.

As noticed above, only one τ derivative needs to be applied to $-V_j$:

$$\partial_{\tau_s}(-V_j) = i \frac{\lambda^2}{\sqrt{2}} \operatorname{Tr}_{c_s} [(Q_{0,j})_{c_s c_s} \mathbf{e}_{m_s n_s}].$$

We now concentrate on the σ derivatives. Since $\partial_{\sigma_s} \mathcal{R}_{\leq j} = \mathcal{R}_{\leq j} \Delta_{\leq j}^s \mathcal{R}_{\leq j}$, we get

$$(3.4) \quad \partial_{\sigma_s}(-V_j) = -\lambda^2 \operatorname{Tr}_{c_s} [\mathbf{e}_{m_s n_s} (Q_j \vec{\sigma})_{c_s}] + \int_0^1 dt_j \operatorname{Tr} [\Delta_j^s U_{\leq j}^2 \mathcal{R}_{\leq j} + U'_j \Delta_{\leq j}^s U_{\leq j} \mathcal{R}_{\leq j} + U'_j U_{\leq j} \Delta_{\leq j}^s \mathcal{R}_{\leq j} + U'_j U_{\leq j}^2 \mathcal{R}_{\leq j} \Delta_{\leq j}^s \mathcal{R}_{\leq j} - D'_{1,j} (\Delta_{\leq j}^s \Sigma_{\leq j} + \Sigma_{\leq j} \Delta_{\leq j}^s) - D_{1,\leq j} (\Delta_j^s \Sigma_{\leq j} + \Sigma'_j \Delta_{\leq j}^s + \Delta_{\leq j}^s \Sigma'_j + \Sigma_{\leq j} \Delta_j^s) + 3D'_{2,j} \Delta_{\leq j}^s].$$

In this formula notice the first term which is the σ derivative of $: \vec{\sigma} \cdot Q_j \vec{\sigma} :$, the sum of the next four terms, depending on whether ∂_{σ_s} acts on \mathcal{R} or on one of the three explicit U -like numerators, and also the seven simpler terms with explicit D -like factors.

Notation

From now on, to shorten formulas and since j is fixed, we shall omit most of the time the $\leq j$ subscripts (but not the all-important j subscript).

The explicit formula for $k = 2$ is also straightforward but longer. We give it here for completeness:

$$(3.5) \quad \partial_{\sigma_{s_2}} \partial_{\sigma_{s_1}}(-V_j) = -\lambda^2 \operatorname{Tr} [\mathbf{e}_{m_{s_1} n_{s_1}} (Q_j)_{c_{s_1} c_{s_2}} \mathbf{e}_{m_{s_2} n_{s_2}}] + \int_0^1 dt_j \operatorname{Tr} [\Delta_j^{s_1} \Delta^{s_2} U \mathcal{R} + \Delta_j^{s_1} U \Delta^{s_2} \mathcal{R} + \Delta_j^{s_1} U^2 \mathcal{R} \Delta^{s_2} \mathcal{R} + \Delta_j^{s_2} \Delta^{s_1} U \mathcal{R} + U'_j \Delta^{s_1} \Delta^{s_2} \mathcal{R} + U'_j \Delta^{s_1} U \mathcal{R} \Delta^{s_2} \mathcal{R} + \Delta_j^{s_2} U \Delta^{s_1} \mathcal{R} + U'_j \Delta^{s_2} \Delta^{s_1} \mathcal{R} + U'_j U \Delta^{s_1} \mathcal{R} \Delta^{s_2} \mathcal{R} + \Delta_j^{s_2} U^2 \mathcal{R} \Delta^{s_1} \mathcal{R} + U'_j \Delta^{s_2} U \mathcal{R} \Delta^{s_1} \mathcal{R} + U'_j U \Delta^{s_2} \mathcal{R} \Delta^{s_1} \mathcal{R} + U'_j U^2 \mathcal{R} \Delta^{s_2} \mathcal{R} \Delta^{s_1} \mathcal{R} + U'_j U^2 \mathcal{R} \Delta^{s_1} \mathcal{R} \Delta^{s_2} \mathcal{R} - D'_{1,j} (\Delta^{s_1} \Delta^{s_2} + \Delta^{s_2} \Delta^{s_1}) - D_{1,j} (\Delta_j^{s_1} \Delta^{s_2} + \Delta_j^{s_2} \Delta^{s_1} + \Delta^{s_1} \Delta_j^{s_2} + \Delta^{s_2} \Delta_j^{s_1})].$$

⁷We recall that loop vertices are the traces obtained by σ derivatives acting on the intermediate field action [Riv07].

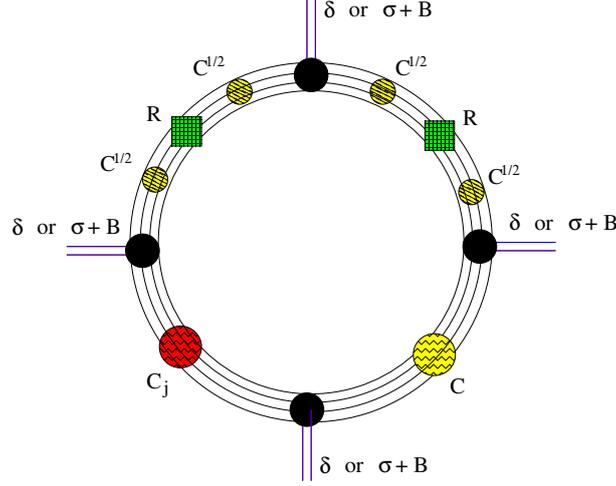


Figure 7. An example of a four-stranded vertex of length four with its typical cycle of insertions. Black (matrix type) dots correspond to δ or $\sigma + B$ operators. Each has its well-defined colour, hence opens a well-defined strand. Any δ insertion is in fact an half edge of the tree $\mathcal{T}_{\mathcal{B}}$, hence pairs with another vertex (not shown in the picture). The marked insertion (pictured a bit larger and in red, together with its neighboring corners) indicates the presence of the slice j propagator C_j . Resolvents are pictured as green squares. The sum is constrained to have exactly k derived insertions of the δ type, the others are $\sigma + B$.

3.2.2. *Wick ordering by the τ field.* — Each $\ell = (a, b)$ for which the τ derivatives have been chosen, see eq. (2.2c), creates exactly a divergent vacuum graph \mathfrak{N}_2 (see fig. 5b) obtained by contracting two quadratic Q_0 factors, one with scale j_a and the other with scale j_b . Fortunately this cancels out with a very special potentially divergent quadratic σ link. To check it, let us perform exactly the remaining τ integral. The result is expressed in the following Lemma.

Lemma 3.2. — *After integrating out the τ field, the expansion is the same as if there had never been any τ fields, but with two modifications:*

- *there exists an exponential of the counterterm*

$$\delta_{\mathfrak{N}_2, \mathcal{B}}(\mathbf{w}) = -\frac{\lambda^4}{4} \sum_{a, b \in \mathcal{B}} X^{\circ 2}(a, b) \text{Tr}[Q_{0, j_a} Q_{0, j_b}],$$

- *each σ link for $\ell = (a, b)$ made of exactly one link between two Q_0 factors, is exactly Wick ordered with respect to the $d\nu_{\mathcal{B}}(\vec{\sigma})$ covariance, namely its value in $A_{\mathcal{G}}$ is $:\vec{\sigma}^a \cdot Q_{0, j_a} Q_{0, j_b} \vec{\sigma}^b:$.*

In other words

$$I_{\mathcal{B}} = \sum_{\mathcal{G}} \int d\nu_{\mathcal{B}}(\vec{\sigma}) e^{\delta_{\mathfrak{N}_2, \mathcal{B}}(\mathbf{w})} \left(\prod_{a \in \mathcal{B}} e^{-V_{j_a}(\vec{\sigma}^a)} \right) :A_{\mathcal{G}}(\vec{\sigma}):,$$

where $:A_{\mathcal{G}}(\vec{\sigma}):$ is obtained by the same formula as if there had never been any τ field, but with one modification: the Wick ordering indicates that each link of the type $\vec{\sigma}^a \cdot Q_{0, j_a} Q_{0, j_b} \vec{\sigma}^b$ is Wick ordered with respect to the $d\nu_{\mathcal{B}}(\vec{\sigma})$ measure.

Proof. — The first part of the statement is obvious: integrating the linear $e^{i \frac{\lambda^2}{\sqrt{2}} Q_0 \cdot \vec{\tau}^a}$ terms with the $d\nu_{\mathcal{B}}(\vec{\tau})$ interpolated covariance must give back the exponential of the full $\delta_{\mathfrak{N}_2}$ counterterm but with the weakening covariance factors $X^{\circ 2}(a, b)$ between nodes a and b . The second statement is also not too

surprising since the counterterm $\delta_{\mathfrak{N}_2}$ should compensate the divergent graphs \mathfrak{N}_2 which are brought down the exponential by the MLVE expansion. But let us check it explicitly. Any tree link $\ell = (a, b)$ in the Faà di Bruno formula either is a τ link hence created a term

$$2w_\ell \left(\frac{i\lambda^2}{\sqrt{2}}\right)^2 \text{Tr}[Q_{0,j_a} Q_{0,j_b}] = -w_\ell \lambda^4 \text{Tr}[Q_{0,j_a} Q_{0,j_b}],$$

or a σ link. In this case it either has joined two Q_0 loop vertices each with one σ field at its free end, or done something else. In the first case, the expectation value of the corresponding term is

$$\int d\nu_{\mathcal{B}}(\vec{\sigma}) \left(\frac{-\lambda^2}{2}\right)^2 2^2 \vec{\sigma}^a \cdot Q_{0,j_a} Q_{0,j_b} \vec{\sigma}^b = w_\ell \lambda^4 \text{Tr}[Q_{0,j_a} Q_{0,j_b}].$$

This proves the second statement: such σ links are exactly Wick-ordered by the τ links. \square

From now on we can therefore forget the auxiliary τ field. Its only purpose was to effectuate the compensations expressed by Lemma 3.2, without disturbing too much the “black box” of the MLVE. Moreover, anticipating on Section 6, notice that the functional integration (with respect to the Gaussian measure $\nu_{\mathcal{B}}$) of the “graphs” \mathcal{G} would result in (perturbative series of) purely convergent Feynman graphs.

3.2.3. Perturbative and non-perturbative contributions. — In all cases (including the single isolated block case) we apply a Hölder inequality with respect to the positive measure $d\nu_{\mathcal{B}}$ to separate four parts: the perturbative part “down from the exponential”, the particular $\frac{\lambda^2}{2}:\vec{\sigma}\cdot Q_{0,j}\vec{\sigma}:_X$ Wick-ordered term (which requires special care, since without the Wick ordering it would lead to a linearly divergent bound which could not be paid for), the other non perturbative quadratic or less than quadratic factors, which we define as

$$(3.8) \quad \tilde{V}_j^{\leq 2} := \frac{\lambda^2}{2}:\vec{\sigma}\cdot Q_{1,j}\vec{\sigma}:_X - 3 \int_0^1 dt_j \text{Tr}[D'_{2,j}\Sigma_{\leq j}]$$

(remember the τ field has been integrated out, hence replaced by the $\delta_{\mathfrak{N}_2, \mathcal{B}}(\mathbf{w})$ counterterm), and finally the higher order non-perturbative factor $V_j^{\geq 3}$. This last factor will require extra care and the full Section 5.2 for its non perturbative bound.

Remark. — The careful reader would have noticed the extra index X associated to the Wick ordering of both $\vec{\sigma}\cdot Q_{0,j}\vec{\sigma}$ and $\vec{\sigma}\cdot Q_{1,j}\vec{\sigma}$. The Wick ordering of those terms were originally defined with respect to the Gaussian measure of covariance \mathbb{I} *i.e.* before the jungle formula and thus before the interpolation of the covariance (see eq. (1.10)). Nevertheless the contraction of the two $\vec{\sigma}$'s (in both expressions) corresponds to a tadpole intermediate graph and is thus never accompanied by weakening factors w . We can therefore equally well consider that the two terms above-mentioned are Wick ordered with respect to the interpolated measure of covariance X .

Finally we write:

$$(3.9) \quad |I_{\mathcal{B}}| \leq |e^{\delta_{\mathfrak{N}_2, \mathcal{B}}(\mathbf{w})}| \underbrace{\left(\int d\nu_{\mathcal{B}} \prod_{a \in \mathcal{B}} e^{-2\Re(\lambda^2):\vec{\sigma}\cdot Q_{0,j_a}\vec{\sigma}:_X}\right)^{1/4}}_{I_1, \text{ non-perturbative}} \underbrace{\left(\int d\nu_{\mathcal{B}} \prod_{a \in \mathcal{B}} e^{-4\Re(\tilde{V}_{j_a}^{\leq 2}(\vec{\sigma}^a))}\right)^{1/4}}_{I_2, \text{ non-perturbative}} \\ \times \underbrace{\left(\int d\nu_{\mathcal{B}} \prod_{a \in \mathcal{B}} e^{4|V_{j_a}^{\geq 3}(\vec{\sigma}^a)|}\right)^{1/4}}_{I_3, \text{ non-perturbative}} \sum_{\mathcal{G}} \underbrace{\left(\int d\nu_{\mathcal{B}} |:A_{\mathcal{G}}(\vec{\sigma}):|^4\right)^{1/4}}_{I_4, \text{ perturbative}}.$$

To bound such expressions, and in particular the “non-perturbative” terms, requires to now work out in more details explicit formulae which in particular show the compensation between the terms of eq. (3.3).

4. Estimates for the interaction

This section is a technical interlude before estimating the non-perturbative terms of eq. (3.9) in Section 5.2. In its first subsection, we make explicit the cancellations at work in V_j and derive a quadratic bound (in Σ) on $|V_j^{\geq 3}|$. It will be used to prove Proposition 5.7 which constitutes one step towards the bound on I_3 of eq. (3.9). In its second subsection, we get a quartic bound on $|V_j^{\geq 3}|$ used both in Section 6 and in Section 5.2 but this time to prove another step of our final bound on I_3 , namely Lemma 5.8.

4.1. Cancellations and quadratic bound. — In this section, we first derive a new expression for $V_j^{\geq 3}$ (eq. (4.1)) in order to explicitly show the cancellation involving the \mathcal{E}_j counterterm. Then we prove a so-called *quadratic bound* on $|V_j^{\geq 3}|$ (Lemma 4.1) in terms of a quadratic form in σ . This estimate will be useful in Section 5.2.

In the sequel, we will be using repeatedly the few following facts:

$$\begin{aligned} [U, \mathcal{R}] &= 0, & \mathbf{1}_j &= \frac{d\mathbf{1}_{\leq j}}{dt}(t_j), & \mathbf{1}_j^2 &= \mathbf{1}_j, \\ [D_1, \mathbf{1}_j] &= [D_2, \mathbf{1}_j] = 0, & D'_j &= D\mathbf{1}_j, & \Sigma'_j &= \mathbf{1}_j\Sigma + \Sigma\mathbf{1}_j. \end{aligned}$$

Notation

From now on, in order to simplify long expressions, we will mainly trade the ' notation (e.g. D', Σ') for the ones with explicit cutoff $\mathbf{1}_j$ (e.g. $D\mathbf{1}_j, \mathbf{1}_j\Sigma + \Sigma\mathbf{1}_j$).

So let us return to eq. (1.16), using cyclicity of the trace, $(\mathbb{I}+U - \mathcal{R}) = (\mathbb{I} - \mathcal{R})U = U(\mathbb{I} - \mathcal{R}) = -URU$, and $D = D_1 + D_2$, we define

$$\begin{aligned} V_j^{\geq 3} &=: \mathcal{E}_j + \int_0^1 dt_j \tilde{v}_j, \\ \tilde{v}_j &= \text{Tr}[U'_j(\mathbb{I}+U_{\leq j} - \mathcal{R}_{\leq j}) + D'_{1,j}\Sigma^2 + D_{1,\leq j}\Sigma'_j\Sigma + D_{1,\leq j}\Sigma\Sigma'_j] \\ &= \text{Tr}[(U\mathbf{1}_j\Sigma + \Sigma\mathbf{1}_jU)(\mathbb{I} - \mathcal{R}) - U\mathbf{1}_jD\mathbf{1}_jUR + 3D_1\mathbf{1}_j\Sigma^2\mathbf{1}_j + 2D_1\Sigma\mathbf{1}_j\Sigma] \\ &= \text{Tr}[(U\mathbf{1}_j\Sigma + \Sigma\mathbf{1}_jU + \Sigma\mathbf{1}_jD\mathbf{1}_j\Sigma)(\mathbb{I} - \mathcal{R}) - D^3\mathbf{1}_j\mathcal{R} - D^2\mathbf{1}_j\Sigma\mathcal{R} - \Sigma\mathbf{1}_jD^2\mathcal{R} \\ &\quad - D_2\mathbf{1}_j\Sigma^2\mathbf{1}_j + 2D_1(\Sigma\mathbf{1}_j\Sigma + \mathbf{1}_j\Sigma^2\mathbf{1}_j)]. \end{aligned}$$

In order to show the compensation involving \mathcal{E}_j , we now expand the $D^3\mathbf{1}_j\mathcal{R}$ term, as

$$\text{Tr}[D^3\mathbf{1}_j\mathcal{R}] = \text{Tr}[D^3\mathbf{1}_j + D^4\mathbf{1}_j + D^5\mathbf{1}_j\mathcal{R} + D^3(\mathbf{1}_j + D\mathbf{1}_j)\Sigma\mathcal{R}].$$

We further expand the pure D terms as $\text{Tr}[D^3\mathbf{1}_j + D^4\mathbf{1}_j] =: \mathcal{D}_{\text{conv},j} + \mathcal{D}_{\text{div},j}$ with

$$\begin{aligned} \mathcal{D}_{\text{conv},j} &:= \mathcal{D}_{\text{conv},\leq j} - \mathcal{D}_{\text{conv},\leq j-1}, & \mathcal{D}_{\text{div},j} &:= \mathcal{D}_{\text{div},\leq j} - \mathcal{D}_{\text{div},\leq j-1}, \\ \mathcal{D}_{\text{conv},\leq j} &:= \text{Tr}\left[\frac{1}{3}D_{2,\leq j}^3 + D_{1,\leq j}D_{2,\leq j}^2 + \frac{1}{4}((D_{1,\leq j} + D_{2,\leq j})^4 - D_{1,\leq j}^4)\right], \\ \mathcal{D}_{\text{div},\leq j} &:= \text{Tr}\left[\frac{1}{3}D_{1,\leq j}^3 + D_{1,\leq j}^2D_{2,\leq j} + \frac{1}{4}D_{1,\leq j}^4\right] = \mathcal{E}_{\leq j}. \end{aligned}$$

Clearly, $\int_0^1 dt_j \mathcal{D}_{\text{div},j} = \mathcal{E}_j$. Hence, redefining $V_j^{\geq 3} =: \int_0^1 dt_j v_j$ and $v_j := v_j^{(0)} + v_j^{(1)} + v_j^{(2)}$, we have

$$(4.1) \quad \begin{cases} v_j^{(0)} = -\text{Tr}[D^5\mathbf{1}_j\mathcal{R}] - \mathcal{D}_{\text{conv},j}, \\ v_j^{(1)} = \text{Tr}[(D\mathbf{1}_j\Sigma + \Sigma\mathbf{1}_jD)(\mathbb{I} - \mathcal{R}) - D^2\mathbf{1}_j\Sigma\mathcal{R} - \Sigma\mathbf{1}_jD^2\mathcal{R} - D^3\mathbf{1}_j\Sigma\mathcal{R} - D^4\mathbf{1}_j\Sigma\mathcal{R}], \\ v_j^{(2)} = \text{Tr}[(2\Sigma\mathbf{1}_j\Sigma + \Sigma\mathbf{1}_jD\mathbf{1}_j\Sigma)(\mathbb{I} - \mathcal{R}) - D_2\mathbf{1}_j\Sigma^2\mathbf{1}_j + 2D_1(\Sigma\mathbf{1}_j\Sigma + \mathbf{1}_j\Sigma^2\mathbf{1}_j)]. \end{cases}$$

This has shown the desired cancellation of the \mathcal{E}_j counterterm with the $-\mathcal{D}_{\text{div},j}$ term.

We now turn to the proof of the following Lemma, suited to a non-perturbative sector of the model analysis, which bounds $|V_j|$ in terms of a quadratic form $\mathcal{Q}_j(\vec{\sigma}) := \frac{1}{|g|} \text{Tr}[\Sigma^*\mathbf{1}_j\Sigma]$, since higher order bounds can certainly not be integrated out with respect to the Gaussian measure $d\nu_{\mathcal{B}}$.

Lemma 4.1 (Quadratic bound). — For g in the cardioid domain Card_ρ , there exists a real positive number k such that

$$|V_j^{\geq 3}| \leq k\rho(1 + \mathcal{Q}_j(\vec{\sigma})).$$

The proof of Lemma 4.1 requires the following upper bounds

Proposition 4.2 (Norms and traces). — For all $0 < \varepsilon < 1$, for any $t_j \in [0, 1]$ and g in the cardioid,

$$\begin{aligned} \|\mathcal{R}\| &\leq 2\rho/|g|, & \text{Tr}[D^4\mathbf{1}_j] &\leq O(1)|g|^4, \\ \|D\| &\leq O(1)|g|, & |\text{Tr}[D^5\mathbf{1}_j]| &\leq O(1)|g|^5 M^{-j}, \\ |\mathcal{D}_{\text{conv},j}| &\leq O(1)|g|^5 M^{-(1-\varepsilon)j}, & \text{Tr}[D^6\mathbf{1}_j] &\leq O(1)|g|^6 M^{-2j}, \\ & & \text{Tr}[D^8\mathbf{1}_j] &\leq O(1)|g|^8 M^{-4j}. \end{aligned}$$

Proof. — Apart from the bound on $\|\mathcal{R}\|$ which uses Lemma 1.4 and the definition of the cardioid domain, the other ones are standard exercises in perturbative power counting. \square

Finally, before we prove Lemma 4.1, let us state the following inequalities that we shall use extensively in this section and the next one.

Proposition 4.3 (Trace inequalities). — Let A, B, C, E be complex square matrices of the same size. Let $\|A\|_2$ denote $(\text{Tr}[AA^*])^{1/2}$ where $*$ denotes the Hermitian conjugation. We have:

1. *Hilbert-Schmidt bound (hereafter HS)*

$$(4.2) \quad |\text{Tr}[AB]| \leq \|A\|_2^2 + \|B\|_2^2.$$

2. *L^1/L^∞ bound: if A is Hermitian (and B bounded),*

$$(4.3) \quad |\text{Tr}[AB]| \leq \|B\| \text{Tr}[|A|]$$

where $\|\cdot\|$ denotes the operator norm.

3. *Cauchy-Schwarz inequality:*

$$(4.4) \quad |\text{Tr}[ABCE]| \leq \|A\| \|C\| \|B\|_2 \|E\|_2.$$

The proofs are very standard and anyway simple enough to be avoided here.

Proof of Lemma 4.1. — We first notice that $|V_j^{\geq 3}| \leq \int_0^1 dt_j |v_j|$. Then $|v_j|$ is smaller than the sum of the modules of each of its terms. As all our bounds will be uniform in t_j , we can simply focus on the modules of each of the terms of v_j . Starting with $v_j^{(0)}$, and according to Proposition 4.2, we have $|v_j^{(0)}| \leq O(1)\rho$.

As $|\text{Tr}[D^2]| = \mathcal{O}(M^{2j})$, a price we cannot afford to pay, we cannot simply apply a HS bound (see eq. (4.2)) to the first two terms of $v_j^{(1)}$. We need to expand the resolvent one step further:

$$\begin{aligned} v_j^{(1)} &= \text{Tr}[-(\Sigma + D)D\mathbf{1}_j\Sigma - \Sigma\mathbf{1}_jD(D + \Sigma)\mathcal{R} - D^2\mathbf{1}_j\Sigma\mathcal{R} - \Sigma\mathbf{1}_jD^2\mathcal{R} - D^3\mathbf{1}_j\Sigma\mathcal{R} - D^4\mathbf{1}_j\Sigma\mathcal{R}] \\ &= -\text{Tr}[2\Sigma\mathbf{1}_jD\mathbf{1}_j\Sigma\mathcal{R} + 2D^2\mathbf{1}_j\Sigma\mathcal{R} + 2\Sigma\mathbf{1}_jD^2\mathcal{R} + D^3\mathbf{1}_j\Sigma\mathcal{R} + D^4\mathbf{1}_j\Sigma\mathcal{R}]. \end{aligned}$$

To the first term we apply the bound (4.4) with $A = \mathcal{R}, B = \Sigma\mathbf{1}_j, C = D, E = \mathbf{1}_j\Sigma$ to get

$$|\text{Tr}[\Sigma\mathbf{1}_jD\mathbf{1}_j\Sigma\mathcal{R}]| \leq \|\mathcal{R}\| \|D\| |\text{Tr}[\Sigma\mathbf{1}_j\Sigma]| \leq O(1)\rho \mathcal{Q}_j(\vec{\sigma}).$$

All the other terms of $v_j^{(1)}$ are bounded the same way: first a HS bound then a L^1/L^∞ one. For example:

$$\begin{aligned} |\mathrm{Tr}[D^2 \mathbf{1}_j \Sigma \mathcal{R}]| &\leq \mathrm{Tr}[\mathcal{R}^* \mathcal{R} D^4 \mathbf{1}_j] - \mathrm{Tr}[\Sigma \mathbf{1}_j \Sigma] \\ &\leq \|\mathcal{R}^* \mathcal{R}\| \mathrm{Tr}[D^4 \mathbf{1}_j] + |g| \mathcal{Q}_j(\vec{\sigma}) \leq O(1) \rho (1 + \mathcal{Q}_j(\vec{\sigma})). \end{aligned}$$

The other terms of $v_j^{(1)}$ are in fact better behaved.

Finally let us turn to $v_j^{(2)}$. For each term, we apply the bound (4.4) with $B = \Sigma \mathbf{1}_j$ and $E = \mathbf{1}_j \Sigma$. We let the reader check that it leads to the desired result. \square

4.2. Convergent loop vertices and quartic bound. — We want to establish a second bound on $|V_j^{\geq 3}|$, more suited to perturbation theory than Lemma 4.1. The idea is to get a bound in a finite number of loop vertices types which have been freed of any resolvent through the successive use of a Hilbert-Schmidt inequality and a L^1/L^∞ bound.

The constraints are many. We want first the loop vertices to be convergent (i.e. any graph built solely out of them must converge). This excludes loop vertices of the type $\mathrm{Tr}[\Sigma^2]$ or $\mathrm{Tr}[D_1 \Sigma^2]$. Another important constraint will be to keep a propagator of scale exactly j in each piece A and B which are to be separated by a HS inequality. This forces us to be careful about the ordering of our operators, to ensure that the HS “cut” keeps one $\mathbf{1}_j$ cutoff *both* in the two halves A and B .

Definition 4.4 (Convergent loop vertices). — Let us define the following convergent and positive loop vertices

$$\begin{aligned} U_j^{0,a} &:= \frac{1}{|g|^6} \mathrm{Tr}[D^6 \mathbf{1}_j], & U_j^{2,a} &:= \frac{1}{|g|^3} \mathrm{Tr}[D^2 \mathbf{1}_j |\Sigma|^2], & U_j^{2,d} &:= \frac{1}{|g|^3} \mathrm{Tr}[D_2 \mathbf{1}_j |\Sigma|^2], \\ U_j^{0,b} &:= \frac{1}{|g|^5} \mathrm{Tr}[D^5 \mathbf{1}_j], & U_j^{2,b} &:= \frac{1}{|g|^3} \mathrm{Tr}[D^2 \Sigma^* \mathbf{1}_j \Sigma], & U_j^{2,e} &:= \frac{1}{|g|^3} \mathrm{Tr}[D_2 \Sigma^* \mathbf{1}_j \Sigma], \\ U_j^{0,c} &:= \frac{1}{|g|^5} \mathcal{D}_{conv,j}, & U_j^{2,c} &:= \frac{1}{|g|^5} \mathrm{Tr}[D^4 \mathbf{1}_j |\Sigma|^2], & U_j^4 &:= \frac{1}{|g|^2} \mathrm{Tr}[|\Sigma|^4 \mathbf{1}_j]. \end{aligned}$$

as well as the following convergent ones

$$U_j^{1,a} := \frac{1}{|g|^{5/2}} \mathrm{Tr}[D^2 \mathbf{1}_j \Sigma], \quad U_j^{1,b} := \frac{1}{|g|^{7/2}} \mathrm{Tr}[D^3 \mathbf{1}_j \Sigma], \quad U_j^3 := \frac{1}{|g|^{3/2}} \mathrm{Tr}[\Sigma^3 \mathbf{1}_j]. \quad \spadesuit$$

Lemma 4.5 (Quartic bound). — Let us define the following finite sets: $A_3 = A_4 := \{a\}$, $A_0 := \{a, b, c\}$, $A_1 := \{a, b\}$ and $A_2 := \{a, b, c, d, e\}$. Let $U_j^{i,a}$ be defined as U_j^i for $i \in \{3, 4\}$. For all $0 \leq i \leq 4$, let \mathfrak{U}_j^i be $\sum_{\alpha \in A_i} |U_j^{i,\alpha}|$. Then, for any g in the cardioid domain,

$$|V_j^{\geq 3}| \leq O(1) (\rho^2 \mathfrak{U}_j^4 + \rho^{3/2} \mathfrak{U}_j^3 + \rho^3 \mathfrak{U}_j^2 + \rho^{5/2} \mathfrak{U}_j^1 + \rho^5 \mathfrak{U}_j^0).$$

Corollary 4.6. — For all $0 < \epsilon < 1$, for any g in the cardioid,

$$|V_j^{\geq 3}|^2 \leq O(1) \rho^3 (M^{-(2-\epsilon)j} + \sum_{i=1}^4 \sum_{\alpha \in A_i} |U_j^{i,\alpha}|^2).$$

Proof. — From Lemma 4.5, we use Proposition 4.2, $\rho \leq 1$ and the Cauchy-Schwarz inequality $(\sum_{i=1}^p a_i)^2 \leq p \sum_{i=1}^p a_i^2$. \square

We postpone the proof of Lemma 4.5 to Appendix A.4 and give here only its main structure. Starting with eq. (4.1), the idea is to apply, to each term of $|V_j^{\geq 3}|$, a HS bound (4.2) (to get positive vertices) followed by a L^1/L^∞ inequality (4.3) (to get rid of the resolvents). The only problem is that not all terms in eq. (4.1) would result in convergent vertices under such a procedure. Thus we need to expand the resolvent until the new terms are ready for a HS bound, always taking great care of the operator order in such a way that both sides of the HS cut receive a cut-off operator $\mathbf{1}_j$. All details are given in Appendix A.4.

