On the Efficiency of Nash Equilibria in the Interference Channel with Noisy Feedback

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***Abstract*—In this paper, the price of anarchy (PoA) and the price of stability (PoS) of the η-Nash equilibrium (η-NE), of the two-user linear deterministic interference channel with noisy channel-output feedback are characterized, with $\eta > 0$ arbitrarily small. The price of anarchy is the ratio between the sum-rate capacity and the smallest sum-rate at an η-NE. The price of stability is the ratio between the sum-rate capacity and the biggest sum-rate at an η-NE. Some of the main conclusions of this work are the following: (a) When both transmitter-receiver pairs are in low interference regime, the PoA can be made arbitrarily close to one as η approaches zero, subject to a particular condition. More specifically, there are scenarios in which even the worst η-NE (in terms of sum-rate) is arbitrarily close to the Pareto boundary of the capacity region. (b) The use of feedback plays a fundamental role on increasing the PoA, in some interference regimes. This is basically because in these regimes, the use of feedback increases the sum-capacity, whereas the smallest sum-rate at an η-NE remains the same. (c) The PoS is equal to one in all interference regimes. This implies that there always exists an η-NE in the Pareto boundary of the capacity region. The ensemble of conclusions of this work reveal the relevance of jointly using equilibrium selection methods and channel-output feedback for reducing the effect of anarchical behavior of the network components in the η-NE sum-rate of the interference channel.**

***Index Terms*—Nash equilibrium, Linear Deterministic Interference Channel, Price of Anarchy, Price of Stability.**

I. Linear Deterministic Interference Channel with Noisy Channel-Output Feedback

Consider the two-user linear deterministic interference channel with noisy channel-output feedback (LD-IC-NOF) described in Figure 1. For all $i \in \{1,2\}$, with $j \in \{1,2\} \setminus \{i\}$, the number of bit-pipes between transmitter i and its corresponding intended receiver is denoted by $\overrightarrow{n}_{ii}$; the number of bit-pipes between transmitter i and its corresponding non-intended receiver is denoted by n_{ji}; and the number of bit-pipes between receiver i and its corresponding transmitter is denoted by $\overleftarrow{n}_{ii}$. These six integer non-negative parameters describe the LD-IC-NOF in Figure 1.

At transmitter i, the channel-input $\boldsymbol{X}_{i,n}$ at channel use n, with $n \in \{1,2,\ldots,N_i\}$, is a q-dimensional binary vector $\boldsymbol{X}_{i,n} = \left(X_{i,n}^{(1)}, X_{i,n}^{(2)}, \ldots, X_{i,n}^{(q)}\right)^{\mathsf{T}} \in \mathcal{X}_i$, with $\mathcal{X}_i = \{0,1\}^q$,

$$q = \max\left(\overrightarrow{n}_{11}, \overrightarrow{n}_{22}, n_{12}, n_{21}\right), \tag{1}$$

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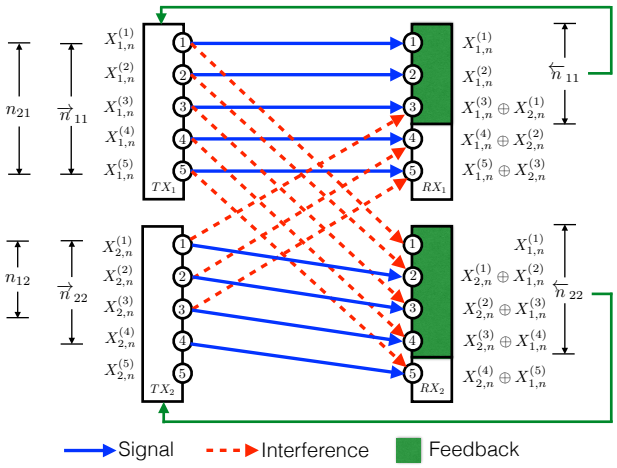

Fig. 1. Two-user linear deterministic interference channel with noisy channel-output feedback at channel use n.

and $N_i \in \mathbb{N}$ the block-length of transmitter-receiver pair i. At receiver i, the channel-output $\overrightarrow{\boldsymbol{Y}}_{i,n}$ at channel use n, with $n \in \{1,2,\ldots,\max(N_1,N_2)\}$, is also a q-dimensional binary vector $\overrightarrow{\boldsymbol{Y}}_{i,n} = \left(\overrightarrow{Y}_{i,n}^{(1)}, \overrightarrow{Y}_{i,n}^{(2)}, \ldots, \overrightarrow{Y}_{i,n}^{(q)}\right)^{\mathsf{T}}$. Let $\boldsymbol{S}$ be a $q \times q$ binary lower shift matrix of the form:

$$\boldsymbol{S} = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 & 0 \end{bmatrix}. \tag{2}$$

The input-output relation during channel use n is given by

$$\overrightarrow{\boldsymbol{Y}}_{i,n} = \boldsymbol{S}^{q-\overrightarrow{n}_{ii}} \boldsymbol{X}_{i,n} + \boldsymbol{S}^{q-n_{ij}} \boldsymbol{X}_{j,n}, \tag{3}$$

where $\boldsymbol{X}_{i,n} = (0,0,\ldots,0)^{\mathsf{T}}$ for all $n > N_i$. The feedback signal $\overleftarrow{\boldsymbol{Y}}_{i,n}$ available at transmitter i at the end of channel use n is

$$\overleftarrow{\boldsymbol{Y}}_{i,n} = \boldsymbol{S}^{\left(\max(\overrightarrow{n}_{ii}, n_{ij}) - \overleftarrow{n}_{ii}\right)^+} \overrightarrow{\boldsymbol{Y}}_{i,n-d}, \tag{4}$$

where d is a finite delay and additions and multiplications are defined over the binary field.

