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Minimal time problem for crowd models with localized vector fields

Michel Duprez¹ and Morgan Morancey¹ and Francesco Rossi²

Abstract—In this work, we study the minimal time to steer a crowd to a desired configuration. The control is a vector field, representing a perturbation of the crowd displacement, localized on a fixed control set. We give a characterization of the minimal time both for discrete and continuous crowds.

I. INTRODUCTION AND MAIN RESULTS

In recent years, the study of systems describing a crowd of interacting autonomous agents has drawn a great interest from the control community. A better understanding of such interaction phenomena can have a strong impact in several key applications, such as road traffic and egress problems for pedestrians. For a few reviews about this topic, see *e.g.* [1]–[8]. Beside the description of interactions, it is now relevant to study problems of control of crowds, *i.e.* of controlling such systems by acting on few agents, or on a small subset of the configuration space.

The nature of the control problem relies on the model used to describe the crowd. Two main classes are widely used.

In discrete models, the position of each agent is clearly identified; the crowd dynamics is described by a large dimensional ordinary differential equation, in which couplings of terms represent interactions. For control of such models, a large literature is available, see *e.g.* reviews [9]–[11], as well as applications, both to pedestrian crowds [12], [13] and to road traffic [14], [15].

In continuous models, instead, the idea is to represent the crowd by the spatial density of agents; in this setting, the evolution of the density solves a partial differential equation of transport type. Nonlocal terms (such as convolutions) model the interactions between the agents. To our knowledge, there exist few studies of control of this family of equations. In [16], the authors provide approximate alignment of a crowd described by the continuous Cucker-Smale model [17]. The control is the acceleration, and it is localized in a control region ω which moves in time. In a similar situation, a stabilization strategy has been established in [18], [19], by generalizing the Jurdjevic-Quinn method to partial differential equations.

In this article, we first study a discrete model, where the crowd is represented by a vector with nd components ($n, d \in$

\mathbb{N}^*) representing the positions of n agents in \mathbb{R}^d . The natural (uncontrolled) vector field is denoted by $v : \mathbb{R}^d \rightarrow \mathbb{R}^d$, assumed Lipschitz and uniformly bounded. We act on the vector field in a fixed portion ω of the space, which will be a nonempty open convex subset of \mathbb{R}^d . The admissible controls are thus functions of the form $\mathbb{1}_\omega u : \mathbb{R}^d \times \mathbb{R}^+ \rightarrow \mathbb{R}^d$. We then consider the following ordinary differential equation (discrete model)

$$\begin{cases} \dot{x}_i(t) = (v + \mathbb{1}_\omega u)(x_i(t), t) \text{ for a.e. } t \geq 0, \\ x_i(0) = x_i^0 \end{cases} \quad (1) \quad \boxed{\text{eq ODE}}$$

for $i \in \{1, \dots, n\}$, where $X^0 := \{x_1^0, \dots, x_n^0\}$ is the initial configuration of the crowd.

We also study a continuous model, where the crowd is represented by its density, that is a time-evolving measure $\mu(t)$ defined on the space \mathbb{R}^d . We consider the same natural vector field v , control region ω and admissible controls $\mathbb{1}_\omega u$. We then consider the following linear transport equation (continuous model)

$$\begin{cases} \partial_t \mu + \nabla \cdot ((v + \mathbb{1}_\omega u)\mu) = 0 & \text{in } \mathbb{R}^d \times \mathbb{R}^+, \\ \mu(\cdot, 0) = \mu^0 & \text{in } \mathbb{R}^d, \end{cases} \quad (2) \quad \boxed{\text{eq:transport}}$$

where μ^0 is the initial density of the crowd. The function $v + \mathbb{1}_\omega u$ represents the vector field acting on μ .

Systems (1) and (2) are first approximations for crowd modeling, since the uncontrolled vector field v is given, and it does not describe interactions between agents. Nevertheless, it is necessary to understand control properties for such simple equations as a first step, before dealing with vector fields depending on the crowd itself. Thus, in a future work, we will study control problems for crowd models with a non-local term $v[\mu]$, based on the results for linear systems presented here.

To a discrete configuration $\{x_1, \dots, x_n\}$, we can associate the empirical measures

$$\mu := \sum_{i=1}^n \frac{1}{n} \delta_{x_i}.$$

With this notation, System (1) is a particular case of System (2). This identification will be used several times in the following, namely to approximate continuous crowds with discrete ones.

We now recall the notion of approximate and exact controllability for Systems (1) and (2). We say that they are *approximately controllable* from the initial configuration from μ^0 to the final one μ^1 on the time interval $[0, T]$ if we can steer the solution from μ^0 at time 0 to a configuration at time T as close to the final configuration as we want with

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an appropriate control $\mathbb{1}_\omega u$. Similarly, *exact controllability* means that we can steer the solution from μ^0 at time 0 exactly to μ^1 at time T . In Definition 5 below, we give a formal definition of the notion of approximate controllability in terms of Wasserstein distance.

In all this paper, we assume that the following geometrical condition is satisfied:

(cond1) **CONDITION 1.1 (Geometrical condition):** Let μ^0, μ^1 be two probability measures on \mathbb{R}^d satisfying:

- (i) For each $x^0 \in \text{supp}(\mu^0)$, there exists $t^0 > 0$ such that $\Phi_{t^0}^v(x^0) \in \omega$, where Φ_t^v is the flow associated to v (see Definition 3 below).
- (ii) For each $x^1 \in \text{supp}(\mu^1)$, there exists $t^1 > 0$ such that $\Phi_{-t^1}^v(x^1) \in \omega$.

Condition 1.1 means that particle crosses the control region. It is the minimal condition that we can expect to steer any initial condition to any target. Indeed, if Item (i) of Condition 1.1 is not satisfied, then there exists a whole subpopulation of the measure μ^0 or μ^1 that never intersects the control region, thus we cannot act on it.

We have proved in [20] that if we consider μ^0, μ^1 two probability measures on \mathbb{R}^d compactly supported, absolutely continuous with respect to the Lebesgue measure and satisfying Condition 1.1, then there exists T such that System (2) is **approximately controllable** at time T from μ^0 to μ^1 with a control $\mathbb{1}_\omega u : \mathbb{R}^d \times \mathbb{R}^+ \rightarrow \mathbb{R}^d$ uniformly bounded, Lipschitz in space and measurable in time.

