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Pedro Montero

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NEWTON-OKOUNKOV BODIES ON PROJECTIVE BUNDLES OVER CURVES

Pedro MONTERO

Abstract. — In this article, we study Newton-Okounkov bodies on projective vector bundles over curves. Inspired by Wolfe’s estimates used to compute the volume function on these varieties, we compute all Newton-Okounkov bodies with respect to linear flags which are compatible with the Harder-Narasimhan filtration of the bundle. Moreover, we characterize semi-stable vector bundles over curves via Newton-Okounkov bodies.

Contents

1. Introduction.....	1
2. Newton-Okounkov bodies and Semi-stability.....	4
2.1. Newton-Okounkov bodies.....	5
2.2. Semi-stability and Harder-Narasimhan filtrations.....	9
3. Divisors on projective bundles over curves.....	11
4. Newton-Okounkov bodies on projective bundles over curves	15
References.....	26

1. Introduction

Let C be a smooth projective curve over an uncountable algebraically closed field of characteristic zero and let E be vector bundle over C of rank $r \geq 2$. It is well-known since Hartshorne’s work [Har71] that numerical information coming from semi-stability properties of E can be translated into positivity conditions. Namely, Hartshorne proved that E is an ample vector bundle if and only if $\mu_{\min}(E)$, the minimal slope of E , is strictly positive. The shape

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of the nef cone of the projective vector bundle $\mathbb{P}(E)$ can be deduced from this result (see [Laz04, §6.4.B], for instance).

Similarly, Miyaoka proved in [Miy87, Theo. 3.1] the following result, which was generalized by Fulger in [Ful11].

Theorem 1.1 (Miyaoka). — *Let C and E be as above. The following conditions are equivalent:*

- (1) E is semi-stable.
- (2) $\text{Nef}(\mathbb{P}(E)) = \text{Psef}(\mathbb{P}(E))$.

In general, the shape of the pseudoeffective cone of $\mathbb{P}(E)$ was determined by Nakayama in [Nak04, Chapter IV] in the course of the proof of the fact that exists a weak Zariski decomposition of pseudoeffective divisors on $\mathbb{P}(E)$ (see also [MDS15]). This cone is spanned by the numerical classes f and $\xi - \mu_{\max}(E)f$, where ξ is the class of the tautological line bundle $\mathcal{O}_{\mathbb{P}(E)}(1)$, f is the class of a fiber of $\pi : \mathbb{P}(E) \rightarrow C$ and $\mu_{\max}(E)$ is the maximal slope of E .

Indirectly, the pseudoeffective cone can be also deduced from the work of Wolfe [Wol05] and Chen [Che08], who explicitly computed the volume function on $\mathbb{P}(E)$. In fact, they showed that for every $t \in \mathbb{R}$, the volume of the numerical class $\xi - tf$ on $\mathbb{P}(E)$ can be expressed in terms of numerical information coming from the Harder-Narasimhan filtration of E . More precisely, their results can be summarized as follows.

Theorem 1.2. — *Let C and E as above. Let*

$$\text{HN}_{\bullet}(E) : 0 = E_{\ell} \subseteq E_{\ell-1} \subseteq \cdots \subseteq E_1 \subseteq E_0 = E$$

be the Harder-Narasimhan filtration of E with successive semi-stable quotients $Q_i = E_{i-1}/E_i$ of rank r_i and slope μ_i . Then,

$$\text{vol}_{\mathbb{P}(E)}(\xi - tf) = r! \cdot \int_{\widehat{\Delta}_{r-1}} \max \left\{ \sum_{j=1}^r s_j \lambda_j - t, 0 \right\} d\lambda,$$

where $\widehat{\Delta}_{r-1} \subseteq \mathbb{R}^r$ is the standard $(r-1)$ -simplex with coordinates $\lambda_1, \dots, \lambda_r$, $d\lambda$ is the standard induced Lebesgue measure for which $\widehat{\Delta}_{r-1}$ has volume $\frac{1}{(r-1)!}$, and $\mathbf{s} = (s_1, \dots, s_r)$ is a vector in \mathbb{R}^r such that the value μ_i appears exactly r_i times in the coordinates of \mathbf{s} , and their appear in increasing order.

Following the idea that numerical information encoded by the Harder-Narasimhan filtration of E should be related to asymptotic numerical invariants of $\mathbb{P}(E)$, we study the geometry of Newton-Okounkov bodies on $\mathbb{P}(E)$. These compact convex bodies were introduced by Okounkov in his original article [Oko96] and they were studied later on by Kaveh and Khovanskii [KK12] and Lazarsfeld and Mustața [LM09], who associated to any big divisor D on

a normal projective variety X of dimension r , and any complete flag of subvarieties Y_\bullet on X satisfying suitable conditions, a convex body $\Delta_{Y_\bullet}(D) \subseteq \mathbb{R}^r$ depending only on the numerical equivalence class of D . Moreover, there exists a *global Newton-Okounkov body* $\Delta_{Y_\bullet}(X) \subseteq \mathbb{R}^r \times \mathbb{N}^1(X)_{\mathbb{R}}$ such that the slice of $\Delta_{Y_\bullet}(X)$ over any big rational class $\eta \in \mathbb{N}^1(X)_{\mathbb{Q}}$ is given by $\Delta_{Y_\bullet}(\eta) \subseteq \mathbb{R}^r \times \{\eta\}$.

Newton-Okounkov bodies of big divisors on geometrically ruled surfaces with respect to *linear flags* (see Definition 4.1) can be computed via Zariski decomposition (see Example 4.2). In higher dimension, we will use methods similar to those used by Wolfe to compute the volume function in [Wol05]. More precisely, we will need to establish bounds on the dimension of global section spaces and understand the Harder-Narasimhan filtration of the symmetric products $S^m E$ for $m \geq 1$. We will restrict ourselves to *linear flags which are compatible with the Harder-Narasimhan filtration of E* (see Definition 4.5).

With the notation of Theorem 1.2 above, let us consider

$$\sigma = \underbrace{(\mu_\ell, \dots, \mu_\ell)}_{r_\ell \text{ times}}, \underbrace{(\mu_{\ell-1}, \dots, \mu_{\ell-1})}_{r_{\ell-1} \text{ times}}, \dots, \underbrace{(\mu_1, \dots, \mu_1)}_{r_1 \text{ times}} \in \mathbb{Q}^r$$

and define for each $t \in \mathbb{R}$ the following polytope inside the full dimensional standard simplex Δ_{r-1} in \mathbb{R}^{r-1}

$$\square_t^{\text{HN}} = \left\{ (\nu_2, \dots, \nu_r) \in \Delta_{r-1} \mid \sum_{i=2}^r \sigma_{i-1} \nu_i + \sigma_r \left(1 - \sum_{i=2}^r \nu_i \right) \geq t \right\}.$$

Then, we prove the following result.

Theorem A. — *Let C be a smooth projective curve and let E be a vector bundle over C of rank $r \geq 2$. Then, for every linear flag Y_\bullet on $\mathbb{P}(E)$ which is compatible with the Harder-Narasimhan filtration and every big rational class $\eta = a(\xi - \mu_\ell f) + bf$ we have that*

$$\Delta_{Y_\bullet}(\eta) = \{(\nu_1, \dots, \nu_r) \in \mathbb{R}_{\geq 0}^r \mid 0 \leq \nu_1 \leq b, (\nu_2, \dots, \nu_r) \in a \square_{\mu_1 + \nu_1 + a(\mu_\ell - \mu_1) - b}^{\text{HN}}\},$$

and hence the global Newton-Okounkov body of $\mathbb{P}(E)$ with respect to Y_\bullet is given by

$$\Delta_{Y_\bullet}(\mathbb{P}(E)) = \left\{ ((a(\xi - \mu_\ell f) + bf), (\nu_1, \dots, \nu_r)) \in \mathbb{N}^1(\mathbb{P}(E))_{\mathbb{R}} \times \mathbb{R}^r \text{ such that } 0 \leq \nu_1 \leq b \text{ and } (\nu_2, \dots, \nu_r) \in a \square_{\mu_1 + \nu_1 + a(\mu_\ell - \mu_1) - b}^{\text{HN}} \right\}.$$

In particular, the global Newton-Okounkov body $\Delta_{Y_\bullet}(\mathbb{P}(E))$ is a rational polyhedral cone and it depends only on $\text{gr}(\text{HN}_\bullet(E))$, the graded vector bundle associated to the Harder-Narasimhan filtration of E .

Moreover, we obtain the following characterization of semi-stability in terms of Newton-Okounkov bodies.

Proposition B. — *Let C be a smooth projective curve and let E be vector bundle over C of rank $r \geq 2$. The following conditions are then equivalent:*

- (1) E is semi-stable.
- (2) For every big rational class $\eta = a(\xi - \mu_\ell f) + bf$ on $\mathbb{P}(E)$ and every linear flag Y_\bullet on $\mathbb{P}(E)$ we have that $\Delta_{Y_\bullet}(\eta) = [0, b] \times a\Delta_{r-1} \subseteq \mathbb{R}^r$.
- (3) There exists a big rational class $\eta = a(\xi - \mu_\ell f) + bf$ on $\mathbb{P}(E)$ and there exists a linear flag Y_\bullet on $\mathbb{P}(E)$ such that $\Delta_{Y_\bullet}(\eta) = [0, b] \times a\Delta_{r-1} \subseteq \mathbb{R}^r$.

Here, $a\Delta_{r-1} = \{(\nu_2, \dots, \nu_r) \in \mathbb{R}_{\geq 0}^{r-1} \mid \sum_{i=2}^r \nu_i \leq a\}$ is the full dimensional standard $(r-1)$ -simplex with side length a .

Outline of the article. — First of all, we recall in §2 definitions and well-known results about Newton-Okounkov bodies and semi-stability of vector bundles over curves. Secondly, §3 is devoted to the different cones of divisors on $\mathbb{P}(E)$ as well as results concerning their volume and restricted volume. Finally, we prove both Theorem A and Proposition B in §4.

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2. Newton-Okounkov bodies and Semi-stability

Throughout this article all varieties will be assumed to be reduced and irreducible schemes of finite type over a fixed uncountable algebraically closed field of characteristic zero k .

In this section, we review the construction of Newton-Okounkov bodies and semi-stability of vector bundles over smooth projective curves. We denote by $N^1(X)$ the group of numerical equivalence classes of Cartier divisors on X , and we define $N^1(X)_\mathbb{k} = N^1(X) \otimes_{\mathbb{Z}} \mathbb{k}$ for $\mathbb{k} = \mathbb{Q}$ or \mathbb{R} . Dually, we denote by $N_1(X)$ the group of numerical equivalence classes of 1-cycles on X . Inside $N_1(X)_\mathbb{R} = N_1(X) \otimes_{\mathbb{Z}} \mathbb{R}$ we distinguish the *Mori cone* $\overline{NE}(X) \subseteq N_1(X)_\mathbb{R}$, which is the closed and convex cone generated by numerical classes of 1-cycles with non-negative real coefficients.

Let E be a locally free sheaf on a variety X . We follow Grothendieck's convention and we define the *projectivization* $\mathbb{P}_X(E) = \mathbb{P}(E)$ of E to be $\mathbf{Proj}_{\mathcal{O}_X} \oplus_{m \geq 0} S^m E$, where $S^m E$ denotes the m th symmetric power of E . This variety is endowed with a natural projection $\pi : \mathbb{P}(E) \rightarrow X$ and a tautological line bundle $\mathcal{O}_{\mathbb{P}(E)}(1)$.