5. Non perturbative functional integral bounds

5.1. Grassmann integrals. — They are identical to those of [GR14; DR16], resulting in the same computation:

$$\begin{aligned} & \int \prod_{\mathcal{B}} \prod_{a \in \mathcal{B}} (d\bar{\chi}_{j_a}^{\mathcal{B}} d\chi_{j_a}^{\mathcal{B}}) e^{-\sum_{a,b=1}^n \bar{\chi}_{j_a}^{\mathcal{B}(a)} Y_{ab} \chi_{j_b}^{\mathcal{B}(b)}} \prod_{\substack{\ell_F \in \mathcal{F}_F \\ \ell_F = (a,b)}} \delta_{j_a j_b} \left(\chi_{j_a}^{\mathcal{B}(a)} \bar{\chi}_{j_b}^{\mathcal{B}(b)} + \chi_{j_b}^{\mathcal{B}(b)} \bar{\chi}_{j_a}^{\mathcal{B}(a)} \right) \\ &= \left(\prod_{\mathcal{B}} \prod_{\substack{a,b \in \mathcal{B} \\ a \neq b}} (1 - \delta_{j_a j_b}) \right) \left(\prod_{\substack{\ell_F \in \mathcal{F}_F \\ \ell_F = (a,b)}} \delta_{j_a j_b} \right) \left(Y_{\hat{a}_1 \dots \hat{a}_k}^{\hat{b}_1 \dots \hat{b}_k} + Y_{\hat{b}_1 \dots \hat{a}_k}^{\hat{a}_1 \dots \hat{b}_k} + \dots + Y_{\hat{b}_1 \dots \hat{b}_k}^{\hat{a}_1 \dots \hat{a}_k} \right), \end{aligned}$$

where $k = |\mathcal{F}_F|$, the sum runs over the 2^k ways to exchange an a_i and a b_i , and the Y factors are (up to a sign) the minors of Y with the lines $b_1 \dots b_k$ and the columns $a_1 \dots a_k$ deleted. The factor $\left(\prod_{\mathcal{B}} \prod_{\substack{a,b \in \mathcal{B} \\ a \neq b}} (1 - \delta_{j_a j_b}) \right)$ ensures that the scales obey a *hard core constraint inside each block*. Positivity of the Y covariance means as usual that the Y minors are all bounded by 1 [AR98; GR14], namely for any a_1, \dots, a_k and b_1, \dots, b_k ,

$$\left| Y_{\hat{b}_1 \dots \hat{a}_k}^{\hat{a}_1 \dots \hat{b}_k} \right| \leq 1.$$

5.2. Bosonic integrals. — This section is devoted to bound the non perturbative terms

$$(5.1) \quad I_{\mathcal{B}}^{NP} := |e^{\delta_{\mathfrak{n}_2, \mathcal{B}}(w)}| \left(\int d\nu_{\mathcal{B}} \prod_{a \in \mathcal{B}} e^{4|V_{j_a}^{\geq 3}(\bar{\sigma}^a)|} \right)^{1/4} \left(\int d\nu_{\mathcal{B}} \prod_{a \in \mathcal{B}} e^{2\Re(\lambda^2) : \bar{\sigma}^a \cdot Q_{0, j_a} \bar{\sigma}^a : X} \right)^{1/4} \\ \times \left(\int d\nu_{\mathcal{B}} \prod_{a \in \mathcal{B}} e^{-4\Re(\tilde{V}_{j_a}^{\leq 2}(\bar{\sigma}^a))} \right)^{1/4}$$

in eq. (3.9). Thus we work within a fixed Bosonic block \mathcal{B} and a fixed set of scales $S_{\mathcal{B}} := \{j_a\}_{a \in \mathcal{B}}$, all *distinct*. To simplify, we put $b = |\mathcal{B}| \leq n$ where n is the order of perturbation in eq. (2.1).

Theorem 5.1. — *For ρ small enough and for any value of the w interpolating parameters, there exist positive $O(1)$ constants such that for $|\mathcal{B}| \geq 2$*

$$I_{\mathcal{B}}^{NP} \leq O(1) e^{O(1)\rho^{3/2}|\mathcal{B}|}.$$

If \mathcal{B} is reduced to a single isolated node a , hence $b = 1$

$$\left| \int d\nu_a(\bar{\sigma}^a) (e^{-V_{j_a}(\bar{\sigma}^a)} - 1) \right| \leq O(1)\rho^{3/2}.$$

Those results are similar to [DR16] but their proof is completely different. Since our theory is more divergent, we need to Taylor expand much farther. The rest of this section is devoted to the proof of Theorem 5.1.

Let us first of all give some definitions:

Definition 5.2 ($Q_1^{(1)}$, $Q_1^{(2)}$ and $Q^{(01)}$). — Let $Q_1^{(1)} \in L(L(\mathcal{H})^{\times})$ be given by its entries in the momentum basis:

$$(Q_1^{(1)})_{cc'; mn, m'n'} = (1 - \delta_{cc'}) \delta_{mn} \delta_{m'n'} \sum_{\mathbf{r} \in [-N, N]^2} \frac{1}{(m^2 + m'^2 + \mathbf{r}^2 + 1)^2}$$

and $\lambda^2 Q_1^{(2)}$ be $Q - Q_0 - Q_1^{(1)}$, see eqs. (1.8) and (1.9) for the definitions of Q and Q_0 . Finally let $Q^{(01)}$ be $Q_0 + Q_1^{(1)}$. ♠

Definition 5.3 (Operators on $V_{\mathcal{B}}$). — Let e_{ab} be the $|\mathcal{B}| \times |\mathcal{B}|$ real matrix the elements of which are $(e_{ab})_{mn} := \delta_{am}\delta_{bn}$. Let \mathcal{A} be a subset of \mathcal{B} and for all $P \in L(L(\mathcal{H})^\times)$, let $P_{\mathcal{A}}$ be the following linear operator on $V_{\mathcal{B}} := \mathbb{R}^{|\mathcal{B}|} \otimes L(\mathcal{H})^\times$:

$$P_{\mathcal{A}} := \sum_{a \in \mathcal{A}} P \mathbf{1}_{j_a} \otimes e_{aa}.$$

Let \tilde{Q}_1 be $(\Re\lambda^2)Q_1^{(1)} + (\Re\lambda^4)Q_1^{(2)}$. ♠

The first step consists in estimating certain determinants:

Proposition 5.4 (Determinants). — Let A_0, A_1, A_2 stand respectively for $\rho X_{\mathcal{B}} Q_{0,\mathcal{A}}, X_{\mathcal{B}} \tilde{Q}_{1,\mathcal{A}}$ and $\rho X_{\mathcal{B}} Q_{\mathcal{A}}^{(01)}$. Then, for ρ small enough, we have

$$\det_2(\mathbb{I} - A_0)^{-1} \leq e^{O(1)\rho^2|\mathcal{A}|}, \quad \det_2(\mathbb{I} - A_1)^{-1} \leq e^{O(1)\rho^2}, \quad \det(\mathbb{I} - A_2)^{-1} \leq e^{O(1)\rho M^{j_1}}$$

where \mathbb{I} is the identity operator on $V_{\mathcal{B}}$, $\det_2(\mathbb{I} - \cdot) := e^{\text{Tr} \log_2(\mathbb{I} - \cdot)}$ and $j_1 := \sup_{a \in \mathcal{A}} j_a$.

Proof. — Let us start with A_2 . Since $Q_{\mathcal{A}}^{(01)} = \sum_{a \in \mathcal{A}} Q_{j_a}^{(01)} \otimes e_{aa}$, we find that

$$\text{Tr} A_2 = \rho \text{Tr}[X_{\mathcal{B}} Q_{\mathcal{A}}^{(01)}] = \rho \sum_{a \in \mathcal{A}} X_{aa}(\mathbf{w}_{\mathcal{B}}) \text{Tr} Q_{j_a}^{(01)} = \rho \sum_{a \in \mathcal{B}'} \text{Tr} Q_{j_a}^{(01)}.$$

Using Lemma A.1, we have

$$\sum_{a \in \mathcal{B}'} \text{Tr} Q_{j_a}^{(01)} \leq O(1) \sum_{a \in \mathcal{A}} M^{j_a} \leq O(1) M^{j_1}$$

where in the last inequality we used that all vertices $a \in \mathcal{B}$ have *different scales* j_a .

Furthermore by the triangular inequality and Lemma A.1 again,

$$\|A_2\| \leq \rho \sum_{a \in \mathcal{A}} \|X(\mathbf{w}_{\mathcal{B}}) e_{aa}\| \|Q_{j_a}^{(01)}\| \leq \rho \sum_{a \in \mathcal{A}} \|Q_{j_a}^{(01)}\| \leq O(1)\rho \sum_{j=0}^{\infty} M^{-j} = O(1)\rho$$

where we used that $\|X(\mathbf{w}_{\mathcal{B}}) e_{aa}\| = 1$ and again that all vertices $a \in \mathcal{B}$ have different scales.

Remarking that by the above upper bounds on $\text{Tr} A_2$ and $\|A_2\|$, for ρ small enough, the series $\sum_{n=1}^{\infty} \frac{1}{n} \text{Tr}[A_2^n]$ converges, we have

$$\begin{aligned} \det(\mathbb{I} - A_2)^{-1} &= e^{-\text{Tr}[\log(\mathbb{I} - A_2)]} = e^{\sum_{n=1}^{\infty} \frac{1}{n} \text{Tr}[A_2^n]} \\ &\leq e^{\text{Tr}[A_2] \sum_{n=1}^{\infty} \|A_2\|^{n-1}} = e^{O(1)\rho M^{j_1}}. \end{aligned}$$

The cases of A_0 and A_1 are very similar. For example,

$$\text{Tr} A_0^2 = \rho^2 \sum_{a, a' \in \mathcal{A}} \text{Tr}[X(\mathbf{w}_{\mathcal{B}}) e_{aa} X(\mathbf{w}_{\mathcal{B}}) e_{a'a'} \otimes Q_{0,j_a} Q_{0,j_{a'}}] = \rho^2 \sum_{a, a'} \delta_{aa'} X_{aa} X_{aa} \text{Tr}[Q_{0,j_a}^2] \leq O(1)\rho^2 |\mathcal{A}|$$

by Lemma A.1. Likewise,

$$\text{Tr} A_1^2 \leq O(1)\rho^2, \quad \|A_0\| \leq O(1)\rho, \quad \|A_1\| \leq O(1)\rho.$$

Finally, using $\det_2(\mathbb{I} - A) \leq e^{\frac{1}{2} \text{Tr}[A^2] \sum_{n \geq 2} \|A\|^{n-2}}$, we conclude the proof. □

We can now treat the easy parts of $I_{\mathcal{B}}^{NP}$. It is obvious that

$$|e^{\delta_{\mathfrak{N}_2, \mathcal{B}}(w)}| \leq O(1) e^{O(1)\rho^2 |\mathcal{B}|},$$

since the counterterm $\delta_{\mathfrak{N}_2}$ is logarithmically divergent, hence it can be bounded by a constant per slice j (times ρ^2 , see Lemma 3.2).

The piece $\left(\int d\nu_{\mathcal{B}} \prod_a e^{2(\Re\lambda^2):\bar{\sigma}^a \cdot Q_{0,j_a} \bar{\sigma}^a}\right)^{1/4}$ can be bounded through an explicit computation:

$$\int d\nu_{\mathcal{B}} \prod_a e^{2(\Re\lambda^2):\bar{\sigma}^a \cdot Q_{0,j_a} \bar{\sigma}^a} = \det_2(\mathbb{I} - A_{\mathcal{B}}^0)^{-1/2}$$

where $A_{\mathcal{B}}^0$ equals $4(\Re\lambda^2)X_{\mathcal{B}}Q_{0,\mathcal{B}}$. Using Proposition 5.4, we get

$$\det_2(\mathbb{I} - A_{\mathcal{B}}^0)^{-1/2} \leq e^{O(1)\rho^2|\mathcal{B}|}$$

which reproduces the desired bound. Remark that the Wick-ordering here is absolutely essential to suppress the $\text{Tr}[A_{\mathcal{B}}^0]$ term, since that term is *linearly* divergent.

The bound on $\int d\nu_{\mathcal{B}} \prod_a e^{-4\Re(\tilde{V}_{j_a}^{\leq 2}(\bar{\sigma}^a))}$ is similar. It consists in an exact Gaussian integration but this time with a source term $\int_0^1 dt_j \text{Tr}[D'_{2,j} \Sigma_{\leq j}]$, see eq. (3.8). Let us define $\mathcal{D}_{2,j}$ as $C^{1/2}D'_{2,j}C^{1/2}$, $\underline{\mathcal{D}}_{2,j}$ as $\frac{1}{\lambda^5} \int_0^1 dt_j \mathcal{D}_{2,j}$ and $\bar{\mathcal{D}}_{2,j}$ such that $(\bar{\mathcal{D}}_{2,j})_c := \text{Tr}_c \underline{\mathcal{D}}_{2,j}$. Then,

$$\int d\nu_{\mathcal{B}} \prod_{a \in \mathcal{B}} e^{-4\Re(\tilde{V}_{j_a}^{\leq 2}(\bar{\sigma}^a))} = \det_2(\mathbb{I} + 4X_{\mathcal{B}}\tilde{Q}_{1,\mathcal{B}})^{-1/2} \exp\left(72[\Re(\lambda^5)]^2 \left(\bar{\mathcal{D}}_{2,\mathcal{B}}, \frac{X_{\mathcal{B}}}{\mathbb{I} + 4X_{\mathcal{B}}\tilde{Q}_{1,\mathcal{B}}} \bar{\mathcal{D}}_{2,\mathcal{B}}\right)\right)$$

where $\bar{\mathcal{D}}_{2,\mathcal{B}}$ is the vector of vectors such that $(\bar{\mathcal{D}}_{2,\mathcal{B}})_a := \bar{\mathcal{D}}_{2,j_a}$ for all $a \in \mathcal{B}$ and (\cdot, \cdot) denotes the natural scalar product on $V_{\mathcal{B}}$ inherited from the one on $L(\mathcal{H})^{\times}$. Using Proposition 5.4 the determinant prefactor is bounded by $\exp(O(1)\rho^2)$. As the norm of $X_{\mathcal{B}}\tilde{Q}_{1,\mathcal{B}}$ is bounded above by $O(1)\rho$ and the one of $X_{\mathcal{B}}$ is not greater than $|\mathcal{B}|$, we have, for ρ small enough,

$$\left| \left(\bar{\mathcal{D}}_{2,\mathcal{B}}, \frac{X_{\mathcal{B}}}{\mathbb{I} + 4X_{\mathcal{B}}\tilde{Q}_{1,\mathcal{B}}} \bar{\mathcal{D}}_{2,\mathcal{B}}\right) \right| \leq O(1)|\mathcal{B}|\|\bar{\mathcal{D}}_{2,\mathcal{B}}\|^2 = O(1)|\mathcal{B}| \sum_{a \in \mathcal{B}} \sum_{c=1}^4 \text{Tr}_c [((\bar{\mathcal{D}}_{2,j_a})_c)^2].$$

From the definition of D_2 , see eq. (1.11), and the bound on $A_{\mathcal{M}_2}^r$ (Lemma 1.1), one easily gets $\|\bar{\mathcal{D}}_{2,\mathcal{B}}\|^2 \leq O(1)$ which implies

$$\int d\nu_{\mathcal{B}} \prod_{a \in \mathcal{B}} e^{-4\Re(\tilde{V}_{j_a}^{\leq 2}(\bar{\sigma}^a))} \leq e^{O(1)\rho^2|\mathcal{B}|}.$$

But by far the lengthiest and most difficult bound is the one for $\int d\nu_{\mathcal{B}} \prod_a e^{4|V_j^{\geq 3}|}$, which we treat now. We will actually bound a slightly more general expression.

Theorem 5.5. — *For all $\mathcal{B}' \subset \mathcal{B}$, for all real number α , for ρ small enough and for any value of the w interpolating parameters, there exist positive numbers $K_{\alpha}^{(1)}$ and $K_{\alpha}^{(2)}$ depending on α such that*

$$I_{\mathcal{B}'}^{(3)}(\alpha) := \int d\nu_{\mathcal{B}} \prod_{a \in \mathcal{B}'} e^{\alpha|V_{j_a}^{\geq 3}(\bar{\sigma}^a)|} \leq K_{\alpha}^{(1)} 2^{|\mathcal{B}'|} e^{K_{\alpha}^{(2)}\rho^{3/2}|\mathcal{B}'|}.$$

Corollary 5.6. — *For ρ small enough and for any value of the w interpolating parameters, if $b \geq 2$*

$$\int d\nu_{\mathcal{B}} \prod_{a \in \mathcal{B}} e^{4|V_{j_a}^{\geq 3}(\bar{\sigma}^a)|} \leq O(1)|\mathcal{B}| e^{O(1)\rho^{3/2}|\mathcal{B}|}.$$

From now on we fix a subset \mathcal{B}' of \mathcal{B} . For any $j \in S_{\mathcal{B}'}$ and any integer $p_j \geq 0$ we write

$$(5.2) \quad e^{\alpha|V_j^{\geq 3}|} = \mathcal{P}_j + \mathcal{R}_j, \quad \mathcal{P}_j := \sum_{k=0}^{p_j} \frac{\alpha|V_j^{\geq 3}|^k}{k!}, \quad \mathcal{R}_j := \int_0^1 dt_j (1-t_j)^{p_j} \frac{\alpha|V_j^{\geq 3}|^{p_j+1}}{p_j!} e^{\alpha t_j |V_j^{\geq 3}|}.$$

We choose $p_j = M^j$ (assuming M integer for simplicity) and, in $\prod_{a \in \mathcal{B}'} e^{\alpha |V_{j_a}^{\geq 3}|}$, we distinguish the set \mathcal{A} of indices in which we choose the remainder term from its complement $\bar{\mathcal{A}} = \mathcal{B}' \setminus \mathcal{A}$. The result is:

$$\prod_{a \in \mathcal{B}'} e^{\alpha |V_{j_a}^{\geq 3}|} = \sum_{\mathcal{A} \subset \mathcal{B}'} \prod_{a \in \mathcal{A}} \mathcal{R}_{j_a} \prod_{a \in \bar{\mathcal{A}}} \mathcal{P}_{j_a} = \sum_{\mathcal{A} \subset \mathcal{B}'} \left(\prod_{a \in \mathcal{A}} \frac{\alpha^{p_{j_a}+1}}{p_{j_a}!} \right) \times \sum_{\{k_a: a \in \bar{\mathcal{A}}\}=0}^{\{p_{j_a}\}} \left(\prod_{a \in \bar{\mathcal{A}}} \frac{\alpha^{k_a}}{k_a!} \right) \mathcal{I}(\mathcal{A}, \{k_a\})$$

with

$$\mathcal{I}(\mathcal{A}, \{k_a\}) = \prod_{a \in \mathcal{A}} \left(\int_0^1 dt_{j_a} (1 - t_{j_a})^{p_{j_a} |V_{j_a}^{\geq 3}| p_{j_a} + 1} e^{\alpha t_{j_a} |V_{j_a}|} \right) \prod_{a \in \bar{\mathcal{A}}} |V_{j_a}^{\geq 3}|^{k_a}.$$

To simplify the notations we put by convention $k_a := p_{j_a} + 1$ for $a \in \mathcal{A}$. Remember there is no sum over k_a for such $a \in \mathcal{A}$. Hence we write

$$\prod_{a \in \mathcal{B}'} e^{\alpha |V_{j_a}^{\geq 3}|} = \sum_{\mathcal{A} \subset \mathcal{B}'} \sum_{\{k_a: a \in \bar{\mathcal{A}}\}=0}^{\{p_{j_a}\}} \left(\prod_{a \in \mathcal{B}'} \frac{\alpha^{k_a}}{k_a!} \right) \left(\prod_{a \in \mathcal{A}} k_a \right) \mathcal{I}(\mathcal{A}, \{k_a\}).$$

Let us fix from now on both the subset \mathcal{A} and the integers $\{k_a\}_{a \in \bar{\mathcal{A}}}$ and bound the remaining integral of $\mathcal{I}(\mathcal{A}, \{k_a\})$ with the measure $d\nu_{\mathcal{B}}$. We bound trivially the t_{j_a} integrals and separate again the perturbative from the non-perturbative terms through a Cauchy-Schwarz inequality:

$$(5.3) \quad \int d\nu_{\mathcal{B}} \mathcal{I}(\mathcal{A}, \{k_a\}) \leq \underbrace{\left(\int d\nu_{\mathcal{B}} \prod_{a \in \mathcal{A}} e^{2\alpha |V_{j_a}^{\geq 3}|} \right)^{1/2}}_{\text{non-perturbative}} \underbrace{\left(\int d\nu_{\mathcal{B}} \prod_{a \in \mathcal{B}'} |V_{j_a}^{\geq 3}|^{2k_a} \right)^{1/2}}_{\text{perturbative}}.$$

Note that the non-perturbative term is $I_{\mathcal{A}}^{(3)}(2\alpha)$. Thus in order to get the bound of Theorem 5.5 on $I_{\mathcal{B}'}^{(3)}(\alpha)$, we need a (fortunately cruder) bound on it. This is the object of Proposition 5.7. This bound is actually much worse than in [DR16], as it is growing with a power M^{j_1} rather than logarithmically. But ultimately it will be controlled by the expansion (5.2).

Proposition 5.7. — *For all $\mathcal{B}' \subset \mathcal{B}$, let j_1 stand for $\sup_{a \in \mathcal{B}'} j_a$. For all real number α , for ρ small enough and for any value of the w interpolating parameters, there exists positive numbers K and K_α (the latter depending on α solely) such that*

$$I_{\mathcal{B}'}^{(3)}(\alpha) = \int d\nu_{\mathcal{B}} \prod_{a \in \mathcal{B}'} e^{\alpha |V_{j_a}^{\geq 3}(\vec{\sigma}^a)|} \leq K^{|\mathcal{B}'|} e^{K_\alpha \rho M^{j_1}}.$$

Proof. — We use the quadratic bound of Lemma 4.1. Note that $Q_j(\vec{\sigma}) = \vec{\sigma} \cdot (Q_{0,j} + Q_{1,j}^{(1)}) \vec{\sigma} =: \vec{\sigma} \cdot Q_j^{(01)} \vec{\sigma}$. Thus

$$\int d\nu_{\mathcal{B}} \prod_{a \in \mathcal{B}'} e^{\alpha |V_{j_a}^{\geq 3}(\vec{\sigma}^a)|} \leq e^{k\alpha\rho|\mathcal{B}'|} \int d\nu_{\mathcal{B}} e^{k\alpha\rho \sum_{a \in \mathcal{B}'} \vec{\sigma}^a \cdot Q_{j_a}^{(01)} \vec{\sigma}^a} =: K^{|\mathcal{B}'|} \int d\nu_{\mathcal{B}} e^{k\alpha\rho(\vec{\underline{\sigma}}, Q_{\mathcal{B}'}^{(01)} \vec{\underline{\sigma}})}$$

where $Q_{\mathcal{B}'}^{(01)}$ is now a linear operator on $V_{\mathcal{B}}$. Defining $A := k\alpha\rho X_{\mathcal{B}} Q_{\mathcal{B}'}^{(01)}$, we have

$$\int d\nu_{\mathcal{B}} e^{k\alpha\rho(\vec{\underline{\sigma}}, Q_{\mathcal{B}'}^{(01)} \vec{\underline{\sigma}})} = [\det(\mathbb{I} - A)]^{-1/2},$$

and we conclude with Proposition 5.4. \square

We turn now to the second (perturbative) factor in eq. (5.3), namely $\int d\nu_{\mathcal{B}} \prod_{a \in \mathcal{B}'} |V_{j_a}^{\geq 3}|^{2k_a}$. We replace each $|V_{j_a}^{\geq 3}|^2$ by its *quartic* bound (see Corollary 4.6)

$$|V_j^{\geq 3}|^2 \leq O(1)\rho^3 (M^{-(2-\epsilon)j} + \sum_{i=1}^4 \sum_{\alpha \in A_i} |U_j^{i,\alpha}|^2)$$

and Wick-contract the result. It is indexed by graphs of order $2 \sum_{a \in \mathcal{A}} k_a$. More precisely, any such graph has, for all $a \in \mathcal{B}$ and all $i \in \{1, 2, 3, 4\}$, $q_{a,i}$ pairs of loop vertices of the $U_{j_a}^i$ type (their subindex α will play no further role), and $q_{a,0}$ pairs of constants $\rho^{10} M^{-(2-\epsilon)j_a}$, with

$$q_{a,4} + q_{a,3} + q_{a,2} + q_{a,1} + q_{a,0} = k_a.$$

Let us put

$$(5.4) \quad \begin{aligned} q_r &:= \sum_{a \in \mathcal{B}} q_{a,r} \text{ for } r \in [4]_0 := \{0, 1, \dots, 4\}, & q &:= \sum_{a \in \mathcal{B}} k_a = \sum_{r=0}^4 q_r, \\ Q_{\text{adm}} &:= \left\{ q_{a,r} \in \mathbb{N}, a \in \mathcal{A}, r \in [4]_0 : \forall a \in \mathcal{A}, \sum_{r=0}^4 q_{a,r} = k_a \right\}, & \varphi &:= \sum_{r=0}^4 2r q_r. \end{aligned}$$

$q = n/2$ is the total number of $|V^{\geq 3}|^2$ vertices in the second factor of eq. (5.3) (and half the order n of our graphs), φ is the number of σ -fields for a given choice of a sequence $(q_{a,r}) \in Q_{\text{adm}}$. Each Wick-contraction results in a graph G equipped with a scale attribution $\nu : V(G) := \{\text{loop vertices}\} \rightarrow [j_{\text{max}}]_0$ which associates to each (loop) vertex $a \in \mathcal{B}$ of G an integer j_a reminding us that exactly one of the propagators C of this vertex a bears a cut-off $\mathbf{1}_{j_a}$. In the sequel such a contraction will be denoted G^ν .

The quartic bound of Corollary 4.6 having exactly ten terms, developing a product of q such factors produces 10^q terms. The number of graphs obtained by Wick contracting $2r$ fields is simply $(2r)!! \leq O(1)^r r!$. But if these graphs have uniformly bounded coordination at each vertex *and* a certain number t of tadpoles (*i.e.* contractions of fields belonging to the same vertex), the combinatorics is lower. Indeed the total number of Wick contractions with $2r$ fields and vertices of maximal degree four leading to graphs with exactly t tadpoles is certainly bounded by $O(1)^r (r-t)!$.

Hence using these remarks we find:

$$(5.5) \quad \int d\nu_{\mathcal{B}} \prod_{a \in \mathcal{B}'} |V_{j_a}^{\geq 3}|^{2k_a} \leq (O(1)\rho^3)^q \sup_{\substack{(q_{a,r}) \in Q_{\text{adm}}, \\ 0 \leq t \leq \varphi/2}} M^{-(2-\epsilon) \sum_a q_{a,0} j_a} (\varphi/2 - t)! \sup_{G, t(G)=t} A_{G^\nu}$$

where the supremum is taken over graphs G with $q_{a,r}$ pairs of loop vertices of length r and highest scale j_a for all $r \in [4]$ and all $a \in \mathcal{A}$, and t is the total number of tadpoles of G . In the right-hand side of eq. (5.5), the scale attribution ν is fixed *i.e.* the supremum is not taken over it. The following lemma gives an estimate of A_{G^ν} .

Lemma 5.8. — *There exists $0 < \epsilon \ll 1$ such that any intermediate field graph G^ν of order n , made of propagators joining $n_{r,j}$ loop vertices U_j^r of length r with $t_{r,j}$ loop vertices U_j^r bearing at least a tadpole, for r in $[4]$, obeys the bound*

$$(5.6) \quad |A_G| \leq O(1)^n \prod_{j \in \nu(V(G))} M^{-\frac{1}{2}j[n_{1,j} + 3n_{2,j} - (1+\epsilon)t_{2,j} + 3n_{3,j} - t_{3,j} + 3n_{4,j} - t_{4,j}]}.$$

Proof. — As usual such a power counting result is obtained thanks to multiscale analysis. Each graph G^ν is already equipped with one scale per loop vertex: for all vertex $a \in \mathcal{B}$ there is exactly one C -propagator C_a of scale $j_a = \nu(a)$ (namely in the trace represented by that vertex we have the combination $C_a \mathbf{1}_{j_a}$). We further decompose all remaining C -propagators ($C \mathbf{1}_{\leq j}$) using $\mathbf{1}_{\leq j} = \sum_{k=0}^j \mathbf{1}_k$. Each graph G^ν is now a sum over scale attributions μ (depending on ν) of graphs G_μ^ν which bear one scale per C -propagator. We will first estimate $A_{G_\mu^\nu}$ and then sum over μ to get eq. (5.6).

