HAL
open science

# Separable infinite harmonic functions in cones

Marie-Françoise Bidaut-Véron, Marta Garcia-Huidobro, Laurent Véron

## To cite this version:

Marie-Françoise Bidaut-Véron, Marta Garcia-Huidobro, Laurent Véron. Separable infinite harmonic functions in cones. 2017. hal-01492362v2

**HAL Id: hal-01492362**

**https://hal.science/hal-01492362v2**

Preprint submitted on 29 Mar 2017 (v2), last revised 19 Jan 2018 (v3)

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# Separable infinite harmonic functions in cones

**Marie-Françoise Bidaut-Véron**
**Marta Garcia-Huidobro**
**Laurent Véron**

**Abstract** We study the existence of separable infinite harmonic functions in any cone of $\mathbb{R}^N$ vanishing on its boundary under the form $u(r,\sigma) = r^{-\beta}\omega(\sigma)$. We prove that such solutions exist, the spherical part $\omega$ satisfies a nonlinear eigenvalue problem on a subdomain of the sphere $S^{N-1}$ and that the exponents $\beta = \beta_+ > 0$ and $\beta = \beta_- < 0$ are uniquely determined if the domain is smooth.

*2010 Mathematics Subject Classification*. 35D40; 35J70; 35J62 .

*Key words*. Infinity-Laplacian operator; large solutions; viscosity solutions; regularity; ergodic constant.

## Contents

**1 Introduction** **1**

**2 The General case** **4**
2.1 Two-sided estimates . . . . . 5
2.2 Gradient estimates . . . . . 8
2.3 Proof of Theorem A . . . . . 11
2.4 Proof of Theorem B . . . . . 13

**3 The spherical cap problem** **14**
3.1 Problem on the circle . . . . . 14
3.2 Proof of Theorem C . . . . . 16
3.3 Proof of Theorem D . . . . . 18

## 1 Introduction

Let $S$ be a $C^3$ subdomain of the unit sphere $S^{N-1}$ of $\mathbb{R}^N$ and $C_S := \{\lambda\sigma \in \mathbb{R}^N : \lambda > 0,\, \sigma \in S\}$ is the positive cone generated by $S$. In this paper we study the existence of positive solutions of

$$\Delta_\infty u := \frac{1}{2}\nabla\left|\nabla u\right|^2.\nabla u = 0 \tag{1.1}$$

in $C_S$ vanishing on $\partial C_S \setminus \{0\}$ under the form

$$u(x) = u(r,\sigma) = r^{-\beta}\omega(\sigma), \tag{1.2}$$

where $\beta \in \mathbb{R}$ and $(r, \sigma) \in \mathbb{R}_+ \times S^{N-1}$ are the spherical coordinates $\mathbb{R}^N$; such a function $u$ is called a *separable infinite harmonic function*. The function $\omega$ satisfies the *spherical infinite harmonic problem in* $S$

$$\begin{aligned} \frac{1}{2}\nabla' \left|\nabla'\omega\right|^2 .\nabla'\omega + \beta(2\beta+1)\left|\nabla'\omega\right|^2 \omega + \beta^3(\beta+1)\omega^3 = 0 \qquad & \text{in } S \\ \omega = 0 \qquad & \text{on } \partial S, \end{aligned} \tag{1.3}$$

where $\nabla'$ is the covariant gradient on $S^{N-1}$ for the canonical metric and $(a, b) \mapsto a.b$ the associated quadratic form. The role of the infinite Laplacian for Lipschitz extension of Lipchitz continuous functions defined in a domain has been pointed out by Aronsson in his seminal paper [1]. When the infinity Laplacian $\Delta_\infty$ is replaced by the $p$-Laplacian, the research of regular ($\beta < 0$) separable $p$-harmonic functions has been carried out by Krol [7] in the 2-dim case and by Tolksdorff [13] in the general case. Following Krol' s method, Kichenassamy and Véron [9] studied the 2-dim singular case ($\beta > 0$). Finally, by a completely different approach and in a more general setting Porretta and Véron [12] studied the general case. In that case, the function $\omega$ satisfies the *spherical $p$-harmonic problem in $S$*

$$\begin{aligned} div'\left((\beta^2\omega^2 + |\nabla'\omega|^2)^{\frac{p-2}{2}}\nabla'\omega\right) + \beta\lambda_\beta(\beta^2\omega^2 + |\nabla'\omega|^2)^{\frac{p-2}{2}}\omega = 0 \qquad & \text{in } S \\ \omega = 0 \qquad & \text{on } \partial S, \end{aligned} \tag{1.4}$$

where $\lambda_\beta = \beta(p-1) + p - N$ and $div'$ is the divergence operator acting on vector fields in $TS^{N-1}$.

Following an idea which was introduced by Lasry and Lions [11], Porretta-Véron's method was to transform the equation $(1.4)$ by setting

$$w = -\frac{1}{\beta}\ln\omega \tag{1.5}$$

in the case $\beta > 0$. The function $w$ satisfies the new problem

$$\begin{aligned} -div'\left(\left(1 + |\nabla' w|^2\right)^{p/2-1}\nabla' w\right) + \left(1 + |\nabla' w|^2\right)^{p/2-1}\left(\beta(p-1)|\nabla' w|^2 + \lambda_\beta\right) = 0 \quad & \text{in } S \\ \lim_{\rho(\sigma)\to 0} w(\sigma) = \infty, \end{aligned} \tag{1.6}$$

where $\rho(\sigma) := \text{dist}\,(\sigma, \partial S)$ is the distance is understood in the sense of the geodesic distance on $S$.

In this article we borrow ideas used in [12] to transform problem $(1.1)$ by introducing the function $w$ defined by $(1.5)$. Then $w$ satisfies, in the viscosity sense,

$$\begin{aligned} -\frac{1}{2}\nabla'\left|\nabla' w\right|^2 .\nabla' w + \beta\left|\nabla' w\right|^4 + (2\beta+1)\left|\nabla' w\right|^2 + \beta + 1 = 0 \quad & \text{in } S \\ \lim_{\rho(\sigma)\to 0} w(\sigma) = \infty. \end{aligned} \tag{1.7}$$

We first prove

**Theorem A**. *Let $S \subset S^{N-1}$ be a proper subdomain of $S^{N-1}$ with a $C^3$ boundary. Then for any $\beta > 0$ there exists a Lipschitz continuous function $w$ and a unique $\lambda(\beta) > 0$ such that*

$$\begin{aligned} -\frac{1}{2}\nabla'\left|\nabla' w\right|^2 .\nabla' w + \beta\left|\nabla' w\right|^4 + (2\beta+1)\left|\nabla' w\right|^2 + \lambda(\beta) = 0 \quad & \textit{in } S \\ \lim_{\rho(\sigma)\to 0} w(\sigma) = \infty. \end{aligned} \tag{1.8}$$

*where $\rho(.)$ is the geodesic distance from points in $S$ to $\partial S$.*

We then prove that there exists a unique $\beta$ such that $\lambda(\beta) = \beta + 1$. In a similar way we study the regular case with $\beta < 0$ and we obtain

**Theorem B**. *Let $S \subset S^{N-1}$ be subdomain of $M$ with a $C^3$ boundary. Then there exist exactly two real numbers $\beta_+ > 0$ and $\beta_- < 0$ and at least two positive functions $\omega_+$ and $\omega_-$ in $C^{1,1}(\overline{S})$ (up to multiplication by constants) such that the two functions $u_+$ and $u_-$ defined in $C_S$ by $u_+(r,\sigma) := r^{-\beta_+}\omega_+(\sigma)$ and $u_-(r,\sigma) := r^{-\beta_-}\omega_-(\sigma)$ are infinite harmonic in $C_S$ and vanish on $\partial C_S \setminus \{0\}$ and $\partial C_S$ respectively. Furthermore $\beta_+$ and $\beta_-$ are decreasing functions of $S$ for the inclusion order relation on sets.*

The previous results can be extended to general regular domains on a Riemannian manifold.

In the special case of a rotationally symmetric domain $S$ we have a more precise result which allows us to characterize all the separable infinite harmonic functions in $C_S$ which keep a constant sign and vanish on $\partial C_S \setminus \{0\}$. We denote by $\phi \in (0,\pi)$ the azimuthal angle from the North pole $N$ on $S^{N-1}$.

**Theorem C**. *Let $S_\alpha$ be the spherical cap with azimuthal opening $\alpha \in (0,\pi]$. Then there exist two positive functions $\omega_+$ and $\omega_-$ in $C^\infty(\overline{S})$, vanishing on $\partial S$, such that the two functions*

$$u_+(r,\sigma) = r^{-\frac{\pi^2}{4\alpha(\pi+\alpha)}}\omega_+(\sigma), \tag{1.9}$$

*and*

$$u_-(r,\sigma) = r^{\frac{\pi^2}{4\alpha(\pi-\alpha)}}\omega_-(\sigma), \tag{1.10}$$

*are infinite harmonic in $C_{S_\alpha}$ and vanish on $\partial C_{S_\alpha} \setminus \{0\}$. The two functions $\omega_+$ and $\omega_-$ are unique up to multiplication by constants and depend only on the variable $\phi \in (0,\alpha]$.*

This study reduced to an ordinary differential equation which has been already treated by T. Bhattacharya in [4] and [5]. But for the sake of completeness we present it in the last section of the present paper.

Using these previous results we prove the existence of separable infinite harmonic functions in almost any cone $C_S$.

**Theorem D**. *Assume $S \subset S^{N-1}$ is an outward accessible domain, that is $\partial S = \partial \overline{S}^c$. Then there exist two positive exponents $\beta_+^M \geq \beta_+^m$ and two positive functions $\omega_+^M$ and $\omega_+^m$ in $C^\infty(\overline{S})$, vanishing on $\partial S$ such that the functions*

$$u_+^M(r,\sigma) = r^{-\beta_+^M}\omega_+^M(\sigma) \quad \text{and} \quad u_+^m(r,\sigma) = r^{-\beta_+^m}\omega_+^m(\sigma) \tag{1.11}$$

*are infinite harmonic in $C_S$ and vanish on $\partial C_S \setminus \{0\}$.*

Note that a similar result holds if one considers regular infinite harmonic functions in $C_S$ which vanish on $\partial C_S$.

**Acknowledgements** This article has been prepared with the support of the collaboration programs ECOS C14E08 and FONDECYT grant 1160540 for the three authors.

## 2 The General case

We assume that $(r,\sigma)\in\mathbb{R}_+\times S^{N-1}$ are the spherical coordinates of $x\in\mathbb{R}^N$. If $u$ is a $C^1$ function, then $\nabla u=u_r\mathbf{e}+\frac{1}{r}\nabla' u$ where $\mathbf{e}=\frac{x}{|x|}$ and $\nabla'$ is the tangential gradient of $u(r,.)$ identified to the covariant gradient thanks to the canonical imbedding of $S^{N-1}$ into $\mathbb{R}^N$. Then $|\nabla u|^2=u_r^2+\frac{1}{r^2}|\nabla' u|^2$, thus

$$-\Delta_\infty u=-\left(u_r^2+\frac{1}{r^2}|\nabla' u|^2\right)_r u_r-\frac{1}{r^2}\nabla'\left(u_r^2+\frac{1}{r^2}|\nabla' u|^2\right).\nabla' u=0.$$

A solution $-\Delta_\infty u=0$ which has the form $u(x)=u(r,\sigma)=r^{-\beta}\omega(\sigma)$ satisfies, in the viscosity sense, the spherical infinite harmonic equation

$$\frac{1}{2}\nabla'\left|\nabla'\omega\right|^2.\nabla'\omega+(2\beta+1)\left|\nabla'\omega\right|^2\omega+\beta^3(\beta+1)\omega^3=0. \tag{2.1}$$

**Theorem 2.1.** *For any $C^3$ domain $S\subset S^{N-1}$ there exists a unique $\beta_s>0$ and one nonnegative function $\omega\in C^{0,1}(\overline{S})$ solution of*

$$\begin{aligned} -\frac{1}{2}\nabla'|\nabla'\omega|^2.\nabla'\omega=(2\beta+1)|\nabla'\omega|^2w+\beta^3(\beta+1)\omega^3 \qquad & \textit{in } S\\ \omega=0 \qquad & \textit{in } \partial S. \end{aligned} \tag{2.2}$$

*such that the function $(r,\sigma)\mapsto u_s(r,\sigma):=r^{-\beta_s}\omega(\sigma)$ is positive and $\infty$-harmonic in the cone $C_s=\{x=\lambda\sigma\in\mathbb{R}^N:\lambda>0,\,\sigma\in S\}$ and vanish on $\partial S\setminus\{0\}$.*

Following Porretta-Veron's method, we transform the eigenvalue problem into a large solution problem with absorption by setting

$$w=-\frac{1}{\beta}\ln\omega. \tag{2.3}$$

Therefore the formal new problem is to prove the existence of a unique $\beta>0$ and of a nonnegative function $w$ such that

$$\begin{aligned} \frac{1}{2}\nabla'|\nabla' w|^2.\nabla' w-\beta|\nabla' w|^4-(2\beta+1)|\nabla' w|^2=\beta+1 \qquad & \text{in } S\\ w=\infty \qquad & \text{in } \partial S. \end{aligned} \tag{2.4}$$

The two problems are clearly equivalent for $C^2$ solutions. Since the mapping $w\mapsto\omega$ is smooth and decreasing, it exchanges supersolutions (resp. subsolutions) into subsolutions (resp. supersolutions). Therefore the two problems $(2.2)$-$(2.4)$ are also equivalent if we deal with continuous viscosity solutions.

