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# Separable infinity harmonic functions in cones

Marie-Françoise Bidaut-Véron  
Marta Garcia-Huidobro  
Laurent Véron

**Abstract** We study the existence of separable infinity harmonic functions in any cone of  $\mathbb{R}^N$  vanishing on its boundary under the form  $u(r, \sigma) = r^{-\beta}\psi(\sigma)$ . We prove that such solutions exist, the spherical part  $\psi$  satisfies a nonlinear eigenvalue problem on a subdomain of the sphere  $S^{N-1}$  and that the exponents  $\beta = \beta_+ > 0$  and  $\beta = \beta_- < 0$  are uniquely determined if the domain is smooth. We extend some of our results to non-smooth domains.

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*Key words.* Infinity-Laplacian operator; large solutions; viscosity solutions; comparison; ergodic constant; boundary Harnack inequality.

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## 1 Introduction

Let  $S$  be a  $C^3$  subdomain of the unit sphere  $S^{N-1}$  of  $\mathbb{R}^N$  and  $C_S := \{\lambda\sigma \in \mathbb{R}^N : \lambda > 0, \sigma \in S\}$  is the positive cone generated by  $S$ . In this paper we study the existence of positive solutions of

$$\Delta_\infty u := \frac{1}{2} \nabla |\nabla u|^2 \cdot \nabla u = 0 \tag{1.1}$$

in  $C_S$  vanishing on  $\partial C_S \setminus \{0\}$  under the form

$$u(x) = u(r, \sigma) = r^{-\beta} \psi(\sigma), \quad (1.2)$$

where  $\beta \in \mathbb{R}$  and  $(r, \sigma) \in \mathbb{R}_+ \times S^{N-1}$  are the spherical coordinates  $\mathbb{R}^N$ ; such a function  $u$  is called a *separable infinity harmonic function*. The function  $\psi$  satisfies the *spherical infinity harmonic problem in  $S$*

$$\begin{aligned} \frac{1}{2} \nabla' |\nabla' \psi|^2 \cdot \nabla' \psi + \beta(2\beta + 1) |\nabla' \psi|^2 \psi + \beta^3 (\beta + 1) \psi^3 &= 0 & \text{in } S \\ \psi &= 0 & \text{on } \partial S, \end{aligned} \quad (1.3)$$

where  $\nabla'$  is the covariant gradient on  $S^{N-1}$  for the canonical metric and  $(a, b) \mapsto a \cdot b$  the associated quadratic form. The role of the infinity Laplacian for Lipschitz extension of Lipschitz continuous functions defined in a domain has been pointed out by Aronsson in his seminal paper [1]. When the infinity Laplacian  $\Delta_\infty$  is replaced by the  $p$ -Laplacian, the research of regular ( $\beta < 0$ ) separable  $p$ -harmonic functions has been carried out by Krol [8] in the 2-dim case and by Tolksdorff [16] in the general case. Following Krol's method, Kichenassamy and Véron [10] studied the 2-dim singular case ( $\beta > 0$ ). Finally, by a completely different approach and in a more general setting Porretta and Véron [15] studied the general case. In that case, the function  $\psi$  satisfies the *spherical  $p$ -harmonic problem in  $S$*

$$\begin{aligned} \operatorname{div}' \left( (\beta^2 \psi^2 + |\nabla' \psi|^2)^{\frac{p-2}{2}} \nabla' \psi \right) + \beta \lambda_\beta (\beta^2 \psi^2 + |\nabla' \psi|^2)^{\frac{p-2}{2}} \psi &= 0 & \text{in } S \\ \psi &= 0 & \text{on } \partial S, \end{aligned} \quad (1.4)$$

where  $\lambda_\beta = \beta(p-1) + p - N$  and  $\operatorname{div}'$  is the divergence operator acting on vector fields in  $TS^{N-1}$ .

Following an idea which was introduced by Lasry and Lions [13], Porretta-Véron's method was to transform the equation (1.4) by setting

$$w = -\frac{1}{\beta} \ln \psi \quad (1.5)$$

in the case  $\beta > 0$ . The function  $w$  satisfies the new problem

$$\begin{aligned} -\operatorname{div}' \left( (1 + |\nabla' w|^2)^{p/2-1} \nabla' w \right) + (1 + |\nabla' w|^2)^{p/2-1} (\beta(p-1) |\nabla' w|^2 + \lambda_\beta) &= 0 & \text{in } S \\ \lim_{\rho(\sigma) \rightarrow 0} w(\sigma) &= \infty, \end{aligned} \quad (1.6)$$

where  $\rho(\sigma) := \operatorname{dist}(\sigma, \partial S)$  is the distance understood in the geodesic sense on  $S$ .

In this article we borrow ideas used in [15] to transform problem (1.1) by introducing the function  $w$  defined by (1.5). Then  $w$  satisfies, in the viscosity sense,

$$\begin{aligned} -\frac{1}{2} \nabla' |\nabla' w|^2 \cdot \nabla' w + \beta |\nabla' w|^4 + (2\beta + 1) |\nabla' w|^2 + \beta + 1 &= 0 & \text{in } S \\ \lim_{\rho(\sigma) \rightarrow 0} w(\sigma) &= \infty. \end{aligned} \quad (1.7)$$

We first prove

**Theorem A.** *Let  $S \subset S^{N-1}$  be a proper subdomain of  $S^{N-1}$  with a  $C^3$  boundary. Then for any  $\beta > 0$  there exists a locally Lipschitz continuous function  $w$  and a unique  $\lambda(\beta) > 0$  satisfying in the viscosity sense*

$$-\frac{1}{2} \nabla' |\nabla' w|^2 \cdot \nabla' w + \beta |\nabla' w|^4 + (2\beta + 1) |\nabla' w|^2 + \lambda(\beta) = 0 \quad \text{in } S \quad (1.8)$$

$$\lim_{\rho(\sigma) \rightarrow 0} w(\sigma) = \infty.$$

where  $\rho(\cdot)$  is the geodesic distance from points in  $S$  to  $\partial S$ .

Then we prove that there exists a unique  $\beta$  such that  $\lambda(\beta) = \beta + 1$ . In a similar way we study the regular case where  $\beta < 0$  in (1.2), (we denote  $-\beta = \mu > 0$ ), and we obtain

**Theorem B.** *Let  $S \subset S^{N-1}$  be subdomain with a  $C^3$  boundary. Then there exist exactly two real positive numbers  $\beta_s$  and  $\mu_s$  and at least two positive functions  $\psi_s$  and  $\omega_s$  in  $C(\bar{S}) \cap C_{loc}^{0,1}(S)$  (up to multiplication by constants) such that the two functions  $u_{s,+}$  and  $u_{s,-}$  defined in  $C_S$  by  $u_{s,+}(r, \sigma) := r^{-\beta_s} \psi_s(\sigma)$  and  $u_{s,-}(r, \sigma) := r^{\mu_s} \omega_s(\sigma)$  are infinity harmonic in  $C_S$  and vanish on  $\partial C_S \setminus \{0\}$  and  $\partial C_S$  respectively. Furthermore  $\beta_s$  and  $\mu_s$  are decreasing functions of  $S$  for the inclusion order relation on sets.*

The previous results can be extended to general regular domains on a Riemannian manifold as in [15]. It is an open problem whether the positive solutions associated to the same exponent  $\beta_s$  (or  $b_s$ ) are proportional, see discussion in Remark p. 15.

In the special case of a rotationally symmetric domain  $S$  we have a more precise result which allows us to characterize all the separable infinity harmonic functions in  $C_S$  which keep a constant sign and vanish on  $\partial C_S \setminus \{0\}$ . We denote by  $\phi \in (0, \pi)$  the azimuthal angle from the North pole  $N$  on  $S^{N-1}$ .

**Theorem C.** *Let  $S_\alpha$  be the spherical cap with azimuthal opening  $\alpha \in (0, \pi]$ . Then there exist two positive functions  $\psi_\alpha$  and  $\omega_\alpha$  in  $C^\infty(\bar{S})$ , vanishing on  $\partial S$ , such that the two functions*

$$u_{\alpha,+}(r, \sigma) = r^{-\frac{\pi^2}{4\alpha(\pi+\alpha)}} \psi_\alpha(\sigma), \quad (1.9)$$

and

$$u_{\alpha,-}(r, \sigma) = r^{\frac{\pi^2}{4\alpha(\pi-\alpha)}} \omega_\alpha(\sigma), \quad (1.10)$$

are infinity harmonic in  $C_{S_\alpha}$  and vanish on  $\partial C_{S_\alpha} \setminus \{0\}$ . The two functions  $\psi_\alpha$  and  $\omega_\alpha$  depend only on the variable  $\phi \in (0, \alpha]$  and are unique in the class of rotationally invariant solutions up to multiplication by constants.

This study reduced to an ordinary differential equation which has been already treated by T. Bhattacharya in [4] and [5]. But for the sake of completeness and for some related problems we present it in Section 3 of the present paper.

Using these previous results we prove the existence of separable infinity harmonic functions in any cone  $C_S$ .

**Theorem D.** *Assume  $S \subsetneq S^{N-1}$  is any domain. Then there exist  $\bar{\beta}_s > 0$  and a positive function  $\bar{\psi}_s$  in  $C(\bar{S})$ , locally Lipschitz continuous in  $S$  and vanishing on  $\partial S$ , such that the function*

$$\bar{u}_{s,+}(r, \sigma) = r^{-\bar{\beta}_s} \bar{\psi}_s(\sigma), \quad (1.11)$$

is infinity harmonic in  $C_S$  and vanishes on  $\partial C_S \setminus \{0\}$ .

When the cone  $C_S$  is a little more regular, the construction of the spherical infinity harmonic functions can be performed via an approximation from outside.

**Theorem E.** *Assume that  $S \subsetneq S^{N-1}$  is an outward accessible domain, i.e.  $\partial S = \partial \bar{S}^c$ . Then there exist  $\underline{\beta}_s \in (0, \bar{\beta}_s]$  and a positive function  $\underline{\psi}_s \in C(\bar{S})$ , locally Lipschitz continuous in  $S$  and vanishing on  $\partial S$ , such that the function*

$$\underline{u}_{s,+}(r, \sigma) = r^{-\underline{\beta}_s} \underline{\psi}_s(\sigma) \quad (1.12)$$

*is infinity harmonic in  $C_S$ , vanishes on  $\partial C_S \setminus \{0\}$  and has the property that for any separable infinity-harmonic function in  $C_S$  under the form  $u(r, \sigma) = r^{-\beta} \psi(\sigma)$  where  $\beta > 0$  and  $\psi$  in  $C(\bar{S})$  vanishes on  $\partial S$ , there holds  $\underline{\beta}_s \leq \beta \leq \bar{\beta}_s$ .*

The uniqueness of the exponent  $\beta_s$  is proved under Lipschitz and geometric conditions on  $S$ .

**Theorem F.** *Assume that  $S \subsetneq S^{N-1}$  is a Lipschitz domain satisfying the interior sphere condition. Then  $\bar{\beta}_s = \underline{\beta}_s$ . Furthermore there exists a constant  $c = c(S, p) > 0$  such that for any two positive functions  $\psi_i$ ,  $i = 1, 2$ , satisfying the spherical  $p$ -harmonic problem (1.4), there holds*

$$\frac{\psi_1(\sigma)}{\psi_2(\sigma)} \leq c \frac{\psi_1(\sigma')}{\psi_2(\sigma')} \quad \forall (\sigma, \sigma') \in S. \quad (1.13)$$

Note that the statements and the proofs of Theorems D, E and F can be easily modified if one considers regular infinity harmonic functions in  $C_S$  which vanish on  $\partial C_S$ .

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## 2 The smooth case

We assume that  $(r, \sigma) \in \mathbb{R}_+ \times S^{N-1}$  are the spherical coordinates of  $x \in \mathbb{R}^N$ . If  $u$  is a  $C^1$  function, then  $\nabla u = u_r \mathbf{e} + \frac{1}{r} \nabla' u$  where  $\mathbf{e} = \frac{x}{|x|}$  and  $\nabla'$  is the tangential gradient of  $u(r, \cdot)$  identified to the covariant gradient thanks to the canonical imbedding of  $S^{N-1}$  into  $\mathbb{R}^N$ . Then  $|\nabla u|^2 = u_r^2 + \frac{1}{r^2} |\nabla' u|^2$ , thus

$$-\Delta_\infty u = - \left( u_r^2 + \frac{1}{r^2} |\nabla' u|^2 \right)_r u_r - \frac{1}{r^2} \nabla' \left( u_r^2 + \frac{1}{r^2} |\nabla' u|^2 \right) \cdot \nabla' u = 0.$$

A solution  $-\Delta_\infty u = 0$  which has the form  $u(x) = u(r, \sigma) = r^{-\beta} \psi(\sigma)$  satisfies, in the viscosity sense, the spherical infinity harmonic equation

$$\frac{1}{2} \nabla' |\nabla' \psi|^2 \cdot \nabla' \psi + \beta(2\beta + 1) |\nabla' \psi|^2 \psi + \beta^3(\beta + 1) \psi^3 = 0. \quad (2.1)$$

**Theorem 2.1.** *For any  $C^3$  domain  $S \subset S^{N-1}$  there exists a unique  $\beta_s > 0$  and one nonnegative function  $\psi \in C^{0,1}(\bar{S})$  solution of*

$$\begin{aligned} -\frac{1}{2} \nabla' |\nabla' \psi|^2 \cdot \nabla' \psi &= \beta(2\beta + 1) |\nabla' \psi|^2 \psi + \beta^3(\beta + 1) \psi^3 && \text{in } S \\ \psi &= 0 && \text{in } \partial S. \end{aligned} \quad (2.2)$$

such that the function  $(r, \sigma) \mapsto u_s(r, \sigma) := r^{-\beta_s} \psi(\sigma)$  is positive and  $\infty$ -harmonic in the cone  $C_s = \{x = \lambda\sigma \in \mathbb{R}^N : \lambda > 0, \sigma \in S\}$  and vanish on  $\partial S \setminus \{0\}$ .

Following Porretta-Veron's method, we transform the eigenvalue problem into a large solution problem with absorption by setting

$$w = -\frac{1}{\beta} \ln \psi. \quad (2.3)$$

Therefore the formal new problem is to prove the existence of a unique  $\beta > 0$  and of a nonnegative function  $w$  such that

$$\begin{aligned} \frac{1}{2} \nabla' |\nabla' w|^2 \cdot \nabla' w - \beta |\nabla' w|^4 - (2\beta + 1) |\nabla' w|^2 &= \beta + 1 & \text{in } S \\ w &= \infty & \text{in } \partial S. \end{aligned} \quad (2.4)$$

The two problems are clearly equivalent for  $C^2$  solutions. Since the mapping  $w \mapsto \psi$  is smooth and decreasing, it exchanges supersolutions (resp. subsolutions) into subsolutions (resp. supersolutions). Therefore the two problems (2.2)-(2.4) are also equivalent if we deal with continuous viscosity solutions.

