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# Generalization of the notion of seniority number

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## Abstract

The concept of seniority number is generalized, as well as that of seniority number operator. It affords to define new hierarchies of configuration interaction spaces. The usefulness of such a hierarchy is illustrated on the buckminsterfullerene system treated at the Hückel level of theory.

# 1 Introduction

The concept of seniority introduced in quantum chemistry by Bytautas et al.<sup>1</sup> has proved very fruitful and has inspired many recent works, see<sup>2-4</sup> to quote a few. It affords to partition the  $n$ -electron Hilbert space into subspaces spanned by sets of Slater determinants having a definite number of unpaired orbitals. For closed-shell systems, it has been observed that the Full Configuration Interaction (FCI) energy is dominated by the contribution of the seniority zero part of the wave function, and that, the higher the seniority number of the determinants, the less important their contribution on average<sup>1</sup>. However, even if one restricts a CI space to the seniority zero subspace, the size of the CI can be prohibitively large. Therefore, it is of interest to push a step further the seniority number partitioning strategy, that is to say, to define other such numbers to further break down the seniority zero subspace into a hierarchy of smaller subspaces. The purpose of this paper is to present a method to define generalized seniority numbers with an illustration to the Hückel model of the Buckminsterfullerene  $C_{60}$  molecule.

Our definition is based on the concepts put forward in Chapter 4 of the Ph. D thesis of M. Vivier entitled "*Sur quelques théorèmes d'algèbre extérieure*"<sup>5</sup> generalized to the case where the shells are not all of the same even dimension. As we shall see, the usual seniority numbers appear in the particular case of the primitive shells, hence the term "generalized seniority", we have coined for the general case.

The paper is organized as follows: In the next section the concept of Generalized Seniority Number (GSN) is defined and explained. Then, we present interesting mathematical results relevant to this concept. Finally, we illustrate its possible use on the  $C_{60}$  example and conclude.

## 2 Generalized seniority numbers

We consider a one-particle Hilbert space  $V$  which is the direct sum (not necessarily orthogonal) of  $n$  vector subspaces  $V_1, \dots, V_n$  of respective dimensions  $2d_1, \dots, 2d_n$ . Each of these subspaces will be called a “shell”, and the set  $\{V_1, \dots, V_n\}$  a “shell partition”. In quantum chemistry, the  $V_i$ ’s can be the vector spaces spanned by sets of spinorbital pairs corresponding to the same atomic orbital. In such a case, the shells will be termed tentatively “primitive shells”, as all the  $d_i$ ’s are equal to 1. Another natural shell partition with larger values for some  $d_i$ ’s occurs when the system has degenerate orbitals. If the sets of degenerate orbitals in increasing energy order are  $d_1$ -,  $\dots$ ,  $d_n$ -fold degenerate, then, the shells  $V_i$ ’s can be defined as the  $2d_i$ -dimensional vector spaces spanned by the associated degenerate pairs of spin-orbitals.

We denote by  $u_i$  the single determinantal function built from a set of  $2d_i$  normalized spin-orbitals,  $(\chi_{i,1}, \dots, \chi_{i,2d_i})_{i=1, \dots, n}$  spanning the shell  $V_i$ :

$$\forall i \in \{1, \dots, n\}, \quad u_i = \chi_{i,1} \wedge \dots \wedge \chi_{i,2d_i} \quad , \quad (1)$$

where  $\wedge$  is the Grassmann (or exterior) product (which is intrinsically antisymmetrical)<sup>6,7</sup>.

**Remark 1:** In quantum chemistry, as mentioned above, the even dimension  $2d_i$  of the  $V_i$ ’s would arise from the fact that there are as many basis spin-orbitals of spin  $+\frac{1}{2}$ , as there are of opposite spin. However, in the following, to alleviate notation, we will not distinguish the spin of the spin-orbitals. That is to say, the spin-orbitals will be labelled by indices running from 1 to  $2d_i$ , irrespective of their spin.

