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# NEUMANN AND MIXED PROBLEMS ON MANIFOLDS WITH BOUNDARY AND BOUNDED GEOMETRY

NADINE GROSSE AND VICTOR NISTOR

ABSTRACT. We prove regularity and well-posedness results for the mixed Dirichlet-Neumann problem for a second order, uniformly strongly elliptic differential operator on a manifold  $M$  with boundary  $\partial M$  and bounded geometry. Our well-posedness result for the Laplacian  $\Delta_g := d^*d \geq 0$  associated to the given metric require the additional assumption that the pair  $(M, \partial_D M)$  be of finite width (in the sense that the distance to  $\partial_D M$  is bounded uniformly on  $M$ , where  $\partial_D M$  is the Dirichlet part of the boundary). The proof is a continuation of the ideas in our previous paper on the Dirichlet problem on manifolds with boundary and bounded geometry (joint with Bernd Ammann). We also obtain regularity results for more general boundary conditions. Our results are formulated in the usual Sobolev spaces defined by the Riemannian metric on  $M$ .

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## 1. INTRODUCTION

We prove regularity and well-posedness results for the mixed Dirichlet-Neumann problem for a second order, uniformly strongly elliptic differential operator on a manifold  $M$  with boundary  $\partial M$  and bounded geometry. For the most part, the results in this paper are based on and extend those of a previous joint paper of ours with Bernd Ammann [5]. The main contribution of this paper is a more thorough

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study of the regularity for more general boundary conditions than the ones studied in [5]. Our regularity and well-posedness results for the mixed Dirichlet-Neumann problem (or system) are formulated in the higher regularity Sobolev spaces  $H^{k+1}$  associated to the given riemannian metric on  $M$ .

Let  $(M, g)$  be a Riemannian manifold with boundary and bounded geometry (see Section 3 and [5, 25]). To formulate our mixed boundary conditions, we fix a subset  $\partial_D M \subset \partial M$  where Dirichlet boundary conditions will be imposed and let  $\partial_N M := \partial M \setminus \partial_D M$ . On  $\partial_N M$ , we will impose Neumann boundary conditions. Both  $\partial_D M$  and  $\partial_N M$  are supposed to be unions of connected components of  $\partial M$ , to avoid the issue of singularities where the type of boundary conditions changes. Let  $P = P_a + Q$  be a uniformly strongly elliptic operator, where  $a$  is a *strongly coercive* bilinear form,  $P_a$  is the operator in divergence form associated to  $a$  and  $Q$  is a first order differential operator, see Section 2.1 for definitions and more details. In this introduction and for our main results, we shall assume that either  $P$  has coefficients in  $W^{k+1, \infty}$  or that  $M$  is a Lie manifold with boundary and structural compactification  $\overline{M}$  and that  $P$  has coefficients in  $C^k(\overline{M})$ . In this paper,  $k \geq 1$  is an arbitrary, but fixed regularity parameter.

Our first result is a regularity result for the problem

$$\begin{cases} Pu = f & \text{in } M \\ u = h_D & \text{on } \partial_D M \\ \partial_\nu^\alpha u = h_N & \text{on } \partial_N M, \end{cases} \quad (1)$$

where  $\partial_\nu^\alpha u$  is the associated Neumann condition, as in Remark 2.3.

**Theorem 1.1.** *Let  $P := P_a + Q$  be a uniformly strongly elliptic second order differential operator. We assume either that  $P$  has coefficients in  $W^{k+1, \infty}$  or that  $M$  is a Lie manifold with boundary with compactification  $\overline{M}$  and  $P$  has coefficients in  $C^k(\overline{M})$ . Then there is  $C > 0$  such that, if  $u \in H^1(M)$ ,  $f := Pu \in H^{k-1}(M)$ ,  $h_D := u|_{\partial_D M} \in H^{k+1/2}(\partial_D M)$ , and  $h_N := \partial_\nu^\alpha u|_{\partial_N M} \in H^{k-1/2}(\partial_N M)$ , then  $u \in H^{k+1}(M)$  and*

$$\|u\|_{H^{k+1}} \leq C (\|f\|_{H^{k-1}} + \|h_D\|_{H^{k+1/2}} + \|h_N\|_{H^{k-1/2}} + \|u\|_{H^1(M)}).$$

In the last equation,  $\|u\|_{H^{k+1}} = \|u\|_{H^{k+1}(M)}$ ,  $\|h_D\|_{H^{k+1/2}} = \|h_D\|_{H^{k+1/2}(\partial_D M)}$ , and so on (we have omitted from the notation of the norms the spaces on which the functions are defined and we will do so accordingly when there is no danger of confusion).

This result follows right away from Theorem 4.5 and Corollary 4.18. Note that in case  $a = g$  (the Riemannian metric), we have that  $P_a = \Delta_g$  and that  $\partial_\nu^\alpha = \partial_\nu$ . We note that if we regard  $d: H_D^1(M) := \{u \in H^1(M) \mid u|_{\partial_D M} = 0\} \rightarrow L^2(M)$ , then  $\Delta_g := d^*d$  has the *right* boundary conditions (Dirichlet on  $\partial_D M$  and Neumann on the complement). Therefore, the above theorem applies to  $P = \Delta_g$  with Dirichlet or Neumann boundary conditions.

Recall from [5] that we say that  $(M, \partial_D M)$  has *finite width* if  $M$  is a manifold with boundary and bounded geometry and there exists  $R > 0$  with  $\text{dist}(x, \partial_D M) \leq R$  for all  $x \in M$ .

**Theorem 1.2.** *Let us use the notation and assumptions of Theorem 1.1 and let us further assume that  $(M, \partial_D M)$  has finite width and that  $Q = 0$ . Then we can choose  $C$  such that*

$$\|u\|_{H^{k+1}(M)} \leq C (\|f\|_{H^{k-1}(M)} + \|h_D\|_{H^{k+1/2}(\partial_D M)} + \|h_N\|_{H^{k-1/2}(\partial_N M)}).$$

Consequently, the same result is true if  $\|Q\|_{H^1 \rightarrow L^2}$  is small enough.

Here  $\|Q\|_{H^1 \rightarrow L^2}$  notes the operator norm of  $Q$  as an operator from  $H^1 \rightarrow L^2$ . The theorem follows directly from Theorems 4.6 and 4.19, see also the discussion following Theorem 3.16. In particular, under the assumptions of Theorem 1.2, we obtain an isomorphism

$$\begin{aligned} \tilde{P}: H^{k+1}(M) &\rightarrow H^{k-1}(M) \oplus H^{k+1/2}(\partial_D M) \oplus H^{k-1/2}(\partial_N M), \\ \tilde{P}(u) &:= (Pu, u|_{\partial_D M}, \partial_\nu^\alpha u|_{\partial_N M}). \end{aligned}$$

Therefore, if  $(M, \partial_D M)$  has finite width, the Laplacian  $P = \Delta_g$  with Dirichlet boundary conditions on  $\partial_D M$  and Neumann boundary conditions on  $\partial_N M$  will satisfy the conditions of the above theorem. The same is true for the Schrödinger operator  $P := \Delta_g + V$ ,  $V \in W^{k, \infty}(M)$ ,  $V \geq 0$ . The proof of the second theorem (Theorem 1.2), follows from the first theorem (Theorem 1.1) by combining it with the isomorphism results in [5]. (More precisely, we use the isomorphism result in [5] that amounts to Theorem 1.2 for  $k = 0$ , if properly formulated to allow for the case  $k = 0$ .)

The proof of our regularity result, Theorem 1.1, is based on uniform local regularity estimates in Fermi coordinate patches near the boundary. These local regularity estimates are well-known in each coordinate patch, our contribution being to show that the resulting estimates can be assumed to be uniform (that is, independent of the patch). The uniform estimates are obtained in Section 2, where we also introduce our notation, recall the definition of some basic analytic concepts (Sobolev spaces, differential operators, manifolds with totally bounded curvature, and so on). Our approach to higher regularity on manifolds with bounded geometry is to use the *compactness* of the associated local problems near the boundary. In Section 3, we recall the definition of manifolds with boundary and bounded geometry and some well-posedness results from [5], such as the Poincaré inequality for these manifolds in case they are with finite width. We also discuss the characterization of Sobolev spaces on manifolds with boundary and bounded geometry using partitions of unity. In the last section, Section 4, we prove that Dirichlet and Neumann boundary conditions satisfy the assumptions of the results in Subsection 2.2, and hence that they provide uniform regularity estimates. We then prove the theorems stated in this introduction (usually in a slightly more general form). Finally, we briefly explain how to obtain some more general results in the case of Lie manifolds. **Acknowledgments.** We would like to thank Bernd Ammann for several useful discussions. We also gladly acknowledge the hospitality of the SFB 1085 Higher Invariants at the Faculty of Mathematics at the University of Regensburg where parts of this article was written.

## 2. PRELIMINARIES

We include here some background material. More precisely, we introduce here the Sobolev spaces and we recall the results from [5] and from other sources that we will use. We refer to [5] for more details.

Throughout this paper,  $M$  will be a (usually non-compact) connected smooth manifold of dimension  $m + 1$  with *smooth* boundary  $\partial M$  and Riemannian metric  $g$ . We shall also assume throughout the paper that we are given a *disjoint union*

decomposition

$$\partial M = \partial_D M \sqcup \partial_N M$$

with  $\partial_D M$  and  $\partial_N M$  open in  $\partial M$ . In particular, each of  $\partial_D M$  and  $\partial_N M$  is closed and a disjoint union of connected components of  $\partial M$ . On the set  $\partial_D M$  we will impose *Dirichlet* boundary conditions, whereas on  $\partial_N M$  we will impose *Neumann* boundary conditions. We call  $\partial_D M$  the *Dirichlet part of the boundary* and  $\partial_N M := \partial M \setminus \partial_D M$  the *Neumann part of the boundary*.

**2.1. Differential operators and Sobolev spaces.** Let  $M$  be a smooth manifold (possibly with boundary) and  $E \rightarrow M$  be a smooth complex vector bundle with connection  $\nabla$ . We shall say that  $(E, \nabla)$  (or simply  $E$ ) has *totally bounded curvature* if its curvature  $R^E$  and all its covariant derivatives  $\nabla^j R^E$  are bounded (on tensor bundles we use the Levi-Civita connection). For simplicity, we shall assume that all vector bundles in this paper (and all connections) have totally bounded curvature.

If  $a_j$  is a (measurable) section of  $\text{End}(E) \otimes_{\mathbb{R}} TM^{\otimes j}$ , for  $0 \leq j \leq \ell$ , and  $a_\ell$  is not identically equal to 0, then

$$Pu := \sum_{j=0}^{\ell} a_j \nabla^j u, \quad (2)$$

is the general form of an *order  $\ell$  differential operator*. Its minimal domain is the space  $\Gamma_c(M; E)$  of *smooth sections of  $E$  with compact support*. It takes values in the space of *measurable sections of  $E$* . The slightly more general case of differential operators acting between sections of *different* vector bundles  $E$  and  $F$  can be, to a large extent, reduced to the setting of a single vector bundle, by using  $E \oplus F$ . Alternatively, we can use projections from trivial vector bundles, see Lemma 3.12, for instance. This more general setting is needed in order to deal with operators such as the chiral Dirac operator, but this will not be the focus of this paper, so we will consider only operators acting on sections of the same vector bundle, for notational simplicity.

