

TIME FRACTIONAL LINEAR PROBLEMS ON $L^2(\mathbb{R}^d)$

HASSAN EMAMIRAD AND ARNAUD ROUGIREL

ABSTRACT. In this paper, we propose a theory for linear time fractional PDEs on $L^2(\mathbb{R}^d)$. The order of the time derivatives under consideration is less than 1. We study well-posedness, regularizing effects and dissipative properties. In particular, we give a necessary and sufficient condition for well-posedness. Regarding regularizing effects, we describe quite precisely the equations that have this effect or not. We highlight that, in purely fractional settings, the regularizing effect acts always only up to finite order; unlike to the standard case.

1. INTRODUCTION

Time fractional differential equations have been the subject of many research in the recent years, both in terms of applications ([GARGBA⁺12], [Las00], [BGMT10]) and mathematical studies ([Kos15], [GKMR14]).

The aim of this paper is to build a theory for linear time fractional PDEs on $L^2(\mathbb{R}^d)$. It is about to extend the well-known theory of first order time derivative equations. Surprisingly, to our knowledge, such extension has not been made yet. However, there are many results regarding abstract time fractional equations: see for instance [Kos14], [Baz98], [KLW16].

For first order time derivative equations of the form

$$u_t = P(D)u, \quad (1.1)$$

where $u = u(x, t)$ and $P(D)$ is a differential operator acting on the space variable x , the well posedness is equivalent to the condition

$$\sup_{\xi \in \mathbb{R}^d} \operatorname{Re} P(\xi) < \infty,$$

where P is the symbol of $P(D)$ (see Subsection 2.3 for notation and definitions).

For time fractional equations of the form

$$\mathbf{D}_{0,t}^\alpha u = P(D)u, \quad (1.2)$$

with $\alpha \in (0, 1)$, the above condition is sufficient (see [ER17]) but no more necessary (see Example 4.3 below). In this paper, we give a quite simple necessary and sufficient condition, namely (4.1), for the above fractional problem to be well posed in $L^2(\mathbb{R}^d)$. See Theorem 4.2.

Roughly speaking, our results state that the critical angle for equation (1.2) is $\frac{\pi}{2}\alpha$ in a similar fashion that $\pi/2$ is the critical angle for (1.1). Indeed, if $|\arg P(\xi)| < \frac{\pi}{2}\alpha$ then the Cauchy problem corresponding to (1.2) is generally ill-posed. If $|\arg P(\xi)| > \frac{\pi}{2}\alpha$ then the problem is well-posed, has regularizing effects and dissipative properties.

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If $|\arg P(\xi)| = \frac{\pi}{2}\alpha$ then the problem is well-posed in $L^2(\mathbb{R}^d)$ but, has no regularizing and dissipative properties.

For instance, the *fractional Schrödinger equation*

$$\mathbf{D}_{0,t}^\alpha u = -i\Delta u, \quad (1.3)$$

is hyperbolic for $\alpha = 1$ in the sense that it has a conservation law and no regularizing effect. In this case, $\arg P(\xi) = \pi/2$.

When $\alpha \in (0, 1)$, the critical angle is $\frac{\pi}{2}\alpha$. Since $\pi/2 > \frac{\pi}{2}\alpha$, there results that (1.3) has regularizing effect and dissipative properties (see Theorem 6.3). In order to recover a hyperbolic behaviour for $\alpha \in (0, 1)$, Equation (1.3) may be modified into

$$\mathbf{D}_{0,t}^\alpha u = -i^\alpha \Delta u,$$

see Example 4.6 for details. This equation appears in the pioneer work of Naber dedicated to fractional extensions of Schrödinger's equation ([Nab04]). It seems that the proper way time fractional Schrödinger's equation should be formulated is still discussed ([NAYH13]). Let us notice that the same issue holds for *transport equations*. We hope that the systematic analysis we have made in this paper will be useful for modeling processes using fractional time derivatives.

Regarding the regularizing effect of (1.2), we would like to emphasize that, contrary to the standard case, it acts only up to a finite order even if the right hand side of (1.2) is the Laplacian operator (see Proposition 6.7 and Example 6.8).

In [ER17], we introduce some “no-initial” value problems. These problems are interesting since (1.2) can by no means be considered as an autonomous equation if $\alpha \in (0, 1)$ (see [ER17] for details). In this paper, we continue the study of this kind of problems. To our knowledge, this study has also not been made at that time.

The paper is organised as follows. In the next section we introduce some preliminaries which are essential in our forthcoming discussions. This includes the definition of fractional derivatives, mild and strong solutions and Fourier multiplier results.

The third section is devoted to analyse some asymptotic properties of *Mittag-Leffler functions*. Sections 4 and 6 are devoted to the study of quantitative and qualitative properties of the solutions to

$$\mathbf{D}_{\tau,t}^\alpha u = P(D)u, \quad u(s) = v,$$

where τ and s are real numbers satisfying $\tau \leq s$. It has to be noticed that the analysis of the above problem differs whether $\tau = s$ or not. In the case $\tau = s$, we give existence and uniqueness results when the initial condition v lies in $L^2(\mathbb{R}^d)$. That extends the results of [ER17] where v was supposed to be smoother— i.e. lying in the domain of the operator $P(D)$. Our qualitative results concern regularizing effects and dissipative properties of the solutions.

In Section 5, we study the three time variable solution operator corresponding to the above equation.

2. PRELIMINARIES

2.1. Fractional derivatives. In this subsection, we recall the required background on *Caputo's fractional derivatives*. Let us introduce the convolution of functions defined on semi-infinite intervals.

Definition 2.1. For each $\tau \in \mathbb{R}$, let g be a function of $L^1_{\text{loc}}([0, \infty))$ and f be an element of $L^1_{\text{loc}}([\tau, \infty), X)$. Then the convolution of g and f is the function of $L^1_{\text{loc}}([\tau, \infty), X)$ defined by

$$g *_\tau f(t) = \int_\tau^t g(t-y)f(y)dy, \quad \text{a.e. } t \in [\tau, \infty).$$

Remark 2.2. (i) If $\tau = 0$ then we will write $g * f$ instead of $g *_\tau f$.

(ii) A function f belongs to $L^1_{\text{loc}}([0, \infty), X)$ iff for each positive T , f lies in $L^1(0, T, X)$.

The following kernel is of fundamental importance in the theory of fractional derivatives.

Definition 2.3. For $\beta \in (0, \infty)$, let us denote by g_β the function of $L^1_{\text{loc}}([0, \infty))$ defined for a.e. $t > 0$ by

$$g_\beta(t) = \frac{1}{\Gamma(\beta)} t^{\beta-1}.$$

Let us notice that $g_1 = 1$ and for each $\alpha \in [0, 1)$, we have

$$g_{1-\alpha}(t) = \frac{1}{\Gamma(1-\alpha)} t^{-\alpha}, \quad \text{a.e. } t > 0.$$

Moreover, for each $\alpha, \beta \in (0, \infty)$, the following identity holds.

$$g_\alpha * g_\beta = g_{\alpha+\beta}, \quad \text{in } L^1_{\text{loc}}([0, \infty)). \quad (2.1)$$

Then we are able to introduce the well known *fractional derivative* of a function in the sense of Caputo.

Definition 2.4. Let $\alpha \in (0, 1)$, $\tau \in \mathbb{R}$ and $f \in C([\tau, \infty), X)$. We say that f admits a (fractional) derivative of order α in $L^1_{\text{loc}}([\tau, \infty), X)$ or that f is α -differentiable in $L^1_{\text{loc}}([\tau, \infty), X)$ if

$$g_{1-\alpha} *_\tau f \in W^{1,1}_{\text{loc}}([\tau, \infty), X).$$

In that case, its (fractional) derivative of order α is the function of $L^1_{\text{loc}}([\tau, \infty), X)$ defined by

$$\mathbf{D}^\alpha_{\tau,t} f := \frac{d}{dt} \{g_{1-\alpha} *_\tau (f - f(\tau))\}.$$

If $\alpha = 1$ then $\mathbf{D}^1_{\tau,t}$ will denote the usual first order time derivative and f is said to be 1-differentiable in $L^1_{\text{loc}}([\tau, \infty), X)$ if f belongs to $W^{1,1}_{\text{loc}}([\tau, \infty), X)$.

Above, the vector space $W^{1,1}_{\text{loc}}([\tau, \infty), X)$ is the set of functions u in $L^1_{\text{loc}}([\tau, \infty), X)$ whose time-derivative (in the sense of distributions) belongs to $L^1_{\text{loc}}([\tau, \infty), X)$.

Remark 2.5. (i) If $f \in C([\tau, \infty), X)$, $\alpha \in (0, 1)$ and

$$g_{1-\alpha} *_\tau f \in W^{1,1}_{\text{loc}}([\tau, \infty), X)$$

then clearly,

$$g_{1-\alpha} *_\tau (f - f(\tau)) \in W^{1,1}_{\text{loc}}([\tau, \infty), X).$$

Hence Definition 2.4 makes sense.

(ii) It is well known that the space $W^{1,1}_{\text{loc}}([\tau, \infty), X)$ is a subset of $C([\tau, \infty), X)$.

(iii) Analog definition of *fractional derivative in $C([\tau, \infty), X)$* can be given; see [ER17, Definition 3.3] for a precise statement.

The following results will be useful in the sequel. The statement of Proposition 2.6 is closed to [ABHN11, Proposition 1.3.6] and can be proved in a same way.

Proposition 2.6. *For any $T > 0$, let $f \in W^{1,1}(0, T, X)$ and $g \in L^1(0, T)$. Then $g * f$ lies in $W^{1,1}(0, T, X)$ and*

$$\frac{d}{dt}\{g * f\} = g * \frac{d}{dt}f + g(\cdot)f(0), \quad \text{in } L^1(0, T, X). \quad (2.2)$$

By combining (2.2) and (2.1), we prove easily the well known following result.

Corollary 2.7. *Let $\alpha \in (0, 1]$, $\tau \in \mathbb{R}$ and $f \in C([\tau, \infty), X)$ be a α -differentiable function in $L^1_{\text{loc}}([\tau, \infty), X)$. Then*

$$g_\alpha *_\tau \mathbf{D}_{\tau,t}^\alpha f = f - f(\tau) \quad \text{in } L^1_{\text{loc}}([\tau, \infty), X).$$

2.2. Abstract linear fractional Problems. Let $A : D(A) \subseteq X \rightarrow X$ be a closed linear operator on a complex Banach space X , whose norm will be denoted by $\|\cdot\|_X$. The domain $D(A)$ is equipped with the standard graph norm, so that it is a Banach space. For $\alpha \in (0, 1]$ and any real number τ , we consider the following homogeneous linear fractional problem

$$\mathbf{D}_{\tau,t}^\alpha u = Au, \quad u(\tau) = v. \quad (2.3)$$

Following [Prü93], we introduce the definition of *strong* and *mild solutions*.

Definition 2.8. Let $\alpha \in (0, 1]$, $\tau \in \mathbb{R}$ and v be in $D(A)$. We say that a function u is a *strong solution* to (2.3) on $[\tau, \infty)$ if

- (i) u belongs to $C([\tau, \infty), D(A))$ and $u(\tau) = v$;
- (ii) u admits a derivative of order α in $C([\tau, \infty), X)$;
- (iii) $\mathbf{D}_{\tau,t}^\alpha u = Au$ in $C([\tau, \infty), X)$.

Similarly to the case where $\alpha = 1$, the fractional differential equation (2.3) can be transformed into an integral equation. More precisely, we have the following well known result, easily proved via Corollary 2.7.

Proposition 2.9. *Let $\alpha \in (0, 1]$, $\tau \in \mathbb{R}$ and v be in $D(A)$. Then the following propositions are equivalent.*

- (i) u is a strong solution to (2.3) on $[\tau, \infty)$.
- (ii) u belongs to $C([\tau, \infty), D(A))$, and

$$u = v + g_\alpha *_\tau Au, \quad \text{in } C([\tau, \infty), X).$$

When v is no more restricted to live in $D(A)$, we are led to consider solutions in a weaker sense, the so-called *mild solutions*.

Definition 2.10. Let $\alpha \in (0, 1]$, $\tau \in \mathbb{R}$ and v be in X . We say that a function u is a *mild solution* to (2.3) on $[\tau, \infty)$ if

- (i) u belongs to $C([\tau, \infty), X)$;
- (ii) $g_\alpha *_\tau u$ belongs to $C([\tau, \infty), D(A))$;
- (iii) $u = v + A(g_\alpha *_\tau u)$ in $C([\tau, \infty), X)$.

Clearly, A being closed, any strong solution to (2.3) is a mild solution to (2.3).

Definition 2.11. Let $\alpha \in (0, 1]$ and $\tau \in \mathbb{R}$. We say that (2.3) is *well posed* on X if for each v in X , (2.3) has a unique mild solution u on $[\tau, \infty)$ and for all positive time T ,

$$\|u(t)\|_X \leq C_T \|v\|_X, \quad \forall t \in [\tau, \tau + T],$$

where C_T is a real constant independent of t and v .

When the two time parameters problem

$$\mathbf{D}_{\tau,t}^\alpha u = Au, \quad u(s) = v \tag{2.4}$$

is considered, we will use Definition 2.12 for solutions to (2.4). Problem (2.4) is a “no-initial” value problems that is generally ill-posed when $\alpha = 1$ and $\tau < s$. We refer to [ER17] for a first approach to (2.4).

