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Abstract. In a general C^1 domain, we study the perturbative Cauchy theory for the Boltzmann equation with Maxwell boundary conditions with an accommodation coefficient α in $(\sqrt{2/3}, 1]$, and discuss this threshold. We consider polynomial or stretched exponential weights $m(v)$ and prove existence, uniqueness and exponential trend to equilibrium around a global Maxwellian in $L^\infty_{x,v}(m)$. Of important note is the fact that the methods do not involve contradiction arguments.

Keywords: Boltzmann equation; Perturbative theory; Maxwell boundary conditions; Specular reflection boundary conditions; Maxwellian diffusion boundary conditions.

Contents

1. Introduction

The Boltzmann equation rules the dynamics of rarefied gas particles moving in a domain Ω of $\mathbb{R}^3$ with velocities in $\mathbb{R}^3$ when the sole interactions taken into account are elastic binary collisions. More precisely, the Boltzmann equation describes the time evolution of $F(t,x,v)$, the distribution of particles in position and velocity, starting from an initial distribution $F_0(x,v)$. It reads

$$\begin{aligned} \forall t \geqslant 0 \quad , \quad & \forall (x,v) \in \Omega \times \mathbb{R}^3, \quad \partial_t F + v \cdot \nabla_x F = Q(F,F), \\ & \forall (x,v) \in \Omega \times \mathbb{R}^3, \quad F(0,x,v) = F_0(x,v). \end{aligned} \tag{1.1}$$

To which one have to add boundary conditions on F. Throughout this work we consider C^1 bounded domains which allows us to decompose the phase space boundary

$$\Lambda = \partial\Omega \times \mathbb{R}^3$$

The authors would like to acknowledge the Division of Applied Mathematics at Brown University, where this work was achieved.

into three sets

$$\begin{aligned}\Lambda^+ &= \left\{(x,v)\in\partial\Omega\times\mathbb{R}^3, \quad n(x)\cdot v>0\right\},\\ \Lambda^- &= \left\{(x,v)\in\partial\Omega\times\mathbb{R}^3, \quad n(x)\cdot v<0\right\},\\ \Lambda_0 &= \left\{(x,v)\in\partial\Omega\times\mathbb{R}^3, \quad n(x)\cdot v=0\right\},\end{aligned}$$

where $n(x)$ is the outward normal at a point x on $\partial\Omega$. The set Λ^+ is the outgoing set, Λ^- is the ingoing set and Λ_0 is called the grazing set.

In the present work, we consider the physically relevant case where the gas interacts with the boundary $\partial\Omega$ *via* two phenomena. Part of the particles touching the wall elastically bounce against it like billiard balls (specular reflection boundary condition) whereas the other part are absorbed by the wall and then emitted back into the domain according to the thermodynamical equilibrium between the wall and the gas (Maxwellian diffusion boundary condition). This very general type of interactions will be referred to as Maxwell boundary condition and they mathematically translate into

$$\begin{aligned}&\exists\alpha\in(0,1],\ \forall t>0, \forall(x,v)\in\Lambda^-,\\ &\qquad F(t,x,v)=(1-\alpha)F(t,x,\mathcal{R}_x(v))+\alpha P_\Lambda(F(t,x,\cdot))(v)\end{aligned}\tag{1.2}$$

where the Maxwellian diffusion is given by

$$P_\Lambda(F(t,x,\cdot))(v)=c_\mu\mu(v)\left[\int_{v_*\cdot n(x)>0}F(t,x,v_*)\,(v_*\cdot n(x))\;dv_*\right]\tag{1.3}$$

with

$$\mu(v)=\frac{1}{(2\pi)^{3/2}}e^{-\frac{|v|^2}{2}}\quad\text{and}\quad c_\mu\int_{v\cdot n(x)>0}\mu(v)\,(v\cdot n(x))\;dv=1.$$

Note that in our study we allow pure Maxwellian diffusion ($\alpha=1$) but not pure specular reflection ($\alpha=0$). The constant α is called the accommodation coefficient.

The operator $Q(F,F)$ encodes the physical properties of the interactions between two particles. This operator is quadratic and local in time and space. It is given by

$$Q(F,F)=\int_{\mathbb{R}^3\times\mathbb{S}^2}B\left(|v-v_*|,\cos\theta\right)\left[F'F'_*-FF_*\right]dv_*d\sigma,$$

where F', F_*, F'_* and F are the values taken by F at v', v_*, v'_* and v respectively. Define:

$$\begin{cases}v' &= \dfrac{v+v_*}{2}+\dfrac{|v-v_*|}{2}\sigma\\ v'_* &= \dfrac{v+v_*}{2}-\dfrac{|v-v_*|}{2}\sigma\end{cases}\quad\text{and}\quad\cos\theta=\langle\frac{v-v_*}{|v-v_*|},\sigma\rangle.$$

We recognise here the conservation of kinetic energy and momentum when two particles of velocities v and v_* collide to give two particles of velocities v' and v'_*.

The collision kernel B contains all the information about the interaction between two particles and is determined by physics. We mention, at this point, that one can derive this type of equations from Newtonian mechanics at least formally [9][10]. The rigorous validity of the Boltzmann equation from Newtonian laws is known for short times (Landford's theorem [28] or more recently [14, 33]).

A very interesting *a priori* property of the Boltzmann equation combined with Maxwell boundary condition is the preservation of mass. Indeed, standard properties

of Q ([10][9][38] among others) imply that if F is solution to the Boltzmann equation then

$$\forall t \geqslant 0, \quad \int_{\Omega\times\mathbb{R}^3} F(t,x,v)\,dxdv = \int_{\Omega\times\mathbb{R}^3} F_0(x,v)\,dxdv, \tag{1.4}$$

which physically means that the mass is preserved along time.

In the present paper we are interested in the well-posedness of the Boltzmann equation (1.1) for fluctuations around the global equilibrium

$$\mu(v) = \frac{1}{(2\pi)^{3/2}} e^{-\frac{|v|^2}{2}}.$$

More precisely, in the perturbative regime $F = \mu + f$ we construct a Cauchy theory in $L^\infty_{x,v}$ spaces endowed with strech exponential or polynomial weights and study the continuity and the positivity of such solutions.

Under the perturbative regime, the Cauchy problem amounts to solving the perturbed Boltzmann equation

$$\partial_t f + v\cdot\nabla_x f = Lf + Q(f,f) \tag{1.5}$$

with L being the linear Boltzmann operator $Lf = 2Q(\mu, f)$ where we considered Q as a symmetric bilinear operator

$$Q(f,g) = \frac{1}{2}\int_{\mathbb{R}^3\times\mathbb{S}^2} B\left(|v-v_*|, \cos\theta\right)\left[f'g'_* + g'f'_* - fg_* - gf_*\right] dv_* d\sigma. \tag{1.6}$$

Note that f also satisfies the Maxwell boundary condition (1.2) since μ does.

1.1. **Notations and assumptions.**

We describe the assumptions and notations we shall use throughout the article.

Function spaces. Define for any $k \geqslant 0$ the functional

$$\forall \langle\cdot\rangle^k = \left(1 + |\cdot|^k\right).$$

The convention we choose is to index the space by the name of the concerned variable so we have, for p in $[1, +\infty]$,

$$L^p_{[0,T]} = L^p\left([0,T]\right), \quad L^p_t = L^p\left(\mathbb{R}^+\right), \quad L^p_x = L^p\left(\Omega\right), \quad L^p_v = L^p\left(\mathbb{R}^3\right).$$

For $m : \mathbb{R}^3 \longrightarrow \mathbb{R}^+$ a positive measurable function we define the following weighted Lebesgue spaces by the norms

$$\begin{aligned} \|f\|_{L^\infty_{x,v}(m)} &= \sup_{(x,v)\in\Omega\times\mathbb{R}^3} \left[|f(x,v)|\ m(v)\right] \\ \|f\|_{L^1_v L^\infty_x(m)} &= \int_{\mathbb{R}^3} \sup_{x\in\Omega} |f(x,v)|\ m(v)\,dv \end{aligned}$$

and in general with p, q in $[1,\infty)$: $\|f\|_{L^p_v L^q_x(m)} = \left\| \|f\|_{L^q_x}\, m(v)\right\|_{L^p_v}$.

We define the Lebesgue spaces on the boundary:

$$\begin{aligned} \|f\|_{L^\infty_\Lambda(m)} &= \sup_{(x,v)\in\Lambda} \left[|f(x,v)|\ m(v)\right] \\ \|f\|_{L^1 L^\infty_\Lambda(m)} &= \int_{\mathbb{R}^3} \sup_{x:\ (x,v)\in\Lambda} |f(x,v)v\cdot n(x)|\ m(v)\,dv \end{aligned}$$

with obvious equivalent definitions for $\Lambda^{\pm}$ or Λ_0. However, when we do not consider the L^∞ setting in the spatial variable we define

$$\|f\|_{L^2_\Lambda(m)} = \left[\int_\Lambda \left|f(x,v)^2 m(v)^2 \left|v\cdot n(x)\right| \, dS(x)dv\right|\right]^{1/2},$$

where $dS(x)$ is the Lebesgue measure on $\partial\Omega$. We emphasize here that when the underlying space in the velocity variable is L^p with $p \neq \infty$, the measure we consider is $|v\cdot n(x)|\, dS(x)$ as it is the natural one when one thinks about Green formula.

Assumptions on the collision kernel. We assume that the collision kernel B can be written as

$$B(v, v_*, \theta) = \Phi\left(|v - v_*|\right) b\left(\cos\theta\right), \tag{1.7}$$

which covers a wide range of physical situations (see for instance [38, Chapter 1]).

Moreover, we will only consider kernels with hard potentials, that is

$$\Phi(z) = C_\Phi z^\gamma \,, \;\; \gamma \in [0,1], \tag{1.8}$$

where $C_\Phi > 0$ is a given constant. Of special note is the case $\gamma = 0$ which is usually referred to as Maxwellian potentials. We will assume that the angular kernel $b \circ \cos$ is positive and continuous on $(0,\pi)$, and that it satisfies a strong form of Grad's angular cut-off:

$$b_\infty = \|b\|_{L^\infty_{[-1,1]}} < \infty \tag{1.9}$$

The latter property implies the usual Grad's cut-off [15]:

$$l_b = \int_{\mathbb{S}^{d-1}} b\left(\cos\theta\right) d\sigma = \left|\mathbb{S}^{d-2}\right| \int_0^\pi b\left(\cos\theta\right) \sin^{d-2}\theta \, d\theta < \infty. \tag{1.10}$$

Such requirements are satisfied by many physically relevant cases. The hard spheres case ($b = \gamma = 1$) is a prime example.

1.2. **Comparison with previous studies.** Few results have been obtained about the perturbative theory for the Boltzmann equation with other boundary condition than the periodicity of the torus. On the torus we can mention [34][18][20][32][5][17] for collision kernels with hard potentials with cutoff, [16] without the assumption of angular cutoff or [19][25] for soft potentials. A good review of the methods and techniques used can be found in the exhaustive [36].

The study of the well-posedness of the Boltzmann equation, as well as the trend to equilibrium, when the spatial domain is bounded with non-periodic boundary condition is scarce and only focuses on hard potential kernels with angular cutoff. In [21], exponential convergence to equilibrium in $L^\infty_{x,v}$ with the important weight $\langle v\rangle^\beta \mu(v)^{-1/2}$ was established. The boundary condition considered in [21] are pure specular reflections with Ω being strictly convex and analytic and pure Maxwellian diffusion with Ω being smooth and convex. Note that the arguments used in the latter work relied on a non-constructive $L^2_{x,v}$ theory.

More recently, the case of pure Maxwellian boundary condition has been resolved by [12] in $L^\infty_{x,v}\left(\langle v\rangle^\beta \mu(v)^{-1/2}\right)$ in Ω smooth but not necessarily convex and, more importantly, with constructive arguments. They also deal with non-global Maxwellian diffusion and gave an explicit domain of continuity for the solutions. We also mention [27] for a perturbative study around a non-local and rotating Maxwellian. At

last, a very recent work by the first author [3] extended the domain of validity of the previous study to $L^\infty_{x,v}(m)$ where m is a less restrictive weight: a stretched exponential or a polynomial; both for specular reflection and Maxwellian diffusion. His methods are constructive from the results described above (but therefore still rely on the contradiction argument in $L^2_{x,v}$ and the analyticity of Ω for specular reflections).

We also mention some works in the framework of renormalized solutions in bounded domains. The existence of such solutions has been obtained in different settings [29][30] with Maxwell boundary condition. The issue of asymptotic convergence for such solutions was investigated in [11] where they proved a trend to equilibrium faster than any polynomial on condition that the solutions has high Sobolev regularity

The present work establishes the perturbative Cauchy theory for Maxwell boundary condition and exponential trend to equilibrium in $L^\infty_{x,v}$ with a stretched exponential and polynomial weight. There are four main contributions in this work. First, we allow mere polynomial weights for the perturbation, which is a significant improvement over the work [21]. Then we deal with more general, and more physically relevant, boundary conditions and we recover the existing results in the case of pure Maxwellian diffusion. Third, delicate uses of the diffusive part, since $\alpha > 0$, gives constructive proofs and there are the first, to our knowledge, entirely constructive arguments when dealing with specular reflections. Finally, we propose a new method to establish an $L^2 - L^\infty$ theory that simplifies both technically and conceptually the existing $L^2 - L^\infty$ theory [21][12]. We indeed estimate the action of the operator K in between two consecutive rebounds against the wall and work with the different weight than all the previous studies, namely μ^{-1-0} where we prove that K almost acts like $3\nu(v)$. Also, with such an estimate we get rid of the strict convexity and analyticity of Ω that was always required when dealing with some specular reflections. We only need Ω to be a C^1 bounded domain but as a drawback we require $\alpha > \sqrt{2/3}$ (this explicit threshold being obtained thanks to the precise control over K).

We conclude by mentioning that our results also give an explicit set of continuity of the aforementioned solutions. This was known only in the case of pure Maxwellian diffusion, in-flow and bounce-back boundary conditions [24]. In the case of Ω convex we recover the fact that the solutions are continuous away from the grazing set Λ_0 [21]. Concerning the regularity of solutions to the Boltzmann equation with boundary conditions we also refer to [22][23].

1.3. **Organisation of the article.** Section 2 is dedicated to the statement and the description of the main results proved in this paper. We also describe our strategy, which mainly consists in four steps that make the skeleton of the present article.

Section 3 is dedicated to the *a priori* exponential decay of the solutions to the linear part of the perturbed equation in the L^2 setting.

In Section 4 we start by giving a brief mathematical description of the specular characteristics. We then study the semigroup generated by the transport part and the collision frequency kernel $G_\nu = -v \cdot \nabla_x - \nu$ along with the Maxwell boundary condition.

We develop an $L^2 - L^\infty$ theory in Section 5 and we prove that $G = -v \cdot \nabla_x + L$ generates a C^0-semigroup in $L^\infty_{x,v}(\langle v \rangle^\beta \mu^{-1/2})$ that decays exponentially.

We prove the existence and uniqueness of solutions for the full Boltzmann equation (1.1) in the perturbative regime $F = \mu + f$ in Section 6.

At last, Section 7 deals with the positivity and the continuity of the solutions to the full Boltzmann equation that we constructed.

2. Main results

The aim of the present work is to prove the following perturbative Cauchy theory for the full Boltzmann equation with Maxwell boundary condition.

Theorem 2.1. *Let Ω be a C^1 bounded domain and let α in $(\sqrt{2/3}, 1]$. Define*

$$k_\infty = 1 + \gamma + \frac{16\pi b_\infty}{l_b}. \tag{2.1}$$

Let $m = e^{\kappa_1 |v|^{\kappa_2}}$ with $\kappa_1 > 0$ and κ_2 in $(0,2)$ or $m = \langle v \rangle^k$ with $k > k_\infty$.
There exists $\eta > 0$ such that for any $F_0 = \mu + f_0$ in $L^\infty_{x,v}(m)$ satisfying the conservation of mass (1.4) with

$$\|F_0 - \mu\|_{L^\infty_{x,v}(m)} \leqslant \eta,$$

there exists a unique solution $F(t,x,v) = \mu(v) + f(t,x,v)$ in $L^\infty_{t,x,v}(m)$ to the Boltzmann equation (1.1) with Maxwell boundary condition (1.2) and with f_0 as an initial datum. Moreover,

- *F preserves the mass (1.4);*
- *There exist C, $\lambda > 0$ such that*

$$\forall t \geqslant 0, \quad \|F(t) - \mu\|_{L^\infty_{x,v}(m)} \leqslant C e^{-\lambda t} \|f_0\|_{L^\infty_{x,v}(m)};$$

- *If $F_0 \geqslant 0$ then $F(t) \geqslant 0$ for all t.*

Remark 2.2. *We make a few comments about the above theorem.*

1. *Notice that we recover the case of pure diffusion [21][12] since $\alpha = 1$ is allowed.*
2. *It is important to emphasize that the uniqueness holds in the pertubative sense, that is in the set of functions of the form $F = \mu + f$ with f small. The uniqueness for the Boltzmann equation in $L^\infty_{t,x,v}(m)$ with Maxwell boundary condition in the general setting would be a very interesting problem to look at.*
3. *Recent results [7][6] established a quantitative lower bound for the solutions in the case of pure specular reflections and pure diffusion respectively. We think that their methods could be directly applicable to the Maxwell boundary problem and the solutions described in the theorem above should have an exponential lower bound, at least when Ω is convex. However, we only give here a qualitative statement about the positivity.*

Remark 2.3 (Remarks about improvement over α)**.** *As we shall mention it in next sections, we can construct an explicit $L^2_{x,v}$ linear theory if $\alpha > 0$ whereas we strongly need $\alpha > \sqrt{2/3}$ to develop an $L^\infty_{x,v}$ linear theory from the L^2 one. However, the $L^1_v L^\infty_x$ nonlinear theory only relies on the $L^\infty_{x,v}$ linear one. Decreasing our expectations on Ω would allow to increase the range for α.*

- Ω ***smooth and convex:*** $\alpha > 0$ ***and constructive.*** *Very recent result* [26] *managed to obtain an* $L^\infty_{x,v}$ *theory for sole specular reflections by iterating Duhamel's form three times (see later). Thus, a convex combination of their methods and ours allow to derive an* $L^2 - L^\infty$ *theory for any* α *in* $[0,1]$ *and it would be entirely constructive thanks to our explicit* L^2 *linear theory.*
- *Unfortunately, a completely constructive* $L^2_{x,v}$ *theory for* $\alpha = 0$ *is still missing at the moment.*

In order to state our result about the continuity of the solutions constructed in Theorem 2.1 we need a more subtle description of $\partial\Omega$. As noticed by Kim [24], some specific points on Λ_0 can offer continuity.

We define the inward inflection grazing boundary

$$\Lambda_0^{(I-)} = \Lambda_0 \cap \left\{ t_{min}(x,v) = 0,\ t_{min}(x,-v) \neq 0 \text{ and } \exists \delta > 0,\ \forall \tau \in [0,\delta],\ x - \tau v \in \overline{\Omega}^c \right\}$$

where $t_{min}(x,v)$ is the first rebound against the boundary of a particle starting at x with a velocity $-v$ (see Subsection 4.1 for rigorous definition). That leads to the boundary continuity set

$$\mathfrak{C}^-_\Lambda = \Lambda^- \cup \Lambda_0^{(I-)}.$$

As we shall see later, the continuity set $\mathfrak{C}^-_\lambda$ describes the set of boundary points in the phase space that lead to continuous specular reflections.

The key idea is to understand that the continuity of the specular reflection at each bounce against the wall will lead to continuity of the solution. We thus define the continuity set

$$\begin{aligned}\mathfrak{C} = &\left\{ \{0\} \times \left\{ \Omega \times \mathbb{R}^3 \cup \left(\Lambda^+ \cup \mathfrak{C}^-_\Lambda \right) \right\} \right\} \cup \left\{ (0,+\infty) \times \mathfrak{C}^-_\Lambda \right\} \\ &\cup \Big\{ (t,x,v) \in (0,+\infty) \times \left(\Omega \times \mathbb{R}^3 \cup \Lambda^+ \right) : \\ &\qquad \forall 1 \leqslant k \leqslant N(t,x,v) \in \mathbb{N}, (X_{k+1}(x,v), V_k(x,v)) \in \mathfrak{C}^-_\Lambda \Big\}.\end{aligned}$$

The sequence $(T_k(x,v), X_k(x,v), V_k(x,v))_{k\in\mathbb{N}}$ is the sequence of footprints of the backward characteristic trajectory starting at (x,v) and overcoming pure specular reflections; $N(t,x,v)$ is almost always finite and satisfies $T_{N(t,x,v)} \leqslant t < T_{N(t,x,v)+1}(x,v)$. We refer to Subsection 4.1 for more details.

Theorem 2.4. *Let* $F(t,x,v) = \mu + f(t,x,v)$ *be the solution associated to* $F_0 = \mu + f_0$ *described in Theorem 2.1. Suppose that* $F_0 = \mu + f_0$ *is continuous on* $\Omega \times \mathbb{R}^3 \cup \left\{ \Lambda^+ \cup \mathfrak{C}^-_\Lambda \right\}$ *and satisfies the Maxwell boundary condition (1.2) then* $F = \mu + f$ *is continuous on the continuity set* $\mathfrak{C}$.

Remark 2.5. *We emphasize here again that the above theorem holds only in the perturbative regime. We also point out the following properties of the continuity set.*

(1) *From* [7, Proposition A.4] *we know that the set of points* (x,v) *in* $\overline{\Omega} \times \mathbb{R}^3$ *that lead to problematic backward characteristics is of Lebesgue measure zero (see later for more details). We infer that* $\mathfrak{C}$ *is non-empty and when we only consider* t *in* $[0,T]$ *for a given* $T > 0$, *its complementary set is of measure zero.*

(2) *In the case of a convex domain* Ω*, we recover the previous results* [21] *for both pure specular reflections and pure diffusion:* $\mathfrak{C} = \mathbb{R}^+ \times \left(\overline{\Omega} \times \mathbb{R}^3 - \Lambda_0\right)$.

2.1. **Description of the strategy.** Our strategy can be decomposed into four main steps and we now describe each of them briefly.

Step 1: A priori exponential decay in $L^2_{x,v}\left(\mu^{-1/2}\right)$ for the full linear operator. The first step is to prove that the existence of a spectral gap for L in the sole velocity variable can be transposed to $L^2_{x,v}\left(\mu^{-1/2}\right)$ when one adds the skew-symmetric transport operator $-v\cdot\nabla_x$. In other words, we prove that solutions to

$$\partial_t f = Gf = Lf - v\cdot\nabla_x f$$

in $L^2_{x,v}\left(\mu^{-1/2}\right)$ decays exponentially fast. Basically, the spectral gap λ_L of L implies that for such a solution

$$\frac{d}{dt}\left\|f\right\|^2_{L^2_{x,v}\left(\mu^{-1/2}\right)} \leqslant -2\lambda_L \left\|f-\pi_L\left(f\right)\right\|^2_{L^2_{x,v}\left(\mu^{-1/2}\right)},$$

where π_L is the orthogonal projection in $L^2_v\left(\mu^{-1/2}\right)$ onto the kernel of the operator L. This inequality exhibits the hypocoercivity of L. Therefore, one would like that the microscopic part $\pi_L^\perp(f) = f - \pi_L(f)$ controls the fluid part which has the following form

$$\pi_L(f)(t,x,v) = \left[a(t,x) + b(t,x)\cdot v + c(t,x)\left|v\right|^2\right]\mu(v).$$

It is known [18][20] that the fluid part has some elliptic regularity; roughly speaking one has

$$\Delta\pi_L(f) \sim \partial^2\pi_L^\perp f + \text{higher order terms}, \tag{2.2}$$

that can be used in Sobolev spaces H^s to recover some coercivity. We follow the idea of [12] for Maxwellian diffusion and construct a weak version of the elliptic regularity of $a(t,x)$, $b(t,x)$ and $c(t,x)$ by multiplying these coordinates by test functions. Basically, the elliptic regularity of $\pi_L\left(f\right)$ will be recovered thanks to the transport part applied to these test functions while, on the other side, L will encode the control by $\pi_L^\perp\left(f\right)$. The test functions we build works with specular reflections but the estimate for b requires the integrability of the function on the boundary. Such a property holds for Maxwellian diffusion and this is why we cannot deal with the specific case $\alpha = 0$.

Step 2: Semigroup generated by the collision frequency kernel. The collision frequency operator $G_\nu = -\nu(v) - v\cdot\nabla_x$ together with Maxwell boundary condition is proved to generate a strongly continuous semigroup with exponential decay in $L^\infty_{x,v}(m)$ with very general weights $m(v)$. The boundary operator associated with the Maxwell condition is of norm exactly one and therefore the standard theory of transport equation in bounded domains [2] fails. The core idea is to obtain an implicit description of the solutions to $\partial_t f = G_\nu f$ along the characteristic trajectories and to prove that the number of trajectories that do not reach the initial plane $\{t = 0\}$ after a large number of rebounds is very small. Such a method has been developed in [21] and extended in [3]; we adapt it to the case of Maxwell characteristics.

Step 3: $L^\infty_{x,v}(\mu^{-\zeta})$ theory for the full nonlinear equation. The underlying $L^2_{x,v}$-norm is not an algebraic norm for the nonlinear operator Q whereas the $L^\infty_{x,v}$-norm is (see [9][10] or [38] for instance). We therefore follow an $L^2 - L^\infty$ theory [21] to pass on the previous semigroup property in L^2 to L^∞ *via* a change of variable along the flow of characteristics.

Basically, L can be written as $L = -\nu(v) + K$ with K a kernel operator. If we denote by $S_G(t)$ the semigroup generated by $G = L - v \cdot \nabla_x$ we have the following implicit Duhamel along the characteristics

$$S_G(t) = e^{-\nu(v)t} + \int_0^t e^{-\nu(v)(t-s)} K\left[S_G(s)\right] \, ds.$$

The standard methods [37][21][12] used an iterated version of this Duhamel's formula to recover some compactness property, thus allowing to bound the solution in L^∞ by its L^2 norm. To do so they require to study the solution $f(t,x,v)$ along all the possible characteristic trajectories $(X_t(x,v), V_t(x,v))$. We propose here a less technical strategy by estimating the action of K in between two consecutive collisions against $\partial\Omega$ thanks to trace theorems. The core contribution, which also gives the threshold $\alpha > \sqrt{2/3}$, is to work in $L^\infty_{x,v}(\mu^{-\zeta})$ as ζ goes to 1 where K is proven to act roughly like $3\nu(v)$.

Step 4: Extension to polynomial weights. To conclude the present study, we develop an analytic and nonlinear version of the recent work [17], also recently adapted in a nonlinear setting [3]. The main strategy is to find a decomposition of the full linear operator G into $G_1 + A$. We shall prove that G_1 acts like a small perturbation of the operator $G_\nu = -v \cdot \nabla_x - \nu(v)$ and is thus hypodissipative, and that A has a regularizing effect. The regularizing property of the operator A allows us to decompose the perturbative equation (1.5) into a system of differential equations

$$\partial_t f_1 + v \cdot \nabla_x f_1 = G_1(f_1) + Q(f_1 + f_2, f_1 + f_2) \tag{2.3}$$

$$\partial_t f_2 + v \cdot \nabla_x f_2 = L(f_2) + A(f_1). \tag{2.4}$$

The first equation is solved in $L^\infty_{x,v}(m)$ with the initial datum f_0 thanks to the hypodissipativity of G_1. The regularity of $A(f_1)$ allows us to use Step 3 and thus solve the second equation with null initial datum in $L^\infty_{x,v}(\mu^{-\zeta})$.

3. $L^2\left(\mu^{-1/2}\right)$ theory for the linear part of the perturbed Boltzmann equation

This section is devoted to the study of the linear perturbed equation

$$\partial_t f + v \cdot \nabla_x f = L(f),$$

with the Maxwell boundary condition (1.2) in the L^2 setting. Note that we only need α in $(0,1]$ in this section. As we shall see in Subsection 3.1, the space $L^2_v\left(\mu^{-1/2}\right)$ is natural for the operator L. In order to avoid carrying the maxwell weight throughout the computations we look at the function $h(t,x,v) = f(t,x,v)\mu(v)^{-1/2}$. We thus study in this section the following equation in $L^2_{x,v}$

$$\partial_t h + v \cdot \nabla_x h = L_\mu(h), \tag{3.1}$$

with the associated boundary conditions

$$\forall t>0,\ \forall (x,v)\in\Lambda^-,\quad h(t,x,v)=(1-\alpha)h(t,x,\mathcal{R}_x(v))+\alpha P_{\Lambda_\mu}(h)(t,x,v) \tag{3.2}$$

where we defined

$$L_\mu(h)=\frac{1}{\sqrt{\mu}}L\left(\sqrt{\mu}h\right)$$

and P_{Λ_μ} can be viewed as a L^2_v-projection with respect to the measure $|v\cdot n(x)|$:

$$\forall (x,v)\in\Lambda^-,\quad P_{\Lambda_\mu}(h)=c_\mu\sqrt{\mu(v)}\left[\int_{v_*\cdot n(x)>0}h(t,x,v_*)\sqrt{\mu(v_*)}\,(v_*\cdot n(x))\,dv_*\right]. \tag{3.3}$$

We also use the shorthand notation $P^\perp_{\Lambda_\mu}=\mathrm{Id}-P^\perp_{\Lambda_\mu}$.

For general domains Ω, the Cauchy theory in $L^p_{x,v}$ $(1\leqslant p<+\infty)$ of equations of the type

$$\partial_t f+v\cdot\nabla_x f=g$$

with boundary conditions

$$\forall (x,v)\in\Lambda^-,\quad f(t,x,v)=P(f)(t,x,v),$$

where $P:L^p_{\Lambda^+}\longrightarrow L^p_{\Lambda^-}$ is a bounded linear operator, is well-defined in $L^p_{x,v}$ when $\|P\|<1$ [2]. The specific case $\|P\|=1$ can still be dealt with ([2] Section 4) but even though the existence of solutions in $L^p_{x,v}$ can be proven, the uniqueness is not always given unless one can prove that the trace of f belongs to $L^2_{\mathrm{loc}}\left(\mathbb{R}^+;L^p_{x,v}(\Lambda)\right)$.

For Maxwell boundary conditions, the boundary operator P is of norm exactly one and the general theory fails. The need of a trace in $L^2_{x,v}$ is essential to perform Green's identity and obtain the uniqueness of solutions. The pure Maxwellian boudary conditions with mass conservation can still be dealt with because one can show that $P^\perp_{\Lambda_\mu}(h)$ is in $L^2_{\Lambda^+}$ [12]. Unfortunately, in the case of specular reflections the uniqueness is not true in general due to a possible blow-up of the $L^2_{\mathrm{loc}}\left(\mathbb{R}^+;L^2_{x,v}(\Lambda)\right)$ at the grazing set Λ_0 [35, 2, 10].

Following ideas from [21], a sole *a priori* exponential decay of solutions is necessary to obtain a well-posed L^∞ theory provided that we endow the space with a strong weight. This section is thus dedicated to the proof of the following theorem.