Without any loss of generality, the feedback delay is assumed to be equal to one channel use. Transmitter i sends the message index W_i by transmitting the codeword $\boldsymbol{X}_i = (\boldsymbol{X}_{i,1}, \boldsymbol{X}_{i,2}, \ldots, \boldsymbol{X}_{i,N_i})^{\mathsf{T}} \in \mathcal{X}_i^{N_i}$, which is a binary $q \times N_i$ matrix. The encoder of transmitter i can be modeled as a set of deterministic mappings $f_{i,1}^{(N)}, f_{i,2}^{(N)}, \ldots, f_{i,N_i}^{(N)}$, with

$f_{i,1}^{(N)} : \mathcal{W}_i \times \mathbb{N} \to \{0,1\}^q$ and for all $n \in \{2,3,\ldots,N_i\}$, $f_{i,n}^{(N)} : \mathcal{W}_i \times \mathbb{N} \times \{0,1\}^{q\times(n-1)} \to \{0,1\}^q$, such that

$$\boldsymbol{X}_{i,1} = f_{i,1}^{(N)}\left(W_i, \Omega_i\right) \text{ and} \tag{5a}$$

$$\boldsymbol{X}_{i,n} = f_{i,n}^{(N)}\left(W_i, \Omega_i, \overleftarrow{\boldsymbol{Y}}_{i,1}, \overleftarrow{\boldsymbol{Y}}_{i,2}, \ldots, \overleftarrow{\boldsymbol{Y}}_{i,n-1}\right), \tag{5b}$$

where Ω_i is a randomly generated index known by both transmitter i and receiver i, while unknown by transmitter j and receiver j.

The decoder of receiver i is defined by a deterministic function $\psi_i^{(N)} : \{0,1\}^{q\times N} \to \mathcal{W}_i$. At the end of the communication, receiver i uses the $q \times N$ binary matrix $\left(\overrightarrow{\boldsymbol{Y}}_{i,1}, \overrightarrow{\boldsymbol{Y}}_{i,2}, \ldots, \overrightarrow{\boldsymbol{Y}}_{i,N}\right)$ to obtain an estimate $\widehat{W}_i$ of the message index W_i. The decoding error probability in the two-user LD-IC-NOF, denoted by P_e, is given by

$$P_e = \max\left(\Pr\left(\widehat{W}_1 \neq W_1\right), \Pr\left(\widehat{W}_2 \neq W_2\right)\right). \tag{6}$$

An achievable rate pair $(R_1, R_2) \in \mathbb{R}_+^2$ in the LD-IC-NOF satisfies the following definition.

Definition 1 (*Achievable Rate Pairs*): *A rate pair $(R_1, R_2) \in \mathbb{R}_+^2$ is achievable if there exists at least one pair of binary codebooks with codewords (matrices) of sizes $q \times N_1$ and $q \times N_2$, respectively, and the corresponding encoding functions $f_{1,1}^{(N)}, f_{1,2}^{(N)}, \ldots, f_{1,N_1}^{(N)}$ and $f_{2,1}^{(N)}, f_{2,2}^{(N)}, \ldots, f_{2,N_2}^{(N)}$ such that the decoding error probability P_e can be made arbitrarily small by letting the block-lengths N_1 and N_2 grow to infinity.*

The aim of transmitter i is to autonomously choose its transmit-receive configuration, denoted by s_i, in order to maximize its achievable rate R_i. More specifically, the transmit-receive configuration s_i can be described in terms of the block-length N_i, the codebook $\mathcal{X}_i^{N_i}$, the encoding functions $f_{i,1}^{(N)}, f_{i,2}^{(N)}, \ldots, f_{i,N_i}^{(N)}$, the decoding function $\psi_i^{(N)}$, etc. Note that the rate achieved by transmitter-receiver i depends on both configurations s_1 and s_2 due to mutual interference. This reveals the competitive interaction between both links in the decentralized interference channel. The following section models this interaction using tools from game theory.

II. The Two-User Interference Channel as a Game

The competitive interaction of the two transmitter-receiver pairs in the decentralized interference channel can be modeled by the following game in normal-form:

$$\mathcal{G} = \left(\mathcal{K}, \{\mathcal{A}_k\}_{k\in\mathcal{K}}, \{u_k\}_{k\in\mathcal{K}}\right). \tag{7}$$

The set $\mathcal{K} = \{1,2\}$ is the set of players, that is, the set of transmitter-receiver pairs. The sets $\mathcal{A}_1$ and $\mathcal{A}_2$ are the sets of actions of players 1 and 2, respectively. An action of a player $i \in \mathcal{K}$, which is denoted by $s_i \in \mathcal{A}_i$, is basically its transmit-receive configuration as described above. The utility function of player i is $u_i : \mathcal{A}_1 \times \mathcal{A}_2 \to \mathbb{R}_+$ and it is defined as the information rate of transmitter i,

$$u_i(s_1, s_2) = \begin{cases} R_i(s_1, s_2), & \text{if} \quad P_e < \epsilon \\ 0, & \text{otherwise,} \end{cases} \tag{8}$$

where $\epsilon > 0$ is an arbitrarily small number and $R_i(s_1, s_2)$ denotes a transmission rate achievable with the configurations s_1 and s_2. Often, the rate $R_i(s_1, s_2)$ is written as R_i for the sake of simplicity. However, every non-negative achievable rate is associated with the particular transmit-receive configuration pair (s_1, s_2) that achieves it. It is worth noting that there might exist several transmit-receive configurations that achieve the same rate pair (R_1, R_2) and distinction between the different transmit-receive configuration is made only when needed.

A class of transmit-receive configurations $\boldsymbol{s}^* = (s_1^*, s_2^*) \in \mathcal{A}_1 \times \mathcal{A}_2$ that are particularly important in the analysis of this game is referred to as the set of η-Nash equilibria (η-NE). This type of configurations satisfy the following definition.

Definition 2 (η-*Nash equilibrium*): *In the game $\mathcal{G} = \left(\mathcal{K}, \{\mathcal{A}_k\}_{k\in\mathcal{K}}, \{u_k\}_{k\in\mathcal{K}}\right)$, an action profile (s_1^*, s_2^*) is an η-Nash equilibrium if for all $i \in \mathcal{K}$ and for all $s_i \in \mathcal{A}_i$, there exits an $\eta > 0$ such that*

$$u_i(s_i, s_j^*) \leqslant u_i(s_i^*, s_j^*) + \eta. \tag{9}$$

Let (s_1^*, s_2^*) be an η-Nash equilibrium action profile. Then none of the transmitters can increase its own transmission rate more than η bits per channel use by changing its own transmit-receive configuration and keeping the average bit error probability arbitrarily close to zero. Note that if $\eta = 0$, then the classical definition of Nash equilibrium is obtained [1]. Hence, at any η-NE, every transmitter-receiver pair's configuration is optimal with respect to the configuration of the other transmitter-receiver pair.