In order to increase the regularity of the solutions and to avoid the difficulties coming from the fact the above problem is invariant if we add a constant to a solution, instead of (2.4) we consider the regularized problem with absorption

$$\begin{aligned} -\delta \Delta w - \frac{1}{2} \nabla' |\nabla' w|^2 \cdot \nabla' w + \gamma |\nabla' w|^4 + (2\gamma + 1) |\nabla' w|^2 + \epsilon w &= 0 & \text{in } S \\ w &= \infty & \text{in } \partial S, \end{aligned} \quad (2.5)$$

where  $\epsilon, \delta$  are two positive parameters. We will obtain below local estimates on  $\nabla' w$  independent of  $\epsilon$  and  $\delta$ . Thanks to these estimates we will let successively  $\delta$  and  $\epsilon$  to 0 and obtain that, up to a constant, the term  $\epsilon w$  converges to some unique  $\lambda(\gamma)$  called *the ergodic constant* although it has a probabilistic interpretation only in the case of the ordinary Laplacian [13]. The limit problem of (2.5) is the following

$$\begin{aligned} -\frac{1}{2} \nabla' |\nabla' w|^2 \cdot \nabla' w + \gamma |\nabla' w|^4 + (2\gamma + 1) |\nabla' w|^2 + \lambda(\gamma) &= 0 & \text{in } S \\ w &= \infty & \text{in } \partial S. \end{aligned} \quad (2.6)$$

## 2.1 Two-sided estimates

We denote the "positive" geodesic distance  $\rho(\sigma) = \text{dist}(\sigma, \partial S)$ . If  $\sigma \in S^c$  we set  $\tilde{\rho}(\sigma) = -\text{dist}(\sigma, \partial S)$ . If  $\sigma_1$  and  $\sigma_2$  are not antipodal points there exists a unique minimizing geodesic between  $\sigma_1$  and  $\sigma_2$ . It is an arc of a Riemannian circle (or great circle). The geodesic distance between  $\sigma_1$  and  $\sigma_2$  is denoted by  $\ell(\sigma_1, \sigma_2)$ . It coincides with the angle determined by the two straight lines from 0 to  $\sigma_1$  and 0 to  $\sigma_2$ . At this point it is convenient to use Fermi coordinates in  $S$  in a neighborhood of  $\partial S$ . We set

$$S_\tau = \{\sigma \in S : \rho(\sigma) < \tau\}, \quad S'_\tau = S \setminus \overline{S}_\tau, \quad \Sigma_\tau = \{\sigma \in S : \rho(\sigma) = \tau\}.$$

If  $\tau \leq \tau_0$  for any  $\sigma \in S_\tau$  there exists a unique  $z_\sigma \in \partial S$  such that  $\ell(\sigma, z_\sigma) = \rho(\sigma)$ . These Fermi coordinates of  $\sigma$  are defined by  $(\tau, z) \in [0, \tau_0) \times \partial S$ . The mapping  $\Pi$  such that

$$\Pi(\sigma) = (\rho(\sigma), z_\sigma) \quad \forall \sigma \in S_{\tau_0},$$

is a  $C^2$  diffeomorphism from  $S_{\tau_0}$  into  $[0, \tau_0) \times \partial S$ . The expression of the Laplace-Beltrami operator in  $S_{\tau_0}$  is given in [3]:

$$\Delta' u(\sigma) = \frac{\partial^2 u}{\partial \tau^2} - (N-2)H \frac{\partial u}{\partial \tau} + \tilde{\Delta}'_z u \quad \forall \sigma = \Pi^{-1}((\tau, z)), \quad (2.7)$$

where  $H = H(\tau, z)$  is the mean curvature of  $\Sigma_\tau$  and  $\tilde{\Delta}'_z$  is a second order elliptic operator acting on functions defined on  $\Sigma_\tau$ . If  $g = (g_{ij})$  is the metric tensor on  $S^{N-1}$  and by convention,  $|g| = \det(g_{ij})$ , this operator admits the following expression

$$\tilde{\Delta}'_z u = \frac{1}{\sqrt{|g|}} \sum_{j=1}^{N-2} \frac{\partial}{\partial z_j} \left( \sqrt{|g|} a_j \frac{\partial u}{\partial z_j} \right),$$

for some  $a_j > 0$  if we take for coordinates curve-frame  $z_j$  a system of orthogonal 1-dim great circles on  $\Gamma$  intersecting at  $z_\sigma$  (these circle corresponds to the  $(N-2)$ -principal curvatures at this points). The coefficients  $a_j$  depend both on  $z$  and  $\tau$ . Thus, if  $u$  depends only on  $\rho$ ,

$$\Delta' u(\sigma) = \frac{\partial^2 u}{\partial \tau^2} - (N-2)H \frac{\partial u}{\partial \tau}. \quad (2.8)$$

The expression of  $H$  is given in [3] and we can assume that  $\tau_0$  is small enough so that  $H$  remains bounded.

We extend the geodesic distance  $\rho(x) = \text{dist}(x, \partial S)$  as a smooth positive function so that  $\tilde{\rho}(x) := \rho(x)$  if  $\rho(x) \leq \tau_0$  and thus, in the same neighborhood of  $\partial S$ ,  $\nabla \tilde{\rho}(x) = \mathbf{n}_{z_x}$ , the unit outward normal vector to  $\partial S$  at the point  $z_x = \text{Proj}_{\partial \Omega}(x)$ .

If  $w$  depends only on  $\rho$ , (2.5) becomes

$$\begin{aligned} \delta w'' - (N-2)Hw' + w'^2 w'' - \gamma w'^4 - (2\gamma+1)w'^2 - \epsilon w &= 0 & \text{in } S_{\tau_0} \\ w &= \infty & \text{in } \partial S. \end{aligned} \quad (2.9)$$

In the sequel we put

$$\mathcal{P}_\delta(u) := -\delta \Delta u - \frac{1}{2} \nabla' |\nabla' u|^2 \cdot \nabla' u + \gamma |\nabla' u|^4 + (2\gamma+1) |\nabla' u|^2 + \epsilon u = \tilde{\mathcal{P}}_\delta(u) + \epsilon u. \quad (2.10)$$

**Proposition 2.2.** *There exist  $\tau_1 \in (0, \tau_0]$ , three positive constants  $M$ ,  $\epsilon_0$  and  $\delta_0$  and two positive functions  $w^*$ ,  $w_* \in C^2(S)$  such that  $w^* > w_*$  in  $S_\tau$ ,  $w^* + \frac{1}{\gamma} \ln \rho \in L^\infty(S)$  and  $w_* + \frac{1}{\gamma} \ln \rho \in L^\infty(S)$  with the property that for any  $\epsilon \in (0, \epsilon_0]$  and  $\delta \in (0, \delta_0]$  the two functions*

$$\bar{w}(\sigma) = w^* + \frac{M}{\epsilon} \quad (2.11)$$

and

$$\underline{w}(\sigma) = w_* - \frac{M}{\epsilon} \quad (2.12)$$

are respectively a supersolution and a subsolution of  $\mathcal{P}_\delta(u) = 0$ . Furthermore any solution  $w$  of problem (2.6) satisfies  $\underline{w} \leq w \leq \bar{w}$ .

*Proof.* Let  $a > 0$ . We first notice by a standard computation that the solutions of the ODE,

$$\begin{aligned} \delta w'' + w'^2 w'' - \gamma w'^4 - a w'^2 &= 0 & \text{in } (0, 1) \\ w'(0) &= -\infty \end{aligned} \quad (2.13)$$

are negative and given implicitly by

$$\frac{\delta}{a w'(\rho)} + \left( \frac{1}{\gamma} - \frac{\delta}{a} \right) \sqrt{\frac{\gamma}{a}} \tan^{-1} \left( \sqrt{\frac{a}{\gamma}} \frac{1}{w'(\rho)} \right) = -\rho. \quad (2.14)$$

In order to have a global estimate, we set  $w' = -z^{-1}$ , thus (2.14) becomes

$$\frac{\delta z}{a} + \left( \frac{1}{\gamma} - \frac{\delta}{a} \right) \sqrt{\frac{\gamma}{a}} \tan^{-1} \left( z \sqrt{\frac{a}{\gamma}} \right) = \rho, \quad (2.15)$$

provided  $a > \gamma\delta$ . Since  $\tan^{-1} \left( z \sqrt{\frac{a}{\gamma}} \right) \leq z \sqrt{\frac{a}{\gamma}}$ , we derive

$$z(\rho) \geq \gamma\rho \iff 0 > w'(\rho) \geq -\frac{1}{\gamma\rho} \quad \forall \rho > 0, \quad (2.16)$$

with equality only if  $\rho = 0$ . Since we can write (2.15) as

$$\sqrt{\frac{\gamma}{a}} \frac{\delta}{a} \left( z \sqrt{\frac{a}{\gamma}} \right) + \left( \frac{1}{\gamma} - \frac{\delta}{a} \right) \sqrt{\frac{\gamma}{a}} \tan^{-1} \left( z \sqrt{\frac{a}{\gamma}} \right) = \rho \geq \tan^{-1} \left( z \sqrt{\frac{a}{\gamma}} \right), \quad (2.17)$$

we obtain

$$z \leq \sqrt{\frac{\gamma}{a}} \tan(\rho\sqrt{a\gamma}) \iff w'(\rho) \leq -\sqrt{\frac{a}{\gamma}} \cot(\rho\sqrt{a\gamma}) \quad \forall \rho > 0. \quad (2.18)$$

Finally

$$-\frac{1}{\gamma\rho} \leq w'(\rho) \leq -\sqrt{\frac{a}{\gamma}} \cot(\rho\sqrt{a\gamma}) \quad \forall \rho \in (0, \tau_0]. \quad (2.19)$$

and in particular, for any  $\tau_1 \in (0, \tau_0]$ ,

$$|w'(\rho)| \geq \sqrt{\frac{a}{\gamma}} \cot(\tau_1\sqrt{a\gamma}) \quad \forall \rho \in (0, \tau_1]. \quad (2.20)$$

From this estimate we derive

$$w(\rho_0) + \frac{1}{\gamma} \ln \left( \frac{\sin \rho_0 \sqrt{a\gamma}}{\sin \rho \sqrt{a\gamma}} \right) \leq w(\rho) \leq w(\rho_0) + \frac{1}{\gamma} \ln \left( \frac{\rho_0}{\rho} \right) \quad \forall \rho \in (0, \tau_1]. \quad (2.21)$$



The solution  $w$  depends on the value of  $a$  and  $\delta$ . Since  $\partial S$  is smooth, we can assume that  $(N-2)|H|$  is bounded by some constant  $m \geq 0$  in  $S_{\tau_0}$ . Denote by  $w_\tau$  a solution satisfying  $w(\tau) = 0$ , then it is positive in  $S_\tau$  and

$$\begin{aligned} \mathcal{P}_\delta(w_\tau) &= \delta(N-2)Hw'_\tau + (2\gamma+1-a)w_\tau'^2 + \epsilon w_\tau \\ &\geq |w'_\tau|((2\gamma+1-a)|w'_\tau| - m) \\ &\geq |w'_\tau|((2\gamma+1-a)\sqrt{\frac{a}{\gamma}} \cot(\tau\sqrt{a\gamma}) - m). \end{aligned}$$

If we take  $1 < a_1 < 2\gamma+1$ , we choose  $\tau := \tau_1 \in (0, \tau_0]$  such that

$$(2\gamma+1-a_1)\sqrt{\frac{a}{\gamma}} \cot(\tau\sqrt{a_1\gamma}) > m, \quad (2.22)$$

which implies that  $w_\tau$  is a supersolution in  $S_\tau$ . In assuming now  $a := a_2 > 2\gamma+1$ , we also have,

$$\begin{aligned} \mathcal{P}_\delta(w_\tau) &\leq w'_\tau((2\gamma+1-a)w'_\tau - m) + \epsilon w_\tau \\ &\leq |w'_\tau|((2\gamma+1-a)|w'_\tau| + m) + \epsilon w_\tau \\ &\leq |w'_\tau| \left( m - (a-2\gamma-1)\sqrt{\frac{a}{\gamma}} \cot(\tau\sqrt{a\gamma}) \right) + \epsilon w_\tau. \end{aligned}$$

We choose  $a_2 = 4\gamma+2-a_1$ , then

$$m - (a-2\gamma-1)\sqrt{\frac{a}{\gamma}} \cot(\tau\sqrt{a\gamma}) \leq -c < 0 \quad \forall \tau \in (0, \tau_1].$$

Therefore

$$\mathcal{P}_\delta(w_\tau) \leq w_\tau \left( \epsilon + \frac{w'_\tau}{w_\tau} \right).$$

Since  $w'_{\tau_1} < 0$  and  $w_{\tau_1}(\tau_1) = 0^+$ , there exists  $\epsilon_0 > 0$ , such that for any  $\epsilon \in (0, \epsilon_0]$

$$\epsilon + \frac{w'_{\tau_1}(\rho)}{w_{\tau_1}(\rho)} \leq -1 \quad \forall \rho \in (0, \tau_1].$$

Therefore  $w_{\tau_1, a_1}$  and  $w_{\tau_1, a_2}$  are respectively supersolution and subsolution of  $\mathcal{P}_\delta(u) = 0$  in  $S_{\tau_1}$ . We extend them in  $S'_{\tau_1}$  as smooth functions  $\tilde{w}_{\tau_1, a_1}$  and  $\tilde{w}_{\tau_1, a_2}$  in order  $|\tilde{\mathcal{P}}_\delta(\tilde{w}_{\tau_1, a_j})|$  to remain bounded by some constant  $M$ . Finally  $\bar{w} = \tilde{w}_{\tau_1, a_1} + M\epsilon^{-1}$  is a supersolution and  $\underline{w} = \tilde{w}_{\tau_1, a_2} + M\epsilon^{-1}$  is a subsolution of  $\mathcal{P}_\delta(u) = 0$ .

Next, we replace  $\underline{w}$  by

$$\underline{w}_h(\delta) = \underline{w}(\delta + h)$$

and  $\bar{w}$  by

$$\bar{w}_h = \bar{w}(\delta - h)$$

for  $h$  small enough, we still have a sub and a super solution of  $\mathcal{P}_\delta(u) = 0$  in  $S_{\tau_1}$  and  $S_{\tau_1} \setminus S_h$ . In the remaining part of  $S$ , we extend smoothly  $\underline{w}_h$  and  $\bar{w}_h$  in order  $\tilde{\mathcal{P}}_\delta(\underline{w}_h)$  and  $\tilde{\mathcal{P}}_\delta(\bar{w}_h)$  be bounded. We can adjust  $M$  in order  $\mathcal{P}_\delta(\underline{w}_h) \leq 0$  and  $\mathcal{P}_\delta(\bar{w}_h) \geq 0$  in whole  $S$ , and all these manipulations can be

done uniformly with respect to  $h$  and  $\epsilon$ . If  $w$  is any  $C^2$  solution of (2.5), we prove that it dominates the subsolution  $\underline{w}_h$  in  $S$ : actually, if we assume that  $\underline{w}_h$  and  $w$  are not ordered in  $S$ , there exists  $\sigma_0 \in S$  such that

$$\underline{w}_h(\sigma_0) - w(\sigma_0) = \max\{\underline{w}_h(\sigma) - w(\sigma) : \sigma \in S\} > 0.$$

Since the two functions are  $C^2$ ,

$$\nabla \underline{w}_h(\sigma_0) = \nabla w(\sigma_0) \quad \text{and} \quad D^2 \underline{w}_h(\sigma_0) \leq D^2 w(\sigma_0),$$

where  $D^2$  is the Hessian form, in the sense of quadratic forms, i.e.

$$D^2 \underline{w}_h(\sigma_0)(\nabla \underline{w}_h(\sigma_0), \nabla \underline{w}_h(\sigma_0)) \leq D^2 w(\sigma_0)(\nabla w(\sigma_0), \nabla w(\sigma_0)).$$

This implies  $\mathcal{P}_\delta(\underline{w}_h)(\sigma_0) > \mathcal{P}_\delta(w)(\sigma_0) = 0$ , contradiction. Therefore

$$\underline{w}_h \leq w \quad \text{in } S, \quad (2.23)$$

uniformly with respect to  $h$ . Similarly

$$\bar{w}_h \geq w \quad \text{in } S. \quad (2.24)$$

Letting  $h$  tend to 0 the claim follows.  $\square$

## 2.2 Gradient estimates

If  $\sigma_0 \in S^{N-1}$  and  $R < \pi$ , we set  $B_R(\sigma_0) = \{\sigma \in S^{N-1} : \ell(\sigma, \sigma_0) < R\}$ .

