Families of deformations of the thirteen peregrine breather solutions to the NLS equation depending on twenty four parameters

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The thirteenth Peregrine breather and its deformations depending on twenty four parameters solutions to the NLS equation.

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Abstract

We go on with the study of the solutions to the focusing one dimensional nonlinear Schrödinger equation (NLS). We construct here the thirteen's Peregrine breather (P_{13} breather) with its twenty four real parameters, creating deformation solutions to the NLS equation. New families of quasi-rational solutions to the NLS equation in terms of explicit ratios of polynomials of degree 182 in x and t multiplied by an exponential depending on t are obtained. We present characteristic patterns of the modulus of these solutions in the $(x; t)$ plane, in function of the different parameters.

1 Introduction

From the first quasi-rational solution to NLS equation constructed in 1983 by Peregrine [3], a lot of works have carried out. Akhmediev, Eleonskii and Kulagin obtained the first higher order analogue of the Peregrine breather [4, 5] in 1986 and constructed also other families of higher order 3 and 4 were constructed in a series of articles [6, 7], using Darboux transformations.
Since the years 2010, many works were published using various methods, in

particular in 2013, it was found in [8] solution in terms of wronskians and in [9], solutions expressed in terms of determinants of order $2N$ depending on $2N - 2$ real parameters. A new representation has been found as a ratio of a determinant of order $N + 1$ by another one of order N by Ling and Zhao in [10]. Very recently in 2014, another approach have been given in [11] using a dressing method where the solutions are expressed as the quotient of a determinant of order $N + 1$ by another one of order N.

We construct here new solutions to the focusing one dimensional nonlinear Schrödinger equation which appear as deformations of the (analogue) Peregrine breather of order 13 with 24 real parameters. These solutions are completely expressed as a ratio of two polynomials of degree 182 in x and t by an exponential depending on t. We cannot give the solutions in terms of polynomials of x and t in this text because of their lengths; only plots in the $(x; t)$ plane of the modulus of the solutions to analyze the evolution of the solutions in function of the different parameters are presented in details here.

2 Solutions to NLS equation as a ratio of two determinants

We recall the following result which have been proved in [12, 13] :

Theorem 2.1 *The function v defined by*

$$v(x,t) = \frac{\det((n_{jk})_{j,k \in [1,2N]})}{\det((d_{jk})_{j,k \in [1,2N]})} e^{(2it - i\varphi)}$$

is a quasi-rational solution to the NLS equation

$$iv_t + v_{xx} + 2|v|^2 v = 0,$$

where

$$n_{j1} = f_{j,1}(x,t,0), \quad n_{jk} = \frac{\partial^{2k-2} f_{j,1}}{\partial \epsilon^{2k-2}}(x,t,0),$$

$$n_{jN+1} = f_{j,N+1}(x,t,0), \quad n_{jN+k} = \frac{\partial^{2k-2} f_{j,N+1}}{\partial \epsilon^{2k-2}}(x,t,0),$$

$$d_{j1} = g_{j,1}(x,t,0), \quad d_{jk} = \frac{\partial^{2k-2} g_{j,1}}{\partial \epsilon^{2k-2}}(x,t,0),$$

$$d_{jN+1} = g_{j,N+1}(x,t,0), \quad d_{jN+k} = \frac{\partial^{2k-2} g_{j,N+1}}{\partial \epsilon^{2k-2}}(x,t,0),$$

$$2 \le k \le N,\ 1 \le j \le 2N$$

2

The functions f and g are defined for $1 \leq k \leq N$ by :

$$
\begin{aligned}
&f_{4j+1,k} = \gamma_k^{4j-1} \sin A_k, \quad f_{4j+2,k} = \gamma_k^{4j} \cos A_k, \\
&f_{4j+3,k} = -\gamma_k^{4j+1} \sin A_k, \quad f_{4j+4,k} = -\gamma_k^{4j+2} \cos A_k, \\
&f_{4j+1,N+k} = \gamma_k^{2N-4j-2} \cos A_{N+k}, \quad f_{4j+2,N+k} = -\gamma_k^{2N-4j-3} \sin A_{N+k}, \\
&f_{4j+3,N+k} = -\gamma_k^{2N-4j-4} \cos A_{N+k}, \quad f_{4j+4,k} = \gamma_k^{2N-4j-5} \sin A_{N+k}, \\
&g_{4j+1,k} = \gamma_k^{4j-1} \sin B_k, \quad g_{4j+2,k} = \gamma_k^{4j} \cos B_k, \\
&g_{4j+3,k} = -\gamma_k^{4j+1} \sin B_k, \quad g_{4j+4,k} = -\gamma_k^{4j+2} \cos B_k, \\
&g_{4j+1,N+k} = \gamma_k^{2N-4j-2} \cos B_{N+k}, \quad g_{4j+2,N+k} = -\gamma_k^{2N-4j-3} \sin B_{N+k}, \\
&g_{4j+3,N+k} = -\gamma_k^{2N-4j-4} \cos B_{N+k}, \quad g_{4j+4,N+k} = \gamma_k^{2N-4j-5} \sin B_{N+k},
\end{aligned}
\tag{1}
$$