For arbitrary continuous measures, one can expect approximate controllability only, since for general measures there exists no homeomorphism sending one to another. Indeed, if we impose the classical Carathéodory condition of $\mathbb{1}_\omega u$ being Lipschitz in space, measurable in time and uniformly bounded, then the flow $\Phi_t^{v+\mathbb{1}_\omega u}$ is an homeomorphism (see [21, Th. 2.1.1]). Similarly, in the discrete case, such control vector field u cannot separate points, due to uniqueness of the solution of (1). We then assume that both the configuration X^0 and X^1 are disjoint, in the following sense.

DEFINITION 1: A configuration $X = \{x_1, \dots, x_n\}$ is said to be disjoint if $x_i \neq x_j$ for all $i \neq j$. (th opt)

Consider the quantity

$$T^* := \sup\{t^i(x) \text{ s.t. } x \in \text{supp}(\mu^i) \text{ and } i = 0, 1\},$$

where, for all $x \in \mathbb{R}^d$,

$$\begin{cases} t^0(x) := \inf\{t \in \mathbb{R}^+ : \Phi_t^v(x) \in \omega\}, \\ t^1(x) := \inf\{t \in \mathbb{R}^+ : \Phi_{-t}^v(x) \in \omega\}. \end{cases}$$



In this article, we aim to study the minimal time problem. We denote by T_a the minimal time to **approx.** steer the initial configuration μ^0 to a final one μ^1 in the following sense: it is the infimum of time $T > T^*$ at which the considered system is approx. controllable from μ^0 to μ^1 with a control $\mathbb{1}_\omega u : \mathbb{R}^d \times \mathbb{R}^+ \rightarrow \mathbb{R}^d$ uniformly bounded, Lipschitz in space and measurable in time. We similarly define the minimal time T_e to exactly steer the initial configuration μ^0 to a final one μ^1 . Since the minimal time is not always reached, we will speak about *infimum time*.

A. Infimum time for discrete crowds

We denote by $t_i^0 := t^0(x_i^0)$ and $t_i^1 := t^1(x_i^1)$, for $i \in \{1, \dots, n\}$. We now state our first main result.

THEOREM 1.1: (Main result - discrete crowd) Let $X^0 := \{x_1^0, \dots, x_n^0\}$ and $X^1 := \{x_1^1, \dots, x_n^1\}$ be disjoint configurations, satisfying Condition 1.1. Arrange the sequences $\{t_i^0\}_i$ and $\{t_j^1\}_j$ to be increasingly and decreasingly ordered, respectively. Then the infimum time for exact control of System (1) satisfies

$$T_e = M(X^0, X^1) := \max_{i \in \{1, \dots, n\}} |t_i^0 + t_i^1|. \quad (3) \text{minimal time}$$

We give a proof of Theorem 1.1 in Section III. We only consider the case $T > T^*$, that is when all particles of X^0 has entered in ω , so we can act on them (idem for X^1). When $T \in (0, T^*)$ or $T = M$, System (1) can be controllable in some specific cases only (see [22, Rem. 3 and 4]).



B. Infimum time for continuous crowds

Introduce the maps \mathcal{F}_0 and \mathcal{F}_1 defined for all $t \geq 0$ by

$$\begin{cases} \mathcal{F}_0(t) := \mu^0(\{x^0 \in \text{supp}(\mu^0) : t^0(x^0) \leq t\}), \\ \mathcal{F}_1(t) := \mu^1(\{x^1 \in \text{supp}(\mu^1) : t^1(x^1) \leq t\}). \end{cases}$$

The function \mathcal{F}_0 (resp. \mathcal{F}_1) gives the quantity of mass coming from μ^0 (resp. the quantity of mass coming from μ^1 backward in time) which has entered in ω at time t . Observe that we do not decrease \mathcal{F}_0 when the mass eventually leaves ω , and similarly for \mathcal{F}_1 . Define the generalised inverse functions \mathcal{F}_0^{-1} and \mathcal{F}_1^{-1} of \mathcal{F}_0 and \mathcal{F}_1 given for all $m \in [0, 1]$ by

$$\begin{cases} \mathcal{F}_0^{-1}(m) := \inf\{t \geq 0 : \mathcal{F}_0(t) \geq m\}, \\ \mathcal{F}_1^{-1}(m) := \inf\{t \geq 0 : \mathcal{F}_1(t) \geq m\}. \end{cases} \quad (4) \text{def } \mathcal{F}_0 \mathcal{F}_1$$

The function \mathcal{F}_0^{-1} is increasing, lower semi-continuous and gives the time at which a mass m has entered in ω , and similarly for \mathcal{F}_1^{-1} . We then have the following main result about infimum time in the continuous case:

THEOREM 1.2 (Main result - continuous crowd): Let μ^0 and μ^1 be two probability measures, with compact support, absolutely continuous with respect to the Lebesgue measure and satisfying Condition 1.1. Then the infimum time T_a to **approx.** steer μ^0 to μ^1 is equal to

$$S(\mu^0, \mu^1) := \sup_{m \in [0, 1]} \{\mathcal{F}_0^{-1}(m) + \mathcal{F}_1^{-1}(1 - m)\}. \quad (5) \text{def } T_0$$

We give a proof of Theorem 1.2 in Section IV. We observe that S in (5) is the continuous equivalent of M in (3). We remark that, in Item (1) of the definition of the minimal time, the interval (T^*, T_c) makes sense since $M \geq \mathcal{F}_0^{-1}(1) \geq T^*$. As in the discrete case, for the same reason, we only consider the case $T > T^*$ and $T \neq M$ (see [22, Rem. 5 and 6]).

This paper is organised as follow. In Section II, we recall basic properties of the Wasserstein distance, the ordinary differential equations and the continuity equation. We prove our main results Theorem 1.1 in Section III and Theorem 1.2 in IV.

II. THE WASSERSTEIN DISTANCE

In this section, we recall some properties of the Wasserstein distance and its connections with dynamics (1) and (2). We denote by $\mathcal{P}_c(\mathbb{R}^d)$ the space of probability measures in \mathbb{R}^d with compact support.