The set of rate pairs that can be achieved at an η-NE is known as the η-Nash equilibrium region.

Definition 3 (η-*NE Region*): *Let $\eta > 0$. An achievable rate pair (R_1, R_2) is said to be in the η-NE region of the game $\mathcal{G} = \left(\mathcal{K}, \{\mathcal{A}_k\}_{k\in\mathcal{K}}, \{u_k\}_{k\in\mathcal{K}}\right)$ if there exists a pair $(s_1^*, s_2^*) \in \mathcal{A}_1 \times \mathcal{A}_2$ that is an η-NE and the following holds:*

$$u_1(s_1^*, s_2^*) = R_1 \quad \textit{and} \quad u_2(s_1^*, s_2^*) = R_2. \tag{10}$$

The η-NE region is characterized in terms of two regions [2]: the capacity region, denoted by $\mathcal{C}(\overrightarrow{n}_{11}, \overrightarrow{n}_{22}, n_{12}, n_{21}, \overleftarrow{n}_{11}, \overleftarrow{n}_{22})$ and a convex closed region, denoted by $\mathcal{B}_\eta(\overrightarrow{n}_{11}, \overrightarrow{n}_{22}, n_{12}, n_{21}, \overleftarrow{n}_{11}, \overleftarrow{n}_{22})$. In the following, the tuple $(\overrightarrow{n}_{11}, \overrightarrow{n}_{22}, n_{12}, n_{21}, \overleftarrow{n}_{11}, \overleftarrow{n}_{22})$ is used only when needed. The capacity region $\mathcal{C}$ of the two-user LD-IC-NOF is described in Lemma 1 (at the top of the next page) and the convex region $\mathcal{B}_\eta$ for all $\eta > 0$ is described in [2] as follows:

$$\mathcal{B}_\eta = \left\{(R_1, R_2) : L_i \leqslant R_i \leqslant U_i, \text{ for all } i \in \{1,2\}\right\}, \tag{12}$$

where,

$$L_i = \left(\left(\overrightarrow{n}_{ii} - n_{ij}\right)^+ - \eta\right)^+ \text{ and} \tag{13a}$$

$$U_i = \max\left(\overrightarrow{n}_{ii}, n_{ij}\right) - \Bigg(\min\left(\left(\overrightarrow{n}_{jj} - n_{ji}\right)^+, n_{ij}\right) - \Big(\min\Big(\left(\overrightarrow{n}_{jj} - n_{ij}\right)^+, n_{ji}\Big) - \left(\max(\overrightarrow{n}_{jj}, n_{ji}) - \overleftarrow{n}_{jj}\right)^+\Big)^+\Bigg)^+ + \eta, \tag{13b}$$

with $i \in \{1,2\}$ and $j \in \{1,2\} \setminus \{i\}$.

Lemma 1 (*Theorem* 1 *in [3]*): *The capacity region* $\mathcal{C}(\overrightarrow{n}_{11}, \overrightarrow{n}_{22}, n_{12}, n_{21}, \overleftarrow{n}_{11}, \overleftarrow{n}_{22})$ *of the two-user LD-IC-NOF is the set of non-negative rate pairs* (R_1, R_2) *that satisfy for all* $i \in \{1,2\}$, *with* $j \in \{1,2\} \setminus \{i\}$*:*

$$R_i \leqslant \min\left(\max\left(\overrightarrow{n}_{ii}, n_{ji}\right), \max\left(\overrightarrow{n}_{ii}, n_{ij}\right)\right), \tag{11a}$$

$$R_i \leqslant \min\left(\max\left(\overrightarrow{n}_{ii}, n_{ji}\right), \max\left(\overrightarrow{n}_{ii}, \overleftarrow{n}_{jj} - \left(\overrightarrow{n}_{jj} - n_{ji}\right)^+\right)\right), \tag{11b}$$

$$R_1 + R_2 \leqslant \min\left(\max\left(\overrightarrow{n}_{22}, n_{12}\right) + \left(\overrightarrow{n}_{11} - n_{12}\right)^+, \max\left(\overrightarrow{n}_{11}, n_{21}\right) + \left(\overrightarrow{n}_{22} - n_{21}\right)^+\right), \tag{11c}$$

$$\begin{aligned} R_1 + R_2 \leqslant & \max\left(\left(\overrightarrow{n}_{11} - n_{12}\right)^+, n_{21}, \overrightarrow{n}_{11} - \left(\max\left(\overrightarrow{n}_{11}, n_{12}\right) - \overleftarrow{n}_{11}\right)^+\right) \\ & + \max\left(\left(\overrightarrow{n}_{22} - n_{21}\right)^+, n_{12}, \overrightarrow{n}_{22} - \left(\max\left(\overrightarrow{n}_{22}, n_{21}\right) - \overleftarrow{n}_{22}\right)^+\right), \end{aligned} \tag{11d}$$

$$2R_i + R_j \leqslant \max\left(\overrightarrow{n}_{ii}, n_{ji}\right) + \left(\overrightarrow{n}_{ii} - n_{ij}\right)^+ + \max\left(\left(\overrightarrow{n}_{jj} - n_{ji}\right)^+, n_{ij}, \overrightarrow{n}_{jj} - \left(\max\left(\overrightarrow{n}_{jj}, n_{ji}\right) - \overleftarrow{n}_{jj}\right)^+\right). \tag{11e}$$

The following lemma characterizes the η-NE region of the two-user LD-IC-NOF.

