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Time optimal control for continuous mass displacement

Michel Duprez and Francesco Rossi

Abstract— In this work, we study a minimal time problem for a Partial Differential Equation of transport type, that arises in crowd models. The control is a Lipschitz vector field localized on a fixed control set ω .

We provide a complete answer for the minimal time problem. After considering the discrete case, we show that the minimal time to steer one initial configuration to another is related to the condition of having enough mass in ω to feed the desired final configuration.

We also give a numerical method to compute the minimal time and to build the corresponding control. These results are illustrated by some numerical simulations.

I. INTRODUCTION

In recent years, the study of systems describing a crowd of interacting autonomous agents has drawn a great interest from the control community (see e.g. the Cucker-Smale model [3]). A better understanding of such interaction phenomena can have a strong impact in several key applications, such as road traffic and egress problems for pedestrians. Beside the description of interaction, it is now relevant to study problems of **control of crowds**, *i.e.* of controlling such systems by acting on few agents, or with a control localized in a small subset of the configuration space.

Two main classes are widely used to model crowds of interacting agents. In **microscopic models**, the position of each agent is clearly identified; the crowd dynamics is described by a large dimensional ordinary differential equation, in which couplings of terms represent interactions. In **macroscopic models**, instead, the idea is to represent the crowd by the spatial density of agents; in this setting, the evolution of the density solves a partial differential equation of transport type. This is an example of a **distributed parameter system**. Nonlocal terms (*i.e.* of convolution) model the interactions between the agents. In this article, we focus on this second approach, *i.e.* macroscopic models.

To our knowledge, there exist few studies of control of this family of equations. In [7], the authors provide approximate alignment of a crowd described by the macroscopic Cucker-Smale model [3]. The control is the acceleration, and it is localized in a control region ω which moves in time. In a similar situation, a stabilization strategy has been established

in [2], by generalizing the Jurdjevic-Quinn method to distributed parameter systems.

In this article, we consider densities μ defined on the space \mathbb{R}^d , with $d \geq 1$. Let ω be a nonempty open connected subset of \mathbb{R}^d ($d \geq 1$), being the portion of the space on which the control is allowed to act. We denote by \mathcal{U} the set of admissible controls, that are functions $u : \mathbb{R}^d \times \mathbb{R}^+ \rightarrow \mathbb{R}^d$ Lipschitz in space, measurable in time and uniformly bounded with $\text{supp}(u) \subset \bar{\omega}$. Let $v : \mathbb{R}^d \rightarrow \mathbb{R}^d$ be a vector field assumed Lipschitz and uniformly bounded. Consider the following linear transport equation

$$\begin{cases} \partial_t \mu + \nabla \cdot ((v + \mathbb{1}_\omega u)\mu) = 0 & \text{in } \mathbb{R}^d \times \mathbb{R}^+, \\ \mu(\cdot, 0) = \mu^0 & \text{in } \mathbb{R}^d, \end{cases} \quad (1)$$

where $\mu(t)$ is the time-evolving measure representing the crowd density, and μ^0 is the initial data. The control function u belongs to \mathcal{U} . The function $v + \mathbb{1}_\omega u$ represents the velocity field acting on μ . System (1) is a first simple approximation for crowd modeling, since the uncontrolled vector field v is given, and it does not describe interactions between agents. Nevertheless, it is necessary to understand controllability properties for such simple equation as a first step, before dealing with a non-local term $v[\mu]$.

The minimal time problem for System (1) consists, for each initial and final configurations μ^0 and μ^1 , in finding the minimal time T such that it exists a control $u : \mathbb{R}^d \times [0, T] \rightarrow \mathbb{R}^d$ in \mathcal{U} steering μ^0 to μ^1 . For Problem (1), one cannot reach precisely a general μ^1 , as we proved in [4]. Indeed, if we impose the classical Caratheodory condition of u being in \mathcal{U} then the flow $\Phi_t^{v+\mathbb{1}_\omega u}$ is an homeomorphism (see [1, Th. 2.1.1]). As a result, one cannot reach any μ^1 , since for general measures there exists no homeomorphism sending one to another.

Thus, we relax the minimal time problem by considering the following problem: for each initial and final configurations μ^0 and μ^1 , finding the minimal time T such that for all ε it exists a control $u : \mathbb{R}^d \times [0, T] \rightarrow \mathbb{R}^d$ in \mathcal{U} steering μ^0 to an ε -neighbourhood of μ^1 with respect to the Wasserstein distance W_p . The definition of the Wasserstein distance is recalled in Section II.

Even in this relaxed version, one cannot hope to steer any μ^0 to μ^1 , in general. Indeed, from a geometrical point of view, the uncontrolled vector field v needs to send the support of μ^0 to ω forward in time and the support of μ^1 to ω backward in time. This idea is formulated in the following Condition:

Condition 1 (Geometrical condition) Let μ^0, μ^1 be two probability measures on \mathbb{R}^d satisfying:

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- (i) For all $x^0 \in \text{supp}(\mu^0)$, there exists $t_0 > 0$ such that $\Phi_{t_0}^v(x^0) \in \omega$, where Φ_t^v is the flow associated to v , i.e. the solution to the Cauchy problem

$$\begin{cases} \dot{x}(t) = v(x(t)) \text{ for a.e. } t > 0, \\ x(0) = x^0. \end{cases}$$

- (ii) For all $x^1 \in \text{supp}(\mu^1)$, there exists $t_1 > 0$ such that $\Phi_{-t_1}^v(x^1) \in \omega$.

Remark 1 Condition 1 is the minimal one that we can expect to steer any initial condition to any targets. Indeed, if the first item of Condition 1 is not satisfied, there exists a whole sub-population of the measure μ_0 that never intersects the control region. Thus, we cannot act on it, and we cannot steer it to any desired target.

We proved in [4] that for all μ^0, μ^1 two probability measures on \mathbb{R}^d with compact support, absolutely continuous with respect to the Lebesgue measure and satisfying Condition 1, there exists $T > 0$ such that we can steer μ^0 to μ^1 on the time interval $(0, T)$ with a control u in \mathcal{U} .