The intermediate-field graph G_μ^ν is made of edges, of faces f and of loop-vertex corners (in short LVC) ℓ which correspond to C -propagators, hence to the edges of the underlying ordinary graph in the standard representation. Each LVC ℓ has exactly one scale index $j(\ell)$, and we can assume that the r LVCs of a loop vertex v of order $r(v) = r$ (in short, a r LV) are labelled as $\ell_1, \ell_2, \dots, \ell_r$ so that $j(\ell_1) =: j_1 \geq j(\ell_2) =: j_2 \geq \dots \geq j(\ell_r) =: j_r$. Each sum over a face index costs therefore $O(1)M^{j_m(f)}$ where $j_m(f)$ is the minimum over indices of all the LVCs through which the face runs. Hence

$$(5.7) \quad A_{G_\mu^\nu} \leq O(1)^n \prod_{\ell} M^{-2j(\ell)} \prod_f M^{j_m(f)}.$$

This bound is optimal but difficult to analyse. In particular it depends on the topology of G , see [BGR12a; OSVT13]. In our context of a *super*-renormalisable model, we can afford to weaken it and consequently get a new bound which will be factorised over the loop vertices of G . It will have the advantage of depending only on the types and number of vertices of G , thus furnishing also an upper bound for the \sup_G in eq. (5.5).

We call a face f local with respect to a loop vertex (hereafter LV) v if it runs only through corners of v . The set of faces and local faces of G are denoted respectively $F(G)$ and $F_{\text{loc}}(G)$. The complement of F_{loc} in F is F_{nl} , the set of non-local faces of G . Let f be a face of G and v be one of the vertices of G . If f is incident with v , we define $j_m^v(f)$ as the minimum over indices of all the LVCs of v through which the face f runs. Otherwise, $j_m^v(f) := 0$. If f is non-local then it visits at least two LVs. In that case, we replace $j_m(f)$ by the bigger factor $\prod_{v \leftarrow f} M^{j_m^v(f)/2}$ where the product runs over the vertices incident with f :

$$\begin{aligned} A_{G_\mu} &\leq O(1)^n \prod_{v \in V(G)} \prod_{i=1}^{r(v)} M^{-2j_i} \prod_{f \in F_{\text{loc}}(G)} M^{j_m(f)} \prod_{f \in F_{\text{nl}}(G)} \prod_{v \leftarrow f} M^{j_m^v(f)/2} \\ &= O(1)^n \prod_{v \in V(G)} \underbrace{\left(\prod_{i=1}^{r(v)} M^{-2j_i} \prod_{\substack{f \in F_{\text{loc}}(G), \\ f \rightarrow v}} M^{j_m^v(f)} \prod_{\substack{f \in F_{\text{nl}}(G), \\ f \rightarrow v}} M^{j_m^v(f)/2} \right)}_{=: W(v)}. \end{aligned}$$

Our bound is now factorised over the loop vertices of G and we can simply bound the contribution $W(v)$ of each vertex v according to its type.

Consider a 3LV; it can be of type c^3 , $c_1^2 c_2$ or $c_1 c_2 c_3$, depending on whether the three lines hooked to it have the same colour c , two different colours c_1, c_2 or three different colours c_1, c_2, c_3 , see fig. 8. Only

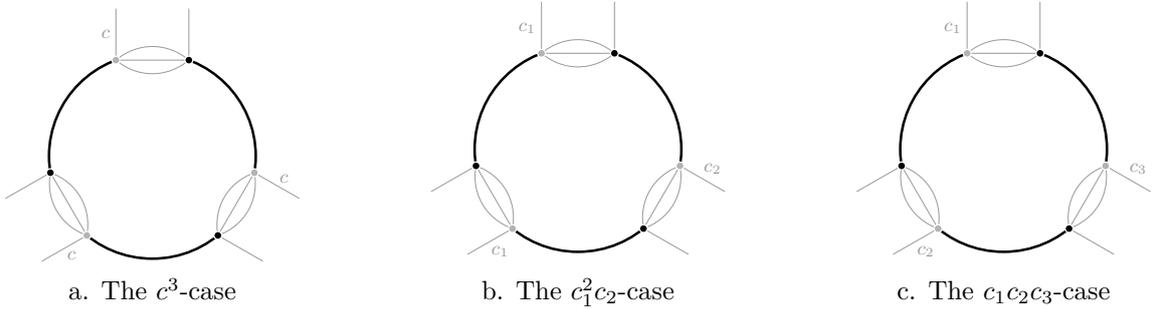


Figure 8. The three coloured versions of a U^3 -loop vertex.

in the two first cases can it have a tadpole, and then one local face incident with a single LVC *i.e.* of length one. Hence:

- In case c^3 , the three faces of length 3 and colour $c' \neq c$ are local, see fig. 9a, and their total cost is M^{3j_3} . In case there is a tadpole (of colour c and LVC $t \in \{1, 2, 3\}$), its local face, see fig. 9c, costs M^{j_t} and the other (non-local) face of colour c , see fig. 9d, costs at most $\inf_{t' \neq t} M^{j_{t'}/2}$. The worst case is when $t = 1$, in which case the total cost of colour c faces is $M^{j_1+j_3/2}$. In case there is no tadpole, the faces of colour c are non-local. There are at most three of them, so their cost is at worst $M^{j_1/2+j_2/2+j_3/2}$. The worst case is therefore the tadpole case with $t = 1$, where the total face cost is $M^{j_1+7j_3/2}$. Joining to the $M^{-2(j_1+j_2+j_3)}$ factor the vertex weight $W(v)$ is therefore bounded in the c^3 case by $M^{-j_1-j_2/2-3(j_2-j_3)/2}$.
- In case $c_1^2 c_2$, the two local faces of length three (and colour $c \neq c_1, c_2$) cost M^{2j_3} and the non-local face of colour c_2 , see fig. 9b, costs $M^{j_3/2}$. In case there is a tadpole (of colour c_1 and LVC $t \in \{1, 2, 3\}$), its face costs M^{j_t} and the other local face of colour c_1 (and length 2) costs $\inf_{t' \neq t} M^{j_{t'}}$; in case there is no tadpole, the single or the two non-local faces of colour c_1 cost at most $M^{j_1/2+j_3/2}$. The worst case is therefore again the tadpole case with $t = 1$, where the total face cost is again $M^{j_1+7j_3/2}$, and the vertex weight $W(v)$ is therefore again bounded in the $c_1^2 c_2$ case by $M^{-j_1-j_2/2-3(j_2-j_3)/2}$.

- Finally the case $c_1 c_2 c_3$ is simpler as there can be no tadpole. The three non-local faces cost in total $M^{3j_3/2}$, the local face costs M^{j_3} , and the vertex weight $W(v)$ is therefore bounded by the better factor $M^{-2j_1-3j_2/2-(j_2-j_3)/2}$.

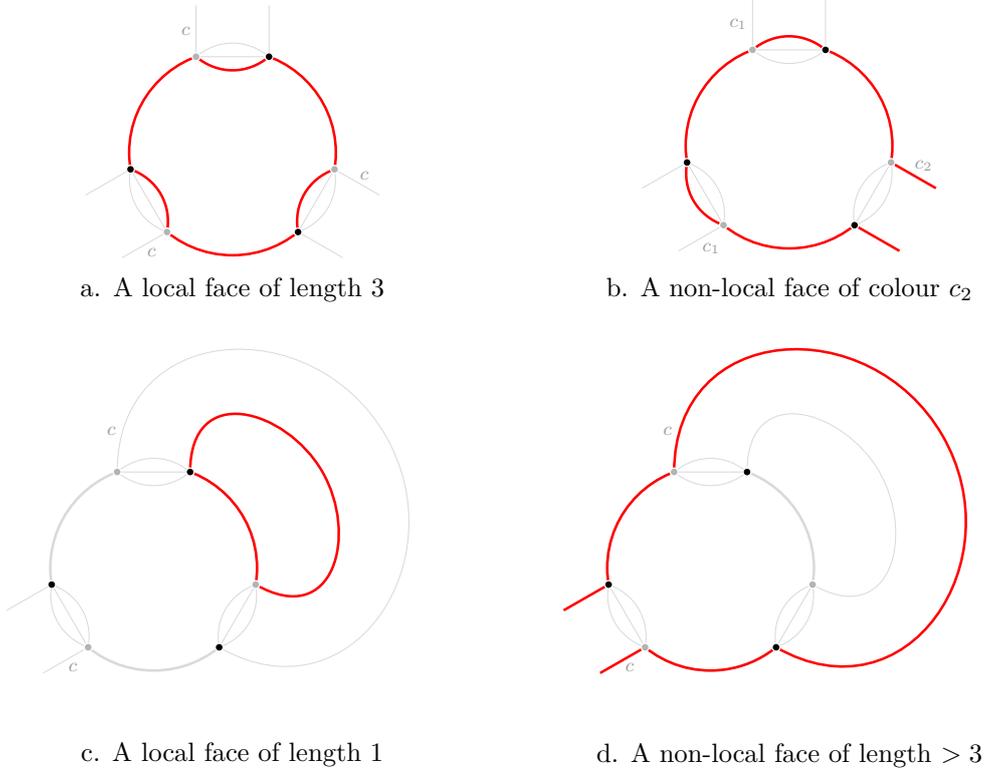


Figure 9. Some faces of a U^3 -vertex

The same analysis can be repeated for 4LV's. As it is somewhat tedious, we postpone it to Appendix A.5.1. There, it can be checked that the worst total face cost is:

- with two tadpoles, $M^{j_1+j_2+4j_4}$,
- with one tadpole, $M^{j_1+j_2/2+7j_4/2}$,
- without tadpole, $M^{(j_1+j_2+j_3+7j_4)/2}$.

The vertex weight $W(v)$ is therefore, when tadpole(s) are present, at worst $M^{-j_1-j_2-2(j_3-j_4)}$, and when they are not $M^{-3j_1/2-3j_2/2-3(j_3-j_4)/2}$. The worst total face costs for loop vertices of degree one and two are available in Appendix A.5.

With a bound on $A_{G^\nu_\mu}$, there remains to sum over μ to get eq. (5.6). We decompose this sum into two parts: first a sum over the relative positions of j_2, \dots, j_r at all vertices of degree $r \geq 2$. This costs at worst $3!^n$. Then a sum over $j_2 \geq \dots \geq j_r$ at each loop vertex. The analysis above has shown that this is convergent and leads to the bound (5.6) and thus to Lemma 5.8. \square

Coming back to the notations of eqs. (5.4) and (5.5) and remembering that the $q_{a,r}$'s are meant for *pairs* of vertices,

$$\sup_{G, t(G)=t} A_{G^\nu} \leq O(1)^n \prod_{a \in \mathcal{B}} M^{-j_a [q_{a,1} + 3q_{a,2} - (\frac{1}{2} + \epsilon)t_{a,2} + 3q_{a,3} - \frac{1}{2}t_{a,3} + 3q_{a,4} - \frac{1}{2}t_{a,4}]},$$

where $t_{a,r} := t_{r,j_a}$, $r = 2, 3, 4$, is the total number of vertices of length r and scale j_a in G which bear at least one tadpole. We put $\tau_{a,r} = t_{a,r}/2$ and $\tau_r = \sum_a \tau_{a,r}/2$. In eq. (5.5) we remark that $t = \sum_{r \geq 2} \sum_a t_{a,r} = 2 \sum_r \tau_r$. Since $q_1 + q_2 + q_3 + q_4 \leq q$, the factor $(\varphi/2 - t)! = (\sum_{r=1}^4 r q_r - t)!$ in

eq. (5.5) is bounded by $O(1)^q \prod_r (q_r!)^r (\tau_r!)^{-2}$ (we put $\tau_1 = 0$ and interpret $n!$ for n not integer as $\Gamma(n)$). Hence the perturbative factor of eq. (5.3) obeys ($\tau_1 = 0$)

$$\left(\int d\nu_{\mathcal{B}} \prod_{a \in \mathcal{B}'} |V_{j_a}^{\geq 3}|^{2k_a} \right)^{1/2} \leq (O(1)\rho^{3/2})^q \sup_{\substack{(q_{a,r}) \in Q_{\text{adm}}, \\ 0 \leq \tau_{a,r} \leq q_{a,r}}} \left(\prod_{r=1}^4 (q_r!)^{r/2} (\tau_r!)^{-1} \right) \\ \times \prod_{a \in \mathcal{B}} M^{-\frac{1}{2}j_a [(2-\epsilon)q_{a,0} + q_{a,1} + \sum_{r=2}^4 (3q_{a,r} - \tau_{a,r}) - \epsilon\tau_{a,2}]}.$$

Joining this last estimate with Proposition 5.7, the term to be bounded in Theorem 5.5 obeys

$$\int d\nu_{\mathcal{B}} \prod_{a \in \mathcal{B}'} e^{\alpha |V_{j_a}^{\geq 3}(\bar{\sigma}^a)|} \leq \sum_{\mathcal{A} \subset \mathcal{B}'} K^{|\mathcal{A}|} e^{K_{\alpha}^{(1)} \rho M^{j_1}} \sum_{\{k_a, a \in \bar{\mathcal{A}}\}=0}^{\{p_{j_a}\}} (O(1)\rho^{3/2})^q \left(\prod_{a \in \mathcal{B}'} \frac{\alpha^{k_a}}{k_a!} \right) \left(\prod_{a \in \mathcal{A}} k_a \right) \\ \sup_{\substack{(q_{a,r}) \in Q_{\text{adm}}, \\ 0 \leq \tau_{a,r} \leq q_{a,r}}} \left(\prod_{r=1}^4 (q_r!)^{r/2} (\tau_r!)^{-1} \right) \prod_{a \in \mathcal{B}} M^{-\frac{1}{2}j_a [(2-\epsilon)q_{a,0} + q_{a,1} + \sum_{r=2}^4 (3q_{a,r} - \tau_{a,r}) - \epsilon\tau_{a,2}]}.$$

where again $j_1 = \sup_{a \in \mathcal{A}} j_a$. Note that we use, and will go on using, the symbols $K, K_{\alpha}, K_{\alpha}^{(1)}, K_{\alpha}^{(2)}$ etc essentially the same way as we do with $O(1)$ *i.e.* to denote generic constants possibly depending on α . In the rest of this proof, our strategy will be to use the power counting namely the powers of M^{-j_a} to compensate both for the large number of Wick contractions (the $q_r!$'s) and for the crude bound of Proposition 5.7.

As $\tau_r = \sum_a \tau_{a,r}$, $(\tau_r!)^{-1} \leq \prod_a (\tau_{a,r}!)^{-1}$. Similarly, since $k_a = \sum_r q_{a,r}$, $(k_a!)^{-1} \leq \prod_r (q_{a,r}!)^{-1}$. Moreover we remark that $\prod_{a \in \mathcal{B}'} \alpha^{k_a} \prod_{a \in \mathcal{A}} k_a \leq (\sup \{2, \alpha\})^q$. Hence

$$(5.8) \quad \int d\nu_{\mathcal{B}} \prod_{a \in \mathcal{B}'} e^{\alpha |V_{j_a}^{\geq 3}|} \leq \sum_{\mathcal{A} \subset \mathcal{B}'} K^{|\mathcal{A}|} e^{K_{\alpha}^{(1)} \rho M^{j_1}} \sum_{\{k_a, a \in \bar{\mathcal{A}}\}=0}^{\{p_{j_a}\}} (K_{\alpha}^{(2)} \rho^{3/2})^q \sup_{\substack{(q_{a,r}) \in Q_{\text{adm}}, \\ 0 \leq \tau_{a,r} \leq q_{a,r}}} \prod_{r=1}^4 \left((q_r!)^{r/2} \prod_{a \in \mathcal{B}'} (q_{a,r}! \tau_{a,r}!)^{-1} \right) \prod_{a \in \mathcal{B}'} M^{-\frac{1}{2}j_a [(2-\epsilon)q_{a,0} + q_{a,1} + \sum_{r=2}^4 (3q_{a,r} - \tau_{a,r}) - \epsilon\tau_{a,2}]}.$$

For $r = 2, 3, 4$ we remark that if $\tau_{a,r} \leq q_{a,r}/2$, we have

$$(\tau_{a,r}!)^{-1} M^{-\frac{1}{2}j_a (3q_{a,r} - \tau_{a,r})} \leq M^{-\frac{5}{4}j_a q_{a,r}},$$

and if $\tau_{a,r} \geq q_{a,r}/2$ (and of course $\tau_{a,r} \leq q_{a,r}$),

$$(\tau_{a,r}!)^{-1} M^{-\frac{1}{2}j_a (3q_{a,r} - \tau_{a,r})} \leq 2^{q_{a,r}} (q_{a,r}!)^{-1/2} M^{-j_a q_{a,r}}.$$

In the sequel we will use the following simple bound several times: for any $\eta \in \mathbb{R}_+^*$,

$$(5.9) \quad M^{-\eta j_a q_{a,r}} \leq K^{\eta q_{a,r}} (q_{a,r}!)^{-\eta},$$

This is an easy consequence of $q_{a,r} \leq k_a \leq M^{j_a+1}$. Thus, using eq. (5.9) with $\eta = 1/4$, we have that for all $\tau_{a,r}$,

$$(\tau_{a,r}!)^{-1} M^{-\frac{1}{2}j_a (3q_{a,r} - \tau_{a,r})} \leq O(1)^{q_{a,r}} (q_{a,r}!)^{-1/4} M^{-j_a q_{a,r}}.$$

Using $\tau_{a,2} \leq q_{a,2}$, eq. (5.8) then becomes

$$\int d\nu_{\mathcal{B}} \prod_{a \in \mathcal{B}'} e^{\alpha |V_{j_a}^{\geq 3}|} \leq \sum_{\mathcal{A} \subset \mathcal{B}'} K^{|\mathcal{A}|} e^{K_{\alpha}^{(1)} \rho M^{j_1}} \sum_{\{k_a, a \in \bar{\mathcal{A}}\}=0}^{\{p_{j_a}\}} (K_{\alpha}^{(2)} \rho^{3/2})^q \sup_{(q_{a,r}) \in Q_{\text{adm}}} (q_1!)^{1/2} \prod_{a \in \mathcal{B}'} (q_{a,1})^{-1} \\ \prod_{r=2}^4 \left((q_r!)^{r/2} \prod_{a \in \mathcal{B}'} (q_{a,r}!)^{-5/4} \right) \prod_{a \in \mathcal{B}'} M^{-j_a [(1-\epsilon)q_{a,0} + \frac{1}{2}q_{a,1} + (1-\epsilon)q_{a,2} + q_{a,3} + q_{a,4}]}.$$

Crude bound versus power counting. We can now take care of the $e^{K_\alpha^{(1)}\rho M^{j_1}}$ factor by using a part of the power counting. Let η be a real positive number. Remembering that for all $a \in \mathcal{A}$, $k_a = M^{j_a+1}$,

$$\prod_{a \in \mathcal{A}} M^{-\eta j_a} \sum_{r=0}^4 q_{a,r} = \prod_{a \in \mathcal{A}} M^{-\eta j_a k_a} \leq \prod_{a \in \mathcal{A}} M^{-\eta j_a M^{j_a}} \leq M^{-\eta j_1 M^{j_1}}.$$

But

$$e^{K_\alpha^{(1)}\rho M^{j_1}} \prod_{a \in \mathcal{A}} M^{-\eta j_a} \sum_{r=0}^4 q_{a,r} \leq e^{K_\alpha^{(1)}\rho M^{j_1}} M^{-\eta j_1 M^{j_1}} \leq K_{\alpha,\eta}$$

so that

$$(5.10) \quad \int d\nu_{\mathcal{B}} \prod_{a \in \mathcal{B}'} e^{\alpha |V_{j_a}^{\geq 3}|} \leq K_{\alpha,\eta} \sum_{\mathcal{A} \subset \mathcal{B}'} K^{|\mathcal{A}|} \sum_{\{k_a, a \in \bar{\mathcal{A}}\}=0}^{\{p_{j_a}\}} (K_\alpha^{(2)} \rho^{3/2})^q \sup_{(q_{a,r}) \in Q_{\text{adm}}} (q_1!)^{1/2} \prod_{a \in \mathcal{B}'} (q_{a,1})^{-1} \\ \prod_{r=2}^4 \left((q_r!)^{r/2} \prod_{a \in \mathcal{B}'} (q_{a,r})^{-5/4} \right) \prod_{a \in \mathcal{B}'} M^{-j_a [(1-\epsilon)q_{a,0} + \frac{1}{2}q_{a,1} + (1-\epsilon)q_{a,2} + q_{a,3} + q_{a,4} - \eta k_a]}.$$

Combinatorics versus power counting. In order to beat the $q_r!$'s, we need to boost the powers of some of the $q_{a,r}!$'s. We use eq. (5.9) for the couples (r, η) equal to $(3, 1/4)$ and $(4, 3/4)$. Eq. (5.10) becomes

$$\int d\nu_{\mathcal{B}} \prod_{a \in \mathcal{B}'} e^{\alpha |V_{j_a}^{\geq 3}|} \leq K_{\alpha,\eta} \sum_{\mathcal{A} \subset \mathcal{B}'} K^{|\mathcal{A}|} \sum_{\{k_a, a \in \bar{\mathcal{A}}\}=0}^{\{p_{j_a}\}} (K_\alpha^{(2)} \rho^{3/2})^q \sup_{(q_{a,r}) \in Q_{\text{adm}}} \\ \prod_{r=1}^4 \left((q_r!)^{r/2} \prod_{a \in \mathcal{B}'} (q_{a,r})^{-r/2} \right) \prod_{a \in \mathcal{B}'} M^{-j_a [(1-\epsilon)q_{a,0} + \frac{1}{2}q_{a,1} + (1-\epsilon)q_{a,2} + \frac{3}{4}q_{a,3} + \frac{1}{4}q_{a,4} - \eta k_a]}.$$

Then for $\epsilon \leq 3/4$ and $\eta < 1/4$,

$$\int d\nu_{\mathcal{B}} \prod_{a \in \mathcal{B}'} e^{\alpha |V_{j_a}^{\geq 3}|} \leq K_{\alpha,\eta} \sum_{\mathcal{A} \subset \mathcal{B}'} K^{|\mathcal{A}|} \sum_{\{k_a, a \in \bar{\mathcal{A}}\}=0}^{\{p_{j_a}\}} (K_\alpha^{(2)} \rho^{3/2})^q \sup_{(q_{a,r}) \in Q_{\text{adm}}} \\ \prod_{r=1}^4 \left(q_r! \prod_{a \in \mathcal{B}'} (q_{a,r})^{-1} M^{-\frac{2}{r}(\frac{1}{4}-\eta)j_a q_{a,r}} \right)^{r/2}.$$

Now we remark that for all r , by the multinomial theorem, $q_r! \prod_{a \in \mathcal{B}'} (q_{a,r})^{-1} M^{-\frac{2}{r}(\frac{1}{4}-\eta)j_a q_{a,r}}$ is one of the terms in the multinomial expansion of $(\sum_{a \in \mathcal{B}'} M^{-\frac{2}{r}(\frac{1}{4}-\eta)j_a})^{q_r}$. Since the j_a 's are all distinct, $(q_r! \prod_{a \in \mathcal{B}'} (q_{a,r})^{-1} M^{-\frac{2}{r}(\frac{1}{4}-\eta)j_a q_{a,r}})^{r/2} \leq (\sum_{j \geq 0} M^{-\frac{2}{r}(\frac{1}{4}-\eta)j})^{q_r r/2} = (K_{r,\eta})^{q_r} \leq O(1)^q$. Hence

$$\int d\nu_{\mathcal{B}} \prod_{a \in \mathcal{B}'} e^{\alpha |V_{j_a}^{\geq 3}|} \leq K_{\alpha,\eta} \sum_{\mathcal{A} \subset \mathcal{B}'} K^{|\mathcal{A}|} \sum_{\{k_a, a \in \bar{\mathcal{A}}\}=0}^{\{p_{j_a}\}} (K_\alpha^{(2)} \rho^{3/2})^q.$$

Let $q_{\mathcal{A}}$ denote $\sum_{a \in \mathcal{A}} k_a$. Then we have

$$\int d\nu_{\mathcal{B}} \prod_{a \in \mathcal{B}'} e^{\alpha |V_{j_a}^{\geq 3}|} \leq K_{\alpha,\eta} \sum_{\mathcal{A} \subset \mathcal{B}'} K^{|\mathcal{A}|} (K_\alpha^{(2)} \rho^{3/2})^{q_{\mathcal{A}}} \prod_{a \in \bar{\mathcal{A}}} \sum_{k_a=0}^{p_{j_a}} (K_\alpha^{(2)} \rho^{3/2})^{k_a} \\ \leq K_{\alpha,\eta} \sum_{\mathcal{A} \subset \mathcal{B}'} K^{|\mathcal{A}|} (K_\alpha^{(2)} \rho^{3/2})^{q_{\mathcal{A}}} \prod_{a \in \bar{\mathcal{A}}} \frac{1 - (K_\alpha^{(2)} \rho^{3/2})^{p_{j_a}+1}}{1 - K_\alpha^{(2)} \rho^{3/2}} \\ \leq K_{\alpha,\eta} \sum_{\mathcal{A} \subset \mathcal{B}'} K^{|\mathcal{A}|} (K_\alpha^{(2)} \rho^{3/2})^{q_{\mathcal{A}}} 2^{|\bar{\mathcal{A}}|} \quad \text{for } \rho \text{ small enough}$$

$$\begin{aligned}
 &\leq K_{\alpha,\eta} \sum_{\mathcal{A} \subset \mathcal{B}'} K^{|\mathcal{A}|} (K_\alpha^{(2)} \rho^{3/2})^{|\mathcal{A}|} 2^{|\bar{\mathcal{A}}|} && k_a \geq 1, a \in \mathcal{A} \\
 &\leq K_{\alpha,\eta} (2 + K_\alpha \rho^{3/2})^{|\mathcal{B}'|} \\
 &\leq K_{\alpha,\eta} 2^{|\mathcal{B}'|} e^{K_\alpha \rho^{3/2} |\mathcal{B}'|}.
 \end{aligned}$$

This completes the proof of Theorem 5.5. \square

To conclude this section, let us briefly comment on the case of a block \mathcal{B} with a single node ($|\mathcal{B}| = 1$) in Corollary 5.6. The proof of the single node case in Theorem 5.1 is very similar, even easier, than the proof of Theorem 5.5 but we need to remember that there is no term with $k = 0$ vertices, because we are dealing with $e^{-V_{j_a}^{\geq 3}(\vec{\sigma}^a)} - 1$ rather than $e^{-V_{j_a}^{\geq 3}(\vec{\sigma}^a)}$.

6. Perturbative functional integral bounds

We still have to bound the fourth ‘‘perturbative’’ factor in eq. (3.9), namely

$$I_4 = \left(\int d\nu_{\mathcal{B}} | :A_{\mathcal{G}}(\vec{\sigma}) :|^4 \right)^{1/4}.$$

It is not fully perturbative though because of the resolvents still present in $A_{\mathcal{G}}$. If $|\mathcal{B}| = 1$, we recall that the graphs \mathcal{G} are either one-vertex maps or one-edge trees. For $|\mathcal{B}| \geq 2$, they are forests with $e(\mathcal{G}) = |\mathcal{B}| - 1$ (coloured) edges joining $n(\mathcal{G}) = c(\mathcal{G}) + e(\mathcal{G})$ (effective) vertices, each of which has a weight given by eqs. (3.4) to (3.6). The number of connected components $c(\mathcal{G})$ is bounded by $|\mathcal{B}| - 1$, hence $n(\mathcal{G}) \leq 2(|\mathcal{B}| - 1)$, see eq. (3.7). I_4^4 can be reexpressed as $\int d\nu_{\mathcal{B}} A_{\mathcal{G}''}(\vec{\sigma})$ where \mathcal{G}'' is the (disjoint) union of two copies of the graph \mathcal{G} and two copies of its mirror conjugate graph \mathcal{G}' of identical structure but on which each operator has been replaced by its Hermitian conjugate. This overall graph \mathcal{G}'' has thus four times as many vertices, edges, resolvents, $\vec{\sigma}^a$ insertions and connected components than the initial graph \mathcal{G} .