**Proposition 2.3.** *Let  $0 \leq \delta, \epsilon \leq 1$  and  $w$  be a smooth solution of*

$$-\frac{1}{2} \nabla' |\nabla' w|^2 \cdot \nabla w - \delta \Delta w + \gamma |\nabla' w|^4 + (2\gamma + 1) |\nabla' w|^2 + \epsilon w = 0 \quad \text{in } B_R(\sigma_0) \subset S, \quad (2.25)$$

where  $S$  is a domain of  $S^{N-1}$ . Then there exists  $c = c(N) > 0$  such that

$$|\nabla' w(\sigma_0)| \leq \frac{c}{\gamma R}. \quad (2.26)$$

*Proof.* We set  $z = |\nabla w|^2$ , then  $2\Delta_\infty w = \nabla' |\nabla' w|^2 \cdot \nabla' w = \nabla' z \cdot \nabla' w$ . We define the linearized operator of  $\Delta_\infty$  at  $w$  following  $h$  by

$$B_w(h) := \frac{d}{dt} \Delta_\infty(w + th)|_{t=0} = \frac{1}{2} \nabla' h \cdot \nabla' z + \nabla' w \cdot \nabla' (\nabla' w \cdot \nabla' h).$$

Thus the linearized operator of  $\Delta_\infty + \delta \Delta$  at  $w$  following  $h$  is

$$\mathcal{L}_w(h) = B_w(h) + \delta \Delta h. \quad (2.27)$$

Thus

$$\mathcal{L}_w(z) = \frac{1}{2} |\nabla' z|^2 + \nabla' w \cdot \nabla' (\nabla' \cdot \nabla z) + \delta \Delta z.$$

We can re-write (2.25) under the form

$$\nabla'w \cdot \nabla z = 2(\gamma z^2 + (2\gamma + 1)z + \epsilon w - \delta \Delta w). \quad (2.28)$$

Hence

$$\nabla'(\nabla'w \cdot \nabla'z) = 2((2\gamma z + 2\gamma + 1)\nabla'z + \epsilon \nabla'w - \delta \nabla'(\Delta w)),$$

and then

$$\nabla w \cdot \nabla'(\nabla'w \cdot \nabla'z) = 2((2\gamma z + 2\gamma + 1)\nabla'z \cdot \nabla'w + \epsilon z - \delta \nabla'(\Delta w) \cdot \nabla'w).$$

By the Weitzenböck formula, since  $\text{Ricc}(S^{N-1}) = (N-2)g_0$  ( $g_0$  is the metric tensor on  $S^{N-1}$ ), we have

$$\begin{aligned} \frac{1}{2}\Delta z &= |D^2w|^2 + \nabla'(\Delta w) \cdot \nabla'w + (N-2)|\nabla'w|^2 \\ &= |D^2w|^2 + \nabla'(\Delta w) \cdot \nabla'w + (N-2)z. \end{aligned}$$

Hence

$$\begin{aligned} \mathcal{L}_w(z) &= \frac{1}{2}|\nabla z|^2 + 2((2\gamma z + 2\gamma + 1)\nabla'z \cdot \nabla'w + 2\epsilon z - 2\delta \nabla'w \cdot \nabla'(\Delta w)) \\ &\quad + 2\delta |D^2w|^2 + 2\delta \nabla'(\Delta w) \cdot \nabla'w + 2\delta(N-2)z. \end{aligned}$$

Expanding the above identity, we see that the terms of order 3 disappear, hence

$$\mathcal{L}_w(z) = \frac{1}{2}|\nabla z|^2 + 2((2\gamma z + 2\gamma + 1)\nabla'z \cdot \nabla'w + 2\epsilon z) + 2\delta |D^2w|^2 + 2\delta(N-2)z. \quad (2.29)$$

If  $\xi \in C_c^2(\overline{B}_R(\sigma_0))$ , we set  $Z = \xi^2 z$  and we derive

$$\begin{aligned} \mathcal{L}_w(Z) &= B_w(\xi^2 z) + \delta \Delta(\xi^2 z) \\ &= \xi^2 \mathcal{L}_w(z) + z \mathcal{L}_w \xi^2 + 2(\nabla'w \cdot \nabla' \xi^2)(\nabla'w \cdot \nabla z) + 2\delta \nabla' \xi^2 \cdot \nabla z, \end{aligned}$$

where

$$\begin{aligned} \mathcal{L}_w \xi^2 &= \frac{1}{2} \nabla'z \cdot \nabla' \xi^2 + \nabla'w \cdot \nabla'(\nabla'w \cdot \nabla' \xi^2) + \delta \Delta \xi^2 \\ &= \frac{1}{2} \nabla'z \cdot \nabla' \xi^2 + D^2w(\nabla' \xi^2) \cdot \nabla'w + D^2 \xi^2(\nabla'w) \cdot \nabla'w + \delta \Delta \xi^2 \\ &= \nabla'z \cdot \nabla' \xi^2 + D^2 \xi^2(\nabla'w) \cdot \nabla'w + \delta \Delta \xi^2 \\ &\geq \nabla'z \cdot \nabla' \xi^2 - z |D^2 \xi^2| + \delta \Delta \xi^2. \end{aligned}$$

By Schwarz inequality,  $(\Delta w)^2 \leq \frac{1}{N-1} |D^2w|^2$ , we derive from (2.27) and (2.28),

$$\begin{aligned} \mathcal{L}_w(z) &\geq \frac{1}{2}|\nabla z|^2 + 4(2\gamma z + 2\gamma + 1)(\gamma z^2 + (2\gamma + 1)z + \epsilon w - \delta \Delta w) + 4\epsilon z \\ &\quad + \frac{2\delta}{N-1}(\Delta w)^2 + 2\delta(N-2)z \\ &\geq \frac{1}{2}|\nabla z|^2 + \frac{\delta}{N-1}(\Delta w)^2 + 4\gamma^2 z^3 - c_0, \end{aligned}$$

for some  $c_0 = c_0(N, \gamma) > 0$ . In the sequel the different positive constants  $c_j$  which will appear below depend only on  $N$  and  $\gamma$ . This implies

$$\begin{aligned} \mathcal{L}_w(Z) \geq & z \left( \nabla' z \cdot \nabla' \xi^2 - z |D^2 \xi^2| + \delta \Delta \xi^2 \right) + 2(\nabla' w \cdot \nabla' \xi^2)(\nabla' w \cdot \nabla z) + 2\delta \nabla' \xi^2 \cdot \nabla z \\ & + \xi^2 \left( \frac{1}{2} |\nabla z|^2 + \frac{\delta}{N-1} (\Delta w)^2 + 4\gamma^2 z^3 - c_0 \right). \end{aligned} \quad (2.30)$$

We choose  $\xi$  such that  $0 \leq \xi \leq 1$ ,  $|\nabla \xi| \leq c_1 R^{-1}$  and  $|D^2 \xi| \leq c_1 R^{-2}$ , then

$$\begin{aligned} (z + 2\delta) |\nabla' z \cdot \nabla' \xi^2| &\leq c_1 \frac{(z + 2\delta)\xi}{R} |\nabla' z| \leq \frac{\xi^2}{8} |\nabla' z|^2 + c_2 \frac{(z + 2\delta)^2}{R^2}, \\ |z(\delta \Delta \xi^2 - z |D^2 \xi^2|)| &\leq \frac{c_3(z + 2\delta)^2}{R^2}, \\ |\nabla' w \cdot \nabla z| &\leq \sqrt{z} |\nabla z|, \\ |\nabla' w \cdot \nabla \xi^2| &\leq 2\xi |\nabla w| |\nabla \xi| \leq \frac{2c_1 \xi \sqrt{z}}{R}, \\ |2(\nabla' w \cdot \nabla' \xi^2)(\nabla' w \cdot \nabla z)| &\leq \frac{4c_1 \xi z |\nabla z|}{R} \leq \frac{\xi^2}{8} |\nabla' z|^2 + c_4 \frac{z^2}{R^2}. \end{aligned}$$

We consider a point  $z_0 \in B_R$  where  $Z$  is maximal, then  $\mathcal{L}_w(Z)(z_0) \leq 0$ , which implies that at this point,

$$\xi^2 \left( \frac{1}{2} |\nabla z|^2 + \frac{\delta}{N-1} (\Delta w)^2 + 4\gamma^2 z^3 - c_0 \right) \leq \frac{\xi^2}{4} |\nabla z|^2 + \frac{c_5(z + 2\delta)^2}{R^2}. \quad (2.31)$$

We assume  $R \leq 1$  and  $2\delta \leq 1$ , we multiply by  $\xi^4$  and obtain

$$\frac{1}{4} |\xi^3 \nabla z|^2 + \frac{\delta \xi^6}{N-1} (\Delta w)^2 + 4\gamma^2 (\xi^2 z)^3 \leq \frac{c_6((\xi^2 z)^2 + 1)}{R^2} + c_0. \quad (2.32)$$

From the inequality

$$4\gamma^2 (\xi^2 z)^3 \leq \frac{c_6(\xi^2 z)^2}{R^2} + \frac{c_7}{R^2},$$

we deduce

$$\xi^2 z \leq \frac{c_8}{R^2} \quad \text{with } c_8 = \max \left\{ c_7, \frac{c_6}{\gamma^2} \right\}. \quad (2.33)$$

If we assume that  $\xi(\sigma_0) = 1$ , we finally infer

$$|\nabla w(\sigma_0)| \leq \frac{\sqrt{c_8}}{R}, \quad (2.34)$$

which is the claim.  $\square$

As an immediate consequence, we have

**Corollary 2.4.** *Let  $0 \leq \epsilon, \delta \leq 1$ . If  $w$  is a solution of (2.5) in  $S$ , it satisfies*

$$|\nabla w(\sigma)| \leq \frac{c}{\gamma \rho(\sigma)} \quad \forall \sigma \in S, \quad (2.35)$$

for some  $c > 0$  depending only of  $N$ .

### 2.3 Proof of Theorem A

We write  $w = w_{\delta,\epsilon,\gamma}$  and

$$-\delta\Delta w - \frac{1}{2}\nabla|\nabla'w|^2\nabla'w + \gamma|\nabla'w|^4 + (2\gamma+1)|\nabla'w|^2 + \epsilon w = 0, \quad (2.36)$$

then  $w_{\delta,\epsilon,\gamma}$  satisfies the estimate (2.35). By Proposition 2.2 it satisfies also

$$-\frac{1}{\gamma}\ln\rho - \frac{M}{\epsilon} \leq w_* \leq w_{\delta,\epsilon,\gamma} \leq w^* + \frac{M}{\epsilon} \leq -\frac{1}{\gamma}\ln\rho + \frac{M}{\epsilon}. \quad (2.37)$$

The set of functions  $\{w_{\delta,\epsilon,\gamma}\}_{\epsilon,\delta}$  is clearly locally equicontinuous in  $S$ . By classical stability results on viscosity solutions (see e.g. [9, Chap 3]), there exist a subsequence  $\{w_{\delta_n,\epsilon,\gamma}\}$  and a function  $w_{\epsilon,\gamma}$  such that  $w_{\delta_n,\epsilon,\gamma} \rightarrow w_{\epsilon,\gamma}$ , and  $w_{\epsilon,\gamma}$  is a viscosity solution of

$$\begin{aligned} -\frac{1}{2}\nabla|\nabla'w|^2\nabla'w + \beta|\nabla'w|^4 + (2\beta+1)|\nabla'w|^2 + \epsilon w &= 0 && \text{in } S \\ w &= \infty && \text{in } \partial S. \end{aligned} \quad (2.38)$$

Furthermore  $w_{\epsilon,\gamma}$  satisfies the same estimates (2.35) and (2.37) as  $w_{\delta,\epsilon,\gamma}$ . Put  $\tilde{w}_{\epsilon,\gamma}(\sigma) = w_{\epsilon,\gamma}(\sigma) - w_{\epsilon,\gamma}(\sigma_0)$  with  $\sigma_0 \in \Omega$ , then  $\tilde{w} := \tilde{w}_{\epsilon,\gamma}$  satisfies

$$-\frac{1}{2}\nabla|\nabla'\tilde{w}|^2\nabla'\tilde{w} + \gamma|\nabla'\tilde{w}|^4 + (2\gamma+1)|\nabla'\tilde{w}|^2 + \epsilon\tilde{w} + \epsilon w(\sigma_0) = 0. \quad (2.39)$$

Moreover

$$|\tilde{w}_{\epsilon,\gamma}(\sigma)| = |w_{\epsilon,\gamma}(\sigma) - w_{\epsilon,\gamma}(\sigma_0)| \leq \max\left\{\frac{c}{\gamma\rho(\tau)} : \tau \in [\sigma, \sigma_0]\right\} |\sigma - \sigma_0|.$$

Thus, as  $\epsilon \rightarrow 0$ ,  $\epsilon\tilde{w}_{\epsilon,\gamma} \rightarrow 0$  locally uniformly in  $S$ . Up to some subsequence  $\{\epsilon_n\}$ ,  $\tilde{w}_{\epsilon_n,\gamma} \rightarrow w_\gamma$  locally uniformly in  $S$  and  $\epsilon_n w_{\epsilon_n,\gamma}(\sigma_0) \rightarrow \lambda(\gamma)$ . As in [15] the expression  $\lambda(\gamma)$  does not depend on  $\sigma_0$ . By analogy with the semilinear case studied in [13], this last limit is called *the ergodic constant*. Furthermore it is easy to check that there exist positive constants  $M_1$  and  $M_2$  such that  $w_1$  and  $w_2$  defined by

$$w_1(x) = -\frac{1}{\gamma}\ln\rho + M_1\rho + M_2 \quad \text{and} \quad w_2(x) = -\frac{1}{\gamma}\ln\rho - M_1\rho - M_2 \quad (2.40)$$

are respectively a supersolution and subsolution of (2.39) in  $S$  and that there holds

$$w_2(x) \leq \tilde{w}_{\epsilon,\gamma}(x) \leq w_1(x) \quad \forall x \in S. \quad (2.41)$$

By the same stability results of viscosity solutions, we infer that  $w_\gamma$  is a positive solution of

$$\begin{aligned} -\frac{1}{2}\nabla|\nabla'w|^2\nabla'w + \gamma|\nabla'w|^4 + (2\gamma+1)|\nabla'w|^2 + \lambda(\gamma) &= 0 && \text{in } S \\ w &= \infty && \text{on } \partial S. \end{aligned} \quad (2.42)$$

Furthermore, there holds from (2.41) and (2.35),

$$\left| w_\gamma + \frac{1}{\gamma} \ln \rho \right| \leq M, \quad (2.43)$$

and

$$|\nabla w_\gamma| \leq \frac{c}{\gamma \rho}. \quad (2.44)$$

**Proposition 2.5.** *For any  $C^3$  domain  $S \subset S^{N-1}$ , the ergodic constant  $\lambda(\gamma) := \lambda(\gamma, S)$  is uniquely determined by  $\gamma$ . Furthermore it is a continuous decreasing function of  $\gamma$  and  $S$  for the order relation of inclusion.*

*Proof.* Assume that the set  $\{\epsilon w_\epsilon(\sigma_0)\}$  of values of the solutions of (2.42) at  $\sigma_0$  admits two different cluster points  $\lambda_1$  and  $\lambda_2$ . Then there exist two locally Lipschitz continuous functions  $w_1$  and  $w_2$  satisfying

$$-\frac{1}{2} \nabla' |\nabla' w_i|^2 \cdot \nabla' w_i + \gamma |\nabla' w_i|^4 + (2\gamma + 1) |\nabla' w_i|^2 + \lambda_i = 0 \quad \text{in } S, \quad (2.45)$$

in the viscosity sense, and such that

$$w_i(\sigma) = -\frac{1}{\gamma} \ln \rho(\sigma) (1 + o(1)) \quad \text{as } \rho(\sigma) \rightarrow 0. \quad (2.46)$$

We can assume that  $\lambda_1 > \lambda_2$ . For  $\epsilon > 0$  let  $v = (1 + \epsilon)w_2$ . Then

$$-\frac{1}{2} \nabla' |\nabla' v|^2 \cdot \nabla' v + (1 + \epsilon)^{-1} \gamma |\nabla' v|^4 + (1 + \epsilon)(2\gamma + 1) |\nabla' v|^2 + (1 + \epsilon)^3 \lambda_2 = 0 \quad \text{in } S. \quad (2.47)$$

For  $X > 0$ , we put

$$f(X) = \frac{\gamma \epsilon}{1 + \epsilon} X^2 - (2\gamma + 1) \epsilon X + \lambda_1 - (1 + \epsilon)^3 \lambda_2.$$

Then

$$f(X) \geq f(X_0) = f\left(\frac{(2\gamma + 1)(1 + \epsilon)}{2\gamma}\right) = -\frac{\epsilon(1 + \epsilon)(2\gamma + 1)^2}{4\gamma} + \lambda_1 - (1 + \epsilon)^3 \lambda_2.$$

Therefore there exists  $\epsilon_0 > 0$  such that for any  $X \geq 0$ ,  $f(X) \geq 0$ , or equivalently

$$(1 + \epsilon)^{-1} \gamma X^2 + (1 + \epsilon)(2\gamma + 1)X + (1 + \epsilon)^3 \lambda_2 \leq \gamma X^2 + (2\gamma + 1)X + \lambda_1. \quad (2.48)$$

This implies that

$$-\frac{1}{2} \nabla' |\nabla' v|^2 \cdot \nabla' v + \gamma |\nabla' v|^4 + (2\gamma + 1) |\nabla' v|^2 + \lambda_1 \geq 0 \quad \text{in } S, \quad (2.49)$$

in the viscosity sense. Since  $w_1 < v$  near  $\partial S$ , it follows from comparison principle that  $w_1 < v$  in  $S$ . Letting  $\epsilon \rightarrow 0$  yields

$$w_1 \leq w_2 \quad \text{in } S. \quad (2.50)$$

Since for any  $k \in \mathbb{R}$ ,  $w_1 + k$  satisfies the same equation as  $w_1$  and the same estimate (2.46) as the  $w_i$  we obtain a contradiction. Thus  $\lambda = \lambda(\gamma)$  is uniquely determined.

