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## Expectile prediction through asymmetric kriging

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## Introduction

Kriging (see Krige (1951)) aims at predicting the conditional mean of a random field  $(Z_t)_{t \in T}$  given the values  $Z_{t_1}, \dots, Z_{t_N}$  of the field at some points  $t_1, \dots, t_N \in T$ , where typically  $T \subset \mathbb{R}^d$ . It seems natural to predict, in the same spirit as Kriging, other functionals. In our study, we focus on expectiles for elliptical random fields. We also did a similar work for quantiles (see Maume-Deschamps et al. (2016b)).

## Elliptical Distributions

Cambanis et al. (1981) give the representation : the random vector  $X \in \mathbb{R}^d$  is elliptical with parameters  $\mu \in \mathbb{R}^d$  and  $\Sigma \in \mathbb{R}^{d \times d}$ , if and only if

$$X = \mu + R\Lambda U^{(d)}, \quad (1)$$

where  $\Lambda\Lambda^T = \Sigma$ ,  $U^{(d)}$  is a  $d$ -dimensional random vector uniformly distributed on  $S^{d-1}$  (the unit disk of dimension  $d$ ), and  $R$  is a non-negative random variable independent of  $U^{(d)}$ . Furthermore,  $X$  is said consistent if :

$$R \stackrel{d}{=} \frac{X^d}{\epsilon} \quad (2)$$

Distribution	$\epsilon$
Gaussian	1
Student, $\nu > 0$	$\frac{X_\nu}{\sqrt{\nu}}$
Unimodal Gaussian Mixture	$\sum_{k=1}^n \pi_k \delta_{\theta_k}$
Laplace, $\lambda > 0$	$\frac{1}{\sqrt{\mathcal{E}(\lambda)}}$
Slash	$\beta(\alpha, 1)$

Table 1: Some consistent distributions

We can now define the notion of elliptical random fields. Indeed, a random field  $(X(t))_{t \in T}$  is  $R$ -elliptical if  $\forall n \in \mathbb{N}$ ,  $\forall t_1, \dots, t_n \in T$ , the vector  $(X(t_1), \dots, X(t_n))$  is  $(R, n)$ -elliptical.