6.1. Contraction process. — To evaluate the amplitude $A_{\mathcal{G}''} = \int d\nu_{\mathcal{B}} | :A_{\mathcal{G}}(\vec{\sigma}) :|^4$, we first replace any isolated vertex of type $V_j^{\geq 3}$ by its quartic bound, Lemma 4.5, and then contract every $\vec{\sigma}^a$ insertion, which means using repeatedly integration by parts until there are no $\vec{\sigma}^a$ numerators left, thanks to the formula

$$(6.1) \quad \int (\vec{\sigma}^a)_{c;mn} F(\vec{\sigma}) d\nu(\vec{\sigma}) = - \sum_{k,l} \int \delta_{ml} \delta_{nk} \frac{\partial F(\vec{\sigma})}{\partial (\vec{\sigma}^a)_{c;kl}} d\nu(\vec{\sigma}),$$

where $d\nu(\vec{\sigma})$ is the standard Gaussian measure of covariance \mathbb{I} . We call this procedure the contraction process. The derivatives $\frac{\partial}{\partial (\vec{\sigma}^a)_c}$ will act on any resolvent $\mathcal{R}_{\leq j_a}$ or any remaining $\vec{\sigma}^a$ insertion of \mathcal{G}'' , creating a new contraction edge⁸. When such a derivative acts on a resolvent,

$$(6.2) \quad \partial_{\sigma_s} \mathcal{R}_{\leq j}^{(\dagger)} = \mathcal{R}_{\leq j}^{(\dagger)} \Delta_{\leq j}^s \mathcal{R}_{\leq j}^{(\dagger)},$$

it creates two new corners representing $\sqrt{C}_{\leq j_a} \mathcal{R}_{\leq j_a} \sqrt{C}_{\leq j_a}$ or $\sqrt{C}_{\leq j_a} \mathcal{R}_{\leq j_a}^\dagger \sqrt{C}_{\leq j_a}$ product of operators. Remark that at the end of this process we have therefore obtained a sum over new *resolvent graphs* \mathcal{G} , the amplitudes of which no longer contain any $\vec{\sigma}^a$ insertion. Nevertheless the number of edges, resolvents and connected components at the end of this contraction process typically has changed. However we have a bound on the number of new edges generated by the contraction process. Since each vertex of \mathcal{G} contains at most three $\vec{\sigma}^a$ insertions⁹, \mathcal{G}'' contains at most $12n(\mathcal{G})$, hence using eq. (3.7) at most $24(|\mathcal{B}| - 1)$ insertions to contract. Each such contraction creates at most one new edge. Therefore each resolvent graph \mathcal{G} contains the initial $4(|\mathcal{B}| - 1)$ coloured edges of \mathcal{G}'' decorated with up to at most $24(|\mathcal{B}| - 1)$

⁸The combinatorics for these contractions will be paid by the small factors earned from the explicit j -th scale propagators, see Section 7.

⁹We focus here on Bosonic blocks with more more than one vertex. The case of isolated vertices will only lead to $O(1)^{|\mathcal{B}|}$ combinatorial factors which will be easily compensated by powers of the coupling constant g .

additional new edges.

Until now, the amplitude $A_{\mathcal{G}}$ contains $\sqrt{C}_{\leq j} = \sum_{j' < j} \sqrt{C}_{j'} + t_j \sqrt{C}_j$ operators. We now develop the product of all such factors as a sum over scale assignments μ , as in [Riv91]. It means that each former $\sqrt{C}_{\leq j}$ is replaced by a fixed scale $\sqrt{C}_{j'}$ operator with scale attribution $j' \leq j$ (the t_j factor being bounded by 1). The amplitude at fixed scale attribution μ is noted $A_{\mathcal{G}_\mu}$. The sum over μ will be standard to bound after the key estimate of Lemma A.2 is established. Similarly the sums over \mathbf{G} and over \mathcal{G} only generate a finite power of $|\mathcal{B}|!$, hence will be no problem using the huge decay factors of Theorem 6.12, see Section 7.

We shall now bound each amplitude $A_{\mathcal{G}_\mu}$. Were it not for the presence of resolvents, the graph \mathcal{G} , which is convergent, would certainly obey the standard bound on convergent amplitudes in super-renormalisable theories. A precise statement can be found in Lemma A.2. The only problem is therefore to get rid of these resolvents, using that their norm is bounded by a constant in the cardioid domain. This can be done through the technique of iterated Cauchy-Schwarz bounds or ICS, introduced for the first time in a similar tensor field theoretic context in [Mag+09].

6.2. Iterated Cauchy-Schwarz estimates. — Let us first give a crude description of the steps necessary to bound the amplitude of a (connected) graph \mathcal{G} by a product of amplitudes freed of resolvents.

6.2.1. ICS algorithm 1.0b. — Let \mathcal{G} be a connected graph in the intermediate field representation obtained after the contraction process *i.e.* a connected component of a resolvent graph. The following steps constitute the core of the ICS method:

1. Write the amplitude $A_{\mathcal{G}}$ of \mathcal{G} as a single trace over $L(\mathcal{H}^{\otimes})$ times a product of Kronecker deltas. This trace contains some resolvents.
2. Write $A_{\mathcal{G}}$ as a scalar product of the form $\langle \alpha, (\mathcal{R} \otimes S \otimes \mathbb{I})\beta \rangle$ or $\langle \alpha, (\mathcal{R} \otimes S \otimes \mathcal{R}^T)\beta \rangle$ where α and β are vectors of an inner product space and S is a permutation operator.
3. Apply Cauchy-Schwarz inequality to the previous expression to get

$$|A_{\mathcal{G}}| \leq \|\mathcal{R}\|^{(2)} \sqrt{\langle \alpha, \alpha \rangle} \sqrt{\langle \beta, \beta \rangle}.$$

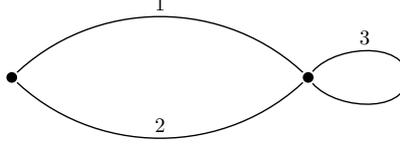
4. Notice that $\langle \alpha, \alpha \rangle$ and $\langle \beta, \beta \rangle$ are also amplitudes of some graphs. If they still contain some resolvents, iterate the process by going back to step 1.

In the rest of this section, we give a bound on the number of iterations of this algorithm before it stops. We also refine it in order to avoid pathological situations. But before that, to give the reader a more concrete idea of the method, we illustrate it now with examples. It will be the occasion to go through all steps of the iterated Cauchy-Schwarz method, and understand why the rough algorithm given above needs to be modified.

6.2.2. Concrete examples. — Let us consider the convergent graph \mathcal{G} of fig. 10, in intermediate field representation, obtained after the contraction process. *Stricto sensu* it represents a sum of different amplitudes. As any spanning tree of it contains a single edge, the possible vertices associated to this graph can be found in eq. (3.4). Let us choose to study the following expression

$$(6.3) \quad A_{\mathcal{G}} = \left(\prod_{i=1}^3 \sum_{m_i, n_i, m'_i, n'_i \in \mathbb{Z}} \right) \text{Tr} \left[(e_{m_1 n_1}^{c_1} \otimes \mathbb{I}_{\hat{c}_1}) C(e_{m_2 n_2}^{c_2} \otimes \mathbb{I}_{\hat{c}_2}) C \right. \\ \left. \times \text{Tr} [\sqrt{C}(e_{m'_2 n'_2}^{c_2} \otimes \mathbb{I}_{\hat{c}_2}) C(e_{m_3 n_3}^{c_3} \otimes \mathbb{I}_{\hat{c}_3}) C(e_{m'_3 n'_3}^{c_3} \otimes \mathbb{I}_{\hat{c}_3}) \sqrt{C} \mathcal{R} \sqrt{C}(e_{m'_1 n'_1}^{c_1} \otimes \mathbb{I}_{\hat{c}_1}) \sqrt{C} \mathcal{R}] \right. \\ \left. \times \delta_{m_1 n'_1} \delta_{n_1 m'_1} \delta_{m_2 n'_2} \delta_{n_2 m'_2} \delta_{m_3 n'_3} \delta_{n_3 m'_3} \right].$$

Vertices of \mathcal{G} correspond to traces and edges to pairs of Kronecker deltas, *e.g.* $\delta_{m_1 n'_1} \delta_{n_1 m'_1}$ is represented by edge number 1.


Figure 10. A convergent graph with resolvents

The first step consists in writing $A_{\mathcal{G}}$ as a single trace. To this aim, we apply the following identity twice (for a general graph, we need to apply it several times): let c be any non empty proper subset of $\{1, 2, 3, 4\}$ and e_{mn}^c be the tensor product $\bigotimes_{c \in c} e_{m_c n_c}^c$. Then

$$(6.4) \quad \sum_{m, n, m', n' \in \mathbb{Z}^{|c|}} (e_{mn}^c)_{ab} (e_{m'n'}^c)_{de} \delta_{mn}^{|c|} \delta_{n'm'}^{|c|} = \delta_{ae}^{|c|} \delta_{bd}^{|c|}.$$

We apply it first to $e_{m_1 n_1}^{c_1} e_{m'_1 n'_1}^{c_1}$ in $A_{\mathcal{G}}$ then to the two remaining $\mathbb{I}_{\hat{c}_1}$ factors but in the reverse direction (*i.e.* from right to left in eq. (6.4)). We get

$$(6.5) \quad A_{\mathcal{G}} = \sum_{m_1, n_1, m'_1, n'_1 \in \mathbb{Z}^3} \left(\prod_{i=2}^3 \sum_{m_i, n_i, m'_i, n'_i \in \mathbb{Z}} \right) \text{Tr} \left[(e_{m_1 n_1}^{\hat{c}_1} \otimes \mathbb{I}_{c_1}) \sqrt{C} \mathcal{R} \sqrt{C} (e_{m'_2 n'_2}^{c_2} \otimes \mathbb{I}_{\hat{c}_2}) C \right. \\ \left. (e_{m_3 n_3}^{c_3} \otimes \mathbb{I}_{\hat{c}_3}) C (e_{m'_3 n'_3}^{c_3} \otimes \mathbb{I}_{\hat{c}_3}) \sqrt{C} \mathcal{R} \sqrt{C} (e_{m'_1 n'_1}^{\hat{c}_1} \otimes \mathbb{I}_{c_1}) C (e_{m_2 n_2}^{c_2} \otimes \mathbb{I}_{\hat{c}_2}) C \right] \\ \times \delta_{m_1 n'_1}^3 \delta_{n_1 m'_1}^3 \delta_{m_2 n'_2} \delta_{n_2 m'_2} \delta_{m_3 n'_3} \delta_{n_3 m'_3}.$$

As usual in quantum field theory, we would like to represent this new expression by a graph \mathcal{G}' , a map in fact. It would allow us to understand how to proceed with Step 1 in the case of a general graph. Given that eq. (6.5) contains only one trace, it is natural to guess that \mathcal{G}' has only one vertex, but still three edges. What is the relationship between \mathcal{G} and \mathcal{G}' ? To understand it, we must come back to the Feynman graphs of our original tensor model. Each edge of a graph in the intermediate field representation corresponds to a melonic quartic vertex, somehow stretched in the direction of its distinguished colour, see fig. 11 left. Applying twice identity (6.4) to a given edge ℓ , we first contract it and then re-expand it in the orthogonal direction. This operation bears the name of partial duality with respect to ℓ , see [Chm08] where S. Chmutov introduced that duality relation. It is a generalization of the natural duality of maps which exchanges vertices and faces. Partial duality can be applied with respect to any spanning submap of a map. Natural duality corresponds to partial duality with respect to the full map. The number of vertices of the partial dual $\mathcal{G}^{E'}$ of \mathcal{G} with respect to the spanning submap $F_{E'}$ of edge-set E' equals the number of faces of $F_{E'}$. In our example, we performed partial duality of \mathcal{G} with respect to edge 1. Its spanning submap of edge-set $\{1\}$ has only one face. \mathcal{G}' has consequently only one vertex, which is confirmed by expression (6.5) containing only one trace. Note also that if a direct edge bears a single colour index c , its dual edge has the three colours \hat{c} . This can be seen on the amplitudes themselves: in eq. (6.5) edge 1 corresponds to the two *three-dimensionnal* deltas $\delta_{m_1 n'_1}^3 \delta_{n_1 m'_1}^3$ whereas edge 1 in eq. (6.3) represents the two *one-dimensionnal* deltas $\delta_{m_1 n'_1} \delta_{n_1 m'_1}$.

Given a map \mathcal{G} , how to draw its dual $\mathcal{G}^{E'}$ with respect to the spanning submap of edge-set $E' \subseteq E(\mathcal{G})$? Cut the edges of \mathcal{G} not in E' , making them half-edges. Turning around the faces of $F_{E'}$, one (partial) orders all the half-edges of \mathcal{G} , *i.e.* including those in $E(\mathcal{G}) \setminus E'$. The cycles of half-edges thus obtained constitute the vertices of $\mathcal{G}^{E'}$. Finally, connect in $\mathcal{G}^{E'}$ the half-edges which formed an edge in \mathcal{G} . The result of this construction in the case of the example of fig. 10 with $E' = \{1\}$ is given in fig. 12. Note that we will always represent one-vertex maps as chord diagrams.

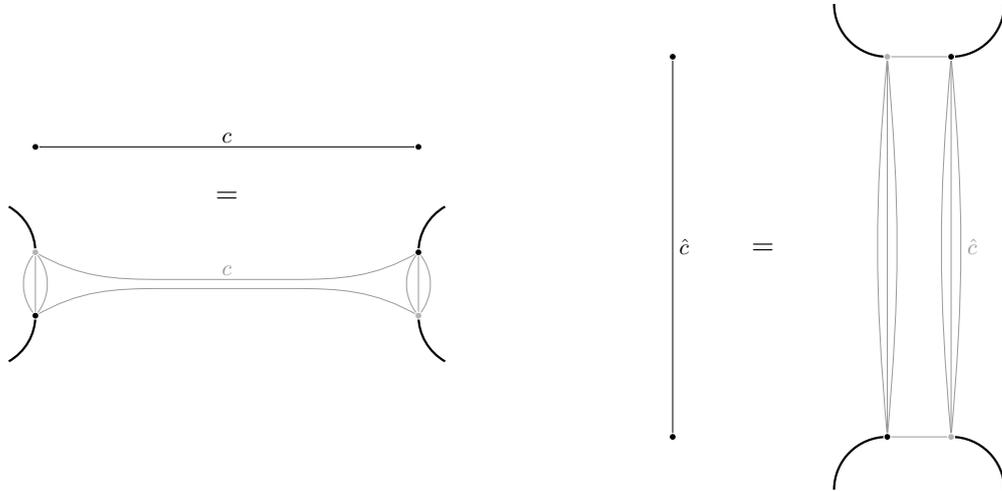


Figure 11. Edges (on the left) and dual edges (on the right) both in the intermediate field and the coloured tensor representations.

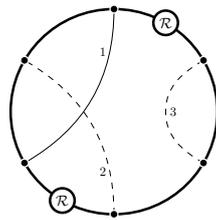


Figure 12. The partial dual $\mathcal{G}^{\{1\}}$ of the map \mathcal{G} of fig. 10, as a chord diagram. In general *i.e.* in the case of the partial dual of \mathcal{G} with respect to E' , edges in E' will be depicted as solid lines and those in $E(\mathcal{G}) \setminus E'$ as dashed lines. Resolvent insertions are explicitly represented. Bold solid line segments on the external circle correspond to propagators (or square roots of propagators around resolvents).

The advantage of writing the amplitude of \mathcal{G} as a single trace is that it allows us to easily identify it with a scalar product. Let us indeed rewrite the amplitude of \mathcal{G} as

$$\begin{aligned}
 A_{\mathcal{G}} = & \sum_{\substack{\mathbf{m}, \mathbf{l} \in \mathbb{Z}^4 \\ m_2, n_2, m'_2, n'_2 \in \mathbb{Z}}} \delta_{m_2 n'_2} \delta_{n_2 m'_2} \\
 & \left(\sum_{\mathbf{n}, \mathbf{k} \in \mathbb{Z}^4} \mathcal{R}_{mn} \mathcal{R}_{lk}^T \sum_{m_3, n_3, m'_3, n'_3 \in \mathbb{Z}} \delta_{m_3 n'_3} \delta_{n_3 m'_3} (\sqrt{C}(e_{m'_2 n'_2}^{c_2} \otimes \mathbb{I}_{\hat{c}_2}) C(e_{m_3 n_3}^{c_3} \otimes \mathbb{I}_{\hat{c}_3}) C(e_{m'_3 n'_3}^{c_3} \otimes \mathbb{I}_{\hat{c}_3}) \sqrt{C})_{\mathbf{nk}} \right) \\
 & \times \left(\sum_{\mathbf{m}_1, \mathbf{n}_1, \mathbf{m}'_1, \mathbf{n}'_1 \in \mathbb{Z}^3} \delta_{\mathbf{m}_1 \mathbf{n}'_1}^3 \delta_{\mathbf{n}_1 \mathbf{m}'_1}^3 (\sqrt{C}(e_{\mathbf{m}'_1 \mathbf{n}'_1}^{\hat{c}_1} \otimes \mathbb{I}_{c_1}) C(e_{m_2 n_2}^{c_2} \otimes \mathbb{I}_{\hat{c}_2}) C(e_{\mathbf{m}_1 \mathbf{n}_1}^{\hat{c}_1} \otimes \mathbb{I}_{c_1}) \sqrt{C})_{l\mathbf{m}} \right).
 \end{aligned}$$

Then the amplitude takes the form of a scalar product in $\mathcal{H}^{\otimes} \otimes L(\mathcal{H}_2) \otimes \mathcal{H}^{\otimes}$:

$$\begin{aligned}
 (6.6) \quad A_{\mathcal{G}} &= \langle \alpha, (\mathcal{R} \otimes \mathcal{R}^T) \beta \rangle, \\
 \alpha &= \sum_{\mathbf{m}_1, \mathbf{n}_1, \mathbf{m}'_1, \mathbf{n}'_1 \in \mathbb{Z}^3} \delta_{\mathbf{m}_1 \mathbf{n}'_1}^3 \delta_{\mathbf{n}_1 \mathbf{m}'_1}^3 (\sqrt{C}(e_{\mathbf{m}'_1 \mathbf{n}'_1}^{\hat{c}_1} \otimes \mathbb{I}_{c_1}) C(e_{m_2 n_2}^{c_2} \otimes \mathbb{I}_{\hat{c}_2}) C(e_{\mathbf{m}_1 \mathbf{n}_1}^{\hat{c}_1} \otimes \mathbb{I}_{c_1}) \sqrt{C})^{\dagger}, \\
 \beta &= \sum_{m_3, n_3, m'_3, n'_3 \in \mathbb{Z}} \delta_{m_3 n'_3} \delta_{n_3 m'_3} (\sqrt{C}(e_{m'_2 n'_2}^{c_2} \otimes \mathbb{I}_{\hat{c}_2}) C(e_{m_3 n_3}^{c_3} \otimes \mathbb{I}_{\hat{c}_3}) C(e_{m'_3 n'_3}^{c_3} \otimes \mathbb{I}_{\hat{c}_3}) \sqrt{C}).
 \end{aligned}$$

The vectors α and β can be pictorially identified: from the graph of fig. 12, one first detaches the two resolvents and then cut along a line joining their former positions, see fig. 13.

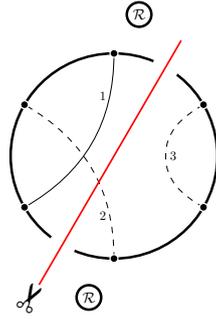


Figure 13. Amplitudes as scalar products.

As can be seen in eq. (6.6), the amplitude of \mathcal{G} does not exhibit any permutation operator. This is due to the fact that the (red) cut of this example crosses only one edge, see fig. 13. A permutation operator appears if and only if there are some crossings among the cut edges. Let us now give a second example, \mathcal{H} , the amplitude of which contains such a permutation, see fig. 14 (left). On the right of \mathcal{H} we have its partial dual with respect to edges 1 and 2. Cutting this diagram through both resolvents, one identifies the two vectors α and β in $\mathcal{H}^{\otimes} \otimes L(\mathcal{H}_{c_3}) \otimes L(\mathcal{H}_{c_2}) \otimes L(\mathcal{H}_{c_1}) \otimes \mathcal{H}^{\otimes}$ (reading counterclockwise) and the permutation operator S (see fig. 14 right) from $L(\mathcal{H}_{c_2}) \otimes L(\mathcal{H}_{c_1}) \otimes L(\mathcal{H}_{c_3})$ to $L(\mathcal{H}_{c_3}) \otimes L(\mathcal{H}_{c_2}) \otimes L(\mathcal{H}_{c_1})$ such that $A_{\mathcal{H}} = \langle \alpha, (\mathcal{R} \otimes S \otimes \mathcal{R}^T) \beta \rangle$.

After having written the amplitude of a graph as a scalar product, we can apply Cauchy-Schwarz inequality which corresponds to Step 3 in the ICS algorithm. Finally there only remains to identify the squares of the norms of α and β as amplitudes of some definite maps. It simply consists in duplicating each half of the cut diagram and glue each piece to its mirror symmetric one *i.e.* its Hermitian conjugate. In the case of graph \mathcal{G} of fig. 13, we get the two chord diagrams of fig. 15. But in general it could happen that $\langle \alpha, \alpha \rangle$ (or $\langle \beta, \beta \rangle$) is infinite that is to say its corresponding chord diagram is dual to a divergent graph. To conclude this section of examples, let us exhibit a graph such that any cut of its chord diagram leads to divergent graphs. Let \mathcal{G} be the graph of fig. 16 (above left), in the intermediate field representation. The gray parts represent renormalized subgraphs. Let us perform partial duality with respect to all its edges and get the chord diagram of fig. 16 (above right). All of its four possible cuts (we *never* cut inside a renormalized block) lead to divergent upper bounds by Cauchy-Schwarz inequality.

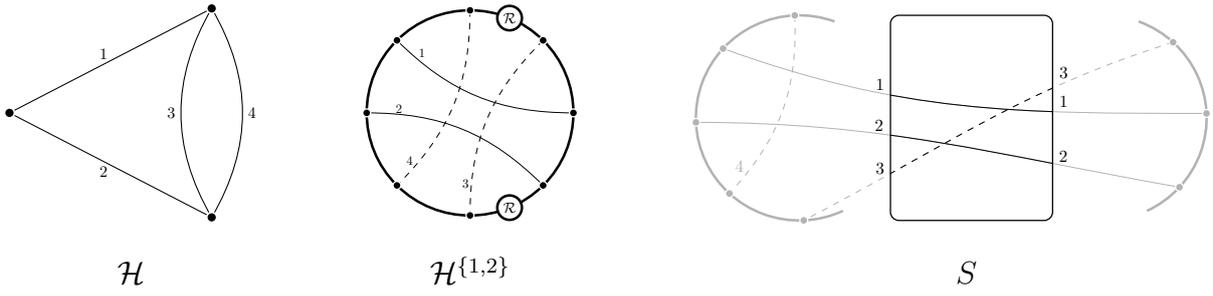


Figure 14. Example of a graph \mathcal{H} (left) the amplitude of which, written as a scalar product, exhibits a permutation operator S (right). The picture in the middle is the partial dual $\mathcal{H}^{\{1,2\}}$ of \mathcal{H} with respect to edges 1 and 2. The vectors whose scalar product equals $A_{\mathcal{H}}$ are identified by cutting the chord diagram of $\mathcal{H}^{\{1,2\}}$ through both resolvents.



Figure 15. $\langle \alpha, \alpha \rangle$ (left) and $\langle \beta, \beta \rangle$ (right) in the case of fig. 13.

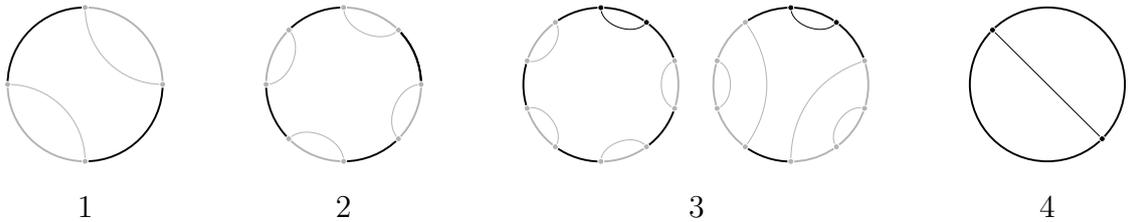


Figure 16. A graph \mathcal{G} with divergent cuts. Gray parts represent renormalized subgraphs. The four possible cuts of $\mathcal{G}^{E(\mathcal{G})}$ are indicated by numbered red segments. On the second line, we display the divergent factors of $\langle \alpha, \alpha \rangle \langle \beta, \beta \rangle$ for the different cuts.

6.2.3. *ICS algorithm 1.0.* — Thus there exist chord diagrams with only divergent cuts. How do we get rid of their resolvents using Cauchy-Schwarz inequality? We can in fact expand some of the resolvents, $\mathcal{R} = \mathbb{I} + U\mathcal{R}$, and get new graphs. In the sequel we will show that for all resolvent graph \mathcal{G} , there is a systematic way of expanding its resolvents such that, for any newly created graph, there exists an iterative cutting scheme which converges itself to a collection of graphs without resolvents.

A more precise (but still not enough) ICS algorithm can be written as follows:

Algorithm 1 ICS 1.0

Require: \mathcal{G} a resolvent graph.

- 1: **Partial duality:** Write $A_{\mathcal{G}}$ as $c(\mathcal{G})$ traces (times Kronecker deltas)
 - 2: **Preparation step:** Expand (some of) the resolvents of $A_{\mathcal{G}}$ conveniently and get a collection S of new resolvent graphs
 - 3: **for** S **in** S **do**
 - 4: **Cutting scheme:** choose a cut and thus write $A_{\mathcal{S}}$ as a scalar product
 - 5: **Cauchy-Schwarz inequality:** apply it to $A_{\mathcal{S}}$
 - 6: Go back to step 4 and iterate sufficiently.
 - 7: **end for**
-

The first step of Algorithm 1 consists in writing the amplitude $A_{\mathcal{G}}$ of a resolvent graph \mathcal{G} as a product of $c(\mathcal{G})$ traces. To this aim, we choose arbitrarily a spanning tree in each connected component and perform partial duality with respect to this set \mathcal{F} of edges. The amplitude of each connected component of \mathcal{G} is then represented by a one-vertex map that we will draw as a chord diagram. The disjoint union of all these chord diagrams form the partial dual $\mathcal{G}^{\mathcal{F}}$ of \mathcal{G} . An edge of colour c in \mathcal{G} still bears colour c in $\mathcal{G}^{\mathcal{F}}$ if it does not belong to \mathcal{F} and bears colours $\hat{c} = \{1, 2, 3, 4\} \setminus \{c\}$ if it is in \mathcal{F} . Tree edges will be represented as plain lines and loop edges as dashed lines in the following pictures.

6.2.4. *The preparation step.* — In order to write the amplitude of (each connected component of) \mathcal{G} as a scalar product we need to choose a cut in the corresponding chord diagram. But as we have seen previously, there exist resolvent graphs such that any Cauchy-Schwartz cut results in divergent amplitudes $\langle \alpha, \alpha \rangle$ and/or $\langle \beta, \beta \rangle$. Nevertheless we can see on fig. 17 that divergent vacuum graphs (which have essentially only one spanning tree and thus a canonical associated chord diagram) have either less than four tree lines and no loops, or one loop line and less than one tree line, or two loops but no tree lines. Thus if a diagram has enough edges, so to speak, between the two resolvents of a cut, the Cauchy-Schwarz bound will be *superficially* convergent. We will ensure it by suitably expanding some resolvents as $\mathcal{R} = \mathbb{I} + \mathcal{R}U$ or $\mathbb{I} + U\mathcal{R}$.

But to ensure finiteness, we also need to find a cut such that no divergent subgraphs pop up in $\langle \alpha, \alpha \rangle$ and/or $\langle \beta, \beta \rangle$. Divergent (2-point) subgraphs appear in chord diagrams as represented in fig. 18. Note that they are absent from resolvent graphs (and from their partial duals) because Multiscale Loop Vertex Expansion produced only renormalized amplitudes. It is easy to convince oneself that if there is no tree line next to corners of cut, there will be no divergent subgraphs in $\langle \alpha, \alpha \rangle$ and $\langle \beta, \beta \rangle$.

We now explain precisely which resolvents will be expanded and how many times. Later on, we will prove that after such expansions there exists a sequence of iterated Cauchy-Schwarz cuts which bounds the amplitude of any resolvent graph by the geometric mean of finite amplitudes, most of them freed of resolvents.

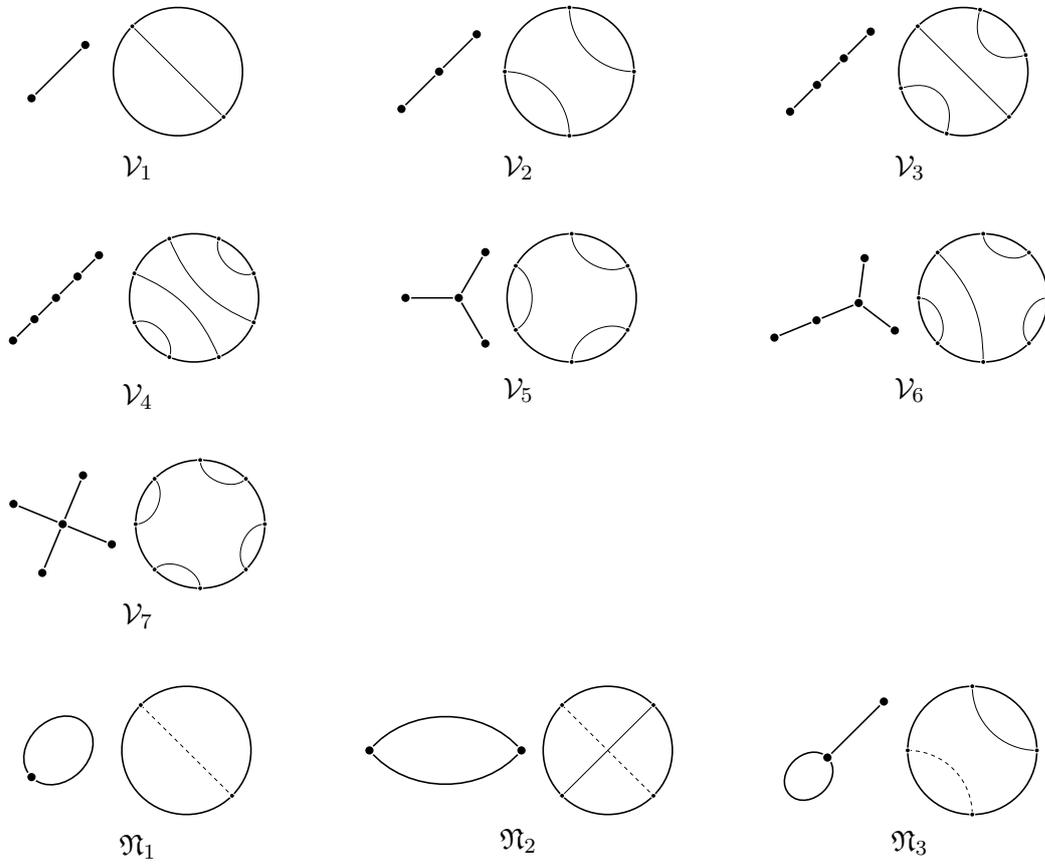


Figure 17. The divergent vacuum graphs in the intermediate field (left) and dual (right) representations.