For proving monotonicity, assume  $\gamma_1 > \gamma_2 > 0$  and let  $w_{\epsilon,1}$  and  $w_{\epsilon,2}$  be solutions of

$$-\frac{1}{2}\nabla'|\nabla'w_{\epsilon,i}|^2 \cdot \nabla'w_{\epsilon,i} + \gamma_i|\nabla'w_{\epsilon,i}|^4 + (2\gamma_i + 1)|\nabla'w_{\epsilon,i}|^2 + \epsilon w_{\epsilon,i} = 0 \quad \text{in } S, \quad (2.51)$$

such that

$$w_{\epsilon,i}(\sigma) = -\frac{1}{\gamma_i} \ln \rho(\sigma) (1 + o(1)) \quad \text{as } \rho(\sigma) \rightarrow 0. \quad (2.52)$$

Then

$$-\frac{1}{2}\nabla'|\nabla'w_{\epsilon,1}|^2 \cdot \nabla'w_{\epsilon,1} + \gamma_2|\nabla'w_{\epsilon,1}|^4 + (2\gamma_2 + 1)|\nabla'w_{\epsilon,1}|^2 + \epsilon w_{\epsilon,1} \leq 0.$$

Since  $w_{\epsilon,1} \leq w_{\epsilon,2}$  near  $\partial S$ , it follows by comparison principle that  $w_{\epsilon,1} \leq w_{\epsilon,2}$  in  $S$  and in particular  $\epsilon w_{\epsilon,1} \leq \epsilon w_{\epsilon,2}$ . Since  $\lambda_1 = \lim_{n \rightarrow \infty} \epsilon w_{\epsilon_n,1}(x_0)$  and  $\lambda_2 = \lim_{n \rightarrow \infty} \epsilon w_{\epsilon_n,2}(x_0)$ , we infer that  $\lambda_1 \leq \lambda_2$ .

For proving the continuity, let  $\{\gamma_n\}$  be a sequence converging to  $\gamma$  and let  $w_n$  be corresponding solutions of

$$-\frac{1}{2}\nabla'|\nabla'w_n|^2 \cdot \nabla'w_n + \gamma_n|\nabla'w_n|^4 + (2\gamma_n + 1)|\nabla'w_n|^2 + \lambda(\gamma_n) = 0 \quad \text{in } S, \quad (2.53)$$

subject to

$$\left| w_n(\sigma) + \frac{1}{\gamma_n} \ln \rho(\sigma) \right| \leq K, \quad (2.54)$$

for some  $K > 0$  independent of  $n$ . Since  $\{w_n\}$  is locally bounded in  $W_{loc}^{1,\infty}(\Omega)$  we can extract sequences, denoted by  $\{w_{n_k}\}$ ,  $\{\lambda(w_{n_k})\}$  such that  $\lambda(w_{n_k}) \rightarrow \bar{\lambda}$  and  $w_{n_k}$  converges locally uniformly to a viscosity solution  $w$  of

$$-\frac{1}{2}\nabla'|\nabla'w|^2 \cdot \nabla'w + \gamma|\nabla'w|^4 + (2\gamma + 1)|\nabla'w|^2 + \bar{\lambda} = 0 \quad \text{in } S, \quad (2.55)$$

subject to  $w(\sigma) = -\frac{1}{\gamma} \ln \rho(\sigma) (1 + o(1))$  as  $\rho(\sigma) \rightarrow 0$ . The existence of such a function implies that  $\bar{\lambda} = \lambda(\gamma)$ . Thus the whole sequence  $\{\lambda(\gamma_n)\}$  converges to  $\lambda(\gamma)$ , a fact which implies the continuity.

Next, let  $S_1 \subset S_2$  be two  $C^3$  subdomains of  $S^{N-1}$ . We denote by  $w_{\delta,\epsilon,\gamma,S_j}$ ,  $j = 1, 2$ , the solutions of (2.5) respectively in  $S_1$  and  $S_2$ . Since these solutions are limit of solutions with finite boundary values and that the maximum principle holds, we infer that  $w_{\delta,\epsilon,\gamma,S_2} \leq w_{\delta,\epsilon,\gamma,S_1}$  in  $S_1$ . Letting  $\delta \rightarrow 0$  yields  $w_{\epsilon,\gamma,S_2} \leq w_{\epsilon,\gamma,S_1}$ . Taking  $\sigma_0 \in S_1$  and since the ergodic constant is uniquely determined, we have  $\epsilon w_{\epsilon,\gamma,S_2}(\sigma_0) \leq \epsilon w_{\epsilon,\gamma,S_1}(\sigma_0)$  and thus  $\lambda(\gamma, S_2) \leq \lambda(\gamma, S_1)$ .  $\square$

## 2.4 Proof of Theorem B

We prove below the following proposition using the result of Theorem C, which proof does not depend on the previous constructions.

**Proposition 2.6.** *For any  $C^3$  domain  $S \subset S^{N-1}$ , there exists a unique  $\beta := \beta_s$  such that  $\lambda(\beta) = \beta + 1$ . Furthermore  $\beta_s$  is a decreasing function of  $S$  for the order relation between spherical domains.*

*Proof.* The function  $\gamma \mapsto \lambda(\gamma, S) - \gamma$  is continuous and decreasing. For  $\epsilon > 0$  we consider two spherical caps  $S_i \subset S \subset S_e$ ; by Proposition 2.5

$$\lambda(\gamma, S_e) \leq \lambda(\gamma, S) \leq \lambda(\gamma, S_i), \quad (2.56)$$

then

$$\lambda(\gamma, S_e) - \gamma \leq \lambda(\gamma, S) - \gamma \leq \lambda(\gamma, S_i) - \gamma. \quad (2.57)$$

By Theorem C, there exists  $\gamma = \beta_{s_e}$  and  $\gamma = \beta_{s_i}$  such that

$$\lambda(\beta_{s_e}, S_e) - \beta_{s_e} = 1 \quad \text{and} \quad \lambda(\beta_{s_i}, S_i) - \beta_{s_i} = 1,$$

and  $\beta_{s_e} < \beta_{s_i}$  unless  $\lambda(\beta_{s_e}, S_e) = \lambda(\beta_{s_i}, S_i)$  and  $S_i = S_e$ . This implies that

$$\lambda(\beta_{s_e}, S) - \beta_{s_e} \geq 1 \quad \text{and} \quad \lambda(\beta_{s_i}, S) - \beta_{s_i} \leq 1. \quad (2.58)$$

By continuity there exists a unique  $\beta = \beta_s \in [\beta_{s_e}, \beta_{s_i}]$  such that  $\lambda(\beta_s, S) - \beta_s = 1$ . To this exponent  $\beta$  corresponds a locally Lipschitz continuous function  $w$  solution of problem (2.4). Then  $\psi = e^{-\beta w}$  is a viscosity solution of (2.2). Notice also that the construction of  $\beta_s$  and the monotonicity of  $S \mapsto \lambda(\gamma, S)$  imply that  $S \mapsto \beta_s$  is decreasing.

Similarly we can consider separable infinity harmonic functions under the form (1.2) with negative  $\beta < 0$ . We set  $\tilde{\beta} = -\beta$ , then (2.2) is replaced by

$$\begin{aligned} -\frac{1}{2} \nabla' |\nabla' \psi|^2 \cdot \nabla' \psi &= \mu(2\mu - 1) |\nabla' \psi|^2 w + \mu^3 (\mu - 1) \psi^3 && \text{in } S \\ \psi &= 0 && \text{in } \partial S. \end{aligned} \quad (2.59)$$

If  $\psi$  is a positive solution of (2.59), we set

$$w = -\frac{1}{\mu} \ln \psi.$$

Then  $w$  satisfies

$$\begin{aligned} -\frac{1}{2} \nabla' |\nabla' w|^2 \cdot \nabla' w + \mu |\nabla' w|^4 + (2\mu - 1) |\nabla' w|^2 &= \mu - 1 && \text{in } S \\ \lim_{\rho(\sigma) \rightarrow 0} w(\sigma) &= \infty, \end{aligned} \quad (2.60)$$

This equation is treated similarly as (2.4). □

*Remark.* It is an open problem whether the positive functions which satisfy (2.2) are unique up to the multiplication by a constant. This is a sharp contrast with the spherical  $p$ -harmonic problem with  $1 < p < \infty$  where uniqueness is proved by the strong maximum principle and Hopf boundary lemma, and this uniqueness result has been extended to Lipschitz domain in [7] using the characterization of the  $p$ -Martin boundary obtained by [14], and a sharp version of boundary Harnack principle. See also Section 3.3 for related results. Notice that uniqueness holds when the solutions are spherically radial (see Section 3.2).



### 3 The general case

#### 3.1 Problem on the circle

We consider here the special case  $N = 2$  and  $S$  is the circle in (2.2). For  $k \in \mathbb{N}_*$ , we set

$$\beta_k = \frac{k^2}{2k+1} \quad \text{and} \quad \mu_k = \frac{k^2}{2k-1}. \quad (3.1)$$

**Proposition 3.1.** *For any  $k \in \mathbb{N}^*$  there exists two  $\frac{\pi}{k}$ -anti-periodic  $C^1$  functions  $\psi_k$  and  $\omega_k$  positive on  $(0, \frac{\pi}{k})$  such that  $x \mapsto |x|^{-\beta_k} \psi_k(\frac{x}{|x|})$  is infinity harmonic and singular in  $\mathbb{R}^2 \setminus \{0\}$  and  $x \mapsto |x|^{\mu_k} \omega_k(\frac{x}{|x|})$  is infinity harmonic and regular in  $\mathbb{R}^2$ .*

*Proof.* We write  $\nabla' \psi = \psi_\sigma e^\perp$  with  $S^1 \sim \mathbb{R}/2\pi$ . Thus (2.2) becomes

$$-\psi_\sigma^2 \psi_{\sigma\sigma} = \beta^3(\beta+1)\psi^3 + \beta(2\beta+1)\psi_\sigma^2 \psi, \quad \psi(0) = \psi(2\pi). \quad (3.2)$$

For  $\psi \neq 0$  the equation in (3.2) can be written as

$$-\frac{\psi_\sigma^2}{\psi^2} \frac{\psi_{\sigma\sigma}}{\psi} = \beta(2\beta+1) \frac{\psi_\sigma^2}{\psi^2} + \beta^3(\beta+1).$$

We set  $Y = \frac{\psi_\sigma}{\psi}$ , then  $Y_\sigma + Y^2 = \frac{\psi_{\sigma\sigma}}{\psi}$  and

$$-Y^2 Y_\sigma = Y^4 + \beta(2\beta+1)Y^2 + \beta^3(\beta+1) = (Y^2 + \beta^2)(Y^2 + \beta(\beta+1)). \quad (3.3)$$

We first search for solutions with  $\beta$  such that  $\beta(\beta+1) \geq 0$ ,  $\beta \neq 0$ . Standard computation yields

$$\left( \frac{\beta}{Y^2 + \beta^2} - \frac{\beta+1}{Y^2 + \beta(\beta+1)} \right) Y_\sigma = 1, \quad (3.4)$$

and this equation is not degenerate and equivalent to (3.2) as long as  $Y \neq 0$ . If  $\beta = -1$  (that is  $\tilde{\beta}_1$  in (3.1)) then (3.4) becomes

$$\frac{Y_\sigma}{Y^2 + 1} = -1, \quad (3.5)$$

thus

$$\tan^{-1} Y(\sigma) = -\sigma \implies Y(\sigma) = -\tan \sigma \implies \psi(\sigma) = \sin \sigma. \quad (3.6)$$

This corresponds to the fact that the coordinate functions are separable and infinity harmonic.

We assume now  $\beta(\beta+1) > 0$ , or equivalently either  $\beta > 0$  or  $\beta < -1$ . We fix  $\psi(0) = 0$  and consider an interval on the right of 0 where  $\psi > 0$ . From the equation  $\psi$  is concave, thus  $\psi_\sigma(0) > 0$ . Because of concavity and periodicity  $\psi$  must change sign. We assume that  $\sigma = \alpha$  is the first critical point of  $\psi$  which is a singular point for (3.2) and (3.4). We integrate (3.4) on a small interval  $(\alpha, \sigma)$  and get

$$\tan^{-1} \left( \frac{Y(\sigma)}{\beta} \right) - \sqrt{\frac{\beta+1}{\beta}} \tan^{-1} \left( \frac{Y(\sigma)}{\sqrt{\beta(\beta+1)}} \right) = \sigma - \alpha.$$

Expanding  $\tan^{-1}(x)$  near  $x = 0$  we obtain

$$Y(\sigma) = -\beta \sqrt[3]{3\beta + 3} \sqrt[3]{\sigma - \alpha} (1 + o(1)) \implies \psi(\sigma) = \psi(\alpha) - C(\beta)\psi(\alpha)(\sigma - \alpha)^{\frac{4}{3}} (1 + o(1)), \quad (3.7)$$

with  $C(\beta) = \frac{3^{\frac{4}{3}}}{4} \beta \sqrt[3]{\beta + 1}$ , and define  $Y(\sigma)$  for  $\sigma \in (\alpha, 2\alpha)$  by imposing  $Y(\sigma) = -Y(2\alpha - \sigma)$  and continue this process in order to construct a  $2\alpha$ -antiperiodic solution belonging to  $C^{1, \frac{1}{3}}(\mathbb{R})$ . Since

$$\left[ \tan^{-1} \left( \frac{Y}{\beta} \right) - \sqrt{\frac{\beta + 1}{\beta}} \tan^{-1} \left( \frac{Y}{\sqrt{\beta(\beta + 1)}} \right) \right]_{\sigma=0}^{\sigma=\alpha} = \alpha,$$

with  $Y(0) = \infty$ ,  $Y(\alpha) = 0$ , the condition for  $\pi$ -antiperiodicity is therefore

$$\left( \sqrt{\frac{\beta + 1}{\beta}} - 1 \right) \frac{\pi}{2} = \alpha \iff \beta = \frac{\pi^2}{4(\alpha^2 + \alpha\pi)}.$$

If  $\beta > 0$ , the periodicity condition yields

$$\sqrt{\frac{\beta + 1}{\beta}} = 1 + \frac{1}{k} \iff \beta = \beta_k = \frac{k^2}{1 + 2k}. \quad (3.8)$$

If  $\beta < -1$

$$\left( 1 - \sqrt{\frac{\beta + 1}{\beta}} \right) \pi = \alpha,$$

and the periodicity condition implies

$$\sqrt{\frac{\beta + 1}{\beta}} = 1 - \frac{1}{k} \iff \beta = \tilde{\beta}_k = \frac{k^2}{1 - 2k}. \quad (3.9)$$

The case  $\beta(\beta + 1) < 0$ , or equivalently  $-1 < \beta < 0$ , is easily ruled out. We find that (3.3) has the constant solution  $Y(\sigma) = \sqrt{-\beta(\beta + 1)}$ , meaning  $\psi(\sigma) = C \exp(\sigma \sqrt{-\beta(\beta + 1)})$ , which is by no mean periodic. On the other hand, in this case we can write (3.4) under the form

$$\frac{d}{d\sigma} \left( \tan^{-1} \left( \frac{Y}{\beta} \right) - \frac{1}{2} \sqrt{\frac{\beta + 1}{-\beta}} \ln \left( \frac{|Y - \sqrt{-\beta(\beta + 1)}|}{Y + \sqrt{-\beta(\beta + 1)}} \right) \right) = 1. \quad (3.10)$$

Since  $Y$  runs from  $Y(0) = \infty$  to  $Y(\alpha) = 0$ , there must be a value  $\sigma_0$  where  $Y(\sigma_0) = \sqrt{-\beta(\beta + 1)}$ . We can integrate (3.10) on  $(0, \sigma_0 - \epsilon)$  and let  $\epsilon \rightarrow 0$ . Since  $\beta < 0$ , it yields

$$\frac{\pi}{2} + \tan^{-1} \left( \frac{Y(\sigma_0 - \epsilon)}{\beta} \right) - \frac{1}{2} \sqrt{\frac{\beta + 1}{-\beta}} \ln \left( \frac{|Y(\sigma_0 - \epsilon) - \sqrt{-\beta(\beta + 1)}|}{Y(\sigma_0 - \epsilon) + \sqrt{-\beta(\beta + 1)}} \right) = \sigma_0 - \epsilon.$$

The left-hand side expression tends to  $\infty$  when  $\epsilon \rightarrow 0$ , a contradiction. Hence there are no solutions with  $\beta \in (-1, 0)$ . This ends the proof of the proposition.  $\square$

*Remark.* When  $k = 1$  the coordinate functions are infinity harmonic and vanish on a straight line. When  $k = 2$ , the regular solution with  $\mu_1 = \frac{4}{3}$  is

$$u(x, y) = x^{\frac{4}{3}} - y^{\frac{4}{3}}.$$

Its existence is due to Aronsson [2]. The corresponding circular function,  $\omega(\sigma) = (\cos \sigma)^{\frac{4}{3}} - (\sin \sigma)^{\frac{4}{3}}$ , admits four nodal sets on  $S^1$ . When  $k = 1$ , then  $\beta_1 = \frac{1}{3}$ . It is proved in [4] that any positive infinity harmonic function in a half-space which vanishes on the boundary except at one point blows-up like the separable infinity harmonic function  $u(r, \sigma) = r^{-\frac{1}{3}}\psi(\sigma)$ .