Let X be a normal projective variety. Following [Laz04], we say that a numerical class $\eta \in N^1(X)_\mathbb{R}$ is *big* if there exists an effective \mathbb{R} -divisor E such that $\eta - E$ is ample. We denote by $\text{Big}(X) \subseteq N^1(X)_\mathbb{R}$ the open convex cone

of big numerical classes. A numerical class $\eta \in N^1(X)_{\mathbb{R}}$ is called *pseudo-effective* if it can be written as the limit of classes of effective \mathbb{R} -divisors. The pseudoeffective cone is the closure of the big cone: $\overline{\mathbf{Big}}(X) = \text{Psef}(X)$ (see [Laz04, Theo. 2.2.26], for instance). Moreover, a numerical class $\eta \in N^1(X)_{\mathbb{R}}$ is *nef* if $\eta \cdot [C] \geq 0$ for every $[C] \in \overline{\mathbf{NE}}(X)$, and is *ample* if it is the numerical class of an \mathbb{R} -divisor that can be written as a finite sum of ample Cartier divisors with positive real coefficients. The cone $\text{Nef}(X) \subseteq N^1(X)_{\mathbb{R}}$ of nef classes is closed convex, and its interior $\text{Amp}(X)$ is the cone of ample classes, by Kleiman's ampleness criterion. A line bundle L is big (resp. pseudo-effective, ample, nef) if and only if its numerical class $c_1(L) \in N^1(X)_{\mathbb{R}}$ is big (resp. pseudo-effective, ample, nef).

Let us recall that the *stable base locus* of a \mathbb{Q} -divisor D on X is the closed set

$$\mathbf{B}(D) = \bigcap_{m>0} \text{Bs}(mD)$$

where $\text{Bs}(mD)$ is the base locus of the complete linear system $|mD|$. Following [ELM⁺06], we define the *augmented base locus* of D to be the closed set

$$\mathbf{B}_+(D) = \bigcap_A \mathbf{B}(D - A),$$

where the intersection runs over all ample \mathbb{Q} -divisors A . Similarly, the *restricted base locus* of D is defined by

$$\mathbf{B}_-(D) = \bigcup_A \mathbf{B}(D + A),$$

where the union runs over all ample \mathbb{Q} -divisors A . By [ELM⁺06, Prop. 1.4, Exam. 1.8, Prop. 1.15, Exam. 1.16], both $\mathbf{B}_-(D)$ and $\mathbf{B}_+(D)$ depend only on the numerical class of D , there is an inclusion $\mathbf{B}_-(D) \subseteq \mathbf{B}_+(D)$, and for any rational number $c > 0$ we have $\mathbf{B}_-(cD) = \mathbf{B}_-(D)$ and $\mathbf{B}_+(cD) = \mathbf{B}_+(D)$. Moreover by [ELM⁺06, Exam. 1.7, Exam. 1.18] we have that $\mathbf{B}_+(D) = \emptyset$ if and only if D is ample, and that $\mathbf{B}_-(D) = \emptyset$ if and only if D is nef.

2.1. Newton-Okounkov bodies. — Let X be a smooth projective variety of dimension n and let L be a big line bundle on X . A full flag of closed subvarieties of X centered at the point $p \in X$

$$Y_{\bullet} : X = Y_0 \supseteq Y_1 \supseteq Y_2 \supseteq \cdots \supseteq Y_{n-1} \supseteq Y_n = \{p\}$$

is an *admissible flag* if $\text{codim}_X(Y_i) = i$, and each Y_i is smooth at the point p . In particular, Y_{i+1} defines a Cartier divisor on Y_i in a neighbourhood of the point p . Following the work of Okounkov [Oko96, Oko03], Kaveh and Khovanskii [KK12] and Lazarsfeld and Mustața [LM09] independently associated to L and Y_{\bullet} a convex body $\Delta_{Y_{\bullet}}(X, L) \subseteq \mathbb{R}^n$ encoding the asymptotic properties of

the complete linear series $|L^{\otimes m}|$. We will follow the presentation of [LM09] and we refer the interested reader to the survey [Bou12] for a comparison of both points of view.

Let D be any divisor on X and let $s = s_1 \in H^0(X, \mathcal{O}_X(D))$ be a non-zero section. We shall compute successive vanishing orders of global sections on a neighbourhood of the point p where all the subvarieties Y_i are smooth. We set $\nu_1(s) = \text{ord}_{Y_1}(s_1)$ the vanishing order of s_1 along Y_1 . Let $y_1 \in \mathcal{O}_{X,p}$ be a local equation for Y_1 on X at the point p , and let $0 \neq s_2 \in H^0(Y_1, \mathcal{O}_{Y_1}(D - \nu_1(s)Y_1))$ be the restriction of $y_1^{-\nu_1(s)} s_1$ to Y_1 . We now set $\nu_2(s) = \text{ord}_{Y_2}(s_2)$, and we proceed inductively in order to get $\nu_{Y_\bullet}(s) = (\nu_1(s), \dots, \nu_n(s)) \in \mathbb{N}^d$, which is independent of the choice of the local equations. This construction leads to a valuation-like function

$$\nu_{Y_\bullet} : H^0(X, \mathcal{O}_X(D)) \setminus \{0\} \rightarrow \mathbb{Z}^n, s \mapsto (\nu_1(s), \dots, \nu_n(s)).$$

We then define the *graded semigroup* of D to be the sub-semigroup of $\mathbb{N}^n \times \mathbb{N}$ defined by

$$\Gamma_{Y_\bullet}(D) = \{(\nu_{Y_\bullet}(s), m) \in \mathbb{N}^n \times \mathbb{N} \mid 0 \neq s \in H^0(X, \mathcal{O}_X(mD))\}.$$

Finally, we define the *Newton-Okounkov body of D with respect to Y_\bullet* to be

$$\Delta_{Y_\bullet}(D) = \text{cone}(\Gamma_{Y_\bullet}(D)) \cap (\mathbb{R}^n \times \{1\}),$$

where $\text{cone}(\Gamma_{Y_\bullet}(D))$ denotes the closed convex cone in $\mathbb{R}^n \times \mathbb{R}$ spanned by $\Gamma_{Y_\bullet}(D)$.

These convex sets $\Delta_{Y_\bullet}(D)$ are compact and they have non-empty interior whenever D is big. Moreover, by [LM09, Theo. A], we have the following identity

$$\text{vol}_{\mathbb{R}^n}(\Delta_{Y_\bullet}(D)) = \frac{1}{n!} \cdot \text{vol}_X(D),$$

where $\text{vol}_X(D) = \lim_{m \rightarrow \infty} \frac{h^0(X, \mathcal{O}_X(mD))}{m^n/n!}$. In particular, if D is big and nef, then $\text{vol}_{\mathbb{R}^n}(\Delta_{Y_\bullet}(D)) = \frac{1}{n!} D^n$, by the Asymptotic Riemann-Roch theorem.

The Newton-Okounkov bodies depend only on numerical classes: if $D \equiv_{\text{num}} D'$ are big divisors then $\Delta_{Y_\bullet}(D) = \Delta_{Y_\bullet}(D')$ for every admissible flag Y_\bullet on X , by [LM09, Prop. 4.1] (see [Jow10, Theo. A] for the converse). This fact, along with the identity $\Delta_{Y_\bullet}(pD) = p \cdot \Delta_{Y_\bullet}(D)$ for every positive integer p , enables us to define an Newton-Okounkov body $\Delta_{Y_\bullet}(\eta) \subseteq \mathbb{R}^n$ for every big rational class $\eta \in \text{Big}(X) \cap \mathbb{N}^1(X)_{\mathbb{Q}}$. Moreover, by [LM09, Theo. B], there exists a *global Newton-Okounkov body*: a closed convex cone

$$\Delta_{Y_\bullet}(X) \subseteq \mathbb{R}^n \times \mathbb{N}^1(X)_{\mathbb{R}}$$

such that for each big rational class $\eta \in \text{Big}(X)_{\mathbb{Q}} = \text{Big}(X) \cap \mathbb{N}^1(X)_{\mathbb{Q}}$ the fiber of the second projection over η is $\Delta_{Y_\bullet}(\eta)$. This enables us to define Newton-Okounkov bodies for big real classes by continuity.

The above construction works for *graded linear series* \mathbf{a}_\bullet associated to a big divisor D on X . A graded linear series is a collection of subspaces $\mathbf{a}_m \subseteq H^0(X, \mathcal{O}_X(mD))$ such that $R(\mathbf{a}_\bullet) = \bigoplus_{m \geq 0} \mathbf{a}_m$ is a graded subalgebra of the section ring $R(D) = \bigoplus_{m \geq 0} H^0(X, \mathcal{O}_X(mD))$. The construction enables us to attach to any graded linear series \mathbf{a}_\bullet a closed and convex set $\Delta_{Y_\bullet}(\mathbf{a}_\bullet) \subseteq \mathbb{R}^n$. This set $\Delta_{Y_\bullet}(\mathbf{a}_\bullet)$ will be compact and will compute the volume of the linear series under some mild conditions listed in [LM09, § 2.3]. We will be specially interested on *restricted complete linear series* of a big divisor D , namely graded linear series of the form

$$\mathbf{a}_m = H^0(X|E, \mathcal{O}_X(mD)) = \text{Im} \left(H^0(X, \mathcal{O}_X(mD)) \xrightarrow{\text{rest}} H^0(E, \mathcal{O}_E(mD)) \right)$$

where $E \subseteq X$ is an irreducible subvariety of dimension $d \geq 1$. Under the hypothesis that $E \not\subseteq \mathbf{B}_+(D)$, the conditions listed in [LM09, § 2.3] are satisfied by [LM09, Lemm. 2.16]. Therefore, the Newton-Okounkov body associated to \mathbf{a}_\bullet above, the *restricted Newton-Okounkov body* (with respect to a fixed admissible flag)

$$\Delta_{X|E}(D) \subseteq \mathbb{R}^d,$$

is compact and

$$\text{vol}_{\mathbb{R}^d}(\Delta_{X|E}(D)) = \frac{1}{d!} \text{vol}_{X|E}(D),$$

where

$$\text{vol}_{X|E}(D) = \lim_{m \rightarrow \infty} \frac{\dim_k \mathbf{a}_m}{m^d/d!}$$

is the *restricted volume* on E of the divisor D . In particular, if D is big and nef, then $\text{vol}_{X|E}(D) = (D^d \cdot E)$, by [ELM⁺09, Cor. 2.17]. Restricted Newton-Okounkov bodies depend only on numerical classes (see [LM09, Rema. 4.25]), so it is meaningful to consider $\Delta_{X|E}(\eta)$ for every big rational class η such that $E \not\subseteq \mathbf{B}_+(\eta)$.

As before, there exists a global Newton-Okounkov body $\Delta_{Y_\bullet}(X|E)$ that enables us to define, by continuity, $\Delta_{X|E}(\eta)$ for any big real numerical class η such that $E \not\subseteq \mathbf{B}_+(\eta)$. See [LM09, Exam. 4.24] for details.

Restricted Newton-Okounkov bodies can be used to describe slices of Newton-Okounkov bodies.