The arguments A_ν and B_ν of these functions are given for $1 \leq \nu \leq 2N$ by

$$
A_\nu = \kappa_\nu x/2 + i\delta_\nu t - ix_{3,\nu}/2 - ie_\nu/2, \quad B_\nu = \kappa_\nu x/2 + i\delta_\nu t - ix_{1,\nu}/2 - ie_\nu/2.
$$

The terms κ_ν, δ_ν, γ_ν are defined by $1 \leq \nu \leq 2N$

$$
\begin{aligned}
&\kappa_j = 2\sqrt{1-\lambda_j^2}, \quad \delta_j = \kappa_j \lambda_j, \quad \gamma_j = \sqrt{\frac{1-\lambda_j}{1+\lambda_j}}, \\
&\kappa_{N+j} = \kappa_j, \quad \delta_{N+j} = -\delta_j, \ \gamma_{N+j} = 1/\gamma_j, \quad 1 \leq j \leq N,
\end{aligned}
\tag{2}
$$

where λ_j are given for $1 \leq j \leq N$ by :

$$
\lambda_j = 1 - 2j^2\epsilon^2, \ \lambda_{N+j} = -\lambda_j.
\tag{3}
$$

The terms $x_{r,\nu}$ $(r = 3, 1)$ are defined for $1 \leq \nu \leq 2N$ by :

$$
x_{r,\nu} = (r-1) \ln \frac{\gamma_\nu - i}{\gamma_\nu + i}.
\tag{4}
$$

The parameters e_ν are given by

$$
\begin{aligned}
&e_j = i \sum_{k=1}^{N-1} a_j \epsilon^{2k+1} j^{2k+1} - \sum_{k=1}^{N-1} b_j \epsilon^{2k+1} j^{2k+1}, \\
&e_{N+j} = i \sum_{k=1}^{N-1} a_j \epsilon^{2k+1} j^{2k+1} + \sum_{k=1}^{N-1} b_j \epsilon^{2k+1} j^{2k+1}, \quad 1 \leq j \leq N,
\end{aligned}
\tag{5}
$$

3 Solutions of order 13 with twenty four parameters

We construct here deformations of the Peregrine breather P_{13} of order 13 depending on 24 parameters. We cannot give the analytic expression of the solution to NLS equation of order 13 with twenty four parameters because of their lengths. For simplicity, we denote

$$
d_3 := \det((n_{jk})_{j,k \in [1,2N]}), \quad d_1 := \det((d_{jk})_{j,k \in [1,2N]}).
$$

In this study, one reaches the current limits of capacity of calculation; all the explicit expressions were calculated for the parameters a_i and b_i for indices i

from 2 to 12. The only expression which was not calculated is that corresponding to i equal to 1 : the calculation of the expression needing 32 core stopped at the end of 6 months.

On the other hand, one could generate the corresponding images in this case i equal to 1 without calculating the determinants explicitly.

Calculations are the more long as indices i are close to 1. Precisely for a_{12}, calculations last approximately 1 hour on 20 cores; for a_2, it is approximately about 3 months on 32 cores.

The number of terms of the polynomials of the numerator d_3 and denominator d_1 of the solutions are shown in the table below (Table 1) when other a_i and b_i are set to 0.

a_i	d_3	d_1
2	1051741	531675
3	559480	282841
4	348403	176084
5	245676	124198
6	183131	92581
7	143053	72324
8	115599	58448
9	94974	48014
10	76243	38529
11	55015	27787
12	30003	15154

Table 1: Number of terms for the polynomials d_3 and d_1 of the solutions to the NLS equation in the case $N = 13$.

We construct figures to show deformations of the Peregrine breather of order 13. We get different types of symmetries in the plots in the (x, t) plane. We give some examples of this fact in the following discussion.

3.1 Peregrine breather of order 13

If we choose $a_i = b_i = 0$ for $1 \leq i \leq 10$, we obtain the classical thirteenth Peregrine breather (here, the peak going until 27 has been truncated)
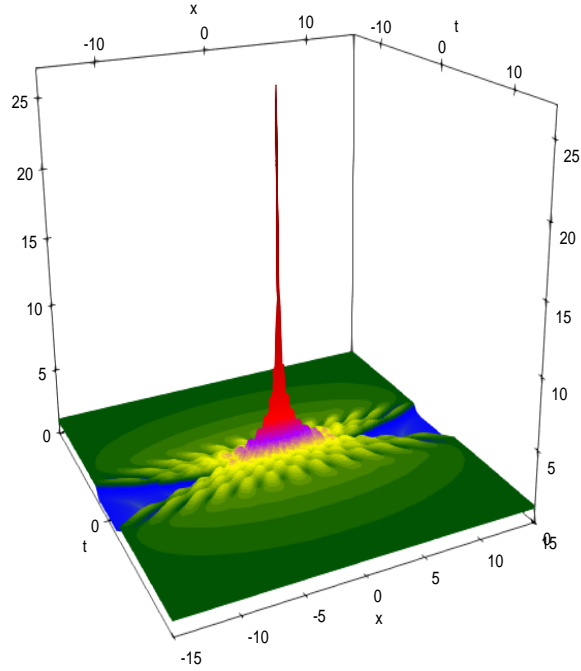
4

Figure 1: Solution to NLS, N=13, all parameters equal to 0, Peregrine breather P_{13}.