Definition 2.12. Let $\alpha \in (0, 1]$, $s, \tau \in \mathbb{R}$ with $\tau \leq s$ and v be in X . We say that a function $u \in C([\tau, \infty), X)$ is a *solution* to (2.4) on $[\tau, \infty)$ if

- (i) u belongs to $L_{\text{loc}}^1([\tau, \infty), D(A))$ and $u(s) = v$;
- (ii) u admits a derivative of order α in $L_{\text{loc}}^1([\tau, \infty), X)$;
- (iii) $\mathbf{D}_{\tau,t}^\alpha u = Au$ in $L_{\text{loc}}^1([\tau, \infty), X)$.

Let us notice that Definition 2.12 encapsulates a regularity effect when $s = \tau$. Indeed, the initial condition v belongs to X whereas $u(t)$ lies in $D(A)$ for almost every t larger than τ .

By considering Problem (2.4), it appears that a time shift simplifies the situation by reducing the study to the case $\tau = 0$. More precisely, for $\tau \leq s$, let us consider the problem

$$\mathbf{D}_{0,t}^\alpha w = Aw, \quad w(s - \tau) = v. \tag{2.5}$$

Then we have the following result.

Proposition 2.13. *Let $\alpha \in (0, 1]$, $\tau \leq s$, $v \in X$, $u \in C([\tau, \infty), X)$ and $w \in C([0, \infty), X)$. We suppose that*

$$u(t + \tau) = w(t), \quad \forall t \geq 0.$$

Then u is a solution to (2.4) on $[\tau, \infty)$ if and only if w is a solution to (2.5) on $[0, \infty)$.

The proof of Proposition 2.13 is easy and analog to the one of [ER17, Proposition 4.2], so we omit it. Notice that the latter is concerned with smoother solutions since the initial condition is supposed to belong to $D(A)$.

Remark 2.14. Statements analog to Proposition 2.13 hold for strong and mild solutions.

When $\tau = 0$ and $A = P(D)$ is a differential operator with constant coefficients on $X := L^2(\mathbb{R}^d)$, the strong solution to (2.3) has a representation of the form

$$u(t) = S_\alpha(t)v,$$

where the *Fourier multiplier* of the solution operator $S_\alpha(t) : D(P) \rightarrow D(P)$ is explicitly computed (see [ER17]). From this computation, it is clear that $S_\alpha(t)$ can be extended into a bounded operator on $L^2(\mathbb{R}^d)$ — see (5.3).

Since, for v in $L^2(\mathbb{R}^d)$, $S_\alpha(t)v$ does not belong generically to $D(P)$, $S_\alpha(\cdot)v$ cannot be a strong solution. Thus the issue is to find an equation whose solution is $S_\alpha(\cdot)v$. Proposition 2.13 below brings an answer by means of *mild solutions*.

Proposition 2.15. *Let $A : D(A) \subseteq X \rightarrow X$ be a closed densely defined linear operator on X . Let us assume that*

(i) *for each v in $D(A)$, there exists a unique strong solution to*

$$\mathbf{D}_{0,t}^\alpha u = Au, \quad u(0) = v; \quad (2.6)$$

(ii) *for each $T > 0$, there exists a constant C_T such that for all v in $D(A)$ and t in $[0, T]$, the solution u to (2.6) satisfies*

$$\|u(t)\|_X \leq C_T \|v\|_X. \quad (2.7)$$

Then for each $\tau \in \mathbb{R}$ and v in X , (2.3) has a unique mild solution u on $[\tau, \infty)$. Moreover, if $(v_n)_{n \geq 0} \subset D(A)$ converges toward v in X then for each positive time T ,

$$u_n(\cdot - \tau) \rightarrow u \quad \text{in } C([\tau, \tau + T], L^2(\mathbb{R}^d)), \quad (2.8)$$

where u_n denotes the strong solution to (2.6) with initial condition v_n .

Proof. It relies on (2.1), Proposition 2.13 and on arguments developed in [Prü93] after Definition 1.2. Let us notice that we do not need to use Titchmarsh's Theorem (see for instance [Dos88]) here. We use (2.1) instead. \square

2.3. Pseudo-differential operators on $L^2(\mathbb{R}^d)$. The Fourier transform on $L^2(\mathbb{R}^d)$, denoted by \mathcal{F} , is defined for $f \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$, by

$$\mathcal{F}f(\xi) = \hat{f}(\xi) := \int_{\mathbb{R}^d} f(x) e^{-i\xi \cdot x} dx, \quad \text{a.e. } \xi \in \mathbb{R}^d.$$

Then its inverse \mathcal{F}^{-1} satisfies

$$\mathcal{F}^{-1}(f)(x) := (2\pi)^{-d} \int_{\mathbb{R}^d} f(\xi) e^{i\xi \cdot x} d\xi, \quad \text{a.e. } x \in \mathbb{R}^d.$$

Moreover, according to Plancherel's identity, we have for all $f, g \in L^2(\mathbb{R}^d)$,

$$\int_{\mathbb{R}^d} \hat{f}(\xi) \overline{\hat{g}(\xi)} d\xi = (2\pi)^d \int_{\mathbb{R}^d} f(x) \overline{g(x)} dx. \quad (2.9)$$

For any function $P : \mathbb{R}^d \rightarrow \mathbb{C}$ continuous on \mathbb{R}^d , let

$$D(P) := \{u \in L^2(\mathbb{R}^d) \mid P(\cdot)\hat{u} \in L^2(\mathbb{R}^d)\}$$

and $P(D) : D(P) \subseteq L^2(\mathbb{R}^d) \rightarrow L^2(\mathbb{R}^d)$, the so-called *pseudo-differential operator with constant coefficients*, defined for all $u \in D(P)$ by

$$P(D)u := \mathcal{F}^{-1}(P(\cdot)\hat{u}).$$

The continuous function P is called the *symbol* of the pseudo-differential operator $P(D)$. In the literature, the symbol of a pseudo-differential operator is supposed to be a smooth function on \mathbb{R}^d . In this paper, we extend this definition to any continuous symbol P . That allows us to encapsulate the case of *fractional Laplacian operators*, nowadays extensively studied – see Example 2.16 below.

The domain $D(P)$ contains $\mathcal{F}^{-1}(\mathcal{D}(\mathbb{R}^d))$, where $\mathcal{D}(\mathbb{R}^d)$ is the space of C^∞ -functions on \mathbb{R}^d with compact support. Since $\mathcal{F}^{-1}(\mathcal{D}(\mathbb{R}^d))$ is dense in $L^2(\mathbb{R}^d)$ (see the proof of Lemma 2.17), there results that $P(D)$ is a closed and densely defined linear operator on $L^2(\mathbb{R}^d)$.

If P is a complex polynomial on \mathbb{R}^d i.e.

$$P : \mathbb{R}^d \rightarrow \mathbb{C}, \quad \xi \mapsto \sum_{|\beta| \leq M} p_\beta \xi^\beta,$$

with

$$\beta = (\beta_1, \dots, \beta_d) \in \mathbb{N}^d, \quad |\beta| = \beta_1 + \dots + \beta_d, \quad M \in \mathbb{N}, \quad p_\beta \in \mathbb{C},$$

then $P(D)$ is a differential operators with constant coefficients. More precisely, let us introduce Hörmander's notation for partial derivatives, that-is-to-say for $k = 1, \dots, d$, put

$$D_k := \frac{1}{i} \partial_{x_k}, \quad D := (D_1, \dots, D_d),$$

where $i^2 = -1$ and ∂_{x_k} denotes the usual partial derivative in the direction x_k . Then

$$P(D) := \sum_{|\beta| \leq M} p_\beta D^\beta.$$

Also

$$\begin{aligned} D(P) &= \{u \in L^2(\mathbb{R}^d) \mid P(D)u \in L^2(\mathbb{R}^d)\} \\ &= \{u \in L^2(\mathbb{R}^d) \mid \int_{\mathbb{R}^d} |P(\xi)\hat{u}(\xi)|^2 d\xi < \infty\}. \end{aligned} \quad (2.10)$$

In (2.10), $P(D)u$ is understood in the sense of distributions.

Example 2.16. let $\beta \in (0, \infty)$ and

$$P_{2\beta} : \mathbb{R}^d \rightarrow \mathbb{C}, \quad \xi = (\xi_1, \dots, \xi_d) \rightarrow -|\xi|^{2\beta} := -\left(\sum_{k=1}^d \xi_k^2\right)^\beta.$$

If $\beta = 1$ then $-P_2(D)$ is the *Laplacian operator*, i.e.

$$-P_2(D) = \sum_{k=0}^d D_k^2 = -\sum_{k=0}^d \partial_{x_k}^2 = -\Delta.$$

If $\beta \in (0, 1)$ then the pseudo-differential operator $-P_{2\beta}(D)$ is called the *fractional Laplacian operator* and is denoted by $(-\Delta)^\beta$. See for instance [DNPV12]. Its domain is the fractional Sobolev space $H^{2\beta}(\mathbb{R}^d) = W^{2\beta, 2}(\mathbb{R}^d)$, and

$$D(-(-\Delta)^\beta) = D(P_{2\beta}) = H^{2\beta}(\mathbb{R}^d) = \{u \in L^2(\mathbb{R}^d) \mid \int_{\mathbb{R}^d} |\xi|^{4\beta} |\hat{u}(\xi)|^2 d\xi < \infty\}.$$

2.4. A Fourier Multiplier Result. We will use the following classical result in *Fourier multiplier Theory*. Recall that $\mathcal{D}(\mathbb{R}^d)$ denotes the space of C^∞ -functions on \mathbb{R}^d with compact support. Then $\mathcal{F}^{-1}(\mathcal{D}(\mathbb{R}^d))$ is the space of L^2 -functions whose Fourier transform is in $\mathcal{D}(\mathbb{R}^d)$. Also, for X a complex Banach space, we denote by $\mathcal{L}(X)$ the space of linear continuous maps from X into X .

Lemma 2.17. *Let $a : \mathbb{R}^d \rightarrow \mathbb{C}$ be a continuous function on \mathbb{R}^d and*

$$A : \mathcal{F}^{-1}(\mathcal{D}(\mathbb{R}^d)) \rightarrow L^2(\mathbb{R}^d), \quad v \mapsto Av := \mathcal{F}^{-1}(a\hat{v}).$$

Then the following propositions are equivalent.

- (i) *A has a unique extension in $L^2(\mathbb{R}^d)$, still labelled A.*
- (ii) *There exists $C \geq 0$ such that $\|Av\|_{L^2(\mathbb{R}^d)} \leq C\|v\|_{L^2(\mathbb{R}^d)}$ for all v in $\mathcal{F}^{-1}(\mathcal{D}(\mathbb{R}^d))$.*
- (iii) *The Fourier multiplier $a(\cdot)$ belongs to $L^\infty(\mathbb{R}^d)$.*

Moreover, if one of the above properties holds then

$$\|A\|_{\mathcal{L}(L^2(\mathbb{R}^d))} = \sup_{\xi \in \mathbb{R}^d} |a(\xi)|. \quad (2.11)$$

Proof. It is clear that (i) implies (ii), with $C := \|A\|_{\mathcal{L}(L^2(\mathbb{R}^d))}$. By [BTW75, Theorem I.2.2 and proof], (ii) implies (iii) and $\sup_{\xi \in \mathbb{R}^d} |a(\xi)| \leq C$. Finally, (iii) implies (i) and $\|A\|_{\mathcal{L}(L^2(\mathbb{R}^d))} \leq \sup_{\xi \in \mathbb{R}^d} |a(\xi)|$. That follows from the fact that $\mathcal{F}^{-1}(\mathcal{D}(\mathbb{R}^d))$ is dense in $L^2(\mathbb{R}^d)$. Indeed, if a function f in $L^2(\mathbb{R}^d)$ satisfies

$$\int_{\mathbb{R}^d} \mathcal{F}^{-1}\varphi(x) \overline{f(x)} dx = 0, \quad \forall \varphi \in \mathcal{D}(\mathbb{R}^d).$$

Then by Plancherel's identity, we have

$$\int_{\mathbb{R}^d} \varphi(\xi) \overline{\widehat{f}(\xi)} d\xi = 0.$$

Thus $f = 0$ and $\mathcal{F}^{-1}(\mathcal{D}(\mathbb{R}^d))$ is dense in $L^2(\mathbb{R}^d)$.

There results that the equivalences hold. Then (2.11) follows easily. \square

3. MITTAG-LEFFLER FUNCTIONS

In fractional differential problems, *Mittag-Leffler functions* play the role of exponential functions in differential equations. In this section, we will give all the material on Mittag-Leffler functions needed to solve the time fractional problems under consideration in this paper. For further information, we refer the reader to [Pod99] or [GKMR14].

3.1. Definition and classical results.

Definition 3.1. For $\alpha > 0$ and $\beta \in \mathbb{R}$, we define the *generalised Mittag-Leffler function*, $E_{\alpha,\beta}$ by

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\alpha + \beta)}, \quad \forall z \in \mathbb{C}.$$

If $\beta = 1$ then we put $E_{\alpha} := E_{\alpha,1}$ and E_{α} is called the *Mittag-Leffler function of order α* .

For all $\alpha > 0$ and $\lambda \in \mathbb{C}$, the *Mittag-Leffler function* E_{α} satisfies

$$\mathbf{D}_{0,t}^{\alpha} E_{\alpha}(t^{\alpha} \lambda) = \lambda E_{\alpha}(t^{\alpha} \lambda), \quad \forall t \geq 0 \quad (3.1)$$

$$\frac{d}{dt} E_{\alpha}(t^{\alpha} \lambda) = \frac{1}{t} E_{\alpha,0}(t^{\alpha} \lambda). \quad (3.2)$$

Moreover, the function E_{α} is increasing and positive on \mathbb{R} (see [Pol48]). Also we have the following deep result that will be extensively used in this paper.