Let us assume from now on that  $E \rightarrow M$  is endowed with a Hermitian form and a connection  $\nabla: \Gamma(M; E) \rightarrow \Gamma(M; E \otimes T^*M)$  that preserves that form and that the tensor bundles  $(TM)^{\otimes i} \otimes (T^*M)^{\otimes j}$  are endowed with the Levi-Civita connection. We define

$$\begin{aligned} W^{k,p}(M; E) &:= \{u \mid \nabla^j u \in L^p(M; E \otimes T^*M^{\otimes j}), 0 \leq j \leq k\} \\ \|u\|_{W^{k,p}(M; E)}^p &:= \sum_{k=0}^k \|\nabla^k u\|_{L^p(M; E \otimes T^*M^{\otimes j})}^p \quad \text{for } p < \infty \text{ and} \\ \|u\|_{W^{k,\infty}(M; E)} &:= \max_{k=0}^k \|\nabla^k u\|_{L^\infty(M; E \otimes T^*M^{\otimes j})}, \end{aligned}$$

where all  $L^p$ -norms are computed with respect to the volume form associated to the metric  $g$ . As usual, we let  $W^{\infty,p} := \cup_k W^{k,p}$  and  $H^k(M; E) := W^{k,2}(M; E)$ . We let  $W_D^{k,\infty}(M; E)$  be the closure in  $W^{k,\infty}(M; E)$  of the space of smooth sections with compact support in  $M \setminus \partial_D M$ . If  $\partial_D M = \partial M \neq \emptyset$ , we shall write  $H_0^k = H_D^k$  and  $W_0^{k,p} = W_D^{k,p}$ , as usual. In this paper, only the cases  $p = 2$  or  $p = \infty$  will be used in a significant way.

If the coefficients in (2) satisfy  $a_j \in W^{k,\infty}(M; \text{End}(E) \otimes_{\mathbb{R}} TM^{\otimes j})$ , we shall say that  $Pu := \sum_{j=0}^{\ell} a_j \nabla^j u$  has *coefficients in  $W^{k,\infty}$* . In particular, a *scalar first order differential operator with coefficients in  $W^{k,\infty}$*  is an operator of the form

$Qu = X(u) + cu$  where  $X$  is a vector field and  $c$  is a measurable function with  $X \in W^{k,\infty}(M; TM)$  and  $c \in W^{k,\infty}(M)$ . By definition the operator  $P := \sum_{j=0}^{\ell} a_j \nabla^j u$  then defines a continuous map

$$P = \sum_{j=0}^{\ell} a_j \nabla^j : W^{k+k,p}(M; E) \rightarrow W^{k,p}(M; E).$$

**Lemma 2.1.** *Let  $k \geq 0$  and let  $P$  be an order  $\ell$  differential operator with coefficients  $a_j \in W^{k+1,\infty}(M; \text{End}(E) \otimes TM^{\otimes j})$ . Let  $\phi \in W^{k+\ell+1,\infty}(M)$ . Then the commutator  $[P, \phi]$  defines a continuous linear map  $H^{k+\ell}(M; E) \rightarrow H^{k+1}(M; E)$ . Moreover, if  $\{\phi_\gamma\}_\gamma$  is a bounded family in  $W^{k+\ell+1,\infty}(M)$ , then operator norms of  $[P, \phi_\gamma] : H^{k+\ell}(M; E) \rightarrow H^{k+1}(M; E)$  are bounded uniformly in  $\gamma$ .*

*Proof.* We have  $[P, \phi]u = \sum_{j=1}^{\ell} a_j \sum_{s=0}^{j-1} \binom{j}{s} \nabla^{j-s} \phi \nabla^s u$ . Thus,

$$\begin{aligned} \|[P, \phi]u\|_{H^{k+1}} &\leq C \sum_{r=0}^{k+1} \sum_{j=0}^{\ell} \|\nabla^r (a_j \sum_{s=0}^{j-1} \nabla^{j-s} \phi \nabla^s u)\|_{L^2} \\ &\leq C \sum_{r=0}^{k+1} \sum_{t=0}^r \sum_{j=0}^{\ell} \|\nabla^{r-t} a_j \nabla^t (\sum_{s=0}^{j-1} \nabla^{j-s} \phi \nabla^s u)\|_{L^2} \end{aligned}$$

and the claim follows by the regularity assumptions on  $a_j$  and  $\phi$ .  $\square$

**Notations 2.2.** *We let  $\bar{V}$  denote the complex conjugate space of a complex vector space  $V$ . Also,  $V'$  will denote the (topological) dual of  $V$  and  $V^* := (\bar{V})'$ .*

Of course, if  $V$  is obtained from the *complexification* of a real, orthogonal vector space  $U$ , that is, if  $V \simeq U \otimes_{\mathbb{R}} \mathbb{C}$ , then we have canonical isomorphisms  $V \simeq V' \simeq \bar{V} \simeq V^*$ , so the distinction between these spaces becomes unnecessary. This is the case in many classical applications.

Let  $\underline{\mathbb{C}}$  denote the trivial, one-dimensional vector bundle  $\mathbb{C} \rightarrow M$  and  $E' = \text{Hom}(E, \underline{\mathbb{C}})$  denote the dual bundle of  $E$ . We define then  $H^{-k}(M; E)$  as the dual of  $H_0^k(M; E')$ . For  $u \in H^{-k}(M; E)$ , the composition  $\Gamma_c(M; E') \hookrightarrow H_0^k(M; E') \xrightarrow{u} \mathbb{C}$  allows us to view  $u$  as a distributional section of  $E$ , and hence  $H^k(M, E) \subset L^2(M, E) \subset H^{-k}(M, E)$ . (Note that we are using the metric  $g$  on  $M$  and its associated volume form to identify functions with distributions.) Using the Hermitian form to identify  $E'$  with  $\bar{E}$ , which leads to  $H_0^k(M; E') \cong H_0^k(M; \bar{E}) \cong \overline{H_0^k(M; E)}$ , we obtain using Notation 2.2 that

$$H^{-k}(M, E) \cong H_0^k(M; E)^* := \overline{H_0^k(M; E)'} \cong H_0^k(M; \bar{E})'. \quad (3)$$

Thus  $H^{-k}(M; E)$  is the (conjugate) dual of  $H_0^k(M; E)$  with pivot  $L^2(M, g)$ . If  $\partial_N M \neq \emptyset$ , we shall proceed by analogy and consider the spaces  $H_D^k(M; E)^* \cong H_D^k(M; \bar{E})'$ .

We denote by  $\mathcal{L}(V, W; \mathbb{C})$  the space of continuous bilinear forms on  $V \times W$ . If  $V$  and  $W$  are finite dimensional, we then have that the space  $\text{Sesq}(V, W)$  of sesquilinear forms  $B : V \times W \rightarrow \mathbb{C}$  is in natural bijection with the spaces  $\mathcal{L}(V, \bar{W}; \mathbb{C})$  of bilinear forms on  $V \times \bar{W}$ , with the space of linear maps  $V \otimes \bar{W} \rightarrow \mathbb{C}$ , and with the spaces of linear maps  $V \rightarrow W^*$  and  $\bar{W} \rightarrow V'$ :

$$\text{Sesq}(V, W) := \mathcal{L}(V, \bar{W}; \mathbb{C}) \simeq \mathcal{L}(V \otimes \bar{W}; \mathbb{C}) \simeq \mathcal{L}(V; W^*) \simeq \mathcal{L}(\bar{W}; V'). \quad (4)$$

In case  $V$  and  $W$  are infinite dimensional *Banach* spaces, we still have continuous maps  $\text{Sesq}(V, W) \simeq \mathcal{L}(V, \overline{W}; \mathbb{C}) \rightarrow \mathcal{L}(V; W^*)$ . In the following, we shall use these isomorphisms to sometimes identify these spaces without further comment. In particular, these identifications allow us to introduce operators “in divergence form” as follows: Let

$$a \in W^{k, \infty} \left( M; [(T^*M \otimes_{\mathbb{R}} E) \otimes_{\mathbb{C}} (T^*M \otimes_{\mathbb{R}} \overline{E})]' \right),$$

that is,  $a$  is a suitably bounded family of sesquilinear forms  $a_x$  on  $T_x^*M \otimes_{\mathbb{R}} E_x$ . We shall say that  $a$  is a  $W^{k, \infty}$ -*sesquilinear form* on  $T^*M \otimes_{\mathbb{R}} E$ . The *Dirichlet form*  $B_a$  on  $H_D^1(M; E)$  associated to  $a$  is

$$B_a(u, v) := \int_M a(\nabla u, \nabla v) \, \text{dvol}_g,$$

where  $\nabla: H_D^1(M; E) \rightarrow L^2(M; T^*M \otimes_{\mathbb{R}} E)$  is the connection. In particular

$$B_a: H_D^1(M; E) \times H_D^1(M; E) \rightarrow \mathbb{C}$$

is a continuous sesquilinear form, and hence it defines a continuous linear map

$$\tilde{P}_a: H_D^1(M; E) \rightarrow H_D^1(M; E)^* \cong H_D^1(M; \overline{E})'$$

that is uniquely determined by the relation  $\tilde{P}_a(v)(w) := B_a(v, w)$  for all  $v \in H_D^1(M; E)$  and  $w \in H_D^1(M; E)$ . (See Equation (4) and the discussion following it.) More precisely, regard  $a$  as section in  $W^{k, \infty}(M; \text{Hom}[T^*M \otimes_{\mathbb{R}} E, (T^*M \otimes_{\mathbb{R}} E)^*])$  using again Equation (4). Also, let  $\nabla^*: L^2(M; T^*M \otimes_{\mathbb{R}} E) \rightarrow H_D^1(M; E)^*$  be the adjoint of  $\nabla: H_D^1(M; E) \subset L^2(M; E) \rightarrow L^2(M; T^*M \otimes_{\mathbb{R}} E)$ . We have

$$\tilde{P}_a(v)(w) := B_a(v, w) := \int_M a(\nabla v, \nabla w) \, \text{dvol}_g = \int_M \langle a(\nabla v), \nabla w \rangle \, \text{dvol}_g,$$

which gives

$$\tilde{P}_a: H_D^1(M; E) \rightarrow H_D^1(M; E)^*, \quad v \mapsto \tilde{P}_a(v) := \nabla^*(a(\nabla v)). \quad (5)$$

Note that  $\nabla^*$  here is the adjoint  $\nabla_{H^1}^*$  of  $\nabla$  as an operator defined on  $H_D^1(M; E)$ . This operator differs from the adjoint of  $\nabla$  defined on  $H^1(M, E)$  only by its domain of definition. We denote by  $(\nabla_{L^2})^*: H^1(M; T^*M \otimes_{\mathbb{R}} E) \rightarrow L^2(M; E)$  the *formal* adjoint of  $\nabla$ , it is a differential operator uniquely determined by  $\langle (\nabla_{L^2})^* v, w \rangle_M = \langle v, \nabla w \rangle_M$  for  $v, w \in H_0^1$ . The operators  $\nabla^* = \nabla_{H^1}^*$  and  $(\nabla_{L^2})^*$  are related by the equation

$$\nabla^* v(w) - \langle (\nabla_{L^2})^* v, w \rangle_M = \langle v, \nabla w \rangle_M - \langle (\nabla_{L^2})^* v, w \rangle_M = \int_{\partial_N M} \langle \nu \cdot v, w \rangle dS, \quad (6)$$

where  $dS$  is the volume element on  $\partial M$  and  $v \in H_D^1(M; T^*M \otimes_{\mathbb{R}} E)$  and  $w \in H^1(M; E)$ , where  $\nu \cdot v = v(\nu)$  in the sense of the duality pairing between  $T^*M$  and  $TM$ . This is related to the weak formulation of the Neumann problem, see [2, 10, 11, 20, 23, 24].

Let

$$\text{res}: H_D^1(M; E)^* \rightarrow H^{-1}(M; E) := H_0^1(M; E)^*$$

be the natural restriction map. Then, on  $H_D^1(M; T^*M \otimes_{\mathbb{R}} E)$ , we have  $\text{res} \circ \nabla^* = (\nabla_{L^2})^*$  by the discussion above. Therefore, using (3),

$$P_a := \text{res} \circ \tilde{P}_a: H_D^1(M; E) \rightarrow H^{-1}(M; E)$$

is the “usual” differential operator associated to  $a$  and has an extension to  $H^1(M; E)$  that is independent of the boundary conditions.