Theorem 2.1 ([LM09, Theo. 4.26, Cor. 4.27]). — *Let X be a normal projective variety of dimension n , and let $E \subseteq X$ be an irreducible and reduced Cartier divisor on X . Fix an admissible flag*

$$Y_\bullet : X = Y_0 \supseteq E \supseteq Y_2 \supseteq \cdots \supseteq Y_{n-1} \supseteq Y_n = \{p\}$$

with divisorial component $Y_1 = E$. Let $\eta \in \text{Big}(X)_{\mathbb{Q}}$ be a rational big class, and consider the Newton-Okounkov body $\Delta_{Y_\bullet}(\eta) \subseteq \mathbb{R}^n$. Write $\text{pr}_1 : \Delta_{Y_\bullet}(\eta) \rightarrow \mathbb{R}$

for the projection onto the first coordinate, and set

$$\begin{aligned}\Delta_{Y_\bullet}(\eta)_{\nu_1=t} &= \text{pr}_1^{-1}(t) \subseteq \{t\} \times \mathbb{R}^{n-1} \\ \Delta_{Y_\bullet}(\eta)_{\nu_1 \geq t} &= \text{pr}_1^{-1}([t, +\infty)) \subseteq \mathbb{R}^n\end{aligned}$$

Assume that $E \not\subseteq \mathbf{B}_+(\eta)$ and let

$$\tau_E(\eta) = \sup\{s > 0 \mid \eta - s \cdot e \in \text{Big}(X)\},$$

where $e \in N^1(X)$ is the numerical class of E . Then, for any $t \in \mathbb{R}$ with $0 \leq t < \tau_E(\eta)$ we have

1. $\Delta_{Y_\bullet}(\eta)_{\nu_1 \geq t} = \Delta_{Y_\bullet}(\eta - te) + t \cdot \vec{e}_1$, where $\vec{e}_1 = (1, 0, \dots, 0) \in \mathbb{N}^n$ is the first standard unit vector.⁽¹⁾
2. $\Delta_{Y_\bullet}(\eta)_{\nu_1=t} = \Delta_{X|E}(\eta - te)$.
3. The function $t \mapsto \text{vol}_X(\eta + te)$ is differentiable at $t = 0$, and

$$\frac{d}{dt} (\text{vol}_X(\eta + te))|_{t=0} = n \cdot \text{vol}_{X|E}(\eta).$$

Let us finish this section with the case of Newton-Okounkov bodies on surfaces (see [LM09, §6.2] for details). We will use this description in Example 4.2 in order to illustrate the shape of Newton-Okounkov bodies on ruled surfaces.

Example 2.2 (Surfaces). — Let S be a smooth projective surface together with a flag $Y_\bullet : Y_0 = S \supseteq Y_1 = C \supseteq Y_2 = \{p\}$, where $C \subseteq S$ is a smooth curve and $p \in C$.

Let D be a big \mathbb{Q} -divisor on S . Any such divisor admits a *Zariski decomposition*, that is we can uniquely write D as a sum

$$D = P(D) + N(D)$$

of \mathbb{Q} -divisors, with $P(D)$ nef and $N(D)$ either zero or effective with negative definite intersection matrix. Moreover, $P(D) \cdot \Gamma = 0$ for every irreducible component Γ of $N(D)$ and for all $m \geq 0$ there is an isomorphism $H^0(S, \mathcal{O}_S([mP(D)])) \cong H^0(S, \mathcal{O}_S([mD]))$. In this decomposition $P(D)$ is called the *positive part* and $N(D)$ the *negative part*. See [Laz04, §2.3.E] and references therein for proofs and applications.

With the above notation, we have that

$$\Delta_{Y_\bullet}(D) = \{(t, y) \in \mathbb{R}^2 \mid \nu \leq t \leq \tau_C(D), \alpha(t) \leq t \leq \beta(t)\}$$

where

1. $\nu \in \mathbb{Q}$ the coefficient of C in $N(D)$,
2. $\tau_C(D) = \sup\{t > 0 \mid D - tC \text{ is big}\}$,
3. $\alpha(t) = \text{ord}_p(N_t \cdot C)$,

1. In fact, this statement remains true even if we do not assume that $E \not\subseteq \mathbf{B}_+(\eta)$. See [KL15, Prop. 1.6].

$$4. \beta(t) = \text{ord}_p(N_t \cdot C) + P_t \cdot C,$$

where $D - tC = P_t + N_t$ is a Zariski decomposition, P_t being the positive and N_t the negative part. Moreover, these bodies are finite polygons, by [KLM12, Theo. B].

2.2. Semi-stability and Harder-Narasimhan filtrations. — Throughout this section, C is a smooth projective curve and E is a locally free sheaf on C of rank $r > 0$ and degree $d = \deg(E) = \deg(c_1(E))$. Given such a bundle we call the rational number

$$\mu(E) = \frac{d}{r}$$

the *slope* of E .

Definition 2.3 (Semi-stability). — Let F be a vector bundle on C of slope μ . We say that F is *semi-stable* if for every non-zero sub-bundle $G \subseteq F$, we have $\mu(G) \leq \mu$. Equivalently, F is semi-stable if for every locally-free quotient $F \twoheadrightarrow Q$ of non-zero rank, we have $\mu \leq \mu(Q)$.

Following [LP97, Prop. 5.4.2], there is a canonical filtration of E with semi-stable quotients.

Proposition 2.4. — *Let E be a vector bundle on C . Then E has an increasing filtration by sub-bundles*

$$\text{HN}_\bullet(E) : 0 = E_\ell \subseteq E_{\ell-1} \subseteq \cdots \subseteq E_1 \subseteq E_0 = E$$

where the quotients E_{i-1}/E_i satisfies the following conditions:

1. Each quotient E_{i-1}/E_i is a semi-stable vector bundle;
2. $\mu(E_{i-1}/E_i) < \mu(E_i/E_{i+1})$ for $i = 1, \dots, \ell - 1$.

This filtration is unique.

The above filtration is called the *Harder-Narasimhan filtration* of E .

Notation 2.5. — Let E be a vector bundle on a smooth projective curve C . We will denote by $Q_i = E_{i-1}/E_i$ the semi-stable quotients of the Harder-Narasimhan filtration of E , each one of rank $r_i = \text{rank}(Q_i)$, degree $d_i = \deg(c_1(Q_i))$ and slope $\mu_i = \mu(Q_i) = d_i/r_i$. With this notation, μ_1 and μ_ℓ correspond to the minimal and maximal slopes, $\mu_{\min}(E)$ and $\mu_{\max}(E)$, respectively.

From a cohomological point of view, semi-stable vector bundles can be seen as the good higher-rank analogue of line bundles. For instance, we have the following classical properties (see [RR84] or [But94, Lemm. 1.12, Lemm. 2.5]).

Lemma 2.6. — *Let E and F be vector bundles on C and $m \in \mathbb{N}$. Then,*

1. $\mu_{\max}(E \otimes F) = \mu_{\max}(E) + \mu_{\max}(F)$.

2. $\mu_{\min}(E \otimes F) = \mu_{\min}(E) + \mu_{\min}(F)$.
3. $\mu_{\max}(S^m E) = m\mu_{\max}(E)$.
4. $\mu_{\min}(S^m E) = m\mu_{\min}(E)$.
5. If $\mu_{\max}(E) < 0$, then $h^0(C, E) = 0$.
6. If $\mu_{\min}(E) > 2g - 2$, then $h^1(C, E) = 0$.

In particular, if E and F are semi-stable then $S^m E$ and $E \otimes F$ are semi-stable.

If E_1, \dots, E_ℓ are vector bundles on C and m_1, \dots, m_ℓ be non-negative integers. By the splitting principle [Ful84, Rema. 3.2.3] we can prove the following formula:

$$\mu(S^{m_1} E_1 \otimes \dots \otimes S^{m_\ell} E_\ell) = \sum_{i=1}^{\ell} m_i \mu(E_i).$$

Moreover, we have that for every $m \geq 1$ the Harder-Narasimhan filtration of the symmetric product $S^m E$ can be computed in terms of the one for E (see [Che08, Prop. 3.4] and [Wol05, Prop. 5.10], for instance).

Proposition 2.7. — *Let E be a vector bundle on C with Harder-Narasimhan filtration*

$$\text{HN}_\bullet(E) : 0 = E_\ell \subseteq E_{\ell-1} \subseteq \dots \subseteq E_1 \subseteq E_0 = E$$

and semi-stable quotients $Q_i = E_{i-1}/E_i$ with slopes $\mu_i = \mu(Q_i)$, for $i = 1, \dots, \ell$. For every positive integer $m \geq 1$, let us consider the vector bundle $S^m E$ with Harder-Narasimhan filtration

$$\text{HN}_\bullet(S^m E) : 0 = W_M \subseteq W_{M-1} \subseteq \dots \subseteq W_1 \subseteq W_0 = S^m E$$

and semi-stable quotients W_{j-1}/W_j with slopes $\nu_j = \mu(W_{j-1}/W_j)$, for $j = 1, \dots, M$. Then, for every $j = 1, \dots, M$ we have that

$$W_j = \sum_{\sum_i m_i \mu_i \geq \nu_{j+1}} S^{m_1} E_0 \otimes \dots \otimes S^{m_\ell} E_{\ell-1}$$

and

$$W_{j-1}/W_j \cong \bigoplus_{\sum_i m_i \mu_i = \nu_j} S^{m_1} Q_1 \otimes \dots \otimes S^{m_\ell} Q_\ell,$$

where the sums are taken over all partitions $\mathbf{m} = (m_1, \dots, m_\ell) \in \mathbb{N}^\ell$ of m , and $S^{m_1} E_0 \otimes \dots \otimes S^{m_\ell} E_{\ell-1}$ denotes the image of the composite

$$E_0^{\otimes m_1} \otimes \dots \otimes E_{\ell-1}^{\otimes m_\ell} \rightarrow E^{\otimes m} \rightarrow S^m E.$$

In particular, there is a refinement F_\bullet of $\text{HN}_\bullet(S^m E)$ of length $L = L(m)$ and whose respective successive quotients are of the form

$$F_{i-1}/F_i \cong Q_{\mathbf{m}(i)} = S^{m_1} Q_1 \otimes \dots \otimes S^{m_\ell} Q_\ell$$

for some partition $\mathbf{m}(i) = (m_1, \dots, m_\ell) \in \mathbb{N}^\ell$ of m , and such that for every $i \in \{1, \dots, L\}$ we have $\mu(Q_{\mathbf{m}(i)}) \leq \mu(Q_{\mathbf{m}(i+1)})$. Moreover, given any partition $\mathbf{m} \in \mathbb{N}^\ell$ of m , there is one and only one $i \in \{1, \dots, L\}$ such that $\mathbf{m}(i) = \mathbf{m}$.

3. Divisors on projective bundles over curves

Let E be a vector bundle on a smooth projective curve C , of rank $r \geq 2$ and degree d . In this section we study divisors on the projective bundle $\pi : \mathbb{P}(E) \rightarrow C$ of one-dimensional quotients. Let us recall that in this case the Néron-Severi group of $\mathbb{P}(E)$ is of the form

$$N^1(\mathbb{P}(E)) = \mathbb{Z} \cdot f \oplus \mathbb{Z} \cdot \xi,$$

where f is the numerical class of a fiber of π and $\xi = \xi_E$ is the numerical class of a divisor representing the tautological line bundle $\mathcal{O}_{\mathbb{P}(E)}(1)$. Moreover, if $[\text{pt}]$ denotes the class of a point in the ring $N^*(\mathbb{P}(E))$ then we have the following relations:

$$f^2 = 0, \quad \xi^{r-1} f = [\text{pt}], \quad \xi^r = d \cdot [\text{pt}].$$

The cone of nef divisors can be described via Hartshorne's characterization of ample vector bundles over curves [Har71, Theo. 2.4] (cf. [Ful11, Lemm. 2.1]).

Lemma 3.1. — $\text{Nef}(\mathbb{P}(E)) = \langle \xi - \mu_{\min} f, f \rangle$.

The cone of pseudo-effective divisors was obtained by Nakayama in [Nak04, Cor. IV.3.8], and it was indirectly computed by Wolfe [Wol05] and Chen [Che08] who independently obtained the volume function $\text{vol}_{\mathbb{P}(E)}$ on $N^1(\mathbb{P}(E))_{\mathbb{R}}$. A more general result on the cone of effective cycles of arbitrary codimension can be found in [Ful11, Theo. 1.1].

Lemma 3.2. — $\text{Psef}(\mathbb{P}(E)) = \langle \xi - \mu_{\max} f, f \rangle$.

In particular, we recover a result of Miyaoka [Miy87, Theo. 3.1] on semi-stable vector bundles over curves that was generalized by Fulger in [Ful11, Prop. 1.5].