3.2 Variation of parameters

With other choices of parameters, we obtain all types of configurations : triangles and multiple concentric rings configurations with a maximum of 91 peaks. The configuration of the solutions of the NLS equation in the $(x; t)$ plane being the same for a_j or b_j, we restrict ourself to present the solutions only in the case of the parameters a_j, for $1 \leq j \leq 12$. We present figures of deformations with only one parameter non equal to zero, all other parameters are fixed equal to zero. For more accuracy, one presents in the following only the views of top.
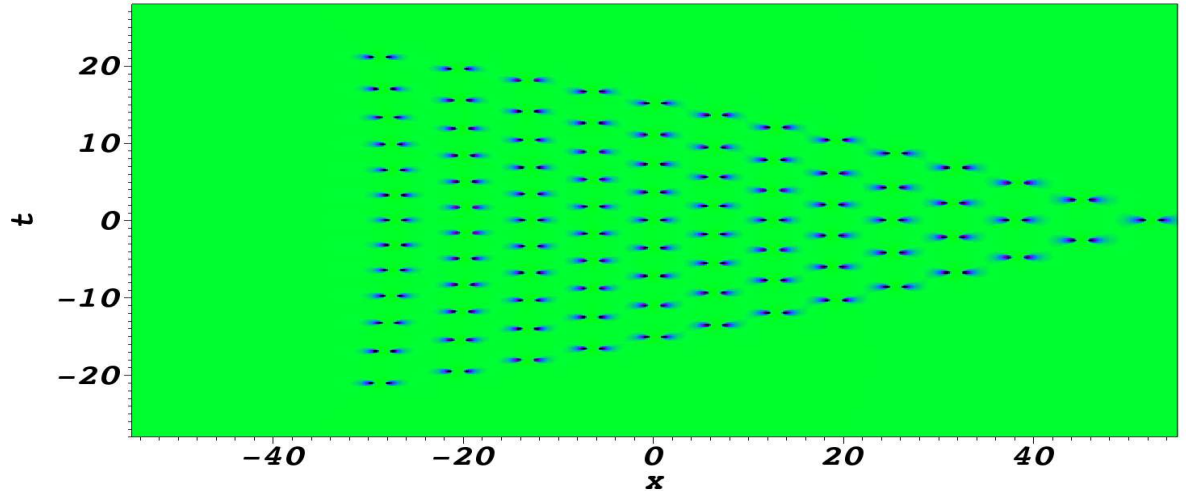
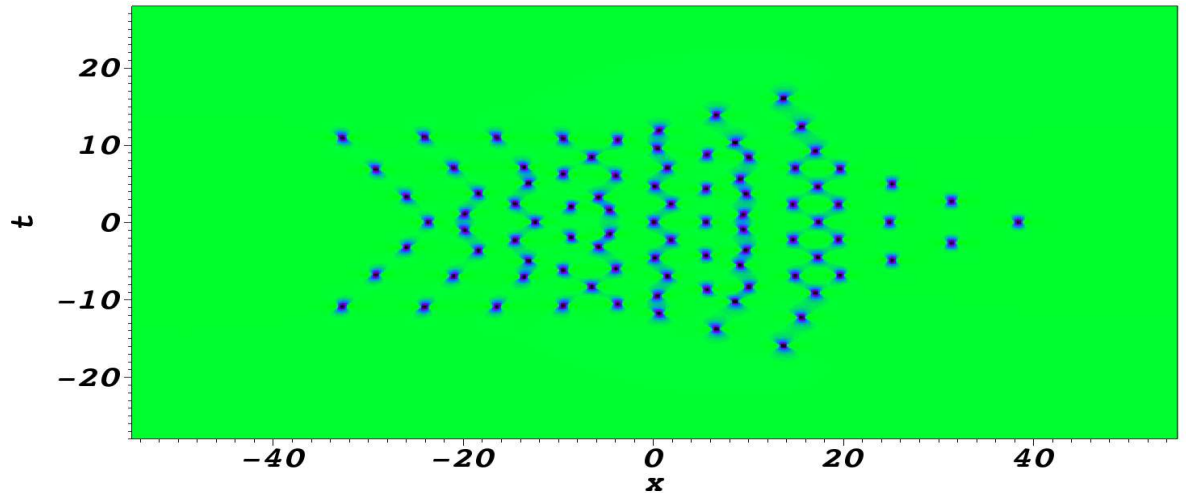Figure 2: Solution to NLS, N=13, $a_1 = 10^3$: triangle with 91 peaks, sight from top.