Theorem 3.1. *Let $\alpha>0$ and let h_0 be in $L^2_{x,v}$ such that h_0 satisfies the preservation of mass*

$$\int_{\Omega\times\mathbb{R}^3}h_0(x,v)\sqrt{\mu(v)}\,dv=0.$$

Suppose that $h(t,x,v)$ in $L^2_{x,v}$ is a mass preserving solution to the linear perturbed Boltzmann equation (3.1) with initial datum h_0 and satisfying the Maxwell boundary condition (3.2). Suppose also that $h|_\Lambda$ belongs to L^2_Λ.
Then there exist explicit C_G, $\lambda_G>0$, independent of h_0 and h, such that

$$\forall t\geqslant 0,\quad \|h(t)\|_{L^2_{x,v}}\leqslant C_Ge^{-\lambda_G t}\|h_0\|_{L^2_{x,v}}.$$

In order to prove Theorem 3.1 we first gather in Subsection 3.1 some well-known properties about the linear operator L. Subsection 3.2 proves a very important

lemma which allows to use the hypocoercivity of $L - v \cdot \nabla_x$ in the case of Maxwell boundary conditions. Finally, the exponential decay is proved in Subsection 3.3.

3.1. **Preliminary properties of L_μ in L^2_v. The linear Boltzmann operator.** We gather some well-known properties of the linear Boltzmann operator L_μ (see [9][10][38][17] for instance).

L_μ is a closed self-adjoint operator in L^2_v with kernel

$$\mathrm{Ker}\,(L_\mu) = \mathrm{Span}\,\{\phi_0(v), \dots, \phi_4(v)\}\,\sqrt{\mu},$$

where $(\phi_i)_{0\leqslant i\leqslant 4}$ is an orthonormal basis of $\mathrm{Ker}\,(L_\mu)$ in L^2_v. More precisely, if we denote π_L to be the orthogonal projection onto $\mathrm{Ker}\,(L_\mu)$ in L^2_v):

$$\begin{cases} \pi_L(h) = \displaystyle\sum_{i=0}^{4} \left(\int_{\mathbb{R}^3} h(v_*)\phi_i(v_*)\sqrt{\mu(v_*)}\,dv_* \right) \phi_i(v)\sqrt{\mu(v)} \\ \\ \phi_0(v) = 1, \quad \phi_i(v) = v_i,\ 1 \leqslant i \leqslant 3, \quad \phi_4(v) = \dfrac{|v|^2 - 3}{\sqrt{6}}, \end{cases} \tag{3.4}$$

and we define $\pi_L^\perp = \mathrm{Id} - \pi_L$. The projection $\pi_L(h(x,\cdot))(v)$ of $h(x,v)$ onto the kernel of L_μ is called its fluid part whereas $\pi_L^\perp(h)$ is its microscopic part.

Also, L_μ can be written under the following form

$$L_\mu = -\nu(v) + K, \tag{3.5}$$

where $\nu(v)$ is the collision frequency

$$\nu(v) = \int_{\mathbb{R}^3\times\mathbb{S}^2} b\,(\cos\theta)\,|v - v_*|^\gamma\,\mu_*\,d\sigma dv_*$$

and K is a bounded and compact operator in L^2_v.

Finally we remind that there exists $\nu_0,\ \nu_1 > 0$ such that

$$\forall v \in \mathbb{R}^3, \quad \nu_0(1 + |v|^\gamma) \leqslant \nu(v) \leqslant \nu_1(1 + |v|^\gamma), \tag{3.6}$$

and that L_μ has a spectral gap $\lambda_L > 0$ in $L^2_{x,v}$ (see [1][31] for explicit proofs)

$$\forall g \in L^2_v, \quad \langle L_\mu(g), g\rangle_{L^2_v} \leqslant -\lambda_L \left\|\pi_L^\perp(g)\right\|^2_{L^2_v}. \tag{3.7}$$

The linear perturbed Boltzmann operator. The linear perturbed Boltzmann operator is the full linear part of the perturbed Boltzmann equation (1.5):

$$G = L - v \cdot \nabla_x$$

or, in our L^2 setting,

$$G_\mu = L_\mu - v \cdot \nabla_x.$$

An important point is that the same computations as to show the *a priori* conservation of mass implies that in $L^2_{x,v}$ the space $\left(\mathrm{Span}\left\{\sqrt{\mu}\right\}\right)^\perp$ is stable under the flow

$$\partial_t h = G_\mu(h)$$

with Maxwell boundary conditions (3.2). Coming back to our general setting $f = h\sqrt{\mu}$ we thus define the $L^2_{x,v}\left(\mu^{-1/2}\right)$ projection onto that space

$$\Pi_G(f) = \left(\int_{\Omega\times\mathbb{R}^3} f(x,v_*)\,dxdv_*\right)\mu(v), \tag{3.8}$$

and its orthogonal projection $\Pi_G^\perp = \mathrm{Id} - \Pi_G$. Note that $\Pi_G^\perp(f) = 0$ amounts to saying that f satisfies the preservation of mass.

3.2. **A priori control of the fluid part by the microscopic part.** As seen in the previous section, the operator L_μ is only coercive on its orthogonal part. The key argument is to show that we recover the full coercivity on the set of solutions to the differential equation. Namely, that for these specific functions, the microscopic part controls the fluid part. This is the purpose of the next lemma.

Lemma 3.2. *Let $h_0(x,v)$ and $g(t,x,v)$ be in $L^2_{x,v}$ such that $\Pi_G(h_0) = \Pi_G(g) = 0$ and let $h(t,x,v)$ in $L^2_{x,v}$ be a mass preserving solution to*

$$\partial_t h + v\cdot\nabla_x h = L_\mu(h) + g \tag{3.9}$$

with initial datum h_0 and satisfying the boundary condition (3.2). Suppose that $h|_\Lambda$ belongs to L^2_Λ. Then there exists an explicit $C_\perp > 0$ and a function $N_h(t)$ such that for all $t \geqslant 0$

(i) $|N_h(t)| \leqslant C_\perp \|h(t)\|^2_{L^2_{x,v}}$;

(ii)

$$\int_0^t \|\pi_L(h)\|^2_{L^2_{x,v}}\,ds \leqslant N_h(t) - N_h(0) + C_\perp\int_0^t\left[\left\|\pi_L^\perp(h)\right\|^2_{L^2_{x,v}} + \left\|P^\perp_{\Lambda_\mu}(h)\right\|^2_{L^2_{\Lambda^+}}\right]ds + C_\perp\int_0^t \|g\|^2_{L^2_{\Lambda^+}}\,ds.$$

The constant $C_\perp$ is independent of h.

The methods of the proof are a technical adaptation of the methods proposed in [12] in the case of purely diffusive boundary condition.

Proof of Lemma 3.2. We recall the definition of π_L (3.4) and we define the function $a(t,x)$, $b(t,x)$ and $c(t,x)$ by

$$\pi_L(h)(t,x,v) = \left[a(t,x) + b(t,x)\cdot v + c(t,x)\frac{|v^2|-3}{2}\right]\sqrt{\mu(v)}. \tag{3.10}$$

The key idea of the proof is to choose suitable test function ψ in $H^1_{x,v}$ that will catch the elliptic regularity of a, b and c and estimate them. Note that for a we strongly use the fact that h preserves the mass.

For a test function $\psi = \psi(t,x,v)$ integrated against the differential equation (3.1) we have by Green's formula

$$\begin{aligned}\int_0^t \frac{d}{dt}\int_{\Omega\times\mathbb{R}^3} \psi h\, dxdvds &= \int_{\Omega\times\mathbb{R}^3} \psi(t)h(t)\, dxdv - \int_{\Omega\times\mathbb{R}^3} \psi_0 h_0\, dxdv \\ &= \int_0^t\int_{\Omega\times\mathbb{R}^3} h\partial_t\psi\, dxdvds + \int_0^t\int_{\Omega\times\mathbb{R}^3} L_\mu[h]\psi\, dxdvds \\ &\quad + \int_0^t\int_{\Omega\times\mathbb{R}^3} hv\cdot\nabla_x\psi\, dxdvds - \int_0^t\int_\Lambda \psi hv\cdot n(x)\, dS(x)dvds \\ &\quad + \int_0^t\int_{\Omega\times\mathbb{R}^3} \psi g\, dxdvds.\end{aligned}$$

We decompose $h = \pi_L(h) + \pi_L^\perp(h)$ in the term involving $v\cdot\nabla_x$ and use the fact that $L_\mu[h] = L_\mu[\pi_L^\perp(h)]$ to obtain the weak formulation

$$-\int_0^t\int_{\Omega\times\mathbb{R}^3} \pi_L(h)v\cdot\nabla_x\psi dxdvds = \Psi_1(t)+\Psi_2(t)+\Psi_3(t)+\Psi_4(t)+\Psi_5(t)+\Psi_6(t) \tag{3.11}$$

with the following definitions

$$\Psi_1(t) = \int_{\Omega\times\mathbb{R}^3} \psi_0 h_0\, dxdv - \int_{\Omega\times\mathbb{R}^3} \psi(t)h(t)\, dxdv, \tag{3.12}$$

$$\Psi_2(t) = \int_0^t\int_{\Omega\times\mathbb{R}^3} \pi_L^\perp(h)v\cdot\nabla_x\psi\, dxdvds, \tag{3.13}$$

$$\Psi_3(t) = \int_0^t\int_{\Omega\times\mathbb{R}^3} L_\mu\left[\pi_L^\perp(h)\right]\psi\, dxdvds, \tag{3.14}$$

$$\Psi_4(t) = -\int_0^t\int_\Lambda \psi hv\cdot n(x)\, dS(x)dvds, \tag{3.15}$$

$$\Psi_5(t) = \int_0^t\int_{\Omega\times\mathbb{R}^3} h\partial_t\psi\, dxdvds, \tag{3.16}$$

$$\Psi_6(t) = \int_0^t\int_{\Omega\times\mathbb{R}^3} \psi g\, dxdvds. \tag{3.17}$$

For each of the functions a, b and c, we shall construct a ψ such that the left-hand side of (3.11) is exactly the L^2_x-norm of the function and the rest of the proof is estimating the six different terms $\Psi_i(t)$. Note that $\Psi_1(t)$ is already under the desired form

$$\Psi_1(t) = N_h(t) - N_h(0) \tag{3.18}$$

with $|N_h(s)| \leqslant C\,\|h\|^2_{L^2_{x,v}}$ if $\psi(x,v)$ is in $L^2_{x,v}$ and its norm is controlled by the one of h (which will be the case for our choices).

Remark 3.3. *The linear perturbed equation (3.9), the Maxwell boundary condition (3.2), and the conservation of mass are invariant under standard time mollification. We therefore consider for simplicity in the rest of the proof that all functions are smooth in the variable t. Exactly the same estimates can be derived for more general functions: study time mollified equation and then take the limit in the smoothing parameter.*

For clarity, every positive constant not depending on h will be denoted by C_i.

Estimate for a. By assumption h is mass-preserving which is equivalent to

$$0 = \int_{\Omega\times\mathbb{R}^3} h(t,x,v)\sqrt{\mu(v)}\,dxdv = \int_\Omega a(t,x)\,dx.$$

We can thus choose the following test function

$$\psi_a(t,x,v) = \left(|v|^2 - \alpha_a\right)\sqrt{\mu}v\cdot\nabla_x\phi_a(t,x)$$

where

$$-\Delta_x\phi_a(t,x) = a(t,x) \quad \text{and} \quad \partial_n\phi_a|_{\partial\Omega} = 0,$$

and $\alpha_a > 0$ is chosen such that for all $1\leqslant i\leqslant 3$

$$\int_{\mathbb{R}^3}\left(|v|^2-\alpha_a\right)\frac{|v|^2-3}{2}v_i^2\mu(v)\,dv = 0.$$

The differential operator ∂_n denotes the tangential derivative at the boundary. The fact that the integral over Ω of $a(t,\cdot)$ is null allows us to use standard elliptic estimate [13]:

$$\forall t\geqslant 0, \quad \|\phi_a(t)\|_{H^2_x} \leqslant C_0\,\|a(t)\|_{L^2_x}\,. \tag{3.19}$$

The latter estimate provides the control of $\Psi_1 = N_h^{(a)}(t) - N_h^{(a)}(0)$, as discussed before, and the control of (3.17), using Cauchy-Schwarz and Young's inequalities,

$$\begin{aligned}|\Psi_6(t)| &\leqslant C\int_0^t \|\phi_a\|^2_{L^2_x}\,\|g\|_{L^2_{x,v}}\,ds\\ &\leqslant \frac{C_1}{4}\int_0^t\|a\|_{L^2_x}\,ds + C_6\int_0^t\|g\|^2_{L^2_{x,v}}\,ds,\end{aligned} \tag{3.20}$$

where $C_1>0$ is given by (3.21) below.

Firstly we compute the term on the right-hand side of (3.11).

$$\begin{aligned}&-\int_0^t\int_{\Omega\times\mathbb{R}^3}\pi_L(h)v\cdot\nabla_x\psi_a\,dxdvds\\ &\quad= -\sum_{1\leqslant i,j\leqslant 3}\int_0^t\int_\Omega a(s,x)\left(\int_{\mathbb{R}^3}\left(|v|^2-\alpha_a\right)v_iv_j\mu(v)\,dv\right)\partial_{x_i}\partial_{x_j}\phi_a(s,x)\,dxds\\ &\qquad-\sum_{1\leqslant i,j\leqslant 3}\int_0^t\int_\Omega b(s,x)\cdot\left(\int_{\mathbb{R}^3}v\left(|v|^2-\alpha_a\right)v_iv_j\mu(v)\,dv\right)\partial_{x_i}\partial_{x_j}\phi_a(s,x)\,dxds\\ &\qquad-\sum_{1\leqslant i,j\leqslant 3}\int_0^t\int_\Omega c(s,x)\left(\int_{\mathbb{R}^3}\left(|v|^2-\alpha_a\right)\frac{|v|^2-3}{2}v_iv_j\mu(v)\,dv\right)\partial_{x_i}\partial_{x_j}\phi_a(s,x).\end{aligned}$$

By oddity the second term is null, as well as the first and last ones are when $i\neq j$. When $i=j$ in the last term we recover exactly our choice of α_a which makes the last term being null too. It only remains the first term when $i=j$

$$\begin{aligned}-\int_0^t\int_{\Omega\times\mathbb{R}^3}\pi_L(h)v\cdot\nabla_x\psi_a\,dxdvds &= -C_1\int_0^t\int_\Omega a(s,x)\Delta_x\phi_a(s,x)\,dxds\\ &= C_1\int_0^t\|a\|^2_{L^2_x}\,ds.\end{aligned} \tag{3.21}$$

Direct computations show $\alpha_a = 10$ and $C_1 > 0$.

We recall $L_\nu = -\nu(v) + K$ where K is a bounded operator and that the H^2_x-norm of $\phi_a(t,x)$ is bounded by the L^2_x-norm of $a(t,x)$. For the terms Ψ_2 (3.13) and Ψ_3 (3.14) a mere Cauchy-Schwarz inequality yields

$$\begin{aligned} \forall i \in \{2,3\}, \quad |\Psi_i(t)| &\leqslant C \int_0^t \|a\|_{L^2_x} \left\|\pi_L^\perp(h)\right\|_{L^2_{x,v}} \, ds \\ &\leqslant \frac{C_1}{4} \int_0^t \|a\|^2_{L^2_x} \, ds + C_2 \int_0^t \left\|\pi_L^\perp(h)\right\|^2_{L^2_{x,v}} \, ds. \end{aligned} \tag{3.22}$$

We used Young's inequality for the last inequality, with C_1 defined in (3.21).

The term Ψ_4 (3.15) deals with boundary so we decompose it into Λ^+ and Λ^-. In the Λ^- integral we apply the Maxwell boundary condition satisfied by h and use the change of variable $v \mapsto \mathcal{R}_x(v)$. Since $|v|^2$, $\mu(v)$, $\phi_a(s,x)$, the specular part and P_{Λ_μ} (3.3) are invariant by this isometric change of variable we get

$$\begin{aligned} \Psi_4(t) = &- \int_0^t \int_{\Lambda^+} h \left(|v|^2 - \alpha_a\right) |v \cdot n(x)| \, \nabla_x \phi_a(s,x) \cdot v \sqrt{\mu} \, dS(x) dv ds \\ &+ (1-\alpha) \int_0^t \int_{\Lambda^+} h \left(|v|^2 - \alpha_a\right) |v \cdot n(x)| \, \nabla_x \phi_a \cdot \mathcal{R}_x(v) \sqrt{\mu} \, dS(x) dv ds \\ &+ \alpha \int_0^t \int_{\Lambda^+} P_{\Lambda_\mu}(h) \left(|v|^2 - \alpha_a\right) |v \cdot n(x)| \, \nabla_x \phi_a \cdot \mathcal{R}_x(v) \sqrt{\mu} \, dS(x) dv ds. \end{aligned}$$

so

$$\begin{aligned} \Psi_4(t) = &- (1-\alpha) \int_0^t \int_{\Lambda^+} h \left(|v|^2 - \alpha_a\right) |v \cdot n(x)| \, \nabla_x \phi_a \cdot \left[v - \mathcal{R}_x(v)\right] \sqrt{\mu} \, dS(x) dv ds \\ &- \alpha \int_0^t \int_{\Lambda^+} \left(|v|^2 - \alpha_a\right) |v \cdot n(x)| \, \nabla_x \phi_a \cdot \left[vh - \mathcal{R}_x(v) P_{\Lambda_\mu}(h)\right] \sqrt{\mu} \end{aligned} \tag{3.23}$$

By definition of the specular reflection and the tangential derivative

$$\begin{aligned} |v \cdot n(x)| \, \nabla_x \phi_a(s,x) \cdot (v - \mathcal{R}_x(v)) &= 2 \left(v \cdot n(x)\right) n \cdot \nabla_x \phi_a(s,x) \\ &= 2 \left(v \cdot n(x)\right) \partial_n \phi_a(s,x). \end{aligned}$$

The contribution of the specular reflection part is therefore null since ϕ_a was chosen such that $\partial_n \phi_a|_{\partial\Omega} = 0$. For the diffusive part we compute

$$vh - \mathcal{R}_x(v) P_{\Lambda_\mu}(h) = v P^\perp_{\Lambda_\mu}(h) + 2 P_{\Lambda_\mu}(h) \left(v \cdot n(x)\right) n(x)$$

and again the term in the direction of $n(x)$ gives a zero contribution since $\partial_n \phi_a|_{\partial\Omega} = 0$. It only remains

$$\Psi_4(t) = -\alpha \int_0^t \int_{\Lambda^+} \left[\left(|v|^2 - \alpha_a\right) |v \cdot n(x)| \sqrt{\mu} \, v \cdot \nabla_x \phi_a\right] P^\perp_{\Lambda_\mu}(h) \, dS(x) dv ds.$$

We apply Cauchy-Schwarz inequality and the control on the H^2 norm of ϕ_a to finally obtain the following estimate

$$
\begin{aligned}
|\Psi_4(t)| &\leqslant C \int_0^t \|a\|_{L^2_x} \left\| P^\perp_{\Lambda_\mu}(h) \right\|_{L^2_{\Lambda^+}} \, ds \\
&\leqslant \frac{C_1}{4} \int_0^t \|a\|^2_{L^2_x} \, ds + C_4 \int_0^t \left\| P^\perp_{\Lambda_\mu}(h) \right\|^2_{L^2_{\Lambda^+}} \, ds,
\end{aligned} \tag{3.24}
$$

where we used Young's inequality with C_1 defined in (3.21).

It remains to estimate the term with time derivatives (3.16). It reads

$$
\begin{aligned}
\Psi_5(t) &= \int_0^t \int_{\Omega\times\mathbb{R}^3} h \left(|v|^2 - \alpha_a \right) v \cdot \left[\partial_t \nabla_x \phi_a \right] \sqrt{\mu} \, dxdvds \\
&= \sum_{i=1}^3 \int_0^t \int_{\Omega\times\mathbb{R}^3} \pi_L(h) \left(|v|^2 - \alpha_a \right) v_i \sqrt{\mu} \, \partial_t \partial_{x_i} \phi_a \, dxdvds \\
&\quad + \int_0^t \int_{\Omega\times\mathbb{R}^3} \pi^\perp_L(h) \left(|v|^2 - \alpha_a \right) \sqrt{\mu} \, v \cdot \left[\partial_t \nabla_x \phi_a \right] \, dxdvds
\end{aligned}
$$

Using oddity properties for the first integral on the right-hand side and then Cauchy-Schwarz and the following bound

$$
\int_{\mathbb{R}^3} \left(|v|^2 - \alpha_a \right)^2 |v|^2 \, \mu(v) \, dv < +\infty
$$

we get

$$
|\Psi_5(t)| \leqslant C \int_0^t \left[\|b\|_{L^2_x} + \left\| \pi^\perp_L(h) \right\|_{L^2_{x,v}} \right] \|\partial_t \nabla_x \phi_a\|_{L^2_x} \, ds. \tag{3.25}
$$

The estimation on $\|\partial_t \nabla_x \phi_a\|_{L^2_x}$ will come from elliptic estimates in negative Sobolev spaces. We use the decomposition of the weak formulation (3.11) between t and $t+\varepsilon$ (instead of between 0 and t) with $\psi(t,x,v) = \phi(x)\sqrt{\mu} \in H^1_x$ with the integral of ϕ on Ω being zero. $\psi(x)\mu(v)$ and $v\psi(x)\mu(v)$ are in $\mathrm{Ker}(L_\mu)$ and therefore are orthogonal to $\pi_L(h)$ and $L_\mu[h]$. Moreover, ψ does not depend on time. Hence,

$$
\Psi_2(t) = \Psi_3(t) = \Psi_5(t) = 0.
$$

At last, with the same computations as before the boundary term is

$$
\Psi_4 = -\alpha \int_0^t \int_{\partial\Omega} \phi(x) \int_{v\cdot n(x)>0} \left(\mathrm{Id} - P_{\Lambda_\mu} \right)(h) \, |v \cdot n(x)| \sqrt{\mu} \, dS(x) dvds = 0.
$$

The weak formulation associated to $\phi(x)\sqrt{\mu}$ is therefore

$$
\begin{aligned}
&\int_{\Omega\times\mathbb{R}^3} \phi(x) h(t+\varepsilon) \sqrt{\mu} \, dxdv - \int_{\Omega\times\mathbb{R}^3} \phi(x) h(t) \sqrt{\mu} \, dxdv \\
&\qquad = \int_t^{t+\varepsilon} \int_{\Omega\times\mathbb{R}^3} \left[\pi_L(h) v \cdot \nabla_x \phi(x) + g\phi(x) \right] \sqrt{\mu} \, dxdvds,
\end{aligned}
$$

which is equal to

$$\int_\Omega \left[a(t+\varepsilon)-a(t)\right]\phi(x)\,dx \\ = C\int_t^{t+\varepsilon}\left[\int_\Omega b(s,x)\cdot\nabla_x\phi(x)\,dxds+\int_{\Omega\times\mathbb{R}^3} g\phi\sqrt{\mu}\,dxdv\right]ds.$$

Dividing by ε and taking the limit as ε goes to 0 yields the following estimates, thanks to a Cauchy-Schwarz inequality,

$$\int_\Omega \partial_t a(t,x)\phi(x)\,dx \leqslant C\left[\|b(t)\|_{L^2_x}\|\nabla_x\phi\|_{L^2_x}+\|g\|_{L^2_{x,v}}\|\phi\|_{L^2_x}\right].$$

Since ϕ has a null integral on Ω we can apply Poincaré inequality.

$$\int_\Omega \partial_t a(t,x)\phi(x)\,dx \leqslant C\left[\|b(t)\|_{L^2_x}+\|g\|_{L^2_{x,v}}\right]\|\nabla_x\phi\|_{L^2_x}.$$

The latter inequality is true for all ϕ in $\mathcal{H}^1_x$ the set of functions in H^1_x with a null integral. Therefore, for all $t\geqslant 0$

$$\|\partial_t a(t,x)\|_{(\mathcal{H}^1_x)^*}\leqslant C\left[\|b(t)\|_{L^2_x}+\|g\|_{L^2_{x,v}}\right] \tag{3.26}$$

where $(\mathcal{H}^1_x)^*$ is the dual of $\mathcal{H}^1_x$.

We fix t and thanks to the conservation of mass we have that the integral of $\partial_t a$ is null on Ω. We can construct $\phi(t,x)$ such that

$$-\Delta_x\phi(t,x)=\partial_t a(t,x)\quad\text{and}\quad \partial_n\phi|_{\partial\Omega}=0.$$

and by standard elliptic estimates [13] and (3.26):

$$\|\phi\|_{\mathcal{H}^1_x}\leqslant\|\partial_t a\|_{(\mathcal{H}^1_x)^*}\leqslant C\left[\|b(t)\|_{L^2_x}+\|g\|_{L^2_{x,v}}\right].$$

Combining this estimate with

$$\|\partial_t\nabla_x\phi_a\|_{L^2_x}=\left\|\nabla_x\Delta^{-1}\partial_t a\right\|_{L^2_x}\leqslant\left\|\Delta^{-1}\partial_t a\right\|_{H^1_x}=\|\phi\|_{\mathcal{H}^1_x}$$

we can further control Ψ_5 in (3.25)

$$|\Psi_5(t)|\leqslant C_5\int_0^t\left[\|b\|^2_{L^2_x}+\left\|\pi^\perp_L(h)\right\|^2_{L^2_{x,v}}+\|g\|^2_{L^2_x}\right]ds. \tag{3.27}$$

We now gather (3.21), (3.18), (3.22), (3.24), (3.27) and (3.20) into (3.11)

$$\int_0^t\|a\|^2_{L^2_x}\,ds\leqslant N^{(a)}_h(t)-N^{(a)}_h(0)+C_{a,b}\int_0^t\|b\|^2_{L^2_x}\,ds \\ +C_a\int_0^t\left[\left\|P^\perp_{\Lambda_\mu}(h)\right\|^2_{L^2_{\Lambda^+}}+\left\|\pi^\perp_L(h)\right\|^2_{L^2_{x,v}}+\|g\|^2_{L^2_{x,v}}\right]ds. \tag{3.28}$$

Estimate for b. The choice of function to integrate against to deal with the b term is more involved.

We emphasize that $b(t,x)$ is a vector $(b_1(t,x),b_2(t,x),b_3(t,x))$. Fix J in $\{1,2,3\}$ and define

$$\psi_J(t,x,v)=\sum_{i=1}^3\varphi_i^{(J)}(t,x,v),$$

with

$$\varphi_i^{(J)}(t,x,v)=\begin{cases}|v|^2\,v_iv_J\sqrt{\mu}\partial_{x_i}\phi_J(t,x)-\dfrac{7}{2}\left(v_i^2-1\right)\sqrt{\mu}\partial_{x_J}\phi_J(t,x), & \text{if } i\neq J\\ \dfrac{7}{2}\left(v_J^2-1\right)\sqrt{\mu}\partial_{x_J}\phi_J(t,x), & \text{if } i=J.\end{cases}$$

where

$$-\Delta_x\phi_J(t,x)=b_J(t,x)\quad\text{and}\quad\phi_J|_{\partial\Omega}=0.$$

Since it will be important we emphasize here that for all $i\neq k$

$$\int_{\mathbb{R}^3}\left(v_i^2-1\right)\mu(v)\,dv=0\quad\text{and}\quad\int_{\mathbb{R}^3}\left(v_i^2-1\right)v_k^2\mu(v)\,dv=0. \tag{3.29}$$

The vanishing of ϕ_J at the boundary implies, by standard elliptic estimate [13],

$$\forall t\geqslant 0,\quad \|\phi_J(t)\|_{H^2_x}\leqslant C_0\,\|b_J(t)\|_{L^2_x}\,. \tag{3.30}$$

Again, this estimate provides the control of $\Psi_1=N_h^{(J)}(t)-N_h^{(J)}(0)$ and of $\Psi_6(t)$ as in (3.20):

$$|\Psi_6(t)|\leqslant\frac{7}{4}\int_0^t\|b_J\|^2_{L^2_x}\,ds+C_6\int_0^t\|g\|^2_{L^2_{x,v}}\,ds. \tag{3.31}$$

We start by the right-hand side of (3.11). By oddity, there is neither contribution from $a(s,x)$ nor from $c(s,x)$. Hence,

$$\begin{aligned}&-\int_0^t\int_{\Omega\times\mathbb{R}^3}\pi_L(h)v\cdot\nabla_x\psi_J\,dxdvds\\&=-\sum_{1\leqslant j,k\leqslant 3}\sum_{\substack{i=1\\ i\neq J}}^3\int_0^t\int_\Omega b_k(s,x)\left(\int_{\mathbb{R}^3}\left|v^2\right|v_kv_iv_jv_J\mu(v)\,dv\right)\partial_{x_j}\partial_{x_i}\phi_J(s,x)\,dxds\\&\quad+\frac{7}{2}\sum_{1\leqslant j,k\leqslant 3}\sum_{\substack{i=1\\ i\neq J}}^3\int_0^t\int_\Omega b_k(s,x)\left(\int_{\mathbb{R}^3}\left(v_i^2-1\right)v_kv_j\mu(v)\,dv\right)\partial_{x_j}\partial_{x_J}\phi_J(s,x)\,dxds\\&\quad-\frac{7}{2}\sum_{1\leqslant j,k\leqslant 3}\int_0^t\int_\Omega b_k(s,x)\left(\int_{\mathbb{R}^3}\left(v_J^2-1\right)v_jv_k\mu(v)\,dv\right)\partial_{x_j}\partial_{x_J}\phi_J(s,x)\,dxds.\end{aligned}$$

The last two integrals on $\mathbb{R}^3$ are zero if $j\neq k$. Moreover, when $j=k$ and $j\neq J$ it is also zero by (3.29). We compute directly for $j=J$

$$\int_{\mathbb{R}^3}\left(v_J^2-1\right)v_J^2\mu(v)\,dv=2.$$

The first term is composed by integrals in v of the form

$$\int_{\mathbb{R}^3}|v|^2\,v_kv_iv_jv_J\mu(v)\,dv$$

which are always null unless two indices are equals to the other two. Therefore if $i=j$ then $k=J$ and if $i\neq j$ we only have two options: $k=i$ and $j=J$ or $k=j$

and $i = J$. Hence,

$$
\begin{aligned}
&-\int_0^t\int_{\Omega\times\mathbb{R}^3}\pi_L(h)v\cdot\nabla_x\psi_J\,dxdvds\\
&=-\sum_{\substack{i=1\\ i\neq J}}^{3}\int_0^t\int_\Omega b_J(s,x)\partial_{x_ix_i}\phi_J\left(\int_{\mathbb{R}^3}|v|^2\,v_i^2v_J^2\mu(v)\,dv\right)dxds\\
&\quad-\sum_{\substack{i=1\\ i\neq J}}^{3}\int_0^t\int_\Omega b_i(s,x)\partial_{x_ix_J}\phi_J\left(\int_{\mathbb{R}^3}|v|^2\,v_i^2v_J^2\mu(v)\,dv\right)dxds\\
&\quad+7\sum_{\substack{i=1\\ i\neq J}}^{3}\int_0^t\int_\Omega b_i(s,x)\partial_{x_ix_J}\phi_J\,dxds-7\int_0^t\int_\Omega b_J(s,x)\partial_{x_J}\partial_{x_J}\phi_J(s,x)\,dxds.
\end{aligned}
$$

To conclude we compute $\int_{\mathbb{R}^3}|v^2|\,v_i^2v_J^2\mu(v)\,dv = 7$ whenever $i\neq J$ and it thus only remains the following equality

$$
\begin{aligned}
-\int_0^t\int_{\Omega\times\mathbb{R}^3}\pi_L(h)v\cdot\nabla_x\psi_a\,dxdvds &= -7\int_0^t\int_\Omega b_J(s,x)\Delta_x\phi_J(s,x)\,dxds\\
&= 7\int_0^t\|b_J\|_{L^2_x}^2\,ds. \qquad (3.32)
\end{aligned}
$$

Then the term Ψ_2 and Ψ_3 are dealt with as in (3.22)

$$
\forall i\in\{2,3\},\quad |\Psi_i(t)|\leqslant\frac{7}{4}\int_0^t\|b_J\|_{L^2_x}^2\,ds+C_2\int_0^t\left\|\pi_L^\perp(h)\right\|_{L^2_{x,v}}^2\,ds. \qquad (3.33)
$$

The boundary term Ψ_4 is divided into Λ^+ and Λ^-, we apply the Maxwell boundary condition (3.2) and the change of variable $v\mapsto\mathcal{R}_x(v)$ on the Λ^- part

$$
\begin{aligned}
\Psi_4(t) =& -\sum_{i=1}^{3}\int_0^t\int_{\Lambda^+}h\,|v\cdot n(x)|\,\varphi_i^{(J)}(s,x,v)\,dS(x)dvds\\
&+(1-\alpha)\sum_{i=1}^{3}\int_0^t\int_{\Lambda^+}h\,|v\cdot n(x)|\,\varphi_i^{(J)}(s,x,\mathcal{R}_x(v))\,dS(x)dvds\\
&+\alpha\sum_{i=1}^{3}\int_0^t\int_{\Lambda^+}P_{\Lambda_\mu}(h)\,|v\cdot n(x)|\,\varphi_i^{(J)}(s,x,\mathcal{R}_x(v))\,dS(x)dvds.
\end{aligned}
$$

We decompose $h = P_{\Lambda_\mu}(h)+P_{\Lambda_\mu}^\perp$ to obtain

$$
\begin{aligned}
\Psi_4(t) =& -\sum_{i=1}^{3}\int_0^t\int_{\Lambda^+}P_{\Lambda_\mu}(h)\,(v\cdot n(x))\left[\varphi_i^{(J)}(s,x,v)-\varphi_i^{(J)}(s,x,\mathcal{R}_x(v))\right]dS(x)dvds\\
&-\sum_{i=1}^{3}\int_0^t\int_{\Lambda^+}P_{\Lambda_\mu}^\perp(h)\,|v\cdot n(x)|\\
&\qquad\times\left[\varphi_i^{(J)}(s,x,v)-(1-\alpha)\varphi_i^{(J)}(s,x,\mathcal{R}_x(v))\right]dS(x)dvds. \qquad (3.34)
\end{aligned}
$$

We apply Cauchy-Schwarz inequality and the elliptic estimate on ϕ_J (3.30) to the second integral obtain the following estimate

(3.35)
$$\begin{aligned}\left|-\sum_{i=1}^{3}\int_0^t\int_{\Lambda^+} P^\perp_{\Lambda_\mu}(h)\,|v\cdot n(x)|\left[\varphi_i^{(J)}(s,x,v)-(1-\alpha)\varphi_i^{(J)}(s,x,\mathcal{R}_x(v))\right]\,dS(x)dvds\right|\\ \leqslant C\int_0^t \|b_J\|\left\|P^\perp_{\Lambda_\mu}(h)\right\|_{L^2_{\Lambda^+}}\,ds\\ \leqslant \frac{7}{4}\int_0^t \|b_J\|^2_{L^2_x}\,ds + C_4\int_0^t \left\|\pi^\perp_L(h)\right\|^2_{L^2_{\Lambda^+}}\,ds,\end{aligned}$$

where we also used Young's inequality.