We aim to study the minimal time problem, i.e. the minimal time to steer μ^0 to μ^1 . We first introduce the following definition:

Definition 1 We say that $T^* > 0$ is an *admissible time* if it satisfies:

- (a) $T^* > T_0^* := \sup\{t^0(x^0) : x^0 \in \text{supp}(\mu^0)\}$, where $t^0(x^0) := \inf\{t \in \mathbb{R}^+ : \Phi_t^v(x^0) \in \omega\}$. (2)
- (b) $T^* > T_1^* := \sup\{t^1(x^1) : x^1 \in \text{supp}(\mu^1)\}$, where $t^1(x^1) := \inf\{t \in \mathbb{R}^+ : \Phi_{-t}^v(x^1) \in \omega\}$.
- (c) $T^* > T_2^*$, where $T_2^* \geq \max\{T_0^*, T_1^*\}$ is the infimum s.t. : There exists a sequence $(u_k)_k$ of \mathcal{C}^∞ -functions equal to 0 in ω^c such that, for all $t \in (0, T_2^*)$,
- $$\lim_{k \rightarrow \infty} [\Phi_t^{v+u_k} \# \mu^0](\omega) \geq 1 - \lim_{k \rightarrow \infty} [\Phi_{t-T_2^*}^{v+u_k} \# \mu^1](\omega).$$

We then have the following main result.

Theorem 1 (Main result) Fix $\varepsilon > 0$ and $p \in (0, \infty]$. Let μ^0, μ^1 be two probability measures, with compact support, absolutely continuous with respect to the Lebesgue measure and satisfying Condition 1. Let T_0 be the infimum of the admissible times T^* . Then T_0 is the optimal time to steer μ^0 to μ^1 , i.e.

- (i) For all $T \geq T_0$, there exists a control $u \in \mathcal{U}$ such that the solution to System (1) satisfies

$$W_p(\mu^1, \mu(T)) \leq \varepsilon. \quad (3)$$

- (ii) For all $T < T_0$, there exists a control $u \in \mathcal{U}$ such that the solution to System (1) satisfies (3).

We give a proof of Theorem 1 in Section IV.

Remark 2 Condition (a) in Definition 1 means that the particles of $\text{supp}(\mu^0)$ need to enter ω if we want to act on it. Idem for Condition (b). The meaning of Condition (c) is the following: functions u_k are used to store the mass in ω . An example of such functions is given in (10). Thus, condition (c) means that at each time t there is more mass

that has entered ω than mass that has left ω . This is the minimal condition that we can expect in this setting, since control can only move masses, without creating them. In this sense, our result shows that this necessary condition is also sufficient.

This paper is organized as follows. In Section II, we recall some properties of the Wasserstein distance and the continuity equation. In Section III, we consider the discrete case. Section IV is devoted to prove the main Theorem 1. In Section V, we propose an algorithm to compute the minimal time and to build the corresponding localized velocity field and give some numerical examples.

II. THE WASSERSTEIN DISTANCE AND THE CONTINUITY EQUATION

In this section, we recall some properties of the continuity equation (1) and the definition of the Wasserstein distance, which will be used all along this paper.

We denote by $\mathcal{P}_c(\mathbb{R}^d)$ the space of probability measures in \mathbb{R}^d with compact support. First of all, we give the definition of the *push-forward* of a measure and the definition of the Wasserstein distance.

Definition 2 Denote by Γ the set of the measurable maps $\gamma : \mathbb{R}^d \rightarrow \mathbb{R}^d$. For a $\gamma \in \Gamma$, we define the *push-forward* $\gamma \# \mu$ of a measure μ of \mathbb{R}^d as follows:

$$(\gamma \# \mu)(E) := \mu(\gamma^{-1}(E)),$$

for all Borel sets E .

For $\mu, \nu \in \mathcal{P}_c(\mathbb{R}^d)$, we denote by $\Pi(\mu, \nu)$ the set of *transference plans* from μ to ν , i.e. the probability measures on $\mathbb{R}^d \times \mathbb{R}^d$ satisfying

$$\int_{\mathbb{R}^d} d\pi(x, \cdot) = d\mu(x) \text{ and } \int_{\mathbb{R}^d} d\pi(\cdot, y) = d\nu(y).$$

Definition 3 Let $p \in [1, \infty)$ and $\mu, \nu \in \mathcal{P}_c(\mathbb{R}^d)$. Define

$$W_p(\mu, \nu) = \inf_{\pi \in \Pi(\mu, \nu)} \left\{ \left(\iint_{\mathbb{R}^d \times \mathbb{R}^d} |x - y|^p d\pi \right)^{1/p} \right\}$$

and

$$W_\infty(\mu, \nu) := \inf\{\pi - \text{esssup}|x - y| : \pi \in \Pi(\mu, \nu)\}.$$

Then, for $p \in [1, \infty]$, W_p is a distance on $\mathcal{P}_c(\mathbb{R}^d)$, called the **Wasserstein distance**. Moreover, the Wasserstein distance can be extended to all pairs of measures $\mu, \nu \in \mathcal{P}_c(\mathbb{R}^d)$ with the same total mass $\mu(\mathbb{R}^d) = \nu(\mathbb{R}^d)$, by the formula

$$W_p(\mu, \nu) = |\mu|^{1/p} W_p \left(\frac{\mu}{|\mu|}, \frac{\nu}{|\nu|} \right).$$

We denote by $\mathcal{P}_c^{ac}(\mathbb{R}^d)$ the subset of $\mathcal{P}_c(\mathbb{R}^d)$ of measures which are absolutely continuous with respect to the Lebesgue measure. On $\mathcal{P}_c^{ac}(\mathbb{R}^d)$, the Wasserstein distance can be reformulated as follows:

Proposition 1 Let $p \in [1, \infty)$ and $\mu, \nu \in \mathcal{P}_c^{ac}(\mathbb{R}^d)$. It holds

$$W_p(\mu, \nu) = \min_{\gamma \in \Gamma} \left\{ \left(\int_{\mathbb{R}^d} |\gamma(x) - x|^p d\mu \right)^{1/p} : \gamma \# \mu = \nu \right\}$$

and

$$W_\infty(\mu, \nu) = \min_{\gamma \in \Gamma} \left\{ \sup_{x \in \mathbb{R}^d} |\gamma(x) - x| : \gamma \# \mu = \nu \right\}.$$

In the rest of the paper, the following properties of the Wasserstein distance will be helpful.