In order to increase the regularity of the solutions and to avoid the difficulties coming form the fact the above problem is invariant if we add a constant to a solution, instead of $(2.4)$ we consider the regularized problem with absorption

$$\begin{aligned} -\delta\Delta w-\frac{1}{2}\nabla'|\nabla' w|^2.\nabla' w+\gamma|\nabla' w|^4+(2\gamma+1)|\nabla' w|^2+\epsilon w=0 \qquad & \text{in } S\\ w=\infty \qquad & \text{in } \partial S, \end{aligned} \tag{2.5}$$

where $\epsilon, \delta$ are two positive parameters. We will obtain below local estimates on $\nabla' w$ independent of $\epsilon$ and $\delta$. Thanks to these estimates we will let successively $\delta$ and $\epsilon$ to 0 and obtain that, up to a constant, the term $\epsilon w$ converges to some unique $\lambda_\beta$ called *the ergodic constant* although it has a probabilist interpretation only in the case of the ordinary Laplacian [11]. The limit problem of $(2.5)$ is the following

$$\begin{aligned} -\frac{1}{2}\nabla'|\nabla' w|^2.\nabla' w + \gamma|\nabla' w|^4 + (2\gamma+1)|\nabla' w|^2 + \lambda_\beta = 0 \qquad & \text{in } S \\ w = \infty \qquad & \text{in } \partial S. \end{aligned} \tag{2.6}$$

## 2.1 Two-sided estimates

We denote the "positive" geodesic distance $\rho(\sigma) = \text{dist}\,(\sigma, \partial S)$. If $\sigma \in S^c$ we set $\tilde{\rho}(\sigma) = -\text{dist}\,(\sigma, \partial S)$. If $\sigma_1$ and $\sigma_2$ are not antipodal points there exists a unique minimizing geodesic between $\sigma_1$ and $\sigma_2$. It is an arc of a Riemannian circle (or great circle). The geodesic distance between $\sigma_1$ and $\sigma_2$ is denoted by $\ell(\sigma_1, \sigma_2)$. It coincides with the angle determined by the two straight lines from 0 to $\sigma_1$ and 0 to$\sigma_2$. At this point it is convenient to use Fermi coordinates in $S$ in a neighborhood of $\partial S$. We set

$$S_\tau = \{\sigma \in S : \rho(\sigma) < \tau\}\,,\ S'_\beta = S \setminus \overline{S}_\tau\,,\ \Sigma_\beta = \{\sigma \in S : \rho(\sigma) = \tau\}.$$

If $\tau \leq \tau_0$ for any $\sigma \in S_\tau$ there exists a unique $z_\sigma \in \partial S$ such that $\ell(\sigma, z_\sigma) = \rho(\sigma)$. These Fermi coordinates of $\sigma$ are defined by $(\tau, z) \in [0, \tau_0) \times \partial S$. The mapping $\Pi$ such that

$$\Pi(\sigma) = (\rho(\sigma), z_\sigma) \qquad \forall \sigma \in S_{\tau_0},$$

is a $C^2$ diffeomorphism from $S_{\tau_0}$ into $[0, \tau_0) \times \partial S$. The expression of the Laplace-Beltrami operator in $S_{\tau_0}$ is given in [3]:

$$\Delta' u(\sigma) = \frac{\partial^2 u}{\partial \tau^2} - (N-2)H\frac{\partial u}{\partial \tau} + \tilde{\Delta}'_z u \qquad \forall \sigma = \Pi^{-1}((\tau, z)), \tag{2.7}$$

where $H = H(\tau, z)$ is the mean curvature of $\Sigma_\tau$ and $\tilde{\Delta}'_z$ is a second order elliptic operator acting on functions defined on $\Sigma_\tau$. If $g = (g_{ij})$ is the metric tensor on $S^{N-1}$ and by convention, $|g| = det(g_{ij})$, this operator admits the following expression

$$\tilde{\Delta}'_z u = \frac{1}{\sqrt{|g|}} \sum_{j=1}^{N-2} \frac{\partial}{\partial z_j}\left(\sqrt{|g|} a_j \frac{\partial u}{\partial z_j}\right),$$

for some $a_j > 0$ if we take for coordinates curve-frame $z_j$ a system of orthogonal 1-dim great circles on $\Gamma$ intersecting at $z_\sigma$ (these circle corresponds to the $(N-2)$-principal curvatures at this points). The coefficients $a_j$ depend both on $z$ and $\tau$. Thus, if $u$ depends only on $\rho$,

$$\Delta' u(\sigma) = \frac{\partial^2 u}{\partial \tau^2} - (N-2)H\frac{\partial u}{\partial \tau}. \tag{2.8}$$

The expression of $H$ is given in [3] and we can assume that $\tau_0$ is small enough so that $H$ remains bounded.

We extend the geodesic distance $\rho(x) = \text{dist}\,(x, \partial S)$ as a smooth positive function so that $\tilde{\rho}(x) := \rho(x)$ if $\rho(x) \leq \tau_0$ and thus, it the same neighborhood of $\partial S$, $\nabla\tilde{\rho}(x) = \mathbf{n}_{z_x}$, the unit outward normal vector to $\partial S$ at the point $z_x = Proj_{\partial\Omega}(x)$.

If $w$ depends only on $\rho$, $(2.6)$ becomes

$$\begin{aligned} \delta w'' - (N-2)Hw' + w'^2 w'' - \gamma w'^4 - (2\gamma+1)w'^2 - \epsilon w = 0 \qquad & \text{in } S_{\tau_0} \\ w = \infty \qquad & \text{in } \partial S. \end{aligned} \tag{2.9}$$

In the sequel we put

$$\mathcal{P}_\delta(u) := -\delta\Delta u - \frac{1}{2}\nabla'|\nabla' u|^2.\nabla' u + \gamma|\nabla' u|^4 + (2\gamma+1)|\nabla' u|^2 + \epsilon u = \tilde{\mathcal{P}}_\delta(u) + \epsilon u. \tag{2.10}$$

**Proposition 2.2.** *There exist $\tau_1 \in (0, \tau_0]$, three positive constants $M$, $\epsilon_0$ and $\delta_0$ and two positive functions $w^*, w_* \in C^2(S)$ such that $w^* > w_*$ in $S_\tau$, $w^* + \frac{1}{\gamma}\ln\rho \in L^\infty(S)$ and $w_* + \frac{1}{\gamma}\ln\rho \in L^\infty(S)$ with the property that for any $\epsilon \in (0, \epsilon_0]$ and $\delta \in (0, \delta_0]$ the two functions*

$$\bar{w}(\sigma) = w^* + \frac{M}{\epsilon} \tag{2.11}$$

*and*

$$\underline{w}(\sigma) = w_* - \frac{M}{\epsilon} \tag{2.12}$$

*are respectively a supersolution and a subsolution of $\mathcal{P}_\delta(u) = 0$. Furthermore any solution $w$ of problem $(2.6)$ satisfies $\underline{w} \leq w \leq \bar{w}$.*

*Proof.* Let $a > 0$. We first notice by a standard computation that the solutions of

$$\begin{aligned} \delta y'' + y'^2 y'' - \gamma y'^4 - a y'^2 = 0 \qquad & \text{in } (0,1) \\ y'(0) = -\infty \qquad & \end{aligned} \tag{2.13}$$

are negative and given implicitely by

$$\frac{\delta}{a y'(\rho)} + \left(\frac{1}{\gamma} - \frac{\delta}{a}\right)\sqrt{\frac{\gamma}{a}}\tan^{-1}\left(\sqrt{\frac{a}{\gamma}}\frac{1}{y'(\rho)}\right) = -\rho. \tag{2.14}$$

In order to have a global estimate, we set $y' = -z^{-1}$, thus $(2.14)$ becomes

$$\frac{\delta z}{a} + \left(\frac{1}{\gamma} - \frac{\delta}{a}\right)\sqrt{\frac{\gamma}{a}}\tan^{-1}\left(z\sqrt{\frac{a}{\gamma}}\right) = \rho, \tag{2.15}$$

provided $a > \gamma\delta$. Since $\tan^{-1}\left(z\sqrt{\frac{a}{\gamma}}\right) \leq z\sqrt{\frac{a}{\gamma}}$, we derive

$$z(\rho) \geq \gamma\rho \Longleftrightarrow 0 > y'(\rho) \geq -\frac{1}{\gamma\rho} \qquad \forall \rho > 0, \tag{2.16}$$

with equality only if $\rho = 0$. Since we can write $(2.15)$ as

$$\sqrt{\frac{\gamma}{a}}\frac{\delta}{a}\left(z\sqrt{\frac{a}{\gamma}}\right) + \left(\frac{1}{\gamma} - \frac{\delta}{a}\right)\sqrt{\frac{\gamma}{a}}\tan^{-1}\left(z\sqrt{\frac{a}{\gamma}}\right) = \rho \geq \tan^{-1}\left(z\sqrt{\frac{a}{\gamma}}\right), \tag{2.17}$$

we obtain

$$z \leq \sqrt{\frac{\gamma}{a}} \tan\left(\rho\sqrt{a\gamma}\right) \Longleftrightarrow y'(\rho) \leq -\sqrt{\frac{a}{\gamma}} \cot\left(\rho\sqrt{a\gamma}\right) \qquad \forall \rho > 0. \tag{2.18}$$

Finally

$$-\frac{1}{\gamma\rho} \leq y'(\rho) \leq -\sqrt{\frac{a}{\gamma}} \cot\left(\rho\sqrt{a\gamma}\right) \qquad \forall \rho \in (0, \tau_0]. \tag{2.19}$$

and in particular, for any $\tau_1 \in (0, \tau_0]$,

$$|y'(\rho)| \geq \sqrt{\frac{a}{\gamma}} \cot\left(\tau_1\sqrt{a\gamma}\right) \qquad \forall \rho \in (0, \tau_1]. \tag{2.20}$$

From this estimate we derive

$$y(\rho_0) + \frac{1}{\gamma} \ln\left(\frac{\sin \rho_0\sqrt{a\gamma}}{\sin \rho\sqrt{a\gamma}}\right) \leq y(\rho) \leq y(\rho_0) + \frac{1}{\gamma} \ln\left(\frac{\rho_0}{\rho}\right) \qquad \forall \rho \in (0, \tau_1]. \tag{2.21}$$

The solution $y$ depends on the value of $a$ and $\delta$. Since $\partial S$ is smooth, we can assume that $(N-2)\,|H|$ is bounded by some constant $m \geq 0$ in $S_{\tau_0}$. Denote by $w_\tau$ a solution satisfying $w(\tau) = 0$, then it is positive in $S_\tau$ and

$$\begin{aligned}
\mathcal{P}_\delta(w_\tau) &= \delta(N-2)Hw'_\tau + (2\gamma + 1 - a)w'^2_\tau + \epsilon w_\tau \\
&\geq |w'_\tau|((2\gamma + 1 - a)|w'_\tau| - m) \\
&\geq |w'_\tau|((2\gamma + 1 - a)\sqrt{\frac{a}{\gamma}} \cot\left(\tau\sqrt{a\gamma}\right) - m).
\end{aligned}$$

If we take $1 < a_1 < 2\gamma + 1$, we choose $\tau := \tau_1 \in (0, \tau_0]$ such that

$$(2\gamma + 1 - a_1)\sqrt{\frac{a}{\gamma}} \cot\left(\tau\sqrt{a_1\gamma}\right) > m, \tag{2.22}$$

which implies that $w_\tau$ is a supersolution in $S_\tau$. In assuming now $a := a_2 > 2\gamma + 1$, we also have,

$$\begin{aligned}
\mathcal{P}_\delta(w_\tau) &\leq w'_\tau((2\gamma + 1 - a)w'_\tau - m) + \epsilon w_\tau \\
&\leq |w'_\tau|\left((2\gamma + 1 - a)|w'_\tau| + m\right) + \epsilon w_\tau \\
&\leq |w'_\tau|\left(m - (a - 2\gamma - 1)\sqrt{\frac{a}{\gamma}} \cot\left(\tau\sqrt{a\gamma}\right)\right) + \epsilon w_\tau.
\end{aligned}$$

We choose $a_2 = 4\gamma + 2 - a_1$, then

$$m - (a - 2\gamma - 1)\sqrt{\frac{a}{\gamma}} \cot\left(\tau\sqrt{a\gamma}\right) \leq -c < 0 \qquad \forall \tau \in (0, \tau_1].$$

Therefore

$$\mathcal{P}_\delta(w_\tau) \leq w_\tau\left(\epsilon + \frac{w'_\tau}{w_\tau}\right).$$

Since $w'_{\tau_1} < 0$ and $w_{\tau_1}(\tau_1) = 0^+$, there exists $\epsilon_0 > 0$, such that for any $\epsilon \in (0, \epsilon_0]$

$$\epsilon + \frac{w'_{\tau_1}(\rho)}{w_{\tau_1}(\rho)} \leq -1 \qquad \forall \rho \in (0, \tau_1].$$

Therefore $w_{\tau_1,a_1}$ and $w_{\tau_1,a_2}$ are respectively supersolution and subsolution of $\mathcal{P}_\delta(u) = 0$ in $S_{\tau_1}$. We extend them in $S'_{\tau_1}$ as smooth functions $\tilde{w}_{\tau_1,a_1}$ and $\tilde{w}_{\tau_1,a_2}$ in order $\left|\tilde{\mathcal{P}}_\delta(\tilde{w}_{\tau_1,a_j})\right|$ to remain bounded by some constant $M$. Finally $\bar{w} = \tilde{w}_{\tau_1,a_1} + M\epsilon^{-1}$ is a supersolution and $\underline{w} = \tilde{w}_{\tau_1,a_2} + M\epsilon^{-1}$ is a subsolution of $\mathcal{P}_\delta(u) = 0$.

Next, we replace $\underline{w}$ by

$$\underline{w}_h(\delta) = \underline{w}(\delta + h)$$

and $\bar{w}$ by

$$\bar{w}_h = \bar{w}(\delta - h)$$

for $h$ small enough, we still have a sub and a super solution of $\mathcal{P}_\delta(u) = 0$ in $S_{\tau_1}$ and $S_{\tau_1} \setminus S_h$. In the remaining part of $S$, we extend smoothly $\underline{w}_h$ and $\bar{w}_h$ in order $\tilde{\mathcal{P}}_\delta(\underline{w}_h)$ and $\tilde{\mathcal{P}}_\delta(\bar{w}_h)$ be bounded. We can adjust $M$ in order $\mathcal{P}_\delta(\underline{w}_h) \leq 0$ and $\mathcal{P}_\delta(\bar{w}_h) \geq 0$ in whole $S$, and all these manipulations can be done uniformly with respect to $h$ and $\epsilon$. If $w$ is any $C^2$ solution of (2.5), we prove that it dominates the subsolution $\underline{w}_h$ in $S$: actually, if we assume that $\underline{w}_h$ and $w$ are not ordered in $S$, there exists $\sigma_0 \in S$ such that

$$\underline{w}_h(\sigma_0) - w(\sigma_0) = \max\{\underline{w}_h(\sigma) - w(\sigma) : \sigma \in S\} > 0.$$

Since the two functions are $C^2$,

$$\nabla \underline{w}_h(\sigma_0) = \nabla w(\sigma_0) \quad \text{and} \quad D^2\underline{w}_h(\sigma_0) \leq D^2 w(\sigma_0),$$

where $D^2$ is the Hessian form, in the sense of quadratic forms, i.e.

$$D^2\underline{w}_h(\sigma_0)(\nabla\underline{w}_h(\sigma_0), \nabla\underline{w}_h(\sigma_0)) \leq D^2 w(\sigma_0)(\nabla w(\sigma_0), \nabla w(\sigma_0)).$$

This implies $\mathcal{P}_\delta(\underline{w}_h)(\sigma_0) > \mathcal{P}_\delta(w)(\sigma_0) = 0$, contradiction. Therefore

$$\underline{w}_h \leq w \qquad \text{in } S, \tag{2.23}$$

uniformly with respect to $h$. Similarly

$$\bar{w}_h \geq w \qquad \text{in } S. \tag{2.24}$$

Letting $h$ tend to 0 the claim follows. □

## 2.2 Gradient estimates

If $\sigma_0 \in S^{N-1}$ and $R < \pi$, we set $B_R(\sigma_0) = \{\sigma \in S^{N-1} : \ell(\sigma, \sigma_0) < R\}$.