The term involving $P_{\Lambda_\mu}(h)$ in (3.34) is computed directly by a change of variable $v\mapsto\mathcal{R}_x(v)$ to come back to the full boundary Λ and the property (3.3) that is

$$P_{\Lambda_\mu}(h)(s,x,v)=z(s,x)\sqrt{\mu(v)}.$$

We also have $\varphi_i^{(J)}$ in the following form

$$\varphi_i^{(J)}(t,x,v)=\widetilde{\varphi}_i^{(J)}(v)\sqrt{\mu(v)}\partial\phi_J(t,x)$$

where ∂_i begin a certain derivative in x and $\widetilde{\varphi}_i^{(J)}$ is an even function. We thus get

$$\begin{aligned}&\int_0^t\int_{\Lambda^+} P_{\Lambda_\mu}(h)\,(v\cdot n(x))\left[\varphi_i^{(J)}(s,x,v)-\varphi_i^{(J)}(s,x,\mathcal{R}_x(v))\right]\,dS(x)dvds\\ &\quad=\int_0^t\int_{\Lambda} P_{\Lambda_\mu}(h)\,(v\cdot n(x))\,\varphi_i^{(J)}(s,x,v)\,dS(x)dvds\\ &\quad=\sum_{k=1}^{3}\int_0^t\int_\Omega z(s,x)n_k(x)\partial_i\phi_J(s,x)\left(\int_{\mathbb{R}^3}\widetilde{\varphi}_i^{(J)}(v)v_k\mu(v)\,dv\right)\,dS(x)ds\\ &\quad=0,\end{aligned}$$

by oddity. Combining the latter with (3.35) inside (3.34) yields

$$|\Psi_4(t)|\leqslant\frac{C_1}{4}\int_0^t\|b_J\|^2_{L^2_x}\,ds+C_4\int_0^t\left\|P^\perp_{\Lambda_\mu}(h)\right\|^2_{L^2_{\Lambda^+}}\,ds. \tag{3.36}$$

It remains to estimate Ψ_5 which involves time derivative (3.16):

$$\begin{aligned}\Psi_5(t) &= \sum_{i=1}^{3}\int_0^t\int_{\Omega\times\mathbb{R}^3} h\partial_t\varphi_i^{(J)}(s,x,v)\,dxdvds\\ &= \sum_{i=1}^{3}\int_0^t\int_{\Omega\times\mathbb{R}^3} \pi^\perp_L(h)\partial_t\varphi_i^{(J)}(s,x,v)\,dxdvds\\ &\quad+\sum_{\substack{i=1\\ i\neq J}}^{3}\int_0^t\int_{\Omega\times\mathbb{R}^3}\pi_L(h)\,|v|^2\,v_iv_J\sqrt{\mu}\partial_{x_i}\phi_J\,dxdvds\\ &\quad+\sum_{i=1}^{3}\pm\frac{7}{2}\int_0^t\int_{\Omega\times\mathbb{R}^3}\pi_L(h)\left(v_i^2-1\right)\sqrt{\mu}\partial_{x_J}\phi_J\,dxdvds.\end{aligned}$$

By oddity arguments, only terms in $a(s,x)$ and $c(s,x)$ can contribute to the last two terms on the right-hand side. However, $i \neq J$ implies that the second term is zero as well as the contribution of $a(s,x)$ in the third term thanks to (3.29). Finally, a Cauchy-Schwarz inequality on both integrals yields as in (3.25)

$$|\Psi_5(t)| \leqslant C \int_0^t \left[\|c\|_{L^2_x} + \left\|\pi_L^\perp(h)\right\|_{L^2_{x,v}} \right] \|\partial_t \nabla_x \phi_J\|_{L^2_x} \, ds. \tag{3.37}$$

To estimate $\|\partial_t \nabla_x \phi_J\|_{L^2_x}$ we follow the idea developed for $a(s,x)$ about negative Sobolev regularity. We apply the weak formulation (3.11) to a specific function between t and $t+\varepsilon$. The test function is $\psi(x,v) = \phi(x) v_J \sqrt{\mu}$ with ϕ in H^1_x and null on the boundary. Note that ψ does not depend on t, vanishes at the boundary and belongs to $\mathrm{Ker}(L)$. Hence,

$$\Psi_3(t) = \Psi_4 = \Psi_5(t) = 0.$$

It remains

$$\begin{aligned} C \int_\Omega \left[b_J(t+\varepsilon) - b_J(t) \right] \phi(x) \, dx \;\; = \;\; & \int_t^{t+\varepsilon} \int_{\Omega \times \mathbb{R}^3} \pi_L(h) v_J v \cdot \nabla_x \phi(x) \sqrt{\mu} \, dx dv ds \\ & + \int_t^{t+\varepsilon} \int_{\Omega \times \mathbb{R}^3} \pi_L^\perp(h) v_J v \cdot \nabla_x \phi(x) \sqrt{\mu} \, dx dv ds \\ & + \int_t^{t+\varepsilon} \int_{\Omega \times \mathbb{R}^3} g \phi(x) v_J \sqrt{\mu} \, dx dv ds. \end{aligned}$$

As for $a(t,x)$ we divide by ε and take the limit as ε goes to 0. By oddity, the first integral on the right-hand side only gives terms with $a(s,x)$ and $c(s,x)$. The second term is dealt with by a Cauchy-Schwarz inequality. Finally, we apply a Cauchy-Schwarz inequality for the last integral with a Poincaré inequality for $\phi(x)$ (ϕ is null on the boundary). This yields

$$\left| \int_\Omega \partial_t b_J(t,x) \phi(x) \, dx \right| \leqslant C \left[\|a\|_{L^2_x} + \|c\|_{L^2_x} + \left\|\pi_L^\perp(h)\right\|_{L^2_{x,v}} + \|g\|_{L^2_x} \right] \|\nabla_x \phi\|_{L^2_x}. \tag{3.38}$$

The latter is true for all $\phi(x)$ in H^1_x vanishing on the boundary. We thus fix t and apply the inequality above to

$$-\Delta_x \phi(t,x) = \partial_t b_J(t,x) \quad \text{and} \quad \phi|_{\partial\Omega} = 0,$$

and obtain

$$\|\partial_t \nabla_x \phi_J\|^2_{L^2_x} = \left\|\nabla_x \Delta^{-1} \partial_t b_J\right\|^2_{L^2_x} = \int_\Omega \left(\nabla_x \Delta^{-1} \partial_t b_J \right) \nabla_x \phi(x) \, dx.$$

We integrate by parts (the boundary term vanishes because of our choice of ϕ).

$$\|\partial_t \nabla_x \phi_J\|^2_{L^2_x} = \int_\Omega \partial_t b_J(t,x) \phi(x) \, dx$$

At last, we use (3.38)

$$\begin{aligned} \|\partial_t \nabla_x \phi_J\|^2_{L^2_x} &\leqslant C \left[\|a\|_{L^2_x} + \|c\|_{L^2_x} + \left\|\pi_L^\perp(h)\right\|_{L^2_{x,v}} + \|g\|_{L^2_x} \right] \|\nabla_x \phi\|_{L^2_x} \\ &= C \left[\|a\|_{L^2_x} + \|c\|_{L^2_x} + \left\|\pi_L^\perp(h)\right\|_{L^2_{x,v}} + \|g\|_{L^2_x} \right] \left\|\nabla_x \Delta_x^{-1} \partial_t b_J\right\|_{L^2_x} \\ &= C \left[\|a\|_{L^2_x} + \|c\|_{L^2_x} + \left\|\pi_L^\perp(h)\right\|_{L^2_{x,v}} + \|g\|_{L^2_x} \right] \|\partial_t \nabla_x \phi_J\|_{L^2_x} \end{aligned}$$

Combining this estimate with (3.37) and using Young's inequality with any $\varepsilon_b > 0$

$$|\Psi_5(t)| \leqslant \varepsilon_b \int_0^t \|a\|_{L^2_x}^2 \, ds + C_5(\varepsilon_b) \int_0^t \left[\|c\|_{L^2_x}^2 + \left\|\pi_L^\perp(h)\right\|_{L^2_{x,v}}^2 + \|g\|_{L^2_x}^2 \right] ds. \tag{3.39}$$

We now gather (3.32), (3.18), (3.33), (3.36) and (3.39)

$$\begin{aligned} \int_0^t \|b_J\|_{L^2_x}^2 \, ds \leqslant & N_h^{(J)}(t) - N_h^{(J)}(0) + \varepsilon_b \int_0^t \|a\|_{L^2_x}^2 \, ds + C_{J,c}(\varepsilon_b) \int_0^t \|c\|_{L^2_x}^2 \, ds \\ & + C_J(\varepsilon_b) \int_0^t \left[\left\| P^\perp_{\Lambda_\mu}(h) \right\|_{L^2_{\Lambda^+}}^2 + \left\|\pi_L^\perp(h)\right\|_{L^2_{x,v}}^2 + \|g\|_{L^2_x}^2 \right] ds. \end{aligned}$$

Finally, summing over all J in $\{1,2,3\}$

$$\begin{aligned} \int_0^t \|b\|_{L^2_x}^2 \, ds \leqslant & N_h^{(b)}(t) - N_h^{(b)}(0) + \varepsilon_b \int_0^t \|a\|_{L^2_x}^2 \, ds + C_{b,c}(\varepsilon_b) \int_0^t \|c\|_{L^2_x}^2 \, ds \\ & + C_b(\varepsilon_b) \int_0^t \left[\left\| P^\perp_{\Lambda_\mu}(h) \right\|_{L^2_{\Lambda^+}}^2 + \left\|\pi_L^\perp(h)\right\|_{L^2_{x,v}}^2 + \|g\|_{L^2_x}^2 \right] ds. \end{aligned} \tag{3.40}$$

Estimate for c. The handling of $c(t,x)$ is quite similar to the one of $a(t,x)$ but it involves a more intricate treatment of the boundary terms as h does not preserves the energy. We choose the following test function

$$\psi_c(t,x,v) = \left(|v|^2 - \alpha_c\right) v \cdot \nabla_x \phi_c(t,x) \sqrt{\mu(v)}$$

where

$$-\Delta_x \phi_c(t,x) = c(t,x) \quad \text{and} \quad \phi_c|_{\partial\Omega} = 0,$$

and $\alpha_c > 0$ is chosen such that for all $1 \leqslant i \leqslant 3$

$$\int_{\mathbb{R}^3} \left(|v|^2 - \alpha_c\right) v_i^2 \mu(v) \, dv = 0.$$

The vanishing of ϕ_c at the boundary implies, by standard elliptic estimate [13],

$$\forall t \geqslant 0, \quad \|\phi_c(t)\|_{H^2_x} \leqslant C_0 \|c(t)\|_{L^2_x}. \tag{3.41}$$

Again, this estimate provides the control of $\Psi_1 = N_h^{(c)}(t) - N_h^{(c)}(0)$ and of $\Psi_6(t)$ as in (3.20):

$$|\Psi_6(t)| \leqslant \frac{C_1}{4} \int_0^t \|c\|_{L^2_x}^2 \, ds + C_6 \int_0^t \|g\|_{L^2_{x,v}}^2 \, ds, \tag{3.42}$$

where C_1 is given by (3.43) below.

We start by the right-hand side of (3.11).

$$
\begin{aligned}
&-\int_0^t\int_{\Omega\times\mathbb{R}^3}\pi_L(h)v\cdot\nabla_x\psi_c\,dxdvds\\
&\quad=-\sum_{1\leqslant i,j\leqslant 3}\int_0^t\int_\Omega a(s,x)\left(\int_{\mathbb{R}^3}\left(|v|^2-\alpha_c\right)v_iv_j\mu(v)\,dv\right)\partial_{x_i}\partial_{x_j}\phi_c(s,x)\,dxds\\
&\qquad-\sum_{1\leqslant i,j\leqslant 3}\int_0^t\int_\Omega b(s,x)\cdot\left(\int_{\mathbb{R}^3}v\left(|v|^2-\alpha_c\right)v_iv_j\mu(v)\,dv\right)\partial_{x_i}\partial_{x_j}\phi_c(s,x)\,dxds\\
&\qquad-\sum_{1\leqslant i,j\leqslant 3}\int_0^t\int_\Omega c(s,x)\left(\int_{\mathbb{R}^3}\left(|v|^2-\alpha_c\right)\frac{|v|^2-3}{2}v_iv_j\mu(v)\,dv\right)\partial_{x_i}\partial_{x_j}\phi_c(s,x).
\end{aligned}
$$

By oddity, the second integral vanishes, as well as all the others if $i\neq j$. Our choice of α_c makes the first integral vanish even for $i=j$. It only remains the last integral with terms $i=j$ and therefore the definition of $\Delta_x\phi_c(t,x)$ gives

$$
-\int_0^t\int_{\Omega\times\mathbb{R}^3}\pi_L(h)v\cdot\nabla_x\psi_c\,dxdvds=C_1\int_0^t\|c\|_{L^2_x}^2\,ds. \tag{3.43}
$$

Again, direct computations show $\alpha_c=5$ and hence $C_1>0$.

Then the term Ψ_2 and Ψ_3 are dealt with as for $a(t,x)$ and $b(t,x)$.

$$
\forall i\in\{2,3\},\quad |\Psi_i(t)|\leqslant\frac{C_1}{4}\int_0^t\|c\|_{L^2_x}^2\,ds+C_2\int_0^t\left\|\pi_L^\perp(h)\right\|_{L^2_{x,v}}^2\,ds, \tag{3.44}
$$

where C_1 is defined in (3.43).

The term Ψ_4 involves integral on the boundary Λ. Again, we divide it into Λ^+ and Λ^-, we use the Maxwell boundary condition (3.2) satisfied by h and we make the change of variable $v\mapsto\mathcal{R}_x(v)$ on Λ^-. As for (3.23) dealing with $a(t,x)$ we obtain

$$
\begin{aligned}
\Psi_4(t)=&-2(1-\alpha)\int_0^t\int_{\Lambda^+}h\left(|v|^2-\alpha_c\right)(v\cdot n(x))^2\,\partial_n\phi_c\sqrt{\mu}\,dS(x)dvds\\
&-\alpha\int_0^t\int_{\Lambda^+}\left(|v|^2-\alpha_c\right)|v\cdot n|\,\nabla_x\phi_c\cdot\left[vP^\perp_{\Lambda_\mu}(h)+2P_{\Lambda_\mu}(h)\,(v\cdot n)n)\right]\sqrt{\mu}.
\end{aligned}
$$

We decompose $h=P_{\Lambda_\mu}(h)+P^\perp_{\Lambda_\mu}(h)$ in the first integral and use (3.3) which says that $P_{\Lambda_\mu}(h)(t,x,v)=z(t,x)\sqrt{\mu(v)}$.

$$
\begin{aligned}
\Psi_4(t)=&-2\int_0^t\int_{\partial\Omega}\partial_n\phi_cz(s,x)\left(\int_{v\cdot n(x)>0}\left(|v|^2-\alpha_c\right)(v\cdot n(x))^2\,\mu(v)\,dv\right)dS(x)ds\\
&-2(1-\alpha)\int_0^t\int_{\Lambda^+}\left[\left(|v|^2-\alpha_c\right)(v\cdot n(x))^2\,\partial_n\phi_c\sqrt{\mu}\right]P^\perp_{\Lambda_\mu}(h)\,dS(x)dvds\\
&-\alpha\int_0^t\int_{\Lambda^+}\left[\left(|v|^2-\alpha_c\right)|v\cdot n|\,\nabla_x\phi_c\cdot v\sqrt{\mu}\right]P^\perp_{\Lambda_\mu}(h)\,dS(x)dvds.
\end{aligned}
$$

Because

$$
(v\cdot n(x))^2=\sum_{1\leqslant i,j\leqslant 3}v_iv_jn_in_j
$$

the first term is null when $i \neq j$ and vanishes for $i = j$ thanks to our choice of α_c. The last two integrals are dealt with by applying Cauchy-Schwarz inequality and the elliptic estimate on ϕ_c in H^2 (3.41). As for the case of $a(t,x)$, we obtain

$$|\Psi_4(t)| \leqslant \frac{C_1}{4}\int_0^t \|c\|^2_{L^2_x}\,ds + C_4\int_0^t \left\|P^\perp_{\Lambda_\mu}(h)\right\|^2_{L^2_{\Lambda^+}}\,ds. \tag{3.45}$$

As for $a(t,x)$ and $b(t,x)$, the estimate on Ψ_5 (3.16) will follow elliptic arguments in negative Sobolev spaces. With exactly the same computations as in (3.25) we have

$$|\Psi_5(t)| \leqslant C\int_0^t \left\|\pi^\perp_L(h)\right\|_{L^2_{x,v}} \|\partial_t\nabla_x\phi_c\|_{L^2_x}\,ds. \tag{3.46}$$

Note that the contribution of π_L vanishes by oddity on the terms involving $a(t,x)$ and $c(t,x)$ and also on the terms involving $b(t,x)$ thanks to our choice of α_c.

To estimate $\|\partial_t\nabla_x\phi_c\|_{L^2_x}$ we use the decomposition of the weak formulation (3.11) between t and $t+\varepsilon$ (instead of between 0 and t) with $\psi(t,x,v) = \sqrt{\mu}\left(|v|^2-3\right)\phi(x)/2$ where ϕ belongs to H^1_x and is null at the boundary. ψ does not depend on t, vanishes at the boundary and $\psi(x)\mu(v)$ is in $\mathrm{Ker}(L)$. Hence,

$$\Psi_3(t) = \Psi_4 = \Psi_5(t) = 0.$$

It remains

$$\begin{aligned} C\int_\Omega \left[c(t+\varepsilon)-c(t)\right]\phi(x)\,dx \;=\; & \int_t^{t+\varepsilon}\int_{\Omega\times\mathbb{R}^3} \pi_L(h)\frac{|v|^2-3}{2}v\cdot\nabla_x\phi(x)\sqrt{\mu}\,dxdvds \\ & + \int_t^{t+\varepsilon}\int_{\Omega\times\mathbb{R}^3} \pi^\perp_L(h)\frac{|v|^2-3}{2}v\cdot\nabla_x\phi(x)\sqrt{\mu}\,dxdvds \\ & \int_t^{t+\varepsilon}\int_{\Omega\times\mathbb{R}^3} g\frac{|v|^2-3}{2}\sqrt{\mu}\phi(x)\,dxdvds. \end{aligned}$$

As for $a(t,x)$ we divide by ε and take the limit as ε goes to 0. By oddity, the first integral on the right-hand side only gives terms with $b(s,x)$. The second and third terms are dealt with by a Cauchy-Schwarz inequality and we apply on ϕ a Poincaré inequality. This yields

$$\left|\int_\Omega \partial_t c(t,x)\phi(x)\,dx\right| \leqslant C\left[\|b\|_{L^2_x} + \left\|\pi^\perp_L(h)\right\|_{L^2_{x,v}} + \|g\|_{L^2_x}\right]\|\nabla_x\phi\|_{L^2_x}.$$

The latter is true for all $\phi(x)$ in H^1_x vanishing on the boundary. We thus fix t and apply the inequality above to

$$-\Delta_x\phi(t,x) = \partial_t c(t,x) \quad \text{and} \quad \phi|_{\partial\Omega} = 0.$$

Exactly the same computation as for b_J we obtain for any $\varepsilon_c > 0$

$$\begin{aligned} |\Psi_5(t)| &\leqslant C\int_0^t \left[\|b\|_{L^2_x} + \left\|\pi^\perp_L(h)\right\|_{L^2_{x,v}} + \|g\|_{L^2_x}\right]\left\|\pi^\perp_L(h)\right\|_{L^2_{x,v}}\,ds \\ &\leqslant \varepsilon_c\int_0^t \|b\|^2_{L^2_x}\,ds + C_5(\varepsilon_c)\int_0^t \left[\left\|\pi^\perp_L(h)\right\|^2_{L^2_{x,v}} + \|g\|^2_{L^2_x}\right]\,ds. \end{aligned} \tag{3.47}$$

We now gather (3.43), (3.18), (3.44), (3.45), (3.47) and (3.42)

$$\begin{aligned}\int_0^t \|c\|_{L^2_x}^2\, ds \leqslant & N_h^{(c)}(t) - N_h^{(c)}(0) + \varepsilon_c \int_0^t \|b\|_{L^2_x}^2\, ds \\ & + C_c(\varepsilon_c) \int_0^t \left[\left\| P^\perp_{\Lambda_\mu}(h) \right\|^2_{L^2_{\Lambda^+}} + \left\| \pi^\perp_L(h) \right\|^2_{L^2_{x,v}} + \|g\|^2_{L^2_x} \right] ds.\end{aligned} \tag{3.48}$$

Conclusion of the proof. We gather the estimates we derived for a, b and c. We compute the linear combination (3.28) $+ \eta \times$ (3.40) $+ \beta \times$ (3.48). For all $\varepsilon_b > 0$ and $\varepsilon_c > 0$ this implies

$$\begin{aligned}\int_0^t \left[\|a\|^2_{L^2_x} + \eta \|b\|^2_{L^2_x} + \beta \|c\|^2_{L^2_x} \right] ds & \\ \leqslant N_h(t) - N_h(0) + C_\perp \int_0^t & \left[\left\| P^\perp_{\Lambda_\mu}(h) \right\|^2_{L^2_{\Lambda^+}} + \left\| \pi^\perp_L(h) \right\|^2_{L^2_{x,v}} + \|g\|^2_{L^2_x} \right] ds \\ + \int_0^t & \left[\eta\varepsilon_b \|a\|^2_{L^2_x} + (C_{a,b} + \beta\varepsilon_c) \|b\|^2_{L^2_x} + \eta C_{b,c}(\varepsilon_b) \|c\|^2_{L^2_x} \right] ds.\end{aligned}$$

We first choose $\eta > C_{a,b}$, then ε_b such that $\eta\varepsilon_b < 1$ and then $\beta > \eta C_{b,c}(\varepsilon_b)$. Finally, we fix ε_c small enough such that $C_{a,b} + \beta\varepsilon_c < \eta$. With such choices we can absorb the last term on the right-hand side by the left-hand side. This concludes the proof of Lemma 3.2. □

3.3. **Exponential decay of the solution.** In this section we show that a solution to (3.1) that preserves mass and has its trace in L^2_Λ decays exponentially fast.

Proof of Theorem 3.1. Let h be a solution described in the statement of the theorem and define for $\lambda > 0$, $\widetilde{h}(t,x,v) = e^{\lambda t} h(t,x,v)$. $\widetilde{h}$ satisfies the conservation of mass and is solution to

$$\partial_t \widetilde{h} + v \cdot \nabla_x \widetilde{h} = L_\mu(\widetilde{h}) + \lambda \widetilde{h}$$

with the Maxwell boundary condition. Moreover, since $h|_\Lambda$ belongs to $L^2_\Lambda\left(\mu^{-1/2}\right)$ so does $\widetilde{h}\Big|_\Lambda$. We can use Green formula and get

$$\frac{1}{2}\frac{d}{dt}\left\|\widetilde{h}\right\|^2_{L^2_{x,v}} = -\frac{1}{2}\int_{\Omega\times\mathbb{R}^3} v\cdot\nabla_x\left(\widetilde{h}^2\right) dxdv + \int_\Omega \langle L_\mu(\widetilde{h})(t,x,\cdot), \widetilde{h}(t,x,\cdot)\rangle_{L^2_v} dx + \lambda \left\|\widetilde{h}\right\|^2_{L^2_{x,v}}$$

Therefore, thnaks to the spectral gap (3.7) of L in L^2_v we get

$$\frac{1}{2}\frac{d}{dt}\left\|\widetilde{h}\right\|^2_{L^2_{x,v}} \leqslant -\frac{1}{2}\int_\Lambda \widetilde{h}^2\, v\cdot n(x)\, dS(x)dv - \lambda_L \left\|\pi^\perp_L(\widetilde{h})\right\|^2_{L^2_{x,v}} + \lambda \left\|\widetilde{h}\right\|^2_{L^2_{x,v}}. \tag{3.49}$$

As we did in previous section, we divide the integral over the boundary and we apply the boundary condition (3.2) followed by the change of variable $v \mapsto \mathcal{R}_x(v)$

that sends Λ^- to Λ^+. At last, we decompose $h|_{\Lambda^+}$ into $P_{\Lambda_\mu}(h) + P^\perp_{\Lambda_\mu}(h)$ and this yields

$$\begin{aligned}
-\int_\Lambda \widetilde{h}^2\, v\cdot n(x)\, dS(x)dv &= -\int_{\Lambda^+}\left[\widetilde{h}^2 - \left((1-\alpha)\widetilde{h} + \alpha P_{\Lambda_\mu}(\widetilde{h})\right)^2\right] v\cdot n(x)\, dS(x)dv \\
&= -(1-(1-\alpha)^2)\left\|P^\perp_{\Lambda_\mu}(\widetilde{h})\right\|^2_{L^2_{\Lambda^+}} \\
&\quad +2\alpha\int_{\Lambda^+} P_{\Lambda_\mu}(\widetilde{h})P^\perp_{\Lambda_\mu}(\widetilde{h}) v\cdot n(x)\, dS(x)dv \\
&= -(1-(1-\alpha)^2)\left\|P^\perp_{\Lambda_\mu}(\widetilde{h})\right\|^2_{L^2_{\Lambda^+}}. \qquad (3.50)
\end{aligned}$$

Combining (3.49) and (3.50) and integrating from 0 to t we get

$$\left\|\widetilde{h}(t)\right\|^2_{L^2_{x,v}} + C\int_0^t\left[\left\|P^\perp_{\Lambda_\mu}(\widetilde{h})\right\|_{L^2_{\Lambda^+}} + \left\|\pi^\perp_L(\widetilde{h})\right\|_{L^2_{x,v}}\right] ds \leqslant \|h_0\|^2_{L^2_{x,v}} + 2\lambda\int_0^t\left\|\widetilde{h}\right\|^2_{L^2_{x,v}} ds. \qquad (3.51)$$

To conclude we use Lemma 3.2 for $\widetilde{h}$ with $g = \lambda\widetilde{h}$:

$$\begin{aligned}
\int_0^t\left\|\pi_L(\widetilde{h})\right\|^2_{L^2_{x,v}} ds \leqslant& N_{\widetilde{h}}(t) - N_{\widetilde{h}}(0) \\
&+ C_\perp\int_0^t\left[\left\|\pi^\perp_L(\widetilde{h})\right\|^2_{L^2_{x,v}} + \left\|P^\perp_{\Lambda_\mu}(\widetilde{h})\right\|^2_{L^2_{\Lambda^+}} + \lambda^2\left\|\widetilde{h}\right\|^2_{L^2_{x,v}}\right] ds \qquad (3.52)
\end{aligned}$$

and we combine $\varepsilon\times(3.52) + (3.51)$ for $\varepsilon > 0$.

$$\begin{aligned}
&\left[\left\|\widetilde{h}\right\|^2_{L^2_{x,v}} - \varepsilon N_{\widetilde{h}}(t)\right] + C_\varepsilon\int_0^t\left(\left\|\pi_L(\widetilde{h})\right\|^2_{L^2_{x,v}} + \left\|\pi^\perp_L(\widetilde{h})\right\|^2_{L^2_{x,v}}\right) ds \\
&\qquad + (C - \varepsilon C_\perp)\int_0^t\left\|P^\perp_{\Lambda_\mu}(\widetilde{h})\right\|^2_{L^2_{\Lambda^+}} ds \\
&\leqslant \|h_0\|^2_{L^2_{x,v}(\mu^{-1/2})} - \varepsilon N_{\widetilde{h}}(0) + \left(\varepsilon C_\perp\lambda^2 + 2\lambda\right)\int_0^t\left\|\widetilde{h}\right\|^2_{L^2_{x,v}} ds
\end{aligned}$$

with $C_\varepsilon = \min\{\varepsilon C_\perp, C - \varepsilon C_\perp\}$. Thanks to the control $\left|N_{\widetilde{h}}(s)\right| \leqslant C\left\|\widetilde{h}(s)\right\|^2_{L^2_{x,v}}$ and the fact that

$$\left\|\pi_L(\widetilde{h})\right\|^2_{L^2_{x,v}} + \left\|\pi^\perp_L(\widetilde{h})\right\|^2_{L^2_{x,v}} = \left\|\widetilde{h}\right\|^2_{L^2_{x,v}}$$

we can choose ε small enough such that $C_\varepsilon > 0$ and then $\lambda > 0$ small enough such that $(\varepsilon C_\perp\lambda^2 + 2\lambda) < C_\varepsilon$. Such choices imply that $\left\|\widetilde{h}\right\|^2_{L^2_{x,v}}$ is uniformly bounded in time by $C\|h_0\|^2_{L^2_{x,v}}$.