Theorem 3.2. [Pod99, Theorem 1.3 and 1.4] *Let $\alpha \in (0, 1]$, $\beta \in \mathbb{R}$ and $\alpha' \in (\alpha, 2)$. Then*

$$E_{\alpha,\beta}(z) = -\frac{1}{\Gamma(\beta - \alpha)} \frac{1}{z} + O\left(\frac{1}{z^2}\right), \quad \text{as } |z| \rightarrow \infty, \quad |\arg(z)| \geq \frac{\pi}{2} \alpha'. \quad (3.3)$$

Moreover, if $\alpha' < 2\alpha$ then

$$E_{\alpha,\beta}(z) = \frac{1}{\alpha} z^{\frac{1-\beta}{\alpha}} \exp(z^{1/\alpha}) + O\left(\frac{1}{z}\right), \quad \text{as } |z| \rightarrow \infty, \quad |\arg(z)| < \frac{\pi}{2} \alpha'. \quad (3.4)$$

In this paper, it is understood that the function $\arg(\cdot)$ ranges in $(-\pi, \pi]$.

3.2. Estimates of the Mittag-Leffler functions. For any continuous function $P : \mathbb{R}^d \rightarrow \mathbb{C}$ and $\alpha \in (0, 1]$, let us consider the following property: there exists $M_P \geq 0$ such that for each $\xi \in \mathbb{R}^d$ satisfying $P(\xi) \neq 0$, the following implication holds:

$$\text{if } |\arg P(\xi)| < \frac{\pi}{2}\alpha \text{ then } \operatorname{Re}(P(\xi)^{1/\alpha}) \leq M_P. \quad (3.5)$$

Of course, for $z \in \mathbb{C}$, we have set

$$z^{1/\alpha} := \begin{cases} |z|^{1/\alpha} \exp\left(i\frac{\arg z}{\alpha}\right) & \text{if } z \neq 0 \\ 0 & \text{if } z = 0 \end{cases},$$

and $\operatorname{Re}(P(\xi)^{1/\alpha})$ denotes the real part of $P(\xi)^{1/\alpha}$.

Lemma 3.3. *Let $P : \mathbb{R}^d \rightarrow \mathbb{C}$ be a continuous function on \mathbb{R}^d and $\alpha \in (0, 1]$. If (3.5) holds then, for each $T > 0$,*

$$\sup_{t \in [0, T]} \sup_{\xi \in \mathbb{R}^d} |E_\alpha(t^\alpha P(\xi))| < \infty. \quad (3.6)$$

If for each $\xi \in \mathbb{R}^d$ satisfying $P(\xi) \neq 0$, one has

$$|\arg P(\xi)| \geq \frac{\pi}{2}\alpha$$

then

$$\sup_{t \geq 0} \sup_{\xi \in \mathbb{R}^d} |E_\alpha(t^\alpha P(\xi))| < \infty. \quad (3.7)$$

Proof. Let $t \in [0, \infty)$ and $\alpha' \in (\alpha, 2\alpha)$. By (3.3), (3.4), there exist $R_1 > 0$ and $C > 0$ such that for each $\xi \in \mathbb{R}^d$,

$$|t^\alpha P(\xi)| > R_1, \quad |\arg P(\xi)| < \frac{\pi}{2}\alpha' \implies |E_\alpha(t^\alpha P(\xi))| \leq C |\exp(tP(\xi)^{1/\alpha})| + \frac{C}{t^\alpha |P(\xi)|} \quad (3.8)$$

and

$$|t^\alpha P(\xi)| > R_1, \quad |\arg P(\xi)| \geq \frac{\pi}{2}\alpha' \implies |E_\alpha(t^\alpha P(\xi))| \leq \frac{C}{t^\alpha |P(\xi)|}. \quad (3.9)$$

We will consider four cases.

(i) If $|t^\alpha P(\xi)| > R_1$ and $|\arg P(\xi)| \geq \frac{\pi}{2}\alpha'$ then by (3.9), one has

$$|E_\alpha(t^\alpha P(\xi))| \leq \frac{C}{R_1}.$$

(ii) If $|t^\alpha P(\xi)| > R_1$ and $|\arg P(\xi)| \in [\frac{\pi}{2}\alpha, \frac{\pi}{2}\alpha')$ then

$$|\exp(tP(\xi)^{1/\alpha})| = \exp\left(t|P(\xi)|^{1/\alpha} \cos\left(\frac{\arg P(\xi)}{\alpha}\right)\right).$$

However,

$$\frac{\pi}{2} \leq \left| \frac{\arg P(\xi)}{\alpha} \right| \leq \frac{\pi}{2} \frac{\alpha'}{\alpha} < \pi,$$

thus

$$|\exp(tP(\xi)^{1/\alpha})| \leq 1.$$

Going back to (3.8), we get

$$|E_\alpha(t^\alpha P(\xi))| \leq C + \frac{C}{R_1}.$$

(iii) If $|t^\alpha P(\xi)| \leq R_1$ then by continuity of the Mittag-Leffler function, we may find a constant $C(R_1)$ such that

$$|E_\alpha(t^\alpha P(\xi))| \leq C(R_1).$$

Hence (3.7) follows from (i), (ii) and (iii).

(iv) Finally, if $t \in [0, T]$, $|t^\alpha P(\xi)| > R_1$ and $|\arg P(\xi)| < \frac{\pi}{2}\alpha$ then, according to (3.5),

$$|\exp(tP(\xi)^{1/\alpha})| = \exp(t\operatorname{Re}(P(\xi)^{1/\alpha})) \leq \exp(TM_P).$$

Hence, with (3.8),

$$|E_\alpha(t^\alpha P(\xi))| \leq C \exp(TM_P) + \frac{C}{R_1}.$$

Hence (3.7) follows from (i)–(iv). \square

It turns out that (3.5) and (3.6) are equivalent if P is continuous. In that case, the following result is the converse of Lemma 3.3.

Lemma 3.4. *Let $P : \mathbb{R}^d \rightarrow \mathbb{C}$ be continuous on \mathbb{R}^d and $\alpha \in (0, 1]$. Suppose that there exists a sequence $(\xi_n)_{n \geq 0} \subset \mathbb{R}^d$ such that for each $n \geq 0$, one has $P(\xi_n) \neq 0$,*

$$|\arg P(\xi_n)| < \frac{\pi}{2}\alpha \quad \text{and} \quad \operatorname{Re}(P(\xi_n)^{1/\alpha}) \xrightarrow[n \rightarrow \infty]{} \infty.$$

Then for each $t > 0$,

$$\sup_{\xi \in \mathbb{R}^d} |E_\alpha(t^\alpha P(\xi))| = \infty. \quad (3.10)$$

Proof. We have

$$|\exp(tP(\xi_n)^{1/\alpha})| = \exp(t\operatorname{Re}(P(\xi_n)^{1/\alpha})) \xrightarrow[n \rightarrow \infty]{} \infty.$$

Moreover, $\operatorname{Re}(P(\xi_n)^{1/\alpha}) \rightarrow \infty$ implies $|P(\xi_n)| \rightarrow \infty$. Thus, with (3.4), we deduce

$$|E_\alpha(t^\alpha P(\xi_n))| \rightarrow \infty.$$

That proves (3.10). \square

Remark 3.5. (i) If $\alpha = 1$ then (3.5) is equivalent to

$$\sup_{\xi \in \mathbb{R}^d} \operatorname{Re}(P(\xi)) < \infty. \quad (3.11)$$

That is the usual assumption on symbols of differential operators with constant coefficients.

(ii) If $\alpha \in (0, 1)$ then (3.11) implies (3.5). Indeed, for each $\xi \in \mathbb{R}^d$ with $P(\xi) \neq 0$, assuming

$$\operatorname{Re}(P(\xi)) \leq M \quad \text{and} \quad |\arg P(\xi)| < \frac{\pi}{2}\alpha,$$

we deduce

$$|P(\xi)| \leq M \sqrt{1 + \tan^2(\frac{\pi}{2}\alpha)}.$$

Thus, for some finite constant M_P ,

$$\operatorname{Re}(P(\xi)^{1/\alpha}) \leq M_P.$$

(iii) In general, (3.5) does not imply (3.11): see Example 4.3 below.

In order to obtain a polynomial decay of Fourier multipliers at infinity, we will assume instead of (3.5), that P satisfies the following stronger assumption: for each α in $(0, 1]$, there exist $R_0 \geq 0$ and $\alpha' \in (\alpha, 2)$ such that for each $\xi \in \mathbb{R}^d$, one has the following implication:

$$\text{if } |P(\xi)| > R_0 \text{ then } |\arg P(\xi)| \geq \frac{\pi}{2}\alpha'. \quad (3.12)$$

Lemma 3.6. *Let $P : \mathbb{R}^d \rightarrow \mathbb{C}$ be continuous on \mathbb{R}^d , $\alpha \in (0, 1]$ and $\beta \in \mathbb{R}$. If (3.12) holds then for all $T > 0$, there exists a constant C_T such that*

$$|E_{\alpha,\beta}(t^\alpha P(\xi))| \leq \frac{C_T}{1 + t^\alpha |P(\xi)|}, \quad \forall t \in [0, T], \quad \forall \xi \in \mathbb{R}^d. \quad (3.13)$$

Moreover, if $R_0 = 0$ in (3.12) then there exists a constant C such that

$$|E_{\alpha,\beta}(t^\alpha P(\xi))| \leq \frac{C}{1 + t^\alpha |P(\xi)|}, \quad \forall t \geq 0, \quad \forall \xi \in \mathbb{R}^d. \quad (3.14)$$

Proof. We start to prove (3.13). Let $T > 0$ be fixed. If $t \in [0, T]$ and $\xi \in \mathbb{R}^d$ are such that $t^\alpha |P(\xi)| > R_0 T^\alpha$ then (3.12) implies

$$|\arg P(\xi)| \geq \frac{\pi}{2}\alpha'.$$

Then, by (3.3), there exists a positive number $R_{1,T}$ larger than $R_0 T^\alpha$ such that for $t^\alpha |P(\xi)| > R_{1,T}$, one has

$$|E_{\alpha,\beta}(t^\alpha P(\xi))| \leq \frac{C}{t^\alpha |P(\xi)|}.$$

Besides, if $t^\alpha |P(\xi)| \leq R_{1,T}$ then, for some constant $C(R_{1,T})$,

$$|E_{\alpha,\beta}(t^\alpha P(\xi))| \leq C(R_{1,T}).$$

Combining the two above estimates, we get (3.13) by choosing

$$C_T := \max\left(C\left(\frac{1}{R_{1,T}} + 1\right), (1 + R_{1,T})C(R_{1,T})\right).$$

If $R_0 = 0$ in (3.12) then $R_{1,T}$ can be chosen independently of T . Hence, C_T becomes independent of T and (3.14) follows. \square

4. MILD SOLUTIONS AND WELL-POSEDNESS ON $L^2(\mathbb{R}^d)$

For $P : \mathbb{R}^d \rightarrow \mathbb{C}$, a continuous function on \mathbb{R}^d , we consider the pseudo-differential operator $P(D)$ whose symbol is the function P as defined in Subsection 2.3. The fundamental hypothesis on P is (3.5) i.e. there exists $M_P \geq 0$ such that for all $\xi \in \mathbb{R}^d$ with $P(\xi) \neq 0$, one has

$$\text{if } |\arg P(\xi)| < \frac{\pi}{2}\alpha \text{ then } \operatorname{Re}(P(\xi)^{1/\alpha}) \leq M_P. \quad (4.1)$$

Theorem 4.1. *Let $\alpha \in (0, 1]$, $s \in \mathbb{R}$ and $P : \mathbb{R}^d \rightarrow \mathbb{C}$ be a continuous function on \mathbb{R}^d . Then for each $v \in L^2(\mathbb{R}^d)$, the problem*

$$\mathbf{D}_{s,t}^\alpha u = P(D)u, \quad u(s) = v, \quad (4.2)$$

has a unique mild solution u . Moreover, u admits the following representation for all t in $[s, \infty)$,

$$u(t) = \mathcal{F}^{-1}\left(E_\alpha((t-s)^\alpha P(\xi))\hat{v}(\xi)\right), \quad \text{in } L^2(\mathbb{R}^d). \quad (4.3)$$

Proof. Let us start to show that, for each v in $D(P)$, the function u defined by

$$u(t) = \mathcal{F}^{-1}\left(E_\alpha(t^\alpha P(\xi))\hat{v}(\xi)\right), \quad (4.4)$$

is the unique solution on $[0, \infty)$ to

$$\begin{cases} u \in C([0, \infty), D(P)) \\ u = v + g_\alpha * P(D)u, \quad \text{in } C([0, \infty), L^2(\mathbb{R}^d)). \end{cases} \quad (4.5)$$

The uniqueness can be established as in the proof of [ER17, Theorem 6.1]. Regarding the existence of solution to (4.5), we may prove with Lemma 3.3 that the function u defined by (4.4), belongs to $C([0, \infty), D(P))$. In order to show that

$$u = v + g_\alpha * P(D)u, \quad \text{in } C([0, \infty), L^2(\mathbb{R}^d)),$$

we introduce for each $\xi \in \mathbb{R}^d$, the function $w : [0, \infty) \rightarrow \mathbb{C}$ defined for all $t \geq 0$ by

$$w(t) := \mathcal{F}(u(t))(\xi) = E_\alpha(t^\alpha P(\xi))\hat{v}(\xi).$$

With (3.1) and Proposition 2.9, we have

$$w(t) = \hat{v}(\xi) + (g_\alpha * P(\xi)w)(t), \quad \forall t \geq 0.$$

Then the issue is to pass from the above equality which holds in $[0, \infty)$ to an equality in $C([0, \infty), L^2(\mathbb{R}^d))$. This is done by using the facts that

- $w(t) - \hat{v}(\xi) = (\mathcal{F}(u(t)) - \hat{v})(\xi)$
- $(g_\alpha * P(\xi)w)(t) = \{g_\alpha * P\mathcal{F}(u(\cdot))\}(t)(\xi)$
- $\mathcal{F}(u(\cdot)) - \hat{v} \in C([0, \infty), L^2(\mathbb{R}^d))$
- $g_\alpha * P\mathcal{F}(u(\cdot)) \in C([0, \infty), L^2(\mathbb{R}^d))$.