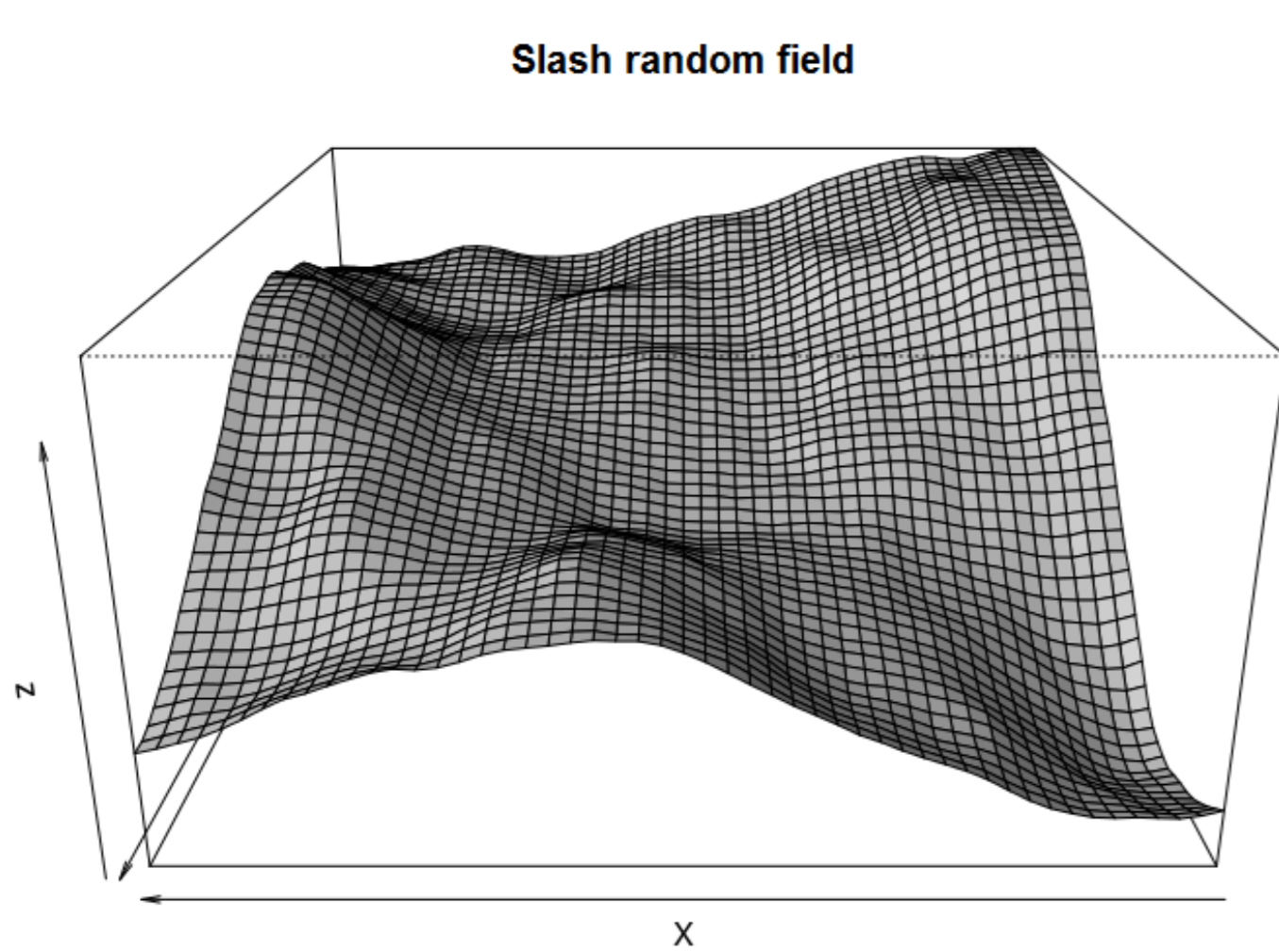


Figure 1: Slash random field with a Matérn  $\frac{3}{2}$  kernel

Furthermore, if  $X = (X_1, X_2)$  is a  $(R, d_1 + d_2)$ -elliptical random vector with parameters  $\mu = (\mu_1, \mu_2)$ , we know that  $X_2|X_1$  is  $(R^*, d_2)$ -elliptical with parameters :

$$\begin{cases} \mu_{2|1} = \mu_2 + \Sigma_{21}\Sigma_{11}^{-1}(x_1 - \mu_1) \\ \Sigma_{2|1} = \Sigma_{22} - \Sigma_{21}\Sigma_{11}^{-1}\Sigma_{12} \end{cases} \quad (3)$$

## Elliptical expectiles

If  $X$  is a random variable, the  $\alpha$ -expectile of  $X$  is given by :

$$e_\alpha(X) = \arg \min_{x \in \mathbb{R}} \mathbb{E}[\phi_\alpha(X - x)]$$

where the scoring function  $\phi_\alpha$  is

$$(1 - \alpha)x^2 \mathbf{1}_{\{x < 0\}} + \alpha x^2 \mathbf{1}_{\{x > 0\}} = \alpha x^2 + (1 - 2\alpha)x^2 \mathbf{1}_{\{x < 0\}}. \quad (4)$$

Let  $X$  be a  $(R, 1)$ -elliptical random variable, with parameters  $\mu$  and  $\sigma^2$ . We define the function  $\Psi_R$  as follows :

$$\Psi_R(x) = \int_{-\infty}^x c_1 g_1(y^2) dy + \frac{1}{x} \int_x^{+\infty} y c_1 g_1(y^2) dy$$

where  $c_1 g_1(t) = \frac{1}{2} f_R(\sqrt{t})$ ,  $f_R(t)$  being the density of  $R$ . Then, the  $\alpha$ -expectile of  $X$  is given by :

$$e_\alpha(X) = \mu + \sigma \Psi_R^{-1} \left( \frac{\alpha}{2\alpha - 1} \right) \quad (5)$$

We propose, in Maume-Deschamps et al. (2016a), several ways to compute  $\Psi_R^{-1} \left( \frac{\alpha}{2\alpha - 1} \right)$ .