By definition of $\widetilde{h}$, this shows an exponential decay for h and concludes the proof of Theorem 3.1. □

4. Semigroup generated by the collision frequency

This section is devoted to proving that the following operator

$$G_\nu = -\nu(v) - v \cdot \nabla_x$$

with the Maxwell boundary condition generates a semigroup $S_{G_\nu}(t)$ with exponential decay in $L^\infty_{x,v}$ endowed with different weights. Such a study has been done for pure specular reflections ($\alpha = 0$) whereas a similar result has been obtained in the purely diffusive case ($\alpha = 1$) (see [21] for maxwellian weights and [3] for more general weights and $L^1_v L^\infty_x$ framework). We adapt the methods of [21][3] in order to fit our boundary condition. They consist in deriving an implicit formulation for the semigroup along the characteristics and then we need to control the characteristic trajectories that do not reach the plane $\{t = 0\}$ in a time t. As we shall see, this number of problematic trajectories is small when the number of rebounds is large and so can be controlled for long times.

Theorem 4.1. *Let $m(v) = m(|v|) \geqslant 0$ be such that*

$$\frac{(1 + |v|)\,\nu(v)}{m(v)} \in L^1_v \quad \text{and} \quad m(v)\mu(v) \in L^\infty_v. \tag{4.1}$$

Then for any f_0 in $L^\infty_{x,v}(m)$ there exists a unique solution $S_{G_\nu}(t)f_0$ in $L^\infty_{x,v}(m)$ to

$$[\partial_t + v \cdot \nabla_x + \nu(v)]\,(S_{G_\nu}(t)f_0) = 0 \tag{4.2}$$

such that $(S_{G_\nu}(t)f_0)|_\Lambda \in L^\infty_\Lambda(m)$ and satisfying the Maxwell boundary condition (1.2) with initial datum f_0. Moreover it satisfies

$$\forall \nu_0' < \nu_0,\ \exists\, C_{m,\nu_0'} > 0,\ \forall t \geqslant 0, \quad \|S_{G_\nu}(t)f_0\|_{L^\infty_{x,v}(m)} \leqslant C_{m,\nu_0'} e^{-\nu_0' t}\, \|f_0\|_{L^\infty_{x,v}(m)},$$

with $\nu_0 = \inf\{\nu(v)\} > 0$.

A corollary of the proof of this theorem is a gain of weight when one integrates in the time variable. This will be of core importance to control the nonlinear operator.

Corollary 4.2. *Let m be such that $m(v)\nu(v)^{-1}$ satisfies the requirements of Theorem 4.1. Then there exists $C_0 > 0$ such that for any $(f_s)_{s\in\mathbb{R}^+}$ in $L^\infty_{x,v}(m)$, any ε in $(0,1)$ and all $t \geqslant 0$,*

$$\left\| \int_0^t S_{G_\nu}(t-s) f_s(x,v)\, ds \right\|_{L^\infty_{x,v}(m)} \leqslant \frac{C_0}{1-\varepsilon} e^{-\varepsilon\nu_0 t} \sup_{s\in[0,t]} \left[e^{\varepsilon\nu_0 s}\, \|f_s\|_{L^\infty_{x,v}(m\nu^{-1})} \right].$$

The rest of this Section is entirely devoted to the proof of these results.

4.1. **Brief description of characteristic trajectories.** The characteristic trajectories of the free transport equation

$$\partial_t f(t,x,v) + v \cdot \nabla_x f(t,x,v) = 0$$

with purely specular reflection boundary condition will play an important role in our proof. Their study has been done in [7, Appendix A] and we describe here the results that we shall use later on.

The description of backward characteristics relies on the time of first rebound against the boundary of Ω. For x in $\overline{\Omega}$ and $v \neq 0$ define

$$t_{min}(x,v) = \max\left\{t \geqslant 0 : \ x - vs \in \overline{\Omega}, \ \forall\, 0 \leqslant s \leqslant t\right\}.$$

Note that for all $(x,v) \notin \Lambda_0 \cup \Lambda^-$, $t_{min}(x,v) > 0$. The characteristic trajectories are straight lines in between two rebounds against the boundary, where the velocity then undergo a specular reflection.

From [7, Appendix A.2], starting from (x,v) in $\overline{\Omega} \times (\mathbb{R}^3 - \{0\})$, one can construct $T_1(x,v) = t_{min}(x,v)$ and the footprint $X_1(x,v)$ on $\partial\Omega$ of the backward trajectory starting from x with velocity v has well as its resulting velocity $V_1(x,v)$:

$$X_1(x,v) = x - T_1(x,v)v \quad \text{and} \quad V_1(x,v) = \mathcal{R}_{X_1(x,v)}(v),$$

where we recall that $\mathcal{R}_y(v)$ is the specular reflection of v at a point $y \in \partial\Omega$. One can iterate the process and construct the second collision with the boundary at time $T_2(x,v) = T_1(x,v) + t_{min}(X_1(x,v), V_1(x,v))$, at the footprint $X_2(x,v) = X_1(X_1(x,v), V_1(x,v))$ and the second reflected velocity $V_2(x,v) = V_1(X_1, V_1)$ and so on so forth to construct a sequence $(T_k(x,v), X_k(x,v), V_k(x,v))$ in $\partial\Omega \times \mathbb{R}^3$. More precisely we have, for almost every (x,v),

$$T_{k+1}(x,v) = T_k + t_{min}(X_k, V_k),\ X_{k+1}(x,v) = X_k - t_{min}(X_k,V_k)V_k,\ V_{k+1} = \mathcal{R}_{X_{k+1}}(V_k).$$

Thanks to [7, Proposition A.4], for a fixed time t and for almost every (x,v) there are a finite number of rebounds. In other terms, there exists $N(t,x,v)$ such that the backward trajectories starting from (x,v) and running for a time t is such that

$$T_{N(t,x,v)}(x,v) \leqslant t < T_{N(t,x,v)+1}(x,v).$$

We conclude this subsection by stating a continuity result about the footprints of characteristics. This is a rewriting of [24, Lemmas 1 and 2].

Lemma 4.3. *Let Ω be a C^1 bounded domain.*

1. *the backward exit time $t_{min}(x,v)$ is lower semi-continuous;*
2. *if $v \cdot n(X_1(x,v)) < 0$ then $t_{min}(x,v)$ and $X_1(x,v)$ are continuous functions of (x,v);*
3. *let (x_0,v_0) be in $\Omega \times \mathbb{R}^3$ with $v_0 \neq 0$ and $t_{min}(x_0,v_0) < \infty$, if $(X_1(x_0,v_0),v_0)$ belongs to Λ_0^{I-} then $t_{min}(x,v)$ is continuous around (x_0,v_0).*

Note that [24, Lemma 2] also gives that if $(X_1(x_0,v_0),v_0)$ belongs to Λ_0^{I+} then $t_{min}(x,v)$ is not continuous around (x_0,v_0). Therefore, points (2) and (3) in Lemma 4.3 imply that $\mathfrak{C}_\Lambda^- = \Lambda^- \cup \Lambda_0^{I-}$ is indeed the boundary continuity set.

4.2. **Proof of Theorem 4.1: uniqueness.** Assume that there exists a solution f of (4.2) in $L^\infty_{x,v}(m)$ satisfying the Maxwell boundary condition and such that $f|_\Lambda$ belongs to $L^\infty_\Lambda(m)$. With the assumptions on the weight $m(v)$ and the following inequalities

$$\|f\|_{L^1_{x,v}} \leqslant \left(\int_{\mathbb{R}^3} \frac{dv}{m(v)}\right) \|f\|_{L^\infty_{x,v}(m)} \quad \text{and} \quad \|f\|_{L^1_\Lambda} \leqslant \left(\int_{\mathbb{R}^3} \frac{|v|}{m(v)}\, dv\right) \|f\|_{L^\infty_\Lambda(m)}$$

we see that f belongs to $L^1_{x,v}$ and its restriction $f|_\Lambda$ belongs to L^1_Λ.

We can therefore use the divergence theorem and the fact that $\nu(v) \geqslant \nu_0 > 0$:

$$\begin{aligned}
\frac{d}{dt}\|f\|_{L^1_{x,v}} &= \int_{\Omega\times\mathbb{R}^3} \operatorname{sgn}(f(t,x,v))\left[-v\cdot\nabla_x - \nu(v)\right] f(t,x,v)\, dxdv \\
&= -\int_{\Omega\times\mathbb{R}^3} v\cdot\nabla_x\left(|f|\right)\, dxdv - \|\nu(v)f\|_{L^1_{x,v}} \\
&\leqslant -\int_\Lambda |f(t,x,v)|\,(v\cdot n(x))\, dS(x)dv - \nu_0\|f\|_{L^1_{x,v}}.
\end{aligned}$$

Using the Maxwell boundary condition (1.2) and then applying the change of variable $v \to \mathcal{R}_x(v)$, which has a unit jacobian since it is an isometry, gives

$$\begin{aligned}
&\int_{\Lambda^-} |f(t,x,v)|\,(v\cdot n(x))\, dS(x)dv \\
&= -\int_{\Lambda^-} |(1-\alpha)f(t,x,\mathcal{R}_x(v)) + \alpha P_\Lambda\left(f(t,x,\cdot)\right)(v)|\,(v\cdot n(x))\, dS(x)dv \\
&= -\int_{\Lambda^+} |(1-\alpha)f(t,x,v) + \alpha P_\Lambda\left(f(t,x,\cdot)\right)(v)|\,(v\cdot n(x))\, dS(x)dv \\
&\leqslant \int_{\Lambda^+} |f(t,x,v)|\,(v\cdot n(x))\, dS(x)dv.
\end{aligned}$$

We used the fact that $P_\Lambda(f)(\mathcal{R}_x(v)) = P_\Lambda(f)(v)$.

The integral over the boundary Λ is therefore positive and so uniqueness follows from a Grönwall lemma.

4.3. **Proof of Theorem 4.1: existence and exponential decay.** Let f_0 be in $L^\infty_{x,v}(m)$. Define the following iterative scheme:

$$\left[\partial_t + v\cdot\nabla_x + \nu\right] f^{(n)} = 0 \quad \text{and} \quad f^{(n)}(0,x,v) = f_0(x,v)\mathbf{1}_{\{|v|\leqslant n\}}$$

with a damped version of the Maxwell boundary condition for $t > 0$ and (x,v) in Λ^-

$$f^{(n)}(t,x,v) = (1-\alpha)f^{(n)}(t,x,\mathcal{R}_x(v)) + \alpha\left(1-\frac{1}{n}\right) P_\Lambda(f^{(n)}|_{\Lambda^+})(t,x,v). \tag{4.3}$$

Denote by $P^{(n)} : \Lambda^+ \longrightarrow \Lambda^-$ the boundary operator associated to (4.3).

Note that $\mu(v)^{-1/2} f_0(x,v)\mathbf{1}_{\{|v|\leqslant n\}}$ is in $L^\infty_{x,v}$ and

$$\forall\, \mu^{-1/2} f \in L^\infty_{\Lambda^+}, \quad \left\|P^{(n)}(\mu^{-1/2}f)\right\|_{L^\infty_{\Lambda^-}} \leqslant \left(1-\frac{\alpha}{n}\right)\left\|\mu^{-1/2}f\right\|_{L^\infty_{\Lambda^+}}.$$

The norm of the operator $P^{(n)}$ is thus strictly smaller than one. We can apply [21, Lemma 14] which implies that $f^{(n)}$ is well-defined in $L^\infty_{x,v}\left(\mu^{-1/2}\right)$ with $f^{(n)}\big|_\Lambda$ in $L^\infty_\Lambda\left(\mu^{-1/2}\right)$.

We shall prove that in fact $f^{(n)}$ decays exponentially fast in $L^\infty_{x,v}(m)$ and that its restriction to the boundary is in $L^\infty_\Lambda(m)$. Finally, we will prove that $f^{(n)}$ converges, up to a subsequence, towards f the desired solution of Theorem 4.1. The proof of Theorem 4.1 consists in the following three steps developed in Subsections 4.3.1, 4.3.2 and 4.3.3.

4.3.1. *Step 1: Implicit formula for $f^{(n)}$.* We use the conservation property that $e^{\nu(v)t}f^{(n)}(t,x,v)$ is constant along the characteristic trajectories. We apply it to the first collision with the boundary (recall Subsection 4.1 for notations) and obtain that for all $(x,v)\notin\Lambda_0\cup\Lambda^-$

$$\begin{aligned}f^{(n)}(t,x,v)=&\mathbf{1}_{\{t-t_{min}(x,v)\leqslant 0\}}e^{-\nu(v)t}f_0(x-tv,v)\mathbf{1}_{\{|v|\leqslant n\}}\\&+\mathbf{1}_{\{t-t_{min}(x,v)>0\}}e^{-\nu(v)t_{min}(x,v)}f^{(n)}\big|_{\Lambda^-}(t-t_{min}(x,v),X_1(x,v),v).\end{aligned}$$

Indeed, either the backward trajectory hits the boundary at $X_1(x,v)$ before time t ($t_{min}<t$) or it reaches the origin plane $\{t=0\}$ before it hits the boundary ($t_{min}\geqslant 0$). Defining $t_1=t_1(t,x,v)=t-t_{min}(x,v)$, and recalling the first footprint $X_1=X_1(x,v)$ and the first change of velocity $V_1(x,v)$), we apply the boundary condition (4.3) and obtain the following implicit formula.

(4.4)

$$\begin{aligned}f^{(n)}(t,x,v)=&\,\mathbf{1}_{\{t_1(x,v)\leqslant 0\}}e^{-\nu(v)t}f_0(x-tv,v)\mathbf{1}_{\{|v|\leqslant n\}}\\&+(1-\alpha)\,\mathbf{1}_{\{t_1(x,v)>0\}}e^{-\nu(v)(t-t_1)}f^{(n)}(t_1,X_1,V_1)\\&+\mathbf{1}_{\{t_1(x,v)>0\}}\left[\Delta_n\mu(v)e^{-\nu(v)(t-t_1)}\int_{v_{1*}\cdot n(x_1)>0}\frac{1}{\mu(v_{1*})}f^{(n)}(t_1,X_1,v_{1*})\,d\sigma_{x_1}(v_{1*})\right],\end{aligned}$$

where we denoted $\Delta_n=\alpha(1-1/n)$ and we defined the probability measure on Λ^+

$$d\sigma_x(v)=c_\mu\mu(v)\,|v\cdot n(x)|\;dv. \tag{4.5}$$

Moreover, once at (t_1,X_1,v_1) with v_1 being either $V_1(x,v)$ or v_{1*} either $t_2\leqslant 0$ (where $t_2=t_1(t_1,X_1,v_1)\leqslant 0$) and the trajectory reaches the initial plane after the first rebound or $t_2>0$ and it can still overcome a collision against the boundary in the time t. Again, the fact that $e^{\nu(v)t}f^{(n)}(t,x,v)$ is constant along the characteristics implies

$$f^{(n)}(t,x,v)=I_1\left(f_0^{(n)}\right)(t,x,v)+R_1\left(f^{(n)}\right)(t,x,v) \tag{4.6}$$

with I_1 accounting for all the trajectories reaching the initial plane in at most 1 rebound in time t

(4.7)

$$\begin{aligned}I_1\left(f_0^{(n)}\right)=&\,\mathbf{1}_{\{t_1\leqslant 0\}}e^{-\nu(v)t}f_0^{(n)}(x-tv,v)\\&+\mathbf{1}_{\{t_1>0\}}\mathbf{1}_{\{t_2\leqslant 0\}}(1-\alpha)e^{-\nu(v)(t-t_1)}e^{-\nu(V_1)t_1}f_0(X_1-t_1V_1,V_1)\\&+\mathbf{1}_{\{t_1>0\}}\left[\Delta_n\mu(v)e^{-\nu(v)(t-t_1)}\int_{v_{1*}\cdot n(x_1)>0}\mathbf{1}_{\{t_2\leqslant 0\}}\frac{e^{-\nu(v_{1*})t_1}}{\mu(v_{1*})}f_0^{(n)}(x_1-t_1v_{1*},v_{1*})\,d\sigma_{x_1}\right],\end{aligned}$$

and $R_1\left(f^{(n)}\right)$ encodes the contribution of all the characteristics that after one rebound are still able to generate new collisions against $\partial\Omega$

$$\begin{aligned} R_1\left(f^{(n)}\right)(t,x,v) &= (1-\alpha)\,\mathbf{1}_{\{t_2>0\}} e^{-\nu(v)(t-t_1)} f^{(n)}(t_1,X_1,V_1) \\ &\quad + \Delta_n \mu(v) e^{-\nu(v)(t-t_1)} \int_{v_{1*}\cdot n(x_1)>0} \mathbf{1}_{\{t_2>0\}} \frac{1}{\mu(v_{1*})} f(t_1,X_1,v_{1*})\, d\sigma_{x_1}(v_{1*}). \end{aligned} \tag{4.8}$$

Of important note, to lighten computations, in each term the value t_2 refers to t_1 of the preceding triple (t_1,x_1,v_1) where v_1 is $V_1(x,v)$ in the first term and v_{1*} in the second. As we are about to iterate (4.6), we shall generate sequences $(t_{k+1},x_{k+1},v_{k+1})$ which have to be understood as $(t_1(t_k^{(l)},x_k^{(l)},v_k^{(l)}),x_1(x_k^{(l)},v_k^{(l)}),v_{k+1})$ and v_{k+1} being either $V_1(x_k^{(l)},v_k^{(l)})$ or an integration variable $v_{(k+1)*}$.

By a straightforward induction we obtain an implicit form for $f^{(n)}$ when one takes into account the contribution of the characteristics reaching $\{t=0\}$ in at most $p\geqslant 1$ rebounds

$$f^{(n)}(t,x,v) = I_p\left(f_0^{(n)}\right)(t,x,v) + R_p\left(f^{(n)}\right)(t,x,v). \tag{4.9}$$

$I_p\left(f_0^{(n)}\right)$ contains all the trajectories reaching the initial plane in at most p rebounds whereas $R_p\left(f^{(n)}\right)$ gathers the contributions of all the trajectories still coming from a collision against the boundary. A more careful induction gives an explicit formula for R_p and this is the purpose of the next Lemma. The main idea is to look at every possible combination of the specular reflections among all the collisions against the boundary, represented by the set ϑ defined below.

Lemma 4.4. *For $p\geqslant 1$ and i in $\{1,\dots,p\}$ define $\vartheta_p(i)$ the set of strictly increasing functions from $\{1,\dots,i\}$ into $\{1,\dots,p\}$. Let $(t_0,x_0,v_0)=(t,x,v)$ in $\mathbb{R}^+\times\Omega\times\mathbb{R}^3$ and $(v_{1*},\dots,v_{p*})$ in R^{3p}. For l in $\vartheta_p(i)$ we define the sequence $(t_k^{(l)},x_k^{(l)},v_k^{(l)})_{1\leqslant k\leqslant p}$ by induction*

$$\begin{aligned} t_k &= t_{k-1}^{(l)} - t_{min}(x_{k-1}^{(l)},v_{k-1}^{(l)}) \quad , \; x_k^{(l)} = X_1(x_{k-1}^{(l)},v_{k-1}^{(l)}) \\ v_k^{(l)} &= \begin{cases} V_1(x_{k-1}^{(l)},v_{k-1}^{(l)}) & \text{if} \quad k\in l\left[\{1,\dots,i\}\right], \\ v_{k*} & \text{otherwise.} \end{cases} \end{aligned}$$

At last, for $1\leqslant k\leqslant p$ define the following measure on $\mathbb{R}^{3k}$

$$d\Sigma_l^k(v_{1*},\dots,v_{p*}) = \frac{\mu(v)}{\mu(v_k^{(l)})}\left[\prod_{j=0}^{k-1} e^{-\nu(v_j^{(l)})(t_j^{(l)}-t_{j+1}^{(l)})}\right] d\sigma_{x_1}(v_{1*})\dots d\sigma_{x_p}(v_{p*})$$

and the following sets

$$\mathcal{V}_j^{(l)} = \left\{v_{j*}\in\mathbb{R}^3, \quad v_{j*}\cdot n(x_j^{(l)})>0\right\}. \tag{4.10}$$

Then we have the following identities

$$\begin{aligned} I_p\left(f_0^{(n)}\right)(t,x,v) &= \sum_{k=0}^{p}\sum_{i=0}^{k}(1-\alpha)^i\Delta_n^{k-i} \\ &\quad\times \sum_{l\in\vartheta_k(i)} \int_{\prod_{1\leqslant j\leqslant p}\mathcal{V}_j^{(l)}} \mathbf{1}_{\left\{t_k^{(l)}>0,\, t_{k+1}^{(l)}\leqslant 0\right\}} e^{-\nu(v_k^{(l)})t_k^{(l)}} f_0^{(n)}(x_k^{(l)}-t_k^{(l)}v_k^{(l)},v_k^{(l)})\, d\Sigma_l^k \end{aligned} \tag{4.11}$$

and

(4.12)

$$R_p\left(f^{(n)}\right)(t,x,v)=\sum_{i=0}^{p}\sum_{l\in\vartheta_p(i)}(1-\alpha)^i\Delta_n^{p-i}\int_{\prod_{1\leqslant j\leqslant p}\mathcal{V}_j^{(l)}}\mathbf{1}_{\left\{t_{p+1}^{(l)}>0\right\}}f^{(n)}(t_p^{(l)},x_p^{(l)},v_p^{(l)})\,d\Sigma_l^p,$$

where we defined $t_{p+1}^{(l)}=t_p^{(l)}-t_{min}(x_p^{(l)},v_p^{(l)})$ *and also by convention* $l\in\vartheta_p(0)$ *means that* $l=0$.

Proof of Lemma 4.4. The proof is done by induction on p and we start with the formula for R_p.

By definition of $R_1\left(f^{(n)}\right)(t,x,v)$ (4.8), the property holds for $p=1$ since on the pure reflection part $\mu(v)=\mu(v_1)$ and $d\sigma_{x_1}$ is a probability measure.

Suppose that the property holds at $p\geqslant 1$. Then we can apply the property (4.6) at rank one to $f^{(n)}(t_p^{(l)},x_p^{(l)},v_p^{(l)})$. In other terms this amounts to applying the preservation of $e^{\nu(v)t}f^{(n)}(t,x,v)$ along characteristics and to keep only the contribution of trajectories still able to generate rebounds. Using the notations $t_{p+1}^{(l)}=t_p^{(l)}-t_{min}(x_p^{(l)},v_p^{(l)})$, $x_{p+1}^{(l)}=X_1(x_p^{(l)},v_p^{(l)})$, and the definition (4.8), it reads

$$\begin{aligned}
&R_1\left(f^{(n)}\right)(t_p^{(l)},x_p^{(l)},v_p^{(l)})\\
&=(1-\alpha)\,\mathbf{1}_{\left\{t_2(t_p^{(l)},x_p^{(l)},v_p^{(l)})>0\right\}}e^{-\nu(v_p^{(l)})(t_p^{(l)}-t_{p+1}^{(l)})}f^{(n)}(t_{p+1}^{(l)},x_{p+1}^{(l)},V_1(x_p^{(l)},v_p^{(l)}))\\
&\quad+\Delta_n\mu(v_p^{(l)})e^{-\nu(v_p^{(l)})(t_p^{(l)}-t_{p+1}^{(l)})}\int_{\mathcal{V}_{p+1}^{(l)}}\mathbf{1}_{\left\{t_2(t_p^{(l)},x_p^{(l)},v_{(p+1)*})>0\right\}}\frac{d\sigma_{x_{p+1}^{(l)}}}{\mu(v_{(p+1)*})}f(t_{p+1}^{(l)},x_{p+1}^{(l)},v_{(p+1)*}).
\end{aligned}$$

Since μ only depends on the norm and since $d\sigma_{x_{p+1}^{(l)}}$ is a probability measure, the specular part above can be rewritten as

$$(1-\alpha)\mu(v_p^{(l)})\int_{\mathcal{V}_{p+1}^{(l)}}\mathbf{1}_{\{t_2>0\}}\frac{e^{-\nu(v_p^{(l)})(t_p^{(l)}-t_{p+1}^{(l)})}}{\mu(V_1(x_p^{(l)},v_p^{(l)}))}f^{(n)}(t_{p+1}^{(l)},x_{p+1}^{(l)},V_1(x_p^{(l)},v_p^{(l)}))\,d\sigma_{x_{p+1}^{(l)}}(v_{(p+1)*}).$$

For each l in $\vartheta_p(i)$ we can generate l_1 in $\vartheta_{p+1}(i+1)$ with $l_1(p+1)=p+1$ (representing the specular reflection case) and $l_2=l$ in $\vartheta_{p+1}(i)$. Plugging $R_1\left(f^{(n)}\right)(t_p^{(l)},x_p^{(l)},v_p^{(l)})$ into $R_p\left(f^{(n)}\right)(t,x,v)$ we obtain for each l the desired integral for l_1 and l_2

$$\mathcal{I}_{l_{1,2}}=\int_{\prod_{1\leqslant j\leqslant p+1}\mathcal{V}_j^{(l_{1,2})}}\mathbf{1}_{\left\{t_{p+1}^{(l_{1,2})}>0\right\}}f^{(n)}(t_{p+1}^{(l_{1,2})},x_{p+1}^{(l_{1,2})},v_{p+1})\,d\Sigma_{l_{1,2}}^p\left(v_{1*},\dots,v_{(p+1)*}\right).$$

Our computations thus lead to

$$R_p\left(f^{(n)}\right)(t,x,v)=\sum_{i=0}^{p}\sum_{\substack{l\in\vartheta_{p+1}(i+1)\\ l(i+1)=p+1}}(1-\alpha)^{i+1}\Delta_n^{p-i}\mathcal{I}_l+\sum_{i=0}^{p}\sum_{\substack{l\in\vartheta_{p+1}(i)\\ l(i)\neq p+1}}(1-\alpha)^i\Delta_n^{p+1-i}\mathcal{I}_l$$

which can be rewritten as

$$R_p\left(f^{(n)}\right)(t,x,v)=\sum_{i=1}^{p+1}\sum_{\substack{l\in\vartheta_{p+1}(i)\\ l(i)=p+1}}(1-\alpha)^i\Delta_n^{p+1-i}\mathcal{I}_l+\sum_{i=0}^{p}\sum_{\substack{l\in\vartheta_{p+1}(i)\\ l(i)\neq p+1}}(1-\alpha)^i\Delta_n^{p+1-i}\mathcal{I}_l$$

For $i = 0$ there can be no l such that $l(i) = p+1$ and for $i = p+1$ the only l in $\vartheta_{p+1}(i)$ is the identity and so there is no l such that $l(i) \neq p+1$. In the end, for $0 \leqslant i \leqslant p+1$, we are summing exactly once every function l in $\vartheta_{p+1}(i)$. This concludes the proof of the lemma for R_p.

At last, I_p could be derived explicitely by the same kind of induction. However, I_p contains all the contributions from characteristics reaching $\{t = 0\}$ in at most p collisions against the boundary. It follows that I_p is the sum of all the possible R_k with k from 0 to p such that $\mathbf{1}_{\{t_{k+1} \leqslant 0\}}$ to which we apply the preservation of $e^{\nu(v)t} f^{(n)}(t,x,v)$ along the backward characteristics starting at $(t_k^{(l)}, x_k^{(l)}, v_k^{(l)})$ up to t. And since $d\sigma_{x_{k+1}}(v_{(k+1)*}) \dots d\sigma_{x_p}(v_{p*})$ is a probability measure on $\mathbb{R}^{3(p-k)}$ we can always have an integral against

$$d\sigma_{x_1}(v_{1*}) \dots d\sigma_{x_p}(v_{p*}).$$

This concludes the proof for I_p. □

4.3.2. *Step 2: Estimates on the operators I_p and R_p.* The next two lemmas give estimates on the operator I_p and R_p. Note that we gain a weight of $\nu(v)$ which will be of great importance when dealing with the bilinear operator.

Lemma 4.5. *There exists $C_m > 0$ only depending on m such that for all $p \geqslant 1$ and all h in $L^\infty_{x,v}(m)$,*

$$\|I_p(h)(t)\|_{L^\infty_{x,v}(m)} \leqslant pC_m e^{-\nu_0 t} \|h\|_{L^\infty_{x,v}(m)}.$$

Moreover we also have the following inequality for all (t,x,v) in $\mathbb{R}^+ \times \overline{\Omega} \times \mathbb{R}^3$

$$m(v)\,|I_p(h)|\,(t,x,v) \leqslant pC_m \left(\nu(v)e^{-\nu(v)t} + e^{-\nu_0 t}\right) \|h\|_{L^\infty_{x,v}(m\nu^{-1})}.$$

Proof of Lemma 4.5. We only prove the second inequality as the first one follows exactly the same computations without multiplying and dividing by $\nu(v_k^{(l)})$.

Bounding by the $L^\infty_{x,v}(m\nu^{-1})$-norm out of the definition (4.11) gives

(4.13)

$$|I_p(h)(t,x,v)| \leqslant \|h\|_{L^\infty_{x,v}(m\nu^{-1})} \sum_{k=0}^{p} \sum_{i=0}^{k} (1-\alpha)^i \Delta_n^{k-i} \sum_{l \in \vartheta_k(i)} \int_{\prod_{1 \leqslant j \leqslant p} \mathcal{V}_j^{(l)}} \mathbf{1}_{\left\{t_k^{(l)} > 0,\, t_{k+1}^{(l)} \leqslant 0\right\}} \frac{\nu(v_k^{(l)})}{m(v_k^{(l)})} e^{-\nu(v_k^{(l)}) t_k^{(l)}}\, d\Sigma_l^k.$$

Fix k, i and l. Then by definition of $(v_k^{(l)})$: either $v_k^{(l)} = V_1(\dots(V_1(x,v)))$ k iterations (case of k specular reflections which means that l is the identity) or there exists J

in $\{1,\dots,p\}$ such that $v_k^{(l)} = V_1(\dots(V_1(x_j,v_{J*})))$ $k-J$ iterations. Since m, ν and μ are radially symmetric this yields

$$\int_{\prod_{1\leqslant j\leqslant p}\mathcal{V}_j^{(l)}} \frac{1}{m(v_k^{(l)})} e^{-\nu(v_k^{(l)})t_k^{(l)}}\, d\Sigma_l^k \tag{4.14}$$
$$= \int_{\prod_{1\leqslant j\leqslant J}\mathcal{V}_j^{(l)}} \frac{\mu(v)\nu(v_k^{(l)})}{m(v_{J*})\mu(v_{J*})} \left[\prod_{j=0}^{k-1} e^{-\nu(v_j^{(l)})(t_j^{(l)}-t_{j+1}^{(l)})}\right] e^{-\nu(v_J^{(l)})t_k^{(l)}}\, d\sigma_{x_1}\dots d\sigma_{x_J}.$$

We use the convention that $v_{0*}=v$ so that this formula holds in both cases.