If  $Q$  is a first order differential operator with coefficients in  $L^\infty$ , then it defines a continuous map  $Q: H_D^1(M; E) \rightarrow L^2(M; E) \subset H_D^1(M; E)^*$ . We shall denote by  $\tilde{Q}$  the resulting operator from  $H_D^1(M; E)$  to  $H_D^1(M; E)^*$  and consider continuous operators of the form

$$\tilde{P} := \tilde{P}_a + \tilde{Q}: H_D^1(M; E) \rightarrow H_D^1(M; E)^*, \quad (7)$$

which we call *second order differential operators in divergence form*. We say that  $\tilde{P}$  has coefficients in  $W^{k, \infty}$  if  $a \in W^{k, \infty}$  and the coefficients of  $Q$  are in  $W^{k, \infty}$ . We set

$$P := \text{res} \circ \tilde{P} := P_a + Q: H_D^1(M; E) \rightarrow H^{-1}(M; E). \quad (8)$$

We further note that in the case of the *pure Dirichlet problem*, the distinction between  $P$  and  $\tilde{P}$  disappears. We need to consider  $\tilde{P}$  only in order to deal with Neumann conditions for  $u \in H^1(M; E)$ , for which the restriction of the normal derivative to the boundary does not make sense (and hence, we shall deal with these low regularity Neumann conditions in a weak sense). Note that  $P$  determines  $\tilde{P}$  as well.

*Remark 2.3.* The best way to see the differences and the similarities between  $P$  and  $\tilde{P}$  is as follows. Let us consider  $h \in H^{k-1/2}(\partial_N M; E)$  and  $f \in H^{k-1}(M; E)$ . Let

$$F(v) := \int_M \langle f, v \rangle \, \text{dvol}_g + \int_{\partial_N M} \langle h, v \rangle \, dS,$$

where  $dS$  is the induced volume form on  $\partial_D M$  and  $\langle \cdot, \cdot \rangle$  denotes the Hermitian form on  $E$ . Then  $F$  defines a linear functional on  $H_D^1(M; \bar{E})$  and hence an element of  $H_D^1(M; E)^*$ . We denote  $\Phi_k(f, h) := F$  the induced map

$$\Phi_k: \check{H}^{k-1}(M; E) := H^{k-1}(M; E) \oplus H^{k-1/2}(\partial_N M; E) \rightarrow H_D^1(M; E)^*.$$

The equation

$$\tilde{P}(u) = F \quad (9)$$

is then equivalent to the mixed boundary value problem

$$\begin{cases} Pu = f & \text{in } M \\ u = 0 & \text{on } \partial_D M \\ \partial_\nu^a u = h & \text{on } \partial_N M \end{cases} \quad (10)$$

that is the boundary problem (1) with  $h_D = 0$ , where  $\partial_\nu^a u := \nu \cdot a(\nabla u)$ . Indeed,  $u = 0$  on  $\partial_D M$  since  $u$  is in the domain of  $\tilde{P}$ . The rest follows from Equation (6), which gives

$$\begin{aligned} \tilde{P}(u)(w) - F(w) &= B_a(u, w) + \langle Qu, w \rangle - \int_M \langle f, w \rangle \, \text{dvol}_g - \int_{\partial_N M} \langle h, w \rangle \, dS \\ &= \langle (\nabla_{H^1})^*(a\nabla u) + Qu - f, w \rangle_M - \int_{\partial_N M} \langle h, w \rangle \, dS \\ &= \langle (\nabla_{L^2})^*(a\nabla u) + Qu - f, w \rangle_M + \int_{\partial_N M} \langle \nu \cdot a(\nabla u) - h, w \rangle \, dS. \end{aligned}$$

This discussion will be useful in treating the mixed boundary value problem (1) (or rather (10)) using a weak formulation in Subsection 4.2.



For second order operators in divergence form, we have the following mapping properties. Recall that  $a \in W^{k,\infty}(M; [(T^*M \otimes_{\mathbb{R}} E) \otimes_{\mathbb{C}} (T^*M \otimes_{\mathbb{R}} \bar{E})]')$  defines a  $W^{k,\infty}$ -sesquilinear form on  $T^*M \otimes_{\mathbb{R}} E$ , and hence a second order differential operator in divergence form  $P_a$ .

**Lemma 2.4.** *Let  $a \in W^{k,\infty}(M; [(T^*M \otimes_{\mathbb{R}} E) \otimes_{\mathbb{C}} (T^*M \otimes_{\mathbb{R}} \bar{E})]')$ , let  $k \geq 1$ , and assume that  $Q$  has coefficients in  $W^{k-1,\infty}$ . Then  $\tilde{P} := \tilde{P}_a + \tilde{Q}$  defines a continuous map  $P: H^{k+1}(M; E) \rightarrow H^{k-1}(M; E)$ . Moreover,  $P \in \mathcal{L}(H^{k+1}(M; E), H^{k-1}(M; E))$  depends continuously on the coefficients.*

The reader should compare the mapping properties of this lemma to Equations (7) and (8), which amount to the statement of the Lemma for  $k = 0$  (in which case, we need to adjust the formulation and use  $\tilde{P}$  instead of  $P$ ).

*Proof of Lemma 2.4.* For  $k \geq 0$ , this follows from the relation (5) and from  $\text{res} \circ \nabla^* = -(\nabla_{L^2})^*$ .  $\square$

In order to study the invertibility of the operator  $P$  in Equation (10), one often uses (strong) 'coercivity'. Recall the following basic concept, where we use the terminology from [?, 26].

**Definition 2.5.** A bounded sesquilinear form  $a$  on  $T^*M \otimes_{\mathbb{R}} E$  will be called *strongly coercive* if there exists  $c_a > 0$  satisfying

$$\Re a(\xi, \xi) \geq c_a \|\xi\|^2, \quad \text{for all } \xi \in T^*M \otimes_{\mathbb{R}} E.$$

Note that this definition depends on the choice of the Riemannian metric  $g$  on  $M$ , because both the strongly coercivity condition and the condition that  $a$  is bounded (i.e.  $a \in W^{0,\infty}(M; [(T^*M \otimes_{\mathbb{R}} E) \otimes_{\mathbb{C}} (T^*M \otimes_{\mathbb{R}} \bar{E})]')$ ) depend on the metric. However, since the metric is fixed in this paper, we will not show the metric in the notation.

We can now introduce the operators in which we are interested.

**Definition 2.6.** Let  $\{P := P_a + Q\}$  be a family of second order differential operator in divergence form with each  $a$  being a bounded sesquilinear form on  $T^*M \otimes_{\mathbb{R}} E$  and each  $Q$  a first order differential operator with coefficients in  $L^\infty$ . We shall say that the family  $\{P\}$  is a *bounded family* if the  $L^\infty$ -bounds on the sesquilinear forms and on the coefficients of the operators  $Q$  can be chosen independently of  $P$ . We say that the bounded family  $\{P\}$  is *uniformly strongly elliptic* if each  $a$  is strongly coercive and we can choose  $c_a$  in Definition 2.5 independent of  $P$ .

In particular, the  $j$ -th coefficients of the differential operators  $P$  in a uniformly strongly elliptic family  $\{P\}$  are uniformly bounded in  $W^{0,\infty}(M; \text{End}(E) \otimes TM^{\otimes j})$ . We shall say that an operator  $P := P_a + Q$  is *uniformly strongly elliptic* if the family  $\{P\}$  consisting of just the operator  $P$  is uniformly strongly elliptic. See [1, 2, 12, 21, 15] for more information on strongly elliptic operators and other related results.

**2.2. Uniform regularity estimates for families.** Let  $(M, g)$  be a Riemannian manifold with smooth boundary  $\partial M$ . We assume that  $M$  has totally bounded curvature (which, we recall, means that the curvature of  $M$  and all its covariant derivatives are bounded) and that the restriction  $H^k(M; E) \rightarrow H^{k-1/2}(\partial M; E)$ , the trace map, is continuous for all  $k \geq 1$ . This assumption is, of course, satisfied if  $M$  has bounded geometry, see Theorem 3.15. Let  $D$  be a differential operator on  $M$

with bounded coefficients (i.e. in  $W^{0,\infty}$ ) acting between sections of two Hermitian vector bundles  $E, E_1 \rightarrow M$ . Let  $C: \Gamma(M; E) \rightarrow \Gamma(\partial M; F)$  be a differential boundary condition, more precisely

$$Cu := (C'u)|_{\partial M}$$

for some differential operator  $C'$  on  $M$ . Typically in our applications, we will have  $E_1 = E$  and  $F = E|_{\partial M}$ , so we will assume that this is the case in what follows, for notational simplicity. The general case is treated in exactly the same way, just by slightly changing the notation. Of course, when dealing with strongly elliptic operators, we need to have  $E_1 = E$ . Similarly, when dealing with Dirichlet or Neumann boundary conditions, we need  $F = E$ . The operator  $C'$  is clearly not uniquely determined, for instance, if  $C$  represents Neumann boundary conditions, then  $C'$  depends on the extension of the unit normal field of the boundary to the interior. We shall say that  $C$  has coefficients in  $W^{k,\infty}$  if we can choose  $C'$  to have coefficients in  $W^{k,\infty}$ . Assume, for simplicity, that  $C$  (or rather a choice of  $C'$  with minimal order) has constant order  $j$  at every point of the boundary and that  $D$  is a second order operator.

**Notations 2.7.** We shall denote by  $\mathcal{D}^{k,j}(M; E)$  the set of pairs  $(D, C)$ , where  $D$  is a second order differential operator defined on sections of  $E \rightarrow M$  and  $C$  is an order  $j$  boundary condition, with both  $D$  and  $C$  assumed to have coefficients in  $W^{k,\infty}$ . In case  $M$  has no boundary and, thus, there are no boundary conditions, we shall denote the resulting space  $\mathcal{D}^{k,0}(M; E)$ . If  $E := \underline{\mathbb{C}}$  (that is, if we are dealing with *scalar* boundary value problems), then we shall drop the vector bundle from the notation.

We endow the space  $\mathcal{D}^{k,j}(M; E)$  with the norm defined by the maximum of the  $W^{k,\infty}$ -norms of the coefficients. Recall the following standard definition:

**Definition 2.8.** We say that  $(D, C) \in \mathcal{D}^{k,j}(M; E)$ ,  $k \geq j$ , satisfies an *order  $k$  regularity estimate on an open subset  $N \subset M$*  if there exists  $c_R > 0$  with the following property: For any  $w \in H^k(M; E)$  such that  $Dw \in H^{k-1}(M; E)$  and  $Cw \in H^{k-j+1/2}(\partial M; E)$ , we have  $w \in H^{k+1}(M; E)$  and

$$\|w\|_{H^{k+1}(M; E)} \leq c_R (\|Dw\|_{H^{k-1}(M; E)} + \|w\|_{H^k(M; E)} + \|Cw\|_{H^{k-j+1/2}(\partial M; E)}).$$

If  $S \subset \mathcal{D}^{k,0}(M; E)$ , we just drop the term  $\|Cw\|_{H^{k-j+1/2}(\partial M; E)}$  from the last equation.

The conditions that  $D$  and  $C$  have coefficients in  $W^{k,\infty}$  are needed so that  $Dw \in H^{k-1}(M; E)$  and  $Cw \in H^{k-j+1/2}(\partial M; E)$  for a large set of sections  $w$ , see Lemma 2.4.

We shall need a *uniform* version of this definition. We consider a family of boundary value problems  $(D, C)$  as follows. Let  $M$  be a Riemannian manifold with (possibly empty) boundary and  $E \rightarrow M$  a Hermitian vector bundle endowed with a metric preserving connection.

**Definition 2.9.** We say that the family  $S \subset \mathcal{D}^{k,j}(M; E)$  satisfies a *uniform order  $k$  regularity estimate on  $N \subset M$*  if each  $(D, C) \in S$  satisfies an order  $k$  regularity estimate on  $N \subset M$  and we can choose the bound  $c_R$  in Definition 2.8 to be independent of the choice of  $(D, C)$ . If  $S \subset \mathcal{D}^{k,0}(M; E)$ , we just consider  $D \in S$ .