The symbol  $(\chi)_1$  will designate the concatenated bases of the  $n$  shells,

$$(\chi)_1 := (\chi_{1,1}, \dots, \chi_{1,2d_1}, \chi_{2,1}, \dots, \chi_{2,2d_2}, \dots, \chi_{n,1}, \dots, \chi_{n,2d_n}), \quad (2)$$

which is a basis of the one-particle Hilbert space  $V$ . We will further denote by  $(\chi)_N$  the  $N$ -particle basis set of Slater determinants induced by  $(\chi)_1$

$$(\chi)_N := (\chi_{i_1, j_1} \wedge \dots \wedge \chi_{i_N, j_N})_{(i_1, j_1) < \dots < (i_N, j_N)}, \quad (3)$$

where the order on the ordered pairs is the lexicographic order:  $(i, j) < (k, l)$  if  $i < k$  or if  $i = k$  and  $j < l$ . The union of all these basis sets,  $(\chi) := \bigcup_i (\chi)_i$ , including  $(\chi)_0 := (1)$ , is a basis of the first quantization equivalent of the Fock space.

In second quantization, the  $(\chi_{i,1}, \dots, \chi_{i,2d_i})$  are created by the operators  $a_{i,1}^\dagger, \dots, a_{i,2d_i}^\dagger$ , respectively, acting on the empty state  $|0\rangle$ :

$$a_{i,j}^\dagger |0\rangle = |\chi_{i,j}\rangle \quad , \quad (4)$$

so that,

$$a_{i,1}^\dagger \dots a_{i,2d_i}^\dagger |0\rangle = |\chi_{i,1} \wedge \dots \wedge \chi_{i,2d_i}\rangle = |u_i\rangle \quad . \quad (5)$$

Since the basis  $(\chi)$  is not orthogonal, the corresponding annihilation operators  $a_{i,j}$ 's, defined by conjugation from Eq (4):  $\langle 0|a_{i,j} = \langle \chi_{i,j}|$  are not very convenient, because  $\langle 0|a_{i,j} a_{k,l}^\dagger |0\rangle = \langle \chi_{i,j} | \chi_{k,l} \rangle \neq \delta_{(i,j), (k,l)}$ . So, we introduce the dual basis  $(\widetilde{\chi})$ , that is the unique basis verifying the following property:

$$\forall i, j, k, l, \quad \langle \widetilde{\chi}_{i,j} | \chi_{k,l} \rangle = \delta_{(i,j), (k,l)} \quad , \quad (6)$$

where  $\delta_{(i,j), (k,l)}$  is the Krönecker symbol for the ordered pair indices  $(i, j)$  and  $(k, l)$ . The

corresponding annihilation operators, denoted by a tilde, that is to say:  $\langle \widetilde{\chi}_{i,j} | = \langle 0 | \widetilde{a}_{i,j}$ , satisfy the desired relationship:

$$\langle 0 | \widetilde{a}_{i,j} a_{k,l}^\dagger | 0 \rangle = \delta_{(i,j),(k,l)} \quad . \quad (7)$$

It is also convenient to extend the notion of creation and annihilation operators to arbitrary quantum states. So, we denote the creation operator,  $a^\dagger(f)$ , of a general state,

$f = \sum_{(i_1,j_1),\dots,(i_k,j_k)} c_{(i_1,j_1),\dots,(i_k,j_k)} \chi_{i_1,j_1} \wedge \cdots \wedge \chi_{i_k,j_k}$ ,  $c_{(i_1,j_1),\dots,(i_k,j_k)} \in \mathbb{C}$ , as follows:

$$a^\dagger(f)|0\rangle = |f\rangle = \sum_{(i_1,j_1),\dots,(i_k,j_k)} c_{(i_1,j_1),\dots,(i_k,j_k)} a_{i_1,j_1}^\dagger \cdots a_{i_k,j_k}^\dagger |0\rangle \quad . \quad (8)$$

For example,  $a^\dagger(\chi_{i,j}) = a_{i,j}^\dagger$  and  $a^\dagger(u_i) = a_{i,1}^\dagger \cdots a_{i,2d_i}^\dagger$ .