DEFINITION 2: For $\mu, \nu \in \mathcal{P}_c(\mathbb{R}^d)$, we denote by $\Pi(\mu, \nu)$ the set of *transference plans* from μ to ν , i.e. the probability measures on $\mathbb{R}^d \times \mathbb{R}^d$ with first marginal μ and second marginal ν . Let $p \in [1, \infty)$ and $\mu, \nu \in \mathcal{P}_c(\mathbb{R}^d)$. Define

$$W_p(\mu, \nu) = \inf_{\pi \in \Pi(\mu, \nu)} \left\{ \left(\iint_{\mathbb{R}^d \times \mathbb{R}^d} |x - y|^p d\pi \right)^{1/p} \right\}, \quad (6)$$

$$W_\infty(\mu, \nu) := \inf \{ \pi - \text{esssup} |x - y| : \pi \in \Pi(\mu, \nu) \}. \quad (7)$$

This is the idea of *optimal transportation*, consisting in finding the optimal way to transport mass from a given measure to another. For a thorough introduction, see e.g. [23]. These distances satisfy some useful properties.

PROPERTY 2.1 (see [23, Chap. 7] and [24]): For all $\mu, \nu \in \mathcal{P}_c(\mathbb{R}^d)$, the infima in (6) or (7) are achieved by at least one minimizer $\pi \in \Pi(\mu, \nu)$.

For $p \in [1, \infty]$, W_p is a distance on $\mathcal{P}_c(\mathbb{R}^d)$, called the **Wasserstein distance**. Moreover, for $p \in [1, \infty)$, the topology induced by the Wasserstein distance W_p on $\mathcal{P}_c(\mathbb{R}^d)$ coincides with the weak topology.

The Wasserstein distance can be extended to all pairs of measures μ, ν compactly supported with the same total mass $\mu(\mathbb{R}^d) = \nu(\mathbb{R}^d) \neq 0$, by the formula

$$W_p(\mu, \nu) = |\mu|^{1/p} W_p \left(\frac{\mu}{|\mu|}, \frac{\nu}{|\nu|} \right).$$

In the rest of the paper, the following properties of the Wasserstein distance will be helpful.

PROPERTY 2.2 (see [23], [25]): Let μ, ρ, ν, η be four positive measures compactly supported satisfying $\mu(\mathbb{R}^d) = \nu(\mathbb{R}^d)$ and $\rho(\mathbb{R}^d) = \eta(\mathbb{R}^d)$. For each $p \in [1, \infty]$, it holds

$$W_p^p(\mu + \rho, \nu + \eta) \leq W_p^p(\mu, \nu) + W_p^p(\rho, \eta).$$

Consider the following system

$$\begin{cases} \partial_t \mu + \nabla \cdot (w\mu) = 0 & \text{in } \mathbb{R}^d \times \mathbb{R}^+, \\ \mu(\cdot, 0) = \mu^0 & \text{in } \mathbb{R}^d, \end{cases} \quad (8)$$

where $w : \mathbb{R}^d \times \mathbb{R}^+ \rightarrow \mathbb{R}^d$. This equation is called the **continuity equation**. We now introduce the flow associated to System (8).

DEFINITION 3: We define the **flow** associated to the vector field w as the application $(x^0, t) \mapsto \Phi_t^w(x^0)$ such that, for all $x^0 \in \mathbb{R}^d$, $t \mapsto \Phi_t^w(x^0)$ is the solution to

$$\begin{cases} \dot{x}(t) = w(x(t), t) \text{ for a.e. } t \geq 0, \\ x(0) = x^0. \end{cases} \quad (9)$$

We denote by Γ the set of the Borel maps $\gamma : \mathbb{R}^d \rightarrow \mathbb{R}^d$. We first recall the definition of the *push-forward* of a measure and of the Wasserstein distance.

DEFINITION 4: For a $\gamma \in \Gamma$, we define the *push-forward* $\gamma\#\mu$ of a measure μ of \mathbb{R}^d as follows:

$$(\gamma\#\mu)(E) := \mu(\gamma^{-1}(E)),$$

for every subset E such that $\gamma^{-1}(E)$ is μ -measurable.

PROPERTY 2.3 (see [25]): Let $\mu, \nu \in \mathcal{P}_c(\mathbb{R}^d)$ and $w : \mathbb{R}^d \times \mathbb{R} \rightarrow \mathbb{R}^d$ be a vector field uniformly bounded, Lipschitz in space and measurable in time with a Lipschitz constant equal to L . For each $t \in \mathbb{R}$ and $p \in [1, \infty)$, it holds

$$W_p(\Phi_t^w\#\mu, \Phi_t^w\#\nu) \leq e^{\frac{(p+1)}{p}L|t|} W_p(\mu, \nu). \quad (10)$$

We denote by “AC measures” the measures which are absolutely continuous with respect to the Lebesgue measure and by $\mathcal{P}_c^{ac}(\mathbb{R}^d)$ the subset of $\mathcal{P}_c(\mathbb{R}^d)$ of AC measures. We now recall a standard result linking (9) and (8), known as the method of characteristics.

THEOREM 2.1 (see [23, Th. 5.34]): Let $T > 0$, $\mu^0 \in \mathcal{P}_c(\mathbb{R}^d)$ and w a vector field uniformly bounded, Lipschitz in space and measurable in time. Then, System (8) admits a unique solution μ in $\mathcal{C}^0([0, T]; \mathcal{P}_c(\mathbb{R}^d))$, where $\mathcal{P}_c(\mathbb{R}^d)$ is equipped with the weak topology. Moreover:

- (i) it holds $\mu(\cdot, t) = \Phi_t^w\#\mu^0$;
- (ii) if $\mu^0 \in \mathcal{P}_c^{ac}(\mathbb{R}^d)$, then $\mu(\cdot, t) \in \mathcal{P}_c^{ac}(\mathbb{R}^d)$.

We now give the precise notions of approximate controllability for System (2) in terms of the Wasserstein distance.