In the case $J=0$, all the collisions against the boundary were specular reflections and so for any j, $v_j^{(l)}$ is a rotation of v and $t_k^{(l)}$ does not depend on any v_{j*}. As ν is rotation invariant the exponential decay inside the integral is exactly $e^{-\nu(v)(t-t_k^{(l)})}e^{-\nu(v)t_k^{(l)}}$. The $d\sigma_{x_j}$ are probability measures and therefore in the case when J is zero

$$(4.14) = \frac{\nu(v)}{m(v)} e^{-\nu(v)t}.$$

In the case $J\neq 0$ we directly bound the exponential decay by $e^{-\nu_0 t}$ and integrate all the variable but v_{J*}. Therefore, by definition (4.5) of $d\sigma_x$

$$(4.14) \leqslant c_\mu e^{-\nu_0 t}\mu(v) \int_{v_{J*}\cdot n(x_J^{(l)})} \frac{\nu(v_{J*})}{m(v_{J*})} \left|v_{J*}\cdot n(x_J^{(l)})\right| dv_{J*} \leqslant \frac{C_m}{m(v)} e^{-\nu_0 t},$$

where we used the boundedness and integrability assumptions on m (4.1).

To conclude we plug our upper bounds on (4.14) inside (4.13) and use

$$\sum_{k=0}^{p}\sum_{i=0}^{k}\sum_{l\in\vartheta_k(i)} (1-\alpha)^i \Delta_n^{k-i} = \sum_{k=0}^{p}\left(1-\frac{\alpha}{n}\right)^k$$

to finally get

$$m(v)\,|I_p(h)(t,x,v)| \leqslant p\left[\nu(v)e^{-\nu(v)t} + C_m e^{-\nu_0 t}\right]\|h\|_{L^\infty_{x,v}(m\nu^{-1})}$$

which concludes the proof. □

The estimate we derive on R_p needs to be more subtle. The main idea behind it is to differentiate the case when the characteristics come from a majority of pure specular reflections, and therefore has a small contribution because of the multiplicative factor $(1-\alpha)^k$, from the case when they come from a majority of diffusions, and therefore has a small contribution because of the small number of such possible composition of diffusive boundary condition.

Lemma 4.6. *There exists $C_m>0$ only depending on m and $\overline{N}$, $\overline{C}>0$ only depending on α and the domain Ω such that for all $T_0>0$, if*

$$p = \overline{N}\left(\left[\overline{C}T_0\right]+1\right)$$

, where $[\cdot]$ *stands for the floor function; then for all* $h = h(t,x,v)$ *and for all* t *in* $[0,T_0]$

$$\sup_{s\in[0,t]}\left[e^{\nu_0 s}\left\|R_p(h)(s)\right\|_{L^\infty_{x,v}(m)}\right] \leqslant C_m\left(\frac{1}{2}\right)^{\left[\overline{C}T_0\right]}\sup_{s\in[0,t]}\left[e^{\nu_0 s}\left\|\mathbf{1}_{\{t_1>0\}}h(s)\right\|_{L^\infty_{\Lambda^+}(m)}\right].$$

Moreover, the following inequality holds for all (t,x,v) *in* $\mathbb{R}^+\times\overline{\Omega}\times\mathbb{R}^3$ *and all* ε *in* $[0,1]$,

$$\begin{aligned} m(v)\left|R_p(h)\right|(t,x,v) \leqslant & C_m e^{-\varepsilon\nu_0 t}\left(\frac{1}{2}\right)^{\left[\overline{C}T_0\right]}\left(\nu(v)e^{-\nu(v)(1-\varepsilon)t}+e^{-\nu_0(1-\varepsilon)t}\right)\\ & \times\sup_{s\in[0,t]}\left[e^{\varepsilon\nu_0 s}\left\|h(s)\right\|_{L^\infty_{\Lambda^+}(m\nu^{-1})}\right]. \end{aligned}$$

Proof of Lemma 4.6. Let (t,x,v) in $\mathbb{R}^\times\Omega\times\mathbb{R}^3$. Again, we shall only prove the second inequality, the first one being dealt with exactly the same way.

First, the exponential decay inside $d\Sigma_l^p$ (see Lemma 4.4) is bounded by $e^{-\nu_0(t-t_p^{(l)})}$ if there is at least one diffusion or by $e^{-\nu(v)(t-t_p^{(l)})}$ if only specular reflections occur in the p rebounds (because then the reflection preserves $|v|$ and ν only depends on the norm), that is $i=p$ and $l=\mathrm{Id}$. Second, by definition of $(t_k^{(l)},x_k^{(l)},v_k^{(l)})$ (see Lemma 4.4) we can bound

$$\begin{aligned} \mathbf{1}_{\left\{t_{p+1}^{(l)}>0\right\}}\left|h(t_p^{(l)},x_p^{(l)},v_p^{(l)})\right|m(v_p^{(l)}) &= \mathbf{1}_{\left\{t_1(t_p^{(l)},x_p^{(l)},v_p^{(l)})>0\right\}}\left|h(t_p^{(l)},x_p^{(l)},v_p^{(l)})\right|m(v_p^{(l)})\\ &\leqslant \mathbf{1}_{\left\{t_p^{(l)}>0\right\}}\nu(v_p^{(l)})\left\|\mathbf{1}_{\{t_1>0\}}h(t_p^{(l)})\right\|_{L^\infty_{\Lambda^+}(m\nu^{-1})}. \end{aligned}$$

We thus obtain the following bound

$$\begin{aligned} \left|R_p(h)(t,x,v)\right| \leqslant & \sum_{i=0}^{p_1}\sum_{l\in\vartheta_p(i)}(1-\alpha)^i\Delta_n^{p-i}\int_{\prod_{1\leqslant j\leqslant p}\mathcal{V}_j^{(l)}}\mathbf{1}_{\left\{t_p^{(l)}>0\right\}}\frac{\mu(v)\nu(v_p^{(l)})}{m(v_p^{(l)})\mu(v_p^{(l)})}\\ &\times e^{-\nu_0(t-t_p^{(l)})}\left\|\mathbf{1}_{\{t_1>0\}}h(t_p^{(l)})\right\|_{L^\infty_{\Lambda^+}(m\nu^{-1})}\,d\sigma_{x_1^{(l)}}\ldots d\sigma_{x_p^{(l)}}\\ &+(1-\alpha)^p\mathbf{1}_{\left\{t_p^{(\mathrm{Id})}>0\right\}}\nu(v)e^{-\nu(v)(t-t_p^{(\mathrm{Id})})}\left\|\mathbf{1}_{\{t_1>0\}}h(t_p^{(\mathrm{Id})})\right\|_{L^\infty_{\Lambda^+}(m\nu^{-1})}. \end{aligned}$$

Which implies for $0\leqslant\varepsilon\leqslant 1$

(4.15)

$$\begin{aligned} & e^{\varepsilon\nu_0 t}m(v)\left|R_p(h)(t,x,v)\right|\\ & \leqslant\left(\sum_{i=0}^{p}\sum_{l\in\vartheta_p(i)}(1-\alpha)^i\Delta_n^{p-i}\int_{\prod_{1\leqslant j\leqslant p}\mathcal{V}_j^{(l)}}\mathbf{1}_{\left\{t_p^{(l)}>0\right\}}\frac{\mu(v)m(v)}{m(v_p^{(l)})\mu(v_p^{(l)})}d\sigma_{x_1^{(l)}}..d\sigma_{x_p^{(l)}}\right)\\ &\quad\times\left(\nu(v)e^{-\nu(v)(1-\varepsilon)t}+e^{-\nu_0(1-\varepsilon)t}\right)\sup_{s\in[0,t]}\left[e^{\varepsilon\nu_0 s}\left\|\mathbf{1}_{\{t_1>0\}}h(s)\right\|_{L^\infty_{\Lambda^+}(m\nu^{-1})}\right]. \end{aligned}$$

By definition, Lemma 4.4, $t_p^{(l)} = t_p^{(l)}(t,x,v,v_1^{(l)},v_2^{(l)},\dots,v_p^{(l)})$ and thus for all j in $\{1,\dots,p\}$,

$$\mathbf{1}_{\left\{t_p^{(l)}>0\right\}} \leqslant \mathbf{1}_{\left\{t_{p-j}^{(l)}>0\right\}}.$$

Following the reasoning of the proof of Lemma 4.5, for fixed i and l, there exists J in $\{0,\dots,p\}$ such that $v_p^{(l)} = V_1(\dots(V_1(x_J,v_{J*}))))$ $p-J$ iterations, with the convention that $v_{0*}=v$. The measures $d\sigma_x$ are probability measures and the functions m, ν and μ are rotation invariant. Therefore

$$\begin{aligned}
&\int_{\prod_{1\leqslant j\leqslant p}\mathcal{V}_j^{(l)}} \mathbf{1}_{\left\{t_p^{(l)}>0\right\}} \frac{\mu(v)m(v)}{m(v_p^{(l)})\mu(v_p^{(l)})} d\sigma_{x_1^{(l)}}\dots d\sigma_{x_p^{(l)}} \\
&\leqslant \int_{\mathcal{V}_j^{(l)}} \frac{\mu(v)m(v)}{m(v_{J*}^{(l)})\mu(v_{J*}^{(l)})}\, d\sigma_{x_J^{(l)}}(v_{j*}) \left(\int_{\prod_{1\leqslant j\leqslant J-1}\left\{v_{j*}\cdot n(x_j^{(l)})>0\right\}} \mathbf{1}_{\left\{t_{(J-1)*}^{(l)}>0\right\}} d\sigma_{x_1^{(l)}}..d\sigma_{x_{J-1}^{(l)}} \right).
\end{aligned}$$

In the case $J=0$ we have $v_{J*}^{(l)} = v$ and therefore the above is exactly one. In the case $J\geqslant 1$, assumption (4.1) on m implies that the integral over $v_{J*}^{(l)}$ is bounded uniformly by C_m. So we have

(4.16)
$$\int_{\prod_{1\leqslant j\leqslant p}\mathcal{V}_j^{(l)}} \mathbf{1}_{\left\{t_p^{(l)}>0\right\}} \frac{\mu(v)m(v)}{m(v_p^{(l)})\mu(v_p^{(l)})}\, d\sigma_{x_1^{(l)}}..d\sigma_{x_p^{(l)}} \leqslant C_m \int_{\prod_{1\leqslant j\leqslant J-1}\mathcal{V}_j^{(l)}} \mathbf{1}_{\left\{t_{(J-1)*}^{(l)}>0\right\}} d\sigma_{x_1^{(l)}}..d\sigma_{x_{J-1}^{(l)}}.$$

Plugging (4.16) into (4.15) gives

$$\begin{aligned}
e^{\varepsilon\nu_0 t} m(v)\left|R_p(h)(t,x,v)\right| \leqslant & C_m F_p(t)\left(\nu(v)e^{-\nu(v)(1-\varepsilon)t} + e^{-\nu_0(1-\varepsilon)t}\right) \\
& \times \sup_{s\in[0,t]}\left[e^{\nu_0 s}\left\|\mathbf{1}_{\{t_1>0\}}h(s)\right\|_{L^\infty_{\Lambda^+}(m\nu^{-1})}\right]
\end{aligned} \tag{4.17}$$

with

$$F_p(t) = \sup_{x,v}\left[\sum_{i=0}^{p}\sum_{l\in\vartheta_p(i)}(1-\alpha)^i\Delta_n^{p-i}\int_{\prod_{1\leqslant j\leqslant J-1}\mathcal{V}_j^{(l)}} \mathbf{1}_{\left\{t_{(J-1)*}^{(l)}>0\right\}} d\sigma_{x_1^{(l)}}..d\sigma_{x_{J-1}^{(l)}}\right].$$

It remains to prove an upper bound on $F_p(t)$ for $0\leqslant t\leqslant T_0$ when T_0 and p are large. Let $T_0>0$, p in $\mathbb{N}$ and $0<\delta<1$ to be determined later.

For any given i in $\{1,\dots,p\}$ and l in $\vartheta_p(i)$ we define the non-grazing sets for all j in $\{1,\dots,p\}$ as

$$\Lambda_j^{(l),\delta} = \left\{v_j^{(l)}\cdot n(x_j^{(l)}) \geqslant \delta\right\} \cap \left\{\left|v_j^{(l)}\right| \leqslant \frac{1}{\delta}\right\}.$$

By definition of the backward characteristics we have $x_j^{(l)} - x_{j+1}^{(l)} = (t_j^{(l)} - t_{j+1}^{(l)})v_j^{(l)}$. Since Ω is a C^1 bounded it is known [21, Lemma 2] that there exists $C_\Omega>0$ such that

$$\forall v_j^{(l)}\in\Lambda_j^{(l),\delta},\quad \left|t_j^{(l)} - t_{j+1}^{(l)}\right| \geqslant \frac{\left|v_j^{(l)}\cdot n(x_j^{(l)})\right|}{C_\Omega\left|v_j^{(l)}\right|} \geqslant \frac{\delta^3}{C_\Omega}.$$

Therefore, for t in $[0,T_0]$, if $t^{(l)}_{(J-1)*}(t,x,v,v^{(l)}_1,v^{(l)}_2,\dots,v^{(l)}_{J-1})>0$ then there can be at most $[C_\Omega T_0\delta^{-3}]+1$ velocities $v^{(l)}_j$ in $\Lambda^{(l),\delta}_j$. Among these, we have exactly k velocities $v^{(l)}_j$ that are integration variables v_{j*} and the rest are specular reflections. Since i represents the total number of specular reflections, it remains exactly $p-i-k$ integration variables that are not in any $\Lambda^{(l),\delta}_j$. Recalling that $d\sigma_x$ is a probability measure, if $v^{(l)}_j$ is a specular reflection we bound the integral in v_{j*} by one. All these thoughts yield

$$
\begin{aligned}
F_p(t)\leqslant&\sup_{x,v}\Big[\sum_{i=0}^{p}\sum_{l\in\vartheta_p(i)}(1-\alpha)^i\Delta_n^{p-i}\\
&\times\sum_{j=0}^{\left[\frac{C_\Omega T_0}{\delta^3}\right]+1}\sum_{k=0}^{j}\int\left\{\begin{array}{l}\text{exactly } k \text{ of } v_*\in\Lambda^{(l),\delta},\\ j-k \text{ of specular in } \Lambda^{(l),\delta},\\ p-i-k \text{ of } v_* \text{ not in } \Lambda^{(l),\delta}\end{array}\right\}\prod_{1\leqslant m\leqslant J-1}d\sigma_{x^{(l)}_m}(v_{m*})\Big]\\
\leqslant&\sum_{i=0}^{p}\sum_{l\in\vartheta_p(i)}(1-\alpha)^i\Delta_n^{p-i}\sum_{j=0}^{\left[\frac{C_\Omega T_0}{\delta^3}\right]+1}\binom{j}{k}\\
&\times\sum_{k=0}^{j}\left(\sup_{t,x,v,i,l,j}\int_{\Lambda^{(l),\delta}}d\sigma_{x^{(l)}_j}(v_*)\right)^k\left(\sup_{t,x,v,i,l,j}\int_{v_*\notin\Lambda^{(l),\delta}}d\sigma_{x^{(l)}_j}(v_*)\right)^{p-i-k}.
\end{aligned}
$$

In what follows we denote by C any positive constant independent of t, x, v, i, l and j. We bound first

$$
\int_{v_*\notin\Lambda^{(l),\delta}}d\sigma_{x^{(l)}_j}(v_*)\leqslant\int_{0<v_*\cdot n(x^{(l)}_j)\leqslant\delta}d\sigma_{x^{(l)}_j)}+\int_{|v_*|\geqslant\delta^{-1}}d\sigma_{x^{(l)}_j)}\leqslant C\delta
$$

and second we bound by one the integrals on $\Lambda^{(l),\delta}$. With $C\delta<1$ we end up with

$$
\begin{aligned}
F_p(t)&\leqslant\sum_{i=0}^{p}\sum_{l\in\vartheta_p(i)}(1-\alpha)^i(C\Delta_n\delta)^{p-i}\sum_{j=0}^{\left[\frac{C_\Omega T_0}{\delta^3}\right]+1}\sum_{k=0}^{j}\binom{j}{k}(C\delta)^{-k}\\
&\leqslant\sum_{i=0}^{p}\sum_{l\in\vartheta_p(i)}(1-\alpha)^i(C\Delta_n\delta)^{p-i}\sum_{j=0}^{\left[\frac{C_\Omega T_0}{\delta^3}\right]+1}\left(1+\frac{1}{C\delta}\right)^j\\
&\leqslant C\delta\left(1+\frac{1}{C\delta}\right)^{\left[\frac{C_\Omega T_0}{\delta^3}\right]+2}\sum_{i=0}^{p}\binom{p}{i}(1-\alpha)^i(C\alpha\delta)^{p-i}\\
&\leqslant 2\left(1+\frac{1}{C\delta}\right)^{\left[\frac{C_\Omega T_0}{\delta^3}\right]+1}((1-\alpha)+C\delta\alpha)^p.
\end{aligned}
$$

Since $\alpha>0$ we can choose $\delta>0$ small enough such that $(1-\alpha)+C\delta\alpha=\alpha_0<1$. Then choose N in $\mathbb{N}$ large enough such that

$$
\left(1+\frac{1}{C\delta}\right)\alpha_0^N\leqslant\frac{1}{2}.
$$

Finally choose $p = N(\left[\frac{C_\Omega T_0}{\delta^3}\right] + 1)$. It follows that

$$F_p(t) \leqslant 2\left[\left(1 + \frac{1}{C\delta}\right)\alpha_0^N\right]^{\left[\frac{C_\Omega T_0}{\delta^3}\right]+1} \leqslant \left(\frac{1}{2}\right)^{\left[\frac{C_\Omega T_0}{\delta^3}\right]}.$$

This inequality with (4.17) concludes the proof of the lemma. □

4.3.3. *Step 3: Exponential decay and convergence of $f^{(n)}$.* Fix $T_0 > 0$ to be chosen later and choose $p = p_R(T_0)$ defined in Lemma 4.6. We have that for all n in $\mathbb{N}$,

- by (4.6), for every (t, x, v) in $\mathbb{R}^+ \times \overline{\Omega} \times \mathbb{R}^3$,

$$\mathbf{1}_{\{t_1(t,x,v)\}\leqslant 0} f^{(n)}(t, x, v) = e^{-\nu(v)t} f_0^{(n)}(x - tv, v)$$

and hence

$$\sup_{s\in[0,t]}\left[e^{\nu_0 s}\left\|\mathbf{1}_{\{t_1(t,x,v)\}\leqslant 0} f^{(n)}(t, x, v)\right\|_{L^\infty_{x,v}(m)}\right] \leqslant \left\|f_0^{(n)}\right\|_{L^\infty_{x,v}(m)} \leqslant \|f_0\|_{L^\infty_{x,v}(m)}; \tag{4.18}$$

- by Lemmas 4.4, 4.5 and 4.6, for every (t, x, v) in $[0, T_0] \times \overline{\Omega} \times \mathbb{R}^3$

$$\begin{aligned}
&\sup_{s\in[0,t]}\left[e^{\nu_0 s}\left\|\mathbf{1}_{\{t_1(s,x,v)>0\}} f^{(n)}(s, x, v)\right\|_{L^\infty_{x,v}(m)}\right] \\
&\quad\leqslant \sup_{s\in[0,t]}\left[e^{\nu_0 s}\left\|I_p(f_0^{(n)})(s)\right\|_{L^\infty_{x,v}(m)}\right] + \sup_{s\in[0,t]}\left[e^{\nu_0 s}\left\|R_p(h)(s)\right\|_{L^\infty_{x,v}(m)}\right] \\
&\quad\leqslant pC_m\|f_0\|_{L^\infty_{x,v}(m)} \\
&\qquad + C_m\left(\frac{1}{2}\right)^{[\overline{C}T_0]}\sup_{s\in[0,t]}\left[e^{\nu_0 s}\left\|\mathbf{1}_{\{t_1(s,x,v)>0\}} f^{(n)}(s, x, v)\right\|_{L^\infty_{\Lambda^+}(m)}\right]
\end{aligned} \tag{4.19}$$

We recall Lemma 4.6 and we have $p_R(T0) \leqslant \overline{N}(\overline{C}T_0+1)$. Let ν_0' in $(0, \nu_0)$. Suppose T_0 was chosen large enough such that

$$C_m\left(\frac{1}{2}\right)^{[\overline{C}T_0]} \leqslant \frac{1}{2} \qquad \text{and} \qquad 2C_m\overline{N}(\overline{C}T_0 + 1)e^{-\nu_0 T_0} \leqslant e^{-\nu_0' T_0}.$$

Applying (4.19) at T_0 gives

$$\left\|\mathbf{1}_{\{t_1(T_0)>0\}} f^{(n)}(T_0)\right\|_{L^\infty_{x,v}(m)} \leqslant 2C_m p_R(T_0)e^{-\nu_0 T_0}\|f_0\|_{L^\infty_{x,v}(m)} \leqslant e^{-\nu_0' T_0}\|f_0\|_{L^\infty_{x,v}(m)},$$

and with (4.18) we finally have

$$\left\|f^{(n)}(T_0)\right\|_{L^\infty_{x,v}(m)} \leqslant e^{-\nu_0' T_0}\|f_0\|_{L^\infty_{x,v}(m)}.$$

We could now start the proof at T_0 up to $2T_0$ and iterating this process we get

$$\begin{aligned}
\forall n \in \mathbb{N}, \quad \left\|f^{(n)}(nT_0)\mathbf{1}_{t_1>0}\right\|_{L^\infty_{x,v}(m)} &\leqslant e^{-\nu_0' T_0}\left\|f^{(n)}((n-1)T_0\right\|_{L^\infty_{x,v}(m)} \\
&\leqslant e^{-2\nu_0' T_0}\left\|f^{(n)}((n-2)T_0)\right\|_{L^\infty_{x,v}(m)} \\
&\leqslant \ldots \leqslant e^{-\nu_0' nT_0}\|f_0\|_{L^\infty_{x,v}(m)}.
\end{aligned}$$

Finally, for all t in $[nT_0,(n+1)T_0]$ we apply (4.19) with the above to get

$$\begin{aligned}\left\|f^{(n)}\mathbf{1}_{t_1>0}(t)\right\|_{L^\infty_{x,v}(m)} &\leqslant 2C_m p_R(T_0)e^{-\nu_0(t-nT_0)}\left\|f^{(n)}(nT_0)\right\|_{L^\infty_{x,v}(m)}\\ &\leqslant 2C_m p_R(T_0)e^{-\nu_0' t}e^{-(\nu_0-\nu_0')(t-nT_0)}\left\|f_0\right\|_{L^\infty_{x,v}(m)}.\end{aligned}$$

Hence the uniform control in t, where $C_0>0$ depends on m, T_0 and ν_0',

$$\exists C_0>0,\ \forall t\geqslant 0,\quad \left\|f^{(n)}\mathbf{1}_{t_1>0}(t)\right\|_{L^\infty_{x,v}}\leqslant C_0 e^{-\nu_0' t}\left\|f_0\right\|_{L^\infty_{x,v}(m)}$$

which combined with (4.18) implies

$$\forall n\in\mathbb{N},\ \forall t\geqslant 0,\quad \left\|f^{(n)}(t)\right\|_{L^\infty_{x,v}(m)}\leqslant \max\{1,C_0\}\,e^{-\nu_0' t}\left\|f_0\right\|_{L^\infty_{x,v}(m)}. \tag{4.20}$$

Since (4.18) and (4.19) holds for x in $\overline{\Omega}$, inequality (4.20) holds in $L^\infty\left(\overline{\Omega}\times\mathbb{R}^3\right)(m)$. Therefore, $\left(f^{(n)}\right)_{n\in\mathbb{N}}$ is bounded in $L^\infty_t L^\infty\left(\overline{\Omega}\times\mathbb{R}^3\right)(m)$ and converges, up to a subsequence, weakly-* towards f in $L^\infty_t L^\infty\left(\overline{\Omega}\times\mathbb{R}^3\right)(m)$ and f is a solution to $\partial_t f = G_\nu f$ satisfying the Maxwell boundary condition and with initial datum f_0. Moreover, we have the expected exponential decay for f thanks to the uniform (4.20). This concludes the proof of Theorem 4.1 and we now prove Corollary 4.2.

Proof of Corollary 4.2. Thanks to the convergence properties of $\left(f^{(n)}\right)_{n\in\mathbb{N}}$, Lemmas 4.4, 4.5 and 4.6 are directly applicable to the semigroup $S_{G_\nu}(t)$ with Δ_n replaced by α. Therefore, as usual, for f_s in $L^\infty_{x,v}(m)$ we decompose into $t-s\leqslant t_{min}(x,v)$ and $t-s\geqslant t_{min}$, which gives thanks to (4.9)

$$\begin{aligned}\int_0^t S_{G_\nu}(t-s)f_s(x,v)\,ds &= \int_{\max\{0,t-t_{min}\}}^t e^{-\nu(v)(t-s)}f_s(x-(t-s)v,v)\,ds\\ &\quad+\int_0^{\max\{0,t-t_{min}\}} I_p(f_s)(t-sx,v)\,ds\\ &\quad+\int_0^{\max\{0,t-t_{min}\}} R_p(S_{G_\nu}f_s)(t-s,x,v)\,ds.\end{aligned} \tag{4.21}$$

Let ε be in $(0,1)$. We bound $e^{-\nu(v)(t-s)}\leqslant e^{-\varepsilon\nu_0 t}e^{-(1-\varepsilon)\nu(v)(t-s)}e^{-\varepsilon\nu_0 s}$ and thus, using the estimate with a gain of weight for I_p in Lemma 4.5, we control from above the absolute value of the first two terms by

$$\begin{aligned}&\left|\int_{\max\{0,t-t_{min}\}}^t e^{-\nu(v)(t-s)}f_s(x-(t-s)v,v)\,ds+\int_0^{\max\{0,t-t_{min}\}} I_p(f_s)(t-sx,v)\,ds\right|\\ &\leqslant pC_m\int_0^t\left(\nu(v)e^{-\nu(v)(t-s)}+e^{-\nu_0(t-s)}\right)\left\|f_s\right\|_{L^\infty_{x,v}(m\nu^{-1})}\,ds\\ &\leqslant pC_m e^{-\varepsilon\nu_0 t}\left(\int_0^t\nu(v)e^{-(1-\varepsilon)\nu(v)(t-s)}+e^{-(1-\varepsilon)\nu_0(t-s)}\right)\sup_{s\in[0,t]}\left[e^{\varepsilon\nu_0 s}\left\|f_s\right\|_{L^\infty_{x,v}(m)}\right]\\ &\leqslant C_m\frac{p}{1-\varepsilon}e^{-\varepsilon\nu_0 t}\sup_{s\in[0,t]}\left[e^{\varepsilon\nu_0 s}\left\|f_s\right\|_{L^\infty_{x,v}(m\nu^{-1})}\right].\end{aligned} \tag{4.22}$$

The third term is treated using Lemma 4.6 with an exponential weight $e^{\varepsilon\nu_0 t}$. This yields

$$\left|\int_0^{\max\{0,t-t_{min}\}} R_p(S_{G_\nu} f_s)(t-s,x,v)\,ds\right|$$
$$\leqslant C_m\left(\frac{1}{2}\right)^{\left[\overline{C}T_0\right]}\int_0^{\max\{0,t-t_{min}\}} e^{-\varepsilon\nu_0(t-s)}\left(\nu(v)e^{-(1-\varepsilon)\nu(v)(t-s)}+e^{-(1-\varepsilon)\nu_0(t-s)}\right)$$
$$\times \sup_{s_*\in[0,t-s]}\left[e^{\varepsilon\nu_0 s_*}\left\|S_{G_\nu}(s_*)(f_s)\right\|_{L^\infty_{x,v}(m\nu^{-1})}\right]\,ds$$

which is further bounded as

$$\leqslant C_m\left(\frac{1}{2}\right)^{\left[\overline{C}T_0\right]} e^{-\varepsilon\nu_0 t}\int_0^t\left(\nu(v)e^{-(1-\varepsilon)\nu(v)(t-s)}+e^{-(1-\varepsilon)\nu_0(t-s)}\right)$$
$$\times \sup_{s_*\in[0,t-s]}\left[\left\|e^{\varepsilon\nu_0(s+s_*)}S_{G_\nu}(s_*)(f_s)\right\|_{L^\infty_{x,v}(m\nu^{-1})}\right]\,ds.$$

Since $m\nu^{-1}$ satisfies the requirements of Theorem 4.1, we can use the exponential decay of $S_{G_\nu}(s_*)$ with the exponential rate being $\varepsilon\nu_0<\nu_0$ and obtain

$$\left|\int_0^{\max\{0,t-t_{min}\}} R_p(S_{G_\nu} f_s)(t-s,x,v)\,ds\right|$$
$$\leqslant \frac{C_m}{1-\varepsilon}\left(\frac{1}{2}\right)^{\left[\overline{C}T_0\right]} e^{-\varepsilon\nu_0 t}\sup_{s\in[0,t]}\left[e^{\varepsilon\nu_0 s}\left\|f_s\right\|_{L^\infty_{x,v}(m\nu^{-1})}\right].\tag{4.23}$$

For any $T_0\geqslant 1$, $2^{\left[\overline{C}T_0\right]}\leqslant 2^{-1}$ and thus (4.23) becomes independent of T_0 and holds for all $t\geqslant 0$. Plugging (4.22) and (4.23) into (4.21) yields the expected gain of weight with exponential decay. □

5. L^∞ theory for the linear operator with Maxwellian weights

As explained in the introduction, the L^2 setting is not algebraic for the bilinear operator Q. We therefore need to work within an L^∞ framework. This section is devoted to the study of the semigroup generated by the full linear operator together with the Maxwell boundary condition in the space $L^\infty_{x,v}\left(\mu^{-\zeta}\right)$ with ζ in $(1/2,1)$. This weight allows us to obtain sharper estimates on the compact operator K and thus extend the validity of our proof up to $\alpha=2/3$. In this section we establish the following theorem.

Theorem 5.1. *Let α be in $(\sqrt{2/3},1]$. There exist ζ_α in $(1/2,1)$ such that for any ζ in $(\zeta_\alpha,1)$, the linear perturbed operator $G=L-v\cdot\nabla_x$, together with Maxwell boundary condition, generates a semigroup $S_G(t)$ on $L^\infty_{x,v}(\mu^{-\zeta})$. Moreover, there exists λ_∞ and $C_\infty>0$ such that*

$$\forall t\geqslant 0,\quad \left\|S_G(t)\left(Id-\Pi_G\right)\right\|_{L^\infty_{x,v}(\mu^{-\zeta})}\leqslant C_\infty e^{-\lambda_\infty t},$$

where Π_G is the orthogonal projection onto $Ker(G)$ in $L^2_{x,v}\left(\mu^{-1/2}\right)$ (see (3.8)). The constants C_∞ and λ_∞ are explicit and depend on α, ζ and the collision kernel.

5.1. **Preliminaries: pointwise estimate on K and $L^2 - L^\infty$ theory.** We recall that $L = -\nu(v) + K$. The following pointwise estimate on K has been proved in [21, Lemma 3] for hard sphere models and [8, Lemma 5.2] for more general kernels.