Hence we deduce

$$\mathcal{F}(u(\cdot)) - \hat{v} = g_\alpha * P\mathcal{F}(u(\cdot)), \quad \text{in } C([0, \infty), L^2(\mathbb{R}^d)).$$

Consequently, u satisfies (4.5).

Then Proposition 2.9 implies that the function u defined by (4.4) is the unique strong solution to

$$\mathbf{D}_{0,t}^\alpha u = P(D)u, \quad u(0) = v.$$

Moreover, the estimate (3.6) in Lemma 3.3 implies that u satisfies the assumption (2.7) in Proposition 2.15. The later proposition gives existence and uniqueness of the mild solution u to (4.2) on $[s, \infty)$. Finally, the estimate (3.6), (4.4) and the limit process (2.8) yield (4.3). \square

In the case where P is continuous, the assumption (4.1) is necessary to have a well posed problem in the sense of Definition 2.11.

Theorem 4.2. *Let $\alpha \in (0, 1]$, $s \in \mathbb{R}$ and $P : \mathbb{R}^d \rightarrow \mathbb{C}$ be a continuous function on \mathbb{R}^d . Then (4.2) is well posed if and only if (4.1) holds.*

Proof. W.l.o.g. we may assume that $s = 0$. Let us assume that (4.1) holds. Then Theorem 4.1 yields that (4.2) has a unique mild solution. The estimate for well-posedness follows from (4.3) and (3.6).

Conversely, for any $v \in \mathcal{F}^{-1}(\mathcal{D}(\mathbb{R}^d))$, we obtain by Fourier transform that the mild solution u to (4.2) satisfies for all $t > 0$,

$$u(t) := \mathcal{F}^{-1}\left(E_\alpha(t^\alpha P(\xi))\hat{v}(\xi)\right), \quad \text{in } L^2(\mathbb{R}^d).$$

By the estimate for well-posedness and Lemma 2.17, we deduce that $E_\alpha(t^\alpha P(\cdot))$ is bounded on \mathbb{R}^d . Then, by Lemma 3.4, we must have (4.1). Recall that for convinence of reading, (4.1) is just a rewriting of (3.5). \square

Example 4.3. For $d = 1$, $a > 0$ and $\theta \in \mathbb{R}$, let us put

$$P(\xi) := ia e^{i\theta} \xi, \quad \forall \xi \in \mathbb{R}.$$

Then, in view of Subsection 2.3, the polynomial P is the symbol of the operator $P(D)$ with domain $D(P) = H^1(\mathbb{R})$ and defined by

$$P(D) = ia e^{i\theta} D = a e^{i\theta} \partial_x.$$

We claim that if

$$|\theta| \leq \frac{\pi}{2}(1 - \alpha) \quad \text{and} \quad \alpha \in (0, 1]$$

then the problem

$$\mathbf{D}_{0,t}^\alpha u = ia e^{i\theta} Du = a e^{i\theta} \partial_x u, \quad u(0) = v \in L^2(\mathbb{R}^d), \quad (4.6)$$

has a unique mild solution u whose Fourier transform satisfies

$$\mathcal{F}(u(t))(\xi) = E_\alpha(t^\alpha ia e^{i\theta} \xi) \hat{v}(\xi), \quad \forall t \geq 0, \quad \forall \xi \in \mathbb{R}^d.$$

Indeed, by Theorem 4.1, it is enough to check (4.1). For, we observe that

$$\arg P(\xi) = \begin{cases} \theta + \frac{\pi}{2} & \text{if } \xi > 0 \\ \theta - \frac{\pi}{2} & \text{if } \xi < 0 \end{cases}.$$

Thus we deduce easily that

$$|\arg P(\xi)| \geq \frac{\pi}{2} \alpha, \quad \forall \xi \neq 0.$$

Hence (4.1) holds.

Let us notice that if $\alpha = 1$ and $\theta = 0$ then we recover the usual *transport equation* on $L^2(\mathbb{R}^d)$, namely

$$u_t = ia Du = a \partial_x u.$$

More generally, if $\alpha \in (0, 1]$ and $\theta = -\frac{\pi}{2}(1 - \alpha)$ then the equation in (4.6) reads

$$\mathbf{D}_{0,t}^\alpha u = ai^\alpha Du = ai^{\alpha-1} \partial_x u.$$

Besides, if $\alpha \in (0, 1)$ then

$$\operatorname{Re} P(\xi) = a \cos\left(\frac{\pi}{2}\alpha\right) \xi \xrightarrow[\xi \rightarrow \infty]{} \infty.$$

Thus (3.11) does not hold. That shows that (4.1) (or (3.5)) and (3.11) are not equivalent— see Remark 3.5 (iii). Also let us notice that, in view of Lemma 2.17, the problem

$$u_t = ai^{\alpha-1} \partial_x u, \quad u(0) = v.$$

is not well posed on $L^2(\mathbb{R}^d)$ in the sense of Definition 2.11.

In the standard case $\alpha = 1$, linear hyperbolic equations like *transport equations* or *Schrödinger equations* induce conservation laws. The symbol P of these equations satisfies $|\arg P(\xi)| = \pi/2$. This is the critical angle since, roughly speaking, if $|\arg P(\xi)| < \pi/2$ then the problem is generally ill-posed. If $|\arg P(\xi)| > \pi/2$ then the problem is well-posed, has regularizing effects and dissipative properties i.e. $u(t) \rightarrow 0$ as $t \rightarrow \infty$. If $|\arg P(\xi)| = \pi/2$ then the problem is well-posed in $L^2(\mathbb{R}^d)$ but, has no regularizing and dissipative properties; in particular $t \mapsto u(t)$ oscillates.

When $\alpha \in (0, 1]$, we will highlight that the critical angle is $\frac{\pi}{2}\alpha$. Indeed, we have seen in Theorem 4.2 that this angle characterizes the well posedness. Also, if $|\arg P(\xi)| = \frac{\pi}{2}\alpha$, Theorem 4.4 below states that the solution oscillates at infinity, like in the case $\alpha = 1$.

For simplicity, we will consider w.l.o.g. the problem

$$\mathbf{D}_{0,t}^\alpha u = P(D)u, \quad u(0) = v \in L^2(\mathbb{R}^d). \quad (4.7)$$

Theorem 4.4. *Let $\alpha \in (0, 1]$, $v \in L^2(\mathbb{R}^d)$ and assume that*

- (i) P is continuous on \mathbb{R}^d ;
- (ii) for almost every $\xi \in \mathbb{R}^d$, one has $P(\xi) \neq 0$ and

$$|\arg P(\xi)| = \frac{\pi}{2}\alpha. \quad (4.8)$$

Then the mild solution u to (4.7) satisfies

$$u(t) - w(t) \xrightarrow[t \rightarrow \infty]{} 0 \quad \text{in } L^2(\mathbb{R}^d), \quad (4.9)$$

where $w : [0, \infty) \rightarrow L^2(\mathbb{R}^d)$ is the function with Fourier transform

$$\mathcal{F}(w(t))(\xi) = \frac{1}{\alpha} \exp(tP(\xi)^{1/\alpha}) \hat{v}(\xi), \quad \forall \xi \in \mathbb{R}^d. \quad (4.10)$$

In particular,

$$\|u(t)\|_{L^2(\mathbb{R}^d)} \xrightarrow[t \rightarrow \infty]{} \frac{1}{\alpha} \|v\|_{L^2(\mathbb{R}^d)}. \quad (4.11)$$

Let us notice that the function w above is the mild solution to

$$\frac{d}{dt} w = P^{1/\alpha}(D)w, \quad w(0) = \frac{1}{\alpha} v.$$

Also, combining (4.8) and (4.10), we may write

$$\mathcal{F}(w(t))(\xi) = \frac{1}{\alpha} \exp(\pm it|P(\xi)|^{1/\alpha}) \hat{v}(\xi), \quad \forall \xi \in \mathbb{R}^d.$$

Proof. According to Theorem 4.1, we know that (4.7) has a unique mild solution given by (4.3) with $s = 0$. Let us first show that

$$\hat{u}(t) - \hat{w}(t) \xrightarrow[t \rightarrow \infty]{} 0 \quad \text{in } L^2(\mathbb{R}^d). \quad (4.12)$$

For, let $\xi \in \mathbb{R}^d$, with $P(\xi) \neq 0$. Then

$$\begin{aligned} \hat{u}(t)(\xi) - \hat{w}(t)(\xi) &= (E_\alpha(t^\alpha P(\xi)) - \frac{1}{\alpha} \exp(tP(\xi)^{1/\alpha})) \hat{v}(\xi) \\ &= O\left(\frac{1}{t^\alpha P(\xi)}\right) \hat{v}(\xi), \end{aligned}$$

according to the asymptotic expansion (3.4) and (4.8). Thus, since $P(\xi) \neq 0$ for a.e. $\xi \in \mathbb{R}^d$, we get

$$\hat{u}(t) - \hat{w}(t) \xrightarrow[t \rightarrow \infty]{} 0 \quad \text{a.e. in } \mathbb{R}^d.$$

Moreover, by estimate (3.7) in Lemma 3.3, there exists a constant C such that

$$|\hat{u}(t)(\xi) - \hat{w}(t)(\xi)| \leq C|\hat{v}(\xi)|, \quad \forall t \geq 0, \quad \text{a.e. } \xi \in \mathbb{R}^d.$$

Then Lebesgue's convergence Theorem implies (4.12); hence (4.9) follows.

In order to prove (4.11), denoting by $\langle \cdot, \cdot \rangle_2$ the inner product of $L^2(\mathbb{R}^d)$, we observe that

$$\begin{aligned} & \left| \|u(t)\|_{L^2(\mathbb{R}^d)}^2 - \|w(t)\|_{L^2(\mathbb{R}^d)}^2 \right| \\ &= \left| \langle u(t) - w(t), (u(t) - w(t)) + 2w(t) \rangle_2 \right| \\ &\leq \|u(t) - w(t)\|_{L^2(\mathbb{R}^d)}^2 + 2\|u(t) - w(t)\|_{L^2(\mathbb{R}^d)}\|w(t)\|_{L^2(\mathbb{R}^d)}. \end{aligned}$$

By (4.9) and $\|w(t)\|_{L^2(\mathbb{R}^d)} = \frac{1}{\alpha}\|v\|_{L^2(\mathbb{R}^d)}$, we obtain (4.11). \square

In the critical case where $|\arg P(\xi)| = \frac{\pi}{2}\alpha$, we cannot expect that the equation in (4.7) possesses a regularizing effect. More precisely, the following result holds.

Proposition 4.5. *Let $\alpha \in (0, 1]$ and assume that*

- (i) $P : \mathbb{R}^d \rightarrow \mathbb{C}$ is continuous on \mathbb{R}^d and satisfies (4.1);
- (ii) there exists a sequence $(\xi_n)_{n \geq 1} \subset \mathbb{R}^d$ and $\beta \in \mathbb{N}^d$ such that

$$|\xi_n^\beta| \rightarrow \infty, \quad |P(\xi_n)| \rightarrow \infty \quad \text{as } n \rightarrow \infty$$

and

$$|P(\xi_n)| \neq 0, \quad |\arg P(\xi_n)| = \frac{\pi}{2}\alpha, \quad \forall n \geq 1.$$

Then for each $t \geq 0$, there is no finite constant $C = C(t, \beta)$ satisfying

$$\|D^\beta u(t)\|_{L^2(\mathbb{R}^d)} \leq C\|v\|_{L^2(\mathbb{R}^d)}, \quad \forall v \in \mathcal{F}^{-1}(\mathcal{D}(\mathbb{R}^d)), \quad (4.13)$$

where u is the mild solution to (4.7) with initial condition v .

Proof. For each $t \geq 0$, let us set

$$a_t(\xi) := \xi^\beta E_\alpha(t^\alpha P(\xi)), \quad \forall \xi \in \mathbb{R}^d.$$

Then $a_t(\cdot)$ is continuous on \mathbb{R}^d and, for each v in $\mathcal{F}^{-1}(\mathcal{D}(\mathbb{R}^d))$, the function $\xi \mapsto a_t(\xi)\hat{v}(\xi)$ belongs to $L^2(\mathbb{R}^d)$. Moreover, by (4.3),

$$D^\beta u(t) = \mathcal{F}^{-1}(a_t(\cdot)\hat{v}),$$

so that $\|D^\beta u(t)\|_{L^2(\mathbb{R}^d)}$ is well defined in (4.13).