**Proposition 2.10.** *Assume that  $S \subset \mathcal{D}^{k,j}(M; E)$  is compact and that each  $(D, C)$  satisfies an order  $k$  regularity estimate on  $M$ . Then  $S$  satisfies a uniform order  $k$  regularity estimate on  $M$ . The same result holds if  $S \subset \mathcal{D}^{k,0}(M; E)$ .*

*Proof.* Consider first the case  $S \subset \mathcal{D}^{k,j}(M; E)$ . Let us assume the contrary and show that we obtain a contradiction. Thus, let us assume that there exist sequences  $(D_i, C_i) \in S$  and  $0 \neq w_i \in H^{k+1}(M; E)$  such that

$$\|w_i\|_{H^{k+1}(M; E)} \geq 2^i (\|D_i w_i\|_{H^{k-1}(M; E)} + \|w_i\|_{H^k(M; E)} + \|C_i w_i\|_{H^{k-j+1/2}(\partial M; E)}).$$

Since  $S$  forms a compact subset in  $\mathcal{D}^{k,j}(M; E)$ , by replacing  $(D_i, C_i)$  with a subsequence, if necessary, we can assume that  $(D_i, C_i)$  converges. Let us denote the limit with  $(D, C) \in S$ . Thus, there is a sequence  $\epsilon_i \rightarrow 0$  with

$$\begin{aligned} \|D_i w\|_{H^{k-1}(M; E)} &\geq \|D w\|_{H^{k-1}(M; E)} - \epsilon_i \|w\|_{H^{k+1}(M; E)}, \\ \|C_i w\|_{H^{k-j+\frac{1}{2}}(\partial M; E)} &\geq \|C w\|_{H^{k-j+\frac{1}{2}}(\partial M; E)} - \epsilon_i \|w\|_{H^{k+\frac{1}{2}}(\partial M; E)}. \end{aligned}$$

Together with the assumed continuity of the trace map, this implies

$$\begin{aligned} \|w_i\|_{H^{k+1}(M; E)} &\geq 2^i (\|D w_i\|_{H^{k-1}(M; E)} + \|w_i\|_{H^k(M; E)} + \|C w_i\|_{H^{k-j+1/2}(\partial M; E)} \\ &\quad - c' \epsilon_i \|w_i\|_{H^{k+1}(M; E)}). \end{aligned}$$

On the other hand,  $(D, C)$  satisfies, by assumption, an order  $k$  regularity estimate. Consequently, there is a  $c > 0$  such that

$$\|D w_i\|_{H^{k-1}(M; E)} + \|w_i\|_{H^k(M; E)} + \|C w_i\|_{H^{k-j+1/2}(\partial M; E)} \geq c^{-1} \|w_i\|_{H^{k+1}(M; E)},$$

and hence we obtain

$$\|w_i\|_{H^{k+1}(M; E)} \geq 2^i (c^{-1} - c' \epsilon_i) \|w_i\|_{H^{k+1}(M; E)}.$$

For  $i \rightarrow \infty$ , this gives the desired contradiction since  $2^i (c^{-1} - c' \epsilon_i) \rightarrow \infty$ , for  $i \rightarrow \infty$ , and  $\|w_i\|_{H^{k+1}(M; E)} \neq 0$ . This completes the proof if  $S \subset \mathcal{D}^{k,j}(M; E)$ .

If  $S \subset \mathcal{D}^{k,0}(M; E)$ , the proof is obtained by simply dropping the terms that contain  $C w$  from the above proof. (We can also replace  $2\epsilon_i$  with  $\epsilon_i$ , but that is not necessary.)  $\square$

Recall that a *relatively compact* subset is a subset whose closure is compact.

**Proposition 2.11.** *Let  $N \subset M$  be a relatively compact open subset and let  $S \subset \mathcal{D}^{k+1,j}(M; E)$  be a bounded subset. Assume that every  $\mathcal{D}^{k,j}(N; E)$ -limit  $(\tilde{D}, \tilde{C})$  (so  $(\tilde{D}, \tilde{C}) \in \mathcal{D}^{k,j}(N; E)$ ) of a sequence  $(D_i, C_i) \in S$  satisfies an order  $k$  regularity estimate on  $N$ . Then  $S$  satisfies a uniform order  $k$  regularity estimate on  $N$ . The same result holds if  $S \subset \mathcal{D}^{k+1,0}(M; E)$ .*

We remark that in this proposition the compactness condition of Proposition 2.10 is replaced by a higher regularity assumptions on the coefficients. This is needed in order to use the Arzela-Ascoli Theorem. Moreover, we note that by choosing a constant sequence, we see that the assumptions imply that each element in  $S$  satisfies an order  $k$  regularity estimate on  $N$ .

*Proof of Proposition 2.11.* We treat explicitly only the case  $S \subset \mathcal{D}^{k+1,j}(M; E)$ , the other one being completely similar. Since the coefficients of all boundary value problems in  $S$  are bounded in  $W^{k+1,\infty}(M)$ , the set of coefficients of the operators  $(D, C) \in S$  is precompact in  $W^{k,\infty}(N)$ , by the Arzela-Ascoli theorem. Let  $K$  be the closure of  $S$  in  $\mathcal{D}^{k,j}(N, E)$ , which is therefore a compact set. Moreover, our

assumptions imply that every element in  $K$  satisfies an order  $k$  regularity estimate. Proposition 2.10 then implies the result.  $\square$

**Corollary 2.12.** *Let  $S \subset \mathcal{D}^{k+1,j}(B_r^m(0) \times [0, r])$  be a bounded, uniformly strongly elliptic family of scalar boundary value problems on  $B_r^m(0) \times [0, r] \subset \mathbb{R}^{m+1}$  equipped with the euclidean metric,  $r \leq \infty$ . We assume that the boundary conditions are either Dirichlet or Neumann (depending on  $j$ ). Then the family  $S$  satisfies a uniform order  $k$  regularity estimate on  $B_{r'}^m(0) \times [0, r')$ ,  $r' < r$ .*

*Proof.* Let  $(D_n, C_n) \in S$  converge to  $(D, C) \in \mathcal{D}^{k,j}(B_{r'}^m(0) \times [0, r')$ . Then  $D$  is a uniformly strongly elliptic operator because the parameter  $c_a$  stays away from 0 on  $S$ , in view of Definition 2.6. Moreover, uniformly strongly elliptic operators with either Dirichlet or Neumann boundary conditions satisfy regularity estimates (of orders up to the regularity of the coefficients), see [20]. See also [1] for smooth coefficients. We can then use Proposition 2.11, for the relatively compact subset  $N := B_{r'}^m(0) \times [0, r')$  of  $M := B_r^m(0) \times [0, r)$ , since the type of boundary conditions (Dirichlet or Neumann) do not change by taking limits.  $\square$

Analogously, (in fact, even easier, since we do not have to take boundary conditions into account), we obtain

**Corollary 2.13.** *Let  $S \subset \mathcal{D}^{k+1,0}(B_r^{m+1}(0); E)$  be a bounded uniformly strongly elliptic family of differential operators on  $B_r^{m+1}(0) \subset \mathbb{R}^{m+1}$ , for some  $0 < r \leq \infty$ . Then the family  $S$  satisfies a uniform order  $k$  regularity estimate on  $B_{r'}^{m+1}(0) \subset \mathbb{R}^{m+1}$  for any  $r' < r$ .*

*Remark 2.14.* The regularity results of this section extend to the  $L^p$ -Sobolev spaces  $W^{k,p}$ ,  $1 < p < \infty$ , with essentially the same proofs by using also the results in [15].

We shall use the weaker result of Proposition 2.10 to deal with Lie manifolds in Subsection 4.3. The slightly weaker results of Corollaries 2.12 and 2.13 will be used to deal with general manifolds of bounded geometry. This is the reason for which we need the stronger regularity on the coefficients for general bounded geometry regularity estimates.

### 3. MANIFOLDS WITH BOUNDARY AND BOUNDED GEOMETRY

We include here some needed definitions and results on manifolds with boundary and bounded geometry. Most of them can be found in [5], where further references are also provided.

**3.1. Definition and Poincaré's inequality.** Let  $(M^{m+1}, g)$  be a Riemannian manifold without boundary. Recall that the *injectivity radius*  $r_{\text{inj}}(M)$  of  $M$  (or, more precisely, of  $(M, g)$ ), is defined as

$$r_{\text{inj}}(M) := \sup\{r > 0 \mid \text{The exponential map } \exp_p^M : B_r^{m+1}(0) \subset T_p M \rightarrow M \\ \text{is a diffeomorphism onto its image for all } p \in M\}.$$

**Definition 3.1.** If  $\partial M = \emptyset$ , we say that a Riemannian manifold  $M$  has *bounded geometry* if  $TM$  has totally bounded curvature and  $r_{\text{inj}}(M) > 0$ .

Recall the following definition from [5, 16].

**Definition 3.2.** Let  $(M^{m+1}, g)$  be a Riemannian manifold of bounded geometry with a hypersurface  $N^m \subset M$  that admits a unit normal field  $\nu$ . We say that a closed subset  $N$  of  $M$  is a bounded geometry hypersurface if the following conditions are satisfied:

- (i) The second fundamental form of  $N$  and all its covariant derivatives are bounded.
- (ii) There is a number  $r_\partial > 0$  such that  $N \times (-r_\partial, r_\partial) \rightarrow M$ ,  $(x, t) \mapsto \exp_x^M(t\nu_x)$  is injective.

Note that if  $N$  is as in the definition above, then  $N$  with the induced metric is itself a manifold with bounded geometry [5, Corollary 6.13]. We have the following definition equivalent to the one in [25], see also [5, Theorem 6.1].

**Definition 3.3.** A Riemannian manifold  $M$  with smooth boundary  $\partial M$  has *bounded geometry* if there is a Riemannian manifold  $\widehat{M}$  with bounded geometry,  $M \subset \widehat{M}$ , such that  $\partial M$  is a bounded geometry hypersurface in  $\widehat{M}$ .

We shall assume from now on that  $M$  is a manifold with boundary and bounded geometry. As unit normal vector field we will always choose the inner unit normal field.

**Definition 3.4.** If  $M$  is a Riemannian manifold with boundary and bounded geometry, if  $\partial_D M$  is an open component of the boundary  $\partial M$ , and if  $M \subset U_R(\partial_D M) := \{p \in M \mid \text{dist}(p, \partial_D M) \leq R\}$  for some  $R > 0$ , then we shall say that  $(M, \partial_D M)$  has *finite width*. If  $\partial_D M = \partial M$ , we shall also say that  $M$  has *finite width*.

Manifolds with finite width satisfy the Poincaré inequality for functions vanishing on  $\partial_D M$ :

**Theorem 3.5.** [5, Theorem 3.10] *Assume that  $(M, \partial_D M)$  has finite width and that  $E$  has totally bounded curvature. Let  $1 \leq p \leq \infty$ . Then there exists  $C_M > 0$  such that*

$$C_M \|f\|_p \leq \|\nabla f\|_p \quad (11)$$

for all  $f \in H_{loc}^1(M; E)$  with  $f|_{\partial_D M} = 0$ .

**3.2. Sobolev spaces and partitions of unity.** We now recall the definition of Sobolev spaces using partitions of unity and ‘‘Fermi coordinates’’ following [16]. See especially Definition 20 of that paper, whose notation we follow here. Recall that  $r_{\text{inj}}(M)$  and  $r_{\text{inj}}(\partial M)$  denote, respectively, the injectivity radii of  $M$  and  $\partial M$ . Also, let  $\nu$  be the inner unit normal vector field of  $\partial M$  and let  $r_\partial$  as in the bounded geometry condition for  $\partial M$  in  $M$ , Definition 3.2. Moreover, for a metric space  $X$ , we shall denote by  $B_r^X(p)$  the open ball of radius  $r$  centered at  $p$  and set  $B_r^m(p) := B_r^{\mathbb{R}^m}(p)$ . Recall that we assume for the rest of the paper that  $M$  is a manifold with boundary and bounded geometry.

In the following, we shall identify  $T_p M$  with  $\mathbb{R}^{m+1}$  and, respectively,  $T_p \partial M$  with  $\mathbb{R}^m$ , using an orthonormal basis, thus obtaining a diffeomorphism  $\exp_p^M : B_r^{m+1}(0) \rightarrow B_r^M(p)$ . For  $r < \frac{1}{2} \min\{r_{\text{inj}}(\partial M), r_{\text{inj}}(M), r_\partial\}$  we define the maps

$$\begin{cases} \kappa_p : B_{2r}^m(0) \times [0, 2r) \rightarrow M, & \kappa_p(x, t) := \exp_q^M(t\nu_q), & \text{if } p \in \partial M, q := \exp_p^{\partial M}(x) \\ \kappa_p : B_r^{m+1}(0) \rightarrow M, & \kappa_p(v) := \exp_p^M(v), & \text{if } \text{dist}(p, \partial M) \geq r, \end{cases}$$

with range

$$U_p(r) := \begin{cases} \kappa_p(B_{2r}^m(0) \times [0, 2r)) \subset M & \text{if } p \in \partial M \\ \kappa_p(B_r^{m+1}(0)) = \exp_p^M(B_r^{m+1}(0)) \subset M & \text{otherwise.} \end{cases} \quad (12)$$

In the next definition we need to consider only the case  $p \in \partial M$ , however, the other case will be useful in the next section when considering partitions of unity.