We define the ‘‘dual’’ annihilation operator of a product state,  $\widetilde{a}(u_i)$ , as the product of the dual annihilation operators,  $\widetilde{a}(\chi_{i,j}) = \widetilde{a}_{i,j}$ , in reverse order:  $\widetilde{a}(u_i) = \widetilde{a}_{i,2d_i} \cdots \widetilde{a}_{i,1}$ , and more generally, by anti-linearity, the ‘‘dual’’ annihilation operator of  $a^\dagger(f)$  as

$$\widetilde{a}(f) = \sum_{(i_1,j_1),\dots,(i_k,j_k)} \bar{c}_{(i_1,j_1),\dots,(i_k,j_k)} \widetilde{a}_{i_k,j_k} \cdots \widetilde{a}_{i_1,j_1} \quad , \quad (9)$$

where the bar  $\bar{c}$  denotes complex conjugation.

**Definition 1:** We say that a  $(2d_i - k)$ -particle Slater determinant  $x$  **is included in**  $u_i$  if there exists a set  $\{h_1, \dots, h_k\}$  such that  $a^\dagger(u_i) = a_{i,h_1}^\dagger \cdots a_{i,h_k}^\dagger a^\dagger(x)$ .

So, for every Slater determinant  $m \in (\chi)_N$  of the  $N$ -particle induced basis set, we can write:

$$a^\dagger(m) = a^\dagger(u_{i_1}) \cdots a^\dagger(u_{i_q}) a^\dagger(x_{j_1}) \cdots a^\dagger(x_{j_r}) \quad , \quad (10)$$

where the  $x_{j_k}$ ’s are strictly included in some  $u_{j_k}$ ’s which are distinct from one another and from  $u_{i_1}, \dots, u_{i_q}$ .

**Definition 2:** We call  $q$  the **degree of  $m$  in the  $u_i$ 's**.

**Definition 3:** The integer  $r$  is called the **generalized seniority number** of  $m$  relative to the  $u_i$ 's. It represents the number of non-empty, non-fully occupied shells.

Note that, when the shells are chosen to be the primitive shells,  $r$  is nothing but the seniority number of the Slater determinant  $m$ .

**Remark 2:** The integer  $p = 2q + r$  is called the **reduced degree** of  $m$ . It coincides with the number of particles of the Slater determinant in the case of  $d_1 = \dots = d_n = 1$  (since, in this case, the  $u_i$ 's are 2-particle states and the  $x_j$ 's are necessarily 1-particle states).

The vector space spanned by all the Slater determinants,  $m$ , of the same GSN  $r$  is noted  $M(r)$ . It only depends upon the shell partition and not upon the choice of the shell basis sets. By extension, all wave functions in subspace  $M(r)$  will be said of GSN  $r$ . The subset of  $M(r)$  containing the wave functions spanned by Slater determinants  $m$  of the same degree  $q$  in the  $u_i$ 's is a subvector space of  $M(r)$ , noted  $M(q, r)$  with  $q \in \{0, \dots, n - r\}$ . For a given  $r$ ,  $M(r)$  is the direct sum of all the  $M(q, r)$ . The  $M(q, r)$  can be further decomposed into their projections onto the  $N$ -particle Hilbert spaces, noted  $M(N, q, r)$ . In the next section, we will introduce a GSN operator, which acts diagonally on the  $M(r)$ 's and whose expectation value on a normalized element of each  $M(r)$  is its GSN.

### 3 Generalized seniority number operator

**Definition 4:** For  $i \in \{1, \dots, n\}$  and any quantum state  $F$ , we consider the decomposition:

$$a^\dagger(F) = \hat{Q}_i(F) + \hat{R}_i(F) \quad , \quad (11)$$

where  $\hat{Q}_i(F)$ , represents the part of the  $a^\dagger(F)$ 's expansion in the  $(\chi)$ -basis containing at least one  $a_{i,j}^\dagger$ , and  $\hat{R}_i(F)$  the part of  $a^\dagger(F)$  which does not contain any creation operator

of a basis spinorbital appearing in  $u_i$ . We call it the **residue** or the **rest of  $F$  relatively to  $u_i$  in the basis**  $(\chi)$ .