Lemma 2 (*Theorem 1 in [2]*): *Let* $\eta > 0$ *be arbitrarily small. The* η*-NE region of the two-user LD-IC-NOF with parameters* $\overrightarrow{n}_{11}$, $\overrightarrow{n}_{22}$, n_{12}, n_{21}, $\overleftarrow{n}_{11}$ *and* $\overleftarrow{n}_{22}$, *is*

$$\mathcal{N}_\eta = \mathcal{C} \cap \mathcal{B}_\eta. \tag{14}$$

III. Preliminaries

A. Definitions

Let $\alpha_i \in \mathbb{Q}$ be the interference parameter of transmitter-receiver pair i, with $i \in \{1,2\}$ and $j \in \{1,2\} \setminus \{i\}$, such that:

$$\alpha_i = \frac{n_{ij}}{\overrightarrow{n}_{ii}}. \tag{15}$$

The scenario in which the desired signal is stronger than the interference ($\alpha_i < 1$) is referred to as low-interference regime (LIR). Alternatively, the scenario in which the desired signal is weaker than or equal to the interference ($\alpha_i \geqslant 1$) is referred to as high-interference regime (HIR). The main results of this paper are presented using a list of events (Boolean variables) that are fully determined by the parameters $\overrightarrow{n}_{11}$, $\overrightarrow{n}_{22}$, n_{12}, and n_{21}. The event in which the number of interference-free bit-pipes at receiver i is bigger than or equal to the number of bit pipes in the cross-interference link in receiver j is denoted by $A_{1,i}$, i.e.,

$$A_{1,i}: \quad \overrightarrow{n}_{ii} - n_{ij} \geqslant n_{ji}. \tag{16}$$

The event in which the number of bit-pipes from transmitter i to receiver i is bigger than or equal to the number of bit pipes in the cross-interference link in receiver j is denoted by $A_{2,i}$, i.e.,

$$A_{2,i}: \quad \overrightarrow{n}_{ii} \geqslant n_{ji}. \tag{17}$$

In the following, given an event, e.g. $A_{2,i} : \overrightarrow{n}_{ii} \geqslant n_{ji}$, the notation $\overline{A}_{2,i}$ implies $\overrightarrow{n}_{ii} < n_{ji}$. Combining the events (16) and (17), eleven different conditions are identified:

$$B_1 : \quad A_{1,1} \wedge A_{1,2}, \tag{18a}$$

$$B_{2,i}: \quad A_{1,i} \wedge \overline{A}_{1,j} \wedge A_{2,j}, \tag{18b}$$

$$B_{3,i}: \quad A_{1,i} \wedge \overline{A}_{1,j} \wedge \overline{A}_{2,j}, \tag{18c}$$

$$B_4 : \quad \overline{A}_{1,1} \wedge \overline{A}_{1,2} \wedge A_{2,1} \wedge A_{2,2}, \tag{18d}$$

$$B_{5,i}: \quad \overline{A}_{1,1} \wedge \overline{A}_{1,2} \wedge \overline{A}_{2,i} \wedge A_{2,j}, \tag{18e}$$

$$B_6 : \quad \overline{A}_{1,1} \wedge \overline{A}_{1,2} \wedge \overline{A}_{2,2} \wedge \overline{A}_{2,2}, \tag{18f}$$

$$B_7 : \quad A_{1,1}, \tag{18g}$$

$$B_8 : \quad \overline{A}_{1,1} \wedge A_{2,1} \wedge A_{2,2}, \tag{18h}$$

$$B_9 : \quad \overline{A}_{1,1} \wedge \overline{A}_{2,1} \wedge A_{2,2}, \tag{18i}$$

$$B_{10}: \quad \overline{A}_{1,1} \wedge A_{2,1} \wedge \overline{A}_{2,2}, \tag{18j}$$

$$B_{11}: \quad \overline{A}_{1,1} \wedge \overline{A}_{2,1} \wedge \overline{A}_{2,2}. \tag{18k}$$

For all $i \in \{1,2\}$, $j \in \{1,2\} \setminus \{i\}$ when both transmitter-receiver pairs are in LIR, i.e., $\overrightarrow{n}_{11} > n_{12}$ and $\overrightarrow{n}_{22} > n_{21}$, the events B_1, $B_{2,i}$, $B_{3,i}$, B_4, $B_{5,i}$, and B_6 exhibit the property stated by the following lemma.

Lemma 3 (*Both transmitter-receiver pairs are in LIR*): *For all* $i \in \{1,2\}$, $j \in \{1,2\} \setminus \{i\}$ *and for all* $(\overrightarrow{n}_{11}, \overrightarrow{n}_{22}, n_{12}, n_{21}) \in \mathbb{N}^4$ *with* $\overrightarrow{n}_{11} > n_{12}$ *and* $\overrightarrow{n}_{22} > n_{21}$, *only one of the events* B_1, $B_{2,i}$, $B_{3,i}$, B_4, $B_{5,i}$, *and* B_6 *holds true.*

Proof: The proof follows from verifying that for all $i \in \{1,2\}$ and $j \in \{1,2\} \setminus \{i\}$, when both transmitter-receiver pairs are in LIR, i.e., $\overrightarrow{n}_{11} > n_{12}$ and $\overrightarrow{n}_{22} > n_{21}$, the events (18a)-(18f) are mutually exclusive. ∎

For all $i \in \{1,2\}$, $j \in \{1,2\} \setminus \{i\}$ when transmitter-receiver pair 1 is in LIR and transmitter-receiver pair 2 is in HIR, i.e., $\overrightarrow{n}_{11} > n_{12}$ and $\overrightarrow{n}_{22} \leqslant n_{21}$, the events B_7, B_8, B_9, B_{10}, and B_{11} exhibit the property stated by the following lemma.