DEFINITION 5: We say that System (2) is **approximately controllable** from μ^0 to μ^1 on the time interval $(0, T)$ if for each $\varepsilon > 0$ there exists a control $\mathbb{1}_\omega u$ such that the corresponding solution μ to System (2) satisfies

$$W_p(\mu^1, \mu(T)) \leq \varepsilon.$$

All the Wasserstein distances W_p are equivalent for $p \in [1, \infty)$, see [23]. Thus, from now on we study approximate controllability with the Wasserstein distance W_1 .

III. INFIMUM TIME IN THE DISCRETE CASE

In this section, we prove Theorem 1.1, i.e. the infimum time in the discrete case. Moreover, we prove some additional results which will be useful in Section IV. We first obtain the following result:

PROPOSITION 3.1: Let $X^0 := \{x_1^0, \dots, x_n^0\} \subset \mathbb{R}^d$ and $X^1 := \{x_1^1, \dots, x_n^1\} \subset \mathbb{R}^d$ be disjoint, satisfying Condition 1.1. Then the infimum time T_e to **exact** control System (1) is equal to

$$\widetilde{M}(X^0, X^1) = \min_{\sigma \in S_n} \max_{i \in \{1, \dots, n\}} |t_i^0 + t_{\sigma(i)}^1|. \quad (11)$$

Proof: Let $T := \widetilde{M}(X^0, X^1) + \delta$ with $\delta > 0$. Consider the sequences $\{t_i^0\}_{i \in \{1, \dots, n\}}$ and $\{t_i^1\}_{i \in \{1, \dots, n\}}$ given at the beginning of section (I-A). For all $i \in \{1, \dots, n\}$, there exist $s_i^0 \in (t_i^0, t_i^0 + \delta/3)$ and $s_i^1 \in (t_i^1, t_i^1 + \delta/3)$ such that

$$y_i^0 := \Phi_{s_i^0}^v(x_i^0) \in \omega \text{ and } y_i^1 := \Phi_{-s_i^1}^v(x_i^1) \in \omega.$$

For all $i, j \in \{1, \dots, n\}$, we define the cost

$$K_{ij} := \begin{cases} \|(y_i^0, s_i^0) - (y_j^1, T - s_j^1)\|_{\mathbb{R}^{d+1}} & \text{if } s_i^0 < T - s_j^1, \\ \infty & \text{otherwise.} \end{cases}$$

Consider the minimization problem:

$$\inf_{\pi \in \mathcal{B}_n} \frac{1}{n} \sum_{i,j=1}^n K_{ij} \pi_{ij}, \quad (12) \text{eq: inf}$$

where \mathcal{B}_n is the set of the bistochastic $n \times n$ matrices, *i.e.* the matrices $\pi := (\pi_{ij})_{1 \leq i,j \leq n}$ satisfying, for all $i, j \in \{1, \dots, n\}$, $\sum_{i=1}^n \pi_{ij} = 1$, $\sum_{j=1}^n \pi_{ij} = 1$, $\pi_{ij} \geq 0$. Using the definition of $\widetilde{M}(X^0, X^1)$, the infimum in (12) is attained. It is a linear minimization problem on the closed convex set \mathcal{B}_n . Hence, as a consequence of Krein-Milman's Theorem (see [26]), the functional (12) admits a minimum at a extremal point, *i.e.* permutation matrix. Let σ be a permutation, for which the associated matrix minimizes (12). Consider the linear applications y_i equal to y_i^0 at time s_i^0 and to $y_{\sigma(i)}^1$ at time $T - s_{\sigma(i)}^1$ defined by

$$y_i(t) := \frac{T - s_{\sigma(i)}^1 - t}{T - s_{\sigma(i)}^1 - s_i^0} y_i^0 + \frac{t - s_i^0}{T - s_{\sigma(i)}^1 - s_i^0} y_{\sigma(i)}^1. \quad (\text{corol T0eps})$$

We now prove by contradiction that these trajectories have no intersection: Assume that there are i and j such that the associated trajectories $y_i(t)$ and $y_j(t)$ intersect. If we associate x_i^0 and x_j^0 to $x_{\sigma(j)}^0$ and $x_{\sigma(i)}^0$, *i.e.* we consider the permutation $\mathcal{T}_{i,j} \circ \sigma$, where $\mathcal{T}_{i,j}$ is the transposition between the i^{th} and the j^{th} elements, then the associated cost (12) is strictly smaller than the cost associated to σ .

This is in contradiction with the fact that σ minimizes (12). We conclude taking a control u around trajectories of class \mathcal{C}^∞ steering X^0 to X^1 .

We now choose $T < \widetilde{M}(X^0, X^1)$. Then, for each permutation σ there exists $i \in \{1, \dots, n\}$ such that $t_i^0 + t_{\sigma(i)}^1 > T$. Hence, for each strategy steering X^0 to X^1 there exists a particle which has not enough time to travel from the initial to the final position, thus one cannot steer X^0 to X^1 . ■

Formula (11) leads to the proof of Theorem 1.1.

Proof of Theorem 1.1: Consider $\widetilde{M}(X^0, X^1)$ given in (11). We assume that the sequence $\{t_i^0\}_{i \in \{1, \dots, n\}}$ is increasingly ordered. Let σ_0 be a minimizing permutation in (11), and k_1 such that $t_{\sigma_0(k_1)}^1$ is a maximiser of $\{t_{\sigma_0(1)}^1, \dots, t_{\sigma_0(n)}^1\}$. We denote by $\sigma_1 := \mathcal{T}_{1,k_1} \circ \sigma_0$. We clearly have

$$\max\{t_1^0 + t_{\sigma_0(1)}^1, t_1^0 + t_{\sigma_0(k_1)}^1, t_{k_1}^0 + t_{\sigma_0(1)}^1\} \leq t_{k_1}^0 + t_{\sigma_0(k_1)}^1.$$

Thus σ_1 minimizes (11) too. We build recursively the sequence of permutations $\sigma_{i+1} = \mathcal{T}_{i+1, k_{i+1}} \circ \sigma_i$, where k_i is a maximizer of $\{t_{\sigma_i(i+1)}^1, \dots, t_{\sigma_i(n)}^1\}$. The sequence $\{t_{\sigma_n(1)}^1, \dots, t_{\sigma_n(n)}^1\}$ is then decreasing and σ_n is a minimizing permutation in (11). We deduce that $\widetilde{M}(X^0, X^1) = M(X^0, X^1)$. ■

IV. INFIMUM TIME FOR AC MEASURES

In this section, we prove main Theorem 1.2 about infimum time for AC measures. We first introduce the auxiliary Corollary 4.1 and Proposition 4.1, that are its natural counterparts for discrete measures. We then prove the main theorem by discrete approximation.