**Definition 3.6.** Let  $r_{FC} := \min \{ \frac{1}{2} r_{\text{inj}}(\partial M), \frac{1}{4} r_{\text{inj}}(M), \frac{1}{2} r_{\partial} \}$  and  $0 < r \leq r_{FC}$ . Then  $\kappa_p: B_r^m(0) \times [0, r) \rightarrow U_p(r)$  is called a *Fermi coordinate chart* at  $p \in \partial M$  (see Figure 1). The charts  $\kappa_p$  for  $\text{dist}(p, \partial M) \geq r$  are called *geodesic normal coordinates*.

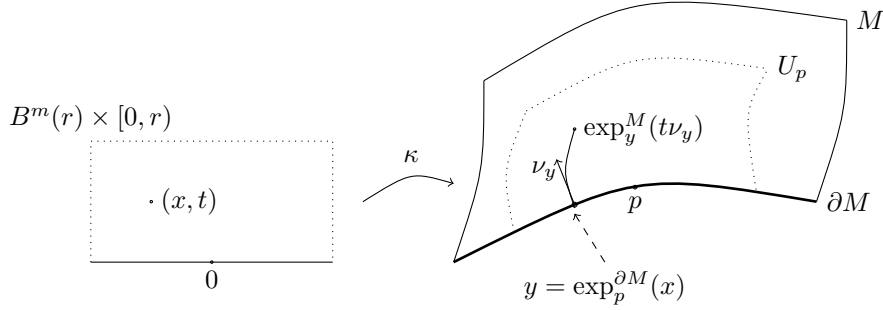


FIGURE 1. Fermi coordinates

To define our Sobolev spaces, we need suitable coverings of our manifold. For the sets in the covering that are away from the boundary, we will use geodesic normal coordinates, whereas for the sets that intersect the boundary, we will use the Fermi coordinates introduced in Definition 3.6.

We shall need suitable coverings and partitions of unity.

**Definition 3.7.** Let  $M$  be a manifold with boundary and bounded geometry and let  $0 < r \leq r_{FC} := \min \{ \frac{1}{2} r_{\text{inj}}(\partial M), \frac{1}{4} r_{\text{inj}}(M), \frac{1}{2} r_{\partial} \}$ , as in Definition 3.6. A subset  $\{p_\gamma\}_{\gamma \in \mathbb{N}}$  is called an  *$r$ -covering subset of  $M$*  if the following conditions are satisfied:

- (i) For each  $R > 0$ , there exists  $N_R \in \mathbb{N}$  such that, for each  $p \in M$ , the set  $\{\gamma \in \mathbb{N} \mid \text{dist}(p_\gamma, p) < R\}$  has at most  $N_R$  elements.
- (ii) For each  $\gamma \in \mathbb{N}$ , we have either  $p_\gamma \in \partial M$  or  $\text{dist}(p_\gamma, \partial M) \geq r$ , so that  $U_\gamma := U_{p_\gamma}(r)$  is defined, compare to (12).
- (iii)  $M \subset \cup_{\gamma=1}^{\infty} U_\gamma$ .

*Remark 3.8.* If  $0 < r < r_{FC}$ , then we can always find an  $r$ -covering subset of  $M$ , since  $M$  is a manifold with boundary and bounded geometry [16, Remark 4.6]. Moreover, it then follows from (i) of Definition 3.7 that the coverings  $\{U_\gamma\}$  of  $M$  and  $\{U_\gamma \cap \partial M\}$  of  $\partial M$  are uniformly locally finite.

We shall need the following class of partitions of unity defined using  $r$ -covering sets. Recall the definition of the sets  $U_\gamma := U_{p_\gamma}(r)$  from Equation (12).

**Definition 3.9.** A partition of unity  $\{\phi_\gamma\}_{\gamma \in \mathbb{N}}$  of  $M$  is called an  *$r$ -uniform partition of unity associated to the  $r$ -covering set  $\{p_\gamma\} \subset M$* , see Definition 3.7, if the support of each  $\phi_\gamma$  is contained in  $U_\gamma$  and  $\sup_\gamma \|\phi_\gamma\|_{W^{\ell, \infty}(M)} < \infty$  for each fixed  $\ell \in \mathbb{Z}_+$ .

*Remark 3.10.* Given an  $r$ -covering set  $S$  with  $r \leq r_{FC}$ , an  $r$ -uniform partition of unity associated to  $S \subset M$  always exists, since  $M$  is a manifold with boundary and bounded geometry [16, Lemma 25].

In order to deal with boundary value problems with values in a vector bundle (systems), we will also need the concept of synchronous trivializations, which we briefly recall here:

**Definition 3.11.** Let  $M$  be a Riemannian manifold with boundary and bounded geometry, and let  $E \rightarrow M$  be a Hermitian vector bundle with metric connection. Let  $(U_\gamma, \kappa_\gamma, \phi_\gamma)$  be Fermi and geodesic normal coordinates on  $M$  together with an associated  $r$ -uniform partition of unity as in the definitions above. If  $p_\gamma \in M \setminus U_r(\partial M)$ , then  $E|_{U_\gamma}$  is trivialized by parallel transport along radial geodesics emanating from  $p_\gamma$ , see [16, Definition 5.2] for a more explicit description. If  $p_\gamma \in \partial M$ , then we trivialize  $E|_{U_\gamma}$  as follows: First we trivialize  $E|_{U_\gamma \cap \partial M}$  along the underlying geodesic normal coordinates on  $\partial M$ . Then, we trivialize by parallel transport along geodesics emanating from  $\partial M$  and being normal to  $\partial M$ , compare [16, Definition 5.12]. The resulting trivializations are called *synchronous trivializations along Fermi coordinates* and are maps

$$\xi_\gamma: \kappa_\gamma^{-1}(U_\gamma) \times \mathbb{C}^t \rightarrow E|_{U_\gamma} \quad (13)$$

where  $t$  is the rank of  $E$ .

In the following, we will always assume that  $E \rightarrow M$  has totally bounded curvature. One way to think of such vector bundles is given by the following lemma:

**Lemma 3.12.** *Let us assume that  $M$  is a manifold with bounded geometry (possibly with boundary) and that  $E \rightarrow M$  is a Hermitian vector bundle of totally bounded curvature. Then there exists an isometric embedding  $E \subset M \times \mathbb{C}^N$  into the trivial  $N$ -dimensional vector bundle with the standard metric such that, if  $e$  denotes the orthogonal projection onto  $E$ , then  $e \in M_N(W^{\infty, \infty}(M))$ . Conversely, if  $e \in M_N(W^{\infty, \infty}(M))$  and  $E := e(M \times \mathbb{C}^N)$ , then  $E$  with the Grassmann (projection) connection has totally bounded curvature.*

*Proof.* Let us consider for each open subset  $U_\gamma$  as above the synchronous trivialization  $\xi_\gamma: E|_{U_\gamma} \rightarrow \kappa_\gamma^{-1}(U_\gamma) \times \mathbb{C}^t$  from Equation (13), with  $\mathbb{C}^t$  the typical fiber above  $p_\gamma$ . Then  $\phi_\gamma^{1/2} \xi_\gamma$  extends to a vector bundle map  $E \rightarrow M \times \mathbb{C}^t$  that is in  $W^{\infty, \infty}$  since  $M$  has bounded geometry and the connection on  $E$  has totally bounded curvature. Let  $N = N_{5r}$ , with  $N_{5r}$  as in Definition 3.7. By the construction of the sets  $U_\gamma$ , we can divide the set of all  $\gamma$ 's into  $N + 1$  disjoint subsets  $\Gamma_k$ , such that, for each fixed  $k$  and any  $\gamma, \gamma' \in \Gamma_k$ , the sets  $U_\gamma$  and  $U_{\gamma'}$  are disjoint, by the construction of the sets  $U_\gamma$ . Let  $\Psi_k := \sum_{\gamma \in \Gamma_k} \phi_\gamma^{1/2} \xi_\gamma$  and  $\Psi := (\Psi_1, \Psi_2, \dots, \Psi_{N+1}): E \rightarrow M \times \mathbb{C}^{t(N+1)}$  be the resulting bundle morphism. Then  $\Psi$  is isometric, it is in  $W^{\infty, \infty}$ , and hence  $e := \Psi \Psi^*$  is the desired projection.  $\square$

This lemma may also be used to reduce differential operators acting on vector bundles to matrices of scalar differential operators.

We have the following proposition that is a direct consequence of Theorems 14 and 26 in [16] (see [5] for more related references). See also [4, 3, 6, 18, 27, 28] for related results, in particular, for the use of the partitions of unity. See [13] and its outgrowth [14] for an introduction to manifolds of bounded geometry.



**Proposition 3.13.** *Let  $M$  be a Riemannian manifold with boundary and bounded geometry. Let  $\{\phi_\gamma\}$  be a uniform partition of unity associated to an  $r$ -covering set  $\{p_\gamma\} \subset M$  and let  $\kappa_\gamma = \kappa_{p_\gamma}$  be as in Definition 3.6. Let  $E \rightarrow M$  be a vector bundle with totally bounded curvature with trivializations  $\xi_\gamma$  as in Definition 3.11. Then*

$$\|u\|^p := \sum_\gamma \|\xi_\gamma^*(\phi_\gamma u)\|_{W^{s,p}}^p$$

defines a norm equivalent to the standard norm on  $W^{s,p}(M; E)$ ,  $s \in \mathbb{R}$ ,  $1 < p < \infty$ .

We immediately obtain

**Corollary 3.14.** *Assume, as before, that  $M$  is a Riemannian manifold with boundary and bounded geometry and that  $E \rightarrow M$  has totally bounded curvature. Then the space  $\Gamma_c(M; E)$  of smooth, compactly supported sections of  $E$  is dense in  $W^{s,p}(M; E)$  for  $s \in \mathbb{R}$  and  $1 < p < \infty$ .*

*Proof.* It is enough to truncate the sum  $u = \sum_\gamma \phi_\gamma u$  and then to use Proposition 3.13.  $\square$

Similarly, we have the following extension of the trace theorem to the case of manifolds with boundary and bounded geometry, see Theorem 27 in [16] (see [5] for more references).

**Theorem 3.15** (Trace theorem). *Let  $M$  be a manifold with boundary and bounded geometry and let  $E \rightarrow M$  have totally bounded curvature. Then, for every  $s > 1/2$ , the restriction to the Dirichlet part of the boundary  $\text{res}: \mathcal{C}_c^\infty(M) \rightarrow \mathcal{C}_c^\infty(\partial_D M)$  extends to a continuous, surjective map*

$$\text{res}: H^s(M; E) \rightarrow H^{s-\frac{1}{2}}(\partial_D M; E).$$

The results of [5], compare Theorem 3.5, give the following (recall that our differential operators are assumed to have coefficients in  $W^{0,\infty}$ ). Let  $C_M \geq 0$  be the (best) constant such that (11) holds and  $c_a$  the best constant defining the strongly coercivity of  $a$ . In particular, if  $(M, \partial_D M)$  does not satisfy a Poincaré inequality as in Theorem 3.5,  $C_M = 0$ .

**Theorem 3.16.** *Let  $M$  be a manifold with boundary and bounded geometry and let  $E \rightarrow M$  have totally bounded curvature. Let  $P = P_a + X$  be a uniformly strongly elliptic second order differential operator on  $M$  with  $X \in L^\infty(M; TM)$ . Let  $c \in W^{0,\infty}(M)$ ,  $\Re(c) \geq c_m \in \mathbb{R}$ ,  $c_m \geq 0$ , and  $\|X\| := \|X\|_{H^1, L^2}$ . We assume that one of the following two conditions is satisfied*

$$\begin{cases} 4c_a c_m > \|X\|^2 & \text{or} \\ 4c_a c_m \geq \|X\|^2 & \text{and } c_m < C_M c_a. \end{cases} \quad (14)$$

Then the operator  $\tilde{P} + c: H_D^1(M; E) \rightarrow H_D^1(M; E)^*$  is an isomorphism.