Let $t > 0$. Since $|\arg P(\xi_n)| = \frac{\pi}{2}\alpha$, we have thanks (3.4),

$$\begin{aligned} a_t(\xi_n) &= \frac{\xi_n^\beta}{\alpha} \left(\exp(tP(\xi_n)^{1/\alpha}) + O\left(\frac{1}{t^\alpha P(\xi_n)}\right) \right) \\ &= \frac{\xi_n^\beta}{\alpha} \left(\exp(\pm it|P(\xi_n)|^{1/\alpha}) + O\left(\frac{1}{t^\alpha P(\xi_n)}\right) \right). \end{aligned}$$

Moreover, $|P(\xi_n)| \rightarrow \infty$, hence

$$|a_t(\xi_n)| \sim \frac{|\xi_n^\beta|}{\alpha}, \quad \text{when } n \rightarrow \infty.$$

Since $|\xi_n^\beta| \rightarrow \infty$ by assumption, we obtain

$$\sup_{\xi \in \mathbb{R}^d} |a_t(\xi)| = \infty, \quad \forall t \geq 0,$$

and the impossibility of (4.13) follows from Lemma 2.17. \square

Example 4.6. Let us consider the following *fractional Schrödinger equation* introduced in [Nab04]

$$\mathbf{D}_{0,t}^\alpha u = -i^\alpha \Delta u = -i^\alpha P_2(D)u. \quad (4.14)$$

By setting $P(\xi) = -i^\alpha P_2(\xi) = i^\alpha |\xi|^2$, we see by Theorem 4.1 that for each $v \in L^2(\mathbb{R}^d)$, Equation (4.14) supplemented with the initial condition $u(0) = v$ has a unique mild solution u . Since

$$\arg P(\xi) = \frac{\pi}{2}\alpha, \quad \forall \xi \neq 0,$$

Theorem 4.4 gives

$$\|u(t)\|_{L^2(\mathbb{R}^d)} \xrightarrow[t \rightarrow \infty]{} \frac{1}{\alpha} \|v\|_{L^2(\mathbb{R}^d)}.$$

This result was originally stated in [Nab04]. More precisely, Theorem 4.4 yields

$$u(t) - w(t) \xrightarrow[t \rightarrow \infty]{} 0 \quad \text{in } L^2(\mathbb{R}^d),$$

where

$$\mathcal{F}(w(t))(\xi) = \frac{1}{\alpha} \exp(it|\xi|^{2/\alpha}) \hat{v}(\xi), \quad \text{a.e. } \xi \in \mathbb{R}^d.$$

Moreover, we claim that Equation (4.14) has no regularizing effect, that is, for each $\beta \in \mathbb{N}^d$, $\beta \neq 0$, there is no finite constant C such that

$$\|D^\beta u(t)\|_{L^2(\mathbb{R}^d)} \leq C \|v\|_{L^2(\mathbb{R}^d)}, \quad \forall v \in \mathcal{F}^{-1}(\mathcal{D}(\mathbb{R}^d)).$$

Indeed, by setting $\xi_n := n(1, \dots, 1) \in \mathbb{R}^d$, we see that $(\xi_n)_{n \geq 1} \subset \mathbb{R}^d$ satisfies the assumption (ii) in Proposition 4.5. Then the claim follows by Proposition 4.5.

The results featured in this example are well known and easily proved in the case $\alpha = 1$. Thus (4.14) is a suitable generalization of Schrödinger's equation.

5. SOLUTION OPERATORS AND REGULARIZING EFFECTS

For $\alpha \in (0, 1]$, $\tau \leq s$, a symbol P and $v \in L^2(\mathbb{R}^d)$, we will study the solution operator of the following fractional problem

$$\mathbf{D}_{\tau,t}^\alpha u = P(D)u, \quad u(s) = v. \quad (5.1)$$

Our main tool is *Fourier multipliers*. Regularizing effects are obtained by assuming (3.12). The study of solution operators to (4.5) differs whether s is equal to τ or not. We will start with the simpler case $\tau = s$.

5.1. The case where $\tau = s$. In Proposition 4.5 below, \mathcal{D}_2 denotes the subset of \mathbb{R}^2 defined by

$$\mathcal{D}_2 := \{(t, s) \in \mathbb{R}^2 \mid t > s\}. \quad (5.2)$$

Proposition 5.1. *Let $\alpha \in (0, 1]$ and P be a continuous function on \mathbb{R}^d satisfying (3.12). Then for t, s in \mathbb{R} with $t \geq s$, the operator $T_\alpha(t, s)$ defined, for each v in $L^2(\mathbb{R}^d)$ by*

$$T_\alpha(t, s)v = \mathcal{F}^{-1}\left(E_\alpha((t-s)^\alpha P(\cdot))\hat{v}(\cdot)\right) \quad (5.3)$$

belongs to $\mathcal{L}(L^2(\mathbb{R}^d))$. Moreover, T_α is continuous on \mathcal{D}_2 with values in $\mathcal{L}(L^2(\mathbb{R}^d), D(P))$, and for each $T > 0$, there exists a constant C_T such that

$$\|T_\alpha(t, s)\|_{\mathcal{L}(L^2(\mathbb{R}^d))} \leq C_T, \quad \forall 0 \leq t - s \leq T \quad (5.4)$$

$$\|P(D)T_\alpha(t, s)\|_{\mathcal{L}(L^2(\mathbb{R}^d))} \leq \frac{C_T}{(t-s)^\alpha}, \quad \forall 0 < t - s \leq T. \quad (5.5)$$

Remark 5.2. According to Theorem 4.1, we know that if $v \in L^2(\mathbb{R}^d)$ then $T_\alpha(\cdot, s)v$ is the unique mild solution to

$$\mathbf{D}_{s,t}^\alpha u = P(D)u, \quad u(s) = v. \quad (5.6)$$

For all purposes, we recall that $\mathcal{L}(L^2(\mathbb{R}^d))$ denotes the complex Banach space of linear and continuous maps from $L^2(\mathbb{R}^d)$ into itself.

Estimate (5.5) shows that the solution operator $T_\alpha(t, s)$ has a regularizing effect. In the case where $\alpha = 1$ and $s = 0$, (5.5) holds for (abstract) analytic semi-groups. Let us notice that when $P \neq 0$ and $\alpha \in (0, 1)$, $S_\alpha(\cdot) := T_\alpha(\cdot, 0)$ is never a semi-group: see [ER17]. However, by (5.3), T_α is *translation invariant* for each $\alpha \in (0, 1]$ i.e.

$$T_\alpha(t, s) = T_\alpha(t - s, 0), \quad \forall t \geq s.$$

Proof of Proposition 5.1. Estimate (5.4) is a consequence of Lemma 2.17 and (3.6). Regarding (5.5), we have for $v \in L^2(\mathbb{R}^d)$ and $0 < t - s \leq T$,

$$\begin{aligned} \|P(D)T_\alpha(t, s)v\|_{L^2(\mathbb{R}^d)} &= (2\pi)^{-d/2} \|P(\xi)E_\alpha((t-s)^\alpha P(\xi))\hat{v}\|_{L^2(\mathbb{R}^d)} \\ &\leq \frac{C_T}{(t-s)^\alpha} (2\pi)^{-d/2} \|\hat{v}\|_{L^2(\mathbb{R}^d)} && \text{(by (3.13))} \\ &= \frac{C_T}{(t-s)^\alpha} \|v\|_{L^2(\mathbb{R}^d)}. \end{aligned}$$

Hence (5.5) follows.

There remains to prove that T_α belongs to $C(\mathcal{D}_2, \mathcal{L}(L^2(\mathbb{R}^d), D(P)))$. For, we will first show that $P(D)T_\alpha$ lies in $C(\mathcal{D}_2, \mathcal{L}(L^2(\mathbb{R}^d)))$. By translation invariance of T_α , it is enough to show that

$$P(D)T_\alpha(\cdot, 0) \in C((\frac{1}{T}, T), \mathcal{L}(L^2(\mathbb{R}^d))),$$

for each $T > 1$. For, let $t \in (\frac{1}{T}, T)$ and $(t_n)_{n \geq 0} \subset (\frac{1}{T}, T)$ be a sequence converging toward t . *Fourier multiplier Theory* yields that

$$\|P(D)(T_\alpha(t, 0) - T_\alpha(t_n, 0))\|_{\mathcal{L}(L^2(\mathbb{R}^d))} = \sup_{\xi \in \mathbb{R}^d} |(f_\xi(t) - f_\xi(t_n))P(\xi)|,$$

where for each $\xi \in \mathbb{R}^d$, f_ξ denotes the function

$$f_\xi : [0, \infty) \rightarrow \mathbb{C}, \quad t \mapsto E_\alpha(t^\alpha P(\xi)). \quad (5.7)$$

Thus there remains to prove that

$$\sup_{\xi \in \mathbb{R}^d} |(f_\xi(t) - f_\xi(t_n))P(\xi)| \xrightarrow{n \rightarrow \infty} 0. \quad (5.8)$$

For, by the mean value theorem, we have for each $\xi \in \mathbb{R}^d$,

$$\begin{aligned}
|(f_\xi(t) - f_\xi(t_n))P(\xi)| &\leq \sup_{\tau \in [\frac{1}{T}, T]} |f'_\xi(\tau)P(\xi)| |t - t_n| \\
&\leq \sup_{\tau \in [\frac{1}{T}, T]} \frac{|E_{\alpha,0}(\tau^\alpha P(\xi))P(\xi)|}{\tau} |t - t_n| && \text{(by (3.2))} \\
&\leq \sup_{\tau \in [\frac{1}{T}, T]} \frac{C_T |P(\xi)|}{\tau(1 + \tau^\alpha |P(\xi)|)} |t - t_n| && \text{(by (3.13))} \\
&\leq C_T T^{1+\alpha} |t - t_n|.
\end{aligned}$$

Hence (5.8) follows. Thus $P(D)T_\alpha(\cdot, \cdot)$ lies in $C(\mathcal{D}_2, \mathcal{L}(L^2(\mathbb{R}^d)))$. In the same way, we may prove that $T_\alpha(\cdot, \cdot)$ belongs to $C(\mathcal{D}_2, \mathcal{L}(L^2(\mathbb{R}^d)))$. That completes the proof of the proposition. \square

5.2. The case where $\tau \leq s$. In that case, the Fourier multiplier is

$$\xi \mapsto \frac{E_\alpha((t - \tau)^\alpha P(\xi))}{E_\alpha((s - \tau)^\alpha P(\xi))}. \quad (5.9)$$

In order to control this multiplier, we will need the following assumptions.

$$P : \mathbb{R}^d \rightarrow \mathbb{R} \text{ is continuous on } \mathbb{R}^d \quad (5.10)$$

$$P(\xi) \leq M_P \text{ for each } \xi \in \mathbb{R}^d. \quad (5.11)$$

The fact that P is real valued warrants that the denominator in (5.9) does not vanish.

Under assumptions (5.10) and (5.11), we deduce from (3.3) the following fundamental estimates about the Mittag-Leffler function: for each α in $(0, 1)$, there exists $R_1 > 0$ such that for $t^\alpha P(\xi) \leq -R_1$, one has

$$\frac{1}{\sqrt{2}\Gamma(1 - \alpha)} \frac{1}{t^\alpha |P(\xi)|} \leq E_\alpha(t^\alpha P(\xi)) \leq \frac{\sqrt{2}}{\Gamma(1 - \alpha)} \frac{1}{t^\alpha |P(\xi)|}. \quad (5.12)$$

It is essential that $\alpha \neq 1$ since the latter inequality is obviously false when $\alpha = 1$.

The following result gives estimates on the Fourier multiplier (5.9).

Lemma 5.3. *Let $\alpha \in (0, 1)$. Under assumptions (5.10) and (5.11), one has*

$$\sup_{\xi \in \mathbb{R}^d} \frac{E_\alpha(t^\alpha P(\xi))}{E_\alpha(s^\alpha P(\xi))} \leq C \max\left(\left(\frac{s}{t}\right)^\alpha, 1, E_\alpha(t^\alpha M_P)\right), \quad \forall t, s > 0 \quad (5.13)$$

$$\sup_{\xi \in \mathbb{R}^d} E_\alpha(t^\alpha P(\xi)) = E_\alpha(t^\alpha M_P), \quad \forall t \geq 0, \quad (5.14)$$

where the constant C in (5.13) is independent of s and t .

Proof. Equality (5.14) is a straightforward consequence of (5.10), (5.11) and the monotonicity of Mittag-Leffler's function on \mathbb{R} . Let s and t be positive and $\xi \in \mathbb{R}^d$. In order to establish (5.13), we will consider four cases.