Figure 3: Solution to NLS, N=13, $a_2 = 10^4$: 11 rings with 5, 10, 10, 10, 10, 10, 10, 5, 5, 10, 5 peaks and one peak in the center, sight from top.
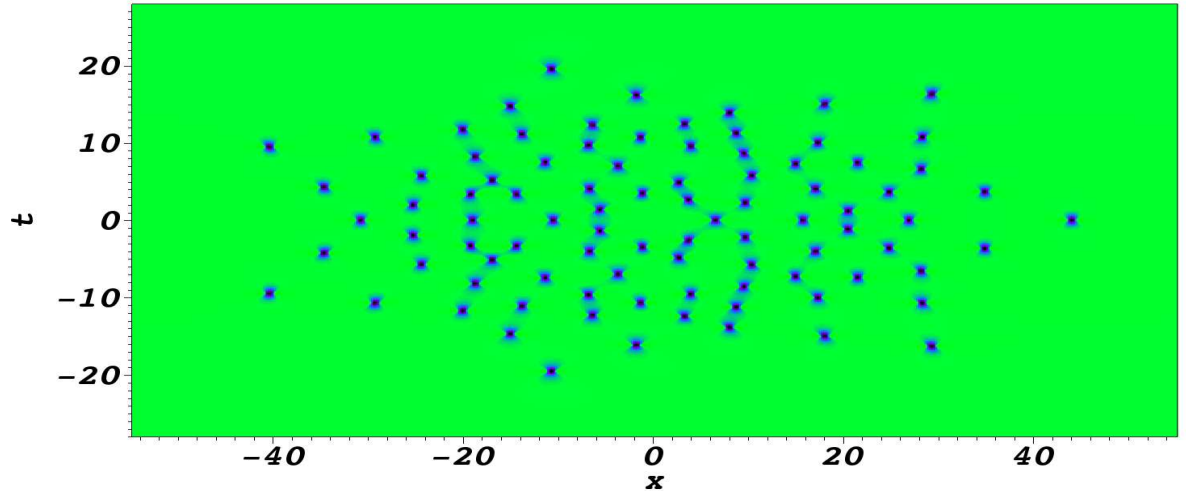
6

Figure 4: Solution to NLS, N=13, $a_3 = 10^6$: 10 rings with 7, 14, 7, 7, 14, 14, 7, 7, 7, 7, peaks, sight from top.
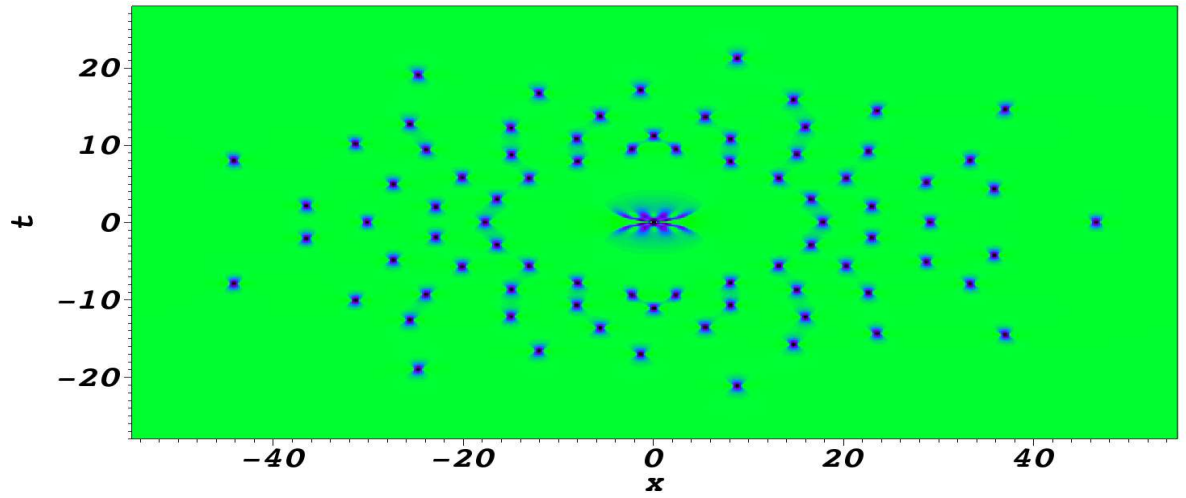