Note that latter case can only occur if  $C_M > 0$ , which entails that  $(M, \partial_D M)$  has finite width; compare with Theorem 3.5.



*Proof of Theorem 3.16.* Let us denote  $|u| := \|\nabla u\|_{L^2(M; T^*M \otimes E)}$ . Then the definitions of  $C_M$  and  $c_a$  give

$$\begin{aligned} \Re\langle(\tilde{P} + c)u, u\rangle_M &= \Re(B_a(u, u) + \langle Xu, u\rangle_M + \langle cu, u\rangle_M) \\ &\geq c_a|u|^2 - \|X\| \|u\| \|u\|_{L^2(M; E)} + c_m \|u\|_{L^2(M; E)}^2 \\ &\geq (c_a - \epsilon)|u|^2 - \|X\| \|u\| \|u\|_{L^2(M; E)} + (c_m + \epsilon C_M) \|u\|_{L^2(M; E)}^2. \end{aligned}$$

If  $\delta := 4(c_a - \epsilon)(c_m + \epsilon C_M) - \|X\|^2 > 0$  for some  $\epsilon \geq 0$  small enough, this gives that there exists  $\eta > 0$  such that  $\Re\langle(\tilde{P} + c)u, u\rangle \geq \eta(|u|^2 + \|u\|_{L^2(M; E)}^2) = \eta \|u\|_{H^2(M; E)}^2$ . We thus obtain that  $\tilde{P} + c$  satisfies the assumptions of the Lax-Milgram Lemma [15] and hence that it is an isomorphism  $H_D^1(M; E) \rightarrow H_D^1(M; E)^*$ . If  $4c_a c_m > \|X\|^2$ , we can take  $\epsilon = 0$  to obtain  $\delta > 0$ . If  $4c_a c_m \geq \|X\|^2$ ,  $(M, \partial_D M)$  has finite width, and  $c_m < C_M c_a$ , it suffices to take  $\epsilon > 0$  small enough to again obtain  $\delta > 0$ .  $\square$

We notice that, the conditions of the above theorem are such that, in general,  $c_a > 0$  and  $\inf_M \Re c \geq \epsilon > 0$ . If, however,  $(M, \partial_D M)$  has finite width and  $X = 0$ , then we can also take  $\inf_M \Re c \geq 0$ .

#### 4. HIGHER REGULARITY AND BOUNDED GEOMETRY

The relevance of uniform regularity conditions introduced in Subsection 2.2 is that it allows us to obtain higher regularity on manifolds with boundary and bounded geometry and suitable boundary conditions as follows.

**4.1. General boundary conditions.** Let  $M$  be a Riemannian manifold with boundary and bounded geometry, as before. Let  $(P, B)$  be a boundary value problem on  $M$ . We assume, for notational simplicity, that  $B$  has constant order  $j$  at the boundary. For the same reasons, we also assume that all vector bundles (domains and ranges of operators) are the same  $E$ . The results of this subsection hold, however, in full generality (when the vector bundles  $E$ ,  $E_1$ , and  $F$  of Subsection 2.2 are distinct). Let  $0 < r \leq r_{FC}$ , as in Definition 3.6, and recall  $U_p$  and  $\kappa_p$  from (12) and  $\xi_p$  from Definition 3.11, where either  $p \in \partial M$  or  $\text{dist}(p, \partial M) \geq r$ . We denote by  $(P_p, B_p)$  the induced boundary value problems on  $\kappa_p^{-1}(U_p) = B_{2r}^m(0) \times [0, 2r)$ , if  $p \in \partial M$ . Then  $P_p = \xi_p^* \circ P \circ (\xi_p)_*$  and  $B_p = \xi_p^* \circ B \circ (\xi_p)_*$ , with the obvious notation, meaning that the operators correspond through the diffeomorphisms  $\xi_p$ . If  $\text{dist}(p, \partial M) \geq r$ , there is no  $B_p$  and we obtain a differential operator  $P_p$  on  $B_r^{m+1}(0)$ . Let us denote the rank of  $E$  by  $t$ . We denote by

$$\begin{aligned} \mathcal{F}_b &:= \{(P_p, B_p) \mid p \in \partial M\} \subset \mathcal{D}^{0,j}(B_{2r}^m(0) \times [0, 2r); \mathbb{C}^t) \\ \mathcal{F}_i &:= \{(P_p) \mid \text{dist}(p, \partial M) \geq r\} \subset \mathcal{D}^{0,\emptyset}(B_r^{m+1}(0); \mathbb{C}^t) \end{aligned} \tag{15}$$

the induced *boundary* and *interior* families of operators. Note that we always equip  $B_{2r}^m(0) \times [0, 2r)$ , (respectively,  $B_r^{m+1}(0)$ ) with the euclidean metric. We tacitly assumed in Equation (15), in order to define the Fermi and normal geodesic coordinates, one considered *all* orthogonal trivializations of the tangent spaces (that is, of  $T_p \partial M$ , if  $p \in \partial M$ , and of  $T_p M$ , if  $\text{dist}(p, \partial M) \geq r$ ). Of course, this yields many equivalent boundary value problems, but it does not affect the boundedness of the families  $\mathcal{F}_b$  and  $\mathcal{F}_i$ , where the boundedness is in the natural norms on  $\mathcal{D}^{k,j}(B_r^{m+1}(0), \mathbb{C}^t)$  and, respectively, on  $\mathcal{D}^{k,j}(B_{2r}^m(0) \times [0, 2r), \mathbb{C}^t)$ .

**Lemma 4.1.** *Assume  $(P, B) \in \mathcal{D}^{k,j}(M, \partial M)$ . Then  $\mathcal{F}_i \subset \mathcal{D}^{k,\emptyset}(B_r^{m+1}(0), \mathbb{C}^t)$  and  $\mathcal{F}_b \subset \mathcal{D}^{k,j}(B_{2r}^m(0) \times [0, 2r), \mathbb{C}^t)$ .*

*Proof.* This follows from the bounded geometry assumption on  $M$ , since the Fermi and normal geodesic coordinates maps  $\kappa_p$  of Equation (12) map  $W^{k,\infty}(M)$  to  $W^{k,\infty}(U_\gamma)$ , since the Christoffel symbols in the chosen coordinate charts are bounded together with all their derivatives, compare with [16].  $\square$

**Theorem 4.2.** *Let  $M$  be a Riemannian manifold with boundary and bounded geometry and let  $E \rightarrow M$  be a Hermitian vector bundle with totally bounded curvature. Let  $(P, B) \in \mathcal{D}^{k,j}(M; E)$  and let each of the families  $\mathcal{F}_b := \{(P_p, B_p) \mid p \in \partial M\}$  and  $\mathcal{F}_i := \{P_p \mid \text{dist}(p, \partial M) \geq r\}$  (see Equation 15 and above) satisfy a uniform order  $k$  regularity estimate. Then  $(P, B)$  satisfies an order  $k$  regularity estimate.*

*Proof.* This follows from Definition 2.8 of uniform order  $k$  regularity estimates, from Proposition 3.13, and from Lemma 2.1:

$$\begin{aligned}
\|u\|_{H_{k+1}}^2 &\lesssim \sum_{\gamma} \|\xi_{\gamma}^*(\phi_{\gamma}u)\|_{H_{k+1}}^2 \\
&\lesssim \sum_{\gamma} \left( \|P_{\gamma}\xi_{\gamma}^*(\phi_{\gamma}u)\|_{H_{k-1}} + \|B_{\gamma}\xi_{\gamma}^*(\phi_{\gamma}u)\|_{H_{k-j+\frac{1}{2}}} + \|\xi_{\gamma}^*(\phi_{\gamma}u)\|_{H_k} \right)^2 \\
&\lesssim \sum_{\gamma} \left( \|\xi_{\gamma}^*P(\phi_{\gamma}u)\|_{H_{k-1}}^2 + \|\xi_{\gamma}^*B(\phi_{\gamma}u)\|_{H_{k-j+\frac{1}{2}}}^2 + \|\xi_{\gamma}^*(\phi_{\gamma}u)\|_{H_k}^2 \right) \\
&\lesssim (\|Pu\|_{H_{k-1}}^2 + \|Bu\|_{H_{k-j+\frac{1}{2}}}^2 + \|u\|_{H_k})^2 + \sum_{\gamma} \|\xi_{\gamma}^*[P, \phi_{\gamma}]u\|_{H_{k-1}}^2 \\
&\quad + \sum_{\gamma} \|\xi_{\gamma}^*[B, \phi_{\gamma}]u\|_{H_{k-j+\frac{1}{2}}}^2 \\
&\lesssim (\|Pu\|_{H_{k-1}}^2 + \|Bu\|_{H_{k-j+\frac{1}{2}}}^2 + \|u\|_{H_k})^2 + \sum_{\gamma} \|[P, \phi_{\gamma}]u\|_{H_{k-1}}^2 \\
&\quad + \sum_{\gamma} \|[B, \phi_{\gamma}]u\|_{H_{k-j+\frac{1}{2}}}^2,
\end{aligned}$$

since the trivializations  $\xi_{\gamma}$  have uniformly bounded norms. Next we notice that  $\sum_{\gamma} \|[P, \phi_{\gamma}]u\|_{H_{k-1}}^2 \leq \|u\|_{H_k}^2$  since the family  $[P, \phi_{\gamma}]u$  is uniformly locally finite. The boundary term is treated similarly by noticing also that if  $j = 0$  it is actually zero.  $\square$

*Remark 4.3.* It is known [1, 26, 17] that a family  $(D_y, C_y)$  of boundary value problems on  $\mathbb{R}_+^{m+1} \subset \mathbb{R}^{m+1}$  satisfies regularity estimates if, and only if,  $(D_y, C_y)$  satisfies the Shapiro-Lopatinski conditions for all  $y$ . It would be interesting to have a *uniform Shapiro-Lopatinski condition* that would imply a uniform, order  $k$  regularity estimate. That would allow us to consider operators with bounded coefficients in  $W^{k,\infty}$  in Corollary 2.12 and in Theorem 4.2, which means that we require less regularity for the coefficients.

*Remark 4.4.* The method of proof of Theorem 4.2 will yield similar global result in other classes of spaces, as long as the local regularity results are available and as long as a local description of these spaces using partitions of unity is available. This is the case for the  $L^p$ -Sobolev spaces,  $1 < p < \infty$ , for which we have both the local description using partitions of unity (Proposition 3.13) and the local regularity results [15].

**4.2. Mixed boundary conditions.** Let us turn now back to the study of mixed boundary value problems on  $M$ . Let  $P = P_a + Q$  be a second order differential operator in divergence form, as in (8), and endow it with the differential boundary conditions  $B: \mathcal{C}^\infty(M) \rightarrow \mathcal{C}^\infty(\partial M)$ , locally of order  $j \in \{0, 1\}$ , that define the mixed boundary value problem (1), namely:

$$Bu = \begin{cases} u & \text{on } \partial_D M \ (j = 0), \\ \partial_\nu^\alpha u & \text{on } \partial_N M \ (j = 1). \end{cases} \quad (16)$$

That is, the boundary conditions  $B$  amount to Dirichlet conditions on  $\partial_D M$  and to the associated Neumann conditions on  $\partial_N M$ . Let

$$\|Bv\|_k := \|v\|_{H^{k+1/2}(\partial_D M)} + \|\partial_\nu^\alpha v\|_{H^{k-1/2}(\partial_N M)}.$$

**Theorem 4.5.** *Let  $P = P_a + Q$  be a uniformly strongly elliptic second order differential operator on  $M$  with coefficients in  $W^{k+1, \infty}$  and let  $B$  be the boundary operator of Equation (16). Then  $(P, B)$  satisfies an order  $k$  regularity estimate in the sense that there exists  $c > 0$  such that, if  $u \in H^k(M)$ ,  $Pu \in H^{k-1}(M)$ ,  $u|_{\partial_D M} \in H^{k+1/2}(\partial_D M)$ , and  $\partial_\nu^\alpha u|_{\partial_N M} \in H^{k-1/2}(\partial_N M)$ , then  $u \in H^{k+1}(M)$  and*

$$\|u\|_{H^{k+1}(M)} \leq c (\|Pu\|_{H^{k-1}(M)} + \|u\|_{H^k(M)} + \|Bu\|_k).$$

Notice that in the statement of the theorem, we have dropped the condition that  $B$  be of constant order (it will be, nevertheless, of *locally constant* order).