**Proposition 2.3.** *Let $0 \leq \delta, \epsilon \leq 1$ and $w$ be a smooth solution of*

$$-\frac{1}{2}\nabla'|\nabla' w|^2.\nabla w - \delta\Delta w + \gamma|\nabla' w|^4 + (2\gamma+1)|\nabla' w|^2 + \epsilon w = 0 \quad \textit{in } B_R(\sigma_0) \subset S, \tag{2.25}$$

*where $S$ is a domain of $S^{N-1}$. Then there exists $c = c(N) > 0$ such that*

$$|\nabla' w(\sigma_0)| \leq \frac{c}{\gamma R}. \tag{2.26}$$

*Proof.* We set $z = |\nabla w|^2$, then $2\Delta_\infty w = \nabla'|\nabla' w|^2.\nabla' w = \nabla' z.\nabla' w$. We define the linearized operator of $\Delta_\infty$ at $w$ following $h$ by

$$B_w(h) := \frac{d}{dt}\Delta_\infty(w+th)\lfloor_{t=0} = \frac{1}{2}\nabla' h.\nabla' z + \nabla' w.\nabla'(\nabla' w.\nabla' h).$$

Thus the linearized operator of $\Delta_\infty + \delta\Delta$ at $w$ following $h$ is

$$\mathcal{L}_w(h) = B_w(h) + \delta\Delta h. \tag{2.27}$$

Thus

$$\mathcal{L}_w(z) = \frac{1}{2}|\nabla' z|^2 + \nabla' w.\nabla'(\nabla'.\nabla z) + \delta\Delta z.$$

We can re-write $(2.25)$ under the form

$$\nabla' w.\nabla z = 2\left(\gamma z^2 + (2\gamma+1)z + \epsilon w - \delta\Delta w\right). \tag{2.28}$$

Hence

$$\nabla'(\nabla' w.\nabla' z) = 2\left((2\gamma z + 2\gamma + 1)\nabla' z + \epsilon\nabla' w - \delta\nabla'(\Delta w)\right),$$

and then

$$\nabla w.\nabla'(\nabla' w.\nabla' z) = 2\left((2\gamma z + 2\gamma + 1)\nabla' z.\nabla' w + \epsilon z - \delta\nabla'(\Delta w).\nabla' w\right).$$

By the Weitznböck formula, since $\mathrm{Ricc}\,(S^{N-1}) = (N-2)g_0$ ($g_0$ is the metric tensor on $S^{N-1}$), we have

$$\begin{aligned}\frac{1}{2}\Delta z &= \left|D^2 w\right|^2 + \nabla'(\Delta w).\nabla' w + (N-2)|\nabla' w|^2\\ &= \left|D^2 w\right|^2 + \nabla'(\Delta w).\nabla' w + (N-2)z.\end{aligned}$$

Hence

$$\begin{aligned}\mathcal{L}_w(z) = \frac{1}{2}\left|\nabla z\right|^2 &+ 2\left((2\gamma z + 2\gamma + 1)\nabla' z.\nabla' w + 2\epsilon z - 2\delta\nabla' w.\nabla'(\Delta w)\right)\\ &+ 2\delta\left|D^2 w\right|^2 + 2\delta\nabla'(\Delta w).\nabla' w + 2\delta(N-2)z.\end{aligned}$$

Expanding the above identity, we see that the terms of order 3 disappear, hence

$$\mathcal{L}_w(z) = \frac{1}{2}\left|\nabla z\right|^2 + 2\left((2\gamma z + 2\gamma + 1)\nabla' z.\nabla' w + 2\epsilon z\right) + 2\delta\left|D^2 w\right|^2 + 2\delta(N-2)z. \tag{2.29}$$

If $\xi \in C_c^2(\overline{B}_R(\sigma_0))$, we set $Z = \xi^2 z$ and we derive

$$\begin{aligned}\mathcal{L}_w(Z) &= B_w(\xi^2 z) + \delta\Delta(\xi^2 z)\\ &= \xi^2\mathcal{L}_w(z) + z\mathcal{L}_w\xi^2 + 2(\nabla' w.\nabla'\xi^2)(\nabla' w.\nabla z) + 2\delta\nabla'\xi^2.\nabla z,\end{aligned}$$

where

$$\begin{aligned}\mathcal{L}_w\xi^2 &= \frac{1}{2}\nabla' z.\nabla'\xi^2 + \nabla' w.\nabla'(\nabla' w.\nabla'\xi^2) + \delta\Delta\xi^2\\ &= \frac{1}{2}\nabla' z.\nabla'\xi^2 + D^2 w(\nabla'\xi^2).\nabla' w + D^2\xi^2(\nabla' w).\nabla' w + \delta\Delta\xi^2\\ &= \nabla' z.\nabla'\xi^2 + D^2\xi^2(\nabla' w).\nabla' w + \delta\Delta\xi^2\\ &\geq \nabla' z.\nabla'\xi^2 - z\left|D^2\xi^2\right| + \delta\Delta\xi^2.\end{aligned}$$

By Schwarz inequality, $(\Delta w)^2 \leq \frac{1}{N-1}\left|D^2 w\right|^2$, we derive from (2.27) and (2.28),

$$\begin{aligned}\mathcal{L}_w(z) &\geq \frac{1}{2}\left|\nabla z\right|^2 + 4\left(2\gamma z + 2\gamma + 1\right)\left(\gamma z^2 + (2\gamma+1)z + \epsilon w - \delta\Delta w\right) + 4\epsilon z \\ &\quad + \frac{2\delta}{N-1}(\Delta w)^2 + 2\delta(N-2)z \\ &\geq \frac{1}{2}\left|\nabla z\right|^2 + \frac{\delta}{N-1}(\Delta w)^2 + 4\gamma^2 z^3 - c_0,\end{aligned}$$

for some $c_0 = c_0(N,\gamma) > 0$. In the sequel the different positive constants $c_j$ which will appear bellow depend only on $N$ and $\gamma$. This implies

$$\begin{aligned}\mathcal{L}_w(Z) &\geq z\left(\nabla' z.\nabla'\xi^2 - z\left|D^2\xi^2\right| + \delta\Delta\xi^2\right) + 2(\nabla' w.\nabla'\xi^2)(\nabla' w.\nabla z) + 2\delta\nabla'\xi^2.\nabla z \\ &\quad + \xi^2\left(\frac{1}{2}\left|\nabla z\right|^2 + \frac{\delta}{N-1}(\Delta w)^2 + 4\gamma^2 z^3 - c_0\right).\end{aligned} \tag{2.30}$$

We choose $\xi$ such that $0 \leq \xi \leq 1$, $|\nabla\xi| \leq c_1 R^{-1}$ and $\left|D^2\xi\right| \leq c_1 R^{-2}$, then

$$(z+2\delta)\left|\nabla' z.\nabla'\xi^2\right| \leq c_1\frac{(z+2\delta)\xi}{R}\left|\nabla' z\right| \leq \frac{\xi^2}{8}\left|\nabla' z\right|^2 + c_2\frac{(z+2\delta)^2}{R^2},$$

$$\left|z\left(\delta\Delta\xi^2 - z\left|D^2\xi^2\right|\right)\right| \leq \frac{c_3(z+2\delta)^2}{R^2},$$

$$\left|\nabla' w.\nabla z\right| \leq \sqrt{z}\left|\nabla z\right|,$$

$$\left|\nabla' w.\nabla\xi^2\right| \leq 2\xi\left|\nabla w\right|\left|\nabla\xi\right| \leq \frac{2c_1\xi\sqrt{z}}{R},$$

$$\left|2(\nabla' w.\nabla'\xi^2)(\nabla' w.\nabla z)\right| \leq \frac{4c_1\xi z\left|\nabla z\right|}{R} \leq \frac{\xi^2}{8}\left|\nabla' z\right|^2 + c_4\frac{z^2}{R^2}.$$

We consider a point $z_0 \in B_R$ where $Z$ is maximal, then $\mathcal{L}_w(Z)(z_0) \leq 0$, which implies that at this point,

$$\xi^2\left(\frac{1}{2}\left|\nabla z\right|^2 + \frac{\delta}{N-1}(\Delta w)^2 + 4\gamma^2 z^3 - c_0\right) \leq \frac{\xi^2}{4}\left|\nabla z\right|^2 + \frac{c_5(z+2\delta)^2}{R^2}. \tag{2.31}$$

We assume $R \leq 1$ and $2\delta \leq 1$, we multiply by $\xi^4$ and obtain

$$\frac{1}{4}\left|\xi^3\nabla z\right|^2 + \frac{\delta\xi^6}{N-1}(\Delta w)^2 + 4\gamma^2(\xi^2 z)^3 \leq \frac{c_6((\xi^2 z)^2+1)}{R^2} + c_0. \tag{2.32}$$

From the inequality

$$4\gamma^2(\xi^2 z)^3 \leq \frac{c_6(\xi^2 z)^2}{R^2} + \frac{c_7}{R^2},$$

we deduce

$$\xi^2 z \leq \frac{c_8}{R^2} \qquad \text{with } c_8 = \max\left\{c_7, \tfrac{c_6}{\gamma^2}\right\}. \tag{2.33}$$

If we assume that $\xi(\sigma_0) = 1$, we finally infer

$$|\nabla w(\sigma_0)| \leq \frac{\sqrt{c_8}}{R}, \tag{2.34}$$

which is the claim. □

As an immediate consequence, we have

**Corollary 2.4.** *Let $0 \leq \epsilon, \delta \leq 1$. If $w$ is a solution of $(2.5)$ in $S$, it satisfies*

$$|\nabla w(\sigma)| \leq \frac{c}{\gamma\rho(\sigma)} \qquad \forall \sigma \in S, \tag{2.35}$$

*for some $c > 0$ depending only of $N$.*

## 2.3 Proof of Theorem A

We write $w = w_{\delta,\epsilon,\gamma}$ and

$$-\delta\Delta w - \frac{1}{2}\nabla\left|\nabla' w\right|^2 \nabla' w + \gamma\left|\nabla' w\right|^4 + (2\gamma+1)\left|\nabla' w\right|^2 + \epsilon w = 0, \tag{2.36}$$

then $w_{\delta,\epsilon,\gamma}$ satisfies the estimate $(2.35)$. By Proposition 2.2 it satisfies also

$$-\frac{1}{\gamma}\ln\rho - \frac{M}{\epsilon} \leq w_* \leq w_{\delta,\epsilon,\gamma} \leq w^* + \frac{M}{\epsilon} \leq -\frac{1}{\gamma}\ln\rho + \frac{M}{\epsilon}. \tag{2.37}$$

The set of functions $\{w_{\delta,\epsilon,\gamma}\}_{\epsilon,\delta}$ is clearly locally equicontinuous in $S$. By classical stability results on viscosity solutions (see e.g. [8, Chap 3]), there exist a subsequence $\{w_{\delta_n,\epsilon,\gamma}\}$ and a function $w_{\epsilon,\gamma}$ such that $w_{\delta_n,\epsilon,\gamma} \to w_{\epsilon,\gamma}$, and $w_{\epsilon,\gamma}$ is a viscosity solution of

$$\begin{aligned} -\frac{1}{2}\nabla'|\nabla' w|^2.\nabla' w + \beta|\nabla' w|^4 + (2\beta+1)|\nabla' w|^2 + \epsilon w = 0 \qquad &\text{in } S \\ w = \infty \qquad &\text{in } \partial S. \end{aligned} \tag{2.38}$$

Furthermore $w_{\epsilon,\gamma}$ satisfies the same estimates $(2.35)$ and $(2.37)$ as $w_{\delta,\epsilon,\gamma}$. Put $\tilde{w}_{\epsilon,\gamma}(\sigma) = w_{\epsilon,\gamma}(\sigma) - w_{\epsilon,\gamma}(\sigma_0)$ with $\sigma_0 \in \Omega$, then $\tilde{w} := \tilde{w}_{\epsilon,\gamma}$ satisfies

$$-\frac{1}{2}\nabla\left|\nabla'\tilde{w}\right|^2 \nabla'\tilde{w} + \gamma\left|\nabla'\tilde{w}\right|^4 + (2\gamma+1)\left|\nabla'\tilde{w}\right|^2 + \epsilon\tilde{w} + \epsilon w(\sigma_0) = 0. \tag{2.39}$$

Moreover

$$|\tilde{w}_{\epsilon,\gamma}(\sigma)| = |w_{\epsilon,\gamma}(\sigma) - w_{\epsilon,\gamma}(\sigma_0)| \leq \max\left\{\frac{c}{\gamma\rho(\tau)} : \tau \in [\sigma,\sigma_0]\right\}|\sigma - \sigma_0|.$$

Thus, as $\epsilon \to 0$, $\epsilon\tilde{w}_{\epsilon,\gamma} \to 0$ locally uniformly in $S$. Up to some subsequence $\{\epsilon_n\}$, $\tilde{w}_{\epsilon_n,\gamma} \to w_\gamma$ locally uniformly in $S$ and $\epsilon_n w_{\epsilon_n,\gamma}(\sigma_0) \to \lambda(\gamma)$. As in [12] the expression $\lambda(\gamma)$ does not depend on $\sigma_0$. By analogy with the semilinear case studied in [11], this last limit is called *the ergodic limit*. By the same stability results of viscosity solutions, we infer that $w_\gamma$ is a positive solution of

$$\begin{aligned} -\frac{1}{2}\nabla\left|\nabla' w\right|^2 \nabla' w + \gamma\left|\nabla' w\right|^4 + (2\gamma+1)\left|\nabla' w\right|^2 + \lambda(\gamma) = 0 \quad &\text{in } S \\ w = \infty \quad &\text{on } \partial S. \end{aligned} \tag{2.40}$$

Furthermore, there holds from $(2.37)$ and $(2.35)$,

$$\left|w_\gamma + \frac{1}{\gamma}\ln\rho\right| \leq M, \tag{2.41}$$

and

$$|\nabla w_\gamma| \leq \frac{c}{\gamma\rho}. \tag{2.42}$$

**Proposition 2.5.** *For any $C^3$ domain $S \subset S^{N-1}$, the ergodic limit $\lambda(\gamma) := \lambda(\gamma, S)$ is uniquely determined by $\gamma$. Furthermore it is a continuous decreasing function of $\gamma$ and $S$ for the order relation of inclusion.*

*Proof.* Assume that the set $\{\epsilon w_\epsilon(\sigma_0)\}$ of values of the solutions of (2.40) at $\sigma_0$ admits two different cluster points $\lambda_1$ and $\lambda_2$. Then there exist two locally Liptchitz continuous functions $w_1$ and $w_2$ satisfying

$$-\frac{1}{2}\nabla'|\nabla' w_i|^2.\nabla' w_i + \gamma|\nabla' w_i|^4 + (2\gamma+1)|\nabla' w_i|^2 + \lambda_i = 0 \qquad \text{in } S, \tag{2.43}$$

in the viscosity sense, and such that

$$w_i(\sigma) = -\frac{1}{\gamma}\ln\rho(\sigma)\,(1+o(1)) \qquad \text{as } \rho(\sigma) \to 0. \tag{2.44}$$

We can assume that $\lambda_1 > \lambda_2$. For $\epsilon > 0$ let $v = (1+\epsilon)w_2$. Then

$$-\frac{1}{2}\nabla'|\nabla' v|^2.\nabla' v + (1+\epsilon)^{-1}\gamma|\nabla' v|^4 + (1+\epsilon)(2\gamma+1)|\nabla' v|^2 + (1+\epsilon)^3\lambda_2 = 0 \qquad \text{in } S. \tag{2.45}$$

For $X > 0$, we put

$$f(X) = \frac{\gamma\epsilon}{1+\epsilon}X^2 - (2\gamma+1)\epsilon X + \lambda_1 - (1+\epsilon)^3\lambda_2.$$

Then

$$f(X) \geq f(X_0) = f\left(\frac{(2\gamma+1)(1+\epsilon)}{2\gamma}\right) = -\frac{\epsilon(1+\epsilon)(2\gamma+1)^2}{4\gamma} + \lambda_1 - (1+\epsilon)^3\lambda_2.$$

Therefore there exists $\epsilon_0 > 0$ such that for any $X \geq 0$, $f(X) \geq 0$, or equivalently

$$(1+\epsilon)^{-1}\gamma X^2 + (1+\epsilon)(2\gamma+1)X + (1+\epsilon)^3\lambda_2 \leq \gamma X^2 + (2\gamma+1)X + \lambda_1. \tag{2.46}$$

This implies that

$$-\frac{1}{2}\nabla'|\nabla' v|^2.\nabla' v + \gamma|\nabla' v|^4 + (2\gamma+1)|\nabla' v|^2 + \lambda_1 \geq 0 \qquad \text{in } S, \tag{2.47}$$

in the viscosity sense. Since $w_1 < v$ near $\partial S$, it follows from comparison principle that $w_1 < v$ in $S$. Letting $\epsilon \to 0$ yields

$$w_1 \leq w_2 \quad \text{in } S. \tag{2.48}$$

Since for any $k \in \mathbb{R}$, $w_1 + k$ satisfies the same equation as $w_1$ and the same estimate (2.44) as the $w_i$ we obtain a contradiction. Thus $\lambda = \lambda(\gamma)$ is uniquely determined.