Corollary 3.3. — *A vector bundle E on a smooth projective curve C is semi-stable if and only if $\text{Nef}(\mathbb{P}(E)) = \text{Psef}(\mathbb{P}(E))$.*

We finish this section by recalling Wolfe's computation of the volume function on $N^1(\mathbb{P}(E))$. See also [Che08, Theo. 1.2].

Notation 3.4. — Let $d \geq 1$ be an integer. We define the *standard d -simplex* Δ_d with $d + 1$ vertices in \mathbb{R}^{d+1} to be

$$\hat{\Delta}_d = \left\{ (x_1, \dots, x_{d+1}) \in \mathbb{R}^{d+1} \mid \sum_{i=1}^{d+1} x_i = 1 \text{ and } x_i \geq 0 \text{ for all } i \right\}.$$

By projecting $\widehat{\Delta}_d$ onto the hyperplane $x_1 = 0$, we can identify $\widehat{\Delta}_d$ with the *full dimensional standard d -simplex* (or just *d -simplex*) in \mathbb{R}^d given by

$$\Delta_d = \left\{ (x_2, \dots, x_{d+1}) \in \mathbb{R}^d \mid \sum_{i=2}^{d+1} x_i \leq 1 \text{ and } x_i \geq 0 \text{ for all } i \right\}.$$

Via the previous identification, we will denote by λ the Lebesgue measure on $\widehat{\Delta}_d$ induced by the standard Lebesgue measure on $\Delta_d \subseteq \mathbb{R}^d$. In particular, we will have $\lambda(\widehat{\Delta}_d) = \frac{1}{d!}$.

Given a positive real number $a > 0$, we define the *d -simplex with side length a* by $a\Delta_d = \{(x_2, \dots, x_{d+1}) \in \mathbb{R}^d \mid \sum_{i=2}^{d+1} x_i \leq a \text{ and } x_i \geq 0 \text{ for all } i\}$. Similar for $a\widehat{\Delta}_d \subseteq \mathbb{R}^{d+1}$, the *standard d -simplex with side length a* .

Theorem 3.5 ([Wol05, Theo. 5.14]). — *Let E be a vector bundle with Harder-Narasimhan filtration of length ℓ and semi-stable quotients Q_i of ranks r_i and slopes μ_i . Then, for any $t \in \mathbb{R}$*

$$\text{vol}_{\mathbb{P}(E)}(\xi - tf) = r! \cdot \int_{\widehat{\Delta}_{\ell-1}} \max \left\{ \sum_{i=1}^{\ell} \mu_i \beta_i - t, 0 \right\} \frac{\beta_1^{r_1-1} \dots \beta_{\ell}^{r_{\ell}-1}}{(r_1-1)! \dots (r_{\ell}-1)!} d\beta$$

where $\widehat{\Delta}_{\ell-1} \subseteq \mathbb{R}^{\ell}$ is the standard $(\ell-1)$ -simplex with coordinates $\beta_1, \dots, \beta_{\ell}$, and β be the standard induced Lebesgue measure.

Remark 3.6. — *Alternatively, Chen computed in [Che08, Theo. 1.2] a similar volume formula, but slightly simplified by integrating in \mathbb{R}^r instead of \mathbb{R}^{ℓ} (cf. [Che08, Prop. 3.5]). More precisely, with the same notation as above*

$$\text{vol}_{\mathbb{P}(E)}(\xi - tf) = r! \cdot \int_{\widehat{\Delta}_{r-1}} \max \left\{ \sum_{j=1}^r s_j \lambda_j - t, 0 \right\} d\lambda$$

where $\widehat{\Delta}_{r-1} \subseteq \mathbb{R}^r$ is the standard $(r-1)$ -simplex with coordinates $\lambda_1, \dots, \lambda_r$, $d\lambda$ is the standard induced Lebesgue measure⁽²⁾, and $\mathbf{s} = (s_1, \dots, s_r)$ is a vector in \mathbb{R}^r such that the value μ_i appears exactly r_i times in the coordinates of \mathbf{s} as in Notation 3.7 below.

Notation 3.7. — *Fix $\ell \geq 1$ and $r \geq 1$ two integers, $(r_1, \dots, r_{\ell}) \in \mathbb{N}^{\ell}$ a partition of r and $t \in \mathbb{R}$. We define for $(\mu_1, \dots, \mu_{\ell}) \in \mathbb{Q}^{\ell}$ the following polytopes:*

$$\widehat{\Pi}_t = \left\{ (\beta_1, \dots, \beta_{\ell}) \in \widehat{\Delta}_{\ell-1} \subseteq \mathbb{R}^{\ell} \mid \sum_{i=1}^{\ell} \mu_i \beta_i \geq t \right\}$$

2. Unlike Chen, we do not normalize the measure in order to have $\lambda(\Delta_{r-1}) = 1$.

and

$$\square_t = \left\{ (\lambda_1, \dots, \lambda_r) \in \widehat{\Delta}_{r-1} \subseteq \mathbb{R}^r \mid \sum_{i=1}^r s_i \lambda_i \geq t \right\},$$

where

$$\mathbf{s} = (\underbrace{\mu_1, \dots, \mu_1}_{r_1 \text{ times}}, \underbrace{\mu_2, \dots, \mu_2}_{r_2 \text{ times}}, \dots, \underbrace{\mu_\ell, \dots, \mu_\ell}_{r_\ell \text{ times}}) \in \mathbb{Q}^r.$$

Similarly, we define

$$\square_t^{\text{HN}} = \left\{ (\lambda_1, \dots, \lambda_r) \in \widehat{\Delta}_{r-1} \subseteq \mathbb{R}^r \mid \sum_{i=2}^r \sigma_{i-1} \lambda_i + \sigma_r \lambda_1 \geq t \right\},$$

where

$$\boldsymbol{\sigma} = (\underbrace{\mu_\ell, \dots, \mu_\ell}_{r_\ell \text{ times}}, \underbrace{\mu_{\ell-1}, \dots, \mu_{\ell-1}}_{r_{\ell-1} \text{ times}}, \dots, \underbrace{\mu_1, \dots, \mu_1}_{r_1 \text{ times}}) \in \mathbb{Q}^r.$$

By abuse of notation, we will also denote by \square_t and \square_t^{HN} the full dimensional polytopes in \mathbb{R}^{r-1} obtained via the projection of $\widehat{\Delta}_{r-1} \subseteq \mathbb{R}^r$ onto $\Delta_{r-1} \subseteq \mathbb{R}^{r-1}$. Explicitly, if (ν_2, \dots, ν_r) are coordinates in \mathbb{R}^{r-1} then

$$\square_t = \left\{ (\nu_2, \dots, \nu_r) \in \Delta_{r-1} \subseteq \mathbb{R}^{r-1} \mid s_1 \left(1 - \sum_{i=2}^r \nu_i \right) + \sum_{i=2}^r s_i \nu_i \geq t \right\}$$

and

$$\square_t^{\text{HN}} = \left\{ (\nu_2, \dots, \nu_r) \in \Delta_{r-1} \mid \sum_{i=2}^r \sigma_{i-1} \nu_i + \sigma_r \left(1 - \sum_{i=2}^r \nu_i \right) \geq t \right\}.$$

Corollary 3.8. — *For any admissible flag Y_\bullet on $\mathbb{P}(E)$ and any big rational class $\xi - tf$ on $\mathbb{P}(E)$ we have that*

$$\text{vol}_{\mathbb{R}^r}(\Delta_{Y_\bullet}(\xi - tf)) = \int_{\widehat{\square}_t} \left(\sum_{i=1}^{\ell} \mu_i \beta_i - t \right) \frac{\beta_1^{r_1-1} \dots \beta_\ell^{r_\ell-1}}{(r_1-1)! \dots (r_\ell-1)!} d\beta,$$

for $\widehat{\square}_t \subseteq \mathbb{R}^\ell$ as in Notation 3.7.

Remark 3.9. — *Let $F = \pi^{-1}(q) \cong \mathbb{P}^{r-1}$ be any fiber of $\pi : \mathbb{P}(E) \rightarrow C$. Then, for any big \mathbb{R} -divisor D on $\mathbb{P}(E)$ we have that $F \not\subseteq \mathbf{B}_+(D)$. In fact, if $D \sim_{\mathbb{R}} A + E$, with A ample \mathbb{R} -divisor and E effective \mathbb{R} -divisor, and if $F \subseteq \text{Supp}(E)$, then we can write $E = aF + E'$ with $a > 0$ and $F \not\subseteq \text{Supp}(E')$, which implies that $F \not\subseteq \mathbf{B}_+(D)$ since $A + aF$ is ample.*

As a direct consequence, we compute the restricted volume function on a fiber $F = \pi^{-1}(q) \cong \mathbb{P}^{r-1}$.

Corollary 3.10. — *Let F be a fiber of $\pi : \mathbb{P}(E) \rightarrow C$ and let $\xi - tf$ be a big rational class. Then,*

$$\mathrm{vol}_{\mathbb{P}(E)|F}(\xi - tf) = (r-1)! \cdot \int_{\widehat{\square}_t} \frac{\beta_1^{r_1-1} \cdots \beta_\ell^{r_\ell-1}}{(r_1-1)! \cdots (r_\ell-1)!} d\beta,$$

for $\widehat{\square}_t \subseteq \mathbb{R}^\ell$ as in Notation 3.7. In particular, if $0 \leq \tau \leq \mu_\ell - t$ then we have that

$$\mathrm{vol}_{\mathbb{R}^{r-1}}(\Delta_{Y_\bullet}(\xi - tf)_{\nu_1=\tau}) = \int_{\widehat{\square}_{t+\tau}} \frac{\beta_1^{r_1-1} \cdots \beta_\ell^{r_\ell-1}}{(r_1-1)! \cdots (r_\ell-1)!} d\beta,$$

where Y_\bullet is any admissible flag on $\mathbb{P}(E)$ with divisorial component $Y_1 = F$. In particular, these volumes depend only on $\mathrm{gr}(\mathrm{HN}_\bullet(E))$, the graded vector bundle associated to the Harder-Narasimhan filtration of E .

Proof. — We consider

$$v(\tau) = \mathrm{vol}_{\mathbb{P}(E)}(\xi - tf + \tau f) = r! \cdot \int_{\widehat{\square}_{t-\tau}} \left(\sum_{i=1}^{\ell} \mu_i \beta_i - t + \tau \right) \frac{\beta_1^{r_1-1} \cdots \beta_\ell^{r_\ell-1}}{(r_1-1)! \cdots (r_\ell-1)!} d\beta.$$

Since $F \not\subseteq \mathbf{B}_+(\xi - tf)$ by Remark 3.9, Theorem 2.1 and the Leibniz integral rule give

$$\mathrm{vol}_{\mathbb{P}(E)|F}(\xi - tf) = \frac{1}{r} \cdot \frac{d}{d\tau} v(\tau) \Big|_{\tau=0} = (r-1)! \cdot \int_{\widehat{\square}_t} \frac{\beta_1^{r_1-1} \cdots \beta_\ell^{r_\ell-1}}{(r_1-1)! \cdots (r_\ell-1)!} d\beta. \quad \square$$

Lemma 3.11. — *Following Notation 3.7, we have that*

$$\mathrm{vol}_{\mathbb{R}^r}(\Delta_{Y_\bullet}(\xi - tf)) = \int_{\square_t} \left(\sum_{j=1}^r s_j \lambda_j - t \right) d\lambda$$

and

$$\mathrm{vol}_{\mathbb{R}^{r-1}}(\Delta_{Y_\bullet}(\xi - tf)_{\nu_1=\tau}) = \int_{\square_{t+\tau}} d\lambda = \mathrm{vol}_{\mathbb{R}^{r-1}}(\square_{t+\tau}).$$

Moreover, $\mathrm{vol}_{\mathbb{R}^{r-1}}(\square_{t+\tau}) = \mathrm{vol}_{\mathbb{R}^{r-1}}(\square_{t+\tau}^{\mathrm{HN}})$.