Figure 5: Solution to NLS, N=13, $a_4 = 10^8$: 5 rings with 9, 18, 18, 18, 18, peaks, in the center P_4, sight from top.
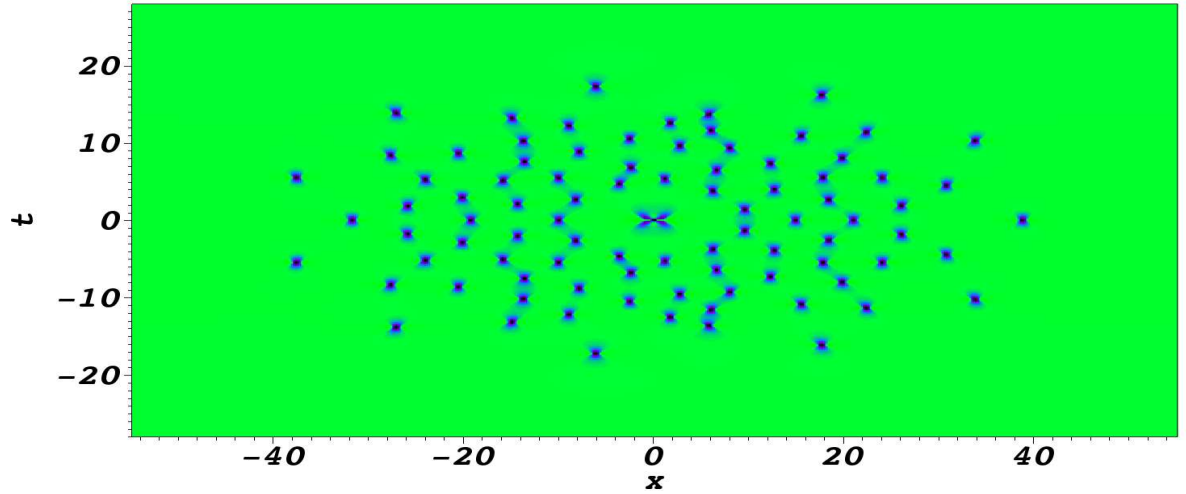
Figure 6: Solution to NLS, N=13, $a_5 = 10^9$: 7 rings of 11, 11, 22, 11, 11, 11, 11 peaks one peak in he center P_2, sight from top.
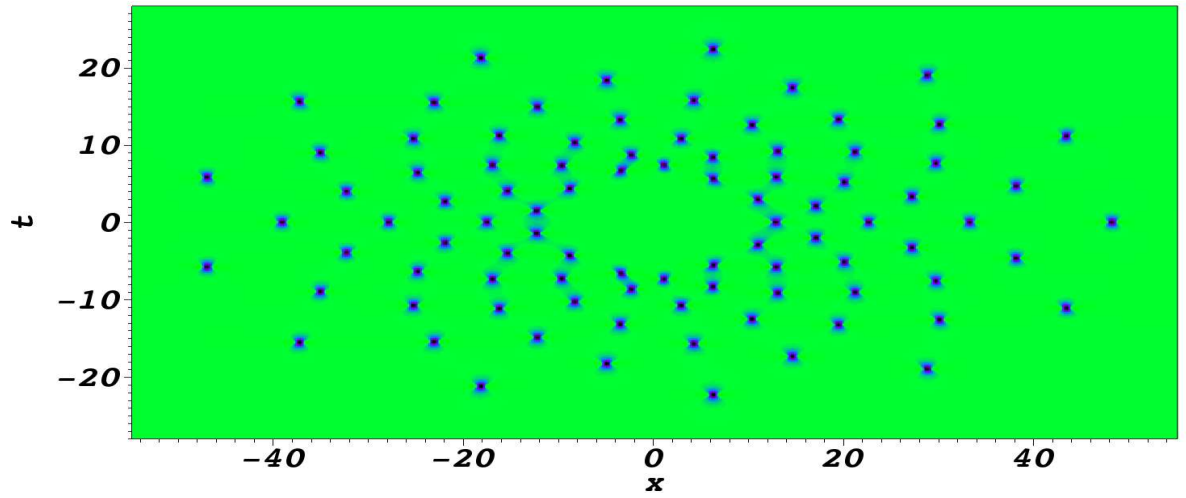

Figure 7: Solution to NLS, N=13, $a_6 = 10^{12}$: 7 rings with 13 peaks without peak in the center, sight of top.
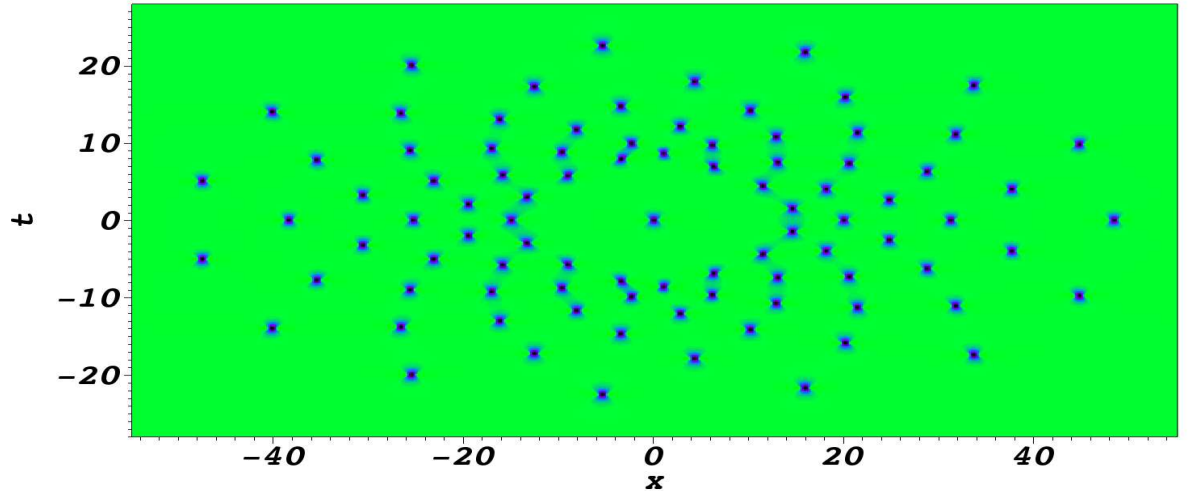
8