Lemma 5.2. *There exists $k(v, v_*) \geqslant 0$ such that for all v in $\mathbb{R}^3$,*

$$K(f)(v) = \int_{\mathbb{R}^3} k(v, v_*) f(v_*)\, dv_*.$$

Moreover, for ζ in $[0, 1)$ there exists $C_\zeta > 0$ and $\varepsilon_\zeta > 0$ such that for all ε in $[0, \varepsilon_\zeta)$,

$$\int_{\mathbb{R}^3} |k(v, v_*)|\, e^{\frac{\varepsilon}{8}|v-v_*|^2 + \frac{\varepsilon}{8}\frac{\left||v|^2-|v_*|^2\right|^2}{|v-v_*|^2}} \frac{\mu(v)^{-\zeta}}{\mu(v_*)^{-\zeta}}\, dv_* \leqslant \frac{C_\zeta}{1+|v|}.$$

We now prove a more precise and more explicit control over the operator K. The idea behind it is that as ζ goes to 1, the operator K gets closer to the collision frequence $3\nu(v)$.

Lemma 5.3. *There exists $C_K > 0$ such that for all ζ in $[1/2, 1]$,*

$$\forall f \in L^\infty_{x,v}(\mu^{-\zeta}), \quad \|K(f)\|_{L^\infty_{x,v}(\nu^{-1}\mu^{-\zeta})} \leqslant C_K(\zeta)\, \|f\|_{L^\infty_{x,v}(\mu^{-\zeta})}$$

where $C_K(\zeta) = 3 + C_K(1-\zeta)$.

Proof of Lemma 5.3. The change of variable $\sigma \to -\sigma$ exchanges v' and v'_* and we can so rewrite

$$K(f)(v) = \int_{R^3\times\mathbb{S}^2} b(\cos\theta)\, |v - v_*|^\gamma \left[2\mu' f'_* - \mu f_*\right] d\sigma dv_* = K_1(f) - K_2(f) \tag{5.1}$$

where K_1 and K_2 are just the integral divide into the two contributions.

We start with K_1. We use the elastic collision identity $\mu\mu_* = \mu'\mu'_*$ to get

$$\begin{aligned}\left|\nu(v)^{-1}\mu(v)^{-\zeta} K_1(f)(v)\right| &\leqslant 2\nu(v)^{-1} \int_{R^3\times\mathbb{S}^2} b(\cos\theta)\, |v - v_*|^\gamma\, \mu_*^\zeta \mu(v')^{1-\zeta} \left|\frac{f'_*}{\mu(v'_*)^\zeta}\right| d\sigma dv_* \\ &\leqslant 2\, \|f\|_{L^\infty_{x,v}(\mu^{-\zeta})}\, \nu(v)^{-1} \int_{R^3\times\mathbb{S}^2} b(\cos\theta)\, |v - v_*|^\gamma\, \mu_*^\zeta\, d\sigma dv_*.\end{aligned}$$

But then by definition of $\nu(v)$,

$$\int_{R^3\times\mathbb{S}^2} b(\cos\theta)\, |v - v_*|^\gamma\, \mu_*^\zeta d\sigma dv_* = \nu(v) + \int_{R^3\times\mathbb{S}^2} b(\cos\theta)\, |v - v_*|^\gamma\, (\mu_*^\zeta - \mu_*) d\sigma dv_*$$

which implies, since b is bounded and $\nu(v) \sim (1 + |v|^\gamma)$ (see (3.6)),

$$\int_{R^3\times\mathbb{S}^2} b(\cos\theta)\, |v - v_*|^\gamma\, \mu_*^\zeta d\sigma dv_* \leqslant \nu(v) + C(1-\zeta)\nu(v) \int_{R^3\times\mathbb{S}^2} (1+|v_*|^\gamma)\, |v_*|^2\, \mu_*^\zeta d\sigma dv_*.$$

To conclude we recall that $\zeta > 1/2$ and the integral above on the right-hand side is uniformly bounded in v_*. Hence,

$$\exists C_K > 0, \quad \|K_1(f)\|_{L^\infty_{x,v}(\mu^{-\zeta})} \leqslant \left(2 + \frac{C_K}{2}(1-\zeta)\right) \|f\|_{L^\infty_{x,v}(\mu^{-\zeta})}\,. \tag{5.2}$$

The term K_2 is similar :

$$\begin{aligned}\left|\nu(v)^{-1}\mu(v)^{-\zeta}K_2(f)(v)\right| &\leqslant \mu^{1-\zeta}\nu^{-1}\int_{R^3\times\mathbb{S}^2} b(\cos\theta)\,|v-v_*|^\gamma\,|f_*|\,d\sigma dv_* \\ &\leqslant \|f\|_{L^\infty_{x,v}(\mu^{-\zeta})}\,\nu(v)^{-1}\int_{R^3\times\mathbb{S}^2} b(\cos\theta)\,|v-v_*|^\gamma\,\mu_*^\zeta d\sigma dv_* \\ &\leqslant \left(1+\frac{C_K}{2}(1-\zeta)\right)\|f\|_{L^\infty_{x,v}(\mu^{-\zeta})}\,.\end{aligned}$$

Plugging the above and (5.2) inside (5.1) concludes the proof. □

We conclude this preliminary section with a statement of the L^2-L^∞ theory that will be at the core of our main proof. It follows the idea developed in [21] that the L^2 theory of previous section could be used to construct a L^∞ one by using the flow of characteristics to transfer pointwise estimates at $x-vt$ into an integral in the space variable. The proof can be found in [21, Lemma 19] and holds as long as $L^\infty_{x,v}(w)\subset L^2_{x,v}(\mu^{-1/2})$.

Proposition 5.4. *Let ζ be in $[1/2,1)$ and assume that there exist $T_0>0$ and C_{T_0}, $\lambda>0$ such that for all $f(t,x,v)$ in $L^\infty_{x,v}(\mu^{-\zeta})$ solution to*

$$\partial_t f + v\cdot\nabla_x f = L(f) \tag{5.3}$$

with Maxwell boundary condition $\alpha>0$ and initial datum f_0, the following holds

$$\forall t\in[0,T_0],\quad \|f(t)\|_{L^\infty_{x,v}(\mu^{-\zeta})}\leqslant e^{\lambda(T_0-2t)}\|f_0\|_{L^\infty_{x,v}(\mu^{-\zeta})}+C_{T_0}\int_0^t\|f(s)\|_{L^2_{x,v}(\mu^{-1/2})}\,ds.$$

Then for all $0<\widetilde{\lambda}<\min\{\lambda,\lambda_G\}$, defined in Theorem 3.1, there exists $C>0$ independent of f_0 such that for all f solution to (5.3) in $L^\infty_{x,v}(\mu^{-\zeta})$ with $\Pi_G(f)=0$,

$$\forall t\geqslant 0,\quad \|f(t)\|_{L^\infty_{x,v}(\mu^{-\zeta})}\leqslant Ce^{-\widetilde{\lambda}t}\|f_0\|_{L^\infty_{x,v}(\mu^{-\zeta})}\,.$$

5.2. **A crucial estimate between two consecutive collisions.** The core of the L^∞ estimate is a delicate control over the action of K in between two rebounds. We define the following operator

$$\mathfrak{K}(f)(t,x,v)=\mu^{-\zeta}(v)\int_{\max\{0,t_{\min}(x,v)\}}^t e^{-\nu(v)(t-s)}K(f(s))(x-(t-s)v,v)\,ds. \tag{5.4}$$

We shall prove the following estimate of this functional along the flow of solutions.

Proposition 5.5. *Let α in $(\sqrt{2/3},1]$. There exists ζ_α in $(1/2,1)$, ε_α in $(0,1)$, $C^{(1)}_\alpha>0$ and $0<C^{(2)}_\alpha<1$ such that for any $T_0>0$ there exists $C_{T_0}>0$ such that if f is solution to $\partial_t f=Gf$ with Maxwell boundary conditions then for all t in $[0,T_0]$*

$$\begin{aligned}\left|e^{\varepsilon\nu_0 t}\mathfrak{K}(f)(t,x,v)\right|\leqslant& C^{(1)}_\alpha\|f_0\|_{L^\infty_{x,v}(\mu^{-\zeta})}+C_{T_0}\int_0^t\|f(s)\|_{L^2_{x,v}(\mu^{-1/2})}\,ds \\ &+C^{(2)}_\alpha\left(1-e^{-\nu(v)(1-\varepsilon)\min\{t,t_{\min}(x,v)\}}\right)\sup_{s\in[0,t]}\left[e^{\varepsilon\nu_0 s}\|f(s)\|_{L^\infty_{x,v}(\mu^{-\zeta})}\right].\end{aligned}$$

Proof of Proposition 5.5. We recall that $G = L - v\cdot\nabla_x = G_\nu + K$. Thanks to Theorem 4.1 with the weight $\mu^{-\zeta}$, G_ν generates a semigroup $S_{G_\nu}(t)$ in $L^\infty_{x,v}(\mu^{-\zeta})$. Moreover, Lemma 5.2 implies that K is a bounded operator in $L^\infty_{x,v}(\mu^{-\zeta})$. We can therefore write a Duhamel's form for the solution f for almost every (s,x,v_*) in $\mathbb{R}^+\times\overline{\Omega}\times\mathbb{R}^3$:

$$f(s,x,v_*) = S_{G_\nu}(s)f_0(x,v_*) + \int_0^s S_{G_\nu}(s-s_*)\left[K(f(s_*))\right](x,v_*)\,ds_*. \tag{5.5}$$

Step 1: New implicit form for f. In what follows, C_r will stand for any positive constant not depending on f but depending on a parameter r. We recall (4.10) the definition of the set of integration $\mathcal{V}_1$. We use the description (4.4), α replacing Δ_n, of $S_{G_\nu}(s-s_*)$ along characteristics until the first collision against the boundary that we denote $(t_1^*,x_1^*,v_1^*) = (t_{\min}(x,v_*),X_1(x,v_*),V_1(x,v_*))$ (see Section 4.1) . We deduce

$$\begin{aligned} f(s,x,v_*) =& J_0(s,x,v_*) + J_K(s,x,v_*) + (1-\alpha)\mathbf{1}_{\{t>t_1^*\}}e^{-\nu(v_*)t_1^*}f(t-t_1^*,x_1^*,v_1^*) \\ &+ \alpha\mathbf{1}_{\{s>t_1^*\}}c_\mu e^{-\nu(v_*)t_1^*}\mu(v_*)\int_{\mathcal{V}_1} f(s-t_1^*,x_1^*,v_{1*})\left(v_{1*}\cdot n(x_1^*)\right)\,dv_{1*}, \end{aligned}$$

where we defined

$$J_0 = S_{G_\nu}(s)f_0(x,v_*) + \mathbf{1}_{\{s>t_1^*\}}e^{-\nu(v_*)t_1^*}S_{G_\nu}(s-t_1^*)f_0(x_1^*,v_*) \tag{5.6}$$

and

$$J_K = \int_{\max\{0,s-t_1^*\}}^s e^{-\nu(v_*)(s-s_*)}K(f(s_*))(x-(s-s_*)v_*,v_*)\,ds. \tag{5.7}$$

We iterate this formula inside the integral over $\mathcal{V}_1$. Using the notation $(\widetilde{t_1},\widetilde{x_1},\widetilde{v_1})$ to denote the first backard collision starting from (x_1^*,v_{1*}) and P_Λ for the diffuse boundary operator (1.3) we end up with a new implicit form for f

$$f(s,x,v_*) = \overline{J_0}(s,x,v_*) + \overline{J_K}(s,x,v_*) + J_f(s,x,v_*) + J_{diff}(s,x,v_*) \tag{5.8}$$

with the following definitions

$$\overline{J_0} = J_0(s,x,v_*) + \alpha\mathbf{1}_{\{s>t_1^*\}}e^{-\nu(v_*)t_1^*}P_\Lambda(J_0(s-t_1^*,x_1^*))(v_*), \tag{5.9}$$

$$\overline{J_K} = J_K(s,x,v_*) + \alpha\mathbf{1}_{\{s>t_1^*\}}e^{-\nu(v_*)t_1^*}P_\Lambda(J_K(s-t_1^*,x_1^*))(v_*), \tag{5.10}$$

(5.11)

$$\begin{aligned} J_f =& (1-\alpha)\mathbf{1}_{\{s>t_1^*\}}\Bigg[e^{-\nu(v_*)t_1^*}f(s-t_1^*,x_1^*,v_1^*) \\ &+ \alpha e^{-\nu(v_*)t_1^*}c_\mu\mu(v_*)\int_{\mathcal{V}_1}\mathbf{1}_{\{s-t_1^*>\widetilde{t_1}\}}e^{-\nu(v_{1*})\widetilde{t_1}}f(s-t_1^*-\widetilde{t_1},\widetilde{x_1},\widetilde{v_1})(v_{1*}\cdot n(x_1^*))dv_{1*}\Bigg], \end{aligned}$$

and at last

(5.12)
$$J_{diff} = \alpha^2 \mathbf{1}_{\{s>t_1^*\}} c_\mu^2 e^{-\nu(v_*)t_1^*}\mu(v_*)\left[\int_{\mathcal{V}_1} \mathbf{1}_{\{s-t_1^*>\widetilde{t_1}\}} e^{-\nu(v_{1*})\widetilde{t_1}}\mu(v_{1*}) \right.$$
$$\left. \times \int_{\widetilde{\mathcal{V}}_1} f(s-t_1^*-\widetilde{t_1},\widetilde{x_1},\widetilde{v}_{1*})\,(\widetilde{v}_{1*}\cdot n(\widetilde{x_1}))\,(v_{1*}\cdot n(x_1))\; d\widetilde{v}_{1*}dv_{1*}\right].$$

We now bound the operator $\mathfrak{K}$ in (5.4) for each of the terms above. We will bound most of the terms uniformly because for any ε in $[0,1]$, from Lemma 5.3 and $e^{-\nu(v)(t-s)} \leqslant e^{-\varepsilon\nu_0 t}e^{-\nu(v)(1-\varepsilon)(t-s)}e^{\varepsilon\nu_0 s}$ the following holds

$$|\mathfrak{K}(F)| \leqslant C_K(\zeta)e^{-\varepsilon\nu_0 t}\left(\int_{\max\{0,t-t_1\}}^t \nu(v)e^{-\nu(v)(1-\varepsilon)(t-s)}ds\right)\sup_{s\in[0,t]}\left[e^{\varepsilon\nu_0 s}\left\|F(s)\right\|_{L^\infty_{x,v}(\mu^{-\zeta})}\right]$$
$$(5.13)\quad \leqslant \frac{C_K(\zeta)}{1-\varepsilon}\left(1-e^{-\nu(v)(1-\varepsilon)\min\{t,t_1\}}\right)e^{-\varepsilon\nu_0 t}\sup_{s\in[0,t]}\left[e^{\varepsilon\nu_0 s}\left\|F(s)\right\|_{L^\infty_{x,v}(\mu^{-\zeta})}\right]$$

Step 2: Estimate for $\overline{J_0}$. We straightforwardly bound J_0 in (5.6) thanks to the exponential decay with rate $\varepsilon\nu_0$ of $S_{G_\nu}(t)$ in Theorem 4.1 (that also holds on the boundary) for all (x,v) in $\overline{\Omega}\times B(0,R)$.

$$|J_0| \leqslant \frac{C_\varepsilon}{\mu^{-\zeta}(v)}e^{-\varepsilon\nu_0 t}\left\|f_0\right\|_{L^\infty_{x,v}(\mu^{-\zeta})}\left[1+\frac{\mu^{-\zeta}(v)}{\mu^{-\zeta}(v_1)}+\left(c_\mu\mu^{-\zeta}(v)\mu(v)\int_{\mathcal{V}_1}\frac{v_{1*}\cdot n(x_1)}{\mu^{-\zeta}(v_{1*})}\,dv_{1*}\right)\right].$$

To conclude we use the fact that $|v_1|=|v|$ and $\mu^{-\zeta}$ is radially symmetric. This yields

$$(5.14)\qquad |J_0| \leqslant C_\varepsilon\mu^{-\zeta}(v)^{-1}e^{-\varepsilon\nu_0 t}\left\|f_0\right\|_{L^\infty_{x,v}(\mu^{-\zeta})}$$

Bounding $J_0(t-t_1^*,x_1,v_{1*})$ exactly the same way yields the same bound for the full $\overline{J_0}$ in (5.9). We conclude thanks to (5.13)

$$(5.15)\qquad \forall 0\leqslant\varepsilon\leqslant 1,\ \forall(t,x,v),\quad |\mathfrak{K}(J_0)(t,x,v)| \leqslant C_{\varepsilon,\zeta}\left\|f_0\right\|_{L^\infty_{x,v}(\mu^{-\zeta})}e^{-\varepsilon\nu_0 t}.$$

Step 3: Estimate for $\overline{J_K}$. We write K under its kernel form with Lemma 5.2 in (5.7). To shorten notations and as we shall legitimate the following change of variable we use $y(v_*)=x-(t-s)v-(s-s_*)v_*$. Since $\nu(v)\geqslant\nu_0$ we see

(5.16)
$$|\mathfrak{K}(J_K)| \leqslant \int_0^t\int_0^s e^{-\nu_0(t-s_*)}\int_{\mathbb{R}^3}|k(v,v_*)|\frac{\mu^{-\zeta}}{\mu_*^{-\zeta}}\mathbf{1}_{\{y(v_*)\in\overline{\Omega}\}}$$
$$\int_{\mathbb{R}^3}|k(v_*,v_{**})|\frac{\mu_*^{-\zeta}}{\mu_{**}^{-\zeta}}\left|f(s_*,y(v_*),v_{**})\mu_{**}^{-\zeta}\right|dv_{**}dv_*ds_*ds$$

We take any $R\geqslant 1$. When $|v|\geqslant R$ we use Lemma 5.2 on $k(v,v_*)$ and when $|v|\leqslant R$ and $|v_*|\geqslant R$ it follows that $|v-v_*|\geqslant R$ and we thus use Lemma 5.2 again with $|k(v,v_*)| \leqslant e^{-\frac{\varepsilon\zeta}{8}R^2}\left|k(v,v_*)e^{\frac{\varepsilon\zeta}{8}|v-v_*|^2}\right|$. At last, if $|v_*|\leqslant 2R$ and $|v_{**}|\geqslant 3R$ we us Lemma 5.2 one more time on $|k(v_*,v_{**})|\leqslant e^{-\frac{\varepsilon\zeta}{8}R^2}\left|k(v_*,v_{**})e^{\frac{\varepsilon\zeta}{8}|v_*-v_{**}|^2}\right|$. Taking the

$L^\infty_{x,v}(\mu^{-\zeta})$-norm of f out of the integral and decomposing the exponential decay as for (5.13), we infer that for all ε in $[0,1]$ and any $R \geqslant 1$:

$$\begin{aligned}&\{\text{terms in (5.16) outside } \{|v| \leqslant R \cap |v_*| \leqslant 2R \cap |v_{**}| \leqslant 3R\}\}\\ &\qquad \leqslant C_{\varepsilon,\zeta}\left(\frac{1}{1+R} + e^{-\frac{\varepsilon_\zeta}{8}R^2}\right) e^{-\varepsilon\nu_0 t} \sup_{s\in[0,t]}\left[e^{\varepsilon\nu_0 s}\left\|f(s)\right\|_{L^\infty_{x,v}(\mu^{-\zeta})}\right]\end{aligned} \tag{5.17}$$

In order to deal with the remaining terms in (5.16) we first approximate $k(\cdot,\cdot)$ by a smooth and compactly supported function k_R uniformly in the following sense:

$$\sup_{|V|\leqslant 3R}\int_{|v_*|\leqslant 3R} |k(V,v_*) - k_R(V,v_*)|\,\mu^{-\zeta}(V)\,dv_* \leqslant \frac{1}{1+R}. \tag{5.18}$$

We decompose $k = k_R + (k - k_R)$ and bound the terms where $k - k_R$ are appearing as before and get (remember that k_R is compactly supported): the terms in (5.16) where $|v| \leqslant R$ and $|v_*| \leqslant 2R$ and $|v_{**}| \leqslant 3R$ are bounded from above by

$$\begin{aligned}&\frac{C_{\varepsilon,\zeta}}{1+R} e^{-\varepsilon\nu_0 t} \sup_{s\in[0,t]}\left[e^{\varepsilon\nu_0 s}\left\|f(s)\right\|_{L^\infty_{x,v}(\mu^{-\zeta})}\right]\\ &\qquad + C_{R,\zeta}\int_0^t\int_0^s e^{-\nu_0(t-s_*)}\int_{\{|v_*|\leqslant 2R\}}\int_{\{|v_{**}\leqslant 3R|\}} \mathbf{1}_{\{y(v_*)\in\overline{\Omega}\}}\,|f(s_*,y(v_*),v_{**})|\end{aligned}$$

At last, we would like to apply the change of variable $v_* \mapsto y(v_*)$ which has Jacobian $(s-s_*)^{-3}$, and so is legitimate if $s - s_* \geqslant \eta > 0$. We thus consider $\eta > 0$ and decompose the integral over s_* into an integral on $[s-\eta, s]$ and an integral on $[0, s-\eta]$. In the first one we bound as before which gives the following upper bound

$$\begin{aligned}&C_{\varepsilon,\zeta}\left(\frac{1}{1+R}+\eta\right) e^{-\varepsilon\nu_0 t} \sup_{s\in[0,t]}\left[e^{\varepsilon\nu_0 s}\left\|f(s)\right\|_{L^\infty_{x,v}(\mu^{-\zeta})}\right]\\ &\qquad + C_{R,\zeta}\int_0^t\int_0^{s-\eta} e^{-\nu_0(t-s_*)}\int_{\substack{\{|v_*|\leqslant 2R\}\\ \{|v_{**}\leqslant 3R|\}}} \mathbf{1}_{\{y(v_*)\in\overline{\Omega}\}}\,|f(s_*,y(v_*),v_{**})|\,.\end{aligned}$$

We now perform $v_* \mapsto y(v_*)$ inside the remaining integral term which makes $\|f(s_*)\|_{L^1_{x,v}}$ appear, which is itself controlled by $\|f(s_*)\|_{L^2_{x,v}(\mu^{-1/2})}$ thanks to Cauchy-Schwarz inequality. We therefore proved that

$$\begin{aligned}&\{\text{terms in (5.16) in } \{|v| \leqslant R \cap |v_*| \leqslant 2R \cap |v_{**}| \leqslant 3R\}\}\\ &\leqslant C_{\varepsilon,\zeta}\left(\frac{1}{1+R}+\eta\right) e^{-\varepsilon\nu_0 t} \sup_{s\in[0,t]}\left[e^{\varepsilon\nu_0 s}\left\|f(s)\right\|_{L^\infty_{x,v}(\mu^{-\zeta})}\right] + C_{R,\zeta,\eta}\int_0^t \left\|f(s)\right\|_{L^2_{x,v}(\mu^{-1/2})}.\end{aligned} \tag{5.19}$$

Gathering (5.17) and (5.19) inside (5.16) finally yields

$$\begin{aligned}|\mathfrak{K}(J_K)| \leqslant& C_{\varepsilon,\zeta}\left(\eta + \frac{1}{1+R} + e^{-\frac{\varepsilon_\zeta}{8}R^2}\right) e^{-\varepsilon\nu_0 t} \sup_{s\in[0,t]}\left[e^{\varepsilon\nu_0 s}\left\|f(s)\right\|_{L^\infty_{x,v}(\mu^{-\zeta})}\right]\\ &+ C_{R,\zeta,\eta}\int_0^t \left\|f(s)\right\|_{L^2_{x,v}(\mu^{-1/2})}.\end{aligned} \tag{5.20}$$

It remains to deal with the second term in $\overline{J_K}$ given by (5.10). However, by definition of the boundary operator P_Λ and since $|v_{1*}\cdot n(x_1^*)| \leqslant C_\zeta \mu^{-\zeta}(v_{1*})$ it follows

directly that

$$\left|\mu^{-\zeta}(v_*)P_\Lambda(J_K(s-t_1^*,x_1^*))(v_*)\right| \leqslant c_\mu C_\zeta \mu_*^{1-\zeta}\int_{\mathbb{R}^3} J_K(s-t_1^*,x_1^*,v_{1*})\mu^{-\zeta}(v_{1*})\,dv_{1*}. \tag{5.21}$$

We bound the integral term uniformly exactly as (5.20) for J_K and since $\zeta<1$ the integral over v_* of $\mu(v_*)^{1-\zeta}$ is finite. As a conclusion, for all ε in $(0,1)$, $R\geqslant 1$ and $\eta>0$,

$$\begin{aligned}\left|\mathfrak{K}(\overline{J_K})\right| \leqslant & C_{\varepsilon,\zeta}\left(\eta+\frac{1}{1+R}+e^{-\frac{\varepsilon_\zeta}{8}R^2}\right)e^{-\varepsilon\nu_0 t}\sup_{s\in[0,t]}\left[e^{\varepsilon\nu_0 s}\left\|f(s)\right\|_{L^\infty_{x,v}(\mu^{-\zeta})}\right]\\ &+C_{R,\zeta,\eta}\int_0^t \left\|f(s)\right\|_{L^2_{x,v}(\mu^{-1/2})}.\end{aligned} \tag{5.22}$$

Step 4: Estimate for $\mathbf{J_f}$. The control of J_f given by (5.11) is straightforward for the first term from (5.13) by taking the $L^\infty_{x,v}(\mu^{-\zeta})$-norm of f (remember that $\mu(v_*)=\mu(v_1^*)$ for a specular reflection). The second term is dealt with the same way and noticing that, in the spirit of Lemma 5.3,

$$c_\mu\mu^{1-\zeta}(v_*)\int_{\mathcal{V}_1}\mu^{-\zeta}(v_{1*})\,(v_{1*}\cdot n(x_1^*))\,dv_{1*}\leqslant 1+C_0(1-\zeta) \tag{5.23}$$

where $C_0>0$ is a universal constant. In the end,

$$\begin{aligned}\left|\mathfrak{K}(J_f)\right| \leqslant & \mathbf{1}_{\{t\geqslant t_1\}}\frac{(1-\alpha)C_K(\zeta)}{1-\varepsilon}\left(1+\alpha(1+C_0(1-\zeta))\right)\left(1-e^{-\nu(v)(1-\varepsilon)\min\{t,t_1\}}\right)\\ &\times e^{-\varepsilon\nu_0 t}\sup_{s\in[0,t]}\left[e^{\varepsilon\nu_0 s}\left\|f(s)\right\|_{L^\infty_{x,v}(\mu^{-\zeta})}\right].\end{aligned} \tag{5.24}$$

Step 5: Estimate for $\mathbf{J_{diff}}$. The last term J_{diff} given by (5.12) is treated thanks to a change of variable on the boundary and a trace theorem from [12]. First,

$$\begin{aligned}\left|\mu_*^{-\zeta}J_{diff}\right| \leqslant & \alpha^2 c_\mu^2\mu^{1-\zeta}(v_*)\\ &\int_{\mathbb{R}^3}d\widetilde{v}_{1*}\int_{\mathbb{R}^3}dv_{1*}e^{-\nu_0(t_1^*+\widetilde{t}_1)}\left|v_{1*}\right|\mu(v_{1*})\left|n(\widetilde{x}_1)\cdot\widetilde{v}_{1*}\right|\left|f(s-t_1^*-\widetilde{t}_1,\widetilde{x}_1,\widetilde{v}_{1*})\right|.\end{aligned} \tag{5.25}$$

Using the spherical coordinate $v_{1*}=r_{1*}u_{1*}$ with u_{1*} in $\mathbb{S}^2$ we have by definition

$$\widetilde{t}_1=t^*_{\min}(x_1,v_{1*})=\frac{t_1^*(x_1,u_{1*})}{r_{1*}}\quad\text{and}\quad\widetilde{x}_1=x_1-\widetilde{t}_1v_{1*}=x_1-t_1^*(x_1,u_{1*})u_{1*} \tag{5.26}$$

and, using the parametrization (θ,ϕ) of the sphere $\mathbb{S}^2$ we compute

$$\begin{aligned}\left|\mu_*^{-\zeta}J_{diff}\right| \leqslant & C\int_{\mathbb{R}^3}d\widetilde{v}_{1*}\left(\int_0^\infty\int_0^{2\pi}\int_0^\pi e^{-\nu_0(t_1^*+\widetilde{t}_1)}\left|f(s-t_1^*-\widetilde{t}_1,\widetilde{x}_1,\widetilde{v}_{1*})\right|\right.\\ &\left.\left|n(\widetilde{x}_1)\cdot\widetilde{v}_{1*}\right|\mu(r_{1*})r_{1*}^3\sin\theta\,dr_{1*}d\theta d\phi\right).\end{aligned} \tag{5.27}$$

We want to perform the change of variable $(r_{1*},\theta,\phi)\mapsto(s-t_1^*-\widetilde{t_1},\widetilde{x_1})$. Thanks to (5.26) we have

$$\partial_{r_{1*}}\left(\widetilde{t_1}\right)=-\frac{1}{r_{1*}^2}t_1^*(x_1,u_1)\quad\text{and}\quad\partial_{r_{1*}}\widetilde{x_1}=0.$$

The jacobian of $(\theta,\phi)\mapsto\widetilde{x_1}$ has been calculated in [12, Proof of Lemma 2.3]. More precisely, from [12, (2.8)] we have, calling $\xi(x)$ a parametrization of $\partial\Omega$

$$\left|\det\left(\partial_{\theta,\phi}\widetilde{x_1}\right)\right|\geqslant\frac{t_1^*(x_1,u_{1*})^2\sin\theta}{|n(\widetilde{x_1})\cdot u_{1*}|}\times\frac{\partial_3\xi(\widetilde{x_1})}{\nabla\xi(\widetilde{x_1})}.$$

Therefore, we bound from below the full jacobian by

$$\left|\det\left(\partial_{r_{1*},\theta,\phi}\left(t-t_1^*-\widetilde{t_1},\widetilde{x_1}\right)\right)\right|\geqslant\frac{t_1^*(x_1,u_{1*})^3\sin\theta}{r_{1*}^2\,|n(\widetilde{x_1})\cdot u_{1*}|}\times\frac{\partial_3\xi(\widetilde{x_1})}{\nabla\xi(\widetilde{x_1})}.$$

The important fact is that x_1 belongs to $\partial\Omega$ and u_{1*} is on $\mathbb{S}^2$. With these conditions, we know from [21, (40)], that $t_1^*(x_1,u_{1*})\geqslant C_\Omega\,|n(x_1)\cdot u_{1*}|$ and hence

$$\left|\det\left(\partial_{r_{1*},\theta,\phi}\left(t-t_1^*-\widetilde{t_1},\widetilde{x_1}\right)\right)\right|\geqslant C_\Omega\frac{|n(x_1)\cdot u_{1*}|^3\sin\theta}{r_{1*}^2}\times\frac{\partial_3\xi(\widetilde{x_1})}{\nabla\xi(\widetilde{x_1})}.$$

We therefore need $|n(x_1)\cdot u_{1*}|$ to be non zero and we thus decompose (5.25) into two integrals. The first one on $\{|n(x_1)\cdot u_{1*}|\leqslant\eta\}$ and the second on $\{|n(x_1)\cdot u_{1*}|\geqslant\eta\}$. On the first one we take the $L^\infty_{x,v}(\mu^{-\zeta})$-norm of f out and on the second one we use the spherical coordinates (5.27) and apply, as announced, the change of variable $(r_{1*},\theta,\phi)\mapsto(s-t_1^*-\widetilde{t_1},\widetilde{x_1})$, which is legitimate on this set. It follows, playing with the exponential decay as for (5.13),

$$\begin{aligned}
&\left|\mu_*^{-\zeta}J_{diff}\right|\\
&\leqslant C_\varepsilon e^{-\varepsilon\nu_0 s}\sup_{s_*\in[0,s]}\left[e^{\varepsilon\nu_0 s_*}\left\|f(s_*)\right\|_{L^\infty_{x,v}(\mu^{-\zeta})}\right]\left(\int_{\mathbb{R}^3}\int_{|n(x_1)\cdot u_{1*}|\leqslant\eta}\frac{|\widetilde{v}_{1*}|\,|v_{1*}|\,\mu(v_{1*})}{\mu^{-\zeta}(\widetilde{v}_{1*})}d\widetilde{v}_{1*}dv_{1*}\right)\\
&\quad+C_{\eta,\varepsilon}e^{-\varepsilon\nu_0 s}\int_{\mathbb{R}^3}d\widetilde{v}_{1*}\int_0^s\int_{\partial\Omega}e^{\varepsilon\nu_0 s_*}\left|f(s_*,y,\widetilde{v}_{1*})\right|\left(r_{1*}^5\mu(r_{1*})\right)|n(y)\cdot\widetilde{v}_{1*}|\;dS(y)ds_*
\end{aligned}$$

where we recall that $dS(y)=|\partial_3\xi(y)|^{-1}\,|\nabla\xi(y)|\,dy$ is the Lebesgue measure on $\partial\Omega$ and we also denoted $r_{1*}=r_{1*}(s,y)$. Since $\zeta\geqslant1/2$, the integral in the first term on the right-hand side above uniformly tends to 0 as η goes to 0. Hence, for any $\eta>0$,

$$\begin{aligned}
\left|\mu_*^{-\zeta}J_{diff}\right|\leqslant&C_\varepsilon\eta e^{-\varepsilon\nu_0 s}\sup_{s_*\in[0,s]}\left[e^{\varepsilon\nu_0 s_*}\left\|f(s_*)\right\|_{L^\infty_{x,v}(\mu^{-\zeta})}\right]\\
&+C_{\varepsilon,\eta}e^{-\varepsilon\nu_0 s}\int_0^s\int_\Lambda e^{\varepsilon\nu_0 s_*}\left|f(s_*,y,\widetilde{v}_{1*})\right|\;d\lambda(y,\widetilde{v}_{1*})ds_*
\end{aligned}\tag{5.28}$$

where $d\lambda(x,v)$ is the boundary measure on the phase space boundary Λ (see Section 1.1).