For proving monotonicity, assume $\gamma_1 > \gamma_2 > 0$ and let $w_{\epsilon,1}$ and $w_{\epsilon,2}$ be solutions of

$$-\frac{1}{2}\nabla'|\nabla' w_{\epsilon,i}|^2.\nabla' w_{\epsilon,i} + \gamma_i|\nabla' w_{\epsilon,i}|^4 + (2\gamma_i+1)|\nabla' w_{\epsilon,i}|^2 + \epsilon w_{\epsilon,i} = 0 \qquad \text{in } S, \tag{2.49}$$

such that

$$w_{\epsilon,i}(\sigma) = -\frac{1}{\gamma_i}\ln\rho(\sigma)\,(1+o(1)) \qquad \text{as } \rho(\sigma)\to 0. \tag{2.50}$$

Then

$$-\frac{1}{2}\nabla'|\nabla' w_{\epsilon,1}|^2.\nabla' w_{\epsilon,1} + \gamma_2|\nabla' w_{\epsilon,1}|^4 + (2\gamma_2+1)|\nabla' w_{\epsilon,1}|^2 + \epsilon w_{\epsilon,1} \le 0.$$

Since $w_{\epsilon,1} \le w_{\epsilon,2}$ near $\partial S$, it follows by comparison principle that $w_{\epsilon,1} \le w_{\epsilon,2}$ in $S$ and in particular $\epsilon w_{\epsilon,1} \le \epsilon w_{\epsilon,2}$. Since $\lambda_1 = \lim_{n\to\infty} \epsilon w_{\epsilon_n,1}(x_0)$ and $\lambda_2 = \lim_{n\to\infty} \epsilon w_{\epsilon_n,2}(x_0)$, we infer that $\lambda_1 \le \lambda_2$.

For proving the continuity, let $\{\gamma_n\}$ be a sequence converging to $\gamma$ and let $w_n$ be corresponding solutions of

$$-\frac{1}{2}\nabla'|\nabla' w_n|^2.\nabla' w_n + \gamma_n|\nabla' w_n|^4 + (2\gamma_n+1)|\nabla' w_n|^2 + \lambda(\gamma_n) = 0 \qquad \text{in } S, \tag{2.51}$$

subject to

$$\left| w_n(\sigma) + \frac{1}{\gamma_n}\ln\rho(\sigma)\right| \le K, \tag{2.52}$$

for some $K>0$ independent of $n$. Since $\{w_n\}$ is locally bounded in $W^{1,\infty}_{loc}(\Omega)$ we can extract sequences, denoted by $\{w_{n_k}\}$, $\{\lambda(w_{n_k})\}$ such that $\lambda(w_{n_k}) \to \bar\lambda$ and $w_{n_k}$ converges locally uniformly to a viscosity solution $w$ of

$$-\frac{1}{2}\nabla'|\nabla' w|^2.\nabla' w + \gamma|\nabla' w|^4 + (2\gamma+1)|\nabla' w|^2 + \bar\lambda = 0 \qquad \text{in } S, \tag{2.53}$$

subject to $w(\sigma) = -\dfrac{1}{\gamma}\ln\rho(\sigma)\,(1+o(1))$ as $\rho(\sigma)\to 0$. The existence of such a function implies that $\bar\lambda = \lambda(\gamma)$. Thus the whole sequence $\{\lambda(\gamma_n)\}$ converges to $\lambda(\gamma)$, a fact which implies the continuity.

Next, let $S_1 \subset S_2$ be two $C^3$ subdomains of $S^{N-1}$. We denote by $w_{\delta,\epsilon,\gamma,S_j}$, $j=1,2$, the solutions of (2.5) respectively in $S_1$ and $S_2$. Since these solutions are limit of solutions with finite boundary values and that the maximum principle holds, we infer that $w_{\delta,\epsilon,\gamma,S_2} \le w_{\delta,\epsilon,\gamma,S_1}$ in $S_1$. Letting $\delta\to 0$ yields $w_{\epsilon,\gamma,S_2} \le w_{\epsilon,\gamma,S_1}$. Taking $\sigma_0 \in S_1$ and since the ergodic constant is uniquely determined, we have $\epsilon w_{\epsilon,\gamma,S_2}(\sigma_0) \le \epsilon w_{\epsilon,\gamma,S_1}(\sigma_0)$ and thus $\lambda(\gamma,S_2) \le \lambda(\gamma,S_1)$. □

## 2.4 Proof of Theorem B

We prove below the following proposition using the result of Theorem C, which proof does not depend on the previous constructions.

**Proposition 2.6.** *For any $C^3$ domain $S \subset S^{N-1}$, there exists a unique $\beta := \beta_S$ such that $\lambda(\beta) = \beta+1$. Furthermore $\beta_S$ is a decreasing function of $S$ for the order relation between spherical domains.*

*Proof.* The function $\gamma \mapsto \lambda(\gamma,S) - \gamma$ is continuous and decreasing. For $\epsilon>0$ we consider two spherical caps $S_i \subset S \subset S_e$; by Proposition 2.5

$$\lambda(\gamma,S_e) \le \lambda(\gamma,S) \le \lambda(\gamma,S_i), \tag{2.54}$$

then

$$\lambda(\gamma,S_e) - \gamma \le \lambda(\gamma,S) - \gamma \le \lambda(\gamma,S_i) - \gamma. \tag{2.55}$$

By Theorem C, there exists $\gamma = \beta_{S_e}$ and $\gamma = \beta_{S_i}$ such that

$$\lambda(\beta_{S_e}, S_e) - \beta_{S_e} = 1 \quad \text{and} \ \lambda(\beta_{S_i}, S_i) - \beta_{S_i} = 1,$$

and $\beta_{S_e} < \beta_{S_i}$ unless $\lambda(\beta_{S_e}, S_e) = \lambda(\beta_{S_i}, S_i)$ and $S_i = S_e$. This implies that

$$\lambda(\beta_{S_e}, S) - \beta_{S_e} \geq 1 \quad \text{and} \ \lambda(\beta_{S_i}, S) - \beta_{S_i} \leq 1. \tag{2.56}$$

By continuity there exists a unique $\beta = \beta_S \in [\beta_{S_e}, \beta_{S_i}]$ such that $\lambda(\beta_S, S) - \beta_S = 1$. To this exponent $\beta$ corresponds a locally Lipschitz continuous function $w$ solution of problem $(2.4)$. Then $\omega = e^{-\beta w}$ is a viscosity solution of $(2.2)$. Notice also that the construction of $\beta_S$ and the monotonicity of $S \mapsto \lambda(\gamma, S)$ imply that $S \mapsto \beta_S$ is decreasing.

Similarly we can consider separable infinite harmonic functions under the form $(1.2)$ with negative $\beta < 0$. We set $\tilde{\beta} = -\beta$, then $(2.2)$ is replaced by

$$\begin{aligned} -\frac{1}{2}\nabla'|\nabla'\omega|^2.\nabla'\omega = \tilde{\beta}(2\tilde{\beta}-1)|\nabla'\omega|^2 w + \tilde{\beta}^3(\tilde{\beta}-1)\omega^3 \qquad & \text{in } S \\ \omega = 0 \qquad & \text{in } \partial S. \end{aligned} \tag{2.57}$$

If $\omega$ is a positive solution of $(2.57)$, we set

$$w = -\frac{1}{\tilde{\beta}} \ln \omega.$$

Then $w$ satisfies

$$\begin{aligned} -\frac{1}{2}\nabla' \left|\nabla' w\right|^2 .\nabla' w + \tilde{\beta} \left|\nabla' w\right|^4 + (2\tilde{\beta}-1) \left|\nabla' w\right|^2 = \tilde{\beta} - 1 \quad \text{in } S \\ \lim_{\rho(\sigma)\to 0} w(\sigma) = \infty, \end{aligned} \tag{2.58}$$

This equation is treated similarly as $(2.4)$. □

*Remark.* It is an open problem whether the positive functions which satisfy $(2.2)$ are unique up to the multiplication by a constant. Up to now the only thing that we can prove is that any two positive solutions $\omega_1$ and $\omega_2$ satisfy

$$0 < \inf_S \frac{\omega_1}{\omega_2} \leq \sup_S \frac{\omega_1}{\omega_2} < \infty. \tag{2.59}$$

# 3 The spherical cap problem

## 3.1 Problem on the circle

We consider here the special case $N = 2$ and $S$ is the circle in (2.2 ). For $k \in \mathbb{N}_*$, we set

$$\beta_k = \frac{k^2}{2k+1} \quad \text{and} \ \tilde{\beta}_k = \frac{k^2}{1-2k}. \tag{3.1}$$

**Proposition 3.1.** *For any $k \in \mathbb{N}^*$ there exists two $\frac{\pi}{k}$-anti-periodic $C^1$ functions $\omega_k$ and $\tilde{\omega}_k$ positive on $(0, \frac{\pi}{k})$ such that $x \mapsto |x|^{-\beta_k}\omega_k(\frac{x}{|x|})$ is infinite harmonic and singular in $\mathbb{R}^2 \setminus \{0\}$ and $x \mapsto |x|^{-\tilde{\beta}_k}\tilde{\omega}_k(\frac{x}{|x|})$ is infinite harmonic and regular in $\mathbb{R}^2$.*

*Proof.* We write $\nabla'\omega = \omega_\sigma \mathbf{e}^\perp$ with $S^1 \sim \mathbb{R}/2\pi$. Thus (2.2) becomes

$$-\omega_\sigma^2\omega_{\sigma\sigma} = \beta^3(\beta+1)\omega^3 + \beta(2\beta+1)\omega_\sigma^2\omega, \qquad \omega(0) = \omega(2\pi). \tag{3.2}$$

For $\omega \neq 0$ the equation in (3.2 ) can be written as

$$-\frac{\omega_\sigma^2}{\omega^2}\frac{\omega_{\sigma\sigma}}{\omega} = \beta(2\beta+1)\frac{\omega_\sigma^2}{\omega^2} + \beta^3(\beta+1).$$

We set $Y = \dfrac{\omega_\sigma}{\omega}$, then $Y_\sigma + Y^2 = \dfrac{\omega_{\sigma\sigma}}{\omega}$ and

$$-Y^2Y_\sigma = Y^4 + \beta(2\beta+1)Y^2 + \beta^3(\beta+1) = (Y^2+\beta^2)(Y^2+\beta(\beta+1)). \tag{3.3}$$

We first search for solutions with $\beta$ such that $\beta(\beta+1) \geq 0$, $\beta \neq 0$. Standard computation yields

$$\left(\frac{\beta}{Y^2+\beta^2} - \frac{\beta+1}{Y^2+\beta(\beta+1)}\right) Y_\sigma = 1, \tag{3.4}$$

and this equation is not degenerate and equivalent to (3.2) as long as $Y \neq 0$. If $\beta = -1$ (that is $\tilde{\beta}_1$ in (3.1 )) then (3.4) becomes

$$\frac{Y_\sigma}{Y^2+1} = -1, \tag{3.5}$$

thus

$$\tan^{-1} Y(\sigma) = -\sigma \Longrightarrow Y(\sigma) = -\tan\sigma \Longrightarrow \omega(\sigma) = \sin\sigma. \tag{3.6}$$

This corresponds to the fact that the coordinate functions are separable and infinite harmonic.
We assume now $\beta(\beta+1) > 0$, or equivalently either $\beta > 0$ or $\beta < -1$. We fix $\omega(0) = 0$ and consider an interval on the right of 0 where $\omega > 0$. From the equation $\omega$ is concave, thus $\omega_\sigma(0) > 0$. Because of concavity and periodicity $\omega$ must change sign. We assume that $\sigma = \alpha$ is the first critical point of $\omega$ which is a singular point for (3.2) and (3.4). We integrate (3.4) on a small interval $(\alpha, \sigma)$ and get

$$\tan^{-1}\left(\frac{Y(\sigma)}{\beta}\right) - \sqrt{\frac{\beta+1}{\beta}}\tan^{-1}\left(\frac{Y(\sigma)}{\sqrt{\beta(\beta+1)}}\right) = \sigma - \alpha.$$

Expanding $\tan^{-1}(x)$ near $x = 0$ we obtain

$$Y(\sigma) = -\beta\sqrt[3]{3\beta+3}\sqrt[3]{\sigma-\alpha}\,(1+o(1)) \Longrightarrow \omega(\sigma) = \omega(\alpha) - C(\beta)\omega(\alpha)(\sigma-\alpha)^{\frac{4}{3}}\,(1+o(1)), \tag{3.7}$$

with $C(\beta) = \frac{3^{\frac{4}{3}}}{4}\beta\sqrt[3]{\beta+1}$, and define $Y(\sigma)$ for $\sigma \in (\alpha, 2\alpha)$ by imposing $Y(\sigma) = -Y(2\alpha-\sigma)$ and continue this process in order to construct a $2\alpha$-antiperiodic solution belonging to $C^{1,\frac{1}{3}}(\mathbb{R})$. Since

$$\left[\tan^{-1}\left(\frac{Y}{\beta}\right) - \sqrt{\frac{\beta+1}{\beta}}\tan^{-1}\left(\frac{Y}{\sqrt{\beta(\beta+1)}}\right)\right]_{\sigma=0}^{\sigma=\alpha} = \alpha,$$

with $Y(0) = \infty$, $Y(\alpha) = 0$, the condition for $\pi$-antiperiodicity is therefore

$$\left(\sqrt{\frac{\beta+1}{\beta}} - 1\right)\frac{\pi}{2} = \alpha \Longleftrightarrow \beta = \frac{\pi^2}{4(\alpha^2+\alpha\pi)}.$$

If $\beta > 0$, the periodicity condition yields

$$\sqrt{\frac{\beta+1}{\beta}} = 1 + \frac{1}{k} \Longleftrightarrow \beta = \beta_k = \frac{k^2}{1+2k}. \tag{3.8}$$