Figure 8: Solution to NLS, N=13, $a_7 = 10^{14}$: 6 rings with 15 peaks and in the center one peak, sight from top.

Figure 9: Solution to NLS, N=13, $a_8 = 10^{15}$: 5 rings with 17 peaks and in the center the Peregrine breather of order 3, sight from top.
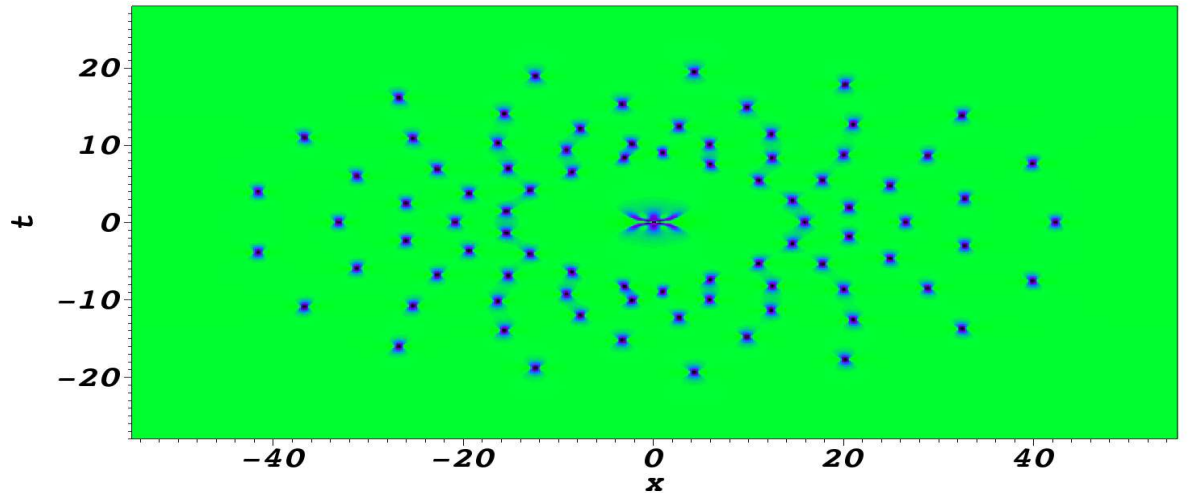
Figure 10: Solution to NLS, N=13, $a_9 = 10^{19}$: 4 rings with 19 peaks and in the center the Peregrine breather of order 5, sight from top.