Property 1 ([6]) Let $w : \mathbb{R}^d \times \mathbb{R} \rightarrow \mathbb{R}^d$ be a vector field uniformly bounded, Lipschitz in space and measurable in time with a Lipschitz constant equal to L . For each $t \in \mathbb{R}$ and $p \in \mathbb{R}$, it holds

$$W_p^p(\Phi_t^w \# \mu, \Phi_t^w \# \nu) \leq e^{(p+1)L|t|} W_p^p(\mu, \nu) \quad (4)$$

and for each $p_1, p_2 \in \mathbb{R}$ with $p_1 \leq p_2$, it holds

$$\begin{cases} W_{p_1}(\mu, \nu) \leq W_{p_2}(\mu, \nu), \\ W_{p_2}(\mu, \nu) \leq |\text{supp}(\mu, \nu)|^{1 - \frac{p_1}{p_2}} W_{p_1}(\mu, \nu). \end{cases}$$

Thus, it is equivalent to steer a initial data μ^0 to a W_p -neighbourhood ($p > 0$) or a W_1 -neighbourhood of the target μ^1 .

We now recall a standard result for the continuity equation:

Theorem 2 ([8]) Let $T \in \mathbb{R}^+$, $\mu^0 \in \mathcal{P}_c^{ac}(\mathbb{R}^d)$ and w be a vector field uniformly bounded, Lipschitz in space and measurable in time. Then the system

$$\begin{cases} \partial_t \mu + \nabla \cdot (w\mu) = 0 & \text{in } \mathbb{R}^d \times \mathbb{R}^+, \\ \mu(\cdot, 0) = \mu^0 & \text{in } \mathbb{R}^d \end{cases}$$

admits a unique solution¹ μ in $\mathcal{C}^0([0, T]; \mathcal{P}_c^{ac}(\mathbb{R}^d))$. Moreover, it holds $\mu(\cdot, t) = \Phi_t^w \# \mu^0$ for all $t \in \mathbb{R}$, where the flow $\Phi_t^w(x^0)$ is the unique solution at time t to

$$\begin{cases} \dot{x}(t) = w(t, x(t)) \text{ for a.e. } t \geq 0, \\ x(0) = x^0. \end{cases}$$

III. OPTIMAL TIME IN THE DISCRETE CASE

In this section, we aim to prove Theorem 1 in the discrete case. We want to send n particles (starting from different positions) to n other particles thanks to the system

$$\begin{cases} \dot{x}(t) = (v + \mathbb{1}_\omega u)(t, x(t)) \text{ for a.e. } t \geq 0, \\ x(0) = x^0. \end{cases} \quad (5)$$

Thus we consider here the initial data μ^0 and the target μ^1 as being sums of Dirac masses.

Proposition 2 Let

$$\mu^0 := \sum_{i=1}^n \frac{1}{n} \delta_{x_i^0} \text{ and } \mu^1 := \sum_{i=1}^n \frac{1}{n} \delta_{x_i^1} \quad (6)$$

be two measures of probability satisfying Condition 1 and $x_i^k \neq x_j^k$ for all $k \in \{0, 1\}$ and $i \neq j$. We define

$$\begin{cases} t_i^0 := \inf\{t \in \mathbb{R}^+ : \Phi_t^v(x_i^0) \in \omega\}, \\ t_i^1 := \inf\{t \in \mathbb{R}^+ : \Phi_{-t}^v(x_i^1) \in \omega\}. \end{cases} \quad (7)$$

Then the minimal time of System (1) is given by:

$$T_1 = \min_{\sigma \in S_n} \max_{i \in \{1, \dots, n\}} |t_i^0 + t_{\sigma(i)}^1|. \quad (8)$$

i.e.

¹Here, $\mathcal{P}_c^{ac}(\mathbb{R}^d)$ is equipped with the weak topology, that coincides with the topology induced by the Wasserstein distance W_p , see [8, Thm 7.12].

- (i) For all $T \geq T_1$, there exists a control $u \in \mathcal{U}$ such that the solution to System (1) satisfies (3).
- (ii) For all $T < T_1$, there exists a control $u \in \mathcal{U}$ such that the solution to System (1) satisfies (3).

Remark 3 In definition (6) of μ^0 and μ^1 , we assume that the points of the family $\{x_1^0, \dots, x_n^0\}$ and $\{x_1^1, \dots, x_n^1\}$ are disjoint. Indeed, if this is not the case, the control vector field u cannot separate points, due to uniqueness of the solution of the ODE. Thus, if $x_i^0 = x_j^0$, the corresponding trajectory of (5) satisfies $x_i(t) = x_j(t)$. For more details, we refer to [4].

Remark 4 Condition (7) can be interpreted as follows: Each particle at point x_i^0 needs to be sent on a target point $x_{\sigma(i)}^1$. T_1 is larger than the minimal time for the particle at x_i^0 to enter in ω plus the minimal time for this particle to go from ω to $x_{\sigma(i)}^1$ (We assume that the particle travels with a quasi infinite velocity in ω).

Proof: Let $T := T_1 + \delta$ with $\delta > 0$. Consider σ the optimal permutation in the definition of T_1 . Consider the sequences $\{t_i^0\}_{i \in \{1, \dots, n\}}$ and $\{t_i^1\}_{i \in \{1, \dots, n\}}$ given in (7) and define, for all $i \in \{1, \dots, n\}$,

$$y_i^0 := \Phi_{t_i^0}^v(x_i^0) \text{ and } y_i^1 := \Phi_{-t_i^1}^v(x_i^1).$$

The proof is divided into two steps:

- (i) In a first step, we build a permutation σ_0 and a flow on ω sending y_i^0 to $y_{\sigma_0(i)}^1$ for all $i \in \{1, \dots, n\}$ with no intersection of these trajectories.
- (ii) In a second step, we deduce a flow sending $\frac{1}{n} \delta_{x_i^0}$ to $\frac{1}{n} \delta_{x_{\sigma_0(i)}^1}$ for all $i \in \{1, \dots, n\}$.