### 3.2 The spherical cap problem

*Proof of Theorem C.* The following representation of  $S^{N-1}$  is classical

$$S^{N-1} = \{ \sigma = (\sin \phi \sigma', \cos \phi) : \sigma' \in S^{N-2}, \phi \in [0, \pi] \}.$$

Then  $\nabla' \psi = \psi_\phi \mathbf{e} + \nabla'_{\sigma'} \psi$  where  $\mathbf{e}$  is a tangent unit downward vector to  $S^{N-1}$  following the great circle going through the point  $\sigma$ . Then  $|\nabla' \psi|^2 = \psi_\phi^2 + |\nabla'_{\sigma'} \psi|^2$ , thus, if  $\psi$  depends only on  $\phi$ , we have

$$\frac{1}{2} \nabla' |\nabla' \psi|^2 \cdot \nabla' \psi = \psi_{\phi\phi} \psi_\phi^2.$$

Therefore such a function  $\psi$ , if it is a  $C^1$  solution of (2.2) in the spherical cap  $S_\alpha$  defined for  $\phi \in (0, \alpha)$ , satisfies

$$\begin{aligned} -\psi_{\phi\phi} \psi_\phi^2 &= \beta(2\beta + 1) \psi_\phi^2 \psi_\phi + \beta^3(\beta + 1) \psi \quad \text{in } (0, \alpha) \\ \psi_\phi(0) &= 0, \quad \psi(\alpha) = 0. \end{aligned} \quad (3.11)$$

The conclusion follows from Proposition 3.1.  $\square$

*Remark.* If  $\alpha = \pi$ , the exponent  $\beta_+$  is  $\frac{1}{8}$  and  $\psi := \psi_{\Sigma^c}$  is a positive solution of

$$\begin{aligned} -\psi_\sigma^2 \psi_{\sigma\sigma} &= \frac{9}{4096} \psi^3 + \frac{5}{32} \psi_\sigma^2 \psi \quad \text{in } (-\pi, \pi) \\ \psi(-\pi) &= \psi(\pi) = 0. \end{aligned} \quad (3.12)$$

Then the function  $u_{\Sigma^c}(r, \sigma) = r^{-\frac{1}{8}} \psi_{\Sigma^c}(\sigma)$  is an infinity harmonic function in  $\mathbb{R}^N \setminus L_{\Sigma^c}$ , which vanishes on the half line  $L_{\Sigma^c} := \{x = t\Sigma : t \geq 0\}$ . The function  $Y = \frac{\psi_\sigma}{\psi}$  can be computed implicitly on  $(0, \pi)$  thanks to the identity

$$\tan^{-1}(8Y(\sigma)) - 3 \tan^{-1}\left(\frac{8}{3}Y(\sigma)\right) = \sigma. \quad (3.13)$$

This yields, with  $Z = \frac{8Y}{3}$ ,  $\tan^{-1}(3Z(\sigma)) - 3 \tan^{-1}(Z(\sigma)) = \sigma$ , hence

$$\frac{3Z - \tan(3 \tan^{-1}(Z))}{1 + 3Z \tan(3 \tan^{-1}(Z))} = \tan \sigma, \quad (3.14)$$

since

$$\tan(3x) = \frac{3 \tan x - \tan^3 x}{1 - 3 \tan^2 x}.$$

This yields

$$\frac{-8Z^3}{1+6Z^2-3Z^4} = \tan \sigma, \quad (3.15)$$

which gives the value of  $Y$  by solving a fourth degree equation and then  $\psi = \psi_{\Sigma^c}$  by integrating  $Y$ .

Using Theorem C we can prove the existence of a singular infinity harmonic function in a cone  $C_{S_{\kappa,\alpha}}$  generated by a spherical annulus  $S_{\kappa,\alpha}$  of the spherical points with azimuthal angle  $\kappa < \phi < \alpha$ .

**Proposition 3.2.** *Assume  $0 \leq \kappa < \alpha < \pi$  and let  $\nu = \frac{1}{2}(\alpha - \kappa)$ . Then there exists a positive singular infinity harmonic function  $u_{S_{\kappa,\alpha},+}$  and a regular infinity harmonic function  $u_{S_{\kappa,\alpha},-}$  in  $C_{S_{\kappa,\alpha}}$  which vanish respectively on  $\partial C_{S_{\kappa,\alpha}} \setminus \{0\}$  and  $\partial C_{S_{\kappa,\alpha}}$  under the form  $u_{S_{\kappa,\alpha},+}(r, \sigma) = r^{-\beta_{S_{\kappa,\alpha}}} \psi_{S_{\kappa,\alpha}}(\sigma)$  and  $u_{S_{\kappa,\alpha},-}(r, \sigma) = r^{\mu_{S_{\kappa,\alpha}}} \omega_{S_{\kappa,\alpha}}(\sigma)$  where*

$$\beta_{S_{\kappa,\alpha}} = \frac{\pi^2}{4\nu(\pi + \nu)} \quad \text{and} \quad \omega_{S_{\kappa,\alpha}} = \frac{\pi^2}{4\nu(\pi - \nu)}, \quad (3.16)$$

and  $\psi_{S_{\kappa,\alpha}}$  and  $\omega_{S_{\kappa,\alpha}}$  are positive solutions of (3.2) in  $S_{\kappa,\alpha}$  vanishing at  $\kappa$  and  $\alpha$  with  $\beta = \beta_{S_{\kappa,\alpha}}$  and  $\beta = -\mu_{S_{\kappa,\alpha}}$  respectively.

*Proof.* By Theorem C there exists a positive and even solution  $\tilde{\psi}$  of

$$\begin{aligned} \tilde{\psi}_{\phi\phi} \psi_{\phi}^2 &= \beta(2\beta + 1) \tilde{\psi}_{\phi}^2 \tilde{\psi}_{\phi} + \beta^3(\beta + 1) \tilde{\psi} \quad \text{in } (-\nu, \nu) := \left(\frac{1}{2}(\kappa - \alpha), \frac{1}{2}(\alpha - \kappa)\right) \\ \tilde{\psi}(-\nu) &= 0, \quad \tilde{\psi}(\nu) = 0, \end{aligned} \quad (3.17)$$

with  $\beta = \beta_{S_{\kappa,\alpha}}$  or  $\beta = -\mu_{S_{\kappa,\alpha}}$ . Then  $\phi \mapsto \psi(\phi) := \tilde{\psi}(\phi + \frac{1}{2}(\kappa + \alpha))$  is a positive solution of (3.2) in  $(\kappa, \alpha)$ . The proof follows.  $\square$

The next technical lemma is a variant of Theorem C and Proposition 3.2.

**Lemma 3.3.** *Assume  $0 < \alpha < \pi$  and  $\epsilon, \gamma > 0$ . Then the solution  $v = v_{\epsilon,\gamma,\alpha}$  of*

$$\begin{aligned} -v'^2 v'' + \gamma v'^4 + (2\gamma + 1)v'^2 + \epsilon v &= 0 \quad \text{in } (0, \alpha) \\ v(0) &= \infty, \quad v(\alpha) = \infty, \end{aligned} \quad (3.18)$$

is an increasing function of  $\epsilon$ . If  $0 < \sigma_0 < \alpha$ , there exists  $\lambda = \lambda(\alpha, \gamma) = \lim_{\epsilon \rightarrow 0} \epsilon v_{\epsilon,\gamma,\alpha}(\sigma_0)$  and this value is independent of  $\sigma_0$ . The function  $\tilde{v} = \tilde{v}_{\epsilon,\gamma,\alpha} = v_{\epsilon,\gamma,\alpha} - v_{\epsilon,\gamma,\alpha}(\sigma_0)$  converges locally uniformly in  $(0, \alpha)$  to a solution  $v = v_{\gamma,\alpha}$  of

$$\begin{aligned} -v'^2 v'' + \gamma v'^4 + (2\gamma + 1)v'^2 + \lambda &= 0 \quad \text{in } (0, \alpha) \\ v(0) &= \infty, \quad v(\alpha) = \infty, \end{aligned} \quad (3.19)$$

with

$$\lambda(\alpha, \gamma) = \frac{1}{4\gamma^3} \left( \frac{\pi^2}{\alpha^2} - \gamma(2\gamma + 1) \right)^2. \quad (3.20)$$

Furthermore

$$v_{\gamma,\alpha}(\phi) = -\frac{1}{\gamma} \ln \phi \quad \text{as } \phi \rightarrow 0, \quad (3.21)$$

and

$$v_{\gamma,\alpha}(\phi) = -\frac{1}{\gamma} \ln(\alpha - \phi) \quad \text{as } \phi \rightarrow \alpha. \quad (3.22)$$

*Proof.* From the proof of Theorem A, we know that  $v_{\epsilon,\gamma,\alpha}$  is an increasing function of  $\epsilon$ . It satisfies estimates (2.37) and (2.35). Furthermore there exists  $\lambda = \lambda(\alpha, \gamma) = \lim_{\epsilon \rightarrow 0} \epsilon v_{\epsilon,\gamma,\alpha}(\sigma_0) \geq 0$ , which is a value independent of  $\sigma_0 \in (0, \alpha)$ , and  $\tilde{v} = \tilde{v}_{\epsilon,\gamma,\alpha} = v_{\epsilon,\gamma,\alpha} - v_{\epsilon,\gamma,\alpha}(\sigma_0)$  converges locally uniformly in  $(0, \alpha)$  to a solution  $v = v_{\gamma,\alpha}$  of (3.19). We set  $Y = -\gamma v'$ , then

$$\begin{aligned} Y^2 Y' + Y^4 + \gamma(2\gamma + 1)Y^2 + \lambda\gamma^3 &= 0 \quad \text{in } (0, \alpha) \\ Y(0) &= \infty, \quad Y(\alpha) = -\infty. \end{aligned} \quad (3.23)$$

We write it under the separable form

$$\left( \frac{Y^2}{Y^4 + \gamma(2\gamma + 1)Y^2 + \lambda\gamma^3} \right) Y' = -1 \Leftrightarrow \left( \frac{A^2}{A^2 - B^2} \frac{1}{Y^2 + A^2} - \frac{B^2}{A^2 - B^2} \frac{1}{Y^2 + B^2} \right) Y' = -1,$$

for some  $A, B > 0$  and with  $A > B$  if we assume  $2\gamma + 1 > 2\gamma\lambda$ . Actually  $A^2 B^2 = \lambda\gamma^3$  and  $A^2 + B^2 = \gamma(2\gamma + 1)$ . Thus

$$\left( A \tan^{-1} \left( \frac{Y}{A} \right) - B \tan^{-1} \left( \frac{Y}{B} \right) \right)' = B^2 - A^2. \quad (3.24)$$

By integration on  $(0, \alpha)$  we derive the identity

$$A + B = \frac{\pi}{\alpha}. \quad (3.25)$$

Since  $A + B = \sqrt{\gamma(2\gamma + 1) + 2\sqrt{\lambda\gamma^3}}$ , we deduce (3.20) from (3.25). Finally, since

$$\tan^{-1} z = \frac{\pi}{2} - \frac{1}{z} + \frac{1}{3z^3} + O(z^{-5}) \quad \text{when } z \rightarrow \infty,$$

we derive

$$-\frac{1}{\gamma v'(\phi)} = \frac{1}{Y(\phi)} = \phi + O(\phi^3) \quad \text{when } \phi \rightarrow 0$$

from (3.24), which implies (3.21) by l'Hospital rule. Relation (3.22) is proved similarly.  $\square$

Next we denote by  $S_\alpha(a)$  the spherical cap with vertex  $a \in S^{N-1}$  and azimuthal opening  $\alpha$  from  $a$  and  $S_\alpha^*(a) = S_\alpha(a) \setminus \{a\}$ . The next statement is a rephrasing of Lemma 3.3 in a geometric framework.

**Corollary 3.4.** *Let  $\alpha, \epsilon$  and  $\gamma > 0$  be as in Lemma 3.3 and  $a \in S^{N-1}$ . Then there exists a unique solution  $w = w_{a,\alpha,\gamma,\epsilon}$  of*

$$\begin{aligned} -\frac{1}{2} \nabla' |\nabla' w|^2 \cdot \nabla' w + \gamma |\nabla' w|^4 + (2\gamma + 1) |\nabla' w|^2 + \epsilon w &= 0 \quad \text{in } S_\alpha^*(a) \\ \lim_{\ell(\sigma,a) \rightarrow 0} w(\sigma) &= \infty \\ \lim_{\ell(\sigma,a) \rightarrow \alpha} w(\sigma) &= \infty, \end{aligned} \quad (3.26)$$

rotationally invariant with respect to  $a$ . If  $a$  is replaced by  $a' \in S^{N-1}$ , the solution  $w_{a',\alpha,\gamma,\epsilon}$  of (3.26) in  $S_\alpha^*(a')$  is derived from  $w_{a,\alpha,\gamma,\epsilon}$  by an orthogonal transformation exchanging  $a$  and  $a'$ . The mapping  $\epsilon \mapsto w_{a,\alpha,\gamma,\epsilon}$  is decreasing and for any  $\sigma_0 \in S_\alpha^*(a)$

$$\lim_{\epsilon \rightarrow 0} \epsilon w_{a,\alpha,\gamma,\epsilon}(\sigma_0) = \lambda(\gamma, S_\alpha^*(a)) := \lambda(\gamma, S_\alpha^*), \quad (3.27)$$

(this notation is coherent with  $\lambda(\gamma, S)$  already used, furthermore its value does not depend on  $a$ ). The function  $\tilde{w}_{a,\alpha,\gamma,\epsilon} = w_{a,\alpha,\gamma,\epsilon} - w_{a,\alpha,\gamma,\epsilon}(\sigma_0)$  converges locally uniformly in  $S_\alpha^*(a')$  to the unique viscosity solution  $w := w_{a,\alpha,\gamma}$  rotationally invariant with respect to  $a$  and vanishing at  $\sigma_0$  of

$$\begin{aligned} -\frac{1}{2}\nabla'|\nabla'w|^2 \cdot \nabla'w + \gamma|\nabla'w|^4 + (2\gamma + 1)|\nabla'w|^2 + \lambda(\gamma, S_\alpha^*) &= 0 \quad \text{in } S_\alpha^*(a) \\ \lim_{\ell(\sigma,a) \rightarrow 0} w(\sigma) &= \infty \\ \lim_{\ell(\sigma,a) \rightarrow \alpha} w(\sigma) &= \infty. \end{aligned} \quad (3.28)$$

Finally

$$w_{a,\alpha,\gamma}(\sigma) = -\frac{1}{\gamma} \ln(\ell(a, \sigma)) + O(1) \quad \text{as } \sigma \rightarrow a, \quad (3.29)$$

and

$$w_{a,\alpha,\gamma}(\sigma) = -\frac{1}{\gamma} \ln(\alpha - \ell(a, \sigma)) + O(1) \quad \text{as } \ell(\sigma, a) \rightarrow 0. \quad (3.30)$$

The following statement is formally similar to Corollary 3.4. It makes more precise the approximations used in the proof of Theorem A, in the construction of the proof of Theorem B in the case  $\alpha = \pi$ .

**Corollary 3.5.** *Let  $\epsilon$  and  $\gamma > 0$  and  $a \in S^{N-1}$ . Then there exists a unique rotationally invariant with respect to  $a$  solution  $v = v_{a,\gamma,\epsilon}$  of*

$$\begin{aligned} -\frac{1}{2}\nabla'|\nabla'v|^2 \cdot \nabla'v + \gamma|\nabla'v|^4 + (2\gamma + 1)|\nabla'v|^2 + \epsilon v &= 0 \quad \text{in } S^{N-1} \setminus \{a\} \\ \lim_{\ell(\sigma,a) \rightarrow 0} v(\sigma) &= \infty. \end{aligned} \quad (3.31)$$

Furthermore, for any  $\sigma_0 \in S^{N-1} \setminus \{a\}$ ,

$$\lim_{\epsilon \rightarrow 0} \epsilon v_{a,\gamma,\epsilon}(\sigma_0) = \Lambda(\gamma) := \frac{1}{4\gamma^3} \left( \frac{1}{4} - \gamma(2\gamma + 1) \right)^2. \quad (3.32)$$

The function  $\tilde{v}_{a,\gamma,\epsilon} = v_{a,\gamma,\epsilon} - v_{a,\gamma,\epsilon}(\sigma_0)$  converges locally uniformly in  $S^{N-1} \setminus \{a\}$  to the unique viscosity solution  $v := v_{a,\gamma}$  rotationally invariant with respect to  $a$  and vanishing at  $\sigma_0$  of

$$\begin{aligned} -\frac{1}{2}\nabla'|\nabla'v|^2 \cdot \nabla'v + \gamma|\nabla'v|^4 + (2\gamma + 1)|\nabla'v|^2 + \Lambda(\gamma) &= 0 \quad \text{in } S^{N-1} \setminus \{a\} \\ \lim_{\ell(\sigma,a) \rightarrow 0} v(\sigma) &= \infty. \end{aligned} \quad (3.33)$$

Finally

$$v_{a,\gamma}(\sigma) = -\frac{1}{\gamma} \ln(\ell(a, \sigma)) + O(1) \quad \text{as } \sigma \rightarrow a. \quad (3.34)$$

As in Corollary 3.4, if  $a$  is replaced by  $a' \in S^{N-1}$ , the solution  $v_{a',\gamma,\epsilon}$  of (3.31) in  $S^{N-1} \setminus \{a'\}$  is derived from  $v_{a,\gamma,\epsilon}$  by an orthogonal transformation exchanging  $a$  and  $a'$ . The mapping  $\epsilon \mapsto v_{a,\gamma,\epsilon}$  is decreasing.