If $\beta < -1$

$$\left(1 - \sqrt{\frac{\beta+1}{\beta}}\right)\pi = \alpha,$$

and the periodicity condition implies

$$\sqrt{\frac{\beta+1}{\beta}} = 1 - \frac{1}{k} \Longleftrightarrow \beta = \tilde{\beta}_k = \frac{k^2}{1-2k}. \tag{3.9}$$

The case $\beta(\beta+1) < 0$, or equivalently $-1 < \beta < 0$, is easily ruled out. We find that (3.3 ) has the constant solution $Y(\sigma) = \sqrt{-\beta(\beta+1)}$, meaning $\omega(\sigma) = C\exp(\sigma\sqrt{-\beta(\beta+1)})$, which is by no mean periodic. On the other hand, in this case we can write $(3.4)$ under the form

$$\frac{d}{d\sigma}\left(\tan^{-1}\left(\frac{Y}{\beta}\right) - \frac{1}{2}\sqrt{\frac{\beta+1}{-\beta}}\ln\left(\frac{|Y-\sqrt{-\beta(\beta+1)}|}{Y+\sqrt{-\beta(\beta+1)}}\right)\right) = 1. \tag{3.10}$$

Since $Y$ runs from $Y(0) = \infty$ to $Y(\alpha) = 0$, there must be a value $\sigma_0$ where $Y(\sigma_0) = \sqrt{-\beta(\beta+1)}$. We can integrate (3.10 ) on $(0, \sigma_0 - \epsilon)$ and let $\epsilon \to 0$. Since $\beta < 0$, it yields

$$\frac{\pi}{2} + \tan^{-1}\left(\frac{Y(\sigma_0-\epsilon)}{\beta}\right) - \frac{1}{2}\sqrt{\frac{\beta+1}{-\beta}}\ln\left(\frac{|Y(\sigma_0-\epsilon)-\sqrt{-\beta(\beta+1)}|}{Y(\sigma_0-\epsilon)+\sqrt{-\beta(\beta+1)}}\right) = \sigma_0 - \epsilon.$$

The left-hand side expression tends to $\infty$ when $\epsilon \to 0$, a contradiction. Hence there are no solutions with $\beta \in (-1, 0)$. This ends the proof of the proposition. □

*Remark.* When $k = 1$ the coordinate functions are infinite harmonic and vanish on a straight line. When $k = 2$ the regular solution with $-\tilde{\beta}_1 = \frac{4}{3}$

$$u(x, y) = x^{\frac{4}{3}} - y^{\frac{4}{3}},$$

has been discovered by Aronsson [2]. The corresponding circular function, $\omega(\sigma) = (\cos\sigma)^{\frac{4}{3}} - (\sin\sigma)^{\frac{4}{3}}$, admits four nodal sets on $S^1$. When $k = 1$, then $\beta_1 = \frac{1}{3}$. It is proved in [4] that any positive infinite harmonic function in a half-space which vanishes on the boundary except at one point blows-up like the separable infinite harmonic function $u(r, \sigma) = r^{-\frac{1}{3}}\omega(\sigma)$.

## 3.2 Proof of Theorem C

The following representation of $S^{N-1}$ is classical

$$S^{N-1} = \left\{\sigma = (\sin\phi\,\sigma', \cos\phi) : \sigma' \in S^{N-2}, \phi \in [0, \pi]\right\}.$$

Then $\nabla'\omega = \omega_\phi \mathbf{e} + \nabla'_{\sigma'}\omega$ where $\mathbf{e}$ is a tangent unit downward vector to $S^{N-1}$ following the great circle going through the point $\sigma$. Then $|\nabla'\omega|^2 = \omega_\phi^2 + |\nabla'_{\sigma'}\omega|^2$, thus, if $\omega$ depends only on $\phi$, we have

$$\frac{1}{2}\nabla'|\nabla'\omega|^2.\nabla'\omega = \omega_{\phi\phi}\omega_\phi^2.$$

Therefore such a function $\omega$, if it is a $C^1$ solution of (2.2) in the spherical cap $S_\alpha$ defined for $\phi \in (0,\alpha)$, satisfies

$$\begin{aligned} -\omega_{\phi\phi}\omega_\phi^2 &= \beta(2\beta+1)\omega_\phi^2\omega_\phi + \beta^3(\beta+1)\omega \quad \text{in } (0,\alpha) \\ \omega_\phi(0) &= 0\,,\ \omega(\alpha) = 0. \end{aligned} \tag{3.11}$$

The conclusion follows from Proposition 3.1. □

*Remark.* If $\alpha = \pi$, the exponent $\beta_+$ is $\frac{1}{8}$ and $\omega := \omega_{_{\Sigma^c}}$ is a positive solution of

$$\begin{aligned} -\omega_\sigma^2\omega_{\sigma\sigma} &= \frac{9}{4096}\omega^3 + \frac{5}{32}\omega_\sigma^2\omega \quad \text{in } (-\pi,\pi) \\ \omega(-\pi) &= \omega(\pi) = 0. \end{aligned} \tag{3.12}$$

Then the function $u_{_{\Sigma^c}}(r,\sigma) = r^{-\frac{1}{8}}\omega_{_{\Sigma^c}}(\sigma)$ is an infinite harmonic function in $\mathbb{R}^N \setminus L_{_{\Sigma^c}}$, which vanishes on the half line $L_{_{\Sigma^c}} := \{x = t\Sigma : t \geq 0\}$. The function $Y = \frac{\omega_\sigma}{\omega}$ can be computed implicitly on $(0,\pi)$ thanks to the identity

$$\tan^{-1}(8Y(\sigma)) - 3\tan^{-1}(\tfrac{8}{3}Y(\sigma)) = \sigma. \tag{3.13}$$

This yields, with $Z = \frac{8Y}{3}$, $\tan^{-1}(3Z(\sigma)) - 3\tan^{-1}(Z(\sigma)) = \sigma$, hence

$$\frac{3Z - \tan(3\tan^{-1}(Z)))}{1 + 3Z\tan(3\tan^{-1}(Z))} = \tan\sigma, \tag{3.14}$$

since

$$\tan(3x) = \frac{3\tan x - \tan^3 x}{1 - 3\tan^2 x}.$$

This yields

$$\frac{-8Z^3}{1 + 6Z^2 - 3Z^4} = \tan\sigma, \tag{3.15}$$

which gives the value of $Y$ by solving a fourth degree equation and then $\omega = \omega_{_{\Sigma^c}}$ by integrating $Y$.

Using Theorem C we can prove the existence of a singular infinite harmonic function in a cone $C_{_{S_{\kappa,\alpha}}}$ generated by a spherical annulus $S_{\kappa,\alpha}$ of the spherical points with azimuthal angle $\kappa < \phi < \alpha$.

**Proposition 3.2.** *Assume $0 \leq \kappa < \alpha < \pi$ and let $\nu = \frac{1}{2}(\alpha - \kappa)$. Then there exists a positive singular infinite harmonic function $u_{_{S_{\kappa,\alpha}}}$ and a regular infinite harmonic function $u_{_{S_{\kappa,\alpha}}}$ in $C_{_{S_{\kappa,\alpha}}}$ which vanish respectively on $\partial C_{_{S_{\kappa,\alpha}}} \setminus \{0\}$ and $\partial C_{_{S_{\kappa,\alpha}}}$ under the form $u_{_{S_{\kappa,\alpha}}}(r,\sigma) = r^{-\beta_{_{S_{\kappa,\alpha}}}}\omega_{_{S_{\kappa,\alpha}}}$ and $u^\dagger_{_{S_{\kappa,\alpha}}}(r,\sigma) = r^{\beta^\dagger_{_{S_{\kappa,\alpha}}}}\omega^\dagger_{_{S_{\kappa,\alpha}}}$ where*

$$\beta_{_{S_{\kappa,\alpha}}} = \frac{\pi^2}{4\nu(\pi+\nu)} \quad \textit{and} \quad \beta^\dagger_{_{S_{\kappa,\alpha}}} = \frac{\pi^2}{4\nu(\pi-\nu)}, \tag{3.16}$$

*and $\omega_{_{S_{\kappa,\alpha}}}$ and $\omega^\dagger_{_{S_{\kappa,\alpha}}}$ are positive solutions of (3.2) in $S_{\kappa,\alpha}$ with $\beta = \beta_{_{S_{\kappa,\alpha}}}$ and $\beta^\dagger_{_{S_{\kappa,\alpha}}}$ respectively, vanishing at $\kappa$ and $\alpha$.*

*Proof.* By Theorem C there exists a positive and even solution $\tilde{\omega}$ of

$$\begin{array}{l}\tilde{\omega}_{\phi\phi}\omega_\phi^2 = \beta(2\beta+1)\tilde{\omega}_\phi^2\tilde{\omega}_\phi + \beta^3(\beta+1)\tilde{\omega} \quad \text{in } (-\nu,\nu) := \left(\frac{1}{2}(\kappa-\alpha), \frac{1}{2}(\alpha-\kappa)\right) \\ \omega(-\nu) = 0\,,\ \omega(\nu) = 0,\end{array} \tag{3.17}$$

with $\beta = \beta_{S_{\kappa,\alpha}}$ or $\beta = \beta^\dagger_{S_{\kappa,\alpha}}$. Then $\phi \mapsto \omega(\phi) := \tilde{\omega}(\phi + \frac{1}{2}(\kappa+\alpha))$ is a positive solution of (3.2) in $(\kappa,\alpha)$. The proof follows. □

## 3.3 Proof of Theorem D

The next technical lemma is a variant of Theorem C and Proposition 3.2.

**Lemma 3.3.** *Assume $0 < \alpha < \pi$ and $\epsilon, \gamma > 0$. Then the solution $v = v_{\epsilon,\gamma,\alpha}$ of*

$$\begin{array}{c}-v'^2v'' + \gamma v'^4 + (2\gamma+1)v'^2 + \epsilon v = 0 \qquad \textit{in } (0,\alpha) \\ v(0) = \infty\,,\ v(\alpha) = \infty,\end{array} \tag{3.18}$$

*is an increasing function of $\epsilon$. If $0 < \sigma_0 < \alpha$, there exists $\lambda = \lambda(\alpha,\gamma) = \lim_{\epsilon\to 0}\epsilon v_{\epsilon,\gamma,\alpha}(\sigma_0)$ and this value is independent of $\sigma_0$. The function $\tilde{v} = \tilde{v}_{\epsilon,\gamma,\alpha} = v_{\epsilon,\gamma,\alpha} - v_{\epsilon,\gamma,\alpha}(\sigma_0)$ converges locally uniformly in $(0,\alpha)$ to a solution $v = v_{\gamma,\alpha}$ of*

$$\begin{array}{c}-v'^2v'' + \gamma v'^4 + (2\gamma+1)v'^2 + \lambda = 0 \qquad \textit{in } (0,\alpha) \\ v(0) = \infty\,,\ v(\alpha) = \infty,\end{array} \tag{3.19}$$

*with*

$$\lambda(\alpha,\gamma) = \frac{1}{4\gamma^3}\left(\frac{\pi^2}{\alpha^2} - \gamma(2\gamma+1)\right)^2. \tag{3.20}$$

*Furthermore*

$$v_{\gamma,\alpha}(\phi) = -\frac{1}{\gamma}\ln\phi \quad \textit{as} \ \ \phi \to 0, \tag{3.21}$$

*and*

$$v_{\gamma,\alpha}(\phi) = -\frac{1}{\gamma}\ln(\alpha-\phi) \quad \textit{as} \ \ \phi \to \alpha. \tag{3.22}$$

*Proof.* From the proof of Theorem A, we know that $v_{\epsilon,\gamma,\alpha}$ is an increasing function of $\epsilon$. It satisfies estimates (2.37) and (2.35). Furthermore there exists $\lambda = \lambda(\alpha,\gamma) = \lim_{\epsilon\to 0}\epsilon v_{\epsilon,\gamma,\alpha}(\sigma_0) \geq 0$, which is a value independent of $\sigma_0 \in (0,\alpha)$, and $\tilde{v} = \tilde{v}_{\epsilon,\gamma,\alpha} = v_{\epsilon,\gamma,\alpha} - v_{\epsilon,\gamma,\alpha}(\sigma_0)$ converges locally uniformly in $(0,\alpha)$ to a solution $v = v_{\gamma,\alpha}$ of (3.19). We set $Y = -\gamma v'$, then

$$\begin{array}{c}Y^2Y' + Y^4 + \gamma(2\gamma+1)Y^2 + \lambda\gamma^3 = 0 \qquad \text{in } (0,\alpha) \\ Y(0) = \infty\,,\ Y(\alpha) = -\infty.\end{array} \tag{3.23}$$

We write it under the separable form

$$\left(\frac{Y^2}{Y^4 + \gamma(2\gamma+1)Y^2 + \lambda\gamma^3}\right)Y' = -1 \Leftrightarrow \left(\frac{A^2}{A^2-B^2}\frac{1}{Y^2+A^2} - \frac{B^2}{A^2-B^2}\frac{1}{Y^2+B^2}\right)Y' = -1,$$

for some $A, B > 0$ and with $A > B$ if we assume $2\gamma + 1 > 2\gamma\lambda$. Actually $A^2B^2 = \lambda\gamma^3$ and $A^2 + B^2 = \gamma(2\gamma + 1)$. Thus

$$\left(A\tan^{-1}\left(\frac{Y}{A}\right) - B\tan^{-1}\left(\frac{Y}{B}\right)\right)' = B^2 - A^2. \tag{3.24}$$

By integration on $(0, \alpha)$ we derive the identity

$$A + B = \frac{\pi}{\alpha}. \tag{3.25}$$

Since $A + B = \sqrt{\gamma(2\gamma + 1) + 2\sqrt{\lambda\gamma^3}}$, we deduce (3.20) from (3.25). Finally, since

$$\tan^{-1} z = \frac{\pi}{2} - \frac{1}{z} + \frac{1}{3z^3} + O(z^{-5}) \quad \text{when } z \to \infty,$$

we derive

$$-\frac{1}{\gamma v'(\phi)} = \frac{1}{Y(\phi)} = \phi + O(\phi^3) \quad \text{when } \phi \to 0$$

from (3.24), which implies (3.21) by l'Hospital rule. Relation (3.22) is proved similarly.

□

Next we denote by $S_\alpha(a)$ the spherical cap with vertex $a \in S^{N-1}$ and azimuthal opening $\alpha$ from $a$ and $S^*_\alpha(a) = S_\alpha(a) \setminus \{a\}$. The next statement is a rephrasing of Lemma 3.3 in a geometric framework.