Lemma 4 (*Transmitter-receiver pair* 1 *in LIR and transmitter-receiver pair* 2 *in HIR*): *For all* $i \in \{1,2\}$, $j \in \{1,2\} \setminus \{i\}$ *and for all* $(\overrightarrow{n}_{11}, \overrightarrow{n}_{22}, n_{12}, n_{21}) \in \mathbb{N}^4$ *with* $\overrightarrow{n}_{11} > n_{12}$ *and* $\overrightarrow{n}_{22} \leqslant n_{21}$, *only one of the events* B_7, B_8, B_9, B_{10}, *and* B_{11} *holds true.*

Proof: The proof of Lemma 4 follows along the same lines of the proof of Lemma 3. ∎

IV. Main Results: Efficiency of an η-NE

This section characterizes the efficiency of the set of Nash equilibria in the two-user LD-IC-NOF using two metrics: price of anarchy (PoA) and price of stability (PoS). The PoA measures the loss of performance due to decentralization by comparing the maximum sum-rate achieved by a centralized LD-IC-NOF with the minimum sum-rate achieved by a decentralized LD-IC-NOF at an η-NE. Alternatively, the PoS measures the loss of performance due to decentralization by comparing the maximum sum-rate achieved by a centralized LD-IC-NOF with the maximum sum-rate achieved by a decentralized LD-IC-NOF at an η-NE [4].

A. Price of Anarchy

Let $\mathcal{A}=\mathcal{A}_1\times\mathcal{A}_2$ be the set of all possible action profiles and $\mathcal{A}_{\eta-\mathrm{NE}}\subset\mathcal{A}$ be the set of η-NE strategies of the game in (7) (Definition 2).

Definition 4 (*Price of Anarchy [5]*): *Let $\eta>0$. The PoA of the game $\mathcal{G}$, denoted by $\mathrm{PoA}(\eta,\mathcal{G})$, is given by:*

$$\mathrm{PoA}(\eta,\mathcal{G})=\frac{\max\limits_{(s_1,s_2)\in\mathcal{A}}\sum\limits_{i=1}^{2}R_i(s_1,s_2)}{\min\limits_{(s_1^*,s_2^*)\in\mathcal{A}_{\eta-\mathrm{NE}}}\sum\limits_{i=1}^{2}R_i(s_1^*,s_2^*)}. \tag{19}$$

Let $\overline{\Sigma}_{C(\overrightarrow{n}_{11},\overrightarrow{n}_{22},n_{12},n_{21},\overleftarrow{n}_{11},\overleftarrow{n}_{22})}$ denote the solution to the optimization problem in the numerator of (19), which correspond to the maximum sum-rate in the centralized case. Let also $\underline{\Sigma}_{N(\overrightarrow{n}_{11},\overrightarrow{n}_{22},n_{12},n_{21},\overleftarrow{n}_{11},\overleftarrow{n}_{22})}$ denote the solution to the optimization problem in the denominator of (19). Closed-form expressions of the maximum sum-rate in the centralized case, i.e., $\overline{\Sigma}_{C(\overrightarrow{n}_{11},\overrightarrow{n}_{22},n_{12},n_{21},\overleftarrow{n}_{11},\overleftarrow{n}_{22})}$ and the minimum sum-rate in the decentralized case, i.e., $\underline{\Sigma}_{N(\overrightarrow{n}_{11},\overrightarrow{n}_{22},n_{12},n_{21},\overleftarrow{n}_{11},\overleftarrow{n}_{22})}$ are presented in [2] and they can be obtained from Lemma 1 and Lemma 2.

The following theorems describe the $\mathrm{PoA}(\eta,\mathcal{G})$ in particular interference regimes of the LD-IC-NOF. In all the cases, it is assumed that $\overleftarrow{n}_{ii}\leqslant\max(\overrightarrow{n}_{ii},n_{ij})$ for all $i\in\{1,2\}$ and $j\in\{1,2\}\setminus\{i\}$. If $\overleftarrow{n}_{11}>\max(\overrightarrow{n}_{11},n_{12})$ or $\overleftarrow{n}_{22}>\max(\overrightarrow{n}_{22},n_{21})$, the results on the η-NE efficiency metrics are the same as those in the case of perfect channel-output feedback, i.e., $\overleftarrow{n}_{11}=\max(\overrightarrow{n}_{11},n_{12})$ or $\overleftarrow{n}_{22}=\max(\overrightarrow{n}_{22},n_{21})$.

Theorem 1 (*Both transmitter-receiver pairs in LIR*): *For all $i\in\{1,2\}$, $j\in\{1,2\}\setminus\{i\}$ and for all $(\overrightarrow{n}_{11},\overrightarrow{n}_{22},n_{12},n_{21},\overleftarrow{n}_{11},\overleftarrow{n}_{22})\in\mathbb{N}^6$ with $\overrightarrow{n}_{11}>n_{12}$ and $\overrightarrow{n}_{22}>n_{21}$, the $\mathrm{PoA}(\eta,\mathcal{G})$ satisfies:*

$$\mathrm{PoA}(\eta,\mathcal{G})=\begin{cases}\frac{\overline{\Sigma}_{C1}}{\underline{\Sigma}_{N1}} & \text{if}\quad B_1\\ \frac{\overline{\Sigma}_{C2,i}}{\underline{\Sigma}_{N1}} & \text{if}\quad B_{2,i}\\ \frac{\overrightarrow{n}_{ii}}{\underline{\Sigma}_{N1}} & \text{if}\quad B_{3,i}\\ \frac{\overline{\Sigma}_{C3}}{\underline{\Sigma}_{N1}} & \text{if}\quad B_4\\ \frac{\overrightarrow{n}_{ii}}{\underline{\Sigma}_{N1}} & \text{if}\quad B_{5,i}\\ \frac{\min(\overrightarrow{n}_{11},\overrightarrow{n}_{22})}{\underline{\Sigma}_{N1}} & \text{if}\quad B_6,\end{cases} \tag{20}$$

where,

$$\begin{aligned}\overline{\Sigma}_{C1}=\min\Big(&\overrightarrow{n}_{11}+\overrightarrow{n}_{22},\overrightarrow{n}_{22}+\overrightarrow{n}_{11}-n_{12},\overrightarrow{n}_{11}+\overrightarrow{n}_{22}-n_{21},\\&\max\left(\overrightarrow{n}_{11}-n_{12},\overleftarrow{n}_{11}\right)+\max\left(\overrightarrow{n}_{22}-n_{21},\overleftarrow{n}_{22}\right),\\&2\overrightarrow{n}_{11}-n_{12}+\max\left(\overrightarrow{n}_{22}-n_{21},\overleftarrow{n}_{22}\right),\\&2\overrightarrow{n}_{22}-n_{21}+\max\left(\overrightarrow{n}_{11}-n_{12},\overleftarrow{n}_{11}\right)\Big);\end{aligned} \tag{21a}$$