Proof. — The first two equalities follow from Remark 3.6. For the last assertion consider $\square_t \subseteq \mathbb{R}^r$ and $\square_t^{\mathrm{HN}} \subseteq \mathbb{R}^r$ as in Notation 3.7. Then, there is a linear transformation $T : \mathbb{R}^r \rightarrow \mathbb{R}^r$, whose associated matrix in the canonical basis of \mathbb{R}^r is given by a permutation matrix, such that $T(\square_t) = \square_t^{\mathrm{HN}}$ and $|\det(T)| = 1$. \square

4. Newton-Okounkov bodies on projective bundles over curves

Let E be a vector bundle on a smooth projective curve C , of rank $r \geq 2$ and degree d . In this section we study the geometry of Newton-Okounkov bodies of rational big classes in $N^1(\mathbb{P}(E))$ in terms of the numerical information of the Harder-Narasimhan filtration of E . In particular, Theorem A will be a consequence of Lemma 4.4 and Theorem 4.7. We follow Notation 2.5.

Definition 4.1 (Linear flag). — A complete flag of subvarieties Y_\bullet on the projective vector bundle $\mathbb{P}(E) \xrightarrow{\pi} C$ is called a *linear flag centered at $p \in \mathbb{P}(E)$, over the point $q \in C$* (or simply a *linear flag*) if $Y_0 = \mathbb{P}(E)$, $Y_1 = \pi^{-1}(q) \cong \mathbb{P}^{r-1}$ and $Y_i \cong \mathbb{P}^{r-i}$ is a linear subspace of Y_{i-1} for $i = 2, \dots, r$, with $Y_r = \{p\}$.

Let us begin with the following example that illustrates the general case. Namely, that the shape of Newton-Okounkov bodies on $\mathbb{P}(E)$ will depend on the semi-stability of E .

Example 4.2 (Ruled surfaces). — Suppose that $\text{rank}(E) = 2$ and let $\eta = a(\xi - \mu_\ell f) + bf \in N^1(\mathbb{P}(E))_{\mathbb{Q}}$ be a big class. In other words, $a, b \in \mathbb{Q}_{>0}$. Let

$$Y_\bullet : \mathbb{P}(E) \supseteq F = \pi^{-1}(q) \supseteq \{p\}$$

be the linear flag centered at $p \in \mathbb{P}(E)$, over $q \in C$. The Newton-Okounkov body of η can be computed by applying [LM09, Theo. 6.4] (see Example 2.2).

Semi-stable case. If E is semi-stable then Corollary 3.3 implies that every big class is ample. In particular, for every $\eta \in \text{Big}(X)_{\mathbb{Q}}$ and every \mathbb{Q} -divisor D_η with numerical class η , we have that $N(D_\eta) = 0$ and $P(D_\eta) = D_\eta$. It follows that, with the notation as in Example 2.2, $\nu = 0$, $\tau_F(\eta) = b$, $\alpha(t) = 0$ for every $t \in [0, b]$ and $\beta(t) = a$ for every $t \in [0, b]$. The Newton-Okounkov bodies are given by rectangles in this case.

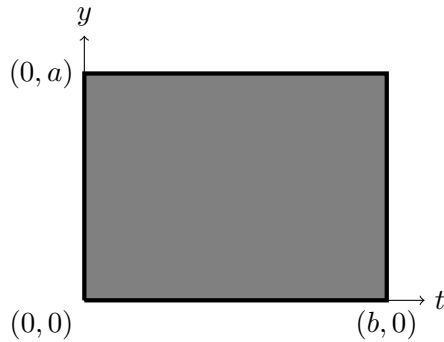


FIGURE 1. Newton-Okounkov body $\Delta_{Y_\bullet}(\eta)$ for E semi-stable

Non semi-stable case. If E is not semi-stable we consider its Harder-Narasimhan filtration

$$0 \rightarrow E_1 \rightarrow E \rightarrow Q_1 \rightarrow 0,$$

and we note that in this case $E_1 = Q_\ell$. The quotient $E \rightarrow Q_1 \rightarrow 0$ corresponds to a section $s : C \rightarrow \mathbb{P}(E)$ with $[s(C)] = \xi - \mu_\ell f$ in $N^1(\mathbb{P}(E))$. The curve $s(C)$ is the only irreducible curve on $\mathbb{P}(E)$ with negative self-intersection. On the other hand, if D_η is any \mathbb{Q} -divisor with numerical class η then, either η is inside the nef cone of $\mathbb{P}(E)$ and thus $P(D_\eta) = D_\eta$ and $N(D_\eta) = 0$, or η is big and not nef in which case we compute that

$$[P(D_\eta)] = \frac{b}{\mu_\ell - \mu_1} (\xi - \mu_1 f) \quad \text{and} \quad [N(D_\eta)] = \left(\frac{a(\mu_\ell - \mu_1) - b}{\mu_\ell - \mu_1} \right) (\xi - \mu_\ell f)$$

in $N^1(\mathbb{P}(E))_{\mathbb{Q}}$.

We notice that $N(D_\eta) = \left(\frac{a(\mu_\ell - \mu_1) - b}{\mu_\ell - \mu_1} \right) \cdot s(C)$ as \mathbb{Q} -divisor by minimality of the negative part and by the fact that the negative part is unique in its numerical equivalence class. See [Bad01, Lemm. 14.10, Cor. 14.13] for details.

Let $t^* = b - a(\mu_\ell - \mu_1)$. We have that η is big and nef if and only if $t^* \geq 0$, in which case the class $\eta - tf$ is big and nef for $0 \leq t \leq t^*$. For $t^* \leq t \leq b$, the same computation above enables us to find the Zariski decomposition of $D_\eta - tF$, which is big and not nef.

We notice that, with the notation as in Example 2.2, $\nu = 0$. On the other hand, the functions $\alpha(t)$ and $\beta(t)$ will depend on whether or not we have $\{p\} = s(C) \cap F$. A straightforward computation shows that the Newton-Okounkov bodies are given by the following finite polygons in \mathbb{R}^2 .

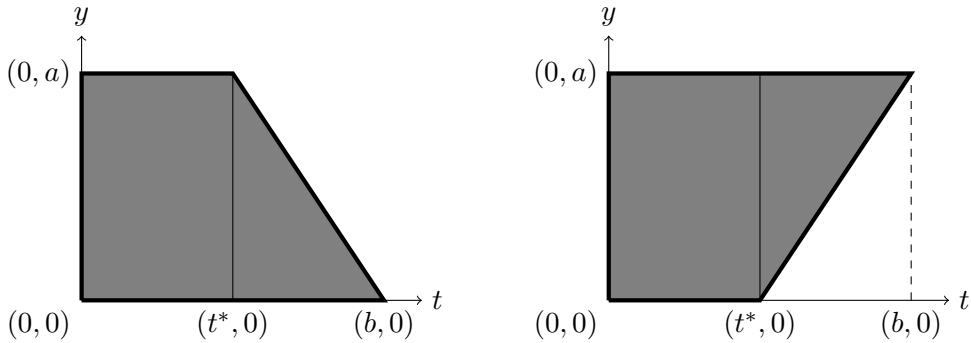


FIGURE 2. $\Delta_{Y_\bullet}(\eta)$ for E not semi-stable and η big and nef
 (a) if $\{p\} \neq s(C) \cap F$ (b) if $\{p\} = s(C) \cap F$

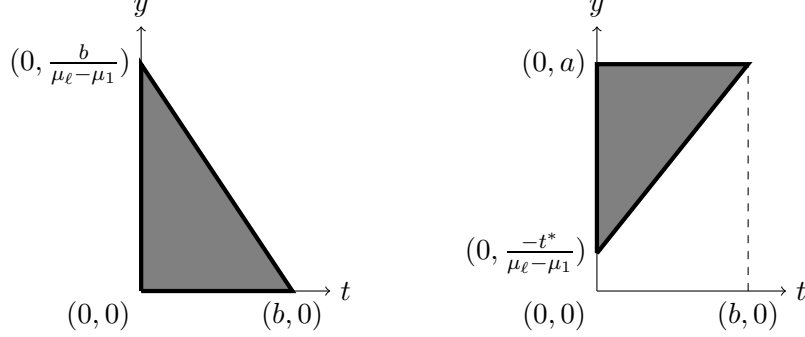


FIGURE 3. $\Delta_{Y_\bullet}(\eta)$ for E not semi-stable and η big and not nef
 (a) if $\{p\} \neq s(C) \cap F$ (b) if $\{p\} = s(C) \cap F$

□

We first observe that Proposition B in §1 states that the shape of Newton-Okounkov bodies on projective semi-stable vector bundles will be similar as in Example 4.2 above. Moreover, this provides a characterization of semi-stability in terms of Newton-Okounkov bodies.

We will need the following observation by Küronya, Lozovanu and Maclean.

Proposition 4.3 ([KLM12, Prop. 3.1]). — *Let X be a normal projective variety together with an admissible flag Y_\bullet . Suppose that D is a big \mathbb{Q} -divisor such that $Y_1 \not\subseteq \mathbf{B}_+(D)$ and that $D - tY_1$ is ample for some $0 \leq t < \tau_{Y_1}(D)$, where $\tau_{Y_1}(D) = \sup\{s > 0 \mid D - sY_1 \text{ is big}\}$. Then*

$$\Delta_{Y_\bullet}(X, D)_{\nu_1=t} = \Delta_{Y_\bullet|Y_1}(Y_1, (D - tY_1)|_{Y_1}),$$

where $Y_\bullet|Y_1 : Y_1 \supseteq Y_2 \supseteq \cdots \supseteq Y_n$ is the induced admissible flag on Y_1 .

In particular, if $\text{Psef}(X) = \text{Nef}(X)$ then we have that the Newton-Okounkov body $\Delta_{Y_\bullet}(D)$ is the closure in \mathbb{R}^n of the following set

$$\{(t, \nu_2, \dots, \nu_n) \in \mathbb{R}^n \mid 0 \leq t < \tau_{Y_1}(D), (\nu_2, \dots, \nu_n) \in \Delta_{Y_\bullet|Y_1}(Y_1, (D - tY_1)|_{Y_1})\}.$$

Proof of Proposition B. — (1) \Rightarrow (2). Let $\eta = a(\xi - \mu_\ell f) + bf$ be a big rational class on $\mathbb{P}(E)$ and $Y_\bullet : X = Y_0 \supseteq Y_1 \supseteq Y_2 \supseteq \cdots \supseteq Y_{r-1} \supseteq Y_r = \{p\}$ be a linear flag centered at $p \in \mathbb{P}(E)$, over the point $q \in C$. Since E is semi-stable we have that $\text{Big}(\mathbb{P}(E)) = \text{Amp}(\mathbb{P}(E))$ by Corollary 3.3. Equivalently, we have that $\mathbf{B}_+(\eta) = \emptyset$ for every big rational class η .

We notice that $\tau_F(\eta) = \sup\{s > 0 \mid \eta - sf \in \text{Big}(\mathbb{P}(E))\} = b$. The implication follows from Proposition 4.3 and [LM09, Exam. 1.1], by noting that if D_η is a \mathbb{Q} -divisor with numerical class η then

$$\Delta_{Y_\bullet}(\eta)_{\nu_1=t} = \Delta_{Y_\bullet|F}(F, (D_\eta - tF)|_F) = a\Delta_{Y_\bullet|F}(F, H) = a\Delta_{r-1} \subseteq \mathbb{R}^{r-1},$$

where $H \subseteq F \cong \mathbb{P}^{r-1}$ is an hyperplane section.