**Corollary 3.4.** *Let $\alpha$, $\epsilon$ and $\gamma > 0$ be as in Lemma 3.3 and $a \in S^{N-1}$. Then there exists a unique solution $w = w_{a,\alpha,\gamma,\epsilon}$ of*

$$\begin{aligned} -\frac{1}{2}\nabla' |\nabla' w|^2 .\nabla' w + \gamma |\nabla' w|^4 + (2\gamma + 1) |\nabla' w|^2 + \epsilon w = 0 \quad \text{in } S^*_\alpha(a) \\ \lim_{\ell(\sigma,a)\to 0} w(\sigma) = \infty \\ \lim_{\ell(\sigma,a)\to \alpha} w(\sigma) = \infty, \end{aligned} \tag{3.26}$$

*rotationally invariant with respect to $a$. If $a$ is replaced by $a' \in S^{N-1}$, the solution $w_{a',\alpha,\gamma,\epsilon}$ of (3.26) in $S^*_\alpha(a')$ is derived from $w_{a,\alpha,\gamma,\epsilon}$ by an orthogonal transformation exchanging $a$ and $a'$. The mapping $\epsilon \mapsto w_{a,\alpha,\gamma,\epsilon}$ is decreasing and for any $\sigma_0 \in S^*_\alpha(a)$*

$$\lim_{\epsilon\to 0} \epsilon w_{a,\alpha,\gamma,\epsilon}(\sigma_0) = \lambda(\alpha, \gamma) := \lambda(\gamma, S^*_\alpha(a)), \tag{3.27}$$

*(this notation is coherent with $\lambda(\gamma, S)$ already used). The function $\tilde{w}_{a,\alpha,\gamma,\epsilon} = w_{a,\alpha,\gamma,\epsilon} - w_{a,\alpha,\gamma,\epsilon}(\sigma_0)$ converges locally uniformly in $S^*_\alpha(a')$ to the unique viscosity solution $w := w_{a,\alpha,\gamma}$ rotationally invariant with respect to $a$ of*

$$\begin{aligned} -\frac{1}{2}\nabla' |\nabla' w|^2 .\nabla' w + \gamma |\nabla' w|^4 + (2\gamma + 1) |\nabla' w|^2 + \lambda(\alpha, \gamma) = 0 \quad \text{in } S^*_\alpha(a) \\ \lim_{\ell(\sigma,a)\to 0} w(\sigma) = \infty \\ \lim_{\ell(\sigma,a)\to \alpha} w(\sigma) = \infty. \end{aligned} \tag{3.28}$$

*Finally*

$$w_{a,\alpha,\gamma}(\sigma) = -\frac{1}{\gamma}\ln\left(\ell(a,\sigma)\right) + O(1) \quad \textit{as} \;\; \sigma \to a, \tag{3.29}$$

*and*

$$w_{a,\alpha,\gamma}(\sigma) = -\frac{1}{\gamma}\ln\left(\alpha - \ell(a,\sigma)\right) + O(1) \quad \textit{as} \;\; \ell(\sigma,a) \to 0. \tag{3.30}$$

The following statement is formally similar to Corollary 3.4. It makes more precise the approximations used in the proof of Theorem A, in the construction of the proof of Theorem B in the case $\alpha = \pi$.

**Corollary 3.5.** *Let $\epsilon$ and $\gamma > 0$ and $a \in S^{N-1}$. Then there exists a unique rotationally invariant with respect to $a$ solution $v = v_{a,\gamma,\epsilon}$, of*

$$\begin{aligned} -\frac{1}{2}\nabla'\left|\nabla' v\right|^2.\nabla' v + \gamma\left|\nabla' v\right|^4 + (2\gamma+1)\left|\nabla' v\right|^2 + \epsilon v = 0 \quad \textit{in } S^{N-1}\setminus\{a\} \\ \lim_{\ell(\sigma,a)\to 0} v(\sigma) = \infty. \end{aligned} \tag{3.31}$$

*Furthermore, for any $\sigma_0 \in S^{N-1}\setminus\{a\}$,*

$$\lim_{\epsilon\to 0}\epsilon v_{a,\gamma,\epsilon}(\sigma_0) = \Lambda(\gamma) := \frac{1}{4\gamma^3}\left(\frac{1}{4} - \gamma(2\gamma+1)\right)^2. \tag{3.32}$$

*The function $\tilde{v}_{a,\gamma,\epsilon} = v_{a,\gamma,\epsilon} - v_{a,\gamma,\epsilon}(\sigma_0)$ converges locally uniformly in $S^{N-1}\setminus\{a\}$ to the unique viscosity solution $v := v_{a,\gamma}$ rotationally invariant with respect to $a$ of*

$$\begin{aligned} -\frac{1}{2}\nabla'\left|\nabla' v\right|^2.\nabla' v + \gamma\left|\nabla' v\right|^4 + (2\gamma+1)\left|\nabla' v\right|^2 + \Lambda(\gamma) = 0 \quad \textit{in } S^{N-1}\setminus\{a\} \\ \lim_{\ell(\sigma,a)\to 0} v(\sigma) = \infty. \end{aligned} \tag{3.33}$$

*Finally*

$$v_{a,\gamma}(\sigma) = -\frac{1}{\gamma}\ln\left(\ell(a,\sigma)\right) + O(1) \quad \textit{as} \;\; \sigma \to a. \tag{3.34}$$

As in Corollary 3.4, if $a$ is replaced by $a' \in S^{N-1}$, the solution $v_{a',\gamma,\epsilon}$ of (3.31) in $S^{N-1}\setminus\{a'\}$ is derived from $v_{a,\gamma,\epsilon}$ by an orthogonal transformation exchanging $a$ and $a'$. The mapping $\epsilon \mapsto v_{a,\gamma,\epsilon}$ is decreasing.

**Proof of Theorem D**. *Step 1: Approximate solutions*. We consider two sequences of smooth spherical domains, $\{S_k\}$ and $\{S'_k\}$ such that

$$S_k \subset \overline{S}_k \subset S_{k+1} \subset S \;\text{ and }\; \bigcup_k S_k = S,$$

$$S \subset S'_{k+1} \subset \overline{S}'_{k+1} \subset S'_k \;\text{ and }\; \mathrm{int}\left(\bigcap_{\mathrm{k}} \mathrm{S}'_{\mathrm{k}}\right) = \mathrm{S}.$$

Such a sequence of domains $\{S'_k\}$ exists since $\partial S = \partial\overline{S}^c$. To each domain we associate the positive exponents $\beta_k := \beta_{S_k}$ and $\beta'_k := \beta_{S'_k}$ and the corresponding spherical p-harmonic functions $\omega_k := \omega_{S_k}$

and $\omega'_k := \omega_{S'_k}$, defined respectively in $S_k$ and $S'_k$, and such that $\omega_k(\sigma_0) = \omega'_k(\sigma_0)$ for some $\sigma_0 \in S_1$, so that the functions $u_k(r,.) = r^{-\beta_k}\omega_k$ and $u'_k(r,.) = r^{-\beta'_k}\omega'_k$ are respectively $p$-harmonic in the cones $C_{S_k}$ and $C_{S'_k}$ and vanish on $\partial C_{S_k} \setminus \{0\}$ and $\partial C_{S'_k} \setminus \{0\}$. For $\gamma, \delta, \epsilon > 0$, we denote by $w_{k,\gamma,\delta,\epsilon}$ the solution of

$$\begin{aligned} -\delta\Delta w - \frac{1}{2}\nabla' \left|\nabla' w\right|^2 .\nabla' w + \gamma \left|\nabla' w\right|^4 + (2\gamma+1)\left|\nabla' w\right|^2 + \epsilon w = 0 \quad \text{in } S_k \\ \lim_{\rho_k(\sigma)\to 0} w(\sigma) = \infty, \end{aligned} \tag{3.35}$$

and by $w'_{k,\gamma,\delta,\epsilon}$ the one of

$$\begin{aligned} -\delta\Delta w' - \frac{1}{2}\nabla' \left|\nabla' w'\right|^2 .\nabla' w' + \gamma \left|\nabla' w'\right|^4 + (2\gamma+1)\left|\nabla' w'\right|^2 + \epsilon w' = 0 \quad \text{in } S'_k \\ \lim_{\rho'_k(\sigma)\to 0} w'(\sigma) = \infty, \end{aligned} \tag{3.36}$$

where $\rho_k(.) = \text{dist}\,(., \partial S_k)$ and $\rho'_k(.) = \text{dist}\,(., \partial S'_k)$. By the maximum principle all the functions $w'_{\ell,\gamma,\delta,\epsilon}$ and $w_{k,\gamma,\delta,\epsilon}$ are positive and the following comparison relations hold:

$$\begin{aligned} &(i) && w'_{\ell,\gamma,\delta,\epsilon} \le w_{k,\gamma,\delta,\epsilon} && \text{in } S'_k && \forall k, \ell > 0, \\ &(ii) && w_{\ell,\gamma,\delta,\epsilon} \le w_{k,\gamma,\delta,\epsilon} && \text{in } S_k && \forall k \le \ell, \\ &(iii) && w'_{\ell,\gamma,\delta,\epsilon} \ge w'_{k,\gamma,\delta,\epsilon} && \text{in } S'_k && \forall k \le \ell, \\ &(iv) && w_{k,\gamma,\delta,\epsilon} \le w_{k,\gamma,\delta,\epsilon'} && \text{in } S_k && \forall \epsilon' \le \epsilon, \\ &(v) && w'_{k,\gamma,\delta,\epsilon} \le w'_{k,\gamma,\delta,\epsilon'} && \text{in } S'_k && \forall \epsilon' \le \epsilon. \end{aligned} \tag{3.37}$$

Furthermore it follows from Corollary 2.4,

$$\begin{aligned} &(i) && \left|\nabla w_{k,\gamma,\delta,\epsilon}(\sigma)\right| \le \frac{c}{\rho_k(\sigma)} \quad \forall \sigma \in S_k, \\ &(ii) && \left|\nabla w'_{k,\gamma,\delta,\epsilon}(\sigma)\right| \le \frac{c}{\rho'_k(\sigma)} \quad \forall \sigma \in S'_k, \end{aligned} \tag{3.38}$$

where $c = c(N)$. Moreover, similarly to in (2.37),

$$\begin{aligned} &(i) && -\frac{1}{\gamma}\ln\rho_k(\sigma) - \frac{M_k}{\epsilon} \le w_{k,\gamma,\delta,\epsilon}(\sigma) \le -\frac{1}{\gamma}\ln\rho_k(\sigma) + \frac{M_k}{\epsilon} && \forall \sigma \in S_k, \\ &(ii) && -\frac{1}{\gamma}\ln\rho'_k(\sigma) - \frac{M'_k}{\epsilon} \le w'_{k,\gamma,\delta,\epsilon}(\sigma) \le -\frac{1}{\gamma}\ln\rho'_k(\sigma) + \frac{M'_k}{\epsilon} && \forall \sigma \in S'_k. \end{aligned} \tag{3.39}$$

We let $\delta \to 0$ and derive that, up to a subsequence, $w_{k,\gamma,\delta_n,\epsilon} \to w_{k,\gamma,\epsilon}$ locally uniformly in $S_k$ and $w'_{k,\gamma,\delta_n,\epsilon} \to w'_{k,\gamma,\epsilon}$ locally uniformly in $S'_k$. The two functions $w_{k,\gamma,\epsilon}$ and $w'_{k,\gamma,\epsilon}$ satisfy (3.37), (3.38) and (3.39) and are viscosity solutions of

$$\begin{aligned} -\frac{1}{2}\nabla' \left|\nabla' w\right|^2 .\nabla' w + \gamma \left|\nabla' w\right|^4 + (2\gamma+1)\left|\nabla' w\right|^2 + \epsilon w = 0 \quad \text{in } S_k \\ \lim_{\rho_k(\sigma)\to 0} w(\sigma) = \infty, \end{aligned} \tag{3.40}$$

and

$$-\frac{1}{2}\nabla'\left|\nabla' w'\right|^2.\nabla' w'+\gamma\left|\nabla' w'\right|^4+(2\gamma+1)\left|\nabla' w'\right|^2+\epsilon w'=0 \quad \text{in } S'_k \qquad \lim_{\rho'_k(\sigma)\to 0} w'(\sigma)=\infty, \tag{3.41}$$

respectively. Furthermore $w'_{\ell,\gamma,\epsilon}\leq w_{k,\gamma,\epsilon}$ in $S_k$ for any $k,\ell>0$, $(k,\epsilon)\mapsto w_{k,\gamma,\epsilon}$ is nonincreasing and $(k,\epsilon)\mapsto w'_{k,\gamma,\epsilon}$ is nondecreasing with respect to $k$ and nonincreasing with respect to $\epsilon$. Next we let $k\to\infty$. Then $w_{k,\gamma,\epsilon}\downarrow w_{\gamma,\epsilon}$ and $w'_{k,\gamma,\epsilon}\uparrow w'_{\gamma,\epsilon}$. The two functions $w'_{\gamma,\epsilon}$ and $w_{\gamma,\epsilon}$ are defined in $S$ and are nonincreasing functions of $\epsilon$. Furthermore there holds

$$\begin{aligned} &(i) \qquad w'_{1,\gamma,\epsilon}\leq w'_{\ell,\gamma,\epsilon}\leq w'_{\gamma,\epsilon}\leq w_{\gamma,\epsilon}\leq w_{k,\gamma,\epsilon}\leq w_{1,\gamma,\epsilon} && \forall k,\ell\geq 1,\\ &(ii) \qquad \max\left\{\left|\nabla w'_{\gamma,\epsilon}(\sigma)\right|,\left|\nabla w_{\gamma,\epsilon}(\sigma)\right|\right\}\leq\frac{c}{\rho(\sigma)} && \forall\sigma\in S, \end{aligned} \tag{3.42}$$

where $c=c(N)$. Estimate (3.42)-(i) can be made more precise in the following way: *for each $\sigma\in S$, there is $k_\sigma\in\mathbb{N}$ such that $\sigma\in S_{k_\sigma}$ and*

$$-\frac{1}{\gamma}\ln\rho(\sigma)-\frac{1}{\epsilon}\max_{k\geq k_\sigma}M_k\leq w_{\gamma,\epsilon}(\sigma)\leq-\frac{1}{\gamma}\ln\rho(\sigma)+\frac{1}{\epsilon}\min_{k\geq k_\sigma}M_k, \tag{3.43}$$

and

$$-\frac{1}{\gamma}\ln\rho(\sigma)-\frac{1}{\epsilon}\max_{k\geq 1}M'_k\leq w'_{\gamma,\epsilon}(\sigma)\leq-\frac{1}{\gamma}\ln\rho(\sigma)+\frac{1}{\epsilon}\min_{k\geq 1}M'_k. \tag{3.44}$$

*Step 2: boundary blow-up.*

The compactness of approximate solutions vanishing at a fixed point in the local uniform convergence topology is easy to obtain thanks to the uniform estimate of the gradient. The main difficulty is to preserve the boundary blow-up when the parameters $k,\gamma,\delta,\epsilon$ tend to their respective limit.