Let $M > 0$ be a positive mass, not necessarily 1, and μ^0, μ^1 be two disjoint measures given by

$$\mu^0 := \sum_{i=1}^n \frac{M}{n} \delta_{x_i^0} \text{ and } \mu^1 := \sum_{i=1}^n \frac{M}{n} \delta_{x_i^1}. \quad (13) \text{def mu dim}$$

It is possible to compute the infimum time to steer μ^0 to μ^1 up to a small mass.

DEFINITION 6 (Minimal time up to small mass): Let $X^0 := \{x_1^0, \dots, x_n^0\} \subset \mathbb{R}^d$ and $X^1 := \{x_1^1, \dots, x_n^1\} \subset \mathbb{R}^d$ be disjoint, and satisfying Condition 1.1. Let $M > 0$ and the corresponding measures. Fix $R \in \{1, \dots, n\}$ and $\varepsilon := MR/n$. We define the minimal time $T_{e,\varepsilon}$ to exactly steer X^0 to X^1 up to R particles as the infimum of time $T \geq T^*$ for which there exists a control $\mathbb{1}_{\omega} u : \mathbb{R}^d \times \mathbb{R}^+ \rightarrow \mathbb{R}^d$ uniformly bounded, Lipschitz in space and measurable in time and $\sigma_0, \sigma_1 \in S_n$ such that for all $i \in \{1, \dots, n - R\}$ it holds $x_{\sigma_0(i)}(T) = x_{\sigma_1(i)}^1$.

Using the definition of $\mathcal{F}_0, \mathcal{F}_1, t_i^0$ and t_i^1 it holds:

COROLLARY 4.1: Let $X^0 := \{x_1^0, \dots, x_n^0\} \subset \mathbb{R}^d$ and $X^1 := \{x_1^1, \dots, x_n^1\} \subset \mathbb{R}^d$ be disjoint, satisfying Condition 1.1. Let μ^0 and μ^1 be the measures given in (13). Fix $\varepsilon := MR/n$ with $R \in \{1, \dots, n\}$. The infimum time $T_{e,\varepsilon}$ to exactly steer X^0 to X^1 up to R particles is equal to

$$S_\varepsilon(\mu^0, \mu^1) := \sup_{m \in [0, 1 - \varepsilon]} \{\mathcal{F}_0^{-1}(m) + \mathcal{F}_1^{-1}(1 - \varepsilon - m)\}, \quad (14) \text{def T0eps}$$

where \mathcal{F}_0^{-1} and \mathcal{F}_1^{-1} are given in (4).

PROPOSITION 4.1: Consider $\mu^0, \mu^1 \in \mathcal{P}_c^{ac}(\mathbb{R}^d)$ satisfying Condition 1.1, sequences $\{\mu_n^0\}_{n \in \mathbb{N}}, \{\mu_n^1\}_{n \in \mathbb{N}}$ of measures compactly supported satisfying Condition 1.1 and two sequences of sets $\{R_n^0\}_{n \in \mathbb{N}}, \{R_n^1\}_{n \in \mathbb{N}}$ of \mathbb{R}^d such that

$$\begin{cases} r_n := \mu^0(R_n^0) = \mu^1(R_n^1) \xrightarrow{n \rightarrow \infty} 0, \\ \mu_n^0(\mathbb{R}^d) = \mu_n^1(\mathbb{R}^d) = 1 - r_n, \\ d_n^0 := W_\infty(\mu_{|(R_n^0)^c}^0, \mu_n^0) \xrightarrow{n \rightarrow \infty} 0, \\ d_n^1 := W_\infty(\mu_{|(R_n^1)^c}^1, \mu_n^1) \xrightarrow{n \rightarrow \infty} 0. \end{cases}$$

Consider the quantity S_ε given in (14). Then for all $\varepsilon, \delta > 0$, there exists $N \in \mathbb{N}^*$ such that for all $n \geq N$, it holds

- (i) $S_{2\varepsilon}(\mu_n^0, \mu_n^1) \leq S_\varepsilon(\mu^0, \mu^1) + \delta$.
- (ii) $S_{2\varepsilon}(\mu^0, \mu^1) \leq S_\varepsilon(\mu_n^0, \mu_n^1) + \delta$.

Proof: There exists $r_n^0 \xrightarrow{n \rightarrow \infty} 0$ and $r_n^1 \xrightarrow{n \rightarrow \infty} 0$ such that

$$\begin{aligned} S_{2\varepsilon}(\mu_n^0, \mu_n^1) &= \sup_{m \in [0, 1 - r_n - 2\varepsilon]} \{\mathcal{F}_{0,n}^{-1}(m) + \mathcal{F}_{1,n}^{-1}(1 - r_n - 2\varepsilon - m)\} \\ &\leq \sup_{m \in [0, 1 - r_n - 2\varepsilon]} \{\mathcal{F}_0^{-1}(m + r_n^0 + r_n) \\ &\quad + \mathcal{F}_1^{-1}(1 + r_n^1 - 2\varepsilon - m)\} + \delta \\ &\leq \sup_{m \in [r_n^0 + r_n, 1 + r_n^0 - 2\varepsilon]} \{\mathcal{F}_0^{-1}(m) \\ &\quad + \mathcal{F}_1^{-1}(1 + r_n^1 + r_n^0 + r_n - 2\varepsilon - m)\} + \delta. \end{aligned}$$

Thus, taking n large enough such that $r_n^1 + r_n^0 + r_n \leq \varepsilon$, we deduce Item (i) by using the fact that $\mathcal{F}_1^{-1}(m_1) \leq \mathcal{F}_1^{-1}(m_2)$, for all $m_1 \leq m_2$.

We similarly prove Item (ii). ■

We now prove Theorem 1.2.