(i) $s^\alpha P(\xi) \leq -R_1$, $t^\alpha P(\xi) \leq -R_1$. Then, by (5.12),

$$\frac{E_\alpha(t^\alpha P(\xi))}{E_\alpha(s^\alpha P(\xi))} \leq 2\left(\frac{s}{t}\right)^\alpha.$$

(ii) $s^\alpha P(\xi) \leq -R_1$, $t^\alpha P(\xi) > -R_1$. Then, by (5.12),

$$\frac{1}{E_\alpha(s^\alpha P(\xi))} \leq \sqrt{2}\Gamma(1-\alpha)s^\alpha |P(\xi)|. \quad (5.15)$$

Moreover, in the present case, we have $-R_1 t^{-\alpha} < P(\xi)$ and

$$-R_1 \left(\frac{s}{t}\right)^\alpha < s^\alpha P(\xi) \leq -R_1.$$

Thus, with (5.15),

$$\frac{1}{E_\alpha(s^\alpha P(\xi))} < \sqrt{2}\Gamma(1-\alpha)R_1 \left(\frac{s}{t}\right)^\alpha.$$

Regarding $E_\alpha(t^\alpha P(\xi))$, we notice that $s^\alpha P(\xi) \leq -R_1$ implies that $P(\xi) \leq 0$. Thus $E_\alpha(t^\alpha P(\xi)) \leq 1$. Then

$$\frac{E_\alpha(t^\alpha P(\xi))}{E_\alpha(s^\alpha P(\xi))} \leq \sqrt{2}\Gamma(1-\alpha)R_1 \left(\frac{s}{t}\right)^\alpha.$$

(iii) $s^\alpha P(\xi) > -R_1$, $t^\alpha P(\xi) \leq -R_1$. Then, since E_α is increasing on \mathbb{R} ,

$$E_\alpha(t^\alpha P(\xi)) \leq E_\alpha(-R_1), \quad E_\alpha(s^\alpha P(\xi)) > E_\alpha(-R_1).$$

Thus

$$\frac{E_\alpha(t^\alpha P(\xi))}{E_\alpha(s^\alpha P(\xi))} \leq 1.$$

(iv) $s^\alpha P(\xi) > -R_1$, $t^\alpha P(\xi) > -R_1$. Then, in view of (5.11),

$$\frac{E_\alpha(t^\alpha P(\xi))}{E_\alpha(s^\alpha P(\xi))} \leq \frac{E_\alpha(t^\alpha M_P)}{E_\alpha(-R_1)}.$$

By gathering these four cases, we get (5.13). \square

By *Fourier multipliers Theory* and Lemma 5.3, we easily obtain the following result. In Corollary 5.4 below, \mathcal{D}_3 is the subset of \mathbb{R}^3 defined by

$$\mathcal{D}_3 := \{(t, s, \tau) \in \mathbb{R}^3 \mid t > \tau, s > \tau\} \cup \{(t, s, s) \in \mathbb{R}^3 \mid t \geq s\}. \quad (5.16)$$

Corollary 5.4. *Let $\alpha \in (0, 1)$ and P satisfy (5.10), (5.11). Then for (t, s, τ) in \mathcal{D}_3 , the function*

$$\xi \mapsto \frac{E_\alpha((t-\tau)^\alpha P(\xi))}{E_\alpha((s-\tau)^\alpha P(\xi))}$$

is a Fourier multiplier on $L^2(\mathbb{R}^d)$. Hence the operator $\mathcal{T}_\alpha(t, s, \tau)$ defined by

$$\mathcal{T}_\alpha(t, s, \tau)v = \mathcal{F}^{-1} \left(\frac{E_\alpha((t-\tau)^\alpha P(\cdot))}{E_\alpha((s-\tau)^\alpha P(\cdot))} \hat{v} \right) \quad \text{in } L^2(\mathbb{R}^d), \quad (5.17)$$

belongs to $\mathcal{L}(L^2(\mathbb{R}^d))$. Moreover, for some constant C , one has

$$\|\mathcal{T}_\alpha(t, s, \tau)\|_{\mathcal{L}(L^2(\mathbb{R}^d))} \leq C \max \left(\left(\frac{s-\tau}{t-\tau}\right)^\alpha, 1, E_\alpha((t-\tau)^\alpha M_P) \right), \quad \forall t > \tau, s \geq \tau \quad (5.18)$$

$$\|\mathcal{T}_\alpha(t, s, s)\|_{\mathcal{L}(L^2(\mathbb{R}^d))} \leq E_\alpha((t-s)^\alpha M_P), \quad \forall t \geq s. \quad (5.19)$$

For $\alpha \in (0, 1)$, let us assume that P satisfies (5.10) and (5.11). Then, with (5.12), we may prove that there exists a constant $C > 0$ such that for each v in $D(P)$ and each (t, s, τ) in $\overline{\mathcal{D}}_3$, we have

$$\left| \frac{E_\alpha((t-\tau)^\alpha P(\xi))}{E_\alpha((s-\tau)^\alpha P(\xi))} \hat{v}(\xi) \right| \leq C E_\alpha((t-\tau)^\alpha M_P) (s-\tau)^\alpha |P(\xi) \hat{v}(\xi)|, \quad \forall \xi \in \mathbb{R}^d.$$

Thus setting

$$\mathcal{T}(t, s, \tau)v = \mathcal{F}^{-1} \left(\frac{E_\alpha((t-\tau)^\alpha P(\cdot))}{E_\alpha((s-\tau)^\alpha P(\cdot))} \hat{v} \right), \quad \forall v \in D(P), \quad (5.20)$$

we define a linear and continuous operator $\mathcal{T}(t, s, \tau)$ from $D(P)$ into $L^2(\mathbb{R}^d)$. Moreover,

$$\|\mathcal{T}(t, s, \tau)\|_{\mathcal{L}(D(P), L^2(\mathbb{R}^d))} \leq C E_\alpha((t-\tau)^\alpha M_P) (s-\tau)^\alpha, \quad (5.21)$$

where C is independent of t, s and τ .

Of course,

$$\mathcal{T}(t, s, \tau)v = \mathcal{T}_\alpha(t, s, \tau)v,$$

for each (t, s, τ) in \mathcal{D}_3 and v in $D(P)$; we refer to Corollary 5.4 for the definition of \mathcal{T}_α . Hence

$$\mathcal{T} : \overline{\mathcal{D}}_3 \times D(P) \rightarrow L^2(\mathbb{R}^d), \quad \mathcal{T}_\alpha : \mathcal{D}_3 \times L^2(\mathbb{R}^d) \rightarrow L^2(\mathbb{R}^d)$$

and

$$\mathcal{T} = \mathcal{T}_\alpha \quad \text{on } \mathcal{D}_3 \times D(P).$$

Let us also precise that $\overline{\mathcal{D}}_3$ is the closure of \mathcal{D}_3 , that is

$$\overline{\mathcal{D}}_3 = \{(t, s, \tau) \in \mathbb{R}^3 \mid t \geq \tau, s \geq \tau\}.$$

The next result states that the operator \mathcal{T}_α has no continuous extension on $\overline{\mathcal{D}}_3$ when $P(D)$ is unbounded.

Proposition 5.5. *Let us assume that P satisfies (5.10), (5.11) and is unbounded from below. Then, for $\alpha \in (0, 1)$ and $s > \tau$,*

$$\|\mathcal{T}_\alpha(t, s, \tau)\|_{\mathcal{L}(L^2(\mathbb{R}^d))} \rightarrow \infty \quad \text{as } t \rightarrow \tau^+. \quad (5.22)$$

Proof. By translation invariance, it is enough to prove (5.22) for $s > 0$ and $\tau = 0$. For any number M greater than $|M_P|$ and $\frac{R_1}{s^\alpha}$, where R_1 is the positive constant involved in (5.12), let $\xi_0 \in \mathbb{R}^d$ be such that $P(\xi_0) = -M$ (recall that P is continuous and unbounded from below). Let also $t_0 > 0$ be such that $t_0^\alpha M = 1$. Then for each $t \in [0, t_0]$, one has

$$\frac{E_\alpha(t^\alpha P(\xi_0))}{E_\alpha(s^\alpha P(\xi_0))} \geq \frac{E_\alpha(-1)}{E_\alpha(-s^\alpha M)}.$$

Since $s^\alpha M > R_1$, (5.12) implies that

$$\frac{E_\alpha(t^\alpha P(\xi_0))}{E_\alpha(s^\alpha P(\xi_0))} \geq \frac{1}{\sqrt{2}} \Gamma(1-\alpha) E_\alpha(-1) s^\alpha M.$$

Thus, since $s > 0$,

$$\sup_{\xi \in \mathbb{R}^d} \frac{E_\alpha(t^\alpha P(\xi))}{E_\alpha(s^\alpha P(\xi))} \rightarrow \infty \quad \text{as } t \rightarrow 0^+.$$

Then (5.22) follows by *Fourier multiplier Theory* (see Lemma 2.17). \square

The following lemma will be useful in the definition of *weak solution* to Problem (5.1).

Lemma 5.6. *Under assumptions (5.10) and (5.11), let $s, \tau \in \mathbb{R}$ be such that $\tau \leq s$. Then the operator $\mathcal{T}(\tau, s, \tau) : D(P) \subseteq L^2(\mathbb{R}^d) \rightarrow L^2(\mathbb{R}^d)$ is self-adjoint.*

Proof. In view of the representation (5.20), it is enough to consider the operator $\mathcal{T}(0, s, 0)$ for $s > 0$. Let us first show that the domain of the adjoint operator $\mathcal{T}(0, s, 0)^*$ is equal to $D(P)$. For, let φ be in $D(\mathcal{T}(0, s, 0)^*)$. Then by Definition of $D(\mathcal{T}(0, s, 0)^*)$, there exists a constant C such that for each $v \in D(P)$, one has

$$|\langle \mathcal{T}(0, s, 0)v, \varphi \rangle_2| \leq C \|v\|_{L^2(\mathbb{R}^d)}. \quad (5.23)$$

In view of (5.20) and Plancherel's identity (2.9), we get

$$\langle \mathcal{T}(0, s, 0)v, \varphi \rangle_2 = (2\pi)^{-d} \int_{\mathbb{R}^d} \frac{\hat{v}(\xi)}{E_\alpha(s^\alpha P(\xi))} \overline{\hat{\varphi}(\xi)} d\xi. \quad (5.24)$$

Since P is real valued, we deduce from these two later relations that

$$\frac{\hat{\varphi}}{E_\alpha(s^\alpha P(\cdot))} \in L^2(\mathbb{R}^d). \quad (5.25)$$

Thus with (5.11), (5.12) and the constant R_1 involved in (5.12), we have

$$\begin{aligned} \int_{\mathbb{R}^d} |s^\alpha P(\xi) \hat{\varphi}(\xi)|^2 d\xi &\leq \int_{[-R_1 \leq s^\alpha P(\xi)]} \max(R_1, s^\alpha M_P)^2 |\hat{\varphi}(\xi)|^2 d\xi \\ &\quad + C \int_{[s^\alpha P(\xi) < -R_1]} \left| \frac{\hat{\varphi}(\xi)}{E_\alpha(s^\alpha P(\xi))} \right|^2 d\xi \\ &\leq C(s) \|\hat{\varphi}\|_{L^2(\mathbb{R}^d)}^2 + C \left\| \frac{\hat{\varphi}}{E_\alpha(s^\alpha P)} \right\|_{L^2(\mathbb{R}^d)}^2. \end{aligned}$$

There results that $\varphi \in D(P)$ since $s > 0$.

Conversely, let $\varphi \in D(P)$. Then, by the monotonicity of E_α ,

$$\begin{aligned} \int_{\mathbb{R}^d} \left| \frac{\hat{\varphi}(\xi)}{E_\alpha(s^\alpha P(\xi))} \right|^2 d\xi &\leq \int_{[-R_1 \leq s^\alpha P(\xi)]} \frac{|\hat{\varphi}(\xi)|^2}{E_\alpha(-R_1)^2} d\xi + C \int_{[s^\alpha P(\xi) < -R_1]} |s^\alpha P(\xi) \hat{\varphi}(\xi)|^2 d\xi \\ &\leq \frac{1}{E_\alpha(-R_1)^2} \|\hat{\varphi}\|_{L^2(\mathbb{R}^d)}^2 + C s^{2\alpha} \|P \hat{\varphi}\|_{L^2(\mathbb{R}^d)}^2. \end{aligned}$$

Thus (5.25) holds, and then (5.23) too, which yields that φ belongs to $D(\mathcal{T}(0, s, 0)^*)$. We have proved that $D(\mathcal{T}(0, s, 0)^*) = D(P)$.

Finally, since P is real valued, (5.24) yields

$$\langle \mathcal{T}(0, s, 0)v, \varphi \rangle_2 = \langle v, \mathcal{T}(0, s, 0)\varphi \rangle_2, \quad \forall v, \varphi \in D(P).$$

Then the proof is complete. \square

6. REGULARIZING EFFECT FOR SOLUTIONS

6.1. The case where $\tau = s$. The problem under consideration becomes

$$\mathbf{D}_{s,t}^\alpha u = P(D)u, \quad u(s) = v \in L^2(\mathbb{R}^d), \quad (6.1)$$

where $P : \mathbb{R}^d \rightarrow \mathbb{C}$ is the symbol of some pseudo-differential operator. In order to get a regularizing effect, we have to strengthen our assumptions on P (see Proposition

4.5). So we will replace (4.1) by (3.12). That will allow us to get that $u(t)$ belongs to $D(P)$ for $t > 0$ even if $u(0)$ does not belong to $D(P)$.

Theorem 6.1. *Let $\alpha \in (0, 1]$, $s \in \mathbb{R}$, $v \in L^2(\mathbb{R}^d)$ and assume that P is continuous on \mathbb{R}^d and satisfies (3.12). Then (6.1) has a unique solution u on $[s, \infty)$, in the sense of Definition 2.12. Moreover, u lies in $C((0, \infty), D(P))$ and admits the following representation for $t \geq s$:*

$$u(t) = \mathcal{F}^{-1} \left(E_\alpha((t-s)^\alpha P(\cdot)) \hat{v}(\cdot) \right). \quad (6.2)$$

Proof. By Proposition 2.13, we may assume w.l.o.g. that $s = 0$. Existence of solution is proved via a continuity argument. More precisely, let $(v_n)_{n \geq 0} \subset D(P)$ be a sequence converging toward v in $L^2(\mathbb{R}^d)$. Recalling the definition of T_α given in Proposition 5.1, we set

$$u := T_\alpha(\cdot, 0)v, \quad u_n := T_\alpha(\cdot, 0)v_n, \quad \text{on } [0, \infty).$$

By (5.5),

$$P(D)u_n \xrightarrow[n \rightarrow \infty]{} P(D)u, \quad \text{in } L^1(0, T, L^2(\mathbb{R}^d)). \quad (6.3)$$

By [ER17, Theorem 6.2], we know that u_n is the strong solution to

$$\mathbf{D}_{0,t}^\alpha u_n = P(D)u_n, \quad u_n(0) = v_n \in D(P). \quad (6.4)$$

Hence u_n lies in $C([0, T], L^2(\mathbb{R}^d))$. Besides, Lebesgue's Theorem and (5.4) yield that u belongs to $C([0, T], L^2(\mathbb{R}^d))$ and

$$u_n \rightarrow u, \quad \text{in } C([0, T], L^2(\mathbb{R}^d)).$$

Then

$$g_{1-\alpha} * (u_n - u_n(0)) \rightarrow g_{1-\alpha} * (u - v), \quad \text{in } L^1(0, T, L^2(\mathbb{R}^d)).$$

Moreover, by (6.4) and (6.3),

$$\frac{d}{dt} \{g_{1-\alpha} * (u_n - u_n(0))\} = P(D)u_n \rightarrow P(D)u, \quad \text{in } L^1(0, T, L^2(\mathbb{R}^d)).$$

Since the operator $\frac{d}{dt}$ with domain $W^{1,1}(0, T, L^2(\mathbb{R}^d))$ is closed in $L^1(0, T, L^2(\mathbb{R}^d))$, the two latter limits yield that $g_{1-\alpha} * (u - v)$ belongs to $W^{1,1}(0, T, L^2(\mathbb{R}^d))$ and

$$\mathbf{D}_{s,t}^\alpha u = P(D)u, \quad \text{in } L^1(0, T, L^2(\mathbb{R}^d)).$$

There results that u is solution to (6.1) in the sense of Definition 2.12.