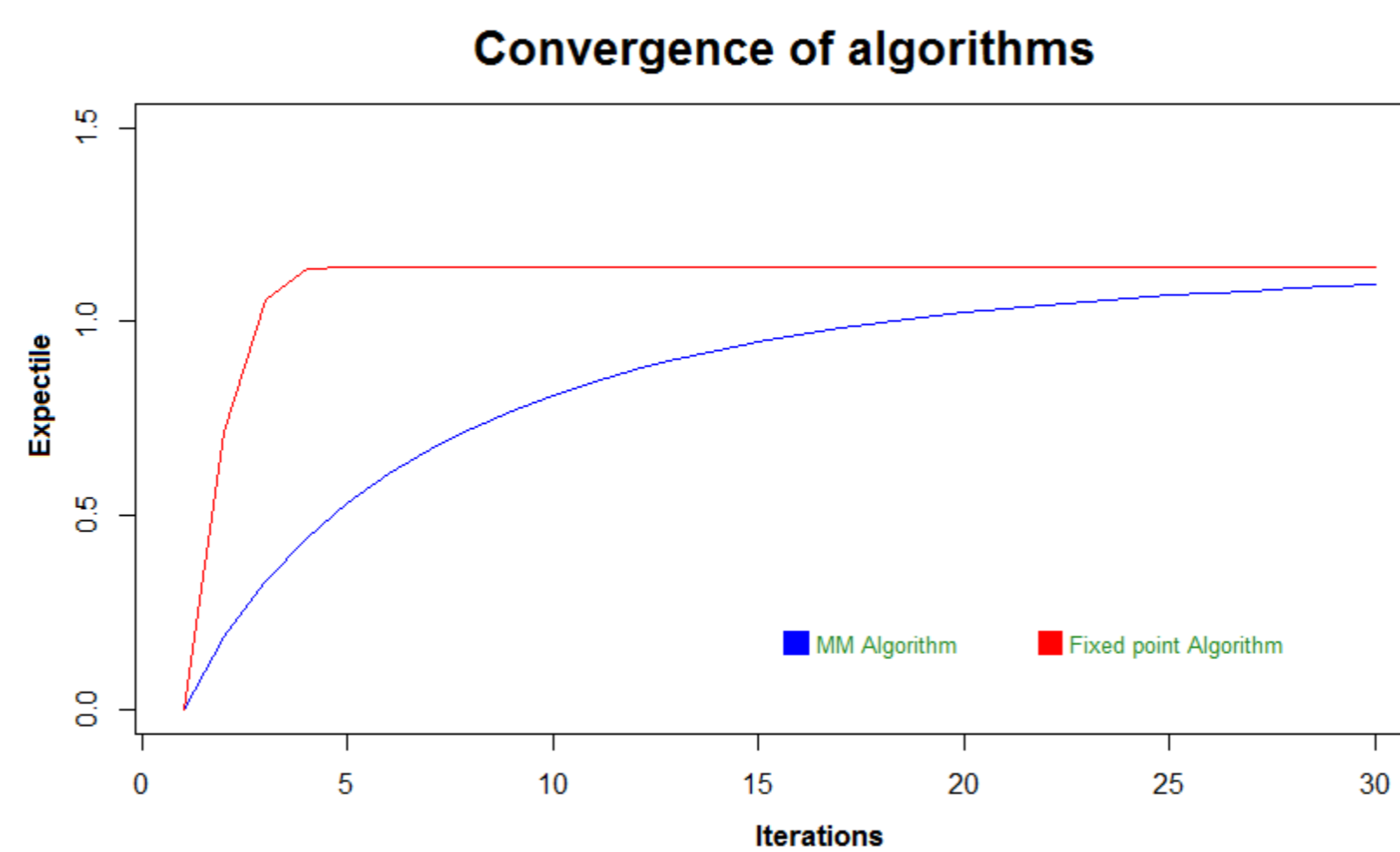


Figure 2: Computation of the 0.95 gaussian expectile

Now, we consider  $X = (X_1, X_2)$ ,  $X_1 \in \mathbb{R}^N$ ,  $X_2 \in \mathbb{R}^a$  a  $(R, N + 1)$ -elliptical random vector. According to (5), the  $\alpha$ -expectile of  $X_2|X_1$  is given by :

$$e_\alpha(X) = \mu_{2|1} + \sqrt{\Sigma_{2|1}} \Psi_{R^*}^{-1} \left( \frac{\alpha}{2\alpha - 1} \right) \quad (6)$$

In the general case, the term  $\Psi_{R^*}^{-1} \left( \frac{\alpha}{2\alpha - 1} \right)$  is difficult to compute. This is why we propose some other predictors.