*Proof.* Let  $\phi_0 \in W^{\infty, \infty}(M)$  be equal to 1 in a neighborhood of  $\partial_D M$  and be equal to 0 in a neighborhood of  $\partial_N M$  and  $\phi_1 := 1 - \phi_0$ . The result follows by applying Corollary 2.12 and Theorem 4.2 to  $\phi_0 u$  together with Dirichlet boundary conditions on the whole of  $\partial M$  and to  $(1 - \phi_0)u$  together with Neumann boundary conditions on the whole of  $\partial M$ :

$$\begin{aligned} \|u\|_{H^{k+1}(M)} &\leq \|\phi_0 u\|_{H^{k+1}(M)} + \|\phi_1 u\|_{H^{k+1}(M)} \\ &\leq c (\|P(\phi_0 u)\|_{H^{k-1}(M)} + \|P(\phi_1 u)\|_{H^{k-1}(M)} + \|\phi_0 u\|_{H^k(M)} \\ &\quad + \|\phi_1 u\|_{H^k(M)} + \|\phi_0 u\|_{H^{k+1/2}(\partial M)} + \|\partial_\nu^\alpha(\phi_1 u)\|_{H^{k-1/2}(\partial M)}) \\ &\leq 2c (\|[P, \phi_0]u\|_{H^{k-1}(M)} + \|Pu\|_{H^{k-1}(M)} + \|u\|_{H^k(M)} + \|Bu\|_k) \\ &\leq c' (\|Pu\|_{H^{k-1}(M)} + \|u\|_{H^k(M)} + \|Bu\|_k) \end{aligned}$$

where  $c$  is the maximum of the bounds appearing in Theorem 4.2 applied first to Dirichlet and then to Neumann boundary conditions and where we used Lemma 2.1. We have also used  $\|\phi_0 u\|_{H^{k+1/2}(\partial M)} + \|\partial_\nu^\alpha(\phi_1 u)\|_{H^{k-1/2}(\partial M)} = \|Bu\|_k$ .  $\square$

Let us now discuss the weak formulation of the Neumann problem, see [2, 20, 23, 24]. Let  $P = P_a + Q$  be a uniformly strongly elliptic second order differential operator on  $M$  with coefficients in  $W^{\infty, \infty}(M)$ , as in Theorem 4.5 above. We define

$$\check{H}^{\ell-1}(M) := \begin{cases} H^{\ell-1}(M) \oplus H^{\ell-1/2}(\partial_N M) & \text{for } \ell \geq 1 \\ H_D^1(M)^* & \text{for } \ell = 0. \end{cases}$$

In particular,  $\check{H}^\ell(M) \subset \check{H}^{\ell-1}(M)$ ,  $\ell \geq 0$ . For  $\ell = 0$ , this inclusion is the map  $\Phi_0$  of Remark 2.3. In general the natural inclusion  $\check{H}^\ell(M) \rightarrow \check{H}^{\ell-1}(M) := H_D^1(M)^*$  is given by  $\Phi_\ell$  of the same remark. Let  $\check{P}_\ell: H_D^{\ell+1}(M) \rightarrow \check{H}_D^{\ell-1}(M)$  be given by

$$\check{P}_\ell(u) := (Pu, \partial_\nu^\alpha u|_{\partial_N M}) \text{ for } \ell \geq 1. \quad (17)$$

Then the relation between  $\tilde{P}_\ell$  and  $\tilde{P}_0 := \tilde{P}$  is by (9) expressed in the commutativity of the diagram

$$\begin{array}{ccc} H_D^{\ell+1}(M) & \xrightarrow{\tilde{P}_\ell} & \check{H}^{\ell-1}(M) \\ \downarrow & & \downarrow \Phi_\ell \\ H_D^1(M) & \xrightarrow{\tilde{P}} & \check{H}^{-1}(M) \end{array} \quad (18)$$

where the vertical arrows are the natural inclusions. Thus, although the definition of  $\tilde{P}_\ell: H_D^{\ell+1}(M) \rightarrow \check{H}_D^{\ell-1}(M)$  for  $\ell = 0$  is different from the definitions of  $\tilde{P}_\ell$  for the other values of  $\ell$ , it fits into a scale of regularity spaces.

**Theorem 4.6.** *Assume that the conditions of Theorem 3.16 are satisfied. In particular,  $P = P_a + Q$  is a uniformly strongly elliptic second order differential operator. Moreover, assume that the coefficients of  $P_a$  are in  $W^{k+1,\infty}(M)$  and  $c \in W^{k,\infty}(M)$  is big enough (as in Equation (14) of that theorem). Then the map  $\tilde{P}_k + c: H_D^{k+1}(M) \rightarrow \check{H}_D^{k-1}(M)$  of Equation (17) is an isomorphism.*

*Proof.* For  $k = 0$ , this is exactly Theorem 3.16. In general, let us replace  $Q$  with  $Q + c$ , and thus assume that  $c = 0$ . Let  $\tilde{P}_0^{-1}: \check{H}^{-1}(M) \rightarrow H_D^1(M)$  denote the inverse of  $\tilde{P}_0: H_D^1(M) \rightarrow \check{H}^{-1}(M)$ , which we have just proved to exist. The regularity result of Theorem 4.5 then gives that, if  $F \in \check{H}_D^{k-1}(M)$  (it is enough to prove first a regularity estimate without  $c$  and then to include it, so we can assume less regularity for  $c$ ), then  $\tilde{P}_0^{-1}(F) \in H_D^{k+1}(M)$ . This proves that  $\tilde{P}_k$  is surjective. Since  $\tilde{P}_k$  is the restriction of  $\tilde{P}_0$  (see Equation (18)), the result follows.  $\square$

Since the trace (or restriction) map  $H^{k+1}(M) \rightarrow H^{k+1/2}(\partial_D M)$  is surjective, we also obtain the following corollary.

**Corollary 4.7.** *Let  $(M, \partial_D M)$ ,  $P$ , and  $c$  be as in Theorem 4.6, then the map  $(P + c, B): H^{k+1}(M) \rightarrow H^{k-1}(M) \oplus H^{k+1/2}(\partial_D M) \oplus H^{k-1/2}(\partial_N M)$ ,  $k \geq 1$ ,*

$$(P + c, B)(u) := ((P + c)u, u|_{\partial_D M}, \partial_\nu^\alpha u|_{\partial_N M})$$

*is an isomorphism.*

The assumptions of the above corollary are satisfied if  $P = \Delta_g$ . Thus the problem (1) is well-posed for  $P = \Delta_g + c$ ,  $c > 0$ , on manifolds with boundary and bounded geometry. If  $(M, \partial_D M)$  has finite width, then this problem is well-posed also for  $c = 0$ .

**4.3. Lie manifolds.** We notice that in Theorem 4.5 we require more regularity for our coefficients (they must belong to a bounded set in  $W^{k+1,\infty}$ ) than what may seem to suffice for order  $k$  regularity estimates. In fact, we can do better in the important case of ‘‘Lie manifolds’’ [7]. We follow the approach in [22].

Recall that a manifold with corners  $\overline{M}$  is a manifold that is locally modeled by  $[0, 1]^n$ . Then  $\overline{M}$  is homeomorphic to a manifold with boundary  $\partial M$ , such that  $\partial M$  is the union of the faces of  $M$ . The *interior* of  $\overline{M}$  is  $\overline{M} \setminus \partial \overline{M}$ . To recall the definition of a ‘‘Lie manifold,’’ we need first to recall that of a ‘‘Lie algebroid:’’

**Definition 4.8.** A *Lie algebroid*  $A \rightarrow \overline{M}$  is a real vector bundle over a manifold with corners  $\overline{M}$  together with a *Lie algebra* bracket  $[\cdot, \cdot]$  on its space of sections  $\Gamma(\overline{M}; A)$  and a vector bundle map  $\varrho: A \rightarrow T\overline{M}$  such that:

$$(i) \quad \varrho_*([X, Y]) = [\varrho_*(X), \varrho_*(Y)] \text{ and}$$

(ii)  $[X, fY] = f[X, Y] + (\varrho_*(X)f)Y$ , for all  $X, Y \in \Gamma(\overline{M}; A)$  and  $f \in \mathcal{C}^\infty(\overline{M})$ .

The map  $\varrho: A \rightarrow T\overline{M}$  is called the *anchor* of  $A$  and  $\varrho_*: \Gamma(\overline{M}; A) \rightarrow \Gamma(T\overline{M})$  is the induced map. We shall denote by  $\text{Diff}(\overline{M}; A)$  the differential operators on  $\overline{M}$  generated by  $\mathcal{C}^\infty(\overline{M})$  and  $\varrho_*(\Gamma(\overline{M}; A))$  and call them *admissible* (differential operators on  $\overline{M}$ ). For further use, let us denote  $\mathcal{V}_b(\overline{M}) \subset \Gamma(\overline{M}; T\overline{M})$  the set of smooth vector fields on  $\overline{M}$  that are tangent to all faces of  $\overline{M}$ .

**Definition 4.9.** A pair  $(\overline{M}, A)$  consisting of a compact manifold with corners  $\overline{M}$  and a Lie algebroid  $A \rightarrow \overline{M}$  with anchor  $\varrho: A \rightarrow T\overline{M}$  is called a *Lie manifold* if it satisfies the following properties:

- (i)  $\varrho: A_x \rightarrow T_x\overline{M}$  is an isomorphism for all  $x \in \overline{M} \setminus \partial\overline{M}$  and
- (ii)  $\varrho_*(\Gamma(\overline{M}; A)) \subset \mathcal{V}_b(\overline{M})$ .

We see therefore that the vector bundle  $A \rightarrow \overline{M}$  extends  $TM$  to  $\overline{M}$ .

*Example 4.10.* One of the simplest examples of a Lie manifold corresponds to  $\Gamma(\overline{M}; A) \simeq \varrho_*(\Gamma(\overline{M}; A)) = \mathcal{V}_b(\overline{M})$ . In case  $\overline{M}$  is a manifold with smooth boundary, this example models manifolds with cylindrical ends.

We shall need the variant of this definition for manifolds with boundary. We first need to introduce Lie submanifolds [7].

**Definition 4.11.** Let  $(\overline{M}, A)$  be a Lie manifold and  $L \subset \overline{M}$  be a submanifold ( $L$  is allowed to have corners). We say that  $L$  is a *sub Lie manifold of  $(\overline{M}, A)$*  if there exists a Lie manifold structure  $(L, B)$  on  $L$  and a tubular neighborhood  $\pi: U \rightarrow L$  of  $L$  in  $\overline{M}$  such that

$$A|_U \cong \{ (v, \xi) \in B \times TU \mid \varrho_B(\xi) = \pi_*(\xi) \in TL \},$$

where  $\varrho_B: B \rightarrow TL$  is the anchor map of  $B$ .

We can now introduce Lie manifolds with boundary.

**Definition 4.12.** A Lie manifold with boundary  $(\overline{M}, A)$  is a pair consisting of a compact manifold with corners  $\overline{M}$  for which there exists a Lie manifold  $(\overline{M}_1, A_1)$  such that  $\overline{M} \subset \overline{M}_1$ ,  $A = A_1|_{\overline{M}}$ , and the closure of  $\partial\overline{M} \setminus \partial\overline{M}_1$  in  $\overline{M}_1$  is a sub Lie manifold of  $\overline{M}_1$ .