Figure 18. Divergent subgraphs in the dual representation.

First of all, we need to define when resolvent expansions should stop *i.e.* when we consider a diagram as secured or said differently when a diagram is ready for the cut process to be defined in the next section.

In the following we will always read a chord diagram counterclockwise. Thus if O_1 and O_2 are operators in $L(\mathcal{H}^{\otimes})$ and appear in the amplitude of \mathcal{C} , we will consider that O_2 is *on the right* of O_1 if O_2 is met just after O_1 counterclockwise around \mathcal{C} or equivalently if $A_{\mathcal{C}}$ contains the product O_1O_2 . We will say symmetrically that O_2 is *on the left* of O_1 if the product O_2O_1 appears in $A_{\mathcal{C}}$. We will denote $r(\mathcal{C})$ the number of resolvents in $A_{\mathcal{C}}$.

Definition 6.1 (Safeness). — Let us consider a chord diagram representing the partial dual of a resolvent graph. A *safe element* is either a half loop edge or a renormalized D -block. ♠

Definition 6.2 (Tree-resolvents). — We say that a resolvent \mathcal{R} is a right (resp. left) *tree-resolvent* if

- the product $\Delta^s S\mathcal{R}$ (resp. $\mathcal{R}S\Delta^s$), where S is itself a possibly empty product of safe elements and s labels a half tree line, appears in $A_{\mathcal{C}}$
- and the number of safe elements in S is less than or equal to six.

A tree-resolvent is a resolvent which is either a right or a left tree-resolvent (or both). Tree-resolvents are the resolvents “closest” to the tree of \mathcal{C} . We also let $t(\mathcal{C})$ be the number of tree-resolvents in $A_{\mathcal{C}}$. ♠

We will need to order the tree-resolvents of a diagram amplitude. In the following if \mathcal{C} is a connected chord diagram, we will write \mathcal{C}_{\bullet} for a pair made of \mathcal{C} and a distinguished tree-resolvent (called root resolvent hereafter). We consider all of its tree-resolvents as ordered counterclockwise starting with the root one and denote them $\mathcal{R}_1, \mathcal{R}_2, \dots, \mathcal{R}_{t(\mathcal{C})}$. If $\mathcal{C} = \sqcup_{i=1}^{c(\mathcal{C})} \mathcal{C}_i$ is a disjoint union of chord diagrams (and $c(\mathcal{C})$ is the number of connected components of \mathcal{C}), \mathcal{C}_{\bullet} stands for a choice of one root resolvent per connected component. In each $\mathcal{C}_{i,\bullet}$, resolvents are ordered from 1 to $t(\mathcal{C}_i)$.

Definition 6.3 (Distance to tree). — Let \mathcal{C} be a connected Feynman chord diagram. Let s be a half tree edge and j an element of $\{1, 2, \dots, t(\mathcal{C})\}$. The pair (s, j) is *admissible* if \mathcal{R}_j is a tree-resolvent and s is separated from \mathcal{R}_j only by safe elements. Said differently, from \mathcal{R}_j to s we meet neither half tree edges nor resolvents. For any admissible pair $p = (s, j)$, let d_p be the number of safe elements in $A_{\mathcal{C}}$ between Δ^s and \mathcal{R}_j . d_p is the *distance* between s and \mathcal{R}_j and is, by Definition 6.2, less than or equal to six. ♠

Definition 6.4 (Secured diagrams). — A connected chord diagram \mathcal{C} is *secured* if either $r(\mathcal{C}) = 0$ or for any admissible pair p , d_p equals six. A possibly disconnected diagram is secured if all its connected components are secured. ♠

We now explain which resolvents of a diagram we expand, and how, in order to reach only secured graphs. Algorithm 2 simply expands on its right a given resolvent of a graph. More precisely it returns the list of graphs representing the various terms of the expansion. A symmetrical algorithm, named EXPANDL, does the same on the left.

Algorithm 2 Right expansion

Require: \mathcal{C}_\bullet a rooted chord diagram, $1 \leq i \leq c(\mathcal{C})$ and $1 \leq j \leq r(\mathcal{C}_i)$.

procedure EXPANDR($\mathcal{C}_\bullet, i, j$) \triangleright Expands once \mathcal{R}_j on its right in $A_{\mathcal{C}_i}$.

$L := []$ \triangleright an empty list

Expand \mathcal{R}_j as $\mathbb{I} + \mathcal{R}_j(D + \Sigma)$

$\mathcal{C}_i^{(0)} := \mathcal{C}_i$ with $\textcircled{\mathcal{R}_j}$ replaced by $\bullet \text{---} \bullet$

5: $\mathcal{C}^{(0)} := \mathcal{C} \sqcup \mathcal{C}_i^{(0)} \setminus \mathcal{C}_i$

$L.append(\mathcal{C}^{(0)})$

$\mathcal{C}_i^{(1)} := \mathcal{C}_i$ with $\textcircled{\mathcal{R}_j}$ replaced by $\bullet \text{---} \textcircled{\mathcal{R}_j} \text{---} \bullet$

$\mathcal{C}^{(1)} := \mathcal{C} \sqcup \mathcal{C}_i^{(1)} \setminus \mathcal{C}_i$

$L.append(\mathcal{C}^{(1)})$

10: Integrate by parts the Σ -term (eq. (6.1)) $\triangleright r(\mathcal{C})$ new graphs.

for k **from** 2 **to** $r(\mathcal{C}) + 1$ **do**

$e_k :=$ the new additional edge

if e_k is a loop **then**

$\mathcal{C}_i^{(k)} := \mathcal{C}_i \cup \{e_k\}$

15: $\mathcal{C}^{(k)} := \mathcal{C} \sqcup \mathcal{C}_i^{(k)} \setminus \mathcal{C}_i$

else $\triangleright e_k$ connects \mathcal{C}_i to $\mathcal{C}_{i'}, i \neq i'$.

$\mathcal{C}_i^{(k)} := (\mathcal{C}_i \cup \mathcal{C}_{i'} \cup \{e_k\})^{\{e_k\}}$

$\mathcal{C}^{(k)} := \mathcal{C} \sqcup \mathcal{C}_i^{(k)} \setminus \{\mathcal{C}_i, \mathcal{C}_{i'}\}$

end if

20: $L.append(\mathcal{C}^{(k)})$

end for

return L

end procedure

Given a non secured connected component \mathcal{C}_i of a Feynman chord diagram, Algorithm 3 decides which resolvent to expand and how many times. Before giving its pseudocode, we need to introduce a few more definitions. Let j be an element of $\{1, 2, \dots, t(\mathcal{C}_i)\}$. We define $\text{RIGHT}_{\mathcal{C}_i, \bullet}(\mathcal{R}_j)$ as the number of consecutive safe elements at the right of \mathcal{R}_j . We define $\text{LEFT}_{\mathcal{C}_i, \bullet}(\mathcal{R}_j)$ symmetrically. We let $\text{RIGHTTREE}_{\mathcal{C}_i, \bullet}(\mathcal{R}_j)$ (resp. $\text{LEFTTREE}_{\mathcal{C}_i, \bullet}(\mathcal{R}_j)$) be True if \mathcal{R}_j is a right (resp. left) tree-resolvent and False otherwise. $\text{ROOT}(\mathcal{C}_i)$ chooses a root resolvent among the tree-resolvents, randomly say.

Finally Algorithm 4 secures all the resolvents of a given diagram \mathcal{C} . More precisely it returns the list of secured diagrams obtained from \mathcal{C} by successive expansions of its resolvents. Algorithm 4 can be thought of as building a rooted tree $T_{\mathcal{C}}$ inductively. At each of the nodes of that tree, there is an associated chord diagram. The root of $T_{\mathcal{C}}$ consists in the input diagram \mathcal{C} . The children of a given node \mathcal{C}' correspond to the $r(\mathcal{C}') + 2$ new graphs obtained by expanding one resolvent of \mathcal{C}' , the one chosen by CHOOSEEXPAND . Algorithm 4 returns the list of totally secured graphs. They correspond to the leaves of $T_{\mathcal{C}}$.

We now prove that Algorithm 4 stops after a finite number of steps and give an upper bound on the number of elements of the list it returns.

Lemma 6.5. — *Let \mathcal{B} be Bosonic block with $n+1$ vertices. Let \mathcal{C} be one of the resolvent graphs obtained from \mathcal{B} by the contraction process. After a finite number of steps, Algorithm 4 applied to \mathcal{C} stops and returns a list of at most $(98n - 28)^{42n-30}$ secured diagrams.*

Proof. — In the computation tree $T_{\mathcal{C}}$ representing Algorithm 4, each new generation corresponds to the expansion of a resolvent and each child of a given node to a term of this expansion (plus integration by parts). Along the branches of $T_{\mathcal{C}}$, from a given node to one of its children, the number of connected

Algorithm 3 Choose & expand

Require: \mathcal{C} a Feynman chord diagram and $1 \leq i \leq c(\mathcal{C})$ such that \mathcal{C}_i not secured.

```

procedure CHOOSEEXPAND( $\mathcal{C}, i$ )
   $\mathcal{C}_{i,\bullet} := (\mathcal{C}_i, \text{ROOT}(\mathcal{C}_i))$ 
   $j := 1$ 
  while  $j \leq t(\mathcal{C}_i)$  do
5:   if  $\text{RIGHTTREE}_{\mathcal{C}_{i,\bullet}}(\mathcal{R}_j)$  and  $\text{LEFT}_{\mathcal{C}_{i,\bullet}}(\mathcal{R}_j) \leq 5$  then
     return EXPANDL( $\mathcal{C}_\bullet, i, j$ )
     else if  $\text{LEFTTREE}_{\mathcal{C}_{i,\bullet}}(\mathcal{R}_j)$  and  $\text{RIGHT}_{\mathcal{C}_{i,\bullet}}(\mathcal{R}_j) \leq 5$  then
       return EXPANDR( $\mathcal{C}_\bullet, i, j$ )
     else
10:     $j := j + 1$ 
     end if
  end while
end procedure

```

Algorithm 4 Securing resolvents

Require: \mathcal{C} a Feynman chord diagram.

```

 $L := [\mathcal{C}]$ 
 $S := []$ 
while  $L$  not empty do
   $D := []$ 
5:  for  $k$  from 0 to  $\text{len}(L) - 1$  do  $\triangleright k$  indexes the graphs in  $L$ .
     if  $L[k]$  secured then
        $S.$ append( $L[k]$ )
        $D.$ append( $L[k]$ )
     else
10:    Pick a non secured connected component  $L[k]_i$  of  $L[k]$ 
        $L := L + \text{CHOOSEEXPAND}(L[k], i)$ 
        $D.$ append( $L[k]$ )
     end if
  end for
15:  for  $\mathcal{G}$  in  $D$  do
      $L.$ remove( $\mathcal{G}$ )
  end for
end while
return  $S$ 

```

components is constant except in case of a new tree edge where it decreases by one. In order to control the maximal number of steps taken by Algorithm 4 we now introduce one more parameter $m(\mathcal{C})$ namely the number of missing safe elements to get \mathcal{C} secured:

$$m(\mathcal{C}) := \sum_{p \text{ admissible}} 6 - d_p.$$

Algorithm 4 stops when $m = 0$.

Let us now inspect the evolution of m along the branches of $T_{\mathcal{C}}$. As Algorithm 4 only expands tree-resolvents, let us consider such an operator \mathcal{R} . Locally, around \mathcal{R} in $A_{\mathcal{C}}$, we have the following situation: $A_1 S_1 \mathcal{R} S_2 A_2$ where both A_1 and A_2 are either half tree edges or resolvents but at least one of them is a half tree edge and S_1, S_2 are possibly empty products of safe elements. If the expansion term of \mathcal{R} is:

- \mathbb{I} and
 - both A_1 and A_2 are half tree edges then m decreases by $12 - |S_1| - |S_2| \geq 1$ if both $|S_1|$ and $|S_2|$ are less than or equal to six, and by $6 - |S_2| \geq 1$ (resp. $6 - |S_1|$) if $|S_1|$ (resp. $|S_2|$) is strictly greater than six,
 - A_1 (resp. A_2) is a resolvent then m decreases by $|S_1|$ (resp. $|S_2|$) if $|S_1| + |S_2| \leq 6$ and by $6 - |S_2|$ (resp. $6 - |S_1|$) otherwise,
- D , m decreases by one,
- a new loop edge, m decreases by one,
- a new tree edge, m increases by $12 + |S_1|$ (resp. $12 + |S_2|$) if \mathcal{R} is left- (resp. right-)expanded.

Thus at each generation, in all cases, the non-negative integer valued linear combination

$$\psi := 18(c - 1) + m$$

strictly decreases. As it is bounded above (at fixed n), Algorithm 4 stops after a finite number of steps.

In order to determinate an upper bound on the number of leaves of $T_{\mathcal{C}}$, we need a bound on its number of generations. As $\psi \geq 0$, the length of a branch of $T_{\mathcal{C}}$ is certainly bounded by $\psi(\mathcal{C})$. The number of children of a node \mathcal{C}' is $r(\mathcal{C}') + 2$. As the number of resolvents increases by 1 with each new added edge, the maximal total number of resolvents over all the nodes of $T_{\mathcal{C}}$ is $r(\mathcal{C}) + \psi(\mathcal{C})$. In conclusion, the number of leaves of $T_{\mathcal{C}}$ is bounded by

$$(r(\mathcal{C}) + \psi(\mathcal{C}) + 2)^{\psi(\mathcal{C})}.$$

As already discussed at the beginning of Section 6, a resolvent graph coming from a Bosonic block with $n + 1$ vertices has at most n connected components, $2n - 1$ tree edges thus at most $4n - 2$ admissible pairs and less than $56n$ resolvents. We get $m(\mathcal{C}) \leq 24n - 12$ and $\psi(\mathcal{C}) \leq 42n - 30$. Consequently, as a function of n , the number of new graphs created by Algorithm 4 is bounded above by $(98n - 28)^{42n - 30}$. \square

6.2.5. Iterative cutting process. — The preparation step has expressed the amplitude of any resolvent graph \mathcal{G} as the sum over the leaves of $T_{\mathcal{G}}$ of the amplitudes of the corresponding secured graphs. Thus, from now on we consider a secured Feynman chord diagram \mathcal{C} , together with a scale attribution μ . We apply Cauchy-Schwarz inequalities to $A_{\mathcal{C}_\mu}$ iteratively until we bound $|A_{\mathcal{C}_\mu}|$ by a geometric mean of convergent resolvent-free amplitudes.

First of all, note that an iterative cutting process can be represented as a rooted binary tree. Its root corresponds to \mathcal{C} and the two children of each node are the result of a Cauchy-Schwarz inequality. It will be convenient to use the Ulam-Harris encoding of rooted plane trees [Mie14]. It identifies the set of vertices of a rooted tree with a subset of the set

$$\mathcal{U} = \bigcup_{n \geq 0} \mathbb{N}^n$$

of integer words, where $\mathbb{N}^0 = \{\emptyset\}$ consists only in the empty word. The root vertex is the word \emptyset . The children of a node represented by a word w are labelled, in our binary case, $w0$ and $w1$.

Definition 6.6 (Odd cut). — Let \mathcal{C} be a secured Feynman chord diagram. Note that all the secured diagrams obtained after the preparation step contain at least one tree line and thus at least one tree-resolvent \mathcal{R}_0 . Thanks to the preparation step, there are at least six safe elements between \mathcal{R}_0 and a half tree edge. An *odd Cauchy-Schwarz cut* starts at \mathcal{R}_0 and ends between the third and fourth safe element situated between \mathcal{R}_0 and the tree in \mathcal{C} . See fig. 19 for a graphical representation. ♠

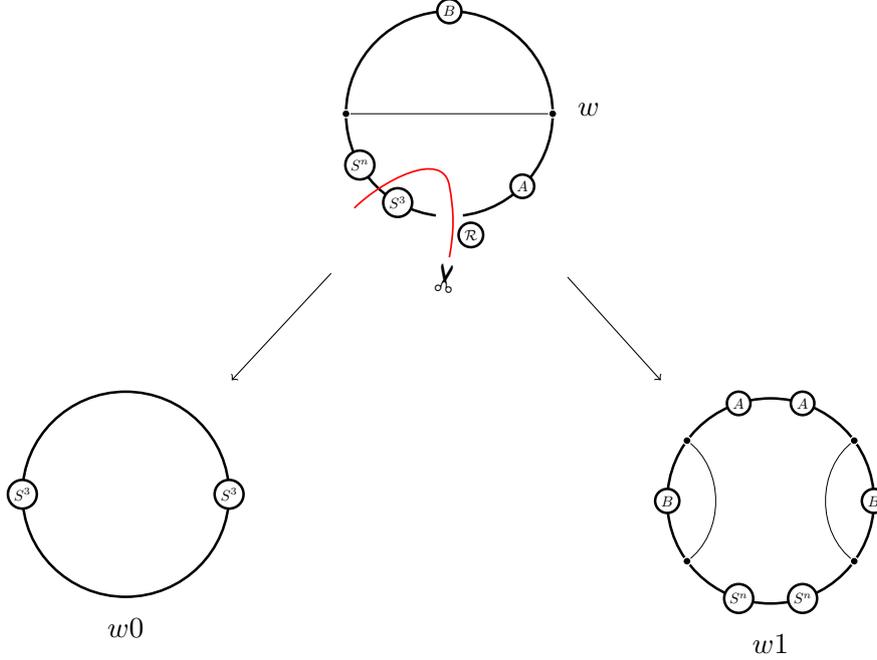


Figure 19. One odd Cauchy-Schwarz iteration ($n \geq 3$). For all $p \geq 0$, S^p represents a product of p safe elements. A and B are (almost) any operators.

Definition 6.7 (Even cut). — Let \mathcal{C} be a secured Feynman chord diagram with an even number, $2k$, of resolvents. An *even Cauchy-Schwarz cut* consists in

1. choosing any of the resolvents in $A_{\mathcal{C}}$, calling it \mathcal{R}_1 and labelling the other ones $\mathcal{R}_2, \dots, \mathcal{R}_{2k}$ (counter)clockwise around the unique vertex of \mathcal{C} ,
2. cutting through \mathcal{R}_1 and \mathcal{R}_{k+1} . ♠

Definition 6.8 (Cutting scheme). — Let \mathcal{C}_s be a secured Feynman chord diagram. We apply Cauchy-Schwarz inequalities iteratively as follows:

0. if $r(\mathcal{C}_s) = 2$ or $2k + 1$, apply an odd cut. $|A_{\mathcal{C}_s}|$ is then bounded by the product of (the square roots of the amplitudes of) a convergent diagram and a secured diagram with an even number (2 or $4k$) of resolvents.
1. For any diagram with an even number of resolvents, perform an even cut and iterate until getting only resolvent-free graphs.

In the following, graphs obtained from secured ones by such a cutting scheme will simply be called *resolvent-free graphs*. ♠

Now, let B_k be the set of binary words (*i.e.* formed from the alphabet $\{0, 1\}$) of length k . According to Definition 6.8 the amplitude of a secured chord diagram is bounded above by the following expressions

$$(6.7) \quad |A_{\mathcal{C}}| = |A_{\emptyset}| \leq \|\mathcal{R}\|^{2e(\mathcal{C})} \begin{cases} \prod_{w \in B_k} |A_w|^{2^{-k}} & \text{if } r(\mathcal{C}) = 2k, k \geq 2, \\ |A_0|^{1/2} |A_{10}|^{1/4} |A_{11}|^{1/4} & \text{if } r(\mathcal{C}) = 2, \\ |A_0|^{1/2} \prod_{w \in B_{2k}} |A_{1w}|^{2^{-2k-1}} & \text{if } r(\mathcal{C}) = 2k + 1. \end{cases}$$

The only (slightly) non trivial factor to explain is $\|\mathcal{R}\|^{2e(\mathcal{C})}$. Each cutting step delivers a factor $\|\mathcal{R}\|^2$ and the number of steps is bounded above by half the maximal possible number of resolvents in $A_{\mathcal{C}}$, namely $2e(\mathcal{C})$.

Our aim is now to get an upper bound on the amplitude of any secured graph. Next Lemma is a first step in this direction as it proves that such amplitudes are finite.

Lemma 6.9 (Convergence of secured graphs). — *Secured graphs are convergent: let \mathcal{G}_s be a secured graph then $|A_{\mathcal{G}_s}| < \infty$.*

To prove it we will need the following

Lemma 6.10 (Between resolvents). — *Between two consecutive resolvents of a secured graph amplitude, there are either at least three D -blocks or at least one half loop edge.*

Proof. — Remark that between two consecutive resolvents of a skeleton graph amplitude, there are either three safe elements or at least one half tree edge. This is obvious from eqs. (3.4) to (3.6). During the contraction process, unmarked half edges can contract to resolvents and thus create graphs such that two consecutive resolvents are only separated by one half loop edge. Thus between two consecutive resolvents of a resolvent graph amplitude, there are either at least three D -blocks or at least one half (tree or loop) edge. Let us now have a look at the preparation step. When a (tree-)resolvent is expanded, it can either merge two intervals between resolvents (if the expansion term is \mathbb{I}) or increase the number of safe elements in an interval if the expansion term is a D operator or half loop edge or create a new tree edge. In consequence, between two consecutive resolvents of a secured graph amplitude, there are either at least three D -blocks or at least one half loop edge or at least one half tree edge. In this last case, as the graph considered is secured, there are at least six safe elements between the two resolvents. \square

Proof of Lemma 6.9. — We prove that for any word w in B_k if $r(\mathcal{G}_s) = 2k$ or in B_{2k} if $r(\mathcal{G}_s) = 2k + 1$, $|A_w| < \infty$. Indeed, note first that the products of a cut of a secured graph, even or odd, are still secured. Thus the cutting scheme of Definition 6.8 cannot create divergent subgraphs as we never cut through a corner adjacent to a tree edge. Then it is enough to check that each resolvent-free map w either contains at least five tree edges or at least two tree edges and one loop edge or at least two loop lines, see fig. 17.

If $r(\mathcal{G}_s) = 1$, we proceed to an odd cut. The resulting resolvent-free graphs, denoted 0 and 1, contain at least six safe elements (see Definition 6.1) and are thus convergent.

If $r(\mathcal{G}_s) = 2$, we split our analysis into two subcases. If the two resolvents in $A_{\mathcal{G}_s}$ are separated by a tree line, and as \mathcal{G}_s is secured, an even cut will produce two resolvent-free graphs the amplitudes of which contain at least twelve safe elements each. They are thus convergent. If one of the two intervals between the two resolvents does not contain half tree edges, it must contain at least one half loop edge or at least three D operators (by Lemma 6.10). In this case, we first perform an odd cut. It results in two secured graphs. One of them is resolvent-free and convergent (see fig. 19). The other one has two resolvents separated either by tree edges (thus at least twelve safe elements) or by at least two half loop edges. An even cut now produces only resolvent-free convergent graphs.

If $r(\mathcal{G}_s) \geq 3$, a resolvent-free graph w necessarily originates from the application of an even cut on a secured graph $\mathcal{G}_s^{(1)}$ with two resolvents. And $\mathcal{G}_s^{(1)}$ itself is the product of an even cut on another secured graph $\mathcal{G}_s^{(0)}$ with four resolvents. By Lemma 6.10, resolvents in $A_{\mathcal{G}_s^{(0)}}$ are separated by at least one half loop edge or at least three D operators. Then resolvents in $A_{\mathcal{G}_s^{(1)}}$ are separated by at least two half loop edges or at least six D operators. An even cut on $\mathcal{G}_s^{(1)}$ thus produces only convergent resolvent-free graphs. \square

6.3. Bounds on secured graphs. — Our next task is to get a better upper bound on the amplitude of a secured graph, in terms of the loop vertex scales. Remember indeed that each node a of a tree in the LVE representation of $\log \mathcal{Z}$, see eq. (2.3), is equipped with a scale j_a *i.e.* an integer between 0 and j_{\max} . Analytically it means that each V_{j_a} in $W_{j_a} = e^{-V_{j_a}} - 1$ contains exactly one $\mathbf{1}_{j_a}$ cutoff adjacent to a \sqrt{C} operator (and all other propagators bear $\mathbf{1}_{\leq j_a}$ cutoffs), see eq. (1.16). Moreover the scales of the nodes of a Bosonic block are all distinct. After applying the derivatives (situated at both ends of each tree edge of a Bosonic block) to the W_{j_a} 's one gets skeleton graphs which are forests with generically more vertices than their corresponding abstract tree. Each duplicated vertex is a derivative of some W_{j_a} and bears consequently a $\mathbf{1}_{j_a}$ cutoff. Thus the (loop) vertices of the skeleton graphs do not have distinct scales but contain at least as many $(\sqrt{C})_{j_a}$'s as the underlying tree.

During the contraction process (*i.e.* integration by parts of the σ fields not contained in the resolvents) no $(\sqrt{C})_j$ operator are created nor destroyed. When two sigmas contract to each other, corners (*i.e.* places where square roots of propagators are situated) do not change. When a sigma field contracts to a resolvent, two new corners are created but both with a $\mathbf{1}_{\leq j}$ cutoff. The potentially adjacent $\mathbf{1}_j$ cutoff is left unchanged. Secured graphs bear thus at least as many $(\sqrt{C})_{j_a}$'s as their original skeleton graphs.

Lemma 6.11. — *Let \mathcal{B} be a Bosonic block and \mathcal{G}_s be a secured graph originating from \mathcal{B} . Then, there exist $K, \rho \in \mathbb{R}_+^*$ such that for any coupling constant g in the cardioid domain Card_ρ ,*

$$(6.8) \quad |A_{\mathcal{G}_s}| \leq K^{|\mathcal{B}|} \rho^{e(\mathcal{G}_s)} \prod_{a \in \mathcal{B}} M^{-\frac{1}{12}j_a}.$$

Proof. — To facilitate the argument we first need to introduce some more notation. We let \tilde{k} be the number of Cauchy-Schwarz iterations in the cutting process of Definition 6.8. Explicitly,

$$\tilde{k}(\mathcal{G}_s) := \begin{cases} k & \text{if } r(\mathcal{G}_s) = 2k \text{ and } k \geq 2, \\ 2 & \text{if } r(\mathcal{G}_s) = 2, \\ 2k + 1 & \text{if } r(\mathcal{G}_s) = 2k + 1. \end{cases}$$

We often drop the dependence on \mathcal{G}_s . In order to track corners which bear loop vertex scales, we also introduce the following: let w be either a secured graph or a resolvent-free graph. For all $a \in \mathcal{B}$, we let $c_a(w)$ be the number of corners of w which bear integer a :

$$c_a(w) := |\{c \in s(w) : i_c = a\}|.$$

For all $k' \in \{0\} \cup [\tilde{k}(\mathcal{G}_s)]$, let us note $F_{k'}(\mathcal{G}_s)$ for the set of maps obtained from \mathcal{G}_s after k' steps of the cutting process of Definition 6.8. For example, if $r(\mathcal{G}_s)$ is even and greater than four, $F_k(\mathcal{G}_s)$ is the set of binary words of length $r/2$. For all $\mathbf{m} \in F_{k'}(\mathcal{G}_s)$, let $\alpha_{k'}(\mathbf{m})$ be the exponent of $|A_{\mathbf{m}}|$ in the corresponding Cauchy-Schwarz bound. Then, according to eq. (6.7), for all $a \in \mathcal{B}$, we define $m_{a,k'}$ as follows:

$$m_{a,k'} := \sum_{\mathbf{m} \in F_{k'}(\mathcal{G}_s)} \alpha_{k'}(\mathbf{m}) c_a(\mathbf{m}).$$

We shall now bound the amplitude of \mathcal{G}_s by a multiscale analysis. It means that for all \mathbf{m} in $F(\mathcal{G}_s)$, we expand each $(\sqrt{C})_{\leq j}$ operator as $\sum_{i=0}^j (\sqrt{C})_i$. Each map \mathbf{m} is then equipped with a scale attribution, namely a given integer per corner of \mathbf{m} . These attributions correspond to the usual scale attributions on edges in the tensor graph representation. Nevertheless, they are here constrained: there exist (marked) corners with a fixed scale j_a (these are the loop vertex scales) and for each corner c of \mathbf{m} , i_c is bounded by some j_a . Let $s(\mathbf{m})$ be the set of marked corners of \mathbf{m} . Using Lemmas 6.9 and A.2, there exists a positive real number K such that

$$(6.9) \quad |A_{\mathbf{m}}| = \left| \sum_{\mu} A_{\mathbf{m},\mu} \right| \leq (K|g|)^{e(\mathbf{m})} \prod_{c \in s(\mathbf{m})} M^{-\frac{1}{24}i_c}.$$

Remember that Lemma A.2 is formulated in the tensor graph representation. Here the edges of a chord diagram correspond to the vertices of a tensor Feynman graph and edges of the latter are the corners in the resolvent-free graphs. Moreover, looking at eqs. (3.4) to (3.6), one notices that each loop vertex

bears one factor $\lambda = g^{1/2}$ per corner (in a resolvent-free graph). This explains the term $g^{e(\mathbf{m})}$ in eq. (6.9). From eq. (6.7), we deduce

$$(6.10) \quad |A_{\mathcal{G}_s}| \leq \|\mathcal{R}\|^{2e(\mathcal{G}_s)} (K|g|)^{\sum_{\mathbf{m} \in F_{\tilde{k}}(\mathcal{G}_s)} \alpha_{\tilde{k}}(\mathbf{m})e(\mathbf{m})} \prod_{a \in \mathcal{B}} M^{-\frac{1}{24}m_{a,\tilde{k}}j_a}.$$

Remark that

$$(6.11) \quad \sum_{\mathbf{m} \in F_{\tilde{k}}(\mathcal{G}_s)} \alpha_{\tilde{k}}(\mathbf{m})e(\mathbf{m}) = e(\mathcal{G}_s).$$

Let us indeed consider $w \in F_{k'}(\mathcal{G}_s)$ with $0 \leq k' \leq \tilde{k}$ and any edge ℓ of w . If the $(k'+1)^{\text{th}}$ Cauchy-Schwarz iteration cuts ℓ , then it appears exactly once both in $w0$ and $w1$. If ℓ is not cut, it appears twice in $w0$ or $w1$ but not in both graphs. In the two cases, $e(w) = \frac{1}{2}(e(w0) + e(w1))$. Induction on k' proves eq. (6.11) as $\sum_{\mathbf{m} \in F_0(\mathcal{G}_s)} \alpha_0(\mathbf{m})e(\mathbf{m}) = e(\mathcal{G}_s)$.