Step 1: For all $i, j \in \{1, \dots, n\}$, we define the cost

$$K_{ij} := \begin{cases} \|(y_i^0, t_i^0) - (y_j^1, t_j^1)\|_{\mathbb{R}^{d+1}} & \text{if } t_i^0 < t_j^1, \\ \infty & \text{otherwise.} \end{cases}$$

Consider the minimisation problem:

$$\inf_{\pi_{ij} \in \mathbb{R}^{n^2}} \left\{ \frac{1}{n} \sum_{i,j=1}^n K_{ij} \pi_{ij} : \pi_{ij} \in \mathcal{B}_n \right\}, \quad (9)$$

where \mathcal{B}_n is the set of the bistochastic $n \times n$ matrices, i.e. the matrices satisfying, for all $i, j \in \{1, \dots, n\}$,

$$\sum_{i=1}^n \pi_{ij} = 1, \quad \sum_{j=1}^n \pi_{ij} = 1, \quad \pi_{ij} \geq 0.$$

Using the definition of T_1 , the infimum is finite. It is a linear minimisation problem on the convex set \mathcal{B}_n . By Krein-Milman's theorem (see [5]) the minima are the permutations matrices. Let σ_0 be a permutation which associated matrix minimizes (9). Consider the linear applications equal to y_i^0 at time t_i^0 and to $y_{\sigma_0(i)}^1$ at time $T - t_{\sigma_0(i)}^1$ defined by

$$y_i(t) := \frac{T - t_{\sigma_0(i)}^1 - t}{T - t_{\sigma_0(i)}^1 - t_i^0} y_i^0 + \frac{t - t_i^0}{T - t_{\sigma_0(i)}^1 - t_i^0} y_{\sigma_0(i)}^1.$$

We can prove by contradiction that these trajectories have no intersection: Assume that there are i and j such that the trajectories $y_i(t)$ and $y_j(t)$ intersect.

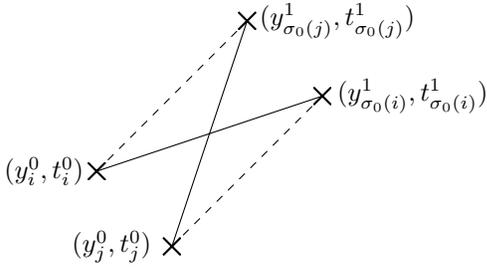


Fig. 1. Optimal permutation

Using some geometrical considerations (see Figure 1), we obtain

$$\begin{cases} \|(y_i^0, t_i^0) - (y_{\sigma_0(j)}^1, t_{\sigma_0(j)}^1)\| < \|(y_i^0, t_i^0) - (y_{\sigma_0(i)}^1, t_{\sigma_0(i)}^1)\|, \\ \|(y_j^0, t_j^0) - (y_{\sigma_0(i)}^1, t_{\sigma_0(i)}^1)\| < \|(y_j^0, t_j^0) - (y_{\sigma_0(j)}^1, t_{\sigma_0(j)}^1)\|. \end{cases}$$

This is in contradiction with the fact that σ_0 minimizes (9).

Step 2: We construct the flow as follows:

$$z_i(t) := \begin{cases} \Phi_t^v(x_i^0) & \text{for all } t \in (0, t_i^0), \\ y_i(t) & \text{for all } t \in (t_i^0, T - t_{\sigma_0(i)}^1), \\ \Phi_{t-T+t_{\sigma_0(i)}^1}^v(y_i^0) & \text{for all } t \in (T - t_{\sigma_0(i)}^1, T). \end{cases}$$

The applications z_i have no intersection. If ω is convex, then, using the definition of the application y_i , the points $y_i(t)$ belongs to ω for all $t \in (t_i^0, T - t_{\sigma_0(i)}^1)$. If ω is not convex, there exists an homeomorphism φ such that $\varphi(y_i(t)) \in \omega$ for all $t \in (t_i^0, T - t_{\sigma_0(i)}^1)$.

The corresponding control can be chosen equal to

$$u(x, t) := \frac{y_{\sigma_0(i)}^1 - y_i^0}{T - t_{\sigma_0(i)}^1 - t_i^0}$$

on a neighbourhood of each trajectory y_i .

Let now $T < T_1$. Then for all permutation σ there exists $i \in \{1, \dots, n\}$ such that $t_i^0 + t_{\sigma(i)}^1 > T$. In other words, for each strategy steering μ^0 to μ^1 there exists a particle which has not enough time to travel from the initial position to the final position. Thus it is not possible to steer μ^0 to μ^1 . ■

We now deduce Theorem 1 in the discrete case:

Corollary 1 *Let μ^0 and μ^1 be the measures given in (6) and satisfying Condition 1. Then the minimal time T_1 given in (8) is equal to T_0 given in Theorem 1.*

Proof: Let K be large enough such that

$$[\Phi_t^{v+u_K} \# \mu^0](\omega) \geq 1 - [\Phi_{t-T^*}^{v+u_K} \# \mu^1](\omega).$$

This means that there are more particles which have entered in ω than particles which have left ω at each time t . Then, there exists σ such that $T_0 \geq \max(t_{\sigma(i)}^1 + t_i^0)$. Thus $T_0 \geq T_1$.

Conversely, let σ be the optimal permutation in the definition of T_1 . Consider the sequence

$$u_k := \theta_k v, \quad (10)$$

where θ_k are some cutoff functions in \mathcal{C}^∞ on ω defined by

$$\begin{cases} 0 \leq \theta_k \leq 1, \\ \theta_k = 0 \text{ in } \omega^c, \\ \theta_k = 1 \text{ in } \omega_k, \end{cases}$$

with $\omega_k := \{x^0 \in \mathbb{R}^d : d(x^0, \omega^c) \geq 1/k\}$. Let $\delta > 0$. For K large enough

$$\inf\{t \in \mathbb{R}^+ : \Phi_s^{v+\theta_k v}(x_i^0) \in \omega \forall s > t\}$$

is smaller that

$$T_1 + \delta - \inf\{t \in \mathbb{R}^+ : \Phi_{-s}^{v+\theta_k v}(x_i^1) \in \omega \forall s > t\}.$$

For more details, we refer to Proposition 2 in [4]. Thus $T_1 \geq T_0$. ■

Formula (8) for the minimal time can be simplified, since one can easily find a permutation σ minimizing the cost:

Corollary 2 *Let μ^0 and μ^1 be the measures given in (6) satisfy Condition 1. Assume that the sequences $\{t_i^0\}_{i \in \{1, \dots, n\}}$ and $\{t_i^1\}_{i \in \{1, \dots, n\}}$ defined in Proposition 2 are, respectively, increasingly and decreasingly ordered. Then the minimal time (8) is given by:*

$$T_2 := \max_{i \in \{1, \dots, n\}} \{t_i^0 + t_i^1\}. \quad (11)$$

Proof: Consider T_1 given in (8). We assume that $\{t_i^0\}_{i \in \{1, \dots, n\}}$ is increasingly ordered. Let σ_0 be a minimising permutation in definition (8). We build recursively a sequence $\{\sigma_1, \dots, \sigma_n\}$ as follows:

- Let k_1 be such that $t_{\sigma_0(k_1)}^1$ is one of the maximisers of $\{t_{\sigma_0(1)}^1, \dots, t_{\sigma_0(n)}^1\}$. We denote by $\sigma_1 := \mathcal{T}_{1, k_1} \circ \sigma_0$ where, for all $i, j \in \{1, \dots, n\}$, $\mathcal{T}_{i, j}$ is the transposition between the first i^{th} and j^{th} elements. As illustrated in Figure 2, we clearly have

$$\begin{cases} t_{k_1}^0 + t_{\sigma_0(k_1)}^1 \geq t_1^0 + t_{\sigma_0(1)}^1, \\ t_{k_1}^0 + t_{\sigma_0(k_1)}^1 \geq t_1^0 + t_{\sigma_0(k_0)}^1, \\ t_{k_1}^0 + t_{\sigma_0(k_1)}^1 \geq t_{k_1}^0 + t_{\sigma_0(1)}^1. \end{cases}$$

Thus σ_1 minimizes also (8).

$$\max_{i \in \{1, \dots, n\}} \{t_i^0 + t_{\sigma_0(i)}^1\} \geq \max_{i \in \{1, \dots, n\}} \{t_i^0 + t_{\sigma_1(i)}^1\}.$$

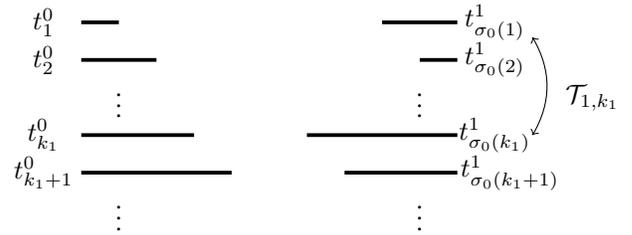


Fig. 2. Computation of the minimal time

- Assume that σ_j is built. Let k_{j+1} be such that $t_{\sigma_j(k_{j+1})}^1$ is a maximizers of $\{t_{\sigma_j(j+1)}^1, \dots, t_{\sigma_j(1)}^1\}$. We denote by $\sigma_{j+1} := \mathcal{T}_{j+1, k_{j+1}} \circ \sigma_j$. Again, we clearly have

$$\begin{cases} t_{k_{j+1}}^0 + t_{\sigma_j(k_{j+1})}^1 \geq t_{j+1}^0 + t_{\sigma_j(j+1)}^1, \\ t_{k_{j+1}}^0 + t_{\sigma_j(k_{j+1})}^1 \geq t_{j+1}^0 + t_{\sigma_j(k_{j+1})}^1, \\ t_{k_{j+1}}^0 + t_{\sigma_j(k_{j+1})}^1 \geq t_{k_{j+1}}^0 + t_{\sigma_j(j+1)}^1. \end{cases}$$

Thus σ_{j+1} minimizes also (8).

$$\max_{i \in \{1, \dots, n\}} \{t_i^0 + t_{\sigma_j(i)}^1\} \geq \max_{i \in \{1, \dots, n\}} \{t_i^0 + t_{\sigma_{j+1}(i)}^1\}.$$

We remark that the sequence $\{t_{\sigma_n(1)}^1, \dots, t_{\sigma_n(n)}^1\}$ is decreasing and σ_n is a minimising permutation. We deduce that $T_1 = T_2$. ■

IV. OPTIMAL TIME FOR CONTINUOUS MASSES

In this section, we prove Theorem 1 about minimal time for continuous masses. To achieve controllability in this setting, one needs to store the mass coming from μ^0 in ω and to send it out with a rate adapted to approximate μ^1 .

Proof of Theorem 1: We first prove item (i):

(i) Fix $\varepsilon > 0$ and $s > 0$. Let T_0 be the infimum satisfying Condition (a)-(c) in Definition 1. Let K be large enough such that

$$[\Phi_t^{v+u_K} \# \mu^0](\omega) \geq 1 - [\Phi_{t-T_0}^{v+u_K} \# \mu^1](\omega) - \varepsilon,$$

for all $t \in (0, T_0)$. We prove that we can steer μ^0 to a W_p -neighbourhood of μ^1 on the time interval $(0, T_0 + s)$. The proof is divided into two steps:

- In a first step, we discretize the supports of μ^0 and μ^1 . We use first a coarse uniform mesh, then on each cell we discretize μ^0 and μ^1 with respect to the mass.
- In a second step, we explain how to connect the different masses of μ^0 and μ^1 .

Step 1:

Let a, b be such that $\text{supp}(\mu^0) \subset (a, a + b)^d$. Consider the sequence of uniform meshes $(\mathcal{T}_k)_{k \in \mathbb{N}^*}$ where, for all $k \in \mathbb{N}^*$, \mathcal{T}_k is defined by

$$\left\{ \prod_{j=1}^d \left(a + \frac{i_j}{2^k} b, a + \frac{i_j + 1}{2^k} b \right) : (i_1, \dots, i_d) \in \{0, \dots, 2^k - 1\}^d \right\}.$$