### 3.3 Proof of Theorem D

*Step 1: Approximate solutions.* We consider an increasing sequence of smooth spherical domains,  $\{S_k\}$  such that

$$S_k \subset \bar{S}_k \subset S_{k+1} \subset S \text{ and } \bigcup_k S_k = S,$$

To each domain we associate the positive exponent  $\beta_k := \beta_{S_k}$  and the corresponding spherical infinity-harmonic function  $\psi_k := \psi_{S_k}$  defined in  $S_k$  and such that  $\psi_k(\sigma_0) = 1$  for some  $\sigma_0 \in S_1$ , so that the function  $u_k(r, \cdot) = r^{-\beta_k} \psi_k$  is infinity-harmonic in the cone  $C_{S_k}$  and vanishes on  $\partial C_{S_k} \setminus \{0\}$ . For  $\gamma, \delta, \epsilon > 0$ , we denote by  $w_{k,\gamma,\delta,\epsilon}$  the solution of

$$\begin{aligned} -\delta \Delta w - \frac{1}{2} \nabla' |\nabla' w|^2 \cdot \nabla' w + \gamma |\nabla' w|^4 + (2\gamma + 1) |\nabla' w|^2 + \epsilon w = 0 \quad \text{in } S_k \\ \lim_{\rho_k(\sigma) \rightarrow 0} w(\sigma) = \infty, \end{aligned} \quad (3.35)$$

where  $\rho_k(\cdot) = \text{dist}(\cdot, \partial S_k)$ . By the maximum principle the functions  $w_{k,\gamma,\delta,\epsilon}$  is positive and the following comparison relations hold:

$$\begin{aligned} (i) \quad & w_{\ell,\gamma,\delta,\epsilon} \leq w_{k,\gamma,\delta,\epsilon} \quad \text{in } S_k \quad \forall k \leq \ell, \\ (ii) \quad & w_{k,\gamma,\delta,\epsilon} \leq w_{k,\gamma,\delta,\epsilon'} \quad \text{in } S_k \quad \forall \epsilon' \leq \epsilon, \\ (iii) \quad & w_{k,\gamma,\delta,\epsilon} \leq w_{k,\gamma',\delta,\epsilon} \quad \text{in } S_k \quad \forall \gamma' \leq \gamma. \end{aligned} \quad (3.36)$$

Furthermore it follows from Corollary 2.4,

$$|\nabla w_{k,\gamma,\delta,\epsilon}(\sigma)| \leq \frac{c}{\rho_k(\sigma)} \quad \forall \sigma \in S_k, \quad (3.37)$$

where  $c = c(N)$ . Moreover, similarly as in (2.37),

$$-\frac{1}{\gamma} \ln \rho_k(\sigma) - \frac{M_k}{\epsilon} \leq w_{k,\gamma,\delta,\epsilon}(\sigma) \leq -\frac{1}{\gamma} \ln \rho_k(\sigma) + \frac{M_k}{\epsilon} \quad \forall \sigma \in S_k. \quad (3.38)$$

We let  $\delta \rightarrow 0$  and derive that, up to a subsequence,  $w_{k,\gamma,\delta_n,\epsilon} \rightarrow w_{k,\gamma,\epsilon}$  locally uniformly in  $S_k$ . The function  $w_{k,\gamma,\epsilon}$  satisfy (3.36), (3.37) and (3.38) and is a viscosity solution of

$$\begin{aligned} -\frac{1}{2} \nabla' |\nabla' w|^2 \cdot \nabla' w + \gamma |\nabla' w|^4 + (2\gamma + 1) |\nabla' w|^2 + \epsilon w = 0 \quad \text{in } S_k \\ \lim_{\rho_k(\sigma) \rightarrow 0} w(\sigma) = \infty. \end{aligned} \quad (3.39)$$

Furthermore the mapping  $(k, \epsilon) \mapsto w_{k,\gamma,\epsilon}$  is nonincreasing, and if we let  $k \rightarrow \infty$ , then  $w_{k,\gamma,\epsilon} \downarrow w_{\gamma,\epsilon}$ . The function  $w_{\gamma,\epsilon}$  is defined in  $S$  and is nonincreasing functions of  $\epsilon$  and  $\gamma$ . Furthermore there holds

$$\begin{aligned} (i) \quad & w_{\gamma,\epsilon} \leq w_{k,\gamma,\epsilon} \leq w_{1,\gamma,\epsilon} \quad \forall k, \ell \geq 1, \\ (ii) \quad & |\nabla w_{\gamma,\epsilon}(\sigma)| \leq \frac{c}{\rho(\sigma)} \quad \forall \sigma \in S, \end{aligned} \quad (3.40)$$

where  $c = c(N)$ . Estimate (3.40)-(i) can be made more precise in the following way: for each  $\sigma \in S$ , there is  $k_\sigma \in \mathbb{N}$  such that  $\sigma \in S_{k_\sigma}$  and

$$-\frac{1}{\gamma} \ln \rho(\sigma) - \frac{1}{\epsilon} \max_{k \geq k_\sigma} M_k \leq w_{\gamma, \epsilon}(\sigma) \leq -\frac{1}{\gamma} \ln \rho(\sigma) + \frac{1}{\epsilon} \min_{k \geq k_\sigma} M_k. \quad (3.41)$$

*Step 2: Boundary blow-up.* The compactness of approximate solutions vanishing at a fixed point in the local uniform convergence topology is easy to obtain thanks to the uniform estimate of the gradient. The main difficulty is to preserve the boundary blow-up when the parameters  $k, \gamma, \delta, \epsilon$  tend to their respective limit.

*Case 1.* We first assume that there exist  $\sigma_0 \in S$ , two decreasing sequences  $\{\epsilon_n\}, \{\delta_\ell\}$  converging to 0 and an increasing sequence  $\{k_j\}$  tending to infinity with the property that

$$\epsilon_n w_{k_j, \gamma, \delta_\ell, \epsilon_n}(\sigma_0) \leq \epsilon_m w_{k_j, \gamma, \delta_\ell, \epsilon_m}(\sigma_0) \quad \text{for all } m < n, j, \ell \in \mathbb{N}. \quad (3.42)$$

Since  $\tilde{w}_{k_j, \gamma, \delta_\ell, \epsilon_n} = w_{k_j, \gamma, \delta_\ell, \epsilon_n} - w_{k_j, \gamma, \delta_\ell, \epsilon_n}(\sigma_0)$  satisfies

$$-\delta_\ell \Delta' \tilde{w} - \frac{1}{2} \nabla' |\nabla' \tilde{w}|^2 \cdot \nabla' \tilde{w} + \gamma |\nabla' \tilde{w}|^4 + (2\gamma + 1) |\nabla' \tilde{w}|^2 + \epsilon_n \tilde{w} + \epsilon_n w_{k_j, \gamma, \delta_\ell, \epsilon_n}(\sigma_0) = 0 \quad \text{in } S_{k_j} \\ \lim_{\rho_{k_j}(\sigma) \rightarrow 0} \tilde{w}(\sigma) = \infty, \quad (3.43)$$

there holds

$$\tilde{w}_{k_j, \gamma, \delta_\ell, \epsilon_n} \geq \tilde{w}_{k_j, \gamma, \delta_\ell, \epsilon_m} \quad \text{for all } m < n, j, \ell \in \mathbb{N}. \quad (3.44)$$

Letting  $\delta_\ell \rightarrow 0$  we derive that  $w_{k_j, \gamma, \delta_\ell, \epsilon_n} \rightarrow w_{k_j, \gamma, \epsilon_n}$  locally uniformly in  $S_{k_j}$  and  $w_{k_j, \gamma, \epsilon_n}$  satisfies

$$-\frac{1}{2} \nabla' |\nabla' w|^2 \cdot \nabla' w + \gamma |\nabla' w|^4 + (2\gamma + 1) |\nabla' w|^2 + \epsilon_n w = 0 \quad \text{in } S_{k_j} \\ \lim_{\rho_{k_j}(\sigma) \rightarrow 0} w(\sigma) = \infty. \quad (3.45)$$

Furthermore, for all  $m < n, j \in \mathbb{N}$ ,

$$\begin{aligned} (i) \quad & w_{k_j, \gamma, \epsilon_n} \geq w_{k_j, \gamma, \epsilon_m} \\ (ii) \quad & w_{k_j, \gamma, \epsilon_n} - w_{k_j, \gamma, \epsilon_n}(\sigma_0) \geq w_{k_j, \gamma, \epsilon_m} - w_{k_j, \gamma, \epsilon_m}(\sigma_0) \\ (iii) \quad & \epsilon_n w_{k_j, \gamma, \epsilon_n}(\sigma_0) \leq \epsilon_m w_{k_j, \gamma, \epsilon_m}(\sigma_0). \end{aligned} \quad (3.46)$$

By monotonicity with respect to  $S_{k_j}$ ,  $w_{k_j, \gamma, \epsilon_n} \downarrow w_{\gamma, \epsilon_n}$  as  $j \rightarrow \infty$ . Let  $a \in \partial S$  and  $v_{a, \gamma, \epsilon_n}$  be the solution of (3.31) with  $\epsilon = \epsilon_n$  which exists by Corollary 3.5. Then

$$v_{a, \gamma, \epsilon_n} \leq w_{k_j, \gamma, \epsilon_n} \quad \text{in } S_{k_j}, \quad (3.47)$$

which yields

$$v_{a, \gamma, \epsilon_n} \leq w_{\gamma, \epsilon_n} \quad \text{in } S. \quad (3.48)$$



This proves that  $w_{\gamma, \epsilon_n}$  is a viscosity solution of

$$-\frac{1}{2} \nabla' |\nabla' w|^2 \cdot \nabla' w + \gamma |\nabla' w|^4 + (2\gamma + 1) |\nabla' w|^2 + \epsilon_n w = 0 \quad \text{in } S$$

$$\lim_{\rho(\sigma) \rightarrow 0} w(\sigma) = \infty, \quad (3.49)$$

and from (3.46),

$$(i) \quad w_{\gamma, \epsilon_n} \geq w_{\gamma, \epsilon_m}$$

$$(ii) \quad w_{\gamma, \epsilon_n} - w_{\gamma, \epsilon_n}(\sigma_0) \geq w_{\gamma, \epsilon_m} - w_{\gamma, \epsilon_m}(\sigma_0)$$

$$(iii) \quad \epsilon_n w_{\gamma, \epsilon_n}(\sigma_0) \leq \epsilon_m w_{\gamma, \epsilon_m}(\sigma_0).$$
(3.50)

Because  $\tilde{w}_{\gamma, \epsilon_n} = w_{\gamma, \epsilon_n} - w_{\gamma, \epsilon_n}(\sigma_0)$  is increasing with respect to  $n$ , locally compact in the topology of local uniform convergence and satisfies

$$-\frac{1}{2} \nabla' |\nabla' \tilde{w}|^2 \cdot \nabla' \tilde{w} + \gamma |\nabla' \tilde{w}|^4 + (2\gamma + 1) |\nabla' \tilde{w}|^2 + \epsilon_n \tilde{w} + \epsilon_n w'_{\gamma, \epsilon_n}(\sigma_0) = 0 \quad \text{in } S$$

$$\lim_{\rho(\sigma) \rightarrow 0} \tilde{w}(\sigma) = \infty, \quad (3.51)$$

and since  $\epsilon_n w_{\gamma, \epsilon_n}(\sigma_0) \rightarrow \lambda(\gamma, S)$  as  $n \rightarrow \infty$ , we infer that  $\tilde{w}_\gamma = \lim_{n \rightarrow \infty} \tilde{w}_{\gamma, \epsilon_n}$  is a locally Lipschitz continuous viscosity solution of

$$-\frac{1}{2} \nabla' |\nabla' \tilde{w}|^2 \cdot \nabla' \tilde{w} + \gamma |\nabla' \tilde{w}|^4 + (2\gamma + 1) |\nabla' \tilde{w}|^2 + \lambda(\gamma, S) = 0 \quad \text{in } S$$

$$\lim_{\rho(\sigma) \rightarrow 0} \tilde{w}(\sigma) = \infty. \quad (3.52)$$

*Case 2.* If the condition of Step 1 does not hold, for any  $\sigma_0 \in S$  there exist two decreasing sequences  $\{\epsilon_n\}$ ,  $\{\delta_\ell\}$  converging to 0 and an increasing sequence  $\{k_j\}$  tending to infinity, all depending on  $\sigma_0$ , such that

$$\lambda(\gamma, S) > \epsilon_n w_{k_j, \gamma, \delta_\ell, \epsilon_n}(\sigma_0) > \epsilon_m w_{k_m, \gamma, \delta_m, \epsilon_m}(\sigma_0) \quad \forall n > m, \quad (3.53)$$

where

$$\lambda(\gamma, S) = \lim_{\substack{n \rightarrow \infty \\ j \rightarrow \infty \\ \ell \rightarrow \infty}} \epsilon_n w_{k_j, \gamma, \delta_\ell, \epsilon_n}(\sigma_0).$$

We fix some  $\sigma_0 \in S$ . Then  $\tilde{w}_{k_j, \gamma, \delta_\ell, \epsilon_n} = w_{k_j, \gamma, \delta_\ell, \epsilon_n} - w_{k_j, \gamma, \delta_\ell, \epsilon_n}(\sigma_0)$  satisfies

$$-\delta_\ell \Delta \tilde{w} - \frac{1}{2} \nabla' |\nabla' \tilde{w}|^2 + \gamma |\nabla' \tilde{w}|^4 + (2\gamma + 1) |\nabla' \tilde{w}|^2 + \epsilon_n \tilde{w} + \lambda(\gamma, S) \geq 0 \quad \text{in } S_{k_j}$$

$$\lim_{\rho_{k_j}(\sigma) \rightarrow 0} \tilde{w}(\sigma) = \infty. \quad (3.54)$$

We introduce the problem

$$-\delta_\ell \Delta Z - \frac{1}{2} \nabla' |\nabla' Z|^2 + \gamma |\nabla' Z|^4 + (2\gamma + 1) |\nabla' Z|^2 + \epsilon_n Z + \lambda(\gamma, S) = 0 \quad \text{in } S_{k_j}$$

$$\lim_{\rho_{k_j}(\sigma) \rightarrow 0} Z(\sigma) = \infty. \quad (3.55)$$

Since (3.55) can be re-written as

$$-\delta_\ell \Delta Z' - \frac{1}{2} \nabla' |\nabla' Z'|^2 + \gamma |\nabla' Z'|^4 + (2\gamma + 1) |\nabla' Z'|^2 + \epsilon_n Z' = 0 \quad \text{in } S_{k_j} \quad (3.56)$$

$$\lim_{\rho_{k_j}(\sigma) \rightarrow 0} Z'(\sigma) = \infty,$$

with  $Z' = Z + \epsilon_n^{-1} \lambda(\gamma, S)$ , existence is ensured by the approximation by finite boundary data as above. We denote by  $Z_{k_j, \gamma, \delta_\ell, \epsilon_n}$  and  $Z'_{k_j, \gamma, \delta_\ell, \epsilon_n}$ , which coincides actually with  $\tilde{w}_{k_j, \gamma, \delta_\ell, \epsilon_n}$ , the solutions of (3.55) and (3.56) obtained by such approximation. Using Corollary 3.5 as in Case 1 and comparison, we obtain the following estimate

$$v_{a, \gamma, \epsilon_n} - \frac{\lambda(\gamma, S)}{\epsilon_n} \leq Z'_{k_j, \gamma, \delta_\ell, \epsilon_n} - \frac{\lambda(\gamma, S)}{\epsilon_n} = Z_{k_j, \gamma, \delta_\ell, \epsilon_n} \leq \tilde{w}_{k_j, \gamma, \delta_\ell, \epsilon_n} \quad \text{in } S_{k_j}, \quad (3.57)$$

where, again  $a \in \partial S$  and  $v_{a, \gamma, \epsilon_n}$  is the solution of (3.31) with  $\epsilon = \epsilon_n$  which exists by Corollary 3.5. Now the sequences  $\{Z_{k_j, \gamma, \delta_\ell, \epsilon_n}\}_{\epsilon_n, k_j}$  and  $\{w_{k_j, \gamma, \delta_\ell, \epsilon_n}\}_{k_j}$  are increasing. Letting successively  $\delta_\ell \rightarrow 0$  and  $k_j \rightarrow \infty$  we infer that, up to a subsequence,  $Z_{k_j, \gamma, \delta_\ell, \epsilon_n}$  converges locally uniformly to some  $Z_{\gamma, \epsilon_n}$  and  $\tilde{w}_{k_j, \gamma, \delta_\ell, \epsilon_n}$  converges locally uniformly to some  $\tilde{w}_{\gamma, \epsilon_n} = w_{\gamma, \epsilon_n} - w_{\gamma, \epsilon_n}(\sigma_0)$  which are respectively viscosity solutions of

$$-\frac{1}{2} \nabla' |\nabla' Z|^2 + \gamma |\nabla' Z|^4 + (2\gamma + 1) |\nabla' Z|^2 + \lambda(\gamma, S) + \epsilon_n Z = 0 \quad \text{in } S \quad (3.58)$$

$$\lim_{\rho(\sigma) \rightarrow 0} Z(\sigma) = \infty,$$

and

$$-\frac{1}{2} \nabla' |\nabla' \tilde{w}|^2 + \gamma |\nabla' \tilde{w}|^4 + (2\gamma + 1) |\nabla' \tilde{w}|^2 + \epsilon_n w_{\gamma, \epsilon_n}(\sigma_0) + \epsilon_n \tilde{w} = 0 \quad \text{in } S \quad (3.59)$$

$$\lim_{\rho(\sigma) \rightarrow 0} \tilde{w}(\sigma) = \infty.$$

Furthermore  $Z_{\gamma, \epsilon_n}$  and  $\tilde{w}_{\gamma, \epsilon_n}$  are locally bounded in  $S$ , relatively compact for the local uniform topology and they satisfy

$$Z_{\gamma, \epsilon_n} \leq \tilde{w}_{\gamma, \epsilon_n} \quad \text{in } S. \quad (3.60)$$