Let us now prove that for all $a \in \mathcal{B}$, $m_{a,\tilde{k}} \geq 2$. Let us consider a fixed a in \mathcal{B} and k' between 0 and \tilde{k} . Let w be a map in $F_{k'}(\mathcal{G}_s)$. We define $c_{a,r}(w)$ as the number of marked corners of w of scale a which are adjacent to a resolvent. We also let $c_{a,f}(w)$ be $c_a(w) - c_{a,r}(w)$. We further decompose $c_{a,r}(w)$ as $c_{a,c}(w) + c_{a,s}(w)$ where $c_{a,c}(w)$ is the number of corners, adjacent to a resolvent, and adjacent to the cut at step k' . Let now c be a marked corner in $s(w)$ such that $i_c = a$. If c is adjacent to a resolvent cut at the $(k')^{\text{th}}$ step, it appears in exactly one graph among $w0$ and $w1$. If c is not adjacent to a resolvent but nevertheless cut (thus by an odd cut), it belongs to both $w0$ and $w1$. If c is not cut, it appears twice either in $w0$ or in $w1$ but not in both. Then

$$\left. \begin{aligned} c_{a,f}(w0) + c_{a,f}(w1) &= 2c_{a,f}(w) + c_{a,c}(w) \\ c_{a,r}(w0) + c_{a,r}(w1) &= 2c_{a,s}(w) \end{aligned} \right\} \Rightarrow c_a(w0) + c_a(w1) = 2c_a(w) - c_{a,c}(w).$$

As $\alpha_{k'+1}(w0) = \alpha_{k'+1}(w1) = \frac{1}{2}\alpha_{k'}(w)$, we have

$$\alpha_{k'+1}(w0)c_a(w0) + \alpha_{k'+1}(w1)c_a(w1) = \alpha_{k'}(w)c_a(w) - \frac{1}{2}c_{a,c}(w).$$

Then, viewing the cutting process of Definition 6.8 as a computation tree T , and resumming $m_{a,\tilde{k}}$ from the leaves to the root of T , one gets

$$m_{a,\tilde{k}} = m_{a,0} - \frac{1}{2} \sum_{k'=0}^{\tilde{k}-1} \sum_{w \in F_{k'}(\mathcal{G}_s)} c_{a,c}(w) = c_a(\mathcal{G}_s) - \frac{1}{2}c_{a,r}(\mathcal{G}_s).$$

As $c_{a,r}(\mathcal{G}_s) \leq c_a(\mathcal{G}_s)$, $m_{a,\tilde{k}} \geq \frac{1}{2}c_a(\mathcal{G}_s)$. Remembering that, as discussed at the beginning of Section 6, any resolvent graph has at least four marked corners of each loop vertex scale (said differently $c_a(\mathcal{G}_s) \geq 4$ for all $a \in \mathcal{B}$), $m_{a,\tilde{k}} \geq 2$. To conclude the proof, we use

- this bound on $m_{a,\tilde{k}}$ as well as eq. (6.11) in eq. (6.10),
- the fact that $e(\mathcal{G}_s)$ grows at most linearly with $|\mathcal{B}|$,
- the fact that $e(\mathcal{G}'') \geq 4$,
- the resolvent bound of Lemma 1.4 and the definition of the cardioid domain Card_ρ . □

The main goal of Section 6 was to give an upper bound on the perturbative term I_4 of eq. (3.9). Here it is:

Theorem 6.12 (Perturbative factor I_4). — *Let \mathcal{B} be a Bosonic block and define n by $|\mathcal{B}| =: n+1$, $n \geq 0$. Then there exists $K \in \mathbb{R}_+^*$ such that the perturbative factor I_4 of eq. (3.9) obeys*

$$I_4(\mathcal{B}; \mathbf{G}) \leq K^n (n!)^{37/2} \rho^{x(\mathbf{G})} \prod_{a \in \mathcal{B}} M^{-\frac{1}{48}j_a}, \quad x(\mathbf{G}) = \begin{cases} e(\mathbf{G}) & \text{if } e(\mathbf{G}) \geq 1 \\ 2 & \text{otherwise.} \end{cases}$$

Proof. — We concentrate here on the case of Bosonic blocks with more than one node. Summing up what we have done in this Section, we have

$$I_4^4 = \int d\nu_{\mathcal{B}} \sum_{\mathcal{G}(\mathbf{G}'')} \sum_{\mathcal{G}_s(\mathcal{G})} A_{\mathcal{G}_s}.$$

The functional integration with respect to the measure $\nu_{\mathcal{B}}$ equals 1 as the integrand does not depend on $\vec{\sigma}$ anymore. Lemma 6.11 gives a bound on $A_{\mathcal{G}_s}$ and Lemma 6.5 a bound on the number of terms in the sum over \mathcal{G}_s . There remains to bound the number of resolvent graphs \mathcal{G} obtained from the contraction process applied to a given skeleton graph \mathbf{G}'' . Then, as already discussed at the beginning of this Section, $n(\mathbf{G}'') \leq 8n$ and $e(\mathbf{G}'') \leq 4n$. Thus $r(\mathbf{G}'') \leq 8n$ and as loop vertices bear at most three sigmas, the total number of sigma fields to be integrated by parts in the contraction process is bounded above by $24n$. We deduce that the number of terms in the sum over $\mathcal{G}(\mathbf{G}'')$ is bounded by $K^n(n!)^{32}$. All in all, we get

$$I_4^4 \leq K^n(n!)^{74} \rho^{e(\mathbf{G}'')} \prod_{a \in \mathcal{B}} M^{-\frac{1}{12}j_a} \Rightarrow I_4 \leq K^n(n!)^{37/2} \rho^{e(\mathbf{G})} \prod_{a \in \mathcal{B}} M^{-\frac{1}{48}j_a}$$

where we used that $e(\mathcal{G}_s) \geq e(\mathbf{G}'') = 4e(\mathbf{G})$. The final bound is obtained by noticing that the possible vertices of a single node Bosonic block bear at least two powers of ρ . \square

7. The final sums

We are now ready to gather the perturbative and non perturbative bounds of Sections 6 and 5 into a unique result on $\log \mathcal{Z}_{\leq j_{\max}}$. Our starting point is the expression of $\log \mathcal{Z}_{\leq j_{\max}}$ obtained after application of the Multiscale Loop Vertex Expansion :

$$(2.3) \quad \mathcal{W}_{\leq j_{\max}}(g) = \log \mathcal{Z}_{\leq j_{\max}}(g) = \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{\mathcal{J} \text{ tree}} \sum_{j_1=1}^{j_{\max}} \cdots \sum_{j_n=1}^{j_{\max}} \int d\mathbf{w}_{\mathcal{J}} \int d\nu_{\mathcal{J}} \partial_{\mathcal{J}} \left[\prod_{\mathcal{B}} \prod_{a \in \mathcal{B}} (-\bar{\chi}_{j_a}^{\mathcal{B}} W_{j_a}(\vec{\sigma}^a, \vec{\tau}^a) \chi_{j_a}^{\mathcal{B}}) \right].$$

Then we need to remember that the functional derivative $\partial_{\mathcal{J}}$, see eq. (2.2), is the product of Fermionic and Bosonic derivatives, eq. (2.2a), and that the latter factor out over the Bosonic components of the tree \mathcal{J} . Then, as in [GR14], we start by estimating the functional integration over the Grassmann variables to get:

$$|\log \mathcal{Z}_{\leq j_{\max}}| \leq \sum_{n=1}^{\infty} \frac{2^n}{n!} \sum_{\mathcal{J} \text{ tree}} \sum_{\{j_a\}} \left(\prod_{\mathcal{B}} \prod_{\substack{a,b \in \mathcal{B} \\ a \neq b}} (1 - \delta_{j_a j_b}) \right) \left(\prod_{\substack{\ell_F \in \mathcal{F}_F \\ \ell_F = (a,b)}} \delta_{j_a j_b} \right) \prod_{\mathcal{B}} |I_{\mathcal{B}}|,$$

$$I_{\mathcal{B}} = \int d\mathbf{w}_{\mathcal{B}} \int d\nu_{\mathcal{B}} \partial_{\mathcal{T}_{\mathcal{B}}} \prod_{a \in \mathcal{B}} (e^{-V_{j_a}} - 1) (\vec{\sigma}^a, \vec{\tau}^a).$$

Using the language of skeleton graphs, applying Hölder inequality (3.9) and using notation of eq. (5.1) and Theorem 6.12, we get

$$|\log \mathcal{Z}_{\leq j_{\max}}| \leq \sum_{n=1}^{\infty} \frac{O(1)^n}{n!} \sum_{\mathcal{J} \text{ tree}} \sum_{\{j_a\}} \left(\prod_{\mathcal{B}} \prod_{\substack{a,b \in \mathcal{B} \\ a \neq b}} (1 - \delta_{j_a j_b}) \right) \prod_{\mathcal{B}} \sum_{\mathbf{G}(\mathcal{B})} I_{\mathcal{B}}^{NP} I_4(\mathcal{B}; \mathbf{G}).$$

Let us introduce $n_{\mathcal{B}} := |\mathcal{B}| \geq 1$, which is therefore 1 plus the integer called n in Theorem 6.12. The sum over skeleton graphs $\mathbf{G}(\mathcal{B})$ can be decomposed into two parts. Due to Faà di Bruno formula, there is a first sum over partitions of the sets of edges incident to each vertex of $\mathcal{T}_{\mathcal{B}}$. The total number of such partitions is bounded above by $O(1)^{n_{\mathcal{B}}} (n_{\mathcal{B}}!)^2$. Given such partitions, there remains to choose appropriate

loop vertices for each vertex of \mathbf{G} . As the number of terms in the k^{th} σ -derivative of $V_j^{\geq 3}$ is bounded by $25k!$ the number of possible choices of loop vertices is bounded by $O(1)^{n_{\mathcal{B}}}(n_{\mathcal{B}}!)^2$. Then,

$$|\log \mathcal{Z}_{\leq j_{\max}}| \leq \sum_{n=1}^{\infty} \frac{O(1)^n}{n!} \sum_{\mathcal{J} \text{ tree}} \sum_{\{j_a\}} \left(\prod_{\mathcal{B}} (n_{\mathcal{B}}!)^4 \prod_{\substack{a,b \in \mathcal{B} \\ a \neq b}} (1 - \delta_{j_a j_b}) \right) \prod_{\mathcal{B}} I_{\mathcal{B}}^{NP} \sup_{\mathbf{G}} I_4(\mathcal{B}; \mathbf{G})$$

and using Theorems 5.1 and 6.12 we have

$$(7.1) \quad |\log \mathcal{Z}_{\leq j_{\max}}| \leq \sum_{n=1}^{\infty} \frac{O(1)^n}{n!} \rho^X \sum_{\mathcal{J} \text{ tree}} \sum_{\{j_a\}} \left(\prod_{\mathcal{B}} (n_{\mathcal{B}}!)^{4+37/2} \prod_{\substack{a,b \in \mathcal{B} \\ a \neq b}} (1 - \delta_{j_a j_b}) \right) \prod_{a=1}^n M^{-\frac{1}{48} j_a}$$

where $X := \sum_{\mathcal{B}} \sup_{\mathbf{G}} x(\mathbf{G})$. As $x(\mathbf{G}) = |\mathcal{B}| - 1$ if $|\mathcal{B}| \geq 2$ and $x(\mathbf{G}) = 2$ if $|\mathcal{B}| = 1$, we have $X \geq \lceil \frac{n}{2} \rceil$.

The factor $\prod_{\mathcal{B}} \prod_{\substack{a,b \in \mathcal{B} \\ a \neq b}} (1 - \delta_{j_a j_b})$ in eq. (7.1) ensures that slice indices j_a are all *different* in each block \mathcal{B} . Therefore

$$\sum_{a \in \mathcal{B}} j_a \geq 1 + 2 + \dots + n_{\mathcal{B}} = \frac{n_{\mathcal{B}}(n_{\mathcal{B}} + 1)}{2},$$

so that

$$\prod_{a=1}^n M^{-j_a/96} \leq \prod_{\mathcal{B}} e^{-O(1)n_{\mathcal{B}}^2}.$$

The number of labeled trees on n vertices is n^{n-2} (the complexity of the complete graph K_n on n vertices), hence the number of two-level trees \mathcal{J} in eq. (7.1) is exactly $2^{n-1}n^{n-2}$. Since $\sum_{\mathcal{B}} n_{\mathcal{B}} = n$, for ρ small enough we have

$$\begin{aligned} |\log \mathcal{Z}_{\leq j_{\max}}| &\leq \sum_{n=1}^{\infty} O(1)^n \rho^{n/2} \sup_{\mathcal{J} \text{ tree}} \left(\prod_{\mathcal{B}} (n_{\mathcal{B}}!)^{4+37/2} e^{-O(1)n_{\mathcal{B}}^2} \right) \sum_{\{j_a\}} \prod_{a=1}^n M^{-\frac{1}{96} j_a} \\ &\leq \sum_{n=1}^{\infty} O(1)^n \rho^{n/2} < +\infty. \end{aligned}$$

Hence for ρ small enough the series (2.3) is absolutely and uniformly convergent in the cardioid domain Card_{ρ} . Analyticity, Taylor remainder bounds and Borel summability follows for each $\mathcal{W}_{j_{\max}}$ (uniformly in j_{\max}) from standard arguments based on Morera's theorem. Similarly, since the sequence $\mathcal{W}_{j_{\max}}$ is easily shown uniformly Cauchy in the cardioid (from the geometric convergence of our bounds in j_{\max} , the limit \mathcal{W}_{∞} exists and its analyticity, uniform Taylor remainder bounds and Borel summability follow again from similar standard application of Morera's theorem. This completes the proof of our main result, Theorem 2.1.

Conclusion

Uniform Taylor remainder estimates at order p are required to complete the proof of Borel summability [Sok80] in Theorem 2.1. They correspond to further Taylor expanding beyond trees up to graphs with *excess* (*i.e.* number of cycles) at most p . The corresponding *mixed expansion* is described in detail in [Gur13a]. The main change is to force for an additional $p!$ factor to bound the cycle edges combinatorics, as expected in the Taylor uniform remainders estimates of a Borel summable function.

The main theorem of this paper clearly also extends to cumulants of the theory, introducing *ciliated* trees and graphs as in [Gur13a]. This is left to the reader.

The next tasks in constructive tensor field theories would be to treat the T_5^4 [BG14] and the $U(1) - T_6^4$ group field theory [OSVT13]. They are both just renormalizable and asymptotically free [BGOS12; Riv15]. Their full construction clearly requires more precise estimates, but at this stage we do not foresee any reason it cannot be done via the strategy of the MLVE.

Appendices

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A.1. Bare and renormalized amplitudes. — Let \mathcal{M}_1^c denote the graph \mathcal{M}_1 with a vertex of colour c . Its regularized bare amplitude $A_{\mathcal{M}_1^c}$ is a function of the incoming index n_c (and appears at first order of the Taylor expansion in g)¹⁰:

$$A_{\mathcal{M}_1^c} = - \sum_{\mathbf{p} \in [-N, N]^4} \frac{\delta(p_c - n_c)}{\mathbf{p}^2 + 1}.$$

The counterterm is minus the value at $n_c = 0$, hence

$$\delta_{\mathcal{M}_1^c} = \sum_{\mathbf{p} \in [-N, N]^4} \frac{\delta(p_c)}{\mathbf{p}^2 + 1} = \sum_{\mathbf{p} \in [-N, N]^3} \frac{1}{\mathbf{p}^2 + 1}.$$

Let us define $A_{\mathcal{M}_1}$ to be $\sum_c A_{\mathcal{M}_1^c}$. Remark that $A_{\mathcal{M}_1^c}$ is independent of c , so that in fact

$$\delta_{\mathcal{M}_1} := \sum_c \delta_{\mathcal{M}_1^c} = 4 \sum_{\mathbf{p} \in [-N, N]^3} \frac{1}{\mathbf{p}^2 + 1}.$$

We can calculate the renormalized amplitude of \mathcal{M}_1 as

$$A^r(\mathcal{M}_1) = - \sum_c \sum_{\mathbf{p} \in \mathbb{Z}^4} \frac{\delta(p_c - n_c) - \delta(p_c)}{\mathbf{p}^2 + 1} = \sum_c \sum_{\mathbf{p} \in \mathbb{Z}^3} \frac{n_c^2}{(n_c^2 + \mathbf{p}^2 + 1)(\mathbf{p}^2 + 1)}.$$

It is now a convergent sum, hence no longer requires the cutoff N .

We now compute the counterterm to the graph \mathcal{M}_2 . Remark that this log divergent mass graph has the tadpole \mathcal{M}_1 as a subgraph, hence its counterterm has to include the subrenormalization of that tadpole. The bare amplitude of \mathcal{M}_2 is

$$A_{\mathcal{M}_2} = \sum_c \sum_{\mathbf{p} \in [-N, N]^4} \frac{\delta(p_c - n_c)}{(\mathbf{p}^2 + 1)^2} \sum_{c'} \sum_{\mathbf{q} \in [-N, N]^4} \frac{\delta(q_{c'} - p_{c'})}{\mathbf{q}^2 + 1}.$$

The partly renormalized amplitude of \mathcal{M}_2 (with only the inner tadpole subtraction) is

$$\begin{aligned} A_{\mathcal{M}_2}^{\text{p. ren}} &= \sum_c \sum_{\mathbf{p} \in [-N, N]^4} \frac{\delta(p_c - n_c)}{(\mathbf{p}^2 + 1)^2} \sum_{c'} \sum_{\mathbf{q} \in \mathbb{Z}^4} \frac{\delta(q_{c'} - p_{c'}) - \delta(q_{c'})}{\mathbf{q}^2 + 1} \\ &= - \sum_c \sum_{\mathbf{p} \in [-N, N]^4} \frac{\delta(p_c - n_c)}{(\mathbf{p}^2 + 1)^2} \sum_{c'} \sum_{\mathbf{q} \in \mathbb{Z}^3} \frac{p_{c'}^2}{(p_{c'}^2 + \mathbf{q}^2 + 1)(\mathbf{q}^2 + 1)}, \end{aligned}$$

where we relaxed the cutoff constraint on the inner tadpole to better show that it is now a convergent

¹⁰By convention, we do not include the powers of the coupling constant g in the amplitudes of the Feynman graphs.

sum. Hence the counterterm for \mathcal{M}_2 is $\delta_{\mathcal{M}_2} := \sum_c \delta_{\mathcal{M}_2^c}$ and

$$\begin{aligned} \delta_{\mathcal{M}_2^c} &= - \sum_{\mathbf{p} \in [-N, N]^4} \frac{\delta(p_c)}{(\mathbf{p}^2 + 1)^2} \sum_{c'} \sum_{\mathbf{q} \in \mathbb{Z}^4} \frac{\delta(q_{c'} - p_{c'}) - \delta(q_{c'})}{\mathbf{q}^2 + 1} \\ &= \sum_{\mathbf{p} \in [-N, N]^4} \frac{\delta(p_c)}{(\mathbf{p}^2 + 1)^2} \sum_{c'} \sum_{\mathbf{q} \in \mathbb{Z}^3} \frac{p_{c'}^2}{(p_{c'}^2 + \mathbf{q}^2 + 1)(\mathbf{q}^2 + 1)} \\ &= 3 \sum_{\mathbf{p} \in [-N, N]^3} \frac{p_1^2}{(\mathbf{p}^2 + 1)^2} \sum_{\mathbf{q} \in \mathbb{Z}^3} \frac{1}{(p_1^2 + \mathbf{q}^2 + 1)(\mathbf{q}^2 + 1)} \end{aligned}$$

where in the last line we used that the value vanishes if $c = c'$, plus again the colour symmetry. The renormalized amplitude for \mathcal{M}_2 is the now fully convergent double sum

$$\begin{aligned} A_{\mathcal{M}_2}^r &= \sum_c \sum_{\mathbf{p} \in \mathbb{Z}^4} \frac{\delta(p_c - n_c) - \delta(p_c)}{(\mathbf{p}^2 + 1)^2} \sum_{c'} \sum_{\mathbf{q} \in \mathbb{Z}^4} \frac{\delta(q_{c'} - p_{c'}) - \delta(q_{c'})}{\mathbf{q}^2 + 1} \\ &= - \sum_c \sum_{\mathbf{p} \in \mathbb{Z}^4} \frac{\delta(p_c - n_c)}{(\mathbf{p}^2 + 1)^2} \sum_{\mathbf{q} \in \mathbb{Z}^3} \frac{p_c^2}{(p_c^2 + \mathbf{q}^2 + 1)(\mathbf{q}^2 + 1)} \\ &\quad - \sum_c \sum_{\mathbf{p} \in \mathbb{Z}^4} \frac{\delta(p_c - n_c) - \delta(p_c)}{(\mathbf{p}^2 + 1)^2} \sum_{c' \neq c} \sum_{\mathbf{q} \in \mathbb{Z}^3} \frac{p_{c'}^2}{(p_{c'}^2 + \mathbf{q}^2 + 1)(\mathbf{q}^2 + 1)} \\ &= \sum_c n_c^2 \sum_{\mathbf{p} \in \mathbb{Z}^3} \sum_{\mathbf{q} \in \mathbb{Z}^3} \left(\frac{3p_1^2[n_c^2 + 2(\mathbf{p}^2 + 1)]}{(n_c^2 + \mathbf{p}^2 + 1)^2(\mathbf{p}^2 + 1)^2(p_1^2 + \mathbf{q}^2 + 1)(\mathbf{q}^2 + 1)} \right. \\ &\quad \left. - \frac{1}{(n_c^2 + \mathbf{p}^2 + 1)^2(n_c^2 + \mathbf{q}^2 + 1)(\mathbf{q}^2 + 1)} \right). \end{aligned}$$

A.2. The Q operator. — We gather here some easy but useful results over the operators Q_0 and Q_1 , which are well-defined bounded operators on $L^2(\{1, 2, 3, 4\} \times \mathbb{Z}^2)$. From their definitions in the momentum basis (which has been used throughout this paper), see eqs. (1.8) and (1.9), it is easy to bound the coefficients of these operators:

$$\begin{aligned} (Q_0)_{c,c';mn,m'n'} &\leq \delta_{cc'} \delta_{mm'} \delta_{nn'} \frac{O(1)}{(m^2 + n^2 + 1)^{1/2}}, \\ (Q_1^{(1)})_{c,c';mn,m'n'} &\leq (1 - \delta_{cc'}) \delta_{mn} \delta_{m'n'} \frac{O(1)}{m^2 + m'^2 + 1}, \\ |(Q_1^{(2)})_{c,c';mn,m'n'}| &\leq \delta_{cc'} \delta_{mm'} \delta_{nn'} \frac{O(1)}{(m^2 + n^2 + 1)^{3/2}} \\ &\quad + (1 - \delta_{cc'}) \delta_{mn} \delta_{m'n'} \frac{O(1)}{(m^2 + m'^2 + 1)^2}. \end{aligned}$$

Hence Q_0 is a bounded operator and *e.g.* $\|gQ_0\| \leq \frac{1}{4}$ for g in the cardioid and ρ small enough. Q_1 is trace class therefore $Q = Q_0 + Q_1$ is bounded with $\|gQ\| \leq \frac{1}{2}$ for ρ small enough. Remark that Q itself, without ultraviolet cutoff, is not a trace class operator, since Q_0 is not trace class: indeed $\sum_{m,n} \frac{O(1)}{\sqrt{m^2 + n^2 + 1}}$ diverges linearly.

Lemma A.1. — *In the cardioid,*

$$\begin{aligned} \|Q_j\| &\leq O(1)M^{-j}, & \text{Tr } Q_j &\leq O(1)M^j, \\ \|Q_{0,j}\| &\leq O(1)M^{-j}, & \text{Tr } Q_{0,j} &\leq O(1)M^j, & \text{Tr}[Q_{0,j}^2] &\leq O(1), \\ \|Q_{1,j}^{(1)}\| &\leq O(1), & \text{Tr } Q_{1,j}^{(1)} &\leq O(1), & \text{Tr}[(Q_{1,j}^{(1)})^2] &\leq O(1)M^{-j}, \\ \|Q_{1,j}^{(2)}\| &\leq O(1)\rho M^{-j}, & \text{Tr } Q_{1,j}^{(2)} &\leq O(1)\rho M^{-j}. \end{aligned}$$

Proof. — Simple exercise by noting that Q_0 is diagonal and that for any bounded operator P , $\|P\| \leq \text{Tr } P$. \square

A.3. Proof of Lemma 1.2. — Recollecting the definitions of $\mathcal{N}, \mathcal{N}_1, \mathcal{N}_2, \mathcal{N}_3, \mathcal{N}_4$ and \mathcal{N}_5 , we get

$$\begin{aligned} \log \mathcal{N}_5 = \sum_{G \in \mathcal{V}'} \frac{(-g)^{|G|}}{S_G} \delta_G + \delta_t + \frac{1}{2} \bar{B}_1^2 - i\lambda^3 \sum_c \delta_{\mathcal{M}_2^c} \text{Tr}_c(\bar{B}_1)_c + \frac{1}{2} \text{Tr} D_1^2 \\ + \frac{1}{2} \bar{B}_2^2 + \frac{1}{2} \text{Tr}[D_2^2] + \frac{1}{3} \text{Tr}[D_1^3] + \text{Tr}[D_1^2 D_2] + \frac{1}{4} \text{Tr}[D_1^4] \end{aligned}$$

where $\mathcal{V}' = \mathcal{V} \setminus \{\mathfrak{N}_1, \mathfrak{N}_2, \mathfrak{N}_3\}$ and $\delta_t = \frac{g}{2}(2N+1) \sum_c (\delta_m^c)^2$. We need to prove that $\log \mathcal{N}_5 = 0$.