Let $x^0 \in \text{supp}(\mu^0)$. Let $t^0(x^0)$ as defined in (2). There exists $t^*(x^0) \in (t^0(x^0), t^0(x^0) + s)$ and a cell $S(x^0)$ of a mesh $\mathcal{T}_k(x^0)$ such that

$$\Phi_{t^*(x^0)}^{v+u_K}(S(x^0)) \subset \omega.$$

By compactness of $\text{supp}(\mu^0)$, there exists $\{x_1^0, \dots, x_n^0\}$ such that $\{S(x_k^0)\}_k$ is a finite subcover of $\text{supp}(\mu^0)$. Without loss of generality, we can assume that $\{S(x_k^0)\}_k$ belong to a mesh \mathcal{T}_K . We discretize the support of μ^1 with the same process. We denote by $\{x_k^0\}_k$ and $\{x_k^1\}_k$ the middlepoints of the cell and by $\{t_k^0\}_k$ and $\{t_k^1\}_k$ the times at which these squares are included in ω . System (1) is conservative, *i.e.* if we denote by μ the solution to System (1) the application $t \mapsto \int_{\mathbb{R}^d} \mu(t)$ is constant. Hence, to send a measure to another, these measures need to have the same total mass. Thus we discretize measure $\mu^0 \times \mathbb{1}_{S(x_k^0)}$ and $\mu^0 \times \mathbb{1}_{S(x_k^1)}$ in some measures with the same total mass. We use a strategy defined in [4]. Fix $k \in \{1, \dots, n\}$ and let $\mu_k^0 := \mu^0 \times \mathbb{1}_{S(x_k^0)}$. To simplify the presentation, assume that $d := 2$ and $\text{supp}(\mu^0) \subset (0, 1)^2$. We denote by

$$\alpha := \mu^0(S(x_k^0)).$$

Fix $n \in \mathbb{N}^*$. Define a_0 the abscissa of the left side of $S(x^0)$. and the points a_i for all $i \in \{1, \dots, \lfloor \alpha n \rfloor\}$ by induction as follows: suppose that for $i \in \{0, \dots, \lfloor \alpha n \rfloor - 1\}$ the points a_i are given, then a_{i+1} is the smallest value satisfying

$$\int_{(a_i, a_{i+1}) \times \mathbb{R}} d\mu_k^0 = \frac{\lfloor \alpha n \rfloor}{n^2}.$$

Let $\alpha_i := \mu^0([(a_i, a_{i+1}) \times \mathbb{R}] \cap S(x_k^0))$. Again, for all $i \in \{0, \dots, \lfloor \alpha n \rfloor - 1\}$, we define $a_{i,0}$ the lowest ordinate of $S(x_k^0)$ and supposing that for a $j \in \{0, \dots, \lfloor \alpha n \rfloor - 1\}$ the points $a_{i,j}$ are already defined, $a_{i,j+1}$ is the smallest values such that

$$\int_{A_{kij}} d\mu^0 = \frac{\lfloor \alpha n \rfloor}{n^3},$$

where $A_{kij} := (a_i, a_{i+1}) \times (a_{ij}, a_{i(j+1)})$. We give in Figure 3 an example of such decomposition.

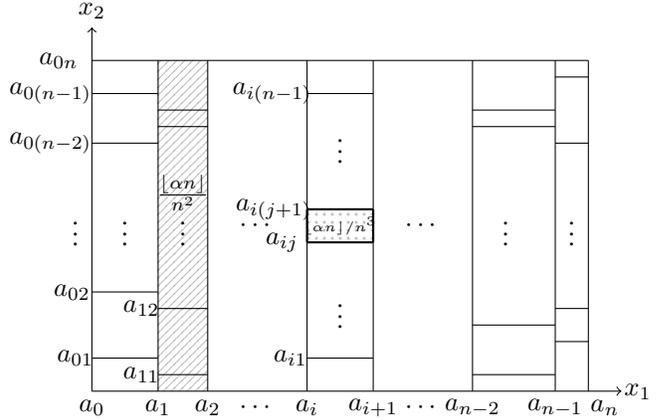


Fig. 3. Example of a decomposition of μ^0 .

We discretize similarly the measure $\mu_k^1 := \mu^1 \times \mathbb{1}_{S(x_k^1)}$ on some sets B_{kij} . To act locally on the sets A_{kij} and $B_{k'i'j'}$, we define sets $C_{kij} \subset A_{kij}$ and $D_{k'i'j'} \subset B_{k'i'j'}$. We then send the mass of μ^0 from each C_{kij} to each $D_{k'i'j'}$, while we do not control the mass in $A_{kij} \setminus C_{kij}$. More precisely, we define, as in Figure 4, $a_i^-, a_i^+, a_{ij}^-, a_{ij}^+$ the smallest values such that

$$\int_{(a_i, a_i^-) \times (a_{ij}, a_{i(j+1)})} d\mu^0 = \int_{(a_i^+, a_{i+1}^+) \times (a_{ij}, a_{i(j+1)})} d\mu^0 = \frac{\lfloor \alpha n \rfloor}{n^4},$$

and

$$\int_{(a_i^-, a_i^+) \times (a_{ij}, a_{ij}^-)} d\mu^0 = \int_{(a_i^-, a_i^+) \times (a_{ij}^+, a_{i(j+1)})} d\mu^0 = \frac{1}{n} \times \left(\frac{\lfloor \alpha n \rfloor}{n^3} - \frac{2\lfloor \alpha n \rfloor}{n^4} \right).$$

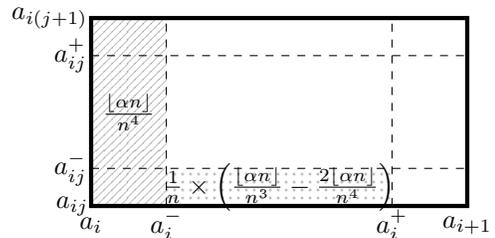


Fig. 4. Example of cell

We define similarly $b_i^+, b_i^-, b_{ij}^+, b_{ij}^-$. We finally define $C_{kij} := [a_i^-, a_i^+] \times [a_{ij}^-, a_{ij}^+]$ and $D_{kij} := [b_i^-, b_i^+] \times [b_{ij}^-, b_{ij}^+]$.

Step 2:

We explain now how to send approximately the measures $\mu^0 \times \mathbb{1}_{C_{kij}}$ to the measures $\mu^1 \times \mathbb{1}_{D_{k'i'j'}}$. Let us first recall the following result:

Proposition 3 (see [4]) Let $\mu^0 \in \mathcal{P}_c^{ac}(\mathbb{R}^d)$ satisfying $\text{supp}(\mu^0) \subset \subset \omega$. Define S a square strictly included in ω and

choose $\delta > 0$. Then there exists a space-dependent velocity field u , Lipschitz and uniformly bounded with $\text{supp}(u) \subset \bar{\omega}$, such that the corresponding solution to System (1) satisfies $\text{supp}(\mu(\delta)) \subset \subset S$.