Figure 11: Solution to NLS, N=13, $a_{10} = 10^{20}$: three rings with 21 peaks and in the center the Peregrine breather of order 7, sight from top.
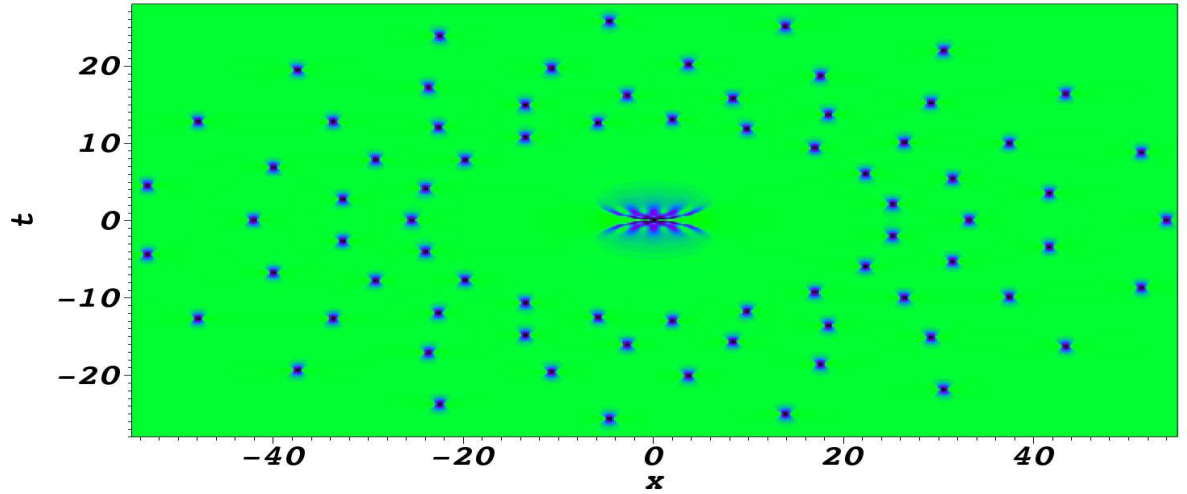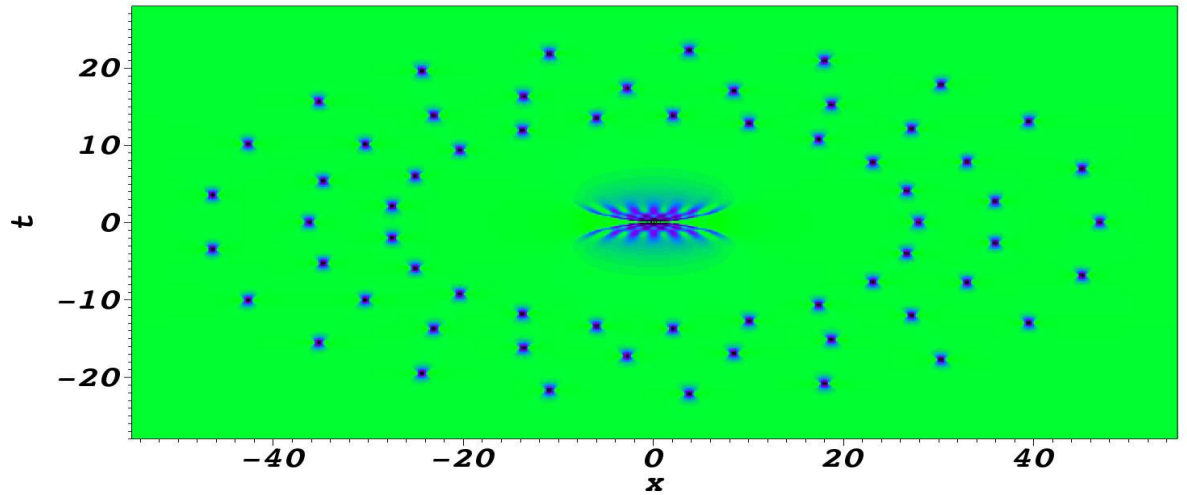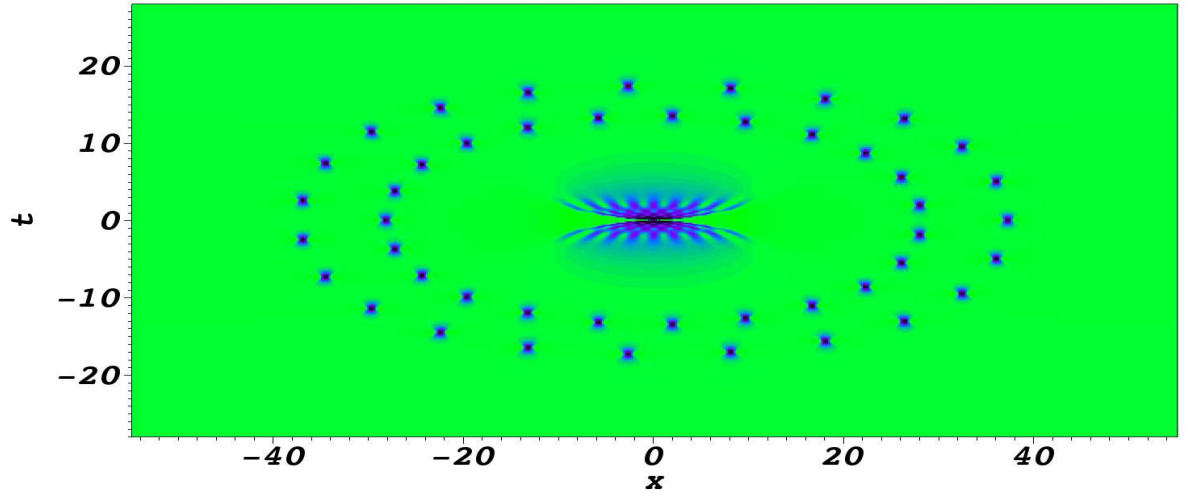
10