Since $T_\alpha(\cdot, 0)$ belongs to $C((0, \infty), \mathcal{L}(L^2(\mathbb{R}^d), D(P)))$ by Proposition 5.1, we derive easily that u lies in $C((0, \infty), D(P))$.

Finally, in view of Corollary 2.7, each solution u to (6.1) satisfies

$$u = v + g_\alpha * P(D)u, \quad \text{in } L_{\text{loc}}^1([0, \infty), L^2(\mathbb{R}^d)).$$

Hence uniqueness is proved as in the case where the initial condition belongs to $D(P)$, so we refer to the uniqueness part of the proof of Theorem 6.1 in [ER17] for details. \square

Example 6.2. Let $\alpha \in (0, 1]$ and $p_0 \in \mathbb{C}$. Under the notation of Example 2.16, we set

$$P(\xi) := P_{2\beta}(\xi) + p_0 = -|\xi|^{2\beta} + p_0, \quad \forall \xi \in \mathbb{R}^d.$$

We consider the following *fractional heat equation*

$$\mathbf{D}_{0,t}^\alpha u = P(D)u = P_{2\beta}(D)u + p_0 u = -(-\Delta)^\beta u + p_0 u. \quad (6.5)$$

Then P satisfies the assumption (3.12). More precisely, it is geometrically clear that for each $\alpha' \in (1, 2)$, there exists $R_0 = R_0(\alpha')$ such that

$$|P(\xi)| > R_0 \implies |\arg P(\xi)| \geq \frac{\pi}{2}\alpha'.$$

Also, if p_0 is a non positive real number then $R_0 = 0$.

Thus Theorem 6.1 yields that Equation (6.5) supplemented with the initial condition $u(0) = v \in L^2(\mathbb{R}^d)$ has a unique solution u . Moreover, u belongs to $C((0, \infty), H^{2\beta}(\mathbb{R}^d))$ and

$$\mathcal{F}(u(t))(\xi) = E_\alpha(t^\alpha(-|\xi|^{2\beta} + p_0))\hat{v}(\xi), \quad \forall t \geq 0, \quad \text{a.e. } \xi \in \mathbb{R}^d.$$

If the initial condition is more regular then the corresponding solution gains regularity. For simplicity we will state this property in the case $s = 0$, that is we will consider the problem

$$\mathbf{D}_{0,t}^\alpha u = P(D)u, \quad u(0) = v. \quad (6.6)$$

Theorem 6.3. *Let $\alpha \in (0, 1]$, and $P, Q : \mathbb{R}^d \rightarrow \mathbb{C}$ be continuous on \mathbb{R}^d . Besides, assume that P fulfills (3.12). Then for each v in $D(Q)$, the solution u to (6.6) belongs to*

$$C((0, \infty), D(PQ)) \cap C([0, \infty), D(Q))$$

and, for each $T > 0$, there exists a constant C_T independent of v such that for all $t \in (0, T]$, one has

$$\|PQ(D)u(t)\|_{L^2(\mathbb{R}^d)} \leq \frac{C_T}{t^\alpha} \|Qv\|_{L^2(\mathbb{R}^d)}. \quad (6.7)$$

Moreover, if $R_0 = 0$ in (3.12), that is

$$|\arg P(\xi)| \geq \frac{\pi}{2}\alpha'$$

for each $\xi \in \mathbb{R}^d$ with $P(\xi) \neq 0$, then there exists a constant C independent of v such that

$$\|PQ(D)u(t)\|_{L^2(\mathbb{R}^d)} \leq \frac{C}{t^\alpha} \|Qv\|_{L^2(\mathbb{R}^d)}, \quad \forall t > 0. \quad (6.8)$$

Proof. By Theorem 6.1, we know that (6.6) has a unique solution u . In particular, u lies in $C([0, \infty), L^2(\mathbb{R}^d))$. Let us show that for each $T > 1$,

$$PQ(D)u \in C((\frac{1}{T}, T), \mathcal{L}(L^2(\mathbb{R}^d))), \quad \forall v \in D(Q). \quad (6.9)$$

For, by (3.13), we have

$$\begin{aligned} |PQ(\xi)E_\alpha(t^\alpha P(\xi))\hat{v}(\xi)| &\leq \frac{C_T}{t^\alpha} |Q(\xi)\hat{v}(\xi)| \\ &\leq C_T T^\alpha |Q(\xi)\hat{v}(\xi)|, \quad \forall t \in (\frac{1}{T}, T), \quad \text{a.e. } \xi \in \mathbb{R}^d. \end{aligned} \quad (6.10)$$

Thus we deduce with Lebesgue's Theorem, that (6.9) holds. Hence u belongs to $C((0, \infty), D(PQ))$. In the same way, we prove that u lies in $C([0, \infty), D(Q))$.

Estimate (6.7) is a consequence of (6.10). (6.8) is obtained as (6.7) by using (3.14) instead of (3.13). That completes the proof of the theorem. \square

Example 6.4. Under the notation of Example 6.2, any solution u to (6.5) on $[0, \infty)$ satisfies for each $T > 0$,

$$\|P(D)u(t)\|_{L^2(\mathbb{R}^d)} \leq \frac{C_T}{t^\alpha} \|u(0)\|_{L^2(\mathbb{R}^d)}, \quad \forall t \in (0, T].$$

Hence Equation (6.5) has a regularizing effect. Moreover, if p_0 is a non positive real number then $R_0 = 0$ according to Example 6.2; hence the following dissipative estimate holds.

$$\|P(D)u(t)\|_{L^2(\mathbb{R}^d)} \leq \frac{C}{t^\alpha} \|u(0)\|_{L^2(\mathbb{R}^d)}, \quad \forall t > 0.$$

That results from Theorem 6.3.

Example 6.5. Let us consider the *fractional transport equation*

$$\mathbf{D}_{0,t}^\alpha u = iae^{i\theta} Du = ae^{i\theta} \partial_x u, \quad u(0) = v \in L^2(\mathbb{R}^d).$$

Under the notation of Example 4.3, let $\alpha \in (0, 1)$ and $\alpha' \in (\alpha, 1]$. Regarding the angle θ , we assume that

$$|\theta| \leq \frac{\pi}{2}(1 - \alpha').$$

Then the symbol $P(\xi) := iae^{i\theta}\xi$ satisfies (3.12) with $R_0 = 0$. Thus Theorem 6.3 yields that

$$\|\partial_x u(t)\|_{L^2(\mathbb{R}^d)} \leq \frac{C}{t^\alpha} \|v\|_{L^2(\mathbb{R}^d)}, \quad \forall t > 0.$$

Thus the above fractional transport equation has a regularizing effect and fullfils a dissipative estimate for $\alpha \in (0, 1)$. In particular, by choosing $\alpha' = 1$, we deduce that the equation

$$\mathbf{D}_{0,t}^\alpha u = a\partial_x u$$

is dissipative and has regularizing effect. That contrasts with the case $\alpha = 1$, since it is well known that the standard transport equation has no regularizing effect.

The following result states that if the order of derivative is small enough, then the partial derivatives of the solution satisfy some dissipative estimates.

Proposition 6.6. *Let us assume that*

- (i) $\alpha \in (0, 1]$; P is continuous on \mathbb{R}^d and satisfies (3.12) with $R_0 = 0$;
- (ii) there exist $m \in \mathbb{N}^*$ and $c > 0$ such that

$$c|\xi|^m \leq |P(\xi)|, \quad \forall \xi \in \mathbb{R}^d; \tag{6.11}$$

- (iii) the initial data v belongs to $L^2(\mathbb{R}^d)$ and u denotes the solution to (6.6);
- (iv) the multi-integer $\beta = (\beta_1, \dots, \beta_d) \in \mathbb{N}^d$ has his length denoted by $|\beta|$.

Then, if $|\beta| \leq m$, one has

$$\|D^\beta u(t)\|_{L^2(\mathbb{R}^d)} \leq \frac{C}{t^{\alpha \frac{|\beta|}{m}}} \|v\|_{L^2(\mathbb{R}^d)}. \tag{6.12}$$

Moreover, if $v \in D(P)$ and $m \leq |\beta| \leq 2m$ then

$$\|D^\beta u(t)\|_{L^2(\mathbb{R}^d)} \leq \frac{C}{t^{\alpha(\frac{|\beta|}{m}-1)}} \|P(D)v\|_{L^2(\mathbb{R}^d)}, \quad \forall t > 0. \tag{6.13}$$

Proof. Let us prove (6.12). For a.e. $\xi \in \mathbb{R}^d$,

$$\begin{aligned} |\xi^\beta \hat{u}(t)| &\leq |\xi|^{|\beta|} |E_\alpha(t^\alpha P(\xi))| |\hat{v}(\xi)| \\ &\leq C \frac{|P(\xi)|^{\frac{|\beta|}{m}}}{1 + t^\alpha |P(\xi)|} |\hat{v}(\xi)| && \text{(by (6.11) and (3.14))} \\ &\leq C \frac{(t^\alpha |P(\xi)|)^{\frac{|\beta|}{m}}}{1 + t^\alpha |P(\xi)|} t^{-\alpha \frac{|\beta|}{m}} |\hat{v}(\xi)| \\ &\leq C t^{-\alpha \frac{|\beta|}{m}} |\hat{v}(\xi)| && \text{(since } |\beta| \leq m). \end{aligned} \tag{6.14}$$

Thus (6.12) follows. In order to prove (6.13), we put $m_1 := |\beta| - m$. Then in view of (6.14), we have

$$\begin{aligned} |\xi^\beta \hat{u}(t)| &\leq C \frac{|P(\xi)|^{\frac{m_1}{m}}}{1 + t^\alpha |P(\xi)|} |P(\xi) \hat{v}(\xi)| \\ &\leq C t^{-\alpha \frac{m_1}{m}} |P(\xi) \hat{v}(\xi)| && \text{(since } m_1 \leq m). \end{aligned}$$

Hence (6.13) follows. \square

Unlike to the standard case $\alpha = 1$, purely fractional equations lack of regularizing effect for high order derivatives even if (3.12) holds.

Proposition 6.7. *Let us assume that*

- (i) $\alpha \in (0, 1)$;
- (ii) P is continuous on \mathbb{R}^d and satisfies (3.12);
- (iii) there exists a sequence $(\xi_n)_{n \geq 1} \subset \mathbb{R}^d$ and $\beta \in \mathbb{N}^d$ such that

$$|P(\xi_n)| \rightarrow \infty, \quad \frac{|\xi_n^\beta|}{|P(\xi_n)|} \rightarrow \infty \quad \text{as } n \rightarrow \infty.$$

Then for each time $t \geq 0$, there is no finite constant $C = C(t)$ satisfying

$$\|D^\beta u(t)\|_{L^2(\mathbb{R}^d)} \leq C \|v\|_{L^2(\mathbb{R}^d)}, \quad \forall v \in \mathcal{F}^{-1}(\mathcal{D}(\mathbb{R}^d)), \tag{6.15}$$

where u is the solution to (6.6) with initial condition v .

Notice that the left hand side of (6.15) is well defined since, by applying Theorem 6.3 with $Q(\xi) = \xi^\beta$, we get that $D^\beta u(\cdot)$ belongs to $C([0, \infty), L^2(\mathbb{R}^d))$, since $v \in \mathcal{F}^{-1}(\mathcal{D}(\mathbb{R}^d)) \subset D(Q)$.

Proof. It relies on the proof of Proposition 4.5. For each $t \geq 0$, let us set

$$a_t(\xi) := \xi^\beta E_\alpha(t^\alpha P(\xi)), \quad \forall \xi \in \mathbb{R}^d.$$

Then $a_t(\cdot)$ is continuous on \mathbb{R}^d for each $t \geq 0$. Thus, by Lemma 2.17, it is enough to prove that $a_t(\cdot)$ is unbounded on \mathbb{R}^d .

If $t = 0$ then $a_t(\cdot)$ is clearly unbounded since $\beta \neq 0$. Thus (6.15) can not hold.