Proof of Theorem 1.2: We first prove Item (i). Fix $\varepsilon, s > 0$. We prove that we can steer μ^0 to a W_1 -neighbourhood of μ^1 of size ε at time $T := S(\mu^0, \mu^1) + s$. We assume that $d := 2$, but the reader will see that the proof can be clearly adapted to any space dimension. The proof is divided into four steps.

Step 1: We first discretize uniformly in space the supports of μ^0 and μ^1 . To simplify the presentation, assume $\text{supp}(\mu^0) \subset (0, 1)^2$ and $\text{supp}(\mu^1) \subset (0, 1)^2$. Consider the sequence of uniform meshes $\mathcal{T}_n := \bigcup_{k \in \{0, \dots, n-1\}^2} S_{n,k}$ with

$$S_{n,k} := \prod_{k \in \{0, \dots, n-1\}^2} \left[\frac{k_j}{n}, \frac{k_j + 1}{n} \right). \quad (\text{fig: mesh})$$

Let $x^0 \in \text{supp}(\mu^0)$. There exists $t^*(x^0) \in (t^0(x^0), t^0(x^0) + s/8)$ such that $\Phi_{t^*(x^0)}^v(x^0) \in \omega$. Consider $r(x^0) > 0$ such that $B_{r(x^0)}(\phi_{t^*(x^0)}^v(x^0)) \subset \subset \omega$. By continuity, for a $\hat{r}(x^0)$, we have

$$\phi_{t^*(x^0)}^v(B_{\hat{r}(x^0)}(x^0)) \subset \subset \omega.$$

By compactness of $\text{supp}(\mu^0)$, for a family $\{x_1^0, \dots, x_M^0\}$, we obtain

$$\text{supp}(\mu^0) \subset \bigcup_{i=1}^M B_{\hat{r}(x_i^0)}(x_i^0). \quad (\text{fig: cell})$$

Thus, for n large enough, the associated mesh \mathcal{T}_n satisfies: for all $k \in \{1, \dots, n-1\}^2$ such that $S_{n,k} \cap \text{supp}(\mu^0) \neq \emptyset$ and all $x^0 \in S_{n,k} \cap \text{supp}(\mu^0)$, there exists a time $t_k^0(x^0) \in (t^0(x^0), t^0(x^0) + s/8)$ such that

$$\Phi_{t_k^0(x^0)}^v(S_{n,k}) \subset \subset \omega$$

and for all $\tilde{x}^0 \in S_{n,k} \cap \text{supp}(\mu^0)$,

$$t^0(\tilde{x}^0) \leq t^0(x^0) + s/8. \quad (15) \quad \text{step 2 cont.}$$

We also take n large enough such that the same condition holds for μ^1 .

Step 2: To send a measure to another, these measures need to have the same total mass. Thus, we discretize measures $\mu^0|_{S_{n,k}}$ and $\mu^1|_{S_{n,k}}$ with some measures with the same total mass. As illustrated in Figure 1, we partition $S_{n,k}$ into some subsets $\{A_{kij}^0\}_i$ with $A_{kij}^0 = [a_i^0, a_{i+1}^0] \times (0, 1)$ such that $\mu^0|_{S_{n,k}}(A_{kij}^0) = 1/n^2$ and for all i we partition A_{kij}^0 into some subsets $\{A_{kij}^0\}_{ij}$ with $A_{kij}^0 = [a_i^0, a_{i+1}^0] \times [a_{ij}^0, a_{i(j+1)}^0]$ such that

$$\mu^0|_{S_{n,k}}(A_{kij}^0) = 1/n^4.$$

For more details, we refer to [20]. If $\mu^0(S_{n,k})$ is not a multiple of $1/n^2$, it remains a small mass that we do not control. We discretize similarly the measure μ^1 on some sets A_{kij}^1 . With the same process (see Figure 2), we build $B_{kij}^0 := [b_i^0, b_{i+1}^0] \times [b_{ij}^0, b_{i(j+1)}^0] \subset \subset A_{kij}^0$ and $B_{kij}^1 := [b_i^1, b_{i+1}^1] \times [b_{ij}^1, b_{i(j+1)}^1] \subset \subset A_{kij}^1$ such that

$$\mu^0(B_{kij}^0) = \mu^1(B_{kij}^1) = (n^2 - 2)^2/n^8.$$

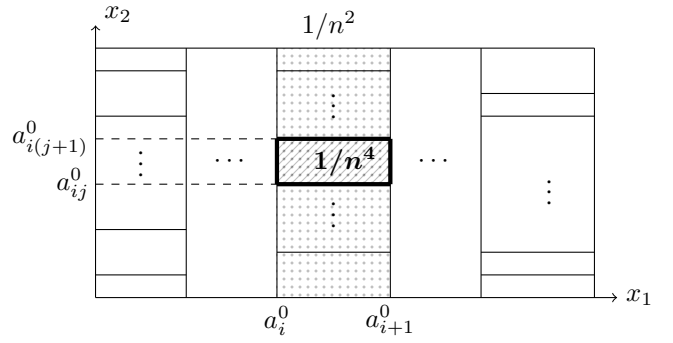


Fig. 1. Example of a partition of $S_{n,k}$ with a cell A_{kij}^0 (hashed).

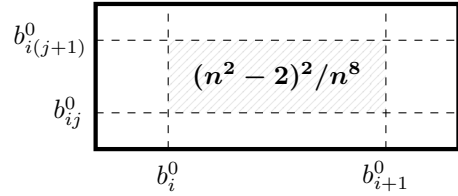


Fig. 2. Example of cells A_{kij}^0 and B_{kij}^0 (hashed).