Let $\underline{\mathcal{F}}(G) := \{\text{divergent forests } \mathcal{F} \text{ of } G : G \notin \mathcal{F}\}$ and $\underline{A}_G := \sum_{\mathcal{F} \in \underline{\mathcal{F}}(G)} (\prod_{g \in \mathcal{F}} -\tau_g) A_G$. Recall that for G a vacuum graph, $\delta_G = -\underline{A}_G$. We will prove that $\log \mathcal{N}_5 - \sum_{G \in \mathcal{V}'} \frac{(-g)^{|G|}}{S_G} \delta_G$ equals $\sum_{G \in \mathcal{V}'} \frac{(-g)^{|G|}}{S_G} \underline{A}_G$. In the following, in order to lighten notations, we will denote \mathcal{M}_1 (resp. \mathcal{M}_2) by \uparrow (resp. \downarrow) or any rotated version of it. Thus we have

$$\underline{\mathcal{F}}(\mathcal{V}_1) = \left\{ \emptyset, \left\{ \uparrow \right\}, \left\{ \downarrow \right\} \right\}, \quad \underline{A}_{\mathcal{V}_1} = (1 - \tau_{\uparrow} - \tau_{\downarrow}) A_G,$$

$$\begin{aligned} \underline{\mathcal{F}}(\mathcal{V}_2) = \left\{ \emptyset, \left\{ \uparrow \right\}, \left\{ \downarrow \right\}, \left\{ \uparrow \downarrow \right\}, \left\{ \downarrow \uparrow \right\}, \left\{ \uparrow \downarrow \uparrow \right\}, \left\{ \downarrow \uparrow \downarrow \right\}, \left\{ \uparrow \downarrow \uparrow \downarrow \right\} \right\}, \\ \underline{A}_{\mathcal{V}_2} = (1 - \tau_{\uparrow})(1 - \tau_{\downarrow}) A_{\mathcal{V}_1} - \tau_{\uparrow \downarrow}(1 - \tau_{\downarrow}) A_{\mathcal{V}_2} - \tau_{\downarrow \uparrow}(1 - \tau_{\uparrow}) A_{\mathcal{V}_2}, \end{aligned}$$

$$\begin{aligned} \underline{\mathcal{F}}(\mathcal{V}_3) = \left\{ \emptyset, \left\{ \uparrow \right\}, \left\{ \downarrow \right\}, \left\{ \uparrow \downarrow \right\}, \left\{ \downarrow \uparrow \right\}, \left\{ \uparrow \downarrow \uparrow \right\}, \left\{ \downarrow \uparrow \downarrow \right\}, \left\{ \uparrow \downarrow \uparrow \downarrow \right\}, \left\{ \downarrow \uparrow \downarrow \uparrow \right\}, \left\{ \uparrow \downarrow \uparrow \downarrow \uparrow \right\}, \left\{ \downarrow \uparrow \downarrow \uparrow \downarrow \right\} \right\}, \\ \underline{A}_{\mathcal{V}_3} = (1 - \tau_{\uparrow})(1 - \tau_{\downarrow})(1 - \tau_{\uparrow \downarrow})(1 - \tau_{\downarrow \uparrow}) A_{\mathcal{V}_3} - \tau_{\uparrow \downarrow \uparrow}(1 - \tau_{\downarrow \uparrow})(1 - \tau_{\uparrow}) A_{\mathcal{V}_3}. \end{aligned}$$

The remaining vacuum graphs are easier to handle since they do not have any overlapping divergences:

$$\begin{aligned} \underline{A}_{\mathcal{V}_4} &= (1 - \tau_{\uparrow})(1 - \tau_{\downarrow})(1 - \tau_{\uparrow \downarrow})(1 - \tau_{\downarrow \uparrow}) A_{\mathcal{V}_4}, \\ \underline{A}_{\mathcal{V}_5} &= (1 - \tau_{\uparrow})(1 - \tau_{\swarrow})(1 - \tau_{\searrow}) A_{\mathcal{V}_5}, \\ \underline{A}_{\mathcal{V}_6} &= (1 - \tau_{\uparrow})(1 - \tau_{\downarrow})(1 - \tau_{\swarrow})(1 - \tau_{\searrow}) A_{\mathcal{V}_6}, \\ \underline{A}_{\mathcal{V}_7} &= (1 - \tau_{\swarrow})(1 - \tau_{\searrow})(1 - \tau_{\nwarrow})(1 - \tau_{\nearrow}) A_{\mathcal{V}_7}. \end{aligned}$$

Using $\delta_m^c = -\delta_{\mathcal{M}_1^c} + \lambda^2 \delta_{\mathcal{M}_2^c}$, it is then easy to check that:

$$\begin{aligned} -\frac{\lambda^2}{2} \underline{A}_{\mathcal{V}_1} &= \frac{1}{2} \bar{B}_1^2 + \frac{\lambda^2}{2} (2N+1) \sum_c (\delta_{\mathcal{M}_1^c})^2, \\ \frac{\lambda^4}{2} \underline{A}_{\mathcal{V}_2} &= \frac{1}{2} \text{Tr} D_1^2 - i\lambda^3 \sum_c \delta_{\mathcal{M}_2^c} \text{Tr}_c(\bar{B}_1)_c - \lambda^4 (2N+1) \sum_c \delta_{\mathcal{M}_1^c} \delta_{\mathcal{M}_2^c}, \\ -\frac{\lambda^6}{2} \underline{A}_{\mathcal{V}_3} &= -\frac{\lambda^6}{2} \bar{A}_{\mathcal{M}_2} \cdot \bar{A}_{\mathcal{M}_2} + \frac{\lambda^6}{2} (2N+1) \sum_c (\delta_{\mathcal{M}_2^c})^2, \\ \frac{\lambda^8}{2} \underline{A}_{\mathcal{V}_4} &= \frac{1}{2} \text{Tr}[D_2^2], \quad -\frac{\lambda^6}{3} \underline{A}_{\mathcal{V}_5} = \frac{1}{3} \text{Tr} D_1^3, \\ \lambda^8 \underline{A}_{\mathcal{V}_6} &= \text{Tr}(D_1^2 D_2), \quad \frac{\lambda^8}{4} \underline{A}_{\mathcal{V}_7} = \frac{1}{4} \text{Tr} D_1^4. \end{aligned}$$

In other words,

$$\log \mathcal{N}_5 = \sum_{G \in \mathcal{V}'} \frac{(-g)^{|G|}}{S_G} \delta_G + \sum_{G \in \mathcal{V}'} \frac{(-g)^{|G|}}{S_G} \underline{A}_G = 0$$

which is the desired result.

A.4. Proof of Lemma 4.5. — We start from expression (4.1) of v_j and first expand $\mathbb{I} - \mathcal{R}$ as $-\mathcal{UR}$ or $-\mathcal{RU}$:

$$\begin{aligned} v_j^{(1)} &= -\operatorname{Tr}[(U\mathbf{1}_j D\mathbf{1}_j \Sigma + \Sigma \mathbf{1}_j D\mathbf{1}_j U)\mathcal{R} + D^2 \mathbf{1}_j \Sigma \mathcal{R} + \Sigma \mathbf{1}_j D^2 \mathcal{R} + D^3 \mathbf{1}_j \Sigma \mathcal{R} + D^4 \mathbf{1}_j \Sigma \mathcal{R}], \\ v_j^{(2)} &= -\operatorname{Tr}[2\Sigma \mathbf{1}_j \Sigma \mathcal{UR} + \Sigma \mathbf{1}_j D\mathbf{1}_j \Sigma \mathcal{UR} + D_2 \mathbf{1}_j \Sigma^2 \mathbf{1}_j - 2D_1(\Sigma \mathbf{1}_j \Sigma + \mathbf{1}_j \Sigma^2 \mathbf{1}_j)]. \end{aligned}$$

Using $D = D_1 + D_2$, we gather in $v_j^{(1)} + v_j^{(2)}$ the six terms quadratic in Σ and linear in D which combine (using trace cyclicity) as (again, ordering is carefully chosen!)

$$\begin{aligned} & -\operatorname{Tr}[2\Sigma \mathbf{1}_j D\mathbf{1}_j \Sigma \mathcal{R} + 2\Sigma \mathbf{1}_j \Sigma D\mathcal{R} + D_2 \mathbf{1}_j \Sigma^2 \mathbf{1}_j - 2D_1(\Sigma \mathbf{1}_j \Sigma + \mathbf{1}_j \Sigma^2 \mathbf{1}_j)] \\ &= \operatorname{Tr}[2\Sigma \mathbf{1}_j D\mathbf{1}_j \Sigma(\mathbb{I} - \mathcal{R}) + 2\Sigma \mathbf{1}_j \Sigma D(\mathbb{I} - \mathcal{R}) - D_2 \mathbf{1}_j \Sigma^2 \mathbf{1}_j - 2D_2(\Sigma \mathbf{1}_j \Sigma + \mathbf{1}_j \Sigma^2 \mathbf{1}_j)] \\ &= -\operatorname{Tr}[2\Sigma \mathbf{1}_j D\mathbf{1}_j \Sigma \mathcal{UR} + 2U\Sigma \mathbf{1}_j \Sigma D\mathcal{R} + D_2 \mathbf{1}_j \Sigma^2 \mathbf{1}_j + 2D_2(\Sigma \mathbf{1}_j \Sigma + \mathbf{1}_j \Sigma^2 \mathbf{1}_j)]. \end{aligned}$$

Then $v_j^{(1)} + v_j^{(2)}$ rewrites as

$$\begin{aligned} v_j^{(1)} + v_j^{(2)} &= -\operatorname{Tr}[2(D^2 \mathbf{1}_j \Sigma + \Sigma \mathbf{1}_j D^2)\mathcal{R} + D^3 \mathbf{1}_j \Sigma \mathcal{R} + 2\Sigma \mathbf{1}_j \Sigma^2 \mathcal{R} \\ &\quad + (D^4 \mathbf{1}_j \Sigma + 3\Sigma \mathbf{1}_j D\mathbf{1}_j \Sigma U + 2U\Sigma \mathbf{1}_j \Sigma D)\mathcal{R} \\ &\quad + 3D_2 \mathbf{1}_j \Sigma^2 \mathbf{1}_j + 2D_2 \Sigma \mathbf{1}_j \Sigma]. \end{aligned}$$

The third line contains only convergent loop vertices and is free of any resolvent. The terms on the second line are ready for a HS bound (and a L^1/L^∞ bound) *i.e.* they will lead to convergent loop vertices. But the first line needs some further expansion of the resolvent factors ($\mathcal{R} = \mathbb{I} + \mathcal{UR} = \mathbb{I} + \mathcal{RU}$):

$$\begin{aligned} \operatorname{Tr}[D^2 \mathbf{1}_j \Sigma \mathcal{R}] &= \operatorname{Tr}[D^2 \mathbf{1}_j \Sigma] + \operatorname{Tr}[(\Sigma + D)D^2 \mathbf{1}_j \Sigma \mathcal{R}] \\ &= \operatorname{Tr}[D^2 \mathbf{1}_j \Sigma] + \operatorname{Tr}[(\Sigma \mathbf{1}_j D)(D\mathbf{1}_j \Sigma \mathcal{R})] + \operatorname{Tr}[D^3 \mathbf{1}_j \Sigma \mathcal{R}] \\ &= \operatorname{Tr}[D^2 \mathbf{1}_j \Sigma] + \operatorname{Tr}[(\Sigma \mathbf{1}_j D)(D\mathbf{1}_j \Sigma \mathcal{R})] + \operatorname{Tr}[D^3 \mathbf{1}_j \Sigma] + \operatorname{Tr}[(\Sigma + D)D^3 \mathbf{1}_j \Sigma \mathcal{R}], \end{aligned}$$

$$\begin{aligned} \operatorname{Tr}[\Sigma \mathbf{1}_j D^2] &= \operatorname{Tr}[D^2 \mathbf{1}_j \Sigma] + \operatorname{Tr}[(\Sigma \mathbf{1}_j D)(D\mathbf{1}_j \Sigma \mathcal{R})] + \operatorname{Tr}[D^3 \mathbf{1}_j \Sigma] \\ &\quad + \operatorname{Tr}[\Sigma \mathbf{1}_j D^3(\Sigma + D)\mathcal{R}], \end{aligned}$$

$$\operatorname{Tr}[D^3 \mathbf{1}_j \Sigma \mathcal{R}] = \operatorname{Tr}[D^3 \mathbf{1}_j \Sigma] + \operatorname{Tr}[(\Sigma + D)D^3 \mathbf{1}_j \Sigma \mathcal{R}],$$

$$\operatorname{Tr}[\Sigma \mathbf{1}_j \Sigma^2 \mathcal{R}] = \operatorname{Tr}[\Sigma^3 \mathbf{1}_j] + \operatorname{Tr}[(\Sigma + D)\Sigma \mathbf{1}_j \Sigma^2 \mathcal{R}].$$

We finally get an expression for v_j which is completely ready for our bounds:

$$\begin{aligned} v_j &= -\operatorname{Tr}[2\Sigma^2 \mathbf{1}_j \Sigma^2 \mathcal{R} + 2D\Sigma \mathbf{1}_j \Sigma^2 \mathcal{R} + 2\Sigma^2 \mathbf{1}_j \Sigma D\mathcal{R} + 3\Sigma D\mathbf{1}_j \Sigma^2 \mathcal{R} \\ &\quad + 4(\Sigma \mathbf{1}_j D)(D\mathbf{1}_j \Sigma \mathcal{R}) + 3(\Sigma \mathbf{1}_j D)(\mathbf{1}_j \Sigma D\mathcal{R}) + 2(D\Sigma \mathbf{1}_j)(\mathbf{1}_j \Sigma D\mathcal{R}) + 5\Sigma D^3 \mathbf{1}_j \Sigma \mathcal{R} \\ (A.1) \quad &\quad + 4D^4 \mathbf{1}_j \Sigma \mathcal{R} + 2\Sigma \mathbf{1}_j D^4 \mathcal{R} \\ &\quad + 2\Sigma^3 \mathbf{1}_j + 3D_2 \mathbf{1}_j \Sigma^2 \mathbf{1}_j + 2D_2 \Sigma \mathbf{1}_j \Sigma + 4D^2 \mathbf{1}_j \Sigma \\ &\quad + D^5 \mathbf{1}_j \mathcal{R}] + \mathcal{D}_{conv,j}. \end{aligned}$$

We need to bound $|V_j| = |\int_0^1 dt_j v_j(t_j)| \leq \int_0^1 dt_j |v_j|$. The bound on $|v_j|$ will be uniform in t_j so that the integral is simply bounded by 1. Let us now show that (the module of) each term of eq. (A.1) is bounded by a sum of (modules of) allowed loop vertices, see Definition 4.4.

$$\begin{aligned} |\operatorname{Tr}[\Sigma^2 \mathbf{1}_j \Sigma^2 \mathcal{R}]| &\leq \operatorname{Tr}[|\Sigma|^4 \mathbf{1}_j] + \operatorname{Tr}[(\Sigma^2)^* \mathbf{1}_j \Sigma^2 \mathcal{R} \mathcal{R}^*] && \text{by eq. (4.2) and } \mathbf{1}_j = \mathbf{1}_j^2 \\ &\leq 2\operatorname{Tr}[|\Sigma|^4 \mathbf{1}_j] = 2\rho^2 U_j^4 && \text{by eq. (4.3) and } |g| \|\mathcal{R}\| \leq \rho. \end{aligned}$$

$$\begin{aligned} |\operatorname{Tr}[D\Sigma \mathbf{1}_j \Sigma^2 \mathcal{R}]| &\leq \operatorname{Tr}[D^2 \Sigma^* \mathbf{1}_j \Sigma] + \operatorname{Tr}[(\Sigma^2)^* \mathbf{1}_j \Sigma^2 \mathcal{R} \mathcal{R}^*] && \text{by eq. (4.2) and } \mathbf{1}_j = \mathbf{1}_j^2 \\ &\leq |g|^3 U_j^{2,b} + \rho^2 U_j^4 && \text{by eq. (4.3) and } |g| \|\mathcal{R}\| \leq \rho \end{aligned}$$

and similarly for the two other terms of the first line of eq. (A.1).

$$|\mathrm{Tr}[(\Sigma \mathbf{1}_j D)(D \mathbf{1}_j \Sigma \mathcal{R})]| \leq \rho^3 U_j^{2,a} + \mathrm{Tr}[\Sigma^* \mathbf{1}_j D^2 \mathbf{1}_j \Sigma \mathcal{R} \mathcal{R}^*] \leq 2\rho^3 U_j^{2,a}.$$

Similarly one gets

$$\begin{aligned} |\mathrm{Tr}[(\Sigma \mathbf{1}_j D)(\mathbf{1}_j \Sigma D \mathcal{R})]| &\leq \rho^3 (U_j^{2,a} + U_j^{2,b}), \\ |\mathrm{Tr}[(D \Sigma \mathbf{1}_j)(\mathbf{1}_j \Sigma D \mathcal{R})]| &\leq 2\rho^3 U_j^{2,b} \\ |\mathrm{Tr}[\Sigma D^3 \mathbf{1}_j \Sigma \mathcal{R}]| &\leq \rho^3 U_j^{2,a} + \rho^5 U_j^{2,c}, \\ |\mathrm{Tr}[D^4 \mathbf{1}_j \Sigma \mathcal{R}]| &\leq \rho^3 U_j^{2,a} + \rho^6 U_j^{0,a} \geq |\mathrm{Tr}[\Sigma \mathbf{1}_j D^4 \mathcal{R}]|. \end{aligned}$$

The remaining terms of v_j , see eq. (A.1), already belong to the list of convergent loop vertices.

A.5. Faces and loop vertices. — We gather here the missing details of the proof of Lemma 5.8.

A.5.1. Quartic loop vertex. — We start by studying the incidence relations between faces and a vertex of type $U_{j_1}^4 = 1/|g|^2 \mathrm{Tr}[|\Sigma|^4 \mathbf{1}_{j_1}]$, see fig. 20a for a graphical representation of it. Making explicit the dependency of $U_{j_1}^4$ on the σ -field, we decompose our analysis into four main cases corresponding to the number of different colours around the vertex.

The c^4 -case. Here all four σ -fields bear the same colour, fig. 20b. A Wick-contraction can create zero, one or two tadpoles. We gather data about possible faces (colour, local or not, length, worst cost) in Table 1. For example, in case of two “planar” tadpoles, there is one local face of length 2 and colour c , depicted in red in fig. 21b. In case of one “non-planar” tadpole, there are two non-local faces of length at least 3, in red in fig. 21c. Also, in any case, there are three local faces of respective colours $c' \neq c$ and length 4, in red in fig. 21a. All in all, the worst cost with tadpole(s) is $M^{j_1+j_2+4j_4}$ and $M^{(j_1+j_2+j_3+7j_4)/2}$ without tadpole.

Tadpoles		Colour	Locality	Length	Worst cost
		$c' \neq c$	local $\times 3$	4	M^{3j_4}
2	planar	c	local $\times 2$	1	$M^{j_1+j_2}$
			local	2	M^{j_4}
	non-planar		local	4	M^{j_4}
1	planar		local	1	M^{j_1}
			non-local	> 1	$M^{j_2/2}$
			non-local	> 2	$M^{j_4/2}$
	non-planar	non-local $\times 2$	> 2	$M^{(j_2+j_4)/2}$	
0			non-local $\times 4$	> 1	$M^{(j_1+j_2+j_3+j_4)/2}$

Table 1. The c^4 -case

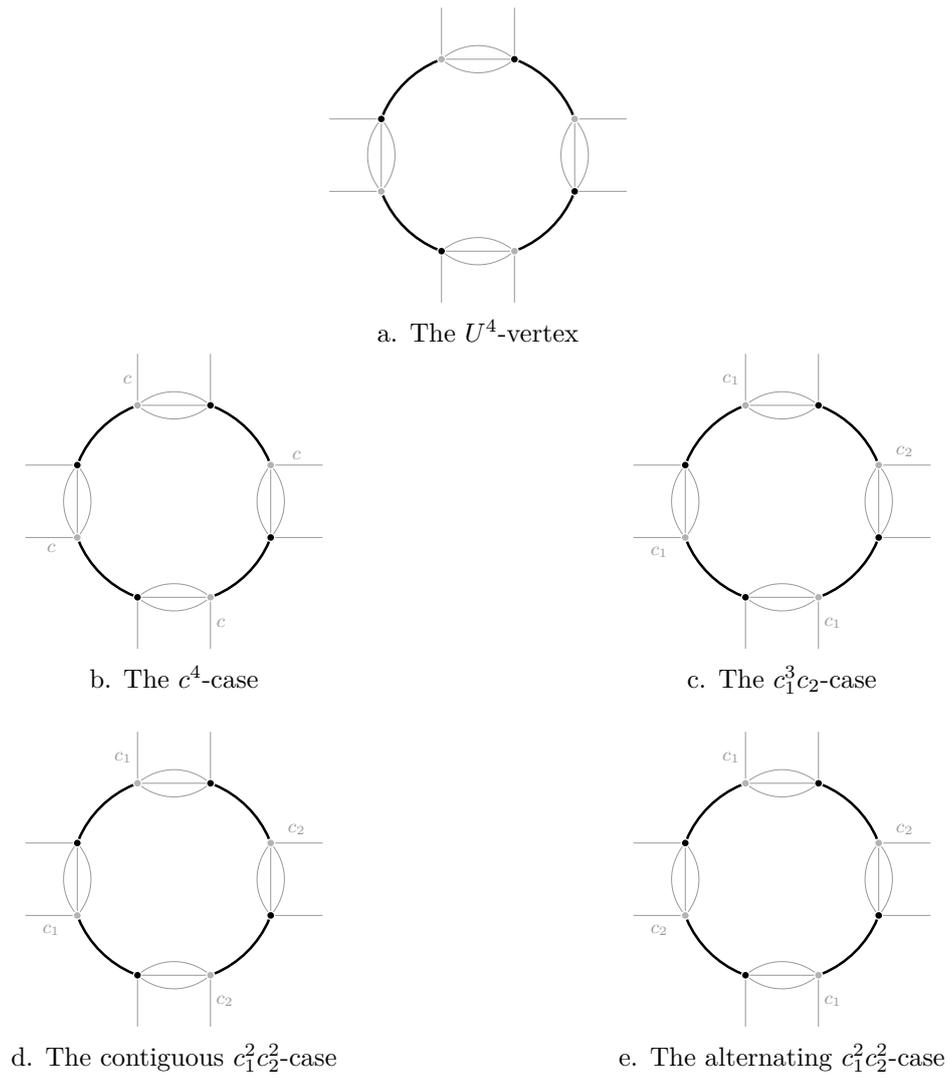
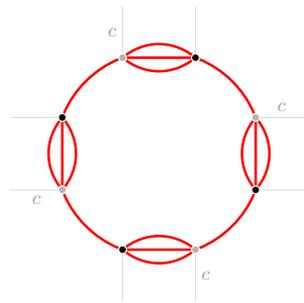
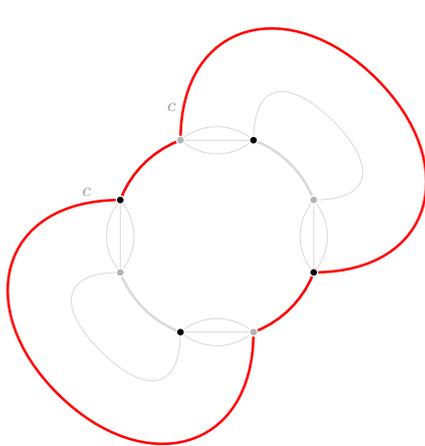


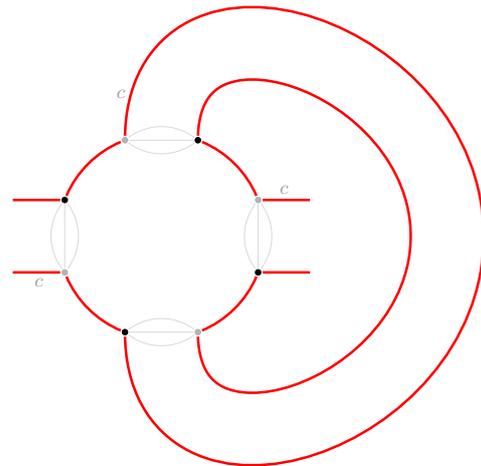
Figure 20. The U^4 -loop vertex and some of its coloured versions.



a. The 3 local faces of colours $c' \neq c$



b. Two planar tadpoles



c. One non-planar tadpole

Figure 21. The c^4 -case and some of its possible faces.

The $c_1^3c_2$ -case. Here there can be zero or one tadpole which could be planar or not, see fig. 20c. The worst cost with tadpole is $M^{j_1+3j_4}$ and $M^{(j_1+j_2+6j_4)/2}$ without, see Table 2.

Tadpoles		Colour	Locality	Length	Worst cost
		$c \neq c_1, c_2$	local $\times 2$	4	M^{2j_4}
		c_2	non-local	> 4	$M^{j_4/2}$
1	planar	c_1	local	1	M^{j_1}
			non-local	> 3	$M^{j_4/2}$
non-planar	non-local $\times 2$		> 2	$M^{(j_2+j_4)/2}$	
	non-local $\times 2$		> 1	$M^{(j_1+j_2)/2}$	
0			non-local	> 2	$M^{j_4/2}$

Table 2. The $c_1^3c_2$ -case

The $c_1^2c_2^2$ -cases. We have two main cases here: either the σ -fields of colour c_1 are contiguous or not, see figs. 20d and 20e. Anyway, there can be again zero, one or two tadpoles after Wick-contraction. All in all, the worst cost with tadpoles is $M^{j_1+j_2+4j_4}$ and $M^{(j_1+j_2+6j_4)/2}$ without, see Table 3.

Tadpoles	Colour	Locality	Length	Worst cost
	$c \neq c_1, c_2$	local $\times 2$	4	M^{2j_4}
2	c_1	local	1	M^{j_1}
		local	3	M^{j_4}
	c_2	local	1	M^{j_2}
		local	3	M^{j_4}
1	c_1	local	1	M^{j_1}
		local	3	M^{j_4}
	c_2	non-local	> 1	$M^{j_2/2}$
		non-local	> 3	$M^{j_4/2}$
0	c_1	non-local	> 1	$M^{j_1/2}$
		non-local	> 3	$M^{j_4/2}$
	c_2	non-local	> 1	$M^{j_2/2}$
		non-local	> 3	$M^{j_4/2}$

a. Contiguous

Tadpoles	Colour	Locality	Length	Worst cost
	$c \neq c_1, c_2$	local $\times 2$	4	M^{2j_4}
2	c_1	local $\times 2$	2	$M^{j_2+j_4}$
	c_2	local $\times 2$	2	$M^{j_3+j_4}$
1	c_1	local $\times 2$	2	$M^{j_2+j_4}$
	c_2	non-local $\times 2$	> 2	$M^{(j_3+j_4)/2}$
0	c_1	non-local $\times 2$	> 2	$M^{(j_2+j_4)/2}$
	c_2	non-local $\times 2$	> 2	$M^{(j_3+j_4)/2}$

b. Alternating

Table 3. The $c_1^2 c_2^2$ -cases

The $c_1^2 c_2 c_3$ -cases. Here again the σ -fields of colour c_1 are either contiguous or not. There can be zero or one tadpole. The worst cost with tadpole is $M^{j_1+3j_4}$ and $M^{(j_1+5j_4)/2}$ without, see Table 4.

Tadpoles	Colour	Locality	Length	Worst cost
	$c \neq c_1, c_2, c_3$	local	4	M^{j_4}
	c_2	non-local	> 4	$M^{j_4/2}$
	c_3	non-local	> 4	$M^{j_4/2}$
1	c_1	local	1	M^{j_1}
		local	3	M^{j_4}
0		non-local	> 1	$M^{j_1/2}$
		non-local	> 3	$M^{j_4/2}$

a. Contiguous

Tadpoles	Colour	Locality	Length	Worst cost
	$c \neq c_1, c_2, c_3$	local	4	M^{j_4}
	c_2	non-local	> 4	$M^{j_4/2}$
	c_3	non-local	> 4	$M^{j_4/2}$
1	c_1	local $\times 2$	2	$M^{j_2+j_4}$
0		non-local $\times 2$	> 2	$M^{(j_2+j_4)/2}$

b. Alternating

Table 4. The $c_1^2 c_2 c_3$ -cases

The $c_1 c_2 c_3 c_4$ -case. All σ -fields bear different colours. No tadpole is possible. The cost is M^{2j_4} .

Tadpoles	Colour	Locality	Length	Worst cost
0	c_1, c_2, c_3, c_4	non-local $\times 4$	> 4	M^{2j_4}

Table 5. The $c_1 c_2 c_3 c_4$ -case

A.5.2. Quadratic loop vertices. — There are five different types of quadratic vertices $U_{j_1}^{2,\alpha}$, see Definition 4.4. They involve D and D_2 operators. Recall that $D = D_1 + D_2$ where $D_1 = C^{1/2} A_{\mathcal{M}_1}^r C^{1/2}$ and $D_2 = C^{1/2} A_{\mathcal{M}_2}^r C^{1/2}$. Both are diagonal operators in the momentum basis. From Lemma 1.1, we get

$$\sup \{ |(D_1)_{m,n}|, |(D)_{m,n}| \} \leq \delta_{mn} \frac{O(1)}{\|m\| + 1}, \quad |(D_2)_{m,n}| \leq \delta_{mn} \frac{O(1)}{\|m\|^{2-\epsilon} + 1}.$$

Each quadratic vertex contains two C -propagators plus a D -type operator. At worst (depending on the position of the $\mathbf{1}_{j_1}$ cutoff in the trace), the D^2 operator brings M^{-2j_2} , D^4 brings M^{-4j_2} and D_2 brings $M^{-(2-\epsilon)j_2}$. Thus the worst vertex is $U_{j_1}^{2,e}$. Note that because of the conservation of the 4-tuple of indices through the D -type insertion, the scales on both of its sides are equal (to j_2). The face data and worst costs for a generic quadratic loop vertex are available in Tables 6 and 7. The worst cost with tadpole is then $M^{j_1+4j_2}$ and $M^{(j_1+7j_2)/2}$ without.

Tadpoles	Colour	Locality	Length	Worst cost
	$c' \neq c$	local $\times 3$	3	M^{3j_2}
1	c	local	1	M^{j_1}
		local	2	M^{j_2}
0		non-local	> 1	$M^{j_1/2}$
		non-local	> 2	$M^{j_2/2}$

Table 6. The c^2 -case

Tadpoles	Colour	Locality	Length	Worst cost
0	$c' \neq c_1, c_2$	local $\times 2$	3	M^{2j_2}
	c_1, c_2	non-local $\times 2$	> 3	M^{j_2}

Table 7. The $c_1 c_2$ -case

A.5.3. Loop vertices of degree one. — The most dangerous case is $U_j^{1,a}$, the D^2 -insertion of which brings M^{-2j} . The cost is $M^{7j/2}$, see Table 8.

Tadpoles	Colour	Locality	Length	Worst cost
0	$c' \neq c$	local $\times 3$	2	M^{3j}
	c	non-local	> 2	$M^{j/2}$

Table 8. The degree one case

A.6. Perturbative bounds. — Multiscale analysis is a powerful tool to bound the Feynman amplitudes of convergent graphs both in the standard [Fel+85] and in the tensor case [BGR12a]. It is especially easy in the superrenormalizable case, as it not only proves uniform bounds on any (renormalized) Feynman amplitude, but it does this and in addition allows to spare a uniform small fraction $\eta > 0$ of the scale factor of every line of the graph, which can then be used for other purposes.