Figure 12: Solution to NLS, N=13, $a_{11} = 10^{20}$: two rings with 23 peaks and in the center the Peregrine breather of order 9, sight from top.
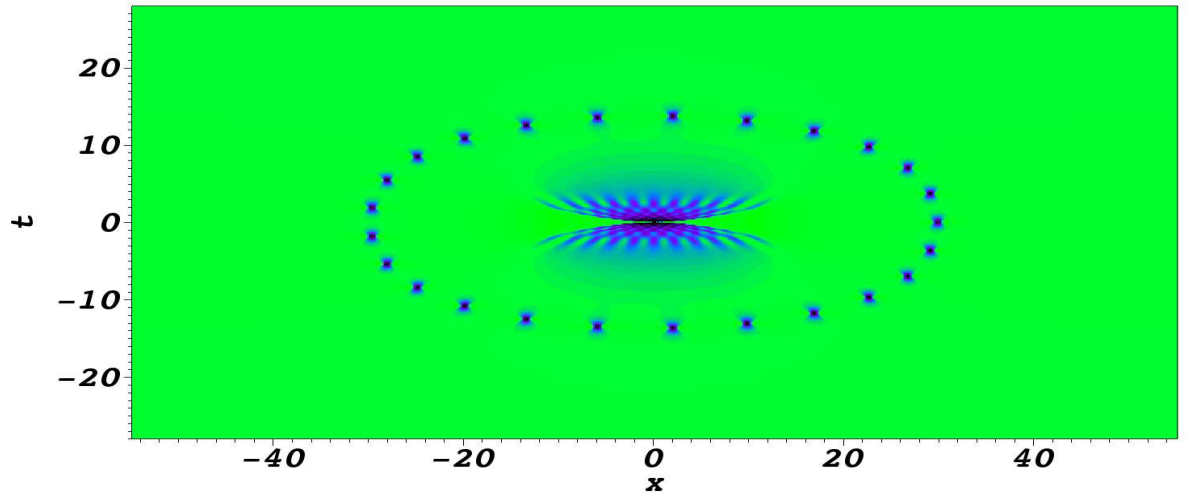

Figure 13: Solution to NLS, N=13, $a_{12} = 10^{20}$: one ring with 25 peaks and in the center the Peregrine breather of order 11, sight from top.

11

4 Conclusion

We have given here an explicit construction of multi-parametric solutions to the NLS equation of order 13 with 24 real parameters. We cannot give the explicit representation in terms of polynomials of degree 182 in x and t because of his length. The properties about the shape of the breather in the (x, t) coordinates, the maximum of amplitude equal to $2N + 1$ and the degree of polynomials in x and t here equal to $N(N+1)$ are checked.

We obtained different patterns in the $(x; t)$ plane by different choices of these parameters. So we obtain a classification of the rogue waves at order 13.

We obtain two types of patterns : the triangles and the concentric rings for the same index i for a_i or b_i non equals to 0.

In the cases $a_1 \neq 0$ or $b_1 \neq 0$ we obtain triangles with a maximum of 91 peaks; for $a_2 \neq 0$ or $b_2 \neq 0$, we have rings with 5 or 10 peaks. For $a_3 \neq 0$ or $b_3 \neq 0$, we obtain rings with 7 or 14 peaks with in the center P_2. For $a_4 \neq 0$ or $b_4 \neq 0$, we have 5 rings with 9, 18, 18, 19, 18, peaks with in the center the Peregrine P_4. For $a_5 \neq 0$ or $b_5 \neq 0$, we have 7 rings of 11, 11, 22, 11, 11, 11, 11 peaks with in the center P_3. For $a_6 \neq 0$ or $b_6 \neq 0$, we have 7 rings with 13 peaks on each of them without peak in the center. For $a_7 \neq 0$ or $b_7 \neq 0$, we have 6 rings with 15 peaks on each of them and in the center the Peregrine breather of order 1 (one peak). For $a_8 \neq 0$ or $b_8 \neq 0$, we have 5 rings with 17 peaks on each of them and in the center the Peregrine breather of order 3. For $a_9 \neq 0$ or $b_9 \neq 0$, we have 4 rings with 19 peaks and in the center the Peregrine breather of order 5. For $a_{10} \neq 0$ or $b_{10} \neq 0$, we have 3 rings with 21 peaks and in the center the Peregrine breather of order 7. For $a_{11} \neq 0$ or $b_{11} \neq 0$, we have only two rings with 23 peaks and in the center the Peregrine breather of order 9. At least, for $a_{12} \neq 0$ or $b_{12} \neq 0$, we have only one ring with 25 peaks and in the center the Peregrine breather of order 11.

We verify in this study the following conjectures :

at order N, for $a_1 \neq 0$ or $b_1 \neq 0$, the modulus of the solution to the NLS equation presents the configuration of a triangle with $N(N+1)/2$ peaks; this has been formulated already by different authors, in particular Akhmediev and Matveev.

at order N, in the case $1 \leq i \leq [\dfrac{N}{2}]$, for $a_{N-i} \neq 0$ or $b_{N-i} \neq 0$, the modulus of the solution to he NLS equation presents i concentric rings with $2N - 2i + 1$ peaks and in the center the P_{N-2i} breather; this conjecture was given for this time by the authors of this paper in 2015.

at order N, in the case $[\dfrac{N}{2}] < i \leq N - 2$, for $a_{N-i} \neq 0$ or $b_{N-i} \neq 0$, the modulus of the solution to the NLS equation presents n_k rings of $k(2N - 2i + 1)$ peaks, for $1 \leq k \leq r$, $rn_r(2N - 2i + 1) < N(N+1)/2$ with

in the center the P_{N-2i} breather, verifying

$$\sum_{n_k=1}^{r} n_k k(2N - 2i + 1) = 2iN - 2i^2 + i.$$

This conjecture was also given for this time by the authors of this paper in 2015.

It would be relevant to prove the preceding conjectures.

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