$\hat{R}_i(F)$  can be expressed as

$$\hat{R}_i(F) = \tilde{a}(u_i)a^\dagger(u_i)a^\dagger(F) \quad , \quad (12)$$

and  $\hat{Q}_i(F)$  can be further decomposed as

$$\hat{Q}_i(F) = \mathring{Q}_i(F) + a^\dagger(u_i)\tilde{a}(u_i)a^\dagger(F) \quad , \quad (13)$$

where  $\mathring{Q}_i(F)$  represents the part of the  $a^\dagger(F)$ 's expansion containing at least one  $a_{i,j}^\dagger$  but not  $a^\dagger(u_i)$  entirely.

Combining Eqs. (11), (12) and (13), we obtain,

$$\mathring{Q}_i(F) = \left(1 - \tilde{a}(u_i)a^\dagger(u_i) - a^\dagger(u_i)\tilde{a}(u_i)\right)a^\dagger(F). \quad (14)$$

**Remark 3:** More generally, we can define  $\mathring{Q}_{i_1, i_2, \dots, i_k}(F) = \mathring{Q}_{i_1}\mathring{Q}_{i_2} \cdots \mathring{Q}_{i_k}(F)$ , (where the order of the  $i_j$ 's is indifferent since the  $\mathring{Q}_{i_j}$ 's commute), which extracts the part of the  $a^\dagger(F)$ 's expansion containing at least one  $a_{i_1, j_1}^\dagger$ , one  $a_{i_2, j_2}^\dagger$ , ... and one  $a_{i_k, j_k}^\dagger$ , without containing entirely  $a^\dagger(u_{i_1})$  nor  $a^\dagger(u_{i_2})$  nor ... nor  $a^\dagger(u_{i_k})$ .

**Definition 5:** The linear operator  $\hat{\Omega} : G \mapsto \hat{\Omega}(G) := \sum_{i=1}^n \mathring{Q}_i(G)$  is called the **generalized seniority number operator**. It acts diagonally on any element of  $M(r)$ :

$$\forall G \in M(r), \quad \hat{\Omega}(G) = r a^\dagger(G) \quad . \quad (15)$$

To prove the latter identity, let  $G \in M(r)$ . The creation operator  $a^\dagger(G)$  can be regarded as a linear combination of  $a^\dagger(m)$ 's, with  $m \in (\chi)$  the induced basis of Slater determi-



nants. For all  $m$ , we can write  $a^\dagger(m) = a^\dagger(u_{i_1}) \cdots a^\dagger(u_{i_q}) a^\dagger(x_{j_1}) \cdots a^\dagger(x_{j_r})$  (for some  $q$ -value) with  $a^\dagger(x_{j_1}) \cdots a^\dagger(x_{j_r})|0\rangle \in M(0, r)$ . Applying  $\hat{\Omega}$  to  $m$ , and using Eq. (14), the only non zero contributions come from  $\hat{Q}_{j_1}(m) = a^\dagger(m), \dots, \hat{Q}_{j_r}(m) = a^\dagger(m)$ , respectively. So, we find exactly  $r$  times  $a^\dagger(m)$  in  $\hat{\Omega}(m)$ . This being true for all the  $a^\dagger(m)$ 's appearing in the expression of  $a^\dagger(G)$ , by linearity of  $\hat{\Omega}$ , we obtain the identity, Eq. (15).

Similarly, a degree operator can be defined as follows:

**Definition 6:** *The linear operator  $\hat{\Xi} : G \mapsto \hat{\Xi}(G) = \sum_{i=1}^n a^\dagger(u_i) \tilde{a}(u_i) a^\dagger(G)$  is called the **degree for the shell partition  $\{V_1, \dots, V_n\}$  operator**. It does not depend upon a change of basis of  $V_i$ , for any  $i$ . It acts diagonally on any element of the  $M(q, r)$ 's:*

$$\forall G \in M(q, r), \quad \hat{\Xi}(G) = q a^\dagger(G) \quad . \quad (16)$$

To prove the latter identity, let  $G \in M(q, r)$ . Applying  $\hat{\Xi}$  to an  $m \in (\chi)$  in the expansion of  $G$ , whose creation operator is necessarily of the form given in Eq. (10), the only contributing terms are  $a^\dagger(u_{i_1}) \tilde{a}(u_{i_1}) a^\dagger(m) = a^\dagger(m), \dots, a^\dagger(u_{i_q}) \tilde{a}(u_{i_q}) a^\dagger(m) = a^\dagger(m)$ , as  $a^\dagger(u_i) \tilde{a}(u_i) a^\dagger(m) = 0$ , for  $i \notin \{i_1, \dots, i_q\}$ . So,  $a^\dagger(m)$  appears exactly  $q$  times in  $\hat{\Xi}(m)$ . This being true for all the  $a^\dagger(m)$ 's appearing in the expression of  $a^\dagger(G)$ , by linearity of  $\hat{\Xi}$ , we obtain the identity, Eq. (16).