Using Proposition 3, it is possible to concentrate the measures $\Phi_{t_k^0}^{v+u_k} \#(\mu^0 \times \mathbb{1}_{C_{kij}})$ and $\Phi_{-t_{k'}^1}^{v+u_k} \#(\mu^1 \times \mathbb{1}_{D_{k'i'j'}})$ in some squares $E_{ijk} \subset \subset C_{ijk}$ and $F_{i'j'k'} \subset \subset D_{i'j'k'}$ (as small as we want). Up to a adaptation of Corollary 1 to the case of arbitrary concentrated masses, if these squares are small enough, then there exists a control u Lipschitz in space, measurable in time and uniformly bounded with $\text{supp}(u) \subset \omega$ such that the support of $\Phi_s^v \# \Phi_{t_k^0}^{v+u_k} \#(\mu^0 \times \mathbb{1}_{C_{kij}})$ is included in the support of $\Phi_{-t_{k'}^1}^{v+u_k} \#(\mu^1 \times \mathbb{1}_{D_{k'i'j'}})$.

Using Step 2 in the proof of Proposition 1 in [4], the refinement of the grid provides convergence to the target μ^1 , i.e.

$$W_1(\mu^1, \mu(T)) \xrightarrow[n \rightarrow \infty]{} 0.$$

If $T = T_0$, then we control at time $T + \varepsilon$ and we conclude using property (4) of the Wasserstein distance.

(ii) We now prove the second item of Theorem 1. Let T_0 be the infimum satisfying condition (a)-(c) of Theorem 1. If $T < T_0^*$ or $T < T_1^*$, then it is clear that System is not controllable at time T , since we cannot act on each particle. For all $T \in (0, T_2^*)$ and sequence $\{u_k\}_k$, there exists $s \in (0, T)$ such that

$$\lim_{k \rightarrow \infty} [\Phi_s^{v+u_k} \# \mu^0](\omega) > 1 - \lim_{k \rightarrow \infty} [\Phi_{s-T}^{v+u_k} \# \mu^1](\omega).$$

(if it is not the case, then T_2^* is not optimal) We can take for example the sequence $\{u_k\}_k$ defined in (10). Hence, at time s , there is less mass which has entered ω than mass which have left ω . Thus we obtain a contradiction. \blacksquare

We have also the following result on the continuity of the minimal time with respect to the initial condition:

Theorem 3 *The application $\mu^0 \mapsto T_0(\mu^0, \mu^1)$ is continuous with respect to the Wasserstein distance W_∞ .*

The definition of W_∞ is recalled in Section II. Theorem 3 will be useful for the approximation of the initial data. As it will be explained in Remark 5, the application $\mu^0 \mapsto T_0(\mu^0, \mu^1)$ is instead not continuous with respect to the L^1 -norm or the other Wasserstein distance W_p with $1 \leq p < \infty$.

Proof of Theorem 3: Consider $\mu^0, \mu^1 \in \mathcal{P}_c^{ac}(\mathbb{R}^d)$ and a sequence $\{\mu_n^0\}_{n \in \mathbb{N}^*} \subset \mathcal{P}_c^{ac}(\mathbb{R}^d)$ satisfying Condition 1 and $W_\infty(\mu^0, \mu_n^0) \leq 1/n$. Let $T_0 = \max\{T_0^*, T_1^*, T_2^*\}$ and $T_{0,n} = \max\{T_{0,n}^*, T_1^*, T_{2,n}^*\}$ be the corresponding minimal time associated to μ^0 and μ_n^0 . In a first step, we prove that $T_{0,n}$ converge to T_0^* and in a second step that $T_{2,n}^*$ converges to T_2^* :

(i) Fix $\varepsilon > 0$. We aim to prove that for n large enough $|T_{0,n} - T_0| \leq \varepsilon$. For all $n \in \mathbb{N}^*$, there exists a measurable map $S_n : \mathbb{R}^d \rightarrow \mathbb{R}^d$ such that $\mu_n^0 = S_n \# \mu^0$. We have $|S_n(x^0) - x^0| \leq 1/n$. For all $x^0 \in \text{supp}(\mu^0)$ there exists $r(x^0)$ such that

$$|x^0 - \tilde{x}^0| \leq r(x^0) \Rightarrow |t_0(x^0) - t^0(\tilde{x}^0)| \leq \varepsilon.$$

By compactness, for a family $\{x_1^0, \dots, x_M^0\} \subset \text{supp}(\mu^0)$ we have $\text{supp}(\mu^0) \subset \cup_i B(x_i^0, r(x_i^0))$. Thus for n such that $1/n < \inf_i r(x_i^0)$, we obtain $|t^0(x^0) - t^0(S_n(x^0))| \leq \varepsilon$.

(ii) Let $T = T_2^* + s$ with $s > 0$ and $\{u_k\}_k$ the corresponding sequence to μ^0 . With the same argument of compactness as above, for n large enough:

$$\lim_{k \rightarrow \infty} [\Phi_t^{v+u_k} \# \mu_n^0](\omega) \geq 1 - \lim_{k \rightarrow \infty} [\Phi_{t-T}^{v+u_k} \# \mu^1](\omega),$$

for all $t \in (s, T)$. Thus $\lim T_{2,n}^* \leq T_2^*$.