At end, the sequence  $\{Z_{\gamma, \epsilon_n}\}_{\epsilon_n}$  is nondecreasing. Hence, up to a subsequence,  $\{\tilde{w}_{\gamma, \epsilon_n}\}$  converges locally uniformly in  $S$  to some  $\tilde{w}_\gamma$  which satisfies

$$\lim_{\epsilon_n \rightarrow 0} Z_{\gamma, \epsilon_n} = Z_\gamma \leq \tilde{w}_\gamma \quad \text{in } S. \quad (3.61)$$

Since

$$\lim_{\rho(\sigma) \rightarrow 0} Z_{\gamma, \epsilon_n}(\sigma) = \infty \leq \lim_{\rho(\sigma) \rightarrow 0} Z_\gamma(\sigma),$$

it follows that  $\tilde{w}_\gamma$  is a locally Lipschitz continuous viscosity solution of (3.52).

*Step 3: End of the proof.* As in the proof of Proposition 2.6,  $\gamma \mapsto \lambda(\gamma, S) - \gamma$  is a non increasing function of  $\gamma$ . We recall that  $\lambda(\gamma, S_\alpha^*) = \lambda(\gamma, S_\alpha^*(a))$ . By formula (3.20), for any  $\alpha > 0$   $\lim_{\gamma \rightarrow 0} \lambda(\gamma, S_\alpha^*) = \infty$ . Since

$$\lambda(\gamma, S) - \gamma \geq \lambda(\gamma, S_\pi^*) - \gamma, \quad (3.62)$$

it follows that  $\lambda(\gamma, S_\alpha^*) - \gamma$  converges to infinity when  $\gamma$  converges to 0. Let  $\alpha > 0$  such that  $\overline{S_\alpha(a)} \subset S$  for some  $a \in S$ . If

$$\gamma = \frac{\pi^2}{4\alpha(\pi + \alpha)},$$

then  $\lambda(\frac{\pi^2}{4\alpha(\pi + \alpha)}, S_\alpha^*) - \frac{\pi^2}{4\alpha(\pi + \alpha)} = 1$ . Since

$$\lambda\left(\frac{\pi^2}{4\alpha(\pi + \alpha)}, S\right) - \frac{\pi^2}{4\alpha(\pi + \alpha)} < \lambda\left(\frac{\pi^2}{4\alpha(\pi + \alpha)}, S_\alpha^*(a)\right) - \frac{\pi^2}{4\alpha(\pi + \alpha)} = 1, \quad (3.63)$$

it follows that

$$\inf \{\lambda(\gamma, S) - \gamma : \gamma > 0\} < 1. \quad (3.64)$$

We set

$$\bar{\beta}_s = \inf \{\gamma : \lambda(\gamma, S) - \gamma < 1\}. \quad (3.65)$$

Let  $\{\gamma_\nu\}$  be a sequence decreasing to  $\bar{\beta}_s$  when  $\nu \rightarrow \infty$  and such that  $\lim_{\nu \rightarrow \infty} \lambda(\gamma_\nu, S) = \bar{\beta}_s + 1$ . As in Step 1 we denote by  $w_{k_j, \gamma_\nu, \delta_\ell, \epsilon_n}$  the solution of (3.35) with  $(k_j, \gamma_\nu, \delta_\ell, \epsilon_n) = (k, \gamma, \delta, \epsilon)$ . There always holds

$$w_{k_j, \gamma_\mu, \delta_\ell, \epsilon_n} \leq w_{k_j, \gamma_\nu, \delta_\ell, \epsilon_n} \quad \text{if } \gamma_\mu \geq \gamma_\nu. \quad (3.66)$$

Again we distinguish two cases

*Case 1.* We assume that there exist  $\sigma_0 \in S$  and monotone sequences  $\{k_j\}$ ,  $\{\delta_\ell\}$  and  $\{\epsilon_n\}$  such that

$$\epsilon_n w_{k_j, \gamma_\nu, \delta_\ell, \epsilon_n}(\sigma_0) \leq \epsilon_m w_{k_j, \gamma_\mu, \delta_\ell, \epsilon_m}(\sigma_0) \quad \text{for all } m < n, \mu \leq \nu, j, \ell \in \mathbb{N}. \quad (3.67)$$

As in Step 2-Case 1 it implies

$$w_{k_j, \gamma_\nu, \delta_\ell, \epsilon_n} - w_{k_j, \gamma_\nu, \delta_\ell, \epsilon_n}(\sigma_0) \geq w_{k_j, \gamma_\mu, \delta_\ell, \epsilon_m} - w_{k_j, \gamma_\mu, \delta_\ell, \epsilon_m}(\sigma_0) \quad \text{for all } m < n, \mu < \nu, j, \ell \in \mathbb{N}. \quad (3.68)$$

Letting  $\delta_\ell \rightarrow 0$  and  $k_j \rightarrow \infty$  we obtain that the limit function  $w_{\gamma_\nu, \epsilon_n}$  satisfies

$$-\frac{1}{2} \nabla' |\nabla' w|^2 \cdot \nabla' w + \gamma_\nu |\nabla' w|^4 + (2\gamma_\nu + 1) |\nabla' w|^2 + \epsilon_n w = 0 \quad \text{in } S \quad (3.69)$$

$$\lim_{\rho(\sigma) \rightarrow 0} w(\sigma) = \infty,$$

and  $\tilde{w}_{\gamma_\nu, \epsilon_n} = w_{\gamma_\nu, \epsilon_n} - w_{\gamma_\nu, \epsilon_n}(\sigma_0)$  is increasing both with respect to  $n$  and  $\nu$ . If  $\epsilon_n \rightarrow 0$  we derive that  $\tilde{w}_{\gamma_\nu} = \lim_{n \rightarrow \infty} \tilde{w}_{\gamma_\nu, \epsilon_n}$  satisfies  $\tilde{w}_{\gamma_\nu}(\sigma_0) = 0$  and

$$-\frac{1}{2} \nabla' |\nabla' \tilde{w}|^2 \cdot \nabla' \tilde{w} + \gamma_\nu |\nabla' \tilde{w}|^4 + (2\gamma_\nu + 1) |\nabla' \tilde{w}|^2 + \lambda'(\gamma_\nu, S) = 0 \quad \text{in } S \quad (3.70)$$

$$\lim_{\rho(\sigma) \rightarrow 0} \tilde{w}(\sigma) = \infty.$$

By gradient estimates and since  $\tilde{w}_{\gamma_\nu}(\sigma_0) = 0$ , the set of functions  $\{\tilde{w}_{\gamma_\nu}\}_\nu$  is relatively compact for the local uniform convergence in  $S$ . Furthermore  $\tilde{w}_{\gamma_\nu}$  is increasing with respect to  $\nu$ , with limit  $\tilde{w}$ . Using (3.65) and the definition of  $\{\gamma_\nu\}$ , we conclude that

$$-\frac{1}{2} \nabla' |\nabla' \tilde{w}|^2 \cdot \nabla' \tilde{w} + \bar{\beta}_s |\nabla' \tilde{w}|^4 + (2\bar{\beta}_s + 1) |\nabla' \tilde{w}|^2 + \bar{\beta}_s + 1 = 0 \quad \text{in } S \quad (3.71)$$

$$\lim_{\rho(\sigma) \rightarrow 0} \tilde{w}(\sigma) = \infty,$$

holds in the viscosity sense.

*Case 2.* We assume that for any  $\sigma_0 \in S$  and  $\nu$  there exist two decreasing sequences  $\{\epsilon_n\}$ ,  $\{\delta_\ell\}$  converging to 0 and an increasing sequence  $\{k_j\}$  tending to infinity such that

$$\bar{\beta}_s + 1 > \epsilon_n w_{k_j, \gamma_\nu, \delta_\ell, \epsilon_n}(\sigma_0) > \epsilon_m w_{k_m, \gamma_\nu, \delta_m, \epsilon_m}(\sigma_0) \quad \forall n > m, \quad (3.72)$$

where

$$\bar{\beta}_s + 1 = \lim_{\nu \rightarrow \infty} \lambda(\gamma_\nu, S) = \lim_{n \rightarrow \infty, j \rightarrow \infty, \ell \rightarrow \infty} \epsilon_n w_{k_j, \gamma_\nu, \delta_\ell, \epsilon_n}(\sigma_0).$$

We follow the ideas in Step 2-Case 2 and consider the problem

$$\begin{aligned} -\delta_\ell \Delta Z - \frac{1}{2} \nabla' |\nabla' Z|^2 + \gamma_\nu |\nabla' Z|^4 + (2\gamma_\nu + 1) |\nabla' Z|^2 + \epsilon_n Z + \bar{\beta}_s + 1 = 0 \quad \text{in } S'_{k_j} \\ \lim_{\rho'_{k_j}(\sigma) \rightarrow 0} Z(\sigma) = \infty. \end{aligned} \quad (3.73)$$

Since (3.73) can be re-written as (3.56) with  $\gamma$  replaced by  $\gamma_\nu$  and setting  $Z' = Z + \epsilon_n^{-1}(\bar{\beta}_s + 1)$ , we have existence and uniqueness of the solution  $Z^*_{k_j, \gamma_\nu, \delta_\ell, \epsilon_n}$  (we do not use the previous notation  $Z_{k_j, \gamma_\nu, \delta_\ell, \epsilon_n}$  since the constant term is not of the form  $\lambda(\gamma_\nu, S)$ ). Then  $Z'^*_{k_j, \gamma_\nu, \delta_\ell, \epsilon_n} = Z^*_{k_j, \gamma_\nu, \delta_\ell, \epsilon_n} + \epsilon_n^{-1}(\bar{\beta}_s + 1)$  satisfies (3.56) with  $\gamma$  replaced by  $\gamma_\nu$ . Then (3.57) is replaced by

$$v_{a, \gamma_\nu, \epsilon_n} - \frac{\bar{\beta}_s + 1}{\epsilon_n} \leq Z'^*_{k_j, \gamma_\nu, \delta_\ell, \epsilon_n} - \frac{\bar{\beta}_s + 1}{\epsilon_n} = Z^*_{k_j, \gamma_\nu, \delta_\ell, \epsilon_n} \leq \tilde{w}'_{k_j, \gamma_\nu, \delta_\ell, \epsilon_n} \quad \text{in } S_{k_j}, \quad (3.74)$$

where  $v_{a, \gamma_\nu, \epsilon_n}$  is as above with obvious modifications. We denote by  $\tilde{w}_{\gamma_\nu, \epsilon_n} = w_{\gamma_\nu, \epsilon_n} - w_{\gamma_\nu, \epsilon_n}(\sigma_0)$  the limit, when  $\delta_\ell \rightarrow 0$  and  $k_j \rightarrow \infty$ , of  $\tilde{w}_{k_j, \gamma_\nu, \delta_\ell, \epsilon_n} = w_{k_j, \gamma_\nu, \delta_\ell, \epsilon_n} - w_{k_j, \gamma_\nu, \delta_\ell, \epsilon_n}(\sigma_0)$  and by  $Z^*_{\gamma_\nu, \epsilon_n}$  the one of  $Z^*_{k_j, \gamma_\nu, \delta_\ell, \epsilon_n}$  under the same conditions. They are respective viscosity solutions of

$$\begin{aligned} -\frac{1}{2} \nabla' |\nabla' \tilde{w}|^2 + \gamma_\nu |\nabla' \tilde{w}|^4 + (2\gamma_\nu + 1) |\nabla' \tilde{w}|^2 + \epsilon_n w'_{\gamma_\nu, \epsilon_n}(\sigma_0) + \epsilon_n \tilde{w} = 0 \quad \text{in } S \\ \lim_{\rho(\sigma) \rightarrow 0} \tilde{w}(\sigma) = \infty. \end{aligned} \quad (3.75)$$

and

$$\begin{aligned} -\frac{1}{2} \nabla' |\nabla' Z|^2 + \gamma_\nu |\nabla' Z|^4 + (2\gamma_\nu + 1) |\nabla' Z|^2 + \bar{\beta}_s + 1 + \epsilon_n Z = 0 \quad \text{in } S \\ \lim_{\rho(\sigma) \rightarrow 0} Z(\sigma) = \infty. \end{aligned} \quad (3.76)$$

Furthermore  $Z^*_{\gamma_\nu, \epsilon_n}$  and  $\tilde{w}_{\gamma_\nu, \epsilon_n}$  are locally bounded in  $S$ , relatively compact for the local uniform topology and they satisfy

$$Z^*_{\gamma_\nu, \epsilon_n} \leq \tilde{w}'_{\gamma_\nu, \epsilon_n} \quad \text{in } S. \quad (3.77)$$

The sequence  $\{Z^*_{\gamma_\nu, \epsilon_n}\}$  is nondecreasing both with respect to  $n$  and  $\nu$ . Therefore the boundary condition is kept. Letting  $\epsilon_n \rightarrow 0$  and  $\gamma_\nu \rightarrow \bar{\beta}_s$  we conclude as in Step 1 that, up to a subsequence  $\{\nu_s\}$  there exists a locally Lipschitz continuous function  $\tilde{w}$  such that  $\tilde{w}_{\gamma_{\nu_s}, \epsilon_n} \rightarrow \tilde{w}$  when  $\nu \rightarrow \infty$  and  $\nu_s \rightarrow \bar{\beta}_s$  successively, and  $\tilde{w}$  is a viscosity solution of (3.71).

We end the proof by setting  $\bar{\psi}_s = e^{-\bar{\beta}_s \tilde{w}}$ .  $\square$

*Mutatis mutandis* in the above proof, one can obtain an existence result of a separable positive regular infinity harmonic function in  $C_S$  vanishing on  $\partial C_S$ .

**Theorem 3.6.** *Assume  $S \subsetneq S^{N-1}$  is any domain. Then there exist  $\bar{\mu}_s > 0$  and a positive function  $\bar{\omega}_s$  in  $C(\bar{S})$ , locally Lipschitz continuous in  $S$  and vanishing on  $\partial S$ , such that the function*

$$\bar{u}_{s,-}(r, \sigma) = r^{\bar{\mu}_s} \bar{\omega}_s(\sigma), \quad (3.78)$$

is infinity harmonic in  $C_S$  and vanishes on  $\partial C_S$ .