Step 3: In the rest of the proof, we send the mass of μ^0 from each B_{kij}^0 to each $B_{k'i'j'}^1$, while we do not control the mass in $A_{kij}^0 \setminus B_{kij}^0$. Consider

$$I_n^0 := \{(k, i, j) : \mu^0(S_{n,k}) > 1/n^4\}.$$

We define similarly I_n^1 . Without loss of generality, we can assume that $|I_n^0| = |I_n^1|$. Indeed, for example in the case $n_0 := |I_n^0| - |I_n^1| > 0$, we remove the n_0 last cells in the set of indices I_n^0 , the total corresponding removed mass is smaller than $1/n^2$, then negligible when $n \rightarrow \infty$. We define

$$R_n^0 := \mathbb{R}^d \setminus \bigcup_{kij \in I_n^0} B_{kij}^0 \quad \text{and} \quad R_n^1 := \mathbb{R}^d \setminus \bigcup_{k'i'j' \in I_n^1} B_{k'i'j'}^1.$$

We remark that, for $i = 0, 1$,

$$\mu^i(R_n^i) \leq 1 - \frac{(n^2 - 2)^2}{n^4} + \frac{1}{n^4} = \frac{5(n^2 - 4)}{n^2} \xrightarrow{n \rightarrow \infty} 0.$$

For $i = 0, 1$, consider μ_n^i defined by

$$\mu_n^i := \sum_{kij \in I_n^i} \frac{(n^2 - 2)^2}{n^8} \delta_{x_{kij}^i},$$

where x_{kij}^i is any point of $B_{kij}^i \cap \text{supp}(\mu^i)$. Moreover, using the definition of $S_{n,k}^i$, for $i \in \{0, 1\}$, we have

$$W_\infty(\mu_n^i|_{(R_n^i)^c}, \mu_n^i) \leq \sqrt{2}/n \xrightarrow{n \rightarrow \infty} 0. \quad (16) \quad \text{eq:wass inf}$$

We remark that the measures μ_n^0, μ_n^1 and the sequences $\{\mu_n^0\}_{n \in \mathbb{N}^*}, \{\mu_n^1\}_{n \in \mathbb{N}^*}, \{R_n^0\}_{n \in \mathbb{N}^*}$ and $\{R_n^1\}_{n \in \mathbb{N}^*}$ satisfy the hypothesis of Proposition 4.1. Since

$$S_{\varepsilon/4}(\mu^0, \mu^1) \leq S(\mu^0, \mu^1),$$

applying Proposition 4.1 for $\delta := s/2$, it holds

$$S_{\varepsilon/2}(\mu_n^0, \mu_n^1) + \frac{s}{2} \leq S_{\varepsilon/4}(\mu^0, \mu^1) + s \leq S(\mu^0, \mu^1) + s = T.$$

Using (15), for $i = 0, 1$, there exists $t_k^i(x_{kij}^i) \in (t^i(x_{kij}^i), t^i(x_{kij}^i) + s/8)$ such that $\Phi_{t_k^i(x_{kij}^i)}^v(B_{kij}^i) \subset \subset \omega$.

Up to replace $t^0(x_{kij}^0)$ by $t_k^0(x_{kij}^0)$, Corollary 4.1 leads to the existence of a control u_n such that for the initial data μ_n^0 the associated solution μ_n to System (2) satisfies

$$W_1(\mu_n^1, \mu_n(T)) \leq \varepsilon/2.$$

$$\Phi_T^{u_n}(x_{kij}^0) = x_{\sigma(kij)}^1.$$

Since we have no intersection of the trajectories $\Phi_t^{u_n}(x_{kij}^0)$ there exist $r, R > 0$ such that for all $t \in (t_k^0(x_{kij}^0), T - t_k^1(x_{kij}^1))$

$$B_r(\Phi_t^{u_n}(x_{kij}^0)) \subset B_R(\Phi_t^{u_n}(x_{kij}^0)) \subset \omega$$

and, for all $t \in (0, T)$,

$$\bigcap_{kij \in I_n^0} B_R(\Phi_t^{u_n}(x_{kij}^0)) = \emptyset.$$

If necessary, the final control u_n concentrates the mass of $\mu_{B^0}^0$ in a ball $B_{\tilde{r}}(x_{kij}^0)$ in the time interval

$(t_k^0(x_{kij}^0), t_k^0(x_{kij}^0) + \delta)$, with $\delta > 0$ small enough, such that

$$\Phi_t^{u_n}(B_{\tilde{r}}(x_{kij}^0)) \subset \subset B_r(\Phi_t^{u_n}(x_{kij}^0)) \cap (\Phi_t^{u_n}(B_{kij}^0))$$

for all $t \in (t_k^0(x_{kij}^0) + \delta, T - t_k^1(x_{kij}^1))$. For details, we refer to [20, Prop. 3.3].

Step 4: We now estimate the Wasserstein distance between $\phi_T^{v+1\omega u} \# \mu^0$ and μ^1 . Using Property 2.2, it holds

$$W_1(\phi_T^{v+1\omega u} \# \mu^0, \mu^1) \leq W_1(\phi_T^{v+1\omega u} \# \mu_{|(R_n^0)^c}^0, \mu_{|(R_n^1)^c}^1) + W_1(\phi_T^{v+1\omega u} \# \mu_{|R_n^0}^0, \mu_{|R_n^1}^1).$$

By triangular inequality,

$$\begin{aligned} & W_1(\phi_T^{v+1\omega u} \# \mu_{|(R_n^0)^c}^0, \mu_{|(R_n^1)^c}^1) \\ & \leq W_1(\phi_T^{v+1\omega u} \# \mu_{|(R_n^0)^c}^0, \phi_T^{v+1\omega u} \# \mu_{|(R_n^0)^c}^0) \\ & \quad + W_1(\phi_T^{v+1\omega u} \# \mu_{|(R_n^0)^c}^0, \mu_{|(R_n^1)^c}^1) + W_1(\mu_{|(R_n^1)^c}^1, \mu_{|(R_n^1)^c}^1). \end{aligned}$$

We now estimate each term in the right-hand side in (19). Using inequalities (10) and (16), we have

$$W_1(\phi_T^{v+1\omega u} \# \mu_{|(R_n^0)^c}^0, \phi_T^{v+1\omega u} \# \mu_{|(R_n^0)^c}^0) \leq e^{2LT} \sqrt{2}/n$$

and

$$W_1(\mu_{|(R_n^0)^c}^1, \mu_{|(R_n^0)^c}^1) \leq W_\infty(\mu_{|(R_n^0)^c}^1, \mu_{|(R_n^0)^c}^1) \leq \sqrt{2}/n.$$

Combining (17), (19), (20) and (21), it holds

$$W_1(\phi_T^{v+1\omega u} \# \mu_{|(R_n^0)^c}^0, \mu_{|(R_n^1)^c}^1) \leq \varepsilon/2 + \frac{e^{2LT} \sqrt{2}}{n}.$$