The implication (2) \Rightarrow (3) is trivial. We only need to prove (3) \Rightarrow (1). We notice that if the Newton-Okounkov body of $\eta = a(\xi - \mu_\ell f) + bf$ is given by $\Delta_{Y_\bullet}(\eta) = [0, b] \times a\Delta_{r-1} \subseteq \mathbb{R}^r$ then η is a big and nef class, by [KL15, Cor. 2.2]. We can therefore compute the volume of η via the top self-intersection $\text{vol}_{\mathbb{P}(E)}(\eta) = \eta^r = ra^{r-1}(b - a(\mu_\ell - \mu(E)))$. On the other hand, we have that

$$\text{vol}_{\mathbb{R}^n}(\Delta_{Y_\bullet}(\eta)) = \text{vol}_{\mathbb{R}^r}([0, b] \times a\Delta_{r-1}) = \frac{a^{r-1}}{(r-1)!}b,$$

and hence [LM09, Theo. A] leads to $\mu_\ell = \mu(E)$, implying the result. \square

We will first reduce our problem to computing the Newton-Okounkov body of the big and nef divisor class $\xi - \mu_1 f$.

Lemma 4.4. — *Let E be a not semi-stable vector bundle on a smooth projective curve C , with $\text{rank}(E) = r \geq 2$. Then the Newton-Okounkov body of any big rational class $\eta \in \text{Big}(\mathbb{P}(E))_{\mathbb{Q}}$ can be expressed in terms of the Newton-Okounkov body $\Delta_{Y_\bullet}(\xi - \mu_1 f) \subseteq \mathbb{R}^r$. More precisely, if $\eta = a(\xi - \mu_\ell f) + bf$ is a big rational class and Y_\bullet is a linear flag, then*

- (1) $\Delta_{Y_\bullet}(\eta) = ([0, t^*] \times a\Delta_{r-1}) \cup (a\Delta_{Y_\bullet}(\xi - \mu_1 f)_{\nu_1 \geq t^*} + t^* \vec{e}_1)$, if η is big and nef;
- (2) $\Delta_{Y_\bullet}(\eta) = a\Delta_{Y_\bullet}(\xi - \mu_1 f)_{\nu_1 \geq -t^*} + t^* \vec{e}_1$, if η is big and not nef.

Here $t^* = b - a(\mu_\ell - \mu_1)$ and $a\Delta_{r-1} \subseteq \mathbb{R}^{r-1}$ is the $(r-1)$ -simplex with side length a .

Proof. — If η is an ample rational class then $\Delta_{Y_\bullet}(\eta)_{\nu_1=t} = a\Delta_{r-1}$ for $0 \leq t \leq t^*$, by Proposition 4.3 and [LM09, Exam. 1.1]. On the other hand, we have that $\Delta_{Y_\bullet}(\eta)_{\nu_1 \geq t^*} = a\Delta_{Y_\bullet}(\xi - \mu_1 f) + t^* \vec{e}_1$ by Theorem 2.1.

If η is big and not nef then $t^* < 0$. Theorem 2.1 implies therefore that $a\Delta_{Y_\bullet}(\xi - \mu_1 f)_{\nu_1 \geq -t^*} = \Delta_{Y_\bullet}(\eta) - t^* \vec{e}_1$, which leads to (2). \square

Let us note that given $q \in C$, the Harder-Narasimhan filtration

$$\text{HN}_\bullet(E) : 0 = E_\ell \subseteq E_{\ell-1} \subseteq \cdots \subseteq E_1 \subseteq E_0 = E$$

induces a (not necessarily complete) flag on $\mathbb{P}(E)$ in the following way: let $Y_0 = \mathbb{P}(E)$ and for every $i = 2, \dots, \ell$ the exact sequence

$$0 \rightarrow Q_i \rightarrow E/E_i \rightarrow E/E_{i-1} \rightarrow 0$$

induces an inclusion $\mathbb{P}((E/E_{i-1})|_q) \hookrightarrow \mathbb{P}((E/E_i)|_q)$. We obtain therefore the following (possibly partial) flag of linear subvarieties

$$\mathbb{P}((E/E_1)|_q) \subseteq \mathbb{P}((E/E_2)|_q) \subseteq \cdots \subseteq \mathbb{P}((E/E_{\ell-1})|_q) \subseteq \mathbb{P}(E|_q) = \pi^{-1}(q) \subseteq \mathbb{P}(E)$$

with $\text{codim}_{\mathbb{P}(E)} \mathbb{P}((E/E_i)|_q) = \text{rank}(E_i) + 1$. We also note that this flag is complete if and only if all the semi-stable quotients Q_i are line bundles.

In general, it will be necessary to choose a complete linear flag on each $\mathbb{P}((E_{i-1}/E_i)|_q) = \mathbb{P}(Q_i|_q)$ in order to complete the flag above.

We shall consider linear flags that are compatible with the Harder-Narasimhan filtration of E in the sense that they complete the previous flag.

Definition 4.5 (Compatible linear flag). — A linear flag Y_\bullet on $\mathbb{P}(E)$ over $q \in C$ is said to be *compatible with the Harder-Narasimhan filtration of E* if

$$Y_{\text{rank } E_{i+1}} = \mathbb{P}((E/E_i)|_q) \cong \mathbb{P}^{r - \text{rank } E_i - 1} \subseteq \mathbb{P}(E)$$

for every $i = 1, \dots, \ell$.

We will need the following estimate due to Wolfe (see [Wol05, Prop. 5.12]). We include a proof for the reader's convenience.

Lemma 4.6 (Wolfe). — *Let C be a smooth projective curve, let E be a non semi-stable vector bundle over C with successive semi-stable quotients Q_1, \dots, Q_ℓ , let $a, b \in \mathbb{Q}$ be two rational numbers such that $a < b$ and let $t \in \mathbb{Z}$ be any integer. Then, for every positive integer $m \geq 1$ and every point $q \in C$, the Harder-Narasimhan filtration of $S^m E \otimes \mathcal{O}_C(-mt \cdot q)$ admits a refinement F_\bullet of length L whose successive quotients are of the form*

$$F_{i-1}/F_i \cong Q_{\mathbf{m}(i),t} = S^{m_1} Q_1 \otimes \dots \otimes S^{m_\ell} Q_\ell \otimes \mathcal{O}_C(-mt \cdot q)$$

for some partition $\mathbf{m}(i) = (m_1, \dots, m_\ell) \in \mathbb{N}^\ell$ of m , and such that for every $i \in \{1, \dots, L\}$ we have $\mu(Q_{\mathbf{m}(i),t}) \leq \mu(Q_{\mathbf{m}(i+1),t})$. Moreover, the number of vector bundles of the form $Q_{\mathbf{m},t}$ appearing as successive quotients with slopes between a and b is $O(m^{\ell-2})$.

Proof. — The existence of such a refinement follows from Proposition 2.7 applied to the bundle $E \otimes \mathcal{O}_C(-t \cdot q)$. For the last assertion we follow almost verbatim Wolfe's argument in the proof of [Wol05, Prop. 5.12].

Following Notation 3.4, let $\widehat{\Delta}_{\ell-1} \subseteq \mathbb{R}^\ell$ be the standard $(\ell - 1)$ -simplex with coordinates (x_1, \dots, x_ℓ) and let $\Delta_{\ell-1} \subseteq \{0\} \times \mathbb{R}^{\ell-1} \cong \mathbb{R}^{\ell-1}$ be the full dimensional $(\ell - 1)$ -simplex obtained by projecting $\widehat{\Delta}_{\ell-1}$ onto the hyperplane $x_1 = 0$. Then, partitions $\mathbf{m} = (m_1, \dots, m_\ell) \in \mathbb{N}^\ell$ of m correspond to integer coordinate points $(m_1, \dots, m_\ell) \in m\widehat{\Delta}_{\ell-1} \subseteq \mathbb{R}^\ell$ and hence to integer coordinate points in $m\Delta_{\ell-1} \subseteq \mathbb{R}^{\ell-1}$.

We note that

$$|m\widehat{\Delta} \cap \mathbb{Z}^\ell| = \binom{m + \ell - 1}{\ell - 1} = O(m^{\ell-1}).$$

If we consider the function $M(x_1, \dots, x_\ell) = \sum_{i=1}^\ell \mu_i x_i - mt$, then $\mu(Q_{\mathbf{m},t}) = M(m_1, \dots, m_\ell)$ and hence vector bundles $Q_{\mathbf{m},t}$ with given slope μ_0

correspond to integer coordinate points on the hypersurface $H_{\mu_0} = M^{-1}(\{\mu_0\})$ in $\widehat{\Delta}_{\ell-1}$. We claim that

$$|m\widehat{\Delta}_{\ell-1} \cap H_{\mu_0} \cap \mathbb{Z}^\ell| \leq |m\widehat{\Delta}_{\ell-2} \cap \mathbb{Z}^{\ell-1}| = O(m^{\ell-2}),$$

where

$$m\widehat{\Delta}_{\ell-2} = \left\{ (0, x_2, \dots, x_\ell) \in \{0\} \times \mathbb{R}^{\ell-1} \mid \sum_{i=2}^{\ell} x_i = m \right\} \subseteq m\Delta_{\ell-1}.$$

In fact, if we consider the restriction to $m\widehat{\Delta}_{\ell-1} \cap H_{\mu_0}$ of the composite function $\varphi = \partial \circ p$ below

$$\begin{array}{ccc} m\widehat{\Delta}_{\ell-1} & & \\ \downarrow p & \searrow \varphi & \\ m\Delta_{\ell-1} & \xrightarrow{\partial} & m\widehat{\Delta}_{\ell-2} \end{array}$$

where

$$p(x_1, x_2, \dots, x_\ell) = (0, x_2, \dots, x_\ell)$$

and

$$\partial(0, x_2, x_3, \dots, x_\ell) = (0, m - \sum_{i=3}^{\ell} x_i, x_3, \dots, x_\ell),$$

then a straightforward computation shows that the equality $\varphi(x_1, \dots, x_\ell) = \varphi(y_1, \dots, y_\ell)$ for $(x_1, \dots, x_\ell), (y_1, \dots, y_\ell) \in m\widehat{\Delta}_{\ell-1} \cap H_{\mu_0}$ is equivalent to the equations $x_i = y_i$ for $i \geq 3$,

$$\mu_1 x_1 + \mu_2 x_2 = \mu_1 y_1 + \mu_2 y_2$$

and

$$x_1 + x_2 = y_1 + y_2,$$

from which we deduce the injectivity of φ , since $\mu_1 < \mu_2$.

We note that there are only finite different values of $M(m_1, \dots, m_\ell)$ between a and b for $(m_1, \dots, m_\ell) \in m\widehat{\Delta} \cap \mathbb{Z}^\ell$ since every partition of m appears only one time in the filtration and each increase in the value of M is at least $1/\min\{k \in \mathbb{N}_{\geq 1} \mid k\mu_j \in \mathbb{Z} \text{ for all } 1 \leq j \leq \ell\} \in \mathbb{Q}_{>0}$, and then only finitely many hypersurfaces contains the integer coordinate points corresponding to vector bundles $Q_{\mathbf{m},t}$ with slopes between a and b . There are therefore at most $O(m^{\ell-2})$ vector bundles $Q_{\mathbf{m},t}$ with slopes between a and b . \square

We can prove now the main reduction step. Namely, compute the Newton-Okounkov body of the big and nef class $\xi - \mu_1 f$ with respect to any linear flag which is compatible with the Harder-Narasimhan filtration.