## Expectile Regression

From now on, we consider the following context:  $(X(t))_{t \in T}$  is an  $R$ -elliptical random field. We consider  $N$  observations at locations  $t_1, \dots, t_N \in T$ , called  $(X(t_1), \dots, X(t_N))$ . In order to predict the value of  $X_2 = X(t)$  given  $X_1 = X(t_1), \dots, X(t_N)$ , we approximate  $X(t)$  by :

$$\hat{e}_\alpha(X_2|X_1 = x_1) = \beta^{*T} x_1 + \beta_0^*, \quad (7)$$

where  $\beta^*$  and  $\beta_0^*$  are solutions of the following minimization problem

$$(\beta^*, \beta_0^*) = \arg \min_{\beta \in \mathbb{R}^N, \beta_0 \in \mathbb{R}} \mathbb{E}[\phi_\alpha(X_2 - \beta^T X_1 - \beta_0)]. \quad (8)$$

In the case  $\alpha = \frac{1}{2}$ , the solution of (8) is exactly the kriging vector. Otherwise, this problem leads to an expectile regression, introduced by Newey and Powell (1987).

In our context of elliptical random fields, we are able to solve this minimization problem, and then define the Expectile Regression Predictor :

$$\hat{e}_\alpha(X_2|X_1 = x_1) = \mu_{2|1} + \sqrt{\Sigma_{2|1}} \Psi_{R^*}^{-1} \left( \frac{\alpha}{2\alpha - 1} \right) \quad (9)$$

Furthermore, its distribution is

$$\hat{e}_\alpha(X_2|X_1) \sim \mathcal{E}_1 \left( \mu_{2|1} + \sqrt{\Sigma_{2|1}} \Psi_{R^*}^{-1} \left( \frac{\alpha}{2\alpha - 1} \right), \Sigma_{2|1} \Sigma_{11}^{-1} \Sigma_{12}, R \right) \quad (10)$$

## Extremal expectiles

In this section, the aim is to establish a relation between  $\Psi_R^{-1}$  and  $\Psi_{R^*}^{-1}$  for extremal values of  $\alpha$ . For that, we do an assumption : their exist  $0 < \ell < +\infty$  and  $\gamma \in \mathbb{R}$  such as :

$$\lim_{x \rightarrow +\infty} \frac{1 - \Psi_{R^*}(x)}{1 - \Psi_R(x^\gamma)} = \ell \quad (11)$$

Under this assumption, we can define Extreme Conditional Expectiles Predictors :

$$\begin{cases} \hat{e}_{\alpha \uparrow}(X_2|X_1 = x_1) = \mu_{2|1} + \sigma_{2|1} \left[ \Psi_{R^*}^{-1} \left( 1 - \frac{\alpha - 1}{(2\alpha - 1)\ell} \right) \right]^\frac{1}{\gamma} \\ \hat{e}_{\alpha \downarrow}(X_2|X_1 = x_1) = \mu_{2|1} - \sigma_{2|1} \left[ \Psi_{R^*}^{-1} \left( 1 - \frac{\alpha - 1}{(2\alpha - 1)\ell} \right) \right]^\frac{1}{\gamma} \end{cases} \quad (12)$$

Distribution	$\gamma$	$\ell$
Gaussian	1	1
Student, $\nu > 0$	$\frac{N}{\nu} + 1$	$\frac{\Gamma(\frac{\nu+N+1}{2})\Gamma(\frac{\nu}{2})}{\Gamma(\frac{\nu+N}{2})\Gamma(\frac{\nu+1}{2})} \left( 1 + \frac{q_1}{\nu} \right)^{\frac{N+\nu}{2}} \frac{\nu^{N+1}}{\nu+N} \frac{\nu-1}{\nu+N-1}$
Unimodal GM	1	$\frac{\min(\theta_1, \dots, \theta_n)^N \exp\left(-\frac{\min(\theta_1, \dots, \theta_n)^2}{2} q_1\right)}{\sum_{k=1}^n \pi_k \theta_k^N \exp\left(-\frac{\theta_k^2}{2} q_1\right)}$
Slash, $\alpha > 0$	$\frac{N}{\alpha} + 1$	$\frac{2^{1-\alpha} (a-1) \Gamma\left(\frac{N+1+a}{2}\right) q_1^{\frac{N+a}{2}}}{(N+a)(N+a-1) \Gamma\left(\frac{N+a}{2}\right) \Gamma\left(\frac{1+a}{2}\right) \chi_{N+a}^2(q_1)}$