We now decompose Λ into

$$\Lambda_\eta=\{(x,v)\in\Lambda,\quad|n(x)\cdot v|\leqslant\eta\text{ or }|v|\leqslant\eta\}\quad\text{and}\quad\Lambda-\Lambda_\eta.$$

Since $\zeta\geqslant1/2$ there exists a uniform $C>0$ such that

$$\int_0^s\int_{\Lambda_\eta}e^{\varepsilon\nu_0 s_*}\left|f(s_*,y,\widetilde{v}_{1*})\right|\;d\lambda(y,\widetilde{v}_{1*})\leqslant Cs\eta\sup_{s_*\in[0,s]}\left[e^{\varepsilon\nu_0 s_*}\left\|f(s_*)\right\|_{L^\infty_{x,v}(\mu^{-\zeta})}\right].\tag{5.29}$$

For the integral on $\Lambda - \Lambda_\eta$ we use the trace lemma [12, Lemma 2.1] that states

$$\int_{\Lambda-\Lambda_\eta} |f(s_*, y, \widetilde{v}_{1*})|\, d\lambda(y, \widetilde{v}_{1*}) \leqslant C_\eta \left\|f(s_*)\right\|_{L^1_{x,v}} + \int_0^{s_*} \left[\left\|f(s_{**})\right\|_{L^1_{x,v}} + \left\|L(f(s_{**}))\right\|_{L^1_{x,v}}\right].$$

Thanks to Cauchy-Schwarz inequality and the boundedness property (see Section 3.1) of L in $L^2_{x,v}(\mu^{-1/2})$ we get

$$\int_0^s \int_{\Lambda-\Lambda_\eta} e^{\varepsilon\nu_0 s_*} \left|f(s_*, y, \widetilde{v}_{1*})\right| \, d\lambda(y, \widetilde{v}_{1*}) \leqslant C_\eta s e^{\varepsilon\nu_0 s} \int_0^s \left\|f(s_*)\right\|_{L^2_{x,v}\left(\mu^{-1/2}\right)} ds_*. \tag{5.30}$$

Gathering (5.29) and (5.30) with (5.28) we conclude

$$\left|\mu_*^{-\zeta} J_{diff}\right| \leqslant C_\varepsilon \eta s e^{-\varepsilon\nu_0 s} \sup_{s_*\in[0,s]} \left[e^{\varepsilon\nu_0 s_*} \left\|f(s_*)\right\|_{L^\infty_{x,v}(\mu^{-\zeta})}\right] + C_{\varepsilon,\eta} s \int_0^s \left\|f(s_*)\right\|_{L^2_{x,v}\left(\mu^{-1/2}\right)}. \tag{5.31}$$

Again, plugging this uniform bound in (5.13) yields for any ε in $(0,1)$ and $\eta > 0$,

$$\left|\mathfrak{K}(J_{diff})\right| \leqslant C_{\varepsilon,\zeta}\eta t e^{-\varepsilon\nu_0 t} \sup_{s\in[0,t]} \left[e^{\varepsilon\nu_0 s} \left\|f(s)\right\|_{L^\infty_{x,v}(\mu^{-\zeta})}\right] + C_{\varepsilon,\eta,\zeta} t \int_0^t \left\|f(s)\right\|_{L^2_{x,v}\left(\mu^{-1/2}\right)}. \tag{5.32}$$

Step 6: Choice of constants and conclusion. We consider $T_0 > 0$ and t in $[0, T_0]$. We bound the full $\mathfrak{K}(f)$ by gathering (5.15), (5.22), (5.24) and (5.32) into (5.8). It yields, for any ε, η in $(0,1)$ and $R \geqslant 1$,

$$\begin{aligned}\left|\mathfrak{K}(f)\right| \leqslant & C_{\varepsilon,\zeta} e^{-\varepsilon\nu_0 t} \left\|f_0\right\|_{L^\infty_{x,v}(\mu^{-\zeta})} + C_{\varepsilon,\zeta,\eta} t \int_0^t \left\|f(s)\right\|_{L^2_{x,v}(\mu^{-1/2})} ds \\ & + \left(\frac{C_\alpha(\zeta)}{1-\varepsilon}(1 - e^{-\nu(v)(1-\varepsilon)\min\{t,t_1\}}) + C_{\varepsilon,\zeta}(\eta(1+t) + \frac{1}{1+R})\right) \\ & \times e^{-\varepsilon\nu_0 t} \sup_{s\in[0,t]} \left[e^{\varepsilon\nu_0 s} \left\|f(s)\right\|_{L^\infty_{x,v}(\mu^{-\zeta})}\right].\end{aligned}$$

We used the following definition

$$C_\alpha(\zeta) = (1-\alpha)\left(3 + C_K(1-\zeta)\right)\left[1 + \alpha(1 + C_0(1-\zeta))\right].$$

For α in $(\sqrt{2/3}, 1]$, $\lim_{\zeta\to 1} C_\alpha(\zeta) = 3(1-\alpha)(1+\alpha) < 1$. We therefore choose our parameters as

(1) ζ sufficiently close to 1 such that $C_\alpha(\zeta) < 1$,
(2) ε sufficiently small so that $C_\alpha(\zeta)/(1-\varepsilon) < 1$,
(3) R large enough and η small enough such that $C_{\varepsilon,\zeta}(\eta(1+T_0) + 1/(1+R)) < 1 - C_\alpha(\zeta)/(1-\varepsilon)$.

Such choices terminates the proof. □

5.3. **Semigroup generated by the linear operator.** This subsection is dedicated to the proof of Theorem 5.1, that is uniqueness and existence of solutions to (5.3) together with the Maxwell boundary condition (1.2) in $L^\infty_{x,v}(\mu^{-\zeta})$. Moreover if f_0 satisfies the conservation laws then f will be proved to decay exponentially.

Proof of Theorem 5.1. Let f_0 be in $L^\infty_{x,v}(\mu^{-\zeta})$ with $1/2 < \zeta < 1$. If f is solution to (5.3) in $L^\infty_{x,v}(\mu^{-\zeta})$ with initial datum f_0 then f belongs to $L^2_{x,v}\left(\mu^{-1/2}\right)$ and $f(t) = S_G(t)f_0$ is in the latter space. This implies first uniqueness and second that $\mathrm{Ker}(G)$ and $(\mathrm{Ker}(G))^\perp$ are stable under the flow of the equation (5.3). It suffices to consider f_0 such that $\Pi_G(f_0) = 0$ and to prove existence and exponential decay of solutions to (5.3) in $L^\infty_{x,v}(\mu^{-\zeta})$ with initial datum f_0.

Thanks to boundedness property of K, the Duhamel's form (5.5) is a contraction, at least for small times. We thus have existence of solutions on small times and proving the exponential decay will also imply global existence. For now on we consider f as described in the Duhamel's expression (5.5).

Looking at previous section and using $t_1 = t_{\min}(x,v)$, f can be implicitely written as (5.8)

$$\begin{aligned} f(t,x,v) =& \overline{J_0}(t,x,v) + \mu^\zeta \mathfrak{K}(f)(t,x,v) + \alpha e^{-\nu(v)t_1} P_\Lambda(J_K(t-t_1,x_1))(t,x,v) \\ &+ \mathbf{1}_{\{t>t_1\}} \left[J_f(t,x,v) + J_{diff}(t,x,v)\right]. \end{aligned}$$

The $L^\infty_{x,v}(\mu^{-\zeta})$-norm of each of these terms has already been estimated. More precisely, $\overline{J_0}$ by (5.14), $\mathfrak{K}(f)$ by Proposition 5.5, $P_\Lambda(J_K)$ by (5.21), J_f is direct from (5.11) and (5.23) and finally J_{diff} by (5.31). With the same kind of choices of constant as in Step 6 of the proof of Proposition 5.5 (note that ζ_α and ε_α are the same) we end up with

$$\begin{aligned} e^{\varepsilon\nu_0 t} \left\|f(t)\right\|_{L^\infty_{x,v}(\mu^{-\zeta})} \leqslant & C_{\varepsilon,\alpha} \left\|f_0\right\|_{L^\infty_{x,v}(\mu^{-\zeta})} + \sup_{x,v}\left(C_\infty(x,v)\right) \sup_{s\in[0,t]} \left[e^{\varepsilon\nu_0 s} \left\|f(s)\right\|_{L^\infty_{x,v}(\mu^{-\zeta})}\right] \\ &+ C_{T_0} \int_0^t \left\|f(s)\right\|_{L^2_{x,v}(\mu^{-1/2})} ds \end{aligned}$$

where

$$C_\infty = C_2^{(\alpha)}(1 - e^{-\nu(v)(1-\varepsilon)\min\{t,t_1\}}) + \mathbf{1}_{\{t>t_1\}}(1-\alpha)\left(1 + \alpha(1 + C_0(1-\zeta))\right) e^{-\nu(v)(1-\varepsilon)t_1}$$

and thus, with our choice of constants,

$$C_\infty(x,v) \leqslant \max\left\{C_2^{(\alpha)},\ (1-\alpha)(1+\alpha)\right\} < 1$$

which implies

$$\forall t \in [0,T_0], \left\|f(t)\right\|_{L^\infty_{x,v}(\mu^{-\zeta})} \leqslant C_1 e^{-\varepsilon\nu_0 t} \left\|f_0\right\|_{L^\infty_{x,v}(\mu^{-\zeta})} + C_{T_0} \int_0^t \left\|f(s)\right\|_{L^2_{x,v}(\mu^{-1/2})} ds.$$

To conclude we choose T_0 large enough such that $C_1 e^{-\frac{\varepsilon}{2}\nu_0 T_0} \leqslant 1$ so that assumptions of the $L^2 - L^\infty$ theory of Proposition 5.4 are fulfilled so we can apply it, thus concluding the proof. □

6. Perturbative Cauchy theory for the full nonlinear equation

This section is dedicated to establishing a Cauchy theory for the perturbed equation

$$\partial_t f + v\cdot\nabla_x f = L(f) + Q(f,f), \tag{6.1}$$

together with the Maxwell boundary condition (1.2) in spaces $L^\infty_{x,v}(m)$, where m is a polynomial or a stretched exponential weight. More precisely we shall prove that for any small initial datum f_0 satisfying the conservation of mass $\Pi_G(f_0)=0$ there exists a unique solution to (6.1) and that this solution decays exponentially fast and also satisfies the conservation of mass.

We divide our study in two different subsections. First, for any small f_0, we build a solution that satisfies the conservation law and decays exponentially fast; this is the purpose of Subsection 6.1. Second, we prove the uniqueness to (6.1) when the initial datum is small in Subsection 6.2.

6.1. Existence of solutions with exponential decay.

The present subsection is dedicated to the following proof of existence.

Theorem 6.1. *Let α be in $(\sqrt{2/3},1]$ and let $m=e^{\kappa_1|v|^{\kappa_2}}$ with $\kappa_1>0$ and κ_2 in $(0,2)$ or $m=\langle v\rangle^k$ with $k>k_\infty$. There exists $\eta>0$ such that for any f_0 in $L^\infty_{x,v}(m)$ with*

$$\Pi_G(f_0)=0 \quad \text{and} \quad \|f_0\|_{L^\infty_{x,v}(m)}\leqslant\eta,$$

there exist at least one solution f to the Boltzmann equation (6.1) with Maxwell boundary condition and with f_0 as an initial datum. Moreover, f satisfies the conservation of mass and there exist C, $\lambda>0$ such that

$$\forall t\geqslant 0,\quad \|f(t)\|_{L^\infty_{x,v}(m)}\leqslant Ce^{-\lambda t}\|f_0\|_{L^\infty_{x,v}(m)}.$$

As explained in Section 2.1, we decompose (6.1) into a system of differential equations. More precisely, we shall decompose $G=L-v\cdot\nabla_x$ as $G=A+B$ in the spirit of [17], where B is "small" compared to $\nu(v)$ and A has a regularising effect. We then shall construct (f_1,f_2) solutions to the following system of equation

$$\partial_t f_1 = Bf_1+Q(f_1+f_2,f_1+f_2) \quad\text{and}\quad f_1(0,x,v)=f_0(x,v), \tag{6.2}$$

$$\partial_t f_2 = Gf_2+Af_1 \quad\text{and}\quad f_2(0,x,v)=0, \tag{6.3}$$

each of the functions satisfying the Maxwell boundary condition. Note that for such functions, the function $f=f_1+f_2$ would be a solution to (6.1) with Maxwell boundary condition and f_0 as initial datum.

Subsection 6.1.1 explicitly describes the decomposition $G=A+B$ and gives some estimates on A, B and Q. Subsections 6.1.2 and 6.1.3 deal with each differential equation (6.2) and (6.3) respectively. Finally, Subsection 6.1.4 combines the previous theories to construct a solution to the full nonlinear perturbed Boltzmann equation.

6.1.1. *Decomposition of the linear operator and first estimates.* We follow the decomposition proposed in [17].

For δ in $(0,1)$, we consider $\Theta_\delta = \Theta_\delta(v, v_*, \sigma)$ in C^∞ that is bounded by one everywhere, is exactly one on the set

$$\left\{|v| \leqslant \delta^{-1} \quad \text{and} \quad 2\delta \leqslant |v - v_*| \leqslant \delta^{-1} \quad \text{and} \quad |\cos\theta| \leqslant 1 - 2\delta\right\}$$

and whose support is included in

$$\left\{|v| \leqslant 2\delta^{-1} \quad \text{and} \quad \delta \leqslant |v - v_*| \leqslant 2\delta^{-1} \quad \text{and} \quad |\cos\theta| \leqslant 1 - \delta\right\}.$$

We define the splitting

$$G = A^{(\delta)} + B^{(\delta)},$$

with

$$A^{(\delta)}h(v) = C_\Phi \int_{\mathbb{R}^3\times\mathbb{S}^2} \Theta_\delta \left[\mu'_* h' + \mu' h'_* - \mu h_*\right] b\left(\cos\theta\right) |v - v_*|^\gamma \, d\sigma dv_*$$

and

$$B^{(\delta)}h(v) = B_2^{(\delta)}h(v) - \nu(v)h(v) - v\cdot\nabla_x h(v) = G_\nu h(v) + B_2^{(\delta)}h(v),$$

where

$$B_2^{(\delta)}h(v) = \int_{\mathbb{R}^3\times\mathbb{S}^2} (1 - \Theta_\delta) \left[\mu'_* h' + \mu' h'_* - \mu h_*\right] b\left(\cos\theta\right) |v - v_*|^\gamma \, d\sigma dv_*.$$

The following lemmas give control over the operators $A^{(\delta)}$ and $B^{(\delta)}$ for $m = e^{\kappa_1|v|^{\kappa_2}}$ with $\kappa_1 > 0$ and κ_2 in $(0,2)$ or $m = \langle v\rangle^k$ with $k > k_\infty$. Their proofs can be found in [17, Section 4] in the specific case of hard sphere ($b = \gamma = 1$) and for more general hard potential with cutoff kernels in [8, Section 6.1.1] (polynomial weight) or [4, Section 2] (exponential weight).

Lemma 6.2. *Let ζ be in $(1/2, 1]$. There exists $C_A > 0$ such that for all f in $L^\infty_{x,v}(m)$*

$$\left\|A^{(\delta)}(f)\right\|_{L^\infty_{x,v}(\mu^{-\zeta})} \leqslant C_A \left\|f\right\|_{L^\infty_{x,v}(m)}.$$

The constant C_A is constructive and only depends on m, ζ, δ and the collision kernel.

Lemma 6.3. *$B_2^{(\delta)}$ satisfies*

$$\forall f \in L^\infty_{x,v}(m), \quad \left\|B_2^{(\delta)}(f)\right\|_{L^\infty_{x,v}(\nu^{-1}m)} \leqslant C_B(\delta)\left\|f\right\|_{L^\infty_{x,v}(m)},$$

where $C_B(\delta) > 0$ is a constructive constant such that

- *if $m = \langle v\rangle^k$ then*

$$\lim_{\delta\to 0} C_B(\delta) = \frac{4}{k - 1 - \gamma}\frac{4\pi b_\infty}{l_b};$$

- *if $m = e^{\kappa_1|v|^{\kappa_2}}$ then*

$$\lim_{\delta\to 0} C_B(\delta) = 0.$$

The operator $B_2^{(\delta)}$ also has a smallness property as an operator from $L^\infty_{x,v}(m)$ to $L^1_v L^\infty_v(|v|^2)$. The following lemma is from [3, Lemma 5.7] in the case of polynomial weights m but is utterly applicable in the case of stretched exponential weights.

Lemma 6.4. *For any $\delta > 0$ there exists $\widetilde{C}_B(\delta)$ such that for all f in $L^\infty_{x,v}(m)$,*

$$\left\| B_2^{(\delta)}(f) \right\|_{L^1_v L^\infty_x(\langle v\rangle^2)} \leqslant \widetilde{C}_B(\delta) \left\| f \right\|_{L^\infty_{x,v}(m)}.$$

Moreover, the following holds: $\lim_{\delta\to 0} \widetilde{C}_B(\delta) = 0$.

Remark 6.5. *We emphasize here that for our choices of weights (see definition of k_∞ (2.1)),* $\lim_{\delta\to 0} C_B(\delta) = C_B(m) < 1$. *Until the end of the present section we only consider $0 < \delta$ small enough such that $C_B(\delta) < 1$.*

We conclude this subsection with a control on the bilinear term in the $L^\infty_{x,v}$ setting.

Lemma 6.6. *For all h and g such that $Q(h,g)$ is well-defined, $Q(h,g)$ belongs to $[Ker(L)]^\perp$ in L^2_v:*

$$\pi_L\left(Q(h,g)\right) = 0.$$

Moreover, there exists $C_Q > 0$ such that for all h and g,

$$\left\| Q(h,g) \right\|_{L^\infty_{x,v}(\nu^{-1}m)} \leqslant C_Q \left\| h \right\|_{L^\infty_{x,v}(m)} \left\| g \right\|_{L^\infty_{x,v}(m)}.$$

The constant C_Q is explicit and depends only on m and the kernel of the collision operator.

Proof of Lemma 6.6. Since we use the symmetric definition of Q (1.6) the orthogonality property can be found in [5, Appendix A.2.1]. The estimate follows directly from [17, Lemma 5.16] and the fact that $\nu(v) \sim \langle v\rangle^\gamma$ (see (3.6)). □

6.1.2. *Study of equation (6.2) in $L^\infty_{x,v}(m)$.* In the section we study the differential equation (6.2). We prove well-posedness for this problem and above all exponential decay as long as the initial datum is small. We deal with the different types of weights m in the same way. Since the "operator norm" of $B_2^{(\delta)}$ tends to zero as δ tends to zero in the case of stretched exponential weight, one has a more direct proof in this case and we refer to [3, Section 5.2.2], where the author dealt with pure diffusion, for the interested reader.

Proposition 6.7. *Let $m = e^{\kappa_1 |v|^{\kappa_2}}$ with $\kappa_1 > 0$ and κ_2 in $(0,2)$ or $m = \langle v\rangle^k$ with $k > k_\infty$. Let f_0 be in $L^\infty_{x,v}(m)$ and $g(t,x,v)$ in $L^\infty_{x,v}(m)$. Then there exists $\delta_m > 0$ such that for any δ in $(0,\delta_m]$ there exist C_1, η_1 and $\lambda_1 > 0$ such that if*

$$\left\| f_0 \right\|_{L^\infty_{x,v}(m)} \leqslant \eta_1 \quad \text{and} \quad \left\| g \right\|_{L^\infty_t L^\infty_{x,v}(m)} \leqslant \eta_1,$$

then there exists a solution f_1 to

$$\partial_t f_1 = G_\nu f_1 + B_2^{(\delta)} f_1 + Q(f_1 + g, f_1 + g), \tag{6.4}$$

with initial datum f_0 and satisfying the Maxwell boundary condition (1.2). Moreover, this solution satisfies

$$\forall t \geqslant 0, \quad \left\| f_1(t) \right\|_{L^\infty_{x,v}(m)} \leqslant C_1 e^{-\lambda_1 t} \left\| f_0 \right\|_{L^\infty_{x,v}(m)}.$$

The constants C_1, η_1 and λ_1 are constructive and only depend on m, δ and the kernel of the collision operator.

Proof of Proposition 6.7. Thanks to Proposition 4.1, G_ν combined with the Maxwell boundary condition generates a semigroup $S_{G_\nu}(t)$ in $L^\infty_{x,v}(m)$. Therefore if f_1 is solution to (6.4) then it has the following Duhamel representation almost everywhere in $\mathbb{R}^+ \times \Omega \times \mathbb{R}^3$

$$\begin{aligned} f_1(t,x,v) =& S_{G_\nu}(t) f_0(x,v) + \int_0^t S_{G_\nu}(t-s) \left[B_2^{(\delta)}(f_1(s)) \right](x,v)\, ds \\ &+ \int_0^t S_{G_\nu}(t-s) \left[Q(f_1(s)+g(s), f_1(s)+g(s)) \right](x,v))\, ds. \end{aligned}$$

To prove existence and exponential decay we use the following iteration scheme starting from $h_0 = 0$.

$$\left\{ \begin{array}{l} h_{l+1} = S_{G_\nu}(t) f_0 + \displaystyle\int_0^t S_{G_\nu}(t-s) \left[B_2^{(\delta)}(h_{l+1}) + Q(h_l+g, h_l+g) \right] ds \\ h_{l+1}(0,x,v) = f_0(x,v). \end{array} \right. \tag{6.5}$$

A contraction argument on the Duhamel representation above would imply that $(h_l)_{l\in\mathbb{N}}$ is well-defined in $L^\infty_{x,v}(m)$ and satisfies the Maxwell boundary condition (because S_{G_ν} does). The computations to prove this contraction property are similar to the ones we are about to develop in order to prove that $(h_l)_{l\in\mathbb{N}}$ is a Cauchy sequence and we therefore only write down the latter.

Considering the difference $h_{l+1} - h_l$ we write, since Q is a symmetric bilinear operator,

$$\begin{aligned} h_{l+1}(t,x,v) - h_l(t,x,v) =& \int_0^t S_{G_\nu}(t-s) \left[B_2^{(\delta)}(h_{l+1}(s) - h_l(s)) \right] ds \\ &+ \int_0^t S_{G_\nu}(t-s) \left[Q(h_l - h_{l-1}, h_l + h_{l-1} + g)(s,x,v) \right] ds \end{aligned} \tag{6.6}$$

As now usual, we write $S_{G_\nu}(t-s)$ under its implicit form after one rebound against the boundary (see (4.4) or Step 1 of the proof of Proposition 5.5). It reads

$$\begin{aligned} &\int_0^t S_{G_\nu}(t-s) \left[B_2^{(\delta)}(h_{l+1}(s) - h_l(s)) \right](x,v)\, ds \\ &= \int_{\max\{0,t-t_1\}}^t e^{-\nu(v)(t-s)} B_2^{(\delta)} \left(h_{l+1}(s) - h_l(s) \right)(x-(t-s)v, v)\, ds \\ &+ (1-\alpha) e^{-\nu(v)t_1} \mathbf{1}_{\{t>t_1\}} \int_0^{t-t_1} S_{G_\nu}((t-t_1)-s) B_2^{(\delta)} \left(h_{l+1}(s) - h_l(s) \right)(x_1, v_1)\, ds \\ &+ \alpha\mu e^{-\nu(v)t_1} \mathbf{1}_{\{t>t_1\}} \int_0^{t-t_1} \int_{\mathcal{V}_1} \frac{1}{\mu(v_{1*})} S_{G_\nu}(t-t_1-s) B_2^{(\delta)} \left(h_{l+1} - h_l \right)(x_1, v_{1*}) d\sigma_{x_1} ds, \end{aligned}$$

where $t_1 = t_{min}(x,v)$, $x_1 = x - t_1 v$ and $v_1 = \mathcal{R}_{x_1}(v)$. Using the decomposition $S_{G_\nu}(f) = I_p(f) + R_p(S_{G_\nu}(f))$ given by Lemma 4.4, we obtain for all $p \geqslant 1$:

$$m(v) \left| h_{l+1} - h_l \right|(t,x,v) = J_B + J_{t_1} + J_{IB} + J_{RB} + J_Q \tag{6.7}$$

with the following definitions

$$J_B = \int_{\max\{0,t-t_1\}}^{t} m(v)e^{-\nu(v)(t-s)} \left|B_2^{(\delta)}\left(h_{l+1}(s)-h_l(s)\right)\right|(x-(t-s)v,v)\,ds, \tag{6.8}$$

$$J_{t_1} = (1-\alpha)e^{-\nu(v)t_1}\mathbf{1}_{\{t>t_1\}}m(v)\left|h_{l+1}-h_l\right|(t-t_1,x_1,v_1), \tag{6.9}$$

$$\begin{aligned} J_{IB} =& \alpha\mu e^{-\nu(v)t_1}\mathbf{1}_{\{t>t_1\}}\int_0^{t-t_1}\int_{\mathcal{V}_1}\frac{m(v)}{\mu(v_{1*})} \\ &\times\left|I_p\left(B_2^{(\delta)}\left(h_{l+1}-h_l\right)\right)(t-t_1-s,x_1,v_{1*})\right|d\sigma_{x_1}(v_{1*})ds, \end{aligned} \tag{6.10}$$

$$\begin{aligned} J_{RB} =& \alpha\mu e^{-\nu(v)t_1}\mathbf{1}_{\{t>t_1\}}\int_0^{t-t_1}\int_{\mathcal{V}_1}\frac{m(v)}{\mu(v_{1*})} \\ &\times\left|R_p\left(S_{G_\nu}B_2^{(\delta)}\left(h_{l+1}-h_l\right)\right)(t-t_1-s,x_1,v_{1*})\right|d\sigma_{x_1}(v_{1*})ds, \end{aligned} \tag{6.11}$$

$$\begin{aligned} J_Q =& \int_0^t m(v)\left|S_{G_\nu}(t-s)\left[Q(h_l-h_{l-1},h_l+h_{l-1}+g)(s,x,v)\right]\right|ds + \alpha\mathbf{1}_{\{t>t_1\}} \\ &\times\mu e^{-\nu(v)t_1}\int_0^{t-t_1}\left|S_{G_\nu}(t-t_1-s)\left[Q(h_l-h_{l-1},h_l+h_{l-1}+g)(s,x_1,v_1)\right]\right|ds. \end{aligned} \tag{6.12}$$

We now estimate each of these terms separately.

Estimate for $\mathbf{J_B}$ and $\mathbf{J_{t_1}}$ These two terms are connected *via* t_1 and it is important to understand that the contributions of J_B and J_{t_1} are interchanging.

By crudely bounding the integrand of (6.8) by the $L^\infty_{x,v}$-norm and controlling $B_2^{(\delta)}$ by Lemma 6.3,

$$\begin{aligned} m(v)\left|B_2^{(\delta)}\left(h_{l+1}-h_l\right)\right|(x-(t-s)v,v) &\leqslant \nu(v)\left\|B_2^{(\delta)}\left(h_{l+1}(s)-h_l(s)\right)\right\|_{L^\infty_{x,v}(m\nu^{-1})} \\ &\leqslant C_B(\delta)\nu(v)\left\|h_{l+1}(s)-h_l(s)\right\|_{L^\infty_{x,v}(m)}. \end{aligned}$$

For ε in $(0,1)$ we have $e^{-\nu(v)(t-s)}\leqslant e^{-\varepsilon\nu_0 t}e^{-\nu(v)(1-\varepsilon)(t-s)}e^{\varepsilon\nu_0 s}$ and thus

$$J_B \leqslant \frac{C_B(\delta)}{1-\varepsilon}\left(1-e^{-\nu(v)(1-\varepsilon)\min\{t,t_1\}}\right)e^{-\varepsilon\nu_0 t}\sup_{s\in[0,t]}\left[e^{\varepsilon\nu_0 s}\left\|h_{l+1}(s)-h_l(s)\right\|_{L^\infty_{x,v}(m)}\right]. \tag{6.13}$$

For J_{t_1} we notice in (6.9) that $|v|=|v_1|$ and therefore $m(v)=m(v_1)$. Also, for $t>t_1$ and all ε in $(0,1)$ we have $e^{-\nu(v)t_1}\leqslant e^{-(1-\varepsilon)\nu(v)t_1}e^{-\varepsilon\nu_0 t}e^{\varepsilon\nu_0(t-t_1)}$ and therefore

$$J_{t_1}\leqslant(1-\alpha)e^{-(1-\varepsilon)\nu(v)t_1}e^{-\varepsilon\nu_0 t}\mathbf{1}_{\{t>t_1\}}\left[e^{\varepsilon\nu_0(t-t_1)}\left\|(h_{l+1}-h_l)(t-t_1)\right\|_{L^\infty_{x,v}(m)}\right]. \tag{6.14}$$

From (6.13) and (6.14) we deduce

$$\forall\varepsilon\in(0,1),\quad J_B+J_{t_1}\leqslant\min\left\{(1-\alpha),\frac{C_B(\delta)}{1-\varepsilon}\right\}e^{-\varepsilon\nu_0 t}\sup_{s\in[0,t]}\left[e^{\varepsilon\nu_0 s}\left\|h_{l+1}-h_l\right\|_{L^\infty_{x,v}(m)}\right]. \tag{6.15}$$

Estimate for $\mathbf{J_{IB}}$ Using Lemma 4.4 (replacing Δ_n by α) to control I_p in (6.10) and bounding the exponential decay in $d\Sigma_l^k$ by $e^{-\nu_0(t-t_1-s-t_k^{(l)}}$ gives the following control

$$J_{IB} \leqslant \int_0^{t-t_1} e^{-\nu_0(t-s)} \int_{\mathcal{V}_1} \sum_{k=0}^{p} \sum_{i=0}^{k} \sum_{l\in\vartheta_k(i)} (1-\alpha)^i \alpha^{k-i} \int_{\prod_{j=1}^p \mathcal{V}_j} \frac{\mu(v)m(v)}{\mu(v_k^{(l)})} \times \left| B_2^{(\delta)} \left(h_{l+1}-h_l\right)\left(s, x_k^{(l)} - t_k^{(l)} v_k^{(l)}, v_k^{(l)}\right)\right| \prod_{0\leqslant j\leqslant k} d\sigma_{x_j^{(l)}}(v_{j*}) ds,$$

where the sequence $(t_k^{(l)}, x_k^{(l)}, v_k^{(l)})$ is associated to the initial point $(t-t_1-s, x_1, v_{1*})$.