$$\begin{aligned}\overline{\Sigma}_{C2,i}=\min\Big(&\overrightarrow{n}_{22}+\overrightarrow{n}_{11}-n_{12},\overrightarrow{n}_{11}+\overrightarrow{n}_{22}-n_{21},\\&\max\left(\overrightarrow{n}_{11}-n_{12},\overleftarrow{n}_{11}\right)+\max\left(\overrightarrow{n}_{22}-n_{21},\overleftarrow{n}_{22}\right),\\&2\overrightarrow{n}_{ii}-n_{ij}+\max\left(n_{ij},\overleftarrow{n}_{jj}\right),\\&2\overrightarrow{n}_{jj}-n_{ji}+\max\left(\overrightarrow{n}_{ii}-n_{ij},\overleftarrow{n}_{ii}\right)\Big);\end{aligned} \tag{21b}$$

$$\begin{aligned}\overline{\Sigma}_{C3}=\min\Big(&\overrightarrow{n}_{22}+\overrightarrow{n}_{11}-n_{12},\overrightarrow{n}_{11}+\overrightarrow{n}_{22}-n_{21},\\&\max\left(n_{21},\overleftarrow{n}_{11}\right)+\max\left(n_{12},\overleftarrow{n}_{22}\right),\\&2\overrightarrow{n}_{11}-n_{12}+\max\left(n_{12},\overleftarrow{n}_{22}\right),\\&2\overrightarrow{n}_{22}-n_{21}+\max\left(n_{21},\overleftarrow{n}_{11}\right)\Big);\ \textit{and}\end{aligned} \tag{21c}$$

$$\underline{\Sigma}_{N1}=\overrightarrow{n}_{11}-n_{12}+\overrightarrow{n}_{22}-n_{21}-2\eta. \tag{21d}$$

Proof: The proof is presented in [2]. ■

From Theorem 1, the following conclusions can be drawn.

Remark 1: When both transmitter-receiver pairs are in LIR, and at least one of the conditions $B_{3,i}$, $B_{5,i}$, or B_6 holds true, with $i\in\{1,2\}$ and $j\in\{1,2\}\setminus\{i\}$, then the $\mathrm{PoA}(\eta,\mathcal{G})$ does not depend on the feedback parameters $\overleftarrow{n}_{11}$ and $\overleftarrow{n}_{22}$.

Corollary 1: For any $(\overrightarrow{n}_{11},\overrightarrow{n}_{22},n_{12},n_{21},\overleftarrow{n}_{11},\overleftarrow{n}_{22})\in\mathbb{N}^6$ with $\overrightarrow{n}_{11}>n_{12}$ and $\overrightarrow{n}_{22}>n_{21}$, that satisfies B_1, it follows that:

$$1<\frac{\overline{\Sigma}_{C4}}{\underline{\Sigma}_{N1}}\leqslant\mathrm{PoA}(\eta,\mathcal{G})\leqslant\frac{\overline{\Sigma}_{C5}}{\underline{\Sigma}_{N1}}, \tag{22}$$

where,

$$\overline{\Sigma}_{C4}=\overrightarrow{n}_{11}+\overrightarrow{n}_{22}-n_{12}-n_{21},\ \text{and} \tag{23}$$

$$\overline{\Sigma}_{C5}=\overrightarrow{n}_{11}+\overrightarrow{n}_{22}-\max(n_{12},n_{21}). \tag{24}$$

The lower bound in (22) is obtained assuming that $\overleftarrow{n}_{11}=0$ and $\overleftarrow{n}_{22}=0$ in (20). That is, when feedback is not available. The upper bound in (22) is obtained assuming that $\overleftarrow{n}_{11}=\max(\overrightarrow{n}_{11},n_{12})=\overrightarrow{n}_{11}$ and $\overleftarrow{n}_{22}=\max(\overrightarrow{n}_{22},n_{21})=\overrightarrow{n}_{22}$ in (20). That is, when perfect channel-output feedback is available al both transmitter-receiver pairs.

Note that for any η arbitrarily small, when both transmitter-receiver pairs are in LIR; condition B_1 holds true; $\overleftarrow{n}_{11}\leqslant\overrightarrow{n}_{11}-n_{12}$; and $\overleftarrow{n}_{22}\leqslant\overrightarrow{n}_{22}-n_{21}$, the sum-rate capacity approaches to the minimum sum-rate at an η-NE ($\mathrm{PoA}(\eta,\mathcal{G})\approx 1$). If $\overleftarrow{n}_{11}>\overrightarrow{n}_{11}-n_{12}$ or $\overleftarrow{n}_{22}>\overrightarrow{n}_{22}-n_{21}$ the use

of feedback enlarges both the capacity region and the η-NE region. However, the PoA increases as the smallest sum-rate at an η-NE remains unchanged with respect to the case without feedback.

Corollary 2: For any $(\overrightarrow{n}_{11}, \overrightarrow{n}_{22}, n_{12}, n_{21}, \overleftarrow{n}_{11}, \overleftarrow{n}_{22}) \in \mathbb{N}^6$ *with* $\overrightarrow{n}_{11} > n_{12}$ *and* $\overrightarrow{n}_{22} > n_{21}$*, that satisfies* $B_{2,i}$ *for all* $i \in \{1,2\}$ *and* $j \in \{1,2\} \setminus \{i\}$*, it follows that:*

$$1 < \frac{\overrightarrow{n}_{ii}}{\underline{\Sigma}_{N1}} \leqslant \mathrm{PoA}\left(\eta, \mathcal{G}\right) \leqslant \frac{\overline{\Sigma}_{C5}}{\underline{\Sigma}_{N1}}. \tag{25}$$