Theorem 4.7. — *Let C be a smooth projective curve and let E be a vector bundle over C of rank $r \geq 2$. Then, for every linear flag Y_\bullet on $\mathbb{P}(E)$ which is compatible with the Harder-Narasimhan filtration we have that*

$$\Delta_{Y_\bullet}(\xi - \mu_1 f) = \{(\nu_1, \dots, \nu_r) \in \mathbb{R}_{\geq 0}^r \mid 0 \leq \nu_1 \leq \mu_\ell - \mu_1, (\nu_2, \dots, \nu_r) \in \square_{\mu_1 + \nu_1}^{\text{HN}}\},$$

where $\square_{\mu_1 + \nu_1}^{\text{HN}} \subseteq \mathbb{R}^{r-1}$ is the full dimensional polytope defined in Notation 3.7.

Proof. — We follow Notation 2.5. We first note that if E is semi-stable then $\mu_1 = \mu_\ell = \mu(E)$ and hence the Theorem follows from Proposition B. Let us suppose from now on that E is not semi-stable.

It follows from Theorem 2.1 and Lemma 3.2 that the projection of the Newton-Okounkov body of $\xi - \mu_1 f$ onto the first coordinate is given by

$$\text{pr}_1(\Delta_{Y_\bullet}(\xi - \mu_1 f)) = [0, \mu_\ell - \mu_1].$$

By continuity of slices of Newton-Okounkov bodies (cf. [KL15, Lemma 1.7]), it suffices to consider a fixed $t \in]0, \mu_\ell - \mu_1[\cap \mathbb{Q}$ and show that the slice of $\Delta_{Y_\bullet}(\xi - \mu_1 f)$ at $t \in]0, \mu_\ell - \mu_1[\cap \mathbb{Q}$ is given by

$$\Delta_{Y_\bullet}(\xi - \mu_1 f)|_{\nu_1=t} = \square_{\mu_1+t}^{\text{HN}} \subseteq \mathbb{R}^{r-1}$$

for any linear flag Y_\bullet on $\mathbb{P}(E)$ which is compatible with the Harder-Narasimhan filtration. Let us fix from now on a linear flag Y_\bullet on $\mathbb{P}(E)$, over $q \in C$, which is compatible with the Harder-Narasimhan filtration and a \mathbb{Q} -divisor D such that $[D] = \xi - \mu_1 f$ in $N^1(\mathbb{P}(E))_{\mathbb{Q}}$. For every integer $m \geq 1$, let us define the subspace

$$\begin{aligned} \mathfrak{a}_m &= \mathfrak{a}_{m,t} = \text{H}^0(\mathbb{P}(E)|_F, \mathcal{O}_{\mathbb{P}(E)}(\lfloor m(D - tF) \rfloor)) \\ &= \text{Im}(\text{H}^0(\mathbb{P}(E), \mathcal{O}_{\mathbb{P}(E)}(\lfloor m(D - tF) \rfloor)) \xrightarrow{\text{rest}} \text{H}^0(F, \mathcal{O}_F(m))) \\ &\subseteq \text{H}^0(F, \mathcal{O}_F(m)). \end{aligned}$$

Let us denote by $Y_\bullet|_F$ the induced flag on F . It follows from [LM09, Prop. 4.1, Rem. 4.25] that the restricted algebra \mathfrak{a}_\bullet above computes the desired slice

$$\Delta_{Y_\bullet|_F}(\mathfrak{a}_\bullet) = \Delta_{Y_\bullet}(\xi - \mu_1 f)|_{\nu_1=t},$$

and that for every integer $n \geq 1$ we have

$$\Delta_{Y_\bullet}(nD)|_{\nu_1=nt} = \Delta_{Y_\bullet|_F}(n(D - tF)) = n\Delta_{Y_\bullet|_F}(D - tF) = n\Delta_{Y_\bullet}(D)|_{\nu_1=t}.$$

We will consider $\mathfrak{a}_{n\bullet} = \{a_{nm}\}_{m \geq 0}$ instead of $\mathfrak{a}_\bullet = \{a_m\}_{m \geq 0}$, for n fixed and divisible enough such that $n\mu_1 \in \mathbb{Z}$ and $nt \in \mathbb{Z}$. We also note that for $0 < t < \mu_\ell - \mu_1$ we have that

$$\mu_{\max}(S^m E \otimes \mathcal{O}_C(-m(\mu_1 + t) \cdot q)) = m(\mu_\ell - \mu_1 - t) > 0$$

and

$$\mu_{\min}(S^m E \otimes \mathcal{O}_C(-m(\mu_1 + t) \cdot q)) = -mt < 0.$$

Therefore, by considering n above large enough we may also assume that $\mu_{\max}(S^{nm}E \otimes \mathcal{O}_C(-nm(\mu_1 + t) \cdot q)) > 2g - 1$ for every $m \geq 1$.

Choose homogeneous coordinates $[x_1 : \cdots : x_r]$ on $F \cong \mathbb{P}^{r-1}$. Since Y_\bullet is a linear flag on $\mathbb{P}(E)$, there is an isomorphism of graded algebras

$$\phi : \bigoplus_{m \geq 0} H^0(F, \mathcal{O}_F(m)) \rightarrow k[x_1, \dots, x_r]$$

such that \mathfrak{a}_m can be regarded as a subspace of $k[x_1, \dots, x_r]_m$, the k -vector space of homogeneous polynomials of degree m in the variables x_1, \dots, x_r , for all $m \geq 0$. Via this identification, \mathfrak{a}_\bullet can be seen as a graded subalgebra of $k[x_1, \dots, x_r]$. Moreover, the projection formula implies that we can identify \mathfrak{a}_m with

$$\text{Im}\left(H^0(C, S^m E \otimes \mathcal{O}_C(-m(\mu_1 + t) \cdot q)) \xrightarrow{\text{rest}} H^0(C, (S^m E \otimes \mathcal{O}_C(-m(\mu_1 + t) \cdot q))|_q)\right).$$

We shall define a graded subalgebra $\mathfrak{b}_{n\bullet} \subseteq \mathfrak{a}_{n\bullet}$ for which we can explicitly compute that

$$\Delta_{Y_\bullet|F}(\mathfrak{b}_{n\bullet}) = n \square_{\mu_1 + t}^{\text{HN}}$$

and we will prove that $\Delta_{Y_\bullet|F}(\mathfrak{b}_{n\bullet}) = \Delta_{Y_\bullet|F}(\mathfrak{a}_{n\bullet}) = n \Delta_{Y_\bullet|F}(\mathfrak{a}_\bullet)$.

Step 1. Construction of $\mathfrak{b}_{n\bullet}$. It follows from Proposition 2.7 (cf. Lemma 4.6) that for every $m \geq 1$ there is a filtration

$$F_\bullet : 0 = F_L \subseteq F_{L-1} \subseteq \cdots \subseteq F_1 \subseteq F_0 = S^{nm}E \otimes \mathcal{O}_C(-nm(\mu_1 + t) \cdot q)$$

whose successive quotients have the form

$$F_{j-1}/F_j \cong Q_{\mathbf{m}(j), \mu_1 + t} = S^{m_1}Q_1 \otimes \cdots \otimes S^{m_\ell}Q_\ell \otimes \mathcal{O}_C(-nm(\mu_1 + t) \cdot q)$$

for some partition $\mathbf{m}(j) \in \mathbb{N}^\ell$ of nm , and $\mu(Q_{\mathbf{m}(j), \mu_1 + t}) \leq \mu(Q_{\mathbf{m}(j+1), \mu_1 + t})$ for every $j \in \{1, \dots, L\}$.

Let us define $J = J(m) \in \{1, \dots, L\}$ to be the largest index such that $\mu(Q_{\mathbf{m}(J), \mu_1 + t}) \leq 2g - 1$. We have that for every $j \in \{J, \dots, L - 1\}$, the short exact sequence

$$0 \rightarrow F_{j+1} \otimes \mathcal{O}_C(-q) \rightarrow F_{j+1} \rightarrow F_{j+1}|_q \rightarrow 0$$

gives an exact sequence in cohomology

$$0 \rightarrow H^0(C, F_{j+1} \otimes \mathcal{O}_C(-q)) \rightarrow H^0(C, F_{j+1}) \rightarrow H^0(C, F_{j+1}|_q) \rightarrow 0,$$

since we have that $h^1(C, F_{j+1} \otimes \mathcal{O}_C(-q)) = 0$, by Lemma 2.6. In particular, we get for every $j \in \{J, \dots, L - 1\}$ a surjection $H^0(C, F_{j+1}) \rightarrow H^0(C, F_{j+1}|_q)$. Therefore, let us consider the subspaces

$$\mathfrak{b}_{nm} = \text{Im}(H^0(C, F_{J+1}) \xrightarrow{\text{rest}} H^0(C, F_{J+1}|_q)) = H^0(C, F_{J+1}|_q) \subseteq \mathfrak{a}_{nm}.$$

It follows that \mathfrak{b}_{n_\bullet} is also a graded subalgebra of $k[x_1, \dots, x_r]$. In fact, it suffices to verify that

$$F_{J(m)+1} \otimes F_{J(m')+1} \subseteq F_{J(m+m')+1}$$

for every $m, m' \geq 1$. To do so, we first note that $F_{J(m)+1}$ and $F_{J(m')+1}$ are sums of vector bundles of the form

$$S^{m_1} E_0 \otimes \cdots \otimes S^{m_\ell} E_{\ell-1}$$

and

$$S^{m'_1} E_0 \otimes \cdots \otimes S^{m'_\ell} E_{\ell-1}$$

where $\mathbf{m} = (m_1, \dots, m_\ell) \in \mathbb{N}^\ell$ and $\mathbf{m}' = (m'_1, \dots, m'_\ell) \in \mathbb{N}^\ell$ are partitions of nm and nm' , respectively, and these partitions are such that

$$\sum_{i=1}^{\ell} \mu_i m_i - nm(\mu_1 + t) > 2g - 1$$

and

$$\sum_{i=1}^{\ell} \mu_i m'_i - nm'(\mu_1 + t) > 2g - 1,$$

respectively. We deduce that $F_{J(m)+1} \otimes F_{J(m')+1} \subseteq F_{J(m+m')+1}$ by adding both inequalities⁽³⁾.

Step 2. The equality $\Delta_{Y_\bullet|F}(\mathfrak{b}_{n_\bullet}) = \Delta_{Y_\bullet|F}(\mathfrak{a}_{n_\bullet})$. Let us note that the Newton-Okounkov body $\Delta_{Y_\bullet|F}(\mathfrak{b}_{n_\bullet})$ of the graded algebra \mathfrak{b}_{n_\bullet} is always defined as a closed and convex subset of $\mathbb{R}^{\ell-1}$, but it might happen that $\Delta_{Y_\bullet|F}(\mathfrak{b}_{n_\bullet})$ is not compact nor computes the volume of the graded linear series \mathfrak{b}_{n_\bullet} (see for instance [LM09, Rem. 1.17]). However, in this case compactness will be ensured since $\Delta_{Y_\bullet|F}(\mathfrak{b}_{n_\bullet}) \subseteq \Delta_{Y_\bullet|F}(\mathfrak{a}_{n_\bullet})$, and the latter is compact.

Let us prove now that $\text{vol}(\mathfrak{a}_{n_\bullet}) = \text{vol}(\mathfrak{b}_{n_\bullet})$. This will imply that the Newton-Okounkov body of \mathfrak{b}_{n_\bullet} is such that

$$\frac{1}{(r-1)!} \text{vol}(\mathfrak{b}_{n_\bullet}) = \text{vol}_{\mathbb{R}^{r-1}}(\Delta_{Y_\bullet|F}(\mathfrak{b}_{n_\bullet})),$$

as desired. Let us define

$$H^0(C|_q, F_j) = \text{Im}(H^0(C, F_j) \xrightarrow{\text{rest}} H^0(C, F_j|_q)) \subseteq \mathfrak{a}_{nm}$$

for every $j \in \{0, \dots, L-1\}$.