**Remark 4:** A third identity follows from the previous two Eqs. (15) and (16),

$$\forall G \in M(q, r), \quad (n - q - r) a^\dagger(G) = \sum_{i=1}^n \hat{R}_i(G) \quad . \quad (17)$$

Indeed, let  $G \in M(q, r)$ . By using Eqs. (11) and (13), we can decompose  $a^\dagger(G)$  in  $n$  different manners as follows:

$$\forall i \in \{1, \dots, n\}, \quad a^\dagger(G) = \hat{Q}_i(G) + a^\dagger(u_i) \tilde{a}(u_i) a^\dagger(G) + \hat{R}_i(G) \quad . \quad (18)$$

Summing these  $n$  equalities and, using Eqs. (15) and (16), we obtain:

$$n a^\dagger(G) = r a^\dagger(G) + q a^\dagger(G) + \sum_{i=1}^n \hat{R}_i(G) \quad , \quad (19)$$

hence the result, Eq. (17).

The operator  $\hat{\Omega}$  can be used to decompose the creation operator of an arbitrary quantum state  $F$  onto the vector spaces of definite GSN,  $M(r)$ 's. This can be achieved by using Löwdin projectors<sup>8</sup>, for example. Let  $a^\dagger(F) = \sum_{r=0}^n a^\dagger(G_r)$  where  $G_r \in M(r)$ . From Eq. (15), we deduce,

$$\hat{\Omega}(F) = \sum_{r=0}^n r a^\dagger(G_r) \quad . \quad (20)$$

For all  $r \neq 0$ , we can extract the  $r a^\dagger(G_r)$  component of this decomposition by projection,

$$r a^\dagger(G_r) = \prod_{\substack{0 \leq j \leq n \\ j \neq r}} \frac{\hat{\Omega}(F) - j a^\dagger(F)}{r - j} \quad . \quad (21)$$

Then, the generalized seniority zero part of  $a^\dagger(F)$  can be obtained by difference,

$$a^\dagger(G_0) = a^\dagger(F) - \sum_{r \neq 0} a^\dagger(G_r) = a^\dagger(F) - \sum_{r \neq 0} \frac{1}{r} \prod_{\substack{0 \leq j \leq n \\ j \neq r}} \frac{\hat{\Omega}(F) - j a^\dagger(F)}{r - j} \quad . \quad (22)$$

## 4 Application to $C_{60}$ Hückel energy levels

The primitive shells partition used to define seniority numbers in quantum chemistry, stems from the fact that spinorbitals of the same spin are degenerate by spin symmetry for the spin-free Hamiltonian usually considered. It is therefore a natural idea to take also into account spatial symmetry, that is to say, to partition the one-particle Hilbert space into subspaces closed with respect to both spin and spatial symmetry operation. The highest known spatial symmetry in molecular system is the icosahedral symmetry. So, we

will illustrate our generalization of the seniority number concept on buckminsterfullerene. It is our take that generalized seniority numbers based on spatial symmetry, can be relevant parameters to limit CI expansions, as already observed for seniority number. This hypothesis relies on known phenomena, where a correlation has been established between the complete filling of a shell of a certain type and an unusual stability property. We have in mind the octet rule, the 18-electron rule or aromaticity, for example. So, we are going to investigate the convergence of CI calculations with respect to fine-tuned variational space hierarchies based on such GSNs.