Note that when both transmitter-receiver pairs are in LIR; condition $B_{2,i}$ holds true for all $i \in \{1,2\}$ and $j \in \{1,2\} \backslash \{i\}$; $\overleftarrow{n}_{ii} \leqslant \overrightarrow{n}_{ii} - n_{ij}$; and $\overleftarrow{n}_{jj} \leqslant n_{ij}$, the use of feedback does not enlarge either the capacity region or the η-NE region. Then, the $\mathrm{PoA}\left(\eta, \mathcal{G}\right)$ is equal to the lower bound in (25), i.e., $\mathrm{PoA}\left(\eta, \mathcal{G}\right) = \frac{\overrightarrow{n}_{ii}}{\underline{\Sigma}_{N1}}$. Conversely, when $\overleftarrow{n}_{ii} > \overrightarrow{n}_{ii} - n_{ij}$ or $\overleftarrow{n}_{jj} > n_{ij}$, the use of feedback enlarges both the capacity region and the η-NE region.

The lower bound in (25) is obtained assuming that $\overleftarrow{n}_{11} = 0$ and $\overleftarrow{n}_{22} = 0$ in (20). That is, when feedback is not available. The upper bound in (25) is obtained assuming that $\overleftarrow{n}_{11} = \max\left(\overrightarrow{n}_{11}, n_{12}\right) = \overrightarrow{n}_{11}$ and $\overleftarrow{n}_{22} = \max\left(\overrightarrow{n}_{22}, n_{21}\right) = \overrightarrow{n}_{22}$ in (20). That is, when perfect channel-output feedback is available at both transmitter-receiver pairs.

Corollary 3: For any $(\overrightarrow{n}_{11}, \overrightarrow{n}_{22}, n_{12}, n_{21}, \overleftarrow{n}_{11}, \overleftarrow{n}_{22}) \in \mathbb{N}^6$ *with* $\overrightarrow{n}_{11} > n_{12}$ *and* $\overrightarrow{n}_{22} > n_{21}$*, that satisfies* B_4*, it follows that:*

$$1 < \frac{\min\left(\overline{\Sigma}_{C5}, n_{12} + n_{21}\right)}{\underline{\Sigma}_{N1}} \leqslant \mathrm{PoA}\left(\eta, \mathcal{G}\right) \leqslant \frac{\overline{\Sigma}_{C5}}{\underline{\Sigma}_{N1}}. \tag{26}$$

Note that when both transmitter-receiver pairs are in LIR; condition B_4 holds true; and $\overline{\Sigma}_{C5} \leqslant n_{12} + n_{21}$, then the $\mathrm{PoA}\left(\eta, \mathcal{G}\right)$ does not depend on the feedback parameters $\overleftarrow{n}_{11}$ and $\overleftarrow{n}_{22}$. When $\overline{\Sigma}_{C5} > n_{12} + n_{21}$, $\overleftarrow{n}_{11} \leqslant n_{21}$, and $\overleftarrow{n}_{22} \leqslant n_{12}$, then the $\mathrm{PoA}\left(\eta, \mathcal{G}\right)$ is equal to the lower bound in (26), i.e., $\mathrm{PoA}\left(\eta, \mathcal{G}\right) = \frac{n_{12}+n_{21}}{\underline{\Sigma}_{N1}}$. Conversely, When $\overline{\Sigma}_{C5} > n_{12} + n_{21}$ and $\overleftarrow{n}_{11} > n_{21}$ or $\overleftarrow{n}_{22} > n_{12}$, the use of feedback enlarges the capacity region and the η-NE region.

Theorem 2 (*Transmitter-receiver pair* 1 *in LIR and transmitter-receiver pair* 2 *in HIR*): *For all* $(\overrightarrow{n}_{11}, \overrightarrow{n}_{22}, n_{12}, n_{21}, \overleftarrow{n}_{11}, \overleftarrow{n}_{22}) \in \mathbb{N}^6$ *with* $\overrightarrow{n}_{11} > n_{12}$ *and* $\overrightarrow{n}_{22} \leqslant n_{21}$*, the* $\mathrm{PoA}\left(\eta, \mathcal{G}\right)$ *satisfies:*

$$\mathrm{PoA}\left(\eta, \mathcal{G}\right) = \begin{cases} \frac{\overrightarrow{n}_{11}}{\overrightarrow{n}_{11} - n_{12} - \eta} & \text{if} \quad B_7 \\ \frac{\overrightarrow{n}_{11}}{\overrightarrow{n}_{11} - n_{12} - \eta} & \text{if} \quad B_8 \\ \frac{\min(\overrightarrow{n}_{22} + \overrightarrow{n}_{11} - n_{12}, n_{21})}{\overrightarrow{n}_{11} - n_{12} - \eta} & \text{if} \quad B_9 \\ \frac{\overrightarrow{n}_{11}}{\overrightarrow{n}_{11} - n_{12} - \eta} & \text{if} \quad B_{10} \\ \frac{\overrightarrow{n}_{11}}{\overrightarrow{n}_{11} - n_{12} - \eta} & \text{if} \quad B_{11}. \end{cases} \tag{27}$$

Note that in the cases in which transmitter-receiver pair 1 is in LIR and transmitter-receiver pair 2 is in HIR, the $\mathrm{PoA}\left(\eta, \mathcal{G}\right)$ does not depend on the feedback parameters. This is basically because the use of feedback in this scenario can enlarge the capacity region but it does not increase the sum-rate capacity (Theorem 4 in [6]).

In the case in which transmitter-receiver pair 1 is in HIR and transmitter-receiver pair 2 is in LIR, i.e., $\overrightarrow{n}_{11} \leqslant n_{12}$ and $\overrightarrow{n}_{22} > n_{21}$, the $\mathrm{PoA}\left(\eta, \mathcal{G}\right)$ for the LD-IC-NOF is characterized as in Theorem 2 interchanging the indices of the parameters.