### 3.4 Proof of Theorems E and F

*Proof of Theorem E. Step 1: Existence of  $\underline{\beta}_s$ .* The proof follows the one of Theorem D, hence we indicate only the main streamlines. We consider a decreasing sequence of smooth spherical domains  $\{S'_k\}$  such that

$$S \subset S'_{k+1} \subset \bar{S}'_{k+1} \subset S'_k \text{ and } \text{int} \left( \bigcap_k S'_k \right) = S.$$

Such a sequence of domains  $\{S'_k\}$  exists since  $\partial S = \partial \bar{S}^c$ . To each domain we associate the positive exponent  $\beta'_k := \beta_{S'_k}$  and the corresponding spherical  $p$ -harmonic function  $\psi'_k := \psi_{S'_k}$ , defined in  $S'_k$ , such that  $\psi'_k(\sigma_*) = 1$  for some  $\sigma_* \in S_1$ , so that the function  $u'_k(r, \cdot) = r^{-\beta'_k} \psi'_k$  is  $p$ -harmonic in  $C_{S'_k}$  and vanish on  $\partial C_{S'_k} \setminus \{0\}$ . For  $\gamma, \delta, \epsilon > 0$ , we denote by  $w'_{k,\gamma,\delta,\epsilon}$  the solution of

$$\begin{aligned} -\delta \Delta w' - \frac{1}{2} \nabla' |\nabla' w'|^2 \cdot \nabla' w' + \gamma |\nabla' w'|^4 + (2\gamma + 1) |\nabla' w'|^2 + \epsilon w' = 0 \quad \text{in } S'_k \\ \lim_{\rho'_k(\sigma) \rightarrow 0} w'(\sigma) = \infty, \end{aligned} \quad (3.79)$$

where  $\rho'_k(\cdot) = \text{dist}(\cdot, \partial S'_k)$ . By the maximum principle all the functions  $w'_{\ell,\gamma,\delta,\epsilon}$  is positive and the following comparison relations hold. Estimates (3.36) are valid the main difference being the fact that  $w'_{\ell,\gamma,\delta,\epsilon} \leq w_{k,\gamma,\delta,\epsilon}$  in  $S_k$  for  $k, \ell > 0$  and that the mapping  $k \mapsto w'_{k,\gamma,\delta,\epsilon}$  is increasing. Similarly  $(\epsilon, \gamma) \mapsto w'_{\ell,\gamma,\delta,\epsilon}$  is nonincreasing. The gradient estimate (3.37) holds for  $w'_{k,\gamma,\delta,\epsilon}$ , provided  $\rho_k(\sigma)$  be replaced by  $\rho'_k(\sigma) = \text{dist}(\sigma, \partial S'_k)$ . Moreover, similarly to in (2.37),

$$-\frac{1}{\gamma} \ln \rho'_k(\sigma) - \frac{M'_k}{\epsilon} \leq w'_{k,\gamma,\delta,\epsilon}(\sigma) \leq -\frac{1}{\gamma} \ln \rho'_k(\sigma) + \frac{M'_k}{\epsilon} \quad \forall \sigma \in S'_k. \quad (3.80)$$

When  $\delta \rightarrow 0$ ,  $w'_{k,\gamma,\delta_n,\epsilon} \rightarrow w'_{k,\gamma,\epsilon}$  locally uniformly in  $S'_k$  to some function  $w'_{k,\gamma,\epsilon}$  which satisfies (3.36), the modified gradient estimate (3.37) (expressed with  $\rho_k$  replaced by  $\rho'_k$ ) and (3.80) and is a viscosity solution of

$$\begin{aligned} -\frac{1}{2} \nabla' |\nabla' w'|^2 \cdot \nabla' w' + \gamma |\nabla' w'|^4 + (2\gamma + 1) |\nabla' w'|^2 + \epsilon w' = 0 \quad \text{in } S'_k \\ \lim_{\rho'_k(\sigma) \rightarrow 0} w'(\sigma) = \infty. \end{aligned} \quad (3.81)$$

When  $k \rightarrow \infty$   $w'_{k,\gamma,\epsilon} \uparrow w'_{\gamma,\epsilon}$  which is a nonincreasing function of  $\epsilon$  and  $\gamma$ .

The proof of the boundary blow-up introduces two cases: either there exists  $\sigma_0 \in S$ , two decreasing sequences  $\{\epsilon_n\}$ ,  $\{\delta_\ell\}$  converging to 0 and an increasing sequence  $\{k_j\}$  tending to infinity with the property that

$$\epsilon_n w'_{k_j,\gamma,\delta_\ell,\epsilon_n}(\sigma_0) \leq \epsilon_m w'_{k_j,\gamma,\delta_\ell,\epsilon_m}(\sigma_0) \quad \text{for all } m < n, j, \ell \in \mathbb{N}. \quad (3.82)$$

Or for any  $\sigma_0 \in S$  there exist two decreasing sequences  $\{\epsilon_n\}$ ,  $\{\delta_\ell\}$  converging to 0 and an increasing sequence  $\{k_j\}$  tending to infinity, all depending on  $\sigma_1$ , such that

$$\lambda'(\gamma, S) > \epsilon_n w'_{k_j, \gamma, \delta_\ell, \epsilon_n}(\sigma_0) > \epsilon_m w'_{k_m, \gamma, \delta_m, \epsilon_m}(\sigma_0) \quad \forall n > m, \quad (3.83)$$

where

$$\lambda'(\gamma, S) = \lim_{\substack{n \rightarrow \infty, \\ j \rightarrow \infty, \\ \ell \rightarrow \infty}} \epsilon_n w'_{k_j, \gamma, \delta_\ell, \epsilon_n}(\sigma_1).$$

In the first case for any  $a \in \partial S$ , there exists a sequence  $\{a_j\}$  such that  $a_j \in S'_{k_j}$  converging to  $a$  (such a sequence exists since  $\partial S = \partial \bar{S}^c$ ). Then

$$v_{a_j, \gamma, \epsilon_n} \leq w'_{k_j, \gamma, \epsilon_n} \quad \text{in } S_{k_j} \quad (3.84)$$

Since  $v_{a, \gamma, \epsilon_n}$  is obtained from  $v_{a_j, \gamma, \epsilon_n}$  by an orthogonal transformation on  $S^{N-1}$ , we derive

$$v_{a, \gamma, \epsilon_n} \leq w'_{\gamma, \epsilon_n} \quad \text{in } S. \quad (3.85)$$

This proves that  $w'_{\gamma, \epsilon_n}$  is a viscosity solution of

$$-\frac{1}{2} \nabla' |\nabla' w|^2 \cdot \nabla' w + \gamma |\nabla' w|^4 + (2\gamma + 1) |\nabla' w|^2 + \epsilon_n w = 0 \quad \text{in } S \quad (3.86)$$

$$\lim_{\rho(\sigma) \rightarrow 0} w(\sigma) = \infty.$$

The proof in the second case is the same as in Theorem D, just replacing  $v_{a, \gamma, \epsilon_n}$  by  $v_{a_j, \gamma, \epsilon_n}$  in (3.57) which becomes

$$v_{a_{k_j}, \gamma, \epsilon_n} - \frac{\lambda(\gamma, S)}{\epsilon_n} \leq Z'_{k_j, \gamma, \delta_\ell, \epsilon_n} - \frac{\lambda(\gamma, S)}{\epsilon_n} = Z_{k_j, \gamma, \delta_\ell, \epsilon_n} \leq \tilde{w}_{k_j, \gamma, \delta_\ell, \epsilon_n} \quad \text{in } S_{k_j}, \quad (3.87)$$

where  $Z'_{k_j, \gamma, \delta_\ell, \epsilon_n}$  and  $Z_{k_j, \gamma, \delta_\ell, \epsilon_n}$  are defined accordingly. This implies again that the limit  $w'_{\gamma, \epsilon_n}$  of  $\{w'_{k_j, \delta_\ell, \epsilon_n}\}$  when  $j \rightarrow \infty$  is a viscosity solution of (3.86).

The proof of the existence of some  $\underline{\beta}_s > 0$  such that  $\lambda'(\underline{\beta}_s, S) = 1 + \underline{\beta}_s$  follows the same dichotomy.

*Step 2: Comparison of exponents.* Since  $w'_{k, \gamma, \delta, \epsilon} \leq w_{k, \gamma, \delta, \epsilon}$  it follows that  $\epsilon w'_{k, \gamma, \delta, \epsilon}(\sigma_0) \leq \epsilon w_{k, \gamma, \delta, \epsilon}(\sigma_0)$  (it is always possible to choose the same  $\sigma_0$  in order to defined the ergodic constant), hence  $\lambda'(\gamma, S) \leq \lambda(\gamma, S)$  and finally  $\underline{\beta}_s \leq \bar{\beta}_s$  by monotonicity. Assume now that  $u(r, \sigma) = r^{-\beta} \psi(\sigma)$  is a positive infinity harmonic function in  $C_S$  which vanishes on  $\partial C_S \setminus \{0\}$ . We proceed by contradiction in assuming that  $\beta > \bar{\beta}_s$ . Hence  $\beta > \beta_k := \beta_{s_k}$  for  $k$  large enough. We set

$$\phi = \psi^\theta \quad \text{where } \theta = \frac{\beta_k}{\beta} < 1.$$

Then

$$\begin{aligned} \nabla \phi &= \theta \psi^{\theta-1} \nabla \psi, \quad |\nabla \phi|^2 = \theta^2 \psi^{2(\theta-1)} |\nabla \psi|^2, \quad |\nabla \phi|^2 \phi = \theta^2 \psi^{3(\theta-1)+1} |\nabla \psi|^2, \\ \nabla |\nabla \phi|^2 \cdot \nabla \phi &= \theta^3 \psi^{3(\theta-1)} \nabla |\nabla \psi|^2 \cdot \nabla \psi + 2\theta^3 (\theta - 1) \psi^{3(\theta-1)-1} |\nabla \psi|^4 \end{aligned}$$

If we denote

$$\mathcal{L}_k \phi = -\frac{1}{2} \nabla' |\nabla' \phi|^2 \cdot \nabla' \phi - \beta_k (2\beta_k + 1) |\nabla' \phi|^2 \phi - \beta_k^3 (\beta_k + 1) \phi^3,$$

then

$$\begin{aligned} \mathcal{L}_k \phi &= \theta^3 \psi^{3(\theta-1)} \left[ \left( \beta(2\beta + 1) - \frac{\beta_k}{\theta} (2\beta_k + 1) \right) \psi |\nabla \psi|^2 + (1 - \theta) \psi^{-1} |\nabla \psi|^4 \right. \\ &\quad \left. + \left( \beta^3 (\beta + 1) - \frac{\beta_k^3}{\theta^3} (\beta_k + 1) \psi^3 \right) \right] \\ &= \frac{\theta^3 \psi^{3(\theta-1)-1} (\beta - \beta_k)}{\beta} (|\nabla \psi|^2 + \beta^2 \psi^2)^2 > 0. \end{aligned} \quad (3.88)$$

Hence the function  $W = -\frac{1}{\beta_k} \ln \phi$  satisfies in the viscosity sense

$$-\frac{1}{2} \nabla' |\nabla' W|^2 \cdot \nabla' W + \beta_k |\nabla W|^4 + (2\beta_k + 1) |\nabla W|^2 + \beta_k + 1 < 0 \quad \text{in } S_k$$

and is bounded in  $S_k$ . By comparison between viscosity solutions,  $W$  is smaller than  $w_k := -\frac{1}{\beta_k} \ln \psi_{s_k}$  where  $\psi_{s_k}$  is a spherical infinity harmonic function in  $S_k$  (see the proof of Theorem D-Step 1). But we can replace  $W$  by  $W + n$  for any  $n \in \mathbb{N}_*$ . This is a contradiction, hence  $\beta \leq \bar{\beta}_s$ . In the same way  $\beta \geq \underline{\beta}_s$ , which ends the proof.  $\square$

*Proof of Theorem F.* If  $\partial S$  is Lipschitz and satisfies the interior sphere condition, it is possible to construct a bounded Lipschitz subdomain  $\Theta$  of  $C_S$  satisfying the interior sphere condition with the following additional properties:

$$\begin{aligned} \Theta &\subset C_S \cap \left( B_2 \setminus \bar{B}_{\frac{1}{2}} \right) \\ \Theta_{\frac{3}{2}, \frac{2}{3}} &:= \Theta \cap \left( B_{\frac{3}{2}} \setminus \bar{B}_{\frac{2}{3}} \right) = C_S \cap \left( B_{\frac{3}{2}} \setminus \bar{B}_{\frac{2}{3}} \right) = \left\{ (r, \sigma) : \frac{2}{3} < r < \frac{3}{2}, \sigma \in S \right\}. \end{aligned} \quad (3.89)$$

and we define the lateral boundary of  $\Theta$  by

$$\partial_\ell \Theta_{\frac{3}{2}, \frac{2}{3}} = \partial C_S \cap \left( B_{\frac{3}{2}} \setminus \bar{B}_{\frac{2}{3}} \right).$$

Assume now that  $u(r, \sigma) = r^{-\beta} \psi(\sigma)$  and  $u'(r, \sigma) = r^{-\beta'} \psi'(\sigma)$ , with  $\beta, \beta' > 0$ , are nonnegative, infinity harmonic in  $C_S$  and vanish on  $\partial C_S \setminus \{0\}$ . Since  $u$  and  $u'$  vanish on  $\partial_\ell \Theta_{\frac{3}{2}, \frac{2}{3}}$ , it follows from [12, Th 1.1], that there exists a constant  $c_1 = c_1(N, \Theta) > 0$  such that for any  $z \in \partial_\ell \Theta_{\frac{3}{2}, \frac{2}{3}} \cap \partial S$  there exists a constant  $\delta_z \in (0, \frac{1}{4})$  such that

$$\frac{u(x)}{u'(x)} \leq c_1 \frac{u(y)}{u'(y)} \quad \forall (x, y) \in \Theta_{\frac{3}{2}, \frac{2}{3}} \cap B_{\delta_z}(z). \quad (3.90)$$

By compactness of  $\partial_\ell \Theta_{\frac{3}{2}, \frac{2}{3}} \cap \partial S$  the constant  $\delta_z$  is actually independent of  $z$  and denoted by  $\delta^*$ . We use now the standard Harnack inequality for infinity harmonic functions in  $\Theta$  (see e.g. [5]) to derive the existence of  $c_2 = c_2(N, \delta^*) > 0$  such that

$$\frac{1}{c_2} \leq \frac{u(x)}{u(y)}, \frac{u'(x)}{u'(y)} \leq c_2 \quad \forall (x, y) \in \Theta_{\frac{3}{2}, \frac{2}{3}} \text{ s.t. } \inf \{ \text{dist}(x, \partial\Theta), \text{dist}(y, \partial\Theta) \} \geq \delta^*. \quad (3.91)$$

Taking  $x = (1, \sigma)$  and  $y = (1, \sigma')$  in (3.90), we derive from (3.90), (3.91) that

$$\frac{\psi(\sigma)}{\psi'(\sigma)} \leq c_3 \frac{\psi(\sigma')}{\psi'(\sigma')} \quad \forall (\sigma, \sigma') \in S.$$

This implies that for a fixed  $\sigma_0 \in S$ , one has

$$\frac{1}{c_3} \frac{\psi(\sigma_0)}{\psi'(\sigma_0)} \leq \frac{\psi(\sigma)}{\psi'(\sigma)} \leq c_3 \frac{\psi(\sigma_0)}{\psi'(\sigma_0)} \quad \forall \sigma \in S. \quad (3.92)$$

We proceed as in Step 1, assuming  $\beta > \beta'$  and defining

$$\phi_* = \psi^\theta \quad \text{where} \quad \theta_* = \frac{\beta'}{\beta} < 1$$

Then

$$-\frac{1}{2} \nabla' |\nabla' \phi_*|^2 \cdot \nabla' \phi_* - \beta'(2\beta' + 1) |\nabla' \phi_*|^2 \phi_* - \beta'^3 (\beta' + 1) \phi_*^3 > 0.$$

The function  $W_* = -\frac{1}{\beta'} \ln \phi_*$  satisfies the inequation

$$-\frac{1}{2} \nabla' |\nabla' W_*|^2 \cdot \nabla' W_* + \beta' |\nabla W_*|^4 + (2\beta' + 1) |\nabla W_*|^2 + \beta' + 1 < 0 \quad \text{in } S, \quad (3.93)$$

while  $w' = -\frac{1}{\beta'} \ln \psi'$  is a solution of the associated equation. From (3.92),  $\psi'(\sigma) = o(\phi(\sigma))$  when  $\rho(\sigma) := \text{dist}(\sigma, \partial S) \rightarrow 0$ . Hence

$$w' - W_* = -\frac{1}{\beta'} \ln \left( \frac{\psi'}{\phi_*} \right) \rightarrow \infty \quad \text{when} \quad \rho(\sigma) \rightarrow 0. \quad (3.94)$$

By comparison there holds  $w' \geq W_*$ . Since for any  $n \in \mathbb{N}_*$ , the function  $W_{*,n} = W_* + n$  satisfies (3.93) and (3.94), it follows that  $w' \geq W_{*,n}$ , contradiction. Hence  $\beta \leq \beta'$ , which ends the proof.  $\square$

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Marie-Françoise Bidaut-Véron  
Laboratoire de Mathématiques et Physique Théorique  
Université François Rabelais, Tours, France  
[veronmf@univ-tours.fr](mailto:veronmf@univ-tours.fr)

Marta Garcia Huidobro  
Departamento de Matemática  
Pontificia Universidad Católica, Santiago, Chile  
[martaghc@gmail.com](mailto:martaghc@gmail.com)

Laurent Véron  
Laboratoire de Mathématiques et Physique Théorique  
Université François Rabelais, Tours, France  
[veronl@univ-tours.fr](mailto:veronl@univ-tours.fr)