*Case 1.* We first assume that there exist $\sigma_0\in S$, two decreasing sequences $\{\epsilon_n\}$, $\{\delta_\ell\}$ converging to 0 and an increasing sequence $\{k_j\}$ tending to infinity with the property that

$$\epsilon_n w'_{k_j,\gamma,\delta_\ell,\epsilon_n}(\sigma_0)\leq\epsilon_m w'_{k_j,\gamma,\delta_\ell,\epsilon_m}(\sigma_0) \quad \text{for all } m<n\,,\ j,\ell\in\mathbb{N}. \tag{3.45}$$

Since $\tilde{w}'_{k_j,\gamma,\delta_\ell,\epsilon_n}=w'_{k_j,\gamma,\delta_\ell,\epsilon_n}-w'_{k_j,\gamma,\delta_\ell,\epsilon_n}(\sigma_0)$ satisfies

$$-\delta_\ell\Delta'\tilde{w}-\frac{1}{2}\nabla'\left|\nabla'\tilde{w}\right|^2.\nabla'\tilde{w}+\gamma\left|\nabla'\tilde{w}\right|^4+(2\gamma+1)\left|\nabla'\tilde{w}\right|^2+\epsilon_n\tilde{w}+\epsilon_n w'_{k_j,\gamma,\delta_\ell,\epsilon_n}(\sigma_0)=0 \quad \text{in } S'_{k_j} \qquad \lim_{\rho'_{k_j}(\sigma)\to 0}\tilde{w}(\sigma)=\infty, \tag{3.46}$$

there holds

$$\tilde{w}'_{k_j,\gamma,\delta_\ell,\epsilon_n}\geq\tilde{w}'_{k_j,\gamma,\delta_\ell,\epsilon_m} \quad \text{for all } m<n\,,\ j,\ell\in\mathbb{N}. \tag{3.47}$$

Letting $\delta_\ell\to 0$ we derive that $w'_{k_j,\gamma,\delta_\ell,\epsilon_n}\to w'_{k_j,\gamma,\epsilon_n}$ locally uniformly in $S'_{k_j}$ and $w'_{k_j,\gamma,\epsilon_n}$ satisfies

$$-\frac{1}{2}\nabla'\left|\nabla' w\right|^2.\nabla' w+\gamma\left|\nabla' w\right|^4+(2\gamma+1)\left|\nabla' w\right|^2+\epsilon_n w=0 \quad \text{in } S'_{k_j} \qquad \lim_{\rho'_{k_j}(\sigma)\to 0}w(\sigma)=\infty. \tag{3.48}$$

Furthermore, for all $m < n\,,\ j \in \mathbb{N}$,

$$\begin{array}{ll} (i) & w'_{k_j,\gamma,\epsilon_n} \geq w'_{k_j,\gamma,\epsilon_m} \\ (ii) & w'_{k_j,\gamma,\epsilon_n} - w'_{k_j,\gamma,\epsilon_n}(\sigma_0) \geq w'_{k_j,\gamma,\epsilon_m} - w'_{k_j,\gamma,\epsilon_m}(\sigma_0) \\ (iii) & \epsilon_n w'_{k_j,\gamma,\epsilon_n}(\sigma_0) \leq \epsilon_m w'_{k_j,\gamma,\epsilon_m}(\sigma_0). \end{array} \tag{3.49}$$

By monotonicity with respect to $S'_{k_j}$, $w'_{k_j,\gamma,\epsilon_n} \uparrow w'_{\gamma,\epsilon_n}$ as $j \to \infty$. Let $a \in \partial S$ and $\{a_j\} \subset \partial S_{k_j}$ converging to $a$ (such a sequence exits since $\partial S = \partial \bar{S}^c$). Let $v_{a_j,\gamma,\epsilon_n}$ be the solution of (3.31) with $\epsilon = \epsilon_n$ and $a = a_j$ which exists by Corollary 3.5. Then

$$v_{a_j,\gamma,\epsilon_n} \leq w'_{k_j,\gamma,\epsilon_n} \qquad \text{in } S_{k_j} \tag{3.50}$$

Since $v_{a,\gamma,\epsilon_n}$ is obtained from $v_{a_j,\gamma,\epsilon_n}$ by an orthogonal transformation on $S^{N-1}$, we derive

$$v_{a,\gamma,\epsilon_n} \leq w'_{\gamma,\epsilon_n} \qquad \text{in } S. \tag{3.51}$$

This proves that $w'_{\gamma,\epsilon_n}$ is a viscosity solution of

$$\begin{array}{r} -\frac{1}{2}\nabla' \left|\nabla' w\right|^2 .\nabla' w + \gamma \left|\nabla' w\right|^4 + (2\gamma+1)\left|\nabla' w\right|^2 + \epsilon_n w = 0 \quad \text{in } S \\ \lim\limits_{\rho(\sigma)\to 0} w(\sigma) = \infty, \end{array} \tag{3.52}$$

and from (3.49),

$$\begin{array}{ll} (i) & w'_{\gamma,\epsilon_n} \geq w'_{\gamma,\epsilon_m} \\ (ii) & w'_{\gamma,\epsilon_n} - w'_{\gamma,\epsilon_n}(\sigma_0) \geq w'_{\gamma,\epsilon_m} - w'_{\gamma,\epsilon_m}(\sigma_0) \\ (iii) & \epsilon_n w'_{\gamma,\epsilon_n}(\sigma_0) \leq \epsilon_m w'_{\gamma,\epsilon_m}(\sigma_0). \end{array} \tag{3.53}$$

Because $\tilde{w}'_{\gamma,\epsilon_n} = w'_{\gamma,\epsilon_n} - w'_{\gamma,\epsilon_n}(\sigma_0)$ is increasing with respect to $n$, locally compact in the topology of local uniform convergence and satisfies

$$\begin{array}{r} -\frac{1}{2}\nabla' \left|\nabla' \tilde{w}\right|^2 .\nabla' \tilde{w} + \gamma \left|\nabla' \tilde{w}\right|^4 + (2\gamma+1)\left|\nabla' \tilde{w}\right|^2 + \epsilon_n \tilde{w} + \epsilon_n w'_{\gamma,\epsilon_n}(\sigma_0) = 0 \quad \text{in } S \\ \lim\limits_{\rho(\sigma)\to 0} \tilde{w}(\sigma) = \infty, \end{array} \tag{3.54}$$

and since $\epsilon_n w'_{\gamma,\epsilon_n}(\sigma_0) \to \lambda'(\gamma, S)$ as $n \to \infty$, we infer that $\tilde{w}'_\gamma = \lim_{n\to\infty} \tilde{w}'_{\gamma,\epsilon_n}$ is a locally Lipschitz continuous viscosity solution of

$$\begin{array}{r} -\frac{1}{2}\nabla' \left|\nabla' \tilde{w}\right|^2 .\nabla' \tilde{w} + \gamma \left|\nabla' \tilde{w}\right|^4 + (2\gamma+1)\left|\nabla' \tilde{w}\right|^2 + \lambda'(\gamma, S) = 0 \quad \text{in } S \\ \lim\limits_{\rho(\sigma)\to 0} \tilde{w}(\sigma) = \infty. \end{array} \tag{3.55}$$

*Case 2*. If the condition of Step 1 does not hold, for any $\sigma_0 \in S$ there exist two decreasing sequences $\{\epsilon_n\}$, $\{\delta_\ell\}$ converging to 0 and an increasing sequence $\{k_j\}$ tending to infinity, all depending on $\sigma_0$, such that

$$\lambda'(\gamma, S) > \epsilon_n w'_{k_j,\gamma,\delta_\ell,\epsilon_n}(\sigma_0) > \epsilon_m w'_{k_m,\gamma,\delta_m,\epsilon_m}(\sigma_0) \qquad \forall\, n > m, \tag{3.56}$$

where

$$\lambda'(\gamma,S)=\lim_{n\to\infty,j\to\infty,\ell\to\infty}\epsilon_n w'_{k_j,\gamma,\delta_\ell,\epsilon_n}(\sigma_0).$$

We fix some $\sigma_0\in S$. Then $\tilde{w}'_{k_j,\gamma,\delta_\ell,\epsilon_n}=w'_{k_j,\gamma,\delta_\ell,\epsilon_n}-w'_{k_j,\gamma,\delta_\ell,\epsilon_n}(\sigma_0)$ satisfies

$$\begin{aligned}-\delta_\ell\Delta\tilde{w}-\frac{1}{2}\nabla'\left|\nabla'\tilde{w}\right|^2+\gamma\left|\nabla'\tilde{w}\right|^4+(2\gamma+1)\left|\nabla'\tilde{w}\right|^2+\epsilon_n\tilde{w}+\lambda'(\gamma,S)\geq 0\quad\text{in } S'_{k_j}\\ \lim_{\rho'_{k_j}(\sigma)\to 0}\tilde{w}(\sigma)=\infty.\end{aligned}\tag{3.57}$$

We introduce the problem

$$\begin{aligned}-\delta_\ell\Delta Z-\frac{1}{2}\nabla'\left|\nabla'Z\right|^2+\gamma\left|\nabla'Z\right|^4+(2\gamma+1)\left|\nabla'Z\right|^2+\epsilon_n Z+\lambda'(\gamma,S)=0\quad\text{in } S'_{k_j}\\ \lim_{\rho'_{k_j}(\sigma)\to 0}Z(\sigma)=\infty.\end{aligned}\tag{3.58}$$

Since $(3.58)$ can be re-written as

$$\begin{aligned}-\delta_\ell\Delta Z'-\frac{1}{2}\nabla'\left|\nabla'Z'\right|^2+\gamma\left|\nabla'Z'\right|^4+(2\gamma+1)\left|\nabla'Z'\right|^2+\epsilon_n Z'=0\quad\text{in } S'_{k_j}\\ \lim_{\rho_{k_j}(\sigma)\to 0}Z'(\sigma)=\infty,\end{aligned}\tag{3.59}$$

with $Z'=Z+\epsilon_n^{-1}\lambda'(\gamma,S)$, existence is ensured by the approximation by finite boundary data as above. We denote by $Z_{k_j,\gamma,\delta_\ell,\epsilon_n}$ and $Z'_{k_j,\gamma,\delta_\ell,\epsilon_n}$, which coincides actually with $\tilde{w}'_{k_j,\gamma,\delta_\ell,\epsilon_n}$, the solutions of $(3.58)$ and $(3.59)$ obtained by such approximation. Using Corollary 3.5 as in Case 1and comparison, we obtain the following estimate

$$v_{a_{k_j},\gamma,\epsilon_n}-\frac{\lambda'(\gamma,S)}{\epsilon_n}\leq Z'_{k_j,\gamma,\delta_\ell,\epsilon_n}-\frac{\lambda'(\gamma,S)}{\epsilon_n}=Z_{k_j,\gamma,\delta_\ell,\epsilon_n}\leq\tilde{w}'_{k_j,\gamma,\delta_\ell,\epsilon_n}\quad\text{in } S'_{k_j},\tag{3.60}$$

where, again $a_{k_j}\in S_{k_j}$ and $v_{a_{k_j},\gamma,\epsilon_n}$ is the solution of $(3.31)$ with $\epsilon=\epsilon_n$ and $a=a_j$ which exists by Corollary 3.5. Now the sequences $\{Z_{k_j,\gamma,\delta_\ell,\epsilon_n}\}_{\epsilon_n,k_j}$ and $\{w_{k_j,\gamma,\delta_\ell,\epsilon_n}\}_{k_j}$ are increasing. Letting successively $\delta_\ell\to 0$ and $k_j\to\infty$ we infer that, up to a subsequence, $Z_{k_j,\gamma,\delta_\ell,\epsilon_n}$ converges locally uniformly to some $Z_{\gamma,\epsilon_n}$ and $\tilde{w}'_{k_j,\gamma,\delta_\ell,\epsilon_n}$ converges locally uniformly to some $\tilde{w}'_{\gamma,\epsilon_n}=w'_{\gamma,\epsilon_n}-w'_{\gamma,\epsilon_n}(\sigma_0)$ which are respectively viscosity solutions of

$$\begin{aligned}-\frac{1}{2}\nabla'\left|\nabla'Z\right|^2+\gamma\left|\nabla'Z\right|^4+(2\gamma+1)\left|\nabla'Z\right|^2+\lambda'(\gamma,S)+\epsilon_n Z=0\quad\text{in } S\\ \lim_{\rho(\sigma)\to 0}Z(\sigma)=\infty.\end{aligned}\tag{3.61}$$

and

$$\begin{aligned}-\frac{1}{2}\nabla'\left|\nabla'\tilde{w}\right|^2+\gamma\left|\nabla'\tilde{w}\right|^4+(2\gamma+1)\left|\nabla'\tilde{w}\right|^2+\epsilon_n w'_{\gamma,\epsilon_n}(\sigma_0)+\epsilon_n\tilde{w}=0\quad\text{in } S\\ \lim_{\rho(\sigma)\to 0}\tilde{w}(\sigma)=\infty.\end{aligned}\tag{3.62}$$

Furthermore $Z_{\gamma,\epsilon_n}$ and $\tilde{w}'_{\gamma,\epsilon_n}$ are locally bounded in $S$, relatively compact for the local uniform topology and they satisfy

$$Z_{\gamma,\epsilon_n}\leq\tilde{w}'_{\gamma,\epsilon_n}\quad\text{in } S.\tag{3.63}$$

At end, the sequence $\{Z_{\gamma,\epsilon_n}\}_{\epsilon_n}$ is nondecreasing. Hence, up to a subsequence, $\{\tilde{w}'_{\gamma,\epsilon_n}\}$ converges locally uniformly in $S$ to some $\tilde{w}'_\gamma$ which satisfies

$$\lim_{\epsilon_n\to 0} Z_{\gamma,\epsilon_n} = Z_\gamma \leq \tilde{w}'_\gamma \quad \text{in } S. \tag{3.64}$$

Since

$$\lim_{\rho(\sigma)\to 0} Z_{\gamma,\epsilon_n}(\sigma) = \infty \leq \lim_{\rho(\sigma)\to 0} Z_\gamma(\sigma),$$

it follows that $\tilde{w}'_\gamma$ is a locally Lipschitz continuous viscosity solution of $(3.55)$.

Because of $(3.37)$ we can easily construct the solution $\tilde{w}_\gamma$ of

$$\begin{aligned} -\frac{1}{2}\nabla' \left|\nabla'\tilde{w}\right|^2 .\nabla'\tilde{w} + \gamma \left|\nabla'\tilde{w}\right|^4 + (2\gamma+1)\left|\nabla'\tilde{w}\right|^2 + \lambda(\gamma,S) = 0 \quad \text{in } S \\ \lim_{\rho(\sigma)\to 0} \tilde{w}(\sigma) = \infty, \end{aligned} \tag{3.65}$$

as approximations from inside, with $\lambda(\gamma,S) \geq \lambda'(\gamma,S)$ and it dominates $\tilde{w}'_\gamma$.