The closure of  $\partial\overline{M} \setminus \partial\overline{M}_1$  in  $\overline{M}_1$  will be denoted  $\partial'\overline{M}$  and will be called the *true boundary of  $\overline{M}$* . It is a Lie manifold on its own, by the definition of a Lie manifold with boundary and of a Lie submanifold. Sometimes, by abuse of notation, we shall identify the Lie manifold  $\overline{M}$  with the manifold with boundary  $M := (\overline{M} \setminus \partial\overline{M}) \cup (\partial\overline{M} \setminus \partial\overline{M}_1)$ . The analysis and regularity will, in fact, happen on  $M$ . We will use the notation introduced in this paragraph throughout the rest of the paper. In particular, the boundary of  $M$  is  $\partial M = \partial\overline{M} \setminus \partial\overline{M}_1 = \partial'\overline{M} \setminus \partial\overline{M}_1$  and  $\partial M$  is the interior of the Lie manifold  $\partial'\overline{M}$ .

*Example 4.13.* Let us consider the following example. Let  $\overline{M}_1$  be a manifold with smooth boundary. Since  $\mathcal{V}_b(\overline{M}_1)$  is a projective  $\mathcal{C}^\infty(\overline{M}_1)$ -module, the Serre-Swan theorem gives that there exists a vector bundle  $A_1 \rightarrow \overline{M}_1$  such that  $\Gamma(\overline{M}_1; A_1) \simeq \mathcal{V}_b(\overline{M}_1)$ . Let  $L \subset \overline{M}_1$  be a submanifold with boundary such that  $\partial L = L \cap \partial\overline{M}_1$  and  $L$  intersects  $\partial\overline{M}_1$  *transversely*. Then  $L$  is a Lie submanifold of  $(\overline{M}_1, A)$ .

Let us now assume that  $L$  has codimension one and that  $L$  divides  $\overline{M}_1$  into two disjoint components. Let  $\overline{M}$  be the closure of one of these components. Then  $\overline{M}$  is a Lie manifold with boundary. Its true boundary is the interior of  $L$ .

For any Lie manifold with boundary  $(\overline{M}, A)$ , the choice of a metric on  $A$  yields a metric on  $M$ , the interior of  $\overline{M}$ , since  $A$  is canonically isomorphic to the tangent space to  $M$ . All the resulting metrics are equivalent and will be called *admissible* metrics. They will have very similar properties, since  $\overline{M}$  is compact, and hence all metrics on  $A$  are (Lipschitz) equivalent.

**Proposition 4.14.** *A Lie manifold with boundary is a manifold with boundary and bounded geometry for any admissible metric.*

*Proof.* Let  $\overline{M}$  be our Lie manifold with boundary and use the notation of Definition 4.12 and in the paragraph following it. We know from [7] that the interior of  $\overline{M}_1 \supset \overline{M}$  has bounded geometry for the metric induced from any metric on  $A_1 \rightarrow \overline{M}_1$ . It follows from the definition of a Lie submanifold (Definition 4.11) that the interior of a Lie submanifold is a submanifold with bounded geometry, since all tensors (including the tubular neighborhood) extend to its compactification, as in [7]. (Recall that a Lie submanifold is a compact manifold with corners.) In particular,  $\partial M$  is a hypersurface with bounded geometry of the interior of  $\overline{M}_1$ . The result then follows from Definition 3.3 of a manifold with boundary and bounded geometry.  $\square$

We continue to use the notation introduced in the paragraph after Definition 4.8 and consider a boundary value problem  $(P, B)$  on  $M$ , with  $B$  of constant order  $j$  on the boundary. We assume that  $P$  and  $B$  are admissible (that is, that they are restrictions of admissible differential operators on  $\overline{M}_1 \supset \overline{M} \supset M$ ). We shall denote by  $\mathcal{D}_{adm}^{\infty, j}(M; E)$  the set of admissible boundary value problems on  $M$ , if  $P$  acts on sections of  $E$  (we do not show the range in the notation). We also denote

$$\begin{aligned} \mathcal{D}_{cont}^{k, j}(M; E) &:= C^k(\overline{M}_1) \mathcal{D}_{adm}^{\infty, j}(M; E) \\ &:= \left\{ \sum_{\text{finite}} a_j P_j \mid a_j \in C^k(\overline{M}_1) \text{ and } P_j \in \mathcal{D}_{adm}^{\infty, j}(M; E) \right\}. \end{aligned}$$

Let  $(P, B) \in \mathcal{D}_{adm}^{\infty, j}(M; E)$ . We shall use the notation and constructions introduced in Subsection 4.1. Namely, we consider the Fermi and geodesic normal coordinate charts  $\kappa_p$  that have as domain the open balls in the tangent space  $T_p M$ . That is, we consider  $\kappa_p: B_r^{T_p M}(p) \rightarrow U_p(r)$  and  $\kappa_p: B_{2r}^{T_p \partial M}(p) \times [0, 2r] \rightarrow U_p(r)$  (analogous to (12)), together with the synchronous trivializations  $\xi_p$  introduced in Definition 3.11. Then we consider the corresponding differential operators  $(P_p, B_p)$  or, simply  $(P_p)$  obtained from  $P$  and  $B$  in the trivialization  $\xi_p$ , as in Subsection 4.1.

The assumption that  $(P, B)$  be admissible means that the definition of  $(P_p, B_p)$  extends to all  $p \in \partial' \overline{M}$  and the definition of  $P_p$  extends to all  $p \in \overline{M}$  such that  $\text{dist}(p, \partial M) \geq r$  [7]. Clearly, this is true if we extend the coefficients to be less regular, that is, for  $(P, B) \in \mathcal{D}_{cont}^{k, j}(M; E)$ . We can then extend the families  $\mathcal{F}_b$  and  $\mathcal{F}_i$  introduced in Subsection 4.1 to subsets (or families)

$$\begin{aligned} \overline{\mathcal{F}}_b &:= \{(P_p, B_p) \mid p \in \partial' \overline{M} \supset \partial M\} \subset \mathcal{D}^{k, j}(B_{2r}^m(0) \times [0, 2r]; \mathbb{C}^t) \\ \overline{\mathcal{F}}_i &:= \{(P_p) \mid p \in \overline{M}, \text{dist}(p, \partial M) \geq r\} \subset \mathcal{D}^{k, \emptyset}(B_r^{m+1}(0); \mathbb{C}^t). \end{aligned} \tag{19}$$

The point here is the following lemma.

**Lemma 4.15.** *The sets  $\partial' \overline{M}$  and  $\{p \in \overline{M}, \text{dist}(p, \partial M)\}$  are compact. Choose a local trivialization of  $A \rightarrow \overline{M}$  over some open subset  $U \subset \overline{M}$ . The families  $(P_p, B_p) \in \mathcal{D}^{k, j}(B_{2r}^m(0) \times [0, 2r]; \mathbb{C}^t)$  and  $P_p \in \mathcal{D}^{k, \emptyset}(B_r^{m+1}(0); \mathbb{C}^t)$  depend continuously on  $p \in U$ .*

*Proof.* The first statement is completely trivial. For the rest, let  $f^t(x) := f(x - t)$  denote the translation, and notice first that the translation  $\mathbb{R} \ni t \mapsto f^t$  and restriction  $W^{k,\infty}(\mathbb{R}^n) \rightarrow W^{k,\infty}(B_r^n(0))$  combine to define a continuous map

$$\mathbb{R}^n \times (\mathcal{C}^k(\mathbb{R}^n) \cap W^{k,\infty}(\mathbb{R}^n)) \ni (t, f) \rightarrow f^t|_{B_r^n(0)} \in W^{k,\infty}(B_r^n(0)).$$

The result then follows by using this observation and local coordinates since the Christoffel symbols in the chosen coordinate charts are bounded together with all their derivatives, compare [16].  $\square$

Lemma 4.15 gives right away the following corollary.

**Corollary 4.16.** *The sets  $\overline{\mathcal{F}}_b \subset \mathcal{D}^{k,j}(B_{2r}^m(0) \times [0, 2r]; \mathbb{C}^t)$  and  $\overline{\mathcal{F}}_i \subset \mathcal{D}^{k,0}(B_r^{m+1}(0); \mathbb{C}^t)$  of Equation (19) are compact.*

Proposition 2.10 and Theorem 4.2 then give the following result. Recall  $(P, B) \in \mathcal{D}_{cont}^{k,j}(M; E)$  consists of admissible boundary value problem with coefficients in  $\mathcal{C}^k(\overline{M})$ .

**Theorem 4.17.** *Let us assume that  $(\overline{M}, A)$  is a Lie manifold with boundary, that  $(P, B) \in \mathcal{D}_{cont}^{k,j}(M; E)$  and that  $P = P_a + Q$  is a strongly elliptic operator such that  $a$  extends to a strongly coercive sesquilinear form on  $A$ . Assume that the family  $(P_p, B_p)$ ,  $p \in \partial\overline{M} = \overline{\partial M}$ , satisfies order  $k$  regularity estimates. Then  $(B, P)$  satisfies an order  $k$  regularity estimate.*

*Proof.* This follows from Corollary 4.16, Proposition 2.10, and Theorem 4.2. Indeed, Corollary 4.16 states that  $\overline{\mathcal{F}}_b$  and  $\overline{\mathcal{F}}_i$  are compact. We have assumed that the set  $\overline{\mathcal{F}}_b$  satisfies order  $k$  regularity estimates. Since it is a compact set, it satisfies a uniform order  $k$  regularity estimate, by Proposition 2.10. The assumption that  $a$  extends to a strongly coercive sesquilinear form the whole of  $A$  shows that the set  $\overline{\mathcal{F}}_i$  satisfies order  $k$  regularity estimates. Since it is compact, it satisfies a uniform order  $k$  regularity estimate, by Proposition 2.10. Consequently, the subsets  $\mathcal{F}_b \subset \overline{\mathcal{F}}_b$  and  $\mathcal{F}_i \subset \overline{\mathcal{F}}_i$  satisfy, each, a uniform order  $k$  regularity estimate. We are thus in position to infer from Theorem 4.2 that  $(P, B)$  satisfies an order  $k$ -regularity estimate, as desired.  $\square$

The point of the above theorem is that we are allowing lower regularity in the coefficients, that is,  $C^k$  instead of  $W^{k+1,\infty}$  and we are not assuming *uniform* order  $k$  regularity estimates. The assumptions of the above theorem are in particular satisfied for mixed boundary conditions introduced in (16).

**Corollary 4.18.** *Consider the setting of Theorem 4.17. In particular,  $M$  is a Lie manifold with boundary,  $P = P_a + Q$  is an adapted uniformly strongly elliptic differential operator with coefficients in  $C^k$  and with a strongly coercive on  $A$ . Assume furthermore that  $B$  gives mixed boundary conditions. Then  $(P, B)$  satisfies an order  $k$  regularity estimate.*

We finally obtain the following result.

**Theorem 4.19.** *Let  $P = P_a + Q$  be as in Corollary 4.18, that is,  $P = P_a + Q$  is strongly elliptic with a strongly coercive on  $A$ ,  $Q$  is a first order operator,  $P$  is admissible with  $C^k$ -coefficients, and  $B$  gives mixible boundary conditions. Let  $c \in W^{k,\infty}(M)$  satisfy the Condition (14) of Theorem 3.16. Then the map  $\tilde{P} + c: H_D^{k+1}(M) \rightarrow \check{H}_D^{k-1}(M)$  is an isomorphism.*



This theorem extends a result from [19]. Its assumptions are satisfied by suitable Schrödinger operators. Indeed, by the results of [7], if we fix a metric on  $A$  and let  $g$  be the induced metric on  $M$ , then the associated Laplacian  $\Delta_g := d^*d$  is an admissible differential operator and we obtain:

**Corollary 4.20.** *Let  $M$  be a Lie manifold with boundary and admissible metric  $g$ . Let  $P = \Delta_g + V$  with  $V \in W^{k,\infty}(\overline{M})$  and assume that either  $\Re(V) \geq 0$  and  $(M, \partial_D M)$  has finite width, or that there exists  $\epsilon > 0$  such that  $\Re(V) \geq \epsilon$ . Then  $\tilde{P}: H_D^{k+1}(M) \rightarrow \check{H}_D^{k-1}(M)$  is an isomorphism.*

It would be interesting to the relation between the results of this paper and those of Karsten Bohlen [9, 8].

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