Using Property 2.1, there exists $\pi \in \Pi(\phi_T^{v+1\omega u} \# \mu_{|R_n^0}^0, \mu_{|R_n^1}^1)$ such that

$$\begin{aligned} W_1(\phi_T^{v+1\omega u} \# \mu_{|R_n^0}^0, \mu_{|R_n^1}^1) & = \int_{(0,1)^2} |x - y| d\pi(x, y) \\ & \leq (2 + T \sup |v|) \times 5(n^2 - 4)/n^4. \end{aligned}$$

Combining (18), (22), (23), we obtain

$$\begin{aligned} & W_1(\phi_T^{v+1\omega u} \# \mu^0, \mu^1) \\ & \leq \varepsilon/2 + \sqrt{2}(1 + e^{2LT})/n + (2 + T \sup |v|)5(n^2 - 4)/n^4, \end{aligned}$$

which leads to the conclusion when $n \rightarrow \infty$.

We similarly prove Item (ii). \blacksquare

REFERENCES

- [1] R. Axelrod, *The Evolution of Cooperation*. Basic Books, 2006.
- [2] N. Bellomo, P. Degond, and E. Tadmor, *Active Particles, Volume 1: Advances in Theory, Models, and Applications*, ser. Mod. and Sim. in Sc., Eng. and Tech. Springer International Publishing, 2017.
- [3] S. Camazine, *Self-organization in Biological Systems*, ser. Princeton studies in complexity. Princeton University Press, 2003.
- [4] E. Cristiani, B. Piccoli, and A. Tosin, *Multiscale modeling of pedestrian dynamics*, ser. MS&A. Modeling, Simulation and Applications. Springer, Cham, 2014, vol. 12. [Online]. Available: <https://doi.org/10.1007/978-3-319-06620-2>
- [5] D. Helbing and R. Calek, *Quantitative Sociodynamics: Stochastic Methods and Models of Social Interaction Processes*, ser. Theory and Decision Library B. Springer Netherlands, 2013.
- [6] M. Jackson, *Social and Economic Networks*. Princ. Uni. Pr., 2010.
- [7] S. Motsch and E. Tadmor, "Heterophilous dynamics enhances consensus," *SIAM Review*, vol. 56, no. 4, pp. 577–621, 2014.
- [8] R. Sepulchre, "Consensus on nonlinear spaces," *Annual reviews in control*, vol. 35, no. 1, pp. 56–64, 2011.
- [9] F. Bullo, J. Cortés, and S. Martínez, *Distributed Control of Robotic Networks*, ser. Princeton Series in Applied Mathematics. Princeton University Press, 2009.
- [10] V. Kumar, N. Leonard, and A. Morse, *Cooperative Control: A Post-Workshop Volume, 2003 Block Island Workshop on Cooperative Control*, ser. Lecture Notes in Control and Information Sciences. Springer Berlin Heidelberg, 2004.
- [11] L. Zhiyun et al., "Leader–follower formation via complex laplacian," *Automatica*, vol. 49, no. 6, pp. 1900–1906, 2013.
- [12] A. Ferscha and K. Zia, "Lifebelt: Crowd evacuation based on vibrotactile guidance," *IEEE Perv. Comp.*, vol. 9, no. 4, pp. 33–42, 2010.
- [13] P. B. Luh, C. T. Wilkie, S.-C. Chang, K. L. Marsh, and N. Olderman, "Modeling and optimization of building emergency evacuation considering blocking effects on crowd movement," *IEEE Trans. on Auto. Sc. and Eng.*, vol. 9, no. 4, pp. 687–700, 2012.
- [14] C. Canudas-de-Wit et al., "Graph constrained-CTM observer design for the Grenoble south ring," *IFAC Proceedings Volumes*, vol. 45, no. 24, pp. 197–202, 2012.
- [15] A. Hegyi et al., "Specialist: A dynamic speed limit control algorithm based on shock wave theory," in *Intelligent Transportation Systems, 2008. ITSC 2008. 11th International IEEE Conference on*. IEEE, 2008, pp. 827–832.
- [16] B. Piccoli, F. Rossi, and E. Trélat, "Control to flocking of the kinetic Cucker-Smale model," *SIAM J. Math. Anal.*, vol. 47, no. 6, pp. 4685–4719, 2015.
- [17] F. Cucker and S. Smale, "Emergent behavior in flocks," *IEEE Trans. Automat. Control*, vol. 52, no. 5, pp. 852–862, 2007.
- [18] M. Caponigro, B. Piccoli, F. Rossi, and E. Trélat, "Mean-field sparse Jurdjevic-Quinn control," *M3AS: Math. Models Meth. Appl. Sc.*, vol. 27, no. 7, pp. 1223–1253, 2017.
- [19] —, "Sparse Jurdjevic-Quinn stabilization of dissipative systems," *Automatica*, vol. 86, pp. 110–120, 2017.
- [20] M. Duprez, M. Morancey, and F. Rossi, "Controllability of the continuity equation with a local vector field," *Submitted*, 2017.
- [21] A. Bressan and B. Piccoli, *Introduction to the mathematical theory of control*, ser. AIMS Series on Applied Mathematics, 2007, vol. 2.
- [22] M. Duprez, M. Morancey, and F. Rossi, "Minimal time problem for crowd models with localized vector fields," *Submitted*, 2017.
- [23] C. Villani, *Topics in optimal transportation*, ser. Graduate Studies in Mathematics. AMS, Providence, RI, 2003, vol. 58.
- [24] T. Champion, L. De Pascale, and P. Jutilinen, "The ∞ -Wasserstein distance: local solutions and existence of optimal transport maps," *SIAM J. Math. Anal.*, vol. 40, no. 1, pp. 1–20, 2008.
- [25] B. Piccoli and F. Rossi, "Transport equation with nonlocal velocity in Wasserstein spaces: convergence of numerical schemes," *Acta Appl. Math.*, vol. 124, pp. 73–105, 2013.

KM40 [26] M. Krein and D. Milman, "On extreme points of regular convex sets," *Studia Math.*, vol. 9, pp. 133–138, 1940.