*Step 3: end of the proof.* As in the proof of Proposition 2.6, the functions $\gamma \mapsto \lambda'(\gamma,S) - \gamma$ and $\gamma \mapsto \lambda(\gamma,S) - \gamma$ are non increasing functions of $\gamma$. We recall that $\lambda(\alpha,\gamma) = \lambda(\gamma, S^*_\alpha(a))$. By formula $(3.20)$, for any $\alpha > 0$ $\lim_{\gamma\to 0}\lambda(\alpha,\gamma) = \infty$. Since

$$\lambda(\gamma,S) - \gamma \geq \lambda'(\gamma,S) - \gamma \geq \lambda(\pi,\gamma) - \gamma, \tag{3.66}$$

it follows that $\lambda(\gamma,S) - \gamma$ and $\lambda'(\gamma,S) - \gamma$ converge to infinity when $\gamma$ converges to $0$. Let $\alpha > 0$ such that $\overline{S_\alpha(a)} \subset S$ for some $a \in S$. If

$$\gamma = \beta_+(\alpha) := \frac{\pi^2}{4\alpha(\pi+\alpha)},$$

then $\lambda(\alpha,\beta_+(\alpha)) - \beta_+(\alpha) = 1$. Since

$$\lambda'(\beta_+(\alpha),S) - \beta_+(\alpha) \leq \lambda(\beta_+(\alpha),S) - \beta_+(\alpha) < \lambda(\beta_+(\alpha), S^*_\alpha(a)) - \beta_+(\alpha) = 1, \tag{3.67}$$

it follows that

$$\inf\left\{\lambda'(\gamma,S) - \gamma : \gamma > 0\right\} \leq \inf\left\{\lambda(\gamma,S) - \gamma : \gamma > 0\right\} < 1. \tag{3.68}$$

We set

$$\beta^M_+ = \inf\left\{\gamma : \lambda(\gamma,S) - \gamma < 1\right\} \quad \text{and} \quad \beta^m_+ = \inf\left\{\gamma : \lambda'(\gamma,S) - \gamma < 1\right\}. \tag{3.69}$$

Let $\{\gamma_\nu\}$ be a sequence decreasing to $\beta^m_+$ when $\nu \to \infty$ and such that $\lim_{\nu\to\infty}\lambda'(\gamma_\nu,S) = \beta^m_+ + 1$. As in Step 1 we denote by $w'_{k_j,\gamma_\nu,\delta_\ell,\epsilon_n}$ the solution of $(3.36)$ with $(k_j,\gamma_\nu,\delta_\ell,\epsilon_n) = (k,\gamma,\delta,\epsilon)$. There always holds

$$w'_{k_j,\gamma_\nu,\delta_\ell,\epsilon_n} \leq w'_{k_j,\gamma_\mu,\delta_\ell,\epsilon_n} \quad \text{if } \nu \geq \mu. \tag{3.70}$$

Again we distinguish two cases

*Case 1.* We assume that there exist $\sigma_0 \in S$ and monotone sequences $\{k_j\}$, $\{\delta_\ell\}$ and $\{\epsilon_n\}$ such that

$$\epsilon_n w'_{k_j,\gamma_\nu,\delta_\ell,\epsilon_n}(\sigma_0) \leq \epsilon_m w'_{k_j,\gamma_\mu,\delta_\ell,\epsilon_m}(\sigma_0) \quad \text{for all } m < n\,,\ \mu \leq \nu\,,\ j,\ell \in \mathbb{N}, \tag{3.71}$$

(notice that the monotonicity with respect to $\nu$ is always satisfied). As in Step 2-Case 1 it implies

$$w'_{k_j,\gamma_\nu,\delta_\ell,\epsilon_n} - w'_{k_j,\gamma_\nu,\delta_\ell,\epsilon_n}(\sigma_0) \geq w'_{k_j,\gamma_\mu,\delta_\ell,\epsilon_m} - w'_{k_j,\gamma_\mu,\delta_\ell,\epsilon_m}(\sigma_0) \quad \text{for all } m < n\,,\ \mu < \nu\,,\ j,\ell \in \mathbb{N}. \tag{3.72}$$

Letting $\delta_\ell \to 0$ and $k_j \to \infty$ we obtain that the limit function $w'_{\gamma_\nu,\epsilon_n}$ satisfies

$$\begin{aligned} -\frac{1}{2}\nabla' \left|\nabla' w\right|^2 .\nabla' w + \gamma_\nu \left|\nabla' w\right|^4 + (2\gamma_\nu + 1)\left|\nabla' w\right|^2 + \epsilon_n w = 0 \quad \text{in } S \\ \lim_{\rho(\sigma)\to 0} w(\sigma) = \infty, \end{aligned} \tag{3.73}$$

and $\tilde{w}'_{\gamma_\nu,\epsilon_n} = w'_{\gamma_\nu,\epsilon_n} - w'_{\gamma_\nu,\epsilon_n}(\sigma_0)$ is increasing both with respect to $n$ and $\nu$. If $\epsilon_n \to 0$ we derive that $\tilde{w}'_{\gamma_\nu} = \lim_{n\to\infty} \tilde{w}'_{\gamma_\nu,\epsilon_n}$ satisfies $\tilde{w}'_{\gamma_\nu}(\sigma_0) = 0$ and

$$\begin{aligned} -\frac{1}{2}\nabla' \left|\nabla' \tilde{w}\right|^2 .\nabla' \tilde{w} + \gamma_\nu \left|\nabla' \tilde{w}\right|^4 + (2\gamma_\nu + 1)\left|\nabla' \tilde{w}\right|^2 + \lambda'(\gamma_\nu, S) = 0 \quad \text{in } S \\ \lim_{\rho(\sigma)\to 0} \tilde{w}(\sigma) = \infty. \end{aligned} \tag{3.74}$$

By gradient estimates and since $\tilde{w}'_{\gamma_\nu}(\sigma_0) = 0$, the set of functions $\{\tilde{w}'_{\gamma_\nu}\}_\nu$ is relatively compact for the local uniform convergence in $S$. Furthermore $\tilde{w}'_{\gamma_\nu}$ is increasing with respect to $\nu$, with limit $\tilde{w}'$. Using (3.69) and the definition of $\{\gamma_\nu\}$, we conclude that

$$\begin{aligned} -\frac{1}{2}\nabla' \left|\nabla' \tilde{w}'\right|^2 .\nabla' \tilde{w}' + \beta_+^m \left|\nabla' \tilde{w}'\right|^4 + (2\beta_+^m + 1)\left|\nabla' \tilde{w}'\right|^2 + \beta_+^m + 1 = 0 \quad \text{in } S \\ \lim_{\rho(\sigma)\to 0} \tilde{w}'(\sigma) = \infty, \end{aligned} \tag{3.75}$$

holds in the viscosity sense.

*Case 2*. We assume that for any $\sigma_0 \in S$ and $\nu$ there exist two decreasing sequences $\{\epsilon_n\}$, $\{\delta_\ell\}$ converging to 0 and an increasing sequence $\{k_j\}$ tending to infinity such that

$$\beta_+^m + 1 > \epsilon_n w'_{k_j,\gamma_\nu,\delta_\ell,\epsilon_n}(\sigma_0) > \epsilon_m w'_{k_m,\gamma_\nu,\delta_m,\epsilon_m}(\sigma_0) \quad \forall\, n > m, \tag{3.76}$$

where

$$\beta_+^m + 1 = \lim_{\nu\to\infty} \lambda'(\gamma_\nu, S) = \lim_{n\to\infty, j\to\infty, \ell\to\infty} \epsilon_n w'_{k_j,\gamma_\nu,\delta_\ell,\epsilon_n}(\sigma_0).$$

We follow the ideas in Step 2-Case 2 and consider the problem

$$\begin{aligned} -\delta_\ell \Delta Z - \frac{1}{2}\nabla' \left|\nabla' Z\right|^2 + \gamma_\nu \left|\nabla' Z\right|^4 + (2\gamma_\nu + 1)\left|\nabla' Z\right|^2 + \epsilon_n Z + \beta_+^m + 1 = 0 \quad \text{in } S'_{k_j} \\ \lim_{\rho'_{k_j}(\sigma)\to 0} Z(\sigma) = \infty. \end{aligned} \tag{3.77}$$

Since (3.77) can be re-written as (3.59) with $\gamma$ replaced by $\gamma_\nu$ and setting $Z' = Z + \epsilon_n^{-1}(\beta_+^m + 1)$, we have existence and uniqueness of the solution $Z^*_{k_j,\gamma_\nu,\delta_\ell,\epsilon_n}$ (we do not use the previous notation $Z_{k_j,\gamma_\nu,\delta_\ell,\epsilon_n}$ since the constant term is not of the form $\lambda(\gamma_\nu, S)$). The function $Z'^*_{k_j,\gamma_\nu,\delta_\ell,\epsilon_n} = Z^*_{k_j,\gamma_\nu,\delta_\ell,\epsilon_n} + \epsilon_n^{-1}(\beta_+^m + 1)$ satisfies (3.59) with $\gamma$ replaced by $\gamma_\nu$. Then (3.60) is replaced by

$$v_{a_{k_j},\gamma_\nu,\epsilon_n} - \frac{\beta_+^m + 1}{\epsilon_n} \leq Z'^*_{k_j,\gamma_\nu,\delta_\ell,\epsilon_n} - \frac{\beta_+^m + 1}{\epsilon_n} = Z^*_{k_j,\gamma_\nu,\delta_\ell,\epsilon_n} \leq \tilde{w}'_{k_j,\gamma_\nu,\delta_\ell,\epsilon_n} \quad \text{in } S'_{k_j}, \tag{3.78}$$

where $v_{a_{k_j},\gamma_\nu,\epsilon_n}$ is as above with obvious modifications. We denote by $\tilde{w}'_{\gamma_\nu,\epsilon_n} = w_{\gamma_\nu,\epsilon_n} - w_{\gamma_\nu,\epsilon_n}(\sigma_0)$ the limit, when $\delta_\ell \to 0$ and $k_j \to \infty$, of $\tilde{w}'_{k_j,\gamma_\nu,\delta_\ell,\epsilon_n} = w_{k_j,\gamma_\nu,\delta_\ell,\epsilon_n} - w_{k_j,\gamma_\nu,\delta_\ell,\epsilon_n}(\sigma_0)$ and by $Z^*_{\gamma_\nu,\epsilon_n}$ the one of $Z^*_{k_j,\gamma_\nu,\delta_\ell,\epsilon_n}$ under the same conditions. They are respective viscosity solutions of

$$\begin{aligned} -\frac{1}{2}\nabla' \left|\nabla'\tilde{w}\right|^2 + \gamma_\nu \left|\nabla'\tilde{w}\right|^4 + (2\gamma_\nu + 1)\left|\nabla'\tilde{w}\right|^2 + \epsilon_n w'_{\gamma_\nu,\epsilon_n}(\sigma_0) + \epsilon_n \tilde{w} = 0 \quad \text{in } S \\ \lim_{\rho(\sigma)\to 0} \tilde{w}(\sigma) = \infty. \end{aligned} \tag{3.79}$$

and

$$\begin{aligned} -\frac{1}{2}\nabla' \left|\nabla' Z\right|^2 + \gamma_\nu \left|\nabla' Z\right|^4 + (2\gamma_\nu + 1)\left|\nabla' Z\right|^2 + \beta^m_+ + 1 + \epsilon_n Z = 0 \quad \text{in } S \\ \lim_{\rho(\sigma)\to 0} Z(\sigma) = \infty. \end{aligned} \tag{3.80}$$

Furthermore $Z^*_{\gamma_\nu,\epsilon_n}$ and $\tilde{w}'_{\gamma_\nu,\epsilon_n}$ are locally bounded in $S$, relatively compact for the local uniform topology and they satisfy

$$Z^*_{\gamma_\nu,\epsilon_n} \leq \tilde{w}'_{\gamma_\nu,\epsilon_n} \quad \text{in } S. \tag{3.81}$$

The sequence $\{Z^*_{\gamma_\nu,\epsilon_n}\}$ is nondecreasing both with respect to $n$ and $\nu$. Therefore the boundary condition is kept. Letting $\epsilon_n \to 0$ and $\gamma_\nu \to \beta^m_+$ we conclude as in Step 1 that , up to a subsequence $\{\nu_s\}$ there exists a locally Lipschitz continuous function $\tilde{w}'$ such that $\tilde{w}'_{\gamma_{\nu_s},\epsilon_n} \to \tilde{w}'$ when $\nu \to \infty$ and $\nu_s \to \beta^m_+$ succesively, and $\tilde{w}'$ is a viscosity solution of $(3.75)$.

We end the proof by setting $\omega^M_+ = e^{-\beta^M_+ w^M_+}$. □

*Mutadis mutandis* in the above proof, we have an existence result for positive regular infinite harmonic function in $C_S$ vanishing on $\partial C_S$.

**Theorem 3.6.** *Assume $S \subset S^{N-1}$ is an outward accessible domain, that is $\partial S = \partial \overline{S}^c$. Then there exist two negative exponents $\beta^M_- \geq \beta^m_-$ and two positive functions $\omega^M_-$ and $\omega^m_-$ in $C^\infty(\overline{S})$, vanishing on $\partial S$ such that*

$$u^M_+(r,\sigma) = r^{-\beta^M_-}\omega^M_-(\sigma) \quad \text{and} \quad u^m_-(r,\sigma) = r^{-\beta^m_-}\omega^m_-(\sigma), \tag{3.82}$$

*are infinite harmonic in $C_S$ and vanish on $\partial C_S$.*

# References


[1] G. Arronson: Extension of functions satisfying Lipschitz conditions. *Ark. Mat.* **6** (1967), 551-561.

[2] G. Arronson: On the equation $u_x u_{xx} + 2u_{xy}u_x u_y + u_y u_{yy} = 0$. *Ark. Mat.* **7** (1968), 395-425.

[3] C. Bandle and M. Marcus: Dependence of Blowup Rate of Large Solutions of Semilinear Elliptic Equations, on the Curvature of the Boundary. *Complex Variables*, **49** (2004), 555-570.

[4] T. Bhattacharya: A note on non-negative singular infinity-harmonic functions in the half-space. *Rev. Mat. Complut.* **18** (2005), 377-385.

[5] T. Bhattacharya: A boundary Harnack principle for infinity-Laplacian and some related results. *Boundary value problems 2007*, 1-17.

[6] E. C. Evans and Ch. K. Smart: Everywhere differentiability of infinity harmonic functions. *Calc. Var. Part. Diff. Eq.* **42** (2011), 289-299.

[7] I. N. Krol: The behaviour of the solutions of a certain quasilinear equation near zero cusps of the boundary. *Proc. Steklov Inst. Math.* **125** (1973), 140-146.

[8] N. Katzourakis: An Introduction to Viscosity Solutions for Fully Nonlinear PDE with Applications to Calculus of Variations in $L^\infty$. Springer Briefs in Mathematics (2015), Springer-Verlag.

[9] S. Kichenassamy and L. Véron: Singular solutions of the $p$-Laplace equation. *Math. Ann.* **275** (1986), 599-615.

[10] P. Juutinen: Principal eigenvalue of a very badly degenerate operator and applications. *J. Diff. Equ.* **236** (2007), 532-550.

[11] J. M. Lasry and P. L. Lions: Nonlinear elliptic equations with singular boundary conditions and stochastic control with state constraints. I. The model problem. *Math. Ann.* **283** (1989), 583-630.

[12] A. Porretta and L. Véron: Separable p-harmonic functions in a cone and related quasilinear equations on manifolds. *J. Europ. Math. Sci.* **11** (2009), 1285-1305.

[13] P. Tolksdorf: On the Dirichlet problem for quasilinear equations in domains with conical boundary points. *Comm. Part. Diff. Eq.* **8** (1983), 773-817.

[14] P. Tolksdorf: Regularity for a more general class of quasilinear elliptic equations. *J. Diff. Eq.* **51** (1984), 126-150.

Marie-Françoise Bidaut-Véron
Laboratoire de Mathématiques et Physique Théorique
CNRS UMR 7350
Université Francois Rabelais, Tours, France
*veronmf@univ-tours.fr*

Marta Garcia Huidobro
Departamento de Mathematicà
Pontificia Università Catolica
Santiago, Chile
*martaghc@gmail.com*

Laurent Véron
Laboratoire de Mathématiques et Physique Théorique
CNRS UMR 7350
Université Francois Rabelais, Tours, France
*veronl@univ-tours.fr*