Table 2: Some examples

Thanks to the paper of Djurčić and Torgašev (2001), we are able to prove that these predictors  $\hat{e}_{\alpha \uparrow}$  and  $\hat{e}_{\alpha \downarrow}$  are asymptotically equivalent to the theoretical expectiles respectively when  $\alpha \rightarrow 1$  and  $\alpha \rightarrow 0$ .

$$\begin{cases} \hat{e}_{\alpha \uparrow}(X_2|X_1 = x_1) \underset{\alpha \rightarrow 1}{\sim} e_\alpha(X_2|X_1 = x_1) \\ \hat{e}_{\alpha \downarrow}(X_2|X_1 = x_1) \underset{\alpha \rightarrow 0}{\sim} e_\alpha(X_2|X_1 = x_1) \end{cases} \quad (13)$$

## Numerical study

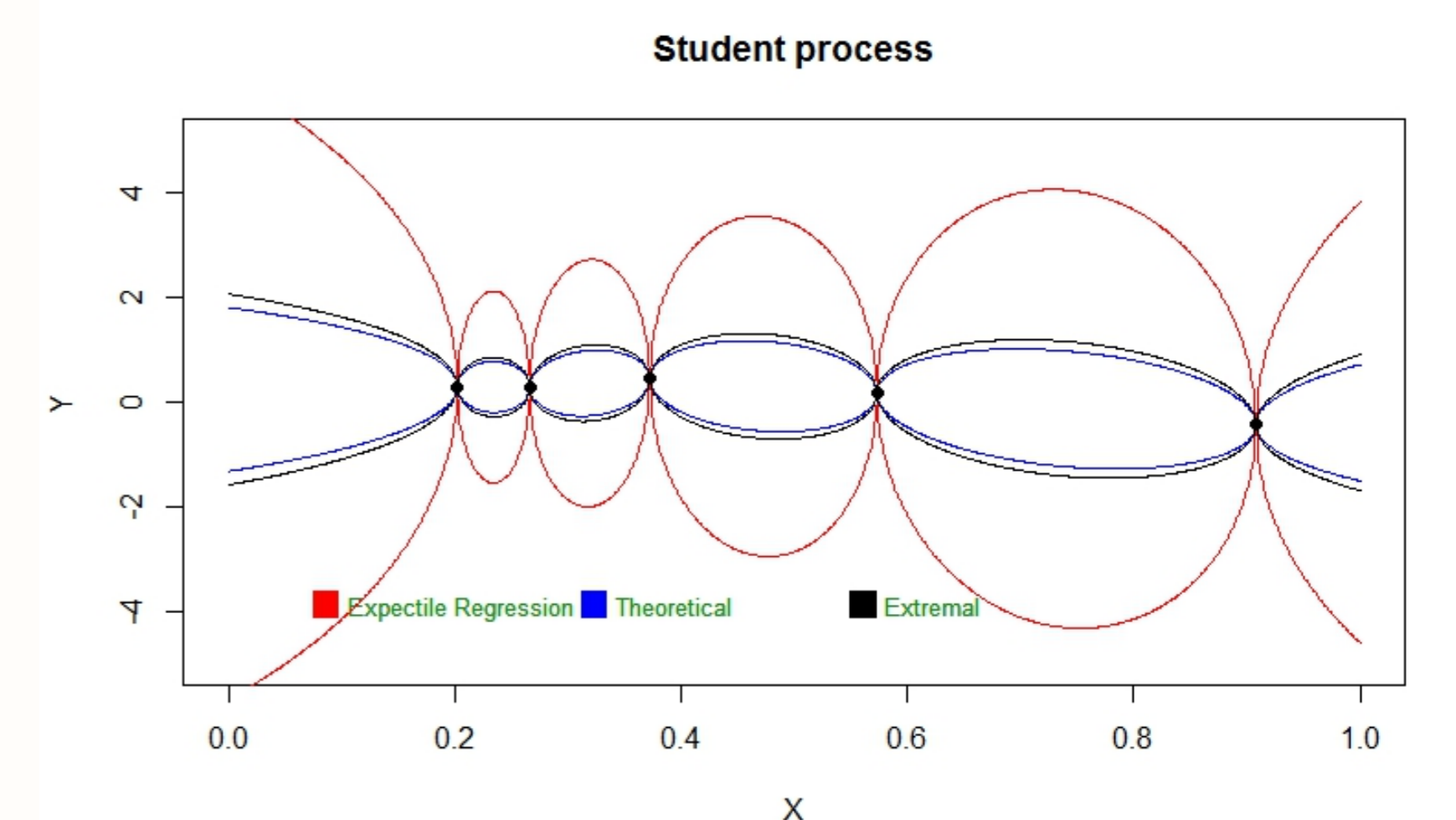


Figure 3: Levels of quantile  $\alpha = 0.9995$  and  $\alpha = 0.0005$

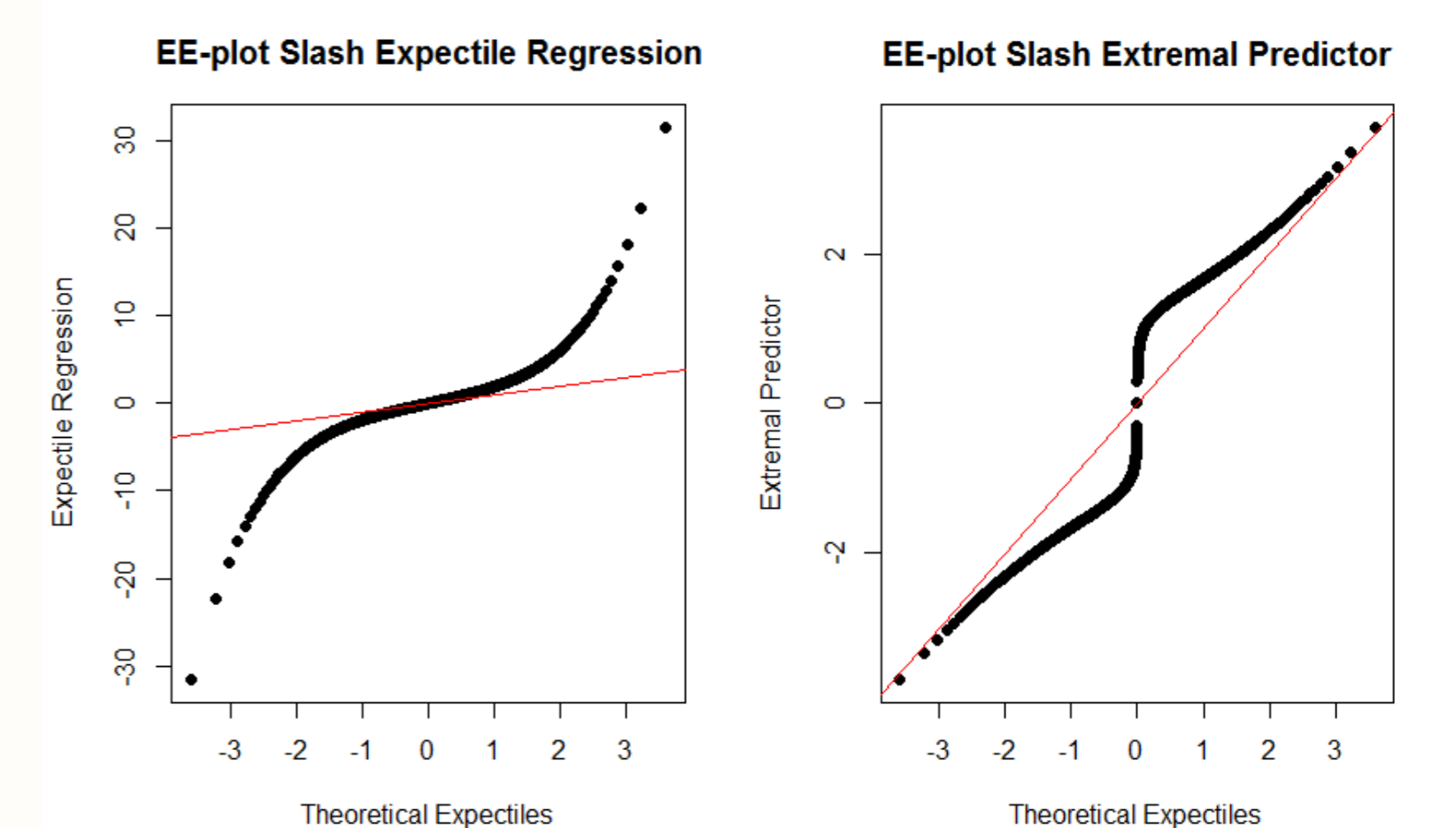


Figure 4: E-E plots for Slash example

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