For a given (k,i,l) there exists a unique J in $\{0,p\}$ such that $v_k^{(l)} = V_1(\dots(V_1(x_j, v_{J*})))$ $k-J$ iterations and thus $(t_k^{(l)}, x_k^{(l)}, v_k^{(l)})$ only depends on $(t-t_1-s, x, v, v_{1*}, \dots, v_{J*})$ and we can integrate the remaining variables. We remind here that $d\sigma_{x_j}$ is a probability measure on $\mathcal{V}_j$ and also

$$d\sigma_{x_J}(v_{J*}) = c_\mu \mu(v_{J*}) v_{J*}\cdot n(x_J) dv_{J*}.$$

Since $\left|v_k^{(l)}\right| = |v_{J*}|$ and $\mu(v)m(v) \leqslant C$, we infer the following,

$$J_{IB} \leqslant C \int_0^{t-t_1} e^{-\nu_0(t-s)} \int_{\mathcal{V}_1} \sum_{k=0}^{p} \sum_{i=0}^{k} \sum_{l\in\vartheta_k(i)} (1-\alpha)^i \alpha^{k-i} \int_{\prod_{j=1}^{J-1}\mathcal{V}_j} ds \prod_{0\leqslant j\leqslant J-1} d\sigma_{x_j^{(l)}}(v_{j*}) \times \left(\int_{\mathbb{R}^3} \left| B_2^{(\delta)} \left(h_{l+1}-h_l\right)\left(s, x_k^{(l)} - t_k^{(l)} v_k^{(l)}, v_k^{(l)}\right)\right| |v_{J*}|\, dv_{J*}\right).$$

We now make the change of variable $v_k^{(l)} \mapsto v_{J*}$ which preserves the norm and then the integral in v_{J*} can be bounded by the $L^1_v L^\infty_x(\langle v\rangle^2)$-norm of $B_2^{(\delta)}$. As in the proof of Lemma 4.5 $\sum_{k,i,l}(1-\alpha)^i\alpha^{k-i} \leqslant p$ and this yields the following estimate

$$J_{IB} \leqslant pC \int_0^t e^{-\nu_0(t-s)} \left\| B_2^{(\delta)} \left(h_{l+1}-h_l\right)(s)\right\|_{L^1_v L^\infty_x(\langle v\rangle^2)} ds.$$

To conclude the estimate on J_{IB} we choose $p = p(T_0)$ defined in Lemma 4.6 (which makes p bounded by $C(1+T_0)$) and we control $B_2^{(\delta)}$ thanks to Lemma 6.4.

$$J_{IB} \leqslant C(1+T_0)\widetilde{C}_B(\delta) e^{-\varepsilon\nu_0 t} \sup_{s\in[0,t]} \left[e^{\varepsilon\nu_0 s} \left\|(h_{l+1}-h_l)(s)\right\|_{L^\infty_{x,v}(m)}\right] \tag{6.16}$$

for all ε in $(0,1)$, $T_0 \geqslant 0$ and t in $[0,1]$.

Estimate for $\mathbf{J_{RB}}$ The term (6.11) is dealt with by crudely bounding the integrand of (6.11) by its $L^\infty_{x,v}$-norm. Using Lemma 4.6 to estimate R_p we get for all t in $[0,T_0]$,

$$\begin{aligned} &\frac{\mu(v)}{\mu(v_{1*})} e^{-\nu(v)t_1} m \left| R_p\left(S_{G_\nu} B_2^{(\delta)}(h_{l+1}-h_l)\right)(t-t_1-s, x_1, v_{1*})\right| \\ &\leqslant C \frac{\mu(v)m(v)}{\mu(v_{1*})m(v_{1*})} \nu(v_{1*}) e^{-\nu_0(t-s)} \left(\frac{1}{2}\right)^{[\overline{C}T_0]} \\ &\quad \times \sup_{s_*\in[0,t-t_1-s]} \left[e^{\nu_0 s_*} \left\| S_{G_\nu}(s_*)\left[B_2^{(\delta)}(h_{l+1}-h_l)\right]\right\|_{L^\infty_{x,v}(m\nu^{-1})}\right]. \end{aligned}$$

Then, we apply Theorem 4.1 about the exponential decay of $S_{G_\nu}(s_*)$ in $L^\infty_{x,v}(m\nu^{-1})$ with a rate $(1-\varepsilon)\nu_0$. At last, we control $B_2^{(\delta)}$ thanks to Lemma 6.3. This yields,

$$J_{RB} \leqslant C\left(\frac{1}{2}\right)^{[\overline{C}T_0]} C_B(\delta) e^{-\varepsilon\nu_0 t} \sup_{s\in[0,t]} \left[e^{\varepsilon\nu_0 s}\, \|(h_{l+1}-h_l)(s)\|_{L^\infty_{x,v}(m)}\right] \tag{6.17}$$

for all ε in $(0,1)$, $T_0 \geqslant 0$ and t in $[0,1]$.

Estimate for $\mathbf{J_Q}$ The term (6.12) is dealt with using the gain of weight of $S_{G_\nu}(t)$ upon integration in time: Corollary 4.2. A direct application of this corollary yields for all ε in $(0,1)$

$$J_Q \leqslant \frac{C_0}{1-\varepsilon} e^{-\varepsilon\nu_0 t} \sup_{s\in[0,t]} \left[e^{\varepsilon\nu_0 s}\, \|Q(h_l-h_{l-1}, h_l+h_{l-1}+g)(s)\|_{L^\infty_{x,v}(m\nu^{-1})}\right].$$

We control Q thanks to Lemma 6.6 and we infer

$$\begin{aligned}\forall \varepsilon\in(0,1),\quad J_Q \leqslant & \frac{C}{1-\varepsilon}\left(\|h_l\|_{L^\infty_{[0,t]}L^\infty_{x,v}(m)} + \|h_{l-1}\|_{L^\infty_{[0,t]}L^\infty_{x,v}(m)} + \|g\|_{L^\infty_t L^\infty_{x,v}(m)}\right)\\ & \times e^{-\varepsilon\nu_0 t} \sup_{s\in[0,t]} \left[e^{\varepsilon\nu_0 s}\, \|(h_l-h_{l-1})(s)\|_{L^\infty_{x,v}(m)}\right].\end{aligned} \tag{6.18}$$

Conclusion of the proof. We gather (6.15), (6.16), (6.17) and (6.12) inside (6.7). This gives for all $0<\varepsilon<1$, all $T_0>0$ and all t in $[0,T_0]$,

$$\begin{aligned}&\sup_{s\in[0,t]} \left[e^{\varepsilon\nu_0 s}\, \|(h_{l+1}-h_l)(s)\|_{L^\infty_{x,v}(m)}\right]\\ &\leqslant \left(\min\left\{(1-\alpha), \frac{C_B(\delta)}{1-\varepsilon}\right\} + C(1+T_0)\widetilde{C}_B(\delta) + 2^{-[\overline{C}T_0]}C\right)\\ &\qquad\times \sup_{s\in[0,t]} \left[e^{\varepsilon\nu_0 s}\, \|h_{l+1}-h_l\|_{L^\infty_{x,v}(m)}\right]\\ &\quad + \frac{C}{1-\varepsilon}\left(\|h_l\|_{L^\infty_{[0,t]}L^\infty_{x,v}(m)} + \|h_{l-1}\|_{L^\infty_{[0,t]}L^\infty_{x,v}(m)} + \|g\|_{L^\infty_t L^\infty_{x,v}(m)}\right)\\ &\qquad\times \sup_{s\in[0,t]} \left[e^{\varepsilon\nu_0 s}\, \|h_l-h_{l-1}\|_{L^\infty_{x,v}(m)}\right]\end{aligned}$$

We choose our constants as follow.

- From Lemma 6.3 we define $\delta_0>0$ such that for all $\delta<\delta_0$, $C_B(\delta)\leqslant C_B(\delta_0)<1$;
- Since $C_B(\delta_0)<1$ we fix ε in $(0,1)$ such that $C_B(\delta_0)+\varepsilon<1$;
- We choose T_0 large enough such that

$$2^{-[\overline{C}T_0]}C \leqslant \frac{1}{4}\left(1-\min\left\{(1-\alpha), \frac{C_B(\delta_0)}{1-\varepsilon}\right\}\right);$$

- At last, we take $\delta<\delta_0$ such that, from Lemma 6.4,

$$C(1+T_0)\widetilde{C}_B(\delta) \leqslant \frac{1}{4}\left(1-\min\left\{(1-\alpha), \frac{C_B(\delta_0)}{1-\varepsilon}\right\}\right).$$

Denoting

$$C_0 = 2C\left(1 - \min\left\{(1-\alpha), \frac{C_B(\delta_0)}{1-\varepsilon}\right\}\right)^{-1},$$

our choice of constants implies that for all t in $[0, T_0]$,

$$\begin{aligned}
&\sup_{s\in[0,t]}\left[e^{\varepsilon\nu_0 s}\left\|h_{l+1}-h_l\right\|_{L^\infty_{x,v}(m)}\right] \\
&\leqslant C_0\left(\left\|h_l\right\|_{L^\infty_{[0,t]}L^\infty_{x,v}(m)} + \left\|h_{l-1}\right\|_{L^\infty_{[0,t]}L^\infty_{x,v}(m)} + \left\|g\right\|_{L^\infty_t L^\infty_{x,v}(m)}\right) \\
&\quad\times \sup_{s\in[0,t]}\left[e^{\varepsilon\nu_0 s}\left\|h_l-h_{l-1}\right\|_{L^\infty_{x,v}(m)}\right].
\end{aligned} \tag{6.19}$$

Of important note is the fact that C_0 and ε do not depend on T_0 and therefore we can iterate the process on $[T_0, 2T_0]$ and so on. The inequality above thus holds for all $t \geqslant 0$.

To conclude, we first prove that $\|h_{l+1}\|_{L^\infty_{[0,t]}L^\infty_{x,v}(m)}$ is uniformly bounded. We could do exactly the same computations but subtracting $S_{G_\nu}(t)f_0(x,v)$ instead of $h_l(t,x,v)$ to $h_l(t,x,v)$ in (6.6). Thus, (6.19) would become for all $t \geqslant 0$,

$$\begin{aligned}
\sup_{s\in[0,t]}\left[e^{\varepsilon\nu_0 s}\left\|(h_{l+1}-S_{G_\nu}f_0)(s)\right\|_{L^\infty_{x,v}(m)}\right] \leqslant & C_0\left(\left\|h_l\right\|_{L^\infty_{[0,t]}L^\infty_{x,v}(m)} + \left\|g\right\|_{L^\infty_t L^\infty_{x,v}(m)}\right) \\
&\times \sup_{s\in[0,t]}\left[e^{\varepsilon\nu_0 s}\left\|h_l\right\|_{L^\infty_{x,v}(m)}\right].
\end{aligned}$$

Using the exponential decay of $S_{G_\nu}(t)f_0$, which is faster than $\varepsilon\nu_0$ (see Theorem 4.1), and assuming that the norms of g and f_0 are bounded by η_1 to be determined later,

$$\begin{aligned}
&\sup_{s\in[0,t]}\left[e^{\varepsilon\nu_0 s}\left\|h_{l+1}(s)\right\|_{L^\infty_{x,v}(m)}\right] \\
&\leqslant C_0^{(1)}\left\|f_0\right\|_{L^\infty_{x,v}(m)} + C_0^{(2)}\left(\left\|h_l\right\|_{L^\infty_{[0,t]}L^\infty_{x,v}(m)} + \eta_1\right)\sup_{s\in[0,t]}\left[e^{\varepsilon\nu_0 s}\left\|h_l(s)\right\|_{L^\infty_{x,v}(m)}\right],
\end{aligned}$$

where $C_0^{(1)}$ and $C_0^{(2)}$ are two positive constants independent of h_{l+1} and η_1.

If we choose $\eta_1 > 0$ small enough such that

$$C_0^{(1)}\eta_1 + C_0^{(2)}\left(2 + C_0^{(1)}\right)\left(1 + C_0^{(1)}\right)\eta_1^2 \leqslant (1 + C_0^{(1)})\eta_1$$

then we obtain by induction that

$$\forall l \in \mathbb{N},\ \forall t \geqslant 0, \quad \sup_{s\in[0,t]}\left[e^{\varepsilon\nu_0 s}\left\|h_l(s)\right\|_{L^\infty_{x,v}(m)}\right] \leqslant \left(1 + C_0^{(1)}\right)\left\|f_0\right\|_{L^\infty_{x,v}(m)}. \tag{6.20}$$

We now plug (6.20) into (6.19) and use the fact that f_0 and g are bounded by η_1. This gives

$$\left\|h_{l+1}-h_l\right\|_{L^\infty_{t,x,v}(m)} \leqslant 3C_0\left(1 + C_0^{(1)}\right)\eta_1\left\|h_l-h_{l-1}\right\|_{L^\infty_{t,x,v}(m)}.$$

The latter implies that for η_1 small enough, $(h_l)_{l\in\mathbb{N}}$ is a Cauchy sequence in $L^\infty_{t,x,v}(m)$ and therefore converges towards f_1 in $L^\infty_{t,x,v}(m)$. Since $\gamma < k$, we can take the limit inside the iterative scheme (6.5) and f_1 is a solution to (6.4). Moreover, by taking the limit inside (6.20), f_1 has the desired exponential decay. This concludes the proof of Proposition 6.7. □

6.1.3. *Study of equation* (6.3) *in* $L^\infty_{x,v}(\mu^{-\zeta})$. We turn to the differential equation (6.3) in $L^\infty_{x,v}(\mu^{-\zeta})$ with ζ in $(1/2,1)$ so that Theorem 5.1 holds.

Proposition 6.8. *Let* $m = e^{\kappa_1|v|^{\kappa_2}}$ *with* $\kappa_1 > 0$ *and* κ_2 *in* $(0,2)$ *or* $m = \langle v\rangle^k$ *with* $k > k_\infty$. *Let* $g = g(t,x,v)$ *be in* $L^\infty_t L^\infty_{x,v}(m)$. *Then there exists a unique function* f_2 *in* $L^\infty_t L^\infty_{x,v}(\mu^{-\zeta})$ *such that*

$$\partial_t f_2 = G(f_2) + A^{(\delta)}(g) \quad \text{and} \quad f_2(0,x,v) = 0.$$

Moreover, if $\Pi_G(f_2+g) = 0$ *and if*

$$\exists\, \lambda_g,\ \eta_g > 0,\ \forall t \geqslant 0,\ \|g(t)\|_{L^\infty_{x,v}(m)} \leqslant \eta_g e^{-\lambda_g t},$$

then for any $0 < \lambda_2 < \min\{\lambda_g,\ \lambda_\infty\}$, *with* λ_∞ *defined in Theorem 5.1, there exist* $C_2 > 0$ *such that*

$$\forall t \geqslant 0, \quad \|f_2(t)\|_{L^\infty_{x,v}(\mu^{-\zeta})} \leqslant C_2 \eta_g e^{-\lambda_2 t}.$$

The constant C_2 *only depends on* λ_2.

Proof of Proposition 6.8. Thanks to the regularising property of A (Lemma 6.2) $A^{(\delta)}(g)$ belongs to $L^\infty_t L^\infty_{x,v}(\mu^{-\zeta})$. Theorem 5.1 implies that there is indeed a unique f_2 solution to the differential equation and it is given by

$$f_2 = \int_0^t S_G(t-s)\left[A^{(\delta)}(g)(s)\right] ds,$$

where $S_G(t)$ is the semigroup generated by $G = L - v\cdot\nabla_x$ in $L^\infty_{x,v}(\mu^{-\zeta})$.

Suppose now that $\Pi_G(f_2+g) = 0$ and that there exists $\eta_2 > 0$ such that $\|g(t)\|_{L^\infty_{x,v}(m)} \leqslant \eta_g e^{-\lambda t}$. Using the definition of Π_G (3.8), the projection part of f_2 is straightforwardly bounded for all $t \geqslant 0$:

$$\begin{aligned} \|\Pi_G(f_2)(t)\|_{L^\infty_{x,v}(\mu^{-\zeta})} &= \|\Pi_G(g)(t)\|_{L^\infty_{x,v}(\mu^{-\zeta})} \leqslant C_{\Pi_G}\|g\|_{L^\infty_{x,v}(m)} \\ &\leqslant C_{\Pi_G}\eta_g\, e^{-\lambda_g t}. \end{aligned} \tag{6.21}$$

Applying $\Pi^\perp_G = \mathrm{Id} - \Pi_G$ to the equation satisfied by f_2 we get, thanks to (3.8),

$$\partial_t\left[\Pi^\perp_G(f_2)\right] = g\left[\Pi^\perp_G(f_2)\right] + \Pi^\perp_G\left(A^{(\delta)}(g)\right).$$

This yields

$$\Pi^\perp_G(f_2) = \int_0^t S_G(t-s)\left[\Pi^\perp_G\left(A^{(\delta)}(g)\right)(s)\right] ds.$$

We use the exponential decay of $S_G(t)$ on $(\mathrm{Ker}(g))^\perp$ (see Theorem 5.1).

$$\left\|\Pi^\perp_G(f_2)\right\|_{L^\infty_{x,v}(\mu^{-\zeta})} \leqslant C_\infty \int_0^t e^{-\lambda_\infty(t-s)}\left\|A^{(\delta)}(g)(s)\right\|_{L^\infty_{x,v}(\mu^{-\zeta})} ds.$$

Using the definition of Π_G (3.8) and then the regularising property of A Lemma 6.2 we further can further bound. Fix $\lambda_2 < \min\{\lambda_\infty,\ \lambda_g\}$,

$$\begin{aligned} \left\|\Pi^\perp_G(f_2)\right\|_{L^\infty_{x,v}(\mu^{-\zeta})} &\leqslant C_G C_\infty C_{\Pi_G} C_A C_g \eta_g \int_0^t e^{-\lambda_\infty(t-s)} e^{-\lambda_g s}\, ds \\ &\leqslant C_G C_\infty C_{\Pi_G} C_A C_g \eta_g\, t e^{-\min\{\lambda_g,\lambda_\infty\}t} \\ &\leqslant C_2(\lambda_2)\eta_g e^{-\lambda_2 t}. \end{aligned} \tag{6.22}$$

Gathering (6.21) and (6.22) yields the desired exponential decay. □

6.1.4. *Proof of Theorem 6.1.* Take f_0 in $L^\infty_{x,v}(m)$ such that $\Pi_G(f_0) = 0$.

The existence will be proved by an iterative scheme. We start with $f_1^{(0)} = f_2^{(0)} = 0$ and we approximate the system of equations (6.2) − (6.3) as follows.

$$\begin{aligned} \partial_t f_1^{(n+1)} &= B^{(\delta)}\left(f_1^{(n+1)}\right) + Q\left(f_1^{(n+1)} + f_2^{(n)}\right) \\ \partial_t f_2^{(n+1)} &= G\left(f_2^{(n+1)}\right) + A^{(\delta)}\left(f_1^{(n+1)}\right), \end{aligned}$$

with the following initial data

$$f_1^{(n+1)}(0,x,v) = f_0(x,v) \quad \text{and} \quad f_2^{(n+1)}(0,x,v) = 0.$$

Assume that $(1 + C_2)\|f_0\| \leqslant \eta_1$, where C_2 was defined in Proposition 6.8 and η_1 was defined in Proposition 6.7. Thanks to Proposition 6.7 and Proposition 6.8, an induction proves first that $\left(f_1^{(n)}\right)_{n\in\mathbb{N}}$ and $\left(f_2^{(n)}\right)_{n\in\mathbb{N}}$ are well-defined sequences and second that for all n in $\mathbb{N}$ and all $t \geqslant 0$

$$\left\|f_1^{(n)}(t)\right\|_{L^\infty_{x,v}(m)} \leqslant e^{-\lambda_1 t}\|f_0\|_{L^\infty_{x,v}(m)} \tag{6.23}$$

$$\left\|f_2^{(n)}(t)\right\|_{L^\infty_{x,v}(\mu^{-\zeta})} \leqslant C_2 e^{-\lambda_2 t}\|f_0\|_{L^\infty_{x,v}(m)}, \tag{6.24}$$

with $\lambda_2 < \min\{\lambda_1, \lambda_\infty\}$. Indeed, if we constructed $f_1^{(n)}$ and $f_2^{(n)}$ satisfying the exponential decay above then we can construct $f_1^{(n+1)}$, which has the required exponential decay (6.23), and then construct $f_2^{(n+1)}$. Finally, we have the following equality

$$\partial_t\left(f_1^{(n+1)} + f_2^{(n+1)}\right) = g\left(f_1^{(n+1)} + f_2^{(n+1)}\right) + Q\left(f_1^{(n+1)} + f_2^{(n)}\right).$$

Thanks to orthogonality property of Q in Lemma 6.6 and the definition of Π_G (3.8) we obtain that the projection is constant with time and thus

$$\Pi_G\left(f_1^{(n+1)} + f_2^{(n+1)}\right) = \Pi_G(f_0) = 0.$$

Applying Proposition 6.8 we obtain the exponential decay (6.24) for $f_2^{(n+1)}$.

We recognize exactly the same iterative scheme for $f_1^{(n+1)}$ as in the proof of Proposition 6.7 with g replaced by $f_2^{(n)}$. Moreover, the uniform bound (6.24) allows us to derive the same estimates as in the latter proof independently of $f_2^{(n)}$. As a conclusion, $\left(f_1^{(n)}\right)_{n\in\mathbb{N}}$ is a Cauchy sequence in $L^\infty_t L^\infty_{x,v}(m)$ and therefore converges strongly towards a function f_1.

By (6.24), the sequence $\left(f_2^{(n)}\right)_{n\in\mathbb{N}}$ is bounded in $L^\infty_t L^\infty_{x,v}(\mu^{-\zeta})$ and is therefore weakly-* compact and therefore converges, up to a subsequence, weakly-* towards f_2 in $L^\infty_t L^\infty_{x,v}(\mu^{-\zeta})$.

Since the kernel inside the collision operator behaves like $|v - v_*|^\gamma$ and that our weight $m(v)$ is either exponetial or of degree $k > 2 > \gamma$, we can take the weak limit inside the iterative scheme. This implies that (f_1, f_2) is solution to the system (6.2) − (6.3) and thus $f = f_1 + f_2$ is solution to the perturbed equation (6.1).

Moreover, taking the limit inside the exponential decays (6.23) and (6.24) yields the expected exponential decay for f.

6.2. **Uniqueness in the perturbative framework.** We conclude the proof of our main Theorem 2.1 stated in Section 2 by proving the uniqueness of solutions in the perturbative regime.

Theorem 6.9. *Let $m = e^{\kappa_1 |v|^{\kappa_2}}$ with $\kappa_1 > 0$ and κ_2 in $(0,2)$ or $m = \langle v \rangle^k$ with $k > k_\infty$. There exits $\eta > 0$ such that for any f_0 in $L^\infty_{x,v}(m)$ such that $\|f_0\|_{L^\infty_{x,v}(m)} \leqslant \eta$ there exists at most one solution $f(t,x,v)$ in $L^\infty_t L^\infty_{x,v}(m)$ to the perturbed Boltzmann equation (6.1) with Maxwell boundary condition and with f_0 as initial datum.*

Proof of Theorem 6.9. Let f_0 be in $L^\infty_{x,v}(m)$ such that $\|f_0\|_{L^\infty_{x,v}(m)} \leqslant \eta$, η to be chosen later. Suppose that there exist two solutions f and $\widetilde{f}$ in $L^\infty_t L^\infty_{x,v}(m)$ associated to the initial datum f_0.

Subtracting the equations satisfied by f and $\widetilde{f}$ we get

$$\partial_t \left(f - \widetilde{f} \right) = G \left(f - \widetilde{f} \right) + Q \left(f - \widetilde{f}, f + \widetilde{f} \right)$$

and following the decomposition of the previous subsection

$$\partial_t \left(f - \widetilde{f} \right) = \left[G_\nu \left(f - \widetilde{f} \right) + B_2^{(\delta} \left(f - \widetilde{f} \right) + Q \left(f - \widetilde{f}, f + \widetilde{f} \right) \right] + A^{(\delta)} \left(f - \widetilde{f} \right).$$

Since G_ν generates a semigroup in $L^\infty_{x,v}(m)$ we can write the equation above under its Duhamel form:

$$\begin{aligned} f - \widetilde{f} = & \int_0^t S_{G_\nu}(t-s) \left[B_2^{(\delta} \left(f - \widetilde{f} \right) + Q \left(f - \widetilde{f}, f + \widetilde{f} \right) \right] ds \\ & + \int_0^t S_{G_\nu}(t-s) \left[A^{(\delta)} \left(f - \widetilde{f} \right) \right] ds. \end{aligned} \tag{6.25}$$

The first term on the right-hand side can be treated the same way as in the proof of Proposition 6.7 and therefore, for δ small enough, there exists $0 < C_1 < 1$ such that

$$\begin{aligned} & \left| \int_0^t S_{G_\nu}(t-s) \left[B_2^{(\delta} \left(f - \widetilde{f} \right) + Q \left(f - \widetilde{f}, f + \widetilde{f} \right) \right] ds \right| \\ & \quad \leqslant \left[(1 - C_1) + C_Q \left(\|f\|_{L^\infty_{[0,t]} L^\infty_{x,v}(m)} + \left\| \widetilde{f} \right\|_{L^\infty_{[0,t]} L^\infty_{x,v}(m)} \right) \right] \left\| f - \widetilde{f} \right\|_{L^\infty_{[0,t]} L^\infty_{x,v}(m)}. \end{aligned} \tag{6.26}$$

Since $S_{G_\nu}(t)$ is bounded on $L^\infty_{t,x,v}(m)$ (see Theorem 4.1), as well as $A^{(\delta)}$ is (see Lemma 6.2, we can bound the second term on the right-hand side of (6.25) by

$$\left| \int_0^t S_{G_\nu}(t-s) \left[A^{(\delta)} \left(f - \widetilde{f} \right) \right] ds \right| \leqslant C_2 t \left\| f - \widetilde{f} \right\|_{L^\infty_{[0,t]} L^\infty_{x,v}(m)}. \tag{6.27}$$

Plugging (6.26) and (6.27) into (6.25) we obtain

(6.28)
$$\left\|f-\widetilde{f}\right\|_{L^\infty_{[0,t]}L^\infty_{x,v}(m)} \leqslant \left[(1-C_1)+C_Q\left(\|f\|_{L^\infty_{[0,t]}L^\infty_{x,v}(m)}+\left\|\widetilde{f}\right\|_{L^\infty_{[0,t]}L^\infty_{x,v}(m)}\right)+C_2t\right] \times \left\|f-\widetilde{f}\right\|_{L^\infty_{[0,t]}L^\infty_{x,v}(m)}.$$

We now need to prove that the $L^\infty_{x,v}(m)$-norm of f and $\widetilde{f}$ must be bounded by the one of f_0. But this follows from (6.28) when one subtracts $S_{G_\nu}(t)f_0$ instead of $\widetilde{f}$ to f. This yields, after controlling $S_{G_\nu}(t)f_0$ by its $L^\infty_{x,v}(m)$-norm,

$$\|f\|_{L^\infty_{[0,t]}L^\infty_{x,v}(m)} \leqslant C_0\|f_0\|_{L^\infty_{x,v}(m)}+\left[(1-C_1)+C_Q\|f\|_{L^\infty_{[0,t]}L^\infty_{x,v}(m)}+C_2t\right]\|f\|_{L^\infty_{[0,t]}L^\infty_{x,v}(m)}.$$

Since $C_1<1$ we fix T_0 such that $C_2T_0<C_1/4$. We deduce that for all t in $[0,T_0]$

$$\forall t\in[0,T_0],\quad \frac{3C_1}{4}\|f\|_{L^\infty_{[0,t]}L^\infty_{x,v}(m)} \leqslant C_0\|f_0\|_{L^\infty_{x,v}(m)}+C_Q\|f\|^2_{L^\infty_{[0,t]}L^\infty_{x,v}(m)}$$

and therefore, if $\|f_0\|_{L^\infty_{x,v}(m)}\leqslant\eta$ with η small enough such that

$$\frac{3C_1}{4}-2\frac{C_QC_0}{C_1}\eta>\frac{C_1}{2}$$

then

$$\forall t\in[0,T_0],\quad \|f\|_{L^\infty_{[0,t]}L^\infty_{x,v}(m)}\leqslant\frac{2C_0}{C_1}\|f_0\|_{L^\infty_{x,v}(m)}. \tag{6.29}$$

To conclude the proof of uniqueness we see that (6.29) is also valid for $\widetilde{f}$ and (6.28) thus becomes

$$\forall t\in[0,T_0],\quad \left\|f-\widetilde{f}\right\|_{L^\infty_{[0,t]}L^\infty_{x,v}(m)}\leqslant\left[\left(1-\frac{3C_1}{4}\right)+4\frac{C_0C_Q}{C_1}\eta\right]\left\|f-\widetilde{f}\right\|_{L^\infty_{[0,t]}L^\infty_{x,v}(m)}.$$

We can choose η even smaller such that the term on the right-hand side can be absorbed by the left-hand side. This implies that $f=\widetilde{f}$ on $[0,T_0]$. Starting at T_0 we can iterate the process and finally getting that $f=\widetilde{f}$ on $\mathbb{R}^+$; which concludes the proof of Theorem 6.9. □

7. Qualitative study of the perturbative solutions to the Boltzmann equation

In this last section, we address the issue of positivity and continuity of the solutions to the Boltzmann equation

$$\partial_tF+v\cdot\nabla_xF=Q(F,F). \tag{7.1}$$

Note that even if our arguments are constructive, we only prove qualitative behaviours and we do not tackle the issue of quantitative estimates. For instance, we prove the positivity of the solutions but do not give any explicit lower bound. Such explicit lower bounds have been recently obtained in the case of pure specular reflections [7] and in the case of pure Maxwellian diffusion [6]. We think that the

proofs can be adapted to fit the case of Maxwell boundary condition as it is a