Theorem 3 (*Both transmitter-receiver pairs in HIR*): *For all* $(\overrightarrow{n}_{11}, \overrightarrow{n}_{22}, n_{12}, n_{21}, \overleftarrow{n}_{11}, \overleftarrow{n}_{22}) \in \mathbb{N}^6$ *with* $\overrightarrow{n}_{11} \leqslant n_{12}$ *and* $\overrightarrow{n}_{22} \leqslant n_{21}$*, the* $\mathrm{PoA}\left(\eta, \mathcal{G}\right)$ *satisfies:*

$$\mathrm{PoA}\left(\eta, \mathcal{G}\right) = \infty. \tag{28}$$

The result on Theorem 3 is due to the fact that $\left(\left(\overrightarrow{n}_{11} - n_{12}\right)^+ - \eta\right)^+ + \left(\left(\overrightarrow{n}_{22} - n_{21}\right)^+ - \eta\right)^+ = 0$. That is, when $\overrightarrow{n}_{11} \leqslant n_{12}$ and $\overrightarrow{n}_{22} \leqslant n_{21}$, none of the transmitter-receiver pairs is able to transmit at a strictly positive rate at the worst η-NE, i.e., $\underline{\Sigma}_{N(\overrightarrow{n}_{11}, \overrightarrow{n}_{22}, n_{12}, n_{21}, \overleftarrow{n}_{11}, \overleftarrow{n}_{22})}$=0.

Note also that in the cases in which the $\mathrm{PoA}\left(\eta, \mathcal{G}\right)$ depends on the feedback parameters $\overleftarrow{n}_{11}$ and $\overleftarrow{n}_{22}$, there exist a value in the feedback parameter $\overleftarrow{n}_{11}$ or the feedback parameter $\overleftarrow{n}_{22}$ beyond which the $\mathrm{PoA}\left(\eta, \mathcal{G}\right)$ increases. The values in the feedback parameters correspond to those values beyond which the sum-rate capacity can be enlarged (Theorem 4 in [6]).

B. Price of Stability

In this section, the efficiency of the η-NEs of the game $\mathcal{G}$ in (7) is analyzed by using the PoS.

Definition 5 (*Price of stability [7]*): *Let* $\eta > 0$*. The PoS of the game* $\mathcal{G}$*, denoted by* $\mathrm{PoS}\left(\eta, \mathcal{G}\right)$*, is given by:*

$$\mathrm{PoS}\left(\eta, \mathcal{G}\right) = \frac{\max\limits_{(s_1, s_2) \in \mathcal{A}} \sum\limits_{i=1}^{2} R_i(s_1, s_2)}{\max\limits_{(s_1^*, s_2^*) \in \mathcal{A}_{\eta-\mathrm{NE}}} \sum\limits_{i=1}^{2} R_i(s_1^*, s_2^*)}. \tag{29}$$

Let $\overline{\Sigma}_{N(\overrightarrow{n}_{11}, \overrightarrow{n}_{22}, n_{12}, n_{21}, \overleftarrow{n}_{11}, \overleftarrow{n}_{22})}$ denote the solution to the optimization problem in the denominator of (29). A closed-form expression of the maximum sum-rate in the decentralized case, i.e., $\overline{\Sigma}_{N(\overrightarrow{n}_{11}, \overrightarrow{n}_{22}, n_{12}, n_{21}, \overleftarrow{n}_{11}, \overleftarrow{n}_{22})}$ is presented in [2] and it can be obtained from Lemma 1 and Lemma 2.

The following proposition characterizes the PoS of the game $\mathcal{G}$ in (7) for the LD-IC-NOF.

Proposition 1 (*PoS*): *For all* $(\overrightarrow{n}_{11}, \overrightarrow{n}_{22}, n_{12}, n_{21}, \overleftarrow{n}_{11}, \overleftarrow{n}_{22}) \in \mathbb{N}^6$ *and for all* $\eta > 0$ *arbitrary small, the PoS in the game* $\mathcal{G}$ *of the LD-IC-NOF is:*

$$\mathrm{PoS}\left(\eta, \mathcal{G}\right) = 1. \tag{30}$$

Note that the fact that the price of stability is equal to one, independently of the parameters $\overrightarrow{n}_{11}$, $\overrightarrow{n}_{22}$, n_{12}, n_{21}, $\overleftarrow{n}_{11}$ and $\overleftarrow{n}_{22}$, implies that despite the anarchical behavior of both transmitter-receiver pairs, the biggest η-NE sum-rate is equal to the sum-rate capacity, i.e., $\overline{\Sigma}_{\mathcal{C}(\overrightarrow{n}_{11}, \overrightarrow{n}_{22}, n_{12}, n_{21}, \overleftarrow{n}_{11}, \overleftarrow{n}_{22})} = \overline{\Sigma}_{\mathcal{N}_\eta(\overrightarrow{n}_{11}, \overrightarrow{n}_{22}, n_{12}, n_{21}, \overleftarrow{n}_{11}, \overleftarrow{n}_{22})}$. This implies that in all interference regimes, there always exist at least one Pareto optimal η-NE.

V. Conclusions

The price of anarchy (PoA) and the price of stability (PoS) of the η-NE of the two-user linear deterministic interference channel with noisy channel-output feedback have been characterized, with $\eta > 0$ arbitrarily small. It has been shown that when both transmitter-receiver pairs are in low interference regime, the PoA can be made arbitrarily close to one as η approaches zero, subject to a particular condition. This immediately implies that in this regime even the worst η-NE (in terms of sum-rate) is arbitrarily close to the Pareto boundary of the capacity region. More importantly, it has been shown that the use of feedback increases the PoA in some interference regimes. This is basically because in these regimes, the use of feedback increases the sum-capacity, where as the smallest sum-rate at an η-NE is not changed. In some cases the PoA can be infinity due to the fact that in high interference regimes, the smallest sum-rate at an η-NE is zero bits per channel use. In other regimes, the use of feedback does not have any impact in the PoA as it does not increase the sum-capacity. Finally, the PoS is shown to be equal to one in all interference regimes. This implies that there always exists an η-NE in the Pareto boundary of the capacity region. The main results of this work highlight the relevance of designing equilibrium selection methods such that decentralized networks can operate at efficient η-NE points. The need of these methods becomes more relevant when channel-output feedback is available as it might increase the PoA.

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