Let now $T < T_2^*$. For all sequence $\{u_k\}_k$, there exists $s \in (0, T)$ and $\delta > 0$ such that

$$\lim_{k \rightarrow \infty} [\Phi_s^{v+u_k} \# \mu^0](\omega) < 1 - \lim_{k \rightarrow \infty} [\Phi_{s-T}^{v+u_k} \# \mu^1](\omega) - \delta.$$

It implies that for n large enough

$$\lim_{k \rightarrow \infty} [\Phi_s^{v+u_k} \# \mu_n^0](\omega) < 1 - \lim_{k \rightarrow \infty} [\Phi_{s-T}^{v+u_k} \# \mu^1](\omega) - \delta/2.$$

Hence $T_{2,n}^* > T$. Thus $\lim_{n \rightarrow \infty} T_{2,n}^* \geq T_2^*$. \blacksquare

Remark 5 The application $\mu^0 \mapsto T_0(\mu^0, \mu^1)$ is not continuous with respect to the L^1 -norm or the Wasserstein distance W_p . Indeed, consider $\mu^0 = \mu^1 := \mathbb{1}_{(0,1)} \in \mathcal{P}_c^{ac}(\mathbb{R}^d)$, $\omega := (0, 1)$ and $v := 2$. The minimal time to send μ^0 to μ^1 is clearly equal to 0. For each $n \in \mathbb{N}^*$, we define the measure

$$\mu_n^0 := \begin{cases} 1 - 1/n & \text{if } x \in (0, 1), \\ 1/n & \text{if } x \in (-2, -1), \\ 0 & \text{otherwise.} \end{cases}$$

We remark that $W_1(\mu_n^0, \mu^0) \rightarrow 0$, $\|\mu_n^0 - \mu^0\|_{L^1} \rightarrow 0$ and $T_0(\mu_n^0, \mu^1) = 1$. Thus the application $\mu^0 \mapsto T_0(\mu_n^0, \mu^1)$ is not continuous with respect to the Wasserstein distance or the L^1 -norm.

V. NUMERICAL SIMULATIONS

In this section, we give a numerical illustration of the algorithm developed in the proof of Theorem 1 to compute the minimal time. We highlight that some concentration of the mass in the control region ω can appear. We first recall this algorithm:

Algorithm 1:

Step 1 : Discretisation of μ^0 and μ^1

- (i) Construction of the uniform mesh
- (ii) Computation of the cell A_{ij} following the mass

Step 2: Computation of the minimal time using Expression (11)

Step 3: Association of the masses of μ^0 and μ^1 minimising Expression (9)

Step 4: Concentration of the masses (if necessary)

Step 5: Final computation

Consider the initial data μ^0 and the target μ^1 defined by

$$\mu^0 := \begin{cases} 0.5 & \text{if } x \in (0, 2), \\ 0 & \text{otherwise} \end{cases}$$

and

$$\mu^1 := \begin{cases} 0.5 & \text{if } x \in (7, 8) \cup (10, 11), \\ 0 & \text{otherwise.} \end{cases}$$

We fix the velocity field $v := 1$ and the control region $\omega := (5, 6)$. Following Algorithm 1, we approximate the minimal time T_0 and a control u such that the corresponding solution μ to System (1) satisfies $W(\mu(T_0 + \delta), \mu^1) \leq \varepsilon$. We use a Lagrangian scheme for simulation (see [6] for more details). Some approximations at some times are given in Figure 5. It is interesting to see that we observe the apparition of concentration of the mass in the control region ω .

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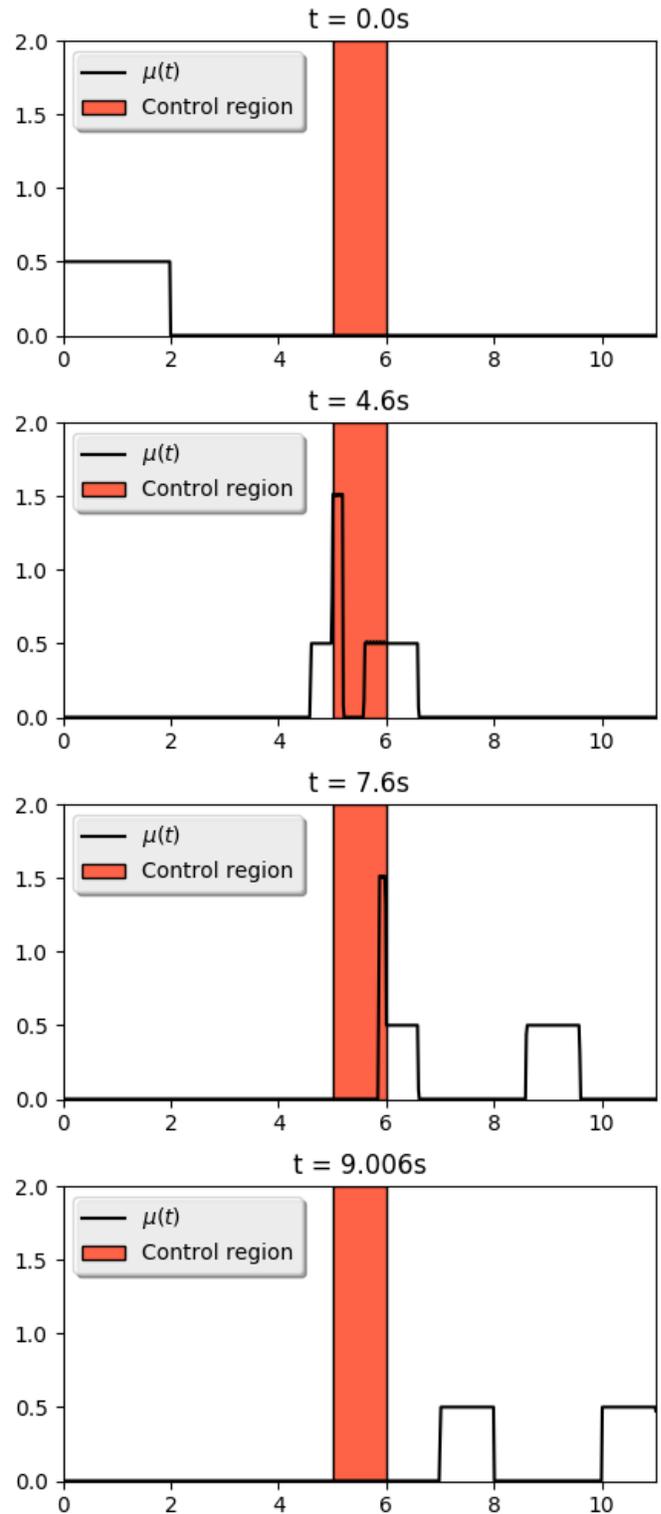


Fig. 5. Solution at time $t = 0$, $t = 4.6$, $t = 7.6$ and $t = T = 9.006$.