Since our goal is just to establish a proof-of-principle, and not to aim at high accuracy prediction, we will perform our study at the Hückel level of theory. As shown in Fig. 1, the one-particle Hilbert space is spanned by 60 orbitals, so  $\dim V = 120$ . If it is partitioned into the corresponding 60 primitive shells, we obtain the usual seniority numbers. However, even if one limits the CI space to seniority 0 Slater determinants, the latter will be of dimension  $\binom{60}{30} \approx 1.18 \times 10^{17}$ , which is clearly untractable. So, to further decompose the seniority 0 space, we are going to use GSNs associated to the shells corresponding to the degenerate orbitals displayed in Fig. 1.

More precisely, the shell partition will consists of 15 shells,  $V_1, V_2, V_3, \dots, V_{14}, V_{15}$  of dimensions  $2, 6, 8, \dots, 8, 6$ , respectively. The largest shell  $V_6$  is of dimension 18. The dimension of the GSN  $r = 0$ -subspace of the seniority 0 space is 1464. For  $r \leq 1$  there are an additional 601594 Slater determinants to include, and for  $r \leq 2$ , another 53141130 Slater determinant set. All of these restricted CI subspaces are amenable to quantum chemistry computations in contrast with the full seniority 0 space. It remains to verify that the  $r$ -based hierarchy is physically meaningful.

## 5 conclusion

A generalized concept of GSN has been introduced with its associated GSN operator. It is based on a partition of the one-particle Hilbert space into shells. A new GSN-based hierarchy of CI-spaces has been proposed.

The usefulness of the hierarchy has been illustrated on a spatial plus spin degeneracy-based partition of the spinorbital basis set. The associated GSN affords to split the seniority 0 space of  $C_{60}$  for a basis set of 60 Hückel molecular orbitals, into CI-subspaces of reduced dimensions, lending themselves to numerical computations for low values of the GSN.

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# FIGURES

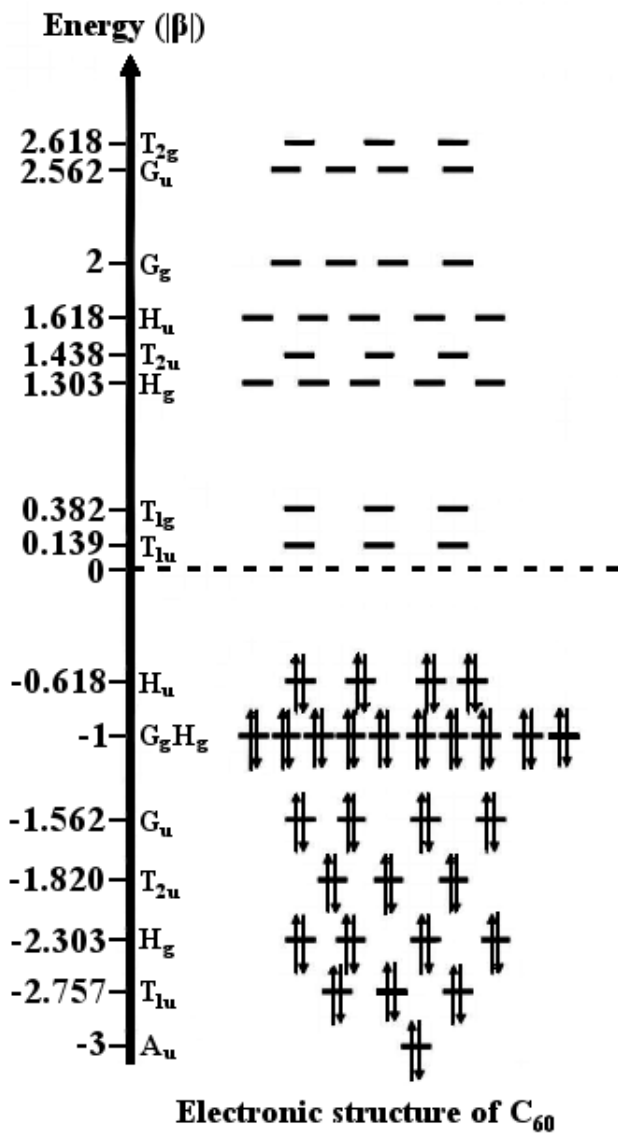


Figure 1: Energy diagram of C<sub>60</sub> Hückel molecular orbitals with electron occupation in the ground state reference configuration.

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