Let us state precisely a Lemma of this type for our model, for which we can take $\eta = \frac{1}{24}$. More precisely

Lemma A.2. — *Let $\mu = \{i_\ell\}$ be a scale attribution for the lines of a vacuum Feynman graph of the T_4^4 theory. There exists $K > 0$ such that*

$$(A.2) \quad \sum_{\mu} A_{G,\mu}^r \left[\prod_{\ell \in G} M^{\frac{i_\ell}{24}} \right] \leq K^{n(G)}$$

where $n(G)$ is the order of G , and $A_{G,\mu}^r$ is the renormalised Feynman amplitude for scale attribution μ .

Proof. — Let us denote $f \bowtie \ell$ if face f runs through line ℓ . When G is convergent, we return to the direct representation and consider eq. (5.7). We have to add the factor $\prod_{\ell} M^{\frac{i_\ell}{24}}$ to the previous factor $\prod_{\ell} M^{-2i_\ell}$. Hence we have to prove

$$\sum_{\mu} \prod_{\ell} M^{-\frac{47}{24}i(\ell)} \prod_f M^{i_m(f)} \leq K^{n(G)}$$

where we recall that $i_m(f) = \inf_{f \bowtie \ell} i_\ell$ is the smallest scale of the edges along face f . We obtain an upper bound by replacing the factor $M^{i_m(f)}$ for each face by $M^{\sum_{\ell \bowtie f} \frac{i_\ell}{L(f)}}$ where $L(f)$ is the length of

the face f . Reordering the product we simply have now to check that

$$\sum_{f \ni \ell} \frac{1}{L(f)} \leq \frac{23}{12} \quad \forall \ell$$

since $-\frac{47}{24} + \frac{23}{12} = -\frac{1}{24}$ and obviously

$$\sum_{\mu} \prod_{\ell} M^{-\frac{i(\ell)}{24}} \leq K^{n(G)}.$$

- If no face of length 1 or 2 runs through ℓ obviously $\sum_{f \ni \ell} \frac{1}{L(f)} \leq \frac{4}{3} < \frac{23}{12}$.
- If a face of length 2 runs through ℓ , but none of length 1, it is easy to check that there can be at most three such faces (not four, otherwise the graph would be \mathfrak{N}_2 in fig. 5). Hence $\sum_{f \ni \ell} \frac{1}{L(f)} \leq \frac{3}{2} + \frac{1}{3} < \frac{23}{12}$.
- Finally if a face of length 1 runs through ℓ it must be a tadpole. It cannot be a melonic tadpole of a \mathcal{M}_1 type (otherwise it would not be a convergent graph). It can be of the non melonic type, but it cannot be the non-melonic tadpole of \mathfrak{N}_1 or \mathfrak{N}_3 in fig. 5 because these graphs diverge. Hence the other faces through f cannot be of length 2 or all of length 3. Hence $\sum_{f \ni \ell} \frac{1}{L(f)} \leq 1 + \frac{2}{3} + \frac{1}{4} = \frac{23}{12}$.

When G contains a divergent subgraph of type \mathcal{M}_1 or \mathcal{M}_2 in fig. 3, renormalization bringing an additional factor M^{-i_ℓ} for the critical melonic tadpoles lines, eq. (A.2) still holds.

Finally when G is one of the ten divergent vacuum graphs, in figs. 4 and 5, its renormalized amplitude is 0 and there is nothing to prove. \square

Index of notation

Feynman Graphs

\mathcal{G} skeleton graph.....	5	\mathcal{B} connected component of a Bosonic forest	17
\mathcal{J} jungle	5	\mathcal{C} chord diagram.....	45
\mathcal{G} resolvent graph.....	6	\mathcal{G}_s secured graph	50
G tensor graphs	9	\mathfrak{m} map	51

Spaces

$\mathcal{H}^\otimes := \bigotimes_{i=1}^4 \mathcal{H}_i$	7	$L(\mathcal{H}_i)$ linear operators on \mathcal{H}_i	10
$\mathcal{H}_i := \ell_2(\mathbb{Z})$	7	$L(L(\mathcal{H})^\times)$	10
\mathcal{M} set of divergent 2-point graphs	9	$\text{Card}_\rho := \{g \in \mathbb{C} : g < \rho \cos(\frac{1}{2} \arg g)\}$	14
\mathcal{V} set of divergent vacuum graphs.....	9	$V_{\mathcal{B}} := \mathbb{R}^{ \mathcal{B} } \otimes L(\mathcal{H})^\times$	28
$L(\mathcal{H})^\times := \prod_{i=1}^4 L(\mathcal{H}_i)$	10		

Tensors

T, \bar{T} original fields	7	$\vec{B}_1 := i\lambda \vec{A}_{\mathcal{M}_1}^r$	11
$\vec{\sigma} := (\sigma_c)_c$, main intermediate field.....	10	$\vec{A}_{\mathcal{M}_2}^r := (A_{\mathcal{M}_2}^r)_c$	12
$\vec{A}_{\mathcal{M}_1}^r := (A_{\mathcal{M}_1}^r)_c$	11	$\vec{\tau}$ intermediate field for \mathfrak{N}_2 counterterm.....	12
		$\vec{B}_2 := -i\lambda^3 \vec{A}_{\mathcal{M}_2}^r$	13

Operators on \mathcal{H}^\otimes

C propagator	7	$D_2 := i\lambda C^{1/2} B_2 C^{1/2}$	13
A^r renormalised amplitude.....	9	$U := \Sigma + D$	13
δ_m mass counterterms	10	$D := D_1 + D_2$	13
δ_t mass counterterms squared.....	10	$\mathcal{R} := (\mathbb{I} - U)^{-1}$	13
$\sigma := \sum_c \sigma_c \otimes \mathbb{I}_{\hat{c}}$	10	$\mathcal{E} := \underline{A}_{\mathcal{V}_5} + \underline{A}_{\mathcal{V}_6} + \underline{A}_{\mathcal{V}_7}$	13
\mathbb{I} identity operator.....	10	$\mathbf{1}_{\leq j} := \mathbf{1}_{1+\ n\ ^2 \leq M^{2j}} \delta_{mn}$	15
$\Sigma := i\lambda \sqrt{C} \sigma \sqrt{C}$	10	$\mathbf{1}_j := \mathbf{1}_{\leq j} - \mathbf{1}_{\leq j-1}$	15
$H := -i\lambda^{-1} \Sigma$	10	$U'_j := \frac{dU}{dt_j}(t_j)$	16
$R := (\mathbb{I} - \Sigma)^{-1}$	10	$D'_{1,j} := \frac{dD_1}{dt_j}(t_j)$	16
$B_1 := i\lambda A_{\mathcal{M}_1}^r$	11	$\Sigma'_j := \frac{d\Sigma}{dt_j}(t_j)$	16
$D_1 := i\lambda C^{1/2} B_1 C^{1/2}$	11	$D'_{2,j} := \frac{dD_2}{dt_j}(t_j)$	16
$U_1 := \Sigma + D_1$	11	$\Delta^s := \frac{\partial U}{\partial \sigma^s}$	21
$R_1 := (\mathbb{I} - U_1)^{-1}$	11		
$B_2 := -i\lambda^3 A_{\mathcal{M}_2}^r$	13		

Operators on $L(\mathcal{H})^\times$

Q	12	Q_1	12
Q_0	12		

Scalars

\mathcal{N} normalization constant	7	\mathcal{N}_2 normalization constant	12
g coupling constant	7	\mathcal{N}_3 normalization constant	13
δ_G counterterm	9	\mathcal{N}_4 normalization constant	13
λ square root of g	10	\mathcal{N}_5 normalization constant	13
b_1 matrix element of B_1	11	\underline{A} prepared amplitude	57
\mathcal{N}_1 normalization constant	11		

Miscellanea

$[n] := \{1, 2, \dots, n\}$	4	$S := \llbracket 1, j_{\max} \rrbracket$	16
\mathcal{Z} partition function	7	\mathbb{I}_S $ S \times S $ identity matrix	16
V_c interaction polynomial	7	$\mathbb{1}$ matrix full of 1's	17
V_c^r renormalised interaction	10	$X(\mathbf{w})$ interpolated Bosonic covariance	18
\mathbb{I} identity element of $L(L(\mathcal{H})^\times)$	10	\mathcal{Q}_j quadratic form on $L(\mathcal{H})^\times$	25
$\mathcal{W} := \log \mathcal{Z}$	13	$X_{\mathcal{B}}$ diagonal block of X	29
W amplitude of a node of a jungle	16		

References

- [ADJ91] J. Ambjorn, B. Durhuus, and T. Jonsson. “Three dimensional simplicial gravity and generalized matrix models”. *Mod. Phys. Lett. A* 6.12 (Apr. 1991), p. 1133.
- [Amb02] J. Ambjorn. “Simplicial Euclidean and Lorentzian Quantum Gravity”. In: *General Relativity and Gravitation*. Ed. by N. T. Bishop and S. D. Maharaj. Proceedings of the 16th International Conference. Durban, South Africa, 15–21 July 2001. World Scientific, Jan. 9, 2002. arXiv: [gr-qc/0201028](#).
- [Amb+13] J. Ambjorn et al. “Causal dynamical triangulations and the search for a theory of quantum gravity”. In: *The Thirteenth Marcel Grossmann Meeting*. Stockholm University, Sweden, 1–7 July 2012. World Scientific, May 29, 2013, pp. 120–137. arXiv: [1305.6680](#) [[gr-qc](#)].
- [AR95] A. Abdesselam and V. Rivasseau. “Trees, forests and jungles: a botanical garden for cluster expansions”. In: *Constructive Physics*. Vol. 446. Lectures Notes in Physics. New York: Springer-Verlag, 1995.
- [AR98] A. Abdesselam and V. Rivasseau. “Explicit Fermionic Tree Expansions”. *Lett. Math. Phys.* 44.1 (1998), pp. 77–88.
- [BDR15] V. Bonzom, T. Delepouve, and V. Rivasseau. “Enhancing non-melonic triangulations: A tensor model mixing melonic and planar maps”. *Nucl. Phys. B* 895 (Apr. 3, 2015), pp. 161–191. arXiv: [1502.01365](#) [[math-ph](#)].
- [BG12a] J. Ben Geloun. “Asymptotic Freedom of Rank 4 Tensor Group Field Theory”. In: *Symmetries and Groups in Contemporary Physics*. Vol. 11. Nankai Series in Pure, Applied Mathematics and Theoretical Physics. Proceedings of the XXIX International Colloquium on Group-Theoretical Methods in Physics. Tianjin, China, 20–26 August 2012. World Scientific, Oct. 2012, pp. 367–372. arXiv: [1210.5490](#) [[hep-th](#)].
- [BG12b] J. Ben Geloun. “Two and four-loop β -functions of rank 4 renormalizable tensor field theories”. *Class. Quant. Grav.* 29 (Oct. 1, 2012), p. 235011. arXiv: [1205.5513](#) [[hep-th](#)].
- [BG14] J. Ben Geloun. “Renormalizable Models in Rank $d \geq 2$ Tensorial Group Field Theory”. *Commun. Math. Phys.* 332.1 (Aug. 23, 2014), pp. 117–188. DOI: [10.1007/s00220-014-2142-6](#). arXiv: [1306.1201](#) [[hep-th](#)].
- [BGMR10] J. Ben Geloun, J. Magnen, and V. Rivasseau. “Bosonic colored group field theory”. *Eur. Phys. J. C* 70.4 (Dec. 2010), pp. 1119–1130. arXiv: [0911.1719](#) [[hep-th](#)].
- [BGOS12] J. Ben Geloun and D. Ousmane Samary. “3D Tensor Field Theory: Renormalization and One-loop β -functions”. *Ann. H. Poincaré* 14.6 (Jan. 2012), pp. 1599–1642. arXiv: [1201.0176](#) [[hep-th](#)].
- [BGR12a] J. Ben Geloun and V. Rivasseau. “A Renormalizable 4-Dimensional Tensor Field Theory”. *Commun. Math. Phys.* 318.1 (Jan. 5, 2012), pp. 69–109. DOI: [10.1007/s00220-012-1549-1](#). arXiv: [1111.4997](#) [[hep-th](#)].
- [BGR12b] V. Bonzom, R. Gurau, and V. Rivasseau. “Random tensor models in the large N limit: Uncoloring the colored tensor models”. *Phys. Rev. D* 85.8 (Apr. 26, 2012), p. 084037. arXiv: [1202.3637](#) [[hep-th](#)].
- [BGR13] J. Ben Geloun and V. Rivasseau. “Addendum to “A Renormalizable 4-Dimensional Tensor Field Theory””. *Commun. Math. Phys.* 322.3 (Sept. 2013), pp. 957–965. arXiv: [1209.4606](#) [[hep-th](#)].
- [BK87] D. C. Brydges and T. Kennedy. “Mayer expansions and the Hamilton-Jacobi equation”. *J. Stat. Phys.* 48.1 (Mar. 3, 1987), pp. 19–49.
- [BLT17] V. Bonzom, L. Lionni, and A. Tanasa. “Diagrammatics of a colored SYK model and of an SYK-like tensor model, leading and next-to-leading orders”. *J. Math. Phys.* 58.5 (Apr. 25, 2017). DOI: [10.1063/1.4983562](#). arXiv: [1702.06944](#) [[hep-th](#)].
- [Bon13] V. Bonzom. “New $1/N$ expansions in random tensor models”. *J. High Energy Phys.* 2013.06 (June 14, 2013), p. 062. DOI: [10.1007/JHEP06\(2013\)062](#). arXiv: [1211.1657](#) [[hep-th](#)].
- [Bon16] V. Bonzom. “Large N Limits in Tensor Models: Towards More Universality Classes of Colored Triangulations in Dimension $d \geq 2$ ”. *SIGMA* 12 (July 20, 2016). Special Issue on Tensor Models, Formalism and Applications, p. 073. DOI: [10.3842/SIGMA.2016.073](#). arXiv: [1603.03570](#) [[math-ph](#)].
- [Bou92] D. V. Boulatov. “A Model of three-dimensional lattice gravity”. *Mod. Phys. Lett. A* 7.18 (June 1992), pp. 1629–1646. arXiv: [hep-th/9202074](#).
- [Chm08] S. Chmutov. “Generalized duality for graphs on surfaces and the signed Bollobás-Riordan polynomial”. *J. Combinatorial Theory, Ser. B* 99.3 (Dec. 18, 2008), pp. 617–638. arXiv: [0711.3490](#) [[math.CO](#)].

- [COR14a] S. Carrozza, D. Oriti, and V. Rivasseau. “Renormalization of an $SU(2)$ tensorial group field theory in three dimensions”. *Commun. Math. Phys.* (2014). arXiv: 1303.6772 [hep-th].
- [COR14b] S. Carrozza, D. Oriti, and V. Rivasseau. “Renormalization of Tensorial Group Field Theories: Abelian $U(1)$ Models in Four Dimensions”. *Commun. Math. Phys.* 327.2 (Apr. 2014), pp. 603–641. arXiv: 1207.6734 [hep-th].
- [Dav85] F. David. “A model of random surfaces with non-trivial critical behaviour”. *Nucl. Phys. B* 257 (1985), pp. 543–576.
- [DFGZJ95] P. Di Francesco, P. Ginsparg, and J. Zinn-Justin. “2D gravity and random matrices”. *Phys. Rept.* 254 (1995), pp. 1–133.
- [DFR94] S. Doplicher, K. Fredenhagen, and J. E. Roberts. “Spacetime quantization induced by classical gravity”. *Phys. Lett. B* 331.1–2 (June 30, 1994), pp. 39–44.
- [DGR14] T. Delepouve, R. Gurau, and V. Rivasseau. “Universality and Borel Summability of Arbitrary Quartic Tensor Models”. *Ann. Inst. H. Poincaré Probab. Statist.* 52.2 (Oct. 8, 2014), pp. 821–848. DOI: 10.1214/14-AIHP655. arXiv: 1403.0170 [hep-th].
- [Dis+06] M. Disertori et al. “Vanishing of Beta Function of Non Commutative Φ_4^4 Theory to all orders”. *Phys. Lett. B* 649.1 (Dec. 2006), pp. 95–102. arXiv: hep-th/0612251.
- [DR06] M. Disertori and V. Rivasseau. “Two- and three-loop beta function of non-commutative Φ_4^4 theory”. *Eur. Phys. J. C* 50 (Oct. 2006), pp. 661–671. arXiv: hep-th/0610224.
- [DR16] T. Delepouve and V. Rivasseau. “Constructive Tensor Field Theory: The T_3^4 Model”. *Commun. Math. Phys.* 345.2 (Apr. 7, 2016), pp. 477–506. DOI: 10.1007/s00220-016-2680-1. arXiv: 1412.5091 [math-ph].
- [EMS74] J.-P. Eckmann, J. Magnen, and R. Sénéor. “Decay properties and borel summability for the Schwinger functions in $P(\phi)_2$ theories”. *Commun. Math. Phys.* 39.4 (July 22, 1974), pp. 251–271.
- [Fel+85] J. Feldman et al. “Bounds on completely convergent Euclidean Feynman graphs”. *Commun. Math. Phys.* 98 (1985), pp. 273–288.
- [Fel+86] J. Feldman et al. “A Renormalizable Field Theory: The Massive Gross-Neveu Model in two Dimensions”. *Commun. Math. Phys.* 103.1 (1986), pp. 67–103.
- [Fer17] F. Ferrari. “The Large D Limit of Planar Diagrams”. Jan. 4, 2017. arXiv: 1701.01171 [hep-th].
- [FO76] J. Feldman and K. Osterwalder. “The Wightman axioms and the mass gap for weakly coupled ϕ_3^4 quantum field theories”. *Ann. Phys.* 97 (1976), pp. 80–135.
- [Fre05] L. Freidel. “Group field theory: An overview”. *Int. J. Theor. Phys.* 44 (May 2005), p. 1769. arXiv: hep-th/0505016.
- [GJ73] J. Glimm and A. Jaffe. “Positivity of the ϕ_3^4 Hamiltonian”. *Fortschr. Phys.* 21 (1973), p. 327.
- [GJ87] J. Glimm and A. Jaffe. *Quantum physics. A functional integral point of view*. Second. New York: Springer-Verlag, 1987, pp. xxii+535. ISBN: 0-387-96476-2.
- [GJS73] J. Glimm, A. Jaffe, and T. Spencer. “The Wightman Axioms and Particle Structure in the $P(\phi)_2$ Quantum Field Model”. *Ann. of Math.* 100.3 (June 28, 1973), pp. 585–632. DOI: 10.2307/1970959.
- [GK15] R. Gurau and T. Krajewski. “Analyticity results for the cumulants in a random matrix model”. *Ann. Inst. Henri Poincaré Comb. Phys. Interact.* 2.2 (2015), pp. 169–228. DOI: 10.4171/AIHPD/17. arXiv: 1409.1705 [math-ph].
- [GK86] K. Gawedzki and A. Kupiainen. “Gross-Neveu model through convergent perturbation expansions”. *Commun. Math. Phys.* 102.1 (1986), pp. 1–30.
- [GR11a] R. Gurau and V. Rivasseau. “The $1/N$ expansion of colored tensor models in arbitrary dimension”. *Eur. Phys. Lett.* 95.5 (Jan. 2011), p. 50004. arXiv: 1101.4182 [gr-qc].
- [GR11b] R. Gurau and J. P. Ryan. “Colored Tensor Models - a Review”. *SIGMA* 8.020 (Sept. 2011), p. 78. arXiv: 1109.4812 [hep-th].
- [GR14] R. Gurau and V. Rivasseau. “The Multiscale Loop Vertex Expansion”. *Ann. H. Poincaré* 16.8 (July 2, 2014), pp. 1869–1897. DOI: 10.1007/s00023-014-0370-0. arXiv: 1312.7226 [math-ph].
- [Gro92] M. Gross. “Tensor models and simplicial quantum gravity in $> 2-D$ ”. *Nucl. Phys. B Proc. Suppl.* 25.1 (Jan. 1992), pp. 144–149.
- [Gur10a] R. Gurau. “Colored Group Field Theory”. *Commun. Math. Phys.* 304 (Dec. 15, 2010), pp. 69–93. DOI: 10.1007/s00220-011-1226-9. arXiv: 0907.2582 [hep-th].

- [Gur10b] R. Gurau. “The $1/N$ expansion of colored tensor models”. *Ann. H. Poincaré* 12.5 (Nov. 2010), pp. 829–847. arXiv: 1011.2726 [gr-qc].
- [Gur11] R. Gurau. “The Complete $1/N$ Expansion of Colored Tensor Models in Arbitrary Dimension”. *Ann. H. Poincaré* 13 (June 2, 2011), pp. 399–423. arXiv: 1102.5759 [gr-qc].
- [Gur13a] R. Gurau. “The $1/N$ Expansion of Tensor Models Beyond Perturbation Theory”. *Commun. Math. Phys.* 330.3 (Oct. 5, 2013), pp. 973–1019. arXiv: 1304.2666 [math-ph].
- [Gur13b] R. Gurau. “Universality for Random Tensors”. *Ann. Inst. H. Poincaré Probab. Statist.* 50.4 (May 5, 2013), pp. 1474–1525. DOI: 10.1214/13-AIHP567. arXiv: 1111.0519 [math.PR].
- [Gur17a] R. Gurau. “Quenched equals annealed at leading order in the colored SYK model”. Feb. 14, 2017. arXiv: 1702.04228 [hep-th].
- [Gur17b] R. Gurau. “The complete $1/N$ expansion of a SYK-like tensor model”. *Nucl. Phys. B* 916 (Jan. 17, 2017), pp. 386–401. DOI: 10.1016/j.nuclphysb.2017.01.015. arXiv: 1611.04032 [hep-th].
- [GW04a] H. Grosse and R. Wulkenhaar. “Renormalisation of ϕ^4 -theory on noncommutative \mathbb{R}^4 in the matrix base”. *Commun. Math. Phys.* 256.2 (Jan. 2004), pp. 305–374. arXiv: hep-th/0401128.
- [GW04b] H. Grosse and R. Wulkenhaar. “The beta-function in duality-covariant noncommutative ϕ^4 -theory”. *Eur. Phys. J. C* 35 (Feb. 2004), pp. 277–282. arXiv: hep-th/0402093.
- [GW09] H. Grosse and R. Wulkenhaar. “Progress in solving a noncommutative quantum field theory in four dimensions”. Sept. 2009. arXiv: 0909.1389 [hep-th].
- [GW13] H. Grosse and R. Wulkenhaar. “Self-dual noncommutative ϕ^4 -theory in four dimensions is a non-perturbatively solvable and non-trivial quantum field theory”. *Commun. Math. Phys.* 329.3 (May 10, 2013), pp. 1069–1130. DOI: 10.1007/s00220-014-1906-3. arXiv: 1205.0465 [math-ph].
- [GW14] H. Grosse and R. Wulkenhaar. “Solvable 4D noncommutative QFT: phase transitions and quest for reflection positivity”. June 30, 2014. arXiv: 1406.7755 [hep-th].
- [GW16] H. Grosse and R. Wulkenhaar. “On the Fixed Point Equation of a Solvable 4D QFT Model”. *Vietnam J. Math.* 44.1 (Mar. 2016), pp. 153–180. arXiv: 1505.05161 [math-ph].
- [Haa96] R. Haag. *Local quantum physics: Fields, Particles, Algebras*. Texts and monographs in physics. Springer, 1996.
- [Hoo74] G. ’t Hooft. “A planar diagram theory for strong interactions”. *Nucl. Phys. B* 72.3 (Apr. 1974), pp. 461–473.
- [Kaz85] V. A. Kazakov. “Bilocal regularization of models of random surfaces”. *Phys. Lett. B* 150.4 (Jan. 10, 1985), pp. 282–284.
- [Kra12] T. Krajewski. “Group field theories”. *PoS QGQGS 2011.005* (Oct. 23, 2012). arXiv: 1210.6257 [gr-qc].
- [KSBS17] C. Krishnan, S. Sanyal, and P. N. Bala Subramanian. “Quantum chaos and holographic tensor models”. *J. High Energ. Phys.* 56 (Mar. 1, 2017). arXiv: 1612.06330 [hep-th].
- [KT17] I. R. Klebanov and G. Tarnopolsky. “Uncolored random tensors, melon diagrams, and the Sachdev-Ye-Kitaev models”. *Phys. Rev. D* 95.046004 (Feb. 13, 2017). arXiv: 1611.08915 [hep-th].
- [Lah15a] V. Lahoche. “Constructive Tensorial Group Field Theory I: The $U(1)$ - T_3^4 Model”. Oct. 16, 2015. arXiv: 1510.05050 [hep-th].
- [Lah15b] V. Lahoche. “Constructive Tensorial Group Field Theory II: The $U(1)$ - T_4^4 Model”. Oct. 16, 2015. arXiv: 1510.05051 [hep-th].
- [LAJ05] R. Loll, J. Ambjorn, and J. Jurkiewicz. “The universe from Scratch”. *Contemp. Phys.* 47 (Sept. 2005), pp. 103–117. arXiv: hep-th/0509010.
- [Mag+09] J. Magnen et al. “Scaling behavior of three-dimensional group field theory”. *Class. Quant. Grav.* 26.18 (July 1, 2009). DOI: 10.1088/0264-9381/26/18/185012. arXiv: 0906.5477 [hep-th].
- [Mie14] G. Miermont. “Aspects of random maps”. Saint-Flour lecture notes. Oct. 9, 2014.
- [MR07] J. Magnen and V. Rivasseau. “Constructive ϕ^4 Field Theory without Tears”. *Ann. H. Poincaré* 9 (Dec. 6, 2007), pp. 403–424. arXiv: 0706.2457 [math-ph].
- [MS76] J. Magnen and R. Sénéor. “The infinite volume limit of the ϕ_3^4 model”. *Ann. Inst. Henri Poincaré* 24.2 (1976), pp. 95–159.
- [MS77] J. Magnen and R. Sénéor. “Phase space cell expansion and Borel summability for the Euclidean ϕ_3^4 theory”. *Commun. Math. Phys.* 56.3 (Oct. 1977), pp. 237–276. DOI: 10.1007/BF01614211.

- [Nel65] E. Nelson. “A quartic interaction in two dimensions”. In: *Mathematical Theory of Elementary Particles*. Ed. by R. Goodman and I. Segal. Endicott House, Dedham, Mass.: MIT press, 1965.
- [OS13] D. Ousmane Samary. “Beta functions of $U(1)^d$ gauge invariant just renormalizable tensor models”. *Phys. Rev. D* 88.105003 (Nov. 5, 2013). arXiv: 1303.7256 [hep-th].
- [OSVT13] D. Ousmane Samary and F. Vignes-Tourneret. “Just Renormalizable TGFT’s on $U(1)^d$ with Gauge Invariance”. *Commun. Math. Phys.* 329.2 (Oct. 26, 2013), pp. 545–578. arXiv: 1211.2618 [hep-th].
- [Reg61] T. Regge. “General relativity without coordinates”. *Il Nuovo Cimento* 19.3 (Feb. 1961), pp. 558–571.
- [Riv07] V. Rivasseau. “Constructive Matrix Theory”. *J. High Energ. Phys.* 09.008 (Aug. 16, 2007). arXiv: 0706.1224 [hep-th].
- [Riv13] V. Rivasseau. “The tensor track, III”. *Fortschr. Phys.* 62.2 (Nov. 13, 2013), pp. 81–107. DOI: 10.1002/prop.201300032. arXiv: 1311.1461 [hep-th].
- [Riv15] V. Rivasseau. “Why are tensor field theories asymptotically free?” *Eur. Phys. Lett.* 111.6 (Sept. 21, 2015). DOI: 10.1209/0295-5075/111/60011. arXiv: 1507.04190 [hep-th].
- [Riv16] V. Rivasseau. “Constructive Tensor Field Theory”. *SIGMA* 12 (Aug. 18, 2016). Special Issue on Tensor Models, Formalism and Applications, p. 085. DOI: 10.3842/SIGMA.2016.085. arXiv: 1603.07312 [math-ph].
- [Riv17] V. Rivasseau. “Loop Vertex Expansion for Higher Order Interactions”. *Lett. Math. Phys.* 108.5 (Dec. 4, 2017), pp. 1147–1162. DOI: 10.1007/s11005-017-1037-9. arXiv: 1702.07602 [math-ph].
- [Riv91] V. Rivasseau. *From Perturbative to Constructive Renormalization*. Princeton series in physics. 336 p. Princeton Univ. Pr., 1991.
- [RS80] M. Reed and B. Simon. *Methods of modern mathematical physics*. Revised and enlarged edition. Vol. 1: Functionnal Analysis. Academic Press, 1980.
- [RW13] V. Rivasseau and Z. Wang. “How to Resum Feynman Graphs”. *Ann. H. Poincaré* 15.11 (Oct. 11, 2013), pp. 2069–2083. arXiv: 1304.5913 [math-ph].
- [Sas91] N. Sasakura. “Tensor model for gravity and orientability of manifold”. *Mod. Phys. Lett.* A6.28 (Sept. 1991), p. 2613.
- [Sim74] B. Simon. *The $P(\phi)_2$ Euclidean (Quantum) Field Theory*. Princeton University Press, 1974.
- [Sok80] A. D. Sokal. “An improvement of Watson’s theorem on Borel summability”. *Journ. Math. Phys.* 21.2 (Feb. 1980), pp. 261–263.
- [SW64] R. F. Streater and A. S. Wightman. *PCT, Spin and Statistics, and all that*. W. A. Benjamin, 1964.
- [VW73] G. Velo and A. S. Wightman, eds. *Constructive Quantum Field Theory*. Vol. 25. Lecture notes in physics. The 1973 “Ettore Majorana” international school of mathematical physics. Springer, 1973.
- [Wit16] E. Witten. “An SYK-Like Model Without Disorder”. Oct. 31, 2016. arXiv: 1610.09758 [hep-th].