3. We note that if $C \cong \mathbb{P}^1$ then all the slopes $\mu_i = \mu(Q_i) = \deg(Q_i)$ are integer numbers and hence the inequality “ $> 2g - 1$ ” becomes “ ≥ 0 ”. If $g \geq 1$ then $4g - 2 > 2g - 1$.

Let $I = I(m) \in \{1, \dots, L\}$ be the smallest index such that $\mu(Q_{\mathbf{m}(I), \mu_1+t}) < 0$. If $I \geq 1$, we have that $H^0(C, F_I) \cong H^0(C, F_{I-1})$ since $H^0(C, Q_{\mathbf{m}(I), \mu_1+t}) = \{0\}$, by Lemma 2.6. By the same reason, we get that

$$H^0(C, F_I) \cong H^0(C, F_{I-1}) \cong \dots \cong H^0(C, F_0).$$

Then we have that

$$\begin{aligned} \dim_k \mathbf{a}_{nm} - \dim_k \mathbf{b}_{nm} &= \dim_k H^0(C|_q, F_0) - \dim_k H^0(C|_q, F_{J+1}) \\ &= \dim_k H^0(C|_q, F_I) - \dim_k H^0(C, F_{J+1}|_q) \\ &\leq \dim_k H^0(C, F_I|_q) - \dim_k H^0(C, F_{J+1}|_q) \\ &\leq \sum_{j=I}^J \text{rank}(Q_{\mathbf{m}(j), \mu_1+t}) = O((nm)^{r-2}), \end{aligned}$$

where the last estimate follows from the fact that the number semi-stable quotients $Q_{\mathbf{m}(j), \mu_1+t}$ with slopes between 0 and $2g-1$ is $O((nm)^{\ell-2})$, by Lemma 4.6, and from the fact that

$$\text{rank}(Q_{\mathbf{m}(j), \mu_1+t}) = \prod_{i=1}^{\ell} \binom{m_i + r_i - 1}{r_i - 1} = O((nm)^{r-\ell}),$$

where $\mathbf{m}(j) = (m_1, \dots, m_{\ell}) \in \mathbb{N}^{\ell}$ and $\sum_{i=1}^{\ell} m_i = nm$. We obtain therefore that

$$\frac{\text{vol}(\mathbf{a}_{n\bullet})}{n^{r-1}} = \lim_{m \rightarrow \infty} \frac{\dim_k \mathbf{a}_{nm}}{(nm)^{r-1}/(r-1)!} = \lim_{m \rightarrow \infty} \frac{\dim_k \mathbf{b}_{nm}}{(nm)^{r-1}/(r-1)!} = \frac{\text{vol}(\mathbf{b}_{n\bullet})}{n^{r-1}}.$$

On the other hand, we have that

$$\begin{aligned} \frac{1}{(r-1)!} \text{vol}(\mathbf{a}_{n\bullet}) &= \frac{1}{(r-1)!} \text{vol}(\mathbf{b}_{n\bullet}) \leq \text{vol}_{\mathbb{R}^{r-1}}(\Delta_{Y_{\bullet}|F}(\mathbf{b}_{n\bullet})) \\ &\leq \text{vol}_{\mathbb{R}^{r-1}}(\Delta_{Y_{\bullet}|F}(\mathbf{a}_{n\bullet})) = \frac{1}{(r-1)!} \text{vol}(\mathbf{a}_{n\bullet}), \end{aligned}$$

and hence $\text{vol}_{\mathbb{R}^{r-1}}(\Delta_{Y_{\bullet}|F}(\mathbf{b}_{n\bullet})) = \text{vol}_{\mathbb{R}^{r-1}}(\Delta_{Y_{\bullet}|F}(\mathbf{a}_{n\bullet}))$. Finally, the inclusion $\Delta_{Y_{\bullet}|F}(\mathbf{b}_{n\bullet}) \subseteq \Delta_{Y_{\bullet}|F}(\mathbf{a}_{n\bullet})$ and the equality of Euclidean volumes lead to $\Delta_{Y_{\bullet}|F}(\mathbf{b}_{n\bullet}) = \Delta_{Y_{\bullet}|F}(\mathbf{a}_{n\bullet})$.

Let us choose homogeneous coordinates $[x_1 : \dots : x_r]$ on F such that the degree 1 part of the isomorphism ϕ ,

$$\phi_1 : H^0(F, \mathcal{O}_F(1)) \cong H^0(C, (E \otimes \mathcal{O}_C(-(\mu_1+t) \cdot q))|_q) \rightarrow k[x_1, \dots, x_r]_1,$$

is such that for every $i = 0, \dots, \ell-1$ the image of the subspace

$$H^0(C, (E_i \otimes \mathcal{O}_C(-(\mu_1+t) \cdot q))|_q) \subseteq H^0(C, (E \otimes \mathcal{O}_C(-(\mu_1+t) \cdot q))|_q)$$

via ϕ_1 coincide with the subspace generated by the variables $x_1, \dots, x_{\text{rank } E_i}$. By taking symmetric powers it follows from Proposition 2.7 that for each $m \geq 1$ we have that $\mathbf{b}_{nm} \subseteq k[x_1, \dots, x_r]_{nm}$, the image of $H^0(C, F_{J+1}|_q) \subseteq$

$H^0(C, F_0|_q)$, corresponds to the subspace of homogeneous polynomials of degree nm generated by polynomials of the form

$$P(\mathbf{x}) = P_1(\mathbf{x}_1) \cdots P_\ell(\mathbf{x}_\ell)$$

where P_i is an homogeneous polynomial of degree $m_i \geq 0$ in the variables $\mathbf{x}_i = (x_{i,1}, \dots, x_{i,r_i})$, where $(x_1, \dots, x_r) = (\mathbf{x}_\ell, \dots, \mathbf{x}_1)$, and the m_i are such that $m_1 + \dots + m_\ell = nm$ and

$$\mu_1 m_1 + \dots + \mu_\ell m_\ell > nm(\mu_1 + t) + 2g - 1.$$

Then, $\mathfrak{b}_{n\bullet}$ is a graded subalgebra of $k[x_1, \dots, x_r]$ whose Newton-Okounkov body, with respect to any linear flag Y_\bullet , computes the desired slice:

$$\Delta_{Y_\bullet|F}(\mathfrak{b}_{n\bullet}) = n\Delta_{Y_\bullet}(\xi - \mu_1 f)|_{\nu_1=t},$$

by the computation above.

Step 3. The Newton-Okounkov body of $\mathfrak{b}_{n\bullet}$. Let us note that since the linear flag Y_\bullet is compatible with the Harder-Narasimhan filtration we have that

$$Y_{\text{rank } E_i+1} = \mathbb{P}((E/E_i)|_q) = \{x_1 = \dots = x_{\text{rank } E_i} = 0\} \subseteq \mathbb{P}(E)$$

for $i = 1, \dots, \ell$.

Let us show that we have the following equality

$$\Delta_{Y_\bullet|F}(\mathfrak{b}_{n\bullet}) = \bigcup_{m \geq 1} \overline{\left\{ \frac{\nu_{Y_\bullet|F}(P)}{m} \mid P \in \mathfrak{b}_{nm} \right\}} = n\Box_{\mu_1+t}^{\text{HN}},$$

where

$$\Box_{\mu_1+t}^{\text{HN}} = \left\{ (\nu_2, \dots, \nu_r) \in \Delta_{r-1} \mid \sum_{i=2}^r \sigma_{i-1} \nu_i + \sigma_r (1 - \sum_{i=2}^r \nu_i) \geq \mu_1 + t \right\}$$

and $\sigma = (\underbrace{\mu_\ell, \dots, \mu_\ell}_{r_\ell \text{ times}}, \underbrace{\mu_{\ell-1}, \dots, \mu_{\ell-1}}_{r_{\ell-1} \text{ times}}, \dots, \underbrace{\mu_1, \dots, \mu_1}_{r_1 \text{ times}}) \in \mathbb{Q}^r$.

We note that since $\text{vol}_{\mathbb{R}^{r-1}}(\Delta_{Y_\bullet|F}(\mathfrak{b}_{n\bullet})) = \text{vol}_{\mathbb{R}^{r-1}}(n\Box_{\mu_1+t}^{\text{HN}})$, by Lemma 3.11 and Step 2 above, it suffices to show that for every $m \geq 1$ and every $P \in \mathfrak{b}_{nm}$ we have that $\frac{\nu_{Y_\bullet|F}(P)}{nm} \in \Box_{\mu_1+t}^{\text{HN}}$. If we write

$$P(x_1, \dots, x_r) = \sum_{\alpha \in \mathbb{N}^{r-1}} c_\alpha x^\alpha \text{ with } c_\alpha \in k[x_r],$$

then it follows from the choice of variables x_i that

$$\nu_{Y_\bullet|F}(P) = \min\{\alpha \in \mathbb{N}^{r-1} \mid c_\alpha \neq 0\},$$

where the minimum is taken with respect to the lexicographic order on \mathbb{Z}^{r-1} . See [Bou12, Example 2.17] for details. In particular, if we write $\nu_{Y_\bullet|F}(P) =$

$\alpha = (\alpha_1, \dots, \alpha_{r-1}) \in \mathbb{N}^{r-1}$ and we denote $\alpha_r = \deg_{x_r} c_\alpha \in \mathbb{N}$, then we have that $\alpha_1 + \dots + \alpha_r = nm$ and

$$\alpha_1 + \dots + \alpha_{r_\ell} = m_\ell, \quad \alpha_{r_\ell+1} + \dots + \alpha_{r_\ell+r_{\ell-1}} = m_{\ell-1}, \dots, \quad \alpha_{r_\ell+\dots+r_2+1} + \dots + \alpha_r = m_1.$$

Hence, if we consider $\frac{\nu_{Y_\bullet} F(P)}{nm} = \left(\frac{\alpha_1}{nm}, \dots, \frac{\alpha_{r-1}}{nm} \right)$ then we have that

1. $\left(\frac{\alpha_1}{nm}, \dots, \frac{\alpha_{r-1}}{nm} \right) \in \Delta_{r-1}$ since $\frac{\alpha_1 + \dots + \alpha_{r-1}}{nm} = 1 - \frac{\alpha_r}{nm} \leq 1$.
2. $\sum_{i=1}^{r-1} \sigma_i \frac{\alpha_i}{nm} + \sigma_r \left(1 - \sum_{i=1}^{r-1} \frac{\alpha_i}{nm} \right) \geq \mu_1 + t$ since

$$\begin{aligned} \sum_{i=1}^{r-1} \sigma_i \frac{\alpha_i}{nm} + \sigma_r \left(1 - \sum_{i=1}^{r-1} \frac{\alpha_i}{nm} \right) &= \sum_{i=1}^r \sigma_i \frac{\alpha_i}{nm} \\ &= \sum_{j=1}^\ell \mu_j \frac{m_j}{nm} > \mu_1 + t + \frac{2g-1}{nm} \end{aligned}$$

by definition of \mathbf{b}_{nm} .

We obtain therefore that $\Delta_{Y_\bullet}(\xi - \mu_1 f)|_{\nu_1=t} = \square_{\mu_1+t}^{\text{HN}}$. \square

We note that Theorem A is a immediate consequence of Theorem 4.7.

Proof of Theorem A. — We note that if $\eta = a(\xi - \mu_\ell f) + bf$ is a big rational class on $\mathbb{P}(E)$ then

$$\Delta_{Y_\bullet}(\eta) = \{(\nu_1, \dots, \nu_r) \in \mathbb{R}_{\geq 0}^r \mid 0 \leq \nu_1 \leq b, (\nu_2, \dots, \nu_r) \in a \square_{\mu_1 + \nu_1 - t^*}^{\text{HN}}\},$$

where $t^* = b - a(\mu_\ell - \mu_1)$, by Lemma 4.4. The result follows. \square

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Pedro MONTERO

Univ. Grenoble Alpes, Institut Fourier, F-38000 Grenoble, France.