If $t > 0$ then the sequence $(\xi_n)_{n \geq 0}$ satisfies by hypothesis

$$|P(\xi_n)| \rightarrow \infty, \quad \frac{|\xi_n^\beta|}{|P(\xi_n)|} \rightarrow \infty \quad \text{as } n \rightarrow \infty.$$

Thus recalling that P fullfils (3.12), we may use (3.3), to derive

$$a_t(\xi_n) \sim \frac{C \xi_n^\beta}{t^\alpha P(\xi_n)} \rightarrow \infty, \quad \text{as } n \rightarrow \infty.$$

Then $a_t(\cdot)$ is unbounded. Hence (6.15) can not hold, which completes the proof of the proposition. \square

Example 6.8. Let us consider the following *time fractional heat equation* on \mathbb{R}^d ,

$$\mathbf{D}_{0,t}^\alpha u = \Delta u, \quad u(0) = v \in L^2(\mathbb{R}^d).$$

This equation has a regularizing effect up to order two when $\alpha \in (0, 1)$. Indeed, Proposition 6.6 yields that, for all indices i, j in $[1, d]$, one has

$$\begin{aligned} \|\partial_{x_i} u(t)\|_{L^2(\mathbb{R}^d)} &\leq \frac{C}{t^{\alpha/2}} \|v\|_{L^2(\mathbb{R}^d)} \\ \|\partial_{x_i x_j} u(t)\|_{L^2(\mathbb{R}^d)} &\leq \frac{C}{t^\alpha} \|v\|_{L^2(\mathbb{R}^d)}, \quad \forall t > 0. \end{aligned}$$

However, there is no regularizing effect at order three. Indeed, for all indices i, j, k in $[1, d]$, Proposition 6.7 with $\xi_n := n(1, \dots, 1)$, yields that

$$\sup_{v \in \mathcal{F}^{-1}(\mathcal{D}(\mathbb{R}^d)), v \neq 0} \frac{\|\partial_{x_i x_j x_k} u(t)\|_{L^2(\mathbb{R}^d)}}{\|v\|_{L^2(\mathbb{R}^d)}} = \infty.$$

Remark 6.9. The lack of regularizing effect featured in Proposition 6.7 is specific to time fractional (non integer) derivatives. Indeed, it is well known that the standard *heat equation* has a regularizing effect at any order and that the following dissipative estimate holds.

$$\|D^\beta u(t)\|_{L^2(\mathbb{R}^d)} \leq \frac{C_\beta}{t^{|\beta|/2}} \|u(0)\|_{L^2(\mathbb{R}^d)},$$

for all multi-integer β .

6.2. The case where $\tau \leq s$. The problem under consideration becomes

$$\mathbf{D}_{\tau,t}^\alpha u = P(D)u, \quad u(s) = v \in L^2(\mathbb{R}^d). \quad (6.16)$$

If v belongs to $D(P)$ then we may prove that $\mathcal{T}(\cdot, s, \tau)v$ is the solution to (6.16) on $[\tau, \infty)$ in the sense of Definition 2.12 (see the proof of [ER17, Theorem 6.1] and use Lemma 5.3). The issue is to solve (6.16) when v belongs more generally to $L^2(\mathbb{R}^d)$. Of course the expected solution is $\mathcal{T}_\alpha(\cdot, s, \tau)v$ (defined in Corollary 5.4).

Because of the behaviour of the Fourier multiplier (5.9) at high frequencies, solutions to (6.16) have to be understood in a weak sense. In this respect, we adapt to our framework Prüss' Definition of weak solutions (see [Prü93, Definition 1.1 Chap. 1]).

Definition 6.10. Let $\alpha \in (0, 1)$, $s, \tau \in \mathbb{R}$ with $\tau \leq s$ and P be a symbol satisfying (5.10), (5.11). Let also $v \in L^2(\mathbb{R}^d)$ and u be in $L_{\text{loc}}^1([\tau, \infty), L^2(\mathbb{R}^d))$. We say that the function u is a *weak solution* to (6.16) on $[\tau, \infty)$ if

$$\langle u(\cdot), \varphi \rangle_2 = \langle v, \mathcal{T}(\tau, s, \tau)\varphi \rangle_2 + \langle g_\alpha *_\tau u, P(D)\varphi \rangle_2, \quad (6.17)$$

in $L_{\text{loc}}^1([\tau, \infty))$, for all $\varphi \in D(P)$.

Remark 6.11. In (6.17), $\mathcal{T}(\tau, s, \tau)$ is defined on $D(P)$ by (5.20), and in view of (5.21), $\mathcal{T}(\tau, s, \tau)\varphi$ belongs to $L^2(\mathbb{R}^d)$ if $\varphi \in D(P)$. It should be clear that, in general, $\mathcal{T}(\tau, s, \tau)\varphi \neq \varphi$ provided $s > \tau$.

A priori, the adjoints of $P(D)$ and $\mathcal{T}(\tau, s, \tau)$ should appear in (6.17). However, these operators are self-adjoint since P is real valued (see Lemma 5.6).

With Corollary 2.7 and Lemma 5.6, we may prove easily that, for each v in $D(P)$, the solution to (6.16) in the sense of Definition 2.12, namely $\mathcal{T}(\cdot, s, \tau)v$ is a weak solution to (6.16).

Theorem 6.12. *Let $\alpha \in (0, 1)$, $P : \mathbb{R}^d \rightarrow \mathbb{R}$ satisfy (5.10) and (5.11). Let also $\tau \leq s$ and $v \in L^2(\mathbb{R}^d)$. Then (6.16) has a unique weak solution u . Moreover,*

$$u(t) = \mathcal{T}_\alpha(t, s, \tau)v = \mathcal{F}^{-1}\left(\frac{E_\alpha((t - \tau)^\alpha P(\cdot))}{E_\alpha((s - \tau)^\alpha P(\cdot))}\hat{v}\right), \quad \text{a.e } t \in (\tau, \infty), \quad (6.18)$$

and $\mathcal{T}_\alpha(\cdot, s, \tau)v$ belongs to $C((\tau, \infty), L^2(\mathbb{R}^d))$.

Proof. Let us start to prove the existence part. Let $T > \tau$ and $(v_n)_{n \geq 0} \subset D(P)$ be a sequence converging toward v in $L^2(\mathbb{R}^d)$. By [ER17, Theorem 6.1], the function u_n defined for each $t > \tau$ by

$$u_n(t) := \mathcal{T}(t, s, \tau)v_n = \mathcal{F}^{-1}\left(\frac{E_\alpha((t - \tau)^\alpha P(\cdot))}{E_\alpha((s - \tau)^\alpha P(\cdot))}\hat{v}_n\right)$$

is the solution to (6.16). Thus, by Remark 6.11, u_n is a weak solution to (6.16); that is to say

$$\langle u_n, \varphi \rangle_2 = \langle v, \mathcal{T}(\tau, s, \tau)\varphi \rangle_2 + \langle g_\alpha *_\tau u_n, P(D)\varphi \rangle_2, \quad \forall \varphi \in D(P), \quad (6.19)$$

in $L^1(\tau, T)$. By (5.18), the function u defined by (6.18) belongs to $L^1(\tau, T, L^2(\mathbb{R}^d))$ and

$$u_n \rightarrow u \quad \text{in } L^1(\tau, T, L^2(\mathbb{R}^d)). \quad (6.20)$$

Moreover,

$$\|g_\alpha *_\tau u\|_{L^1(\tau, T, L^2(\mathbb{R}^d))} \leq \|g_\alpha\|_{L^1(0, T-\tau)} \|u\|_{L^1(\tau, T, L^2(\mathbb{R}^d))}.$$

Thus we may pass to the limit in (6.19) to get that u is a weak solution.

Regarding uniqueness, by linearity it is enough to show that any function $u \in L^1(\tau, T, L^2(\mathbb{R}^d))$ satisfying

$$\langle u, \varphi \rangle_2 = \langle g_\alpha *_\tau u, P(D)\varphi \rangle_2, \quad \forall \varphi \in D(P), \quad (6.21)$$

vanishes. Since $P(D)$ is self-adjoint, we deduce from (6.21), that $g_\alpha *_\tau u$ lies in $L^1(\tau, T, D(P))$ and

$$u = P(D)(g_\alpha *_\tau u), \quad \text{in } L^1(\tau, T, L^2(\mathbb{R}^d)).$$

Taking the Fourier transform, we deduce

$$\hat{u}(t, \xi) = P(\xi)g_\alpha *_\tau \hat{u}(\cdot, \xi)(t) \quad \text{a.e } t \in [\tau, T], \quad \text{a.e } \xi \in \mathbb{R}^d. \quad (6.22)$$

Since $u \in L^1(\tau, T, L^2(\mathbb{R}^d))$, we know that $\hat{u}(\cdot, \xi)$ lies in $L^1(\tau, T)$ for each $\xi \in \mathbb{R}^d$. That fact together with (6.22) allows us to show that $u = 0$: see [ER17, Proof of Theorem 6.1] for details.

Finally, by Lebesgue's Theorem and (5.18), we prove that $\mathcal{T}_\alpha(\cdot, s, \tau)v$ belongs to $C((\tau, T], L^2(\mathbb{R}^d))$. That completes the proof of the Theorem. \square

REFERENCES

- [ABHN11] Wolfgang Arendt, Charles J. K. Batty, Matthias Hieber, and Frank Neubrander. *Vector-valued Laplace transforms and Cauchy problems*, volume 96 of *Monographs in Mathematics*. Birkhäuser/Springer Basel AG, Basel, second edition, 2011.
- [Baz98] Emilia Bazhlekova. The abstract Cauchy problem for the fractional evolution equation. *Fract. Calc. Appl. Anal.*, 1(3):255–270, 1998.
- [BGTM10] D. Baleanu, Z. B. Güvengç, and J. A. Tenreiro Machado, editors. *New trends in nanotechnology and fractional calculus applications*. Springer, New York, 2010. Selected papers from the International Workshop on New Trends in Science and Technology and the International Workshop on Fractional Differentiation and its Applications held at Ankara, 2008.
- [BTW75] Philip Brenner, Vidar Thomée, and Lars B. Wahlbin. *Besov spaces and applications to difference methods for initial value problems*. Lecture Notes in Mathematics, Vol. 434. Springer-Verlag, Berlin-New York, 1975.
- [DNPV12] Eleonora Di Nezza, Giampiero Palatucci, and Enrico Valdinoci. Hitchhiker’s guide to the fractional Sobolev spaces. *Bull. Sci. Math.*, 136(5):521–573, 2012.
- [Dos88] Raouf Doss. An elementary proof of Titchmarsh’s convolution theorem. *Proc. Amer. Math. Soc.*, 104(1):181–184, 1988.
- [ER17] Hassan Emamirad and Arnaud Rougirel. Solution operators of three time variables for fractional linear problems. *Mathematical Methods in the Applied Sciences*, 40(5):1553–1572, 2017.
- [GARGBA⁺12] J.F. Gomez-Aguilar, J.J. Rosales-Garcia, J.J. Bernal-Alvarado, T. Cordova-Fraga, and R. Guzman-Cabrera. Fractional mechanical oscillators. *Revista mexicana de fisica*, 58:348 – 352, 08 2012.
- [GKMR14] Rudolf Gorenflo, Anatoly A. Kilbas, Francesco Mainardi, and Sergei V. Rogosin. *Mittag-Leffler functions, related topics and applications*. Springer Monographs in Mathematics. Springer, Heidelberg, 2014.
- [KLW16] Valentin Keyantuo, Carlos Lizama, and Mahamadi Warma. Existence, regularity and representation of solutions of time fractional diffusion equations. *Adv. Differential Equations*, 21(9-10):837–886, 2016.
- [Kos14] Marko Kostić. Systems of abstract time-fractional equations. *Publ. Inst. Math. (Beograd) (N.S.)*, 95(109):119–132, 2014.
- [Kos15] Marko Kostić. *Abstract Volterra integro-differential equations*. CRC Press, Boca Raton, FL, 2015.
- [Las00] Nick Laskin. Fractional market dynamics. *Physica A: Statistical Mechanics and its Applications*, 287(3–4):482–492, 2000.
- [Nab04] Mark Naber. Time fractional Schrödinger equation. *J. Math. Phys.*, 45(8):3339–3352, 2004.
- [NAYH13] B. N. Narahari Achar, Bradley T. Yale, and John W. Hanneken. Time fractional Schrodinger equation revisited. *Adv. Math. Phys.*, pages Art. ID 290216, 11, 2013.
- [Pod99] Igor Podlubny. *Fractional differential equations*, volume 198 of *Mathematics in Science and Engineering*. Academic Press, Inc., San Diego, CA, 1999. An introduction to fractional derivatives, fractional differential equations, to methods of their solution and some of their applications.
- [Pol48] Harry Pollard. The completely monotonic character of the Mittag-Leffler function $E_a(-x)$. *Bull. Amer. Math. Soc.*, 54:1115–1116, 1948.
- [Prü93] Jan Prüß. *Evolutionary integral equations and applications*, volume 87 of *Monographs in Mathematics*. Birkhäuser Verlag, Basel, 1993.

HASSAN EMAMIRAD
SCHOOL OF MATHEMATICS, INSTITUTE FOR RESEARCH IN FUNDAMENTAL SCIENCES (IPM), P.O.
Box 19395-5746, TEHRAN, IRAN
LABORATOIRE DE MATHÉMATIQUES, UNIVERSITÉ DE POITIERS. TÉLÉPORT 2, BP 179, 86960
CHASSNEUIL DU POITOU CEDEX, FRANCE
E-mail address: emamirad@ipm.ir
E-mail address: emamirad@math.univ-poitiers.fr

ARNAUD ROUGIREL
LABORATOIRE DE MATHÉMATIQUES, UNIVERSITÉ DE POITIERS. TÉLÉPORT 2, BP 179, 86960
CHASSNEUIL DU POITOU CEDEX, FRANCE
E-mail address: rougirel@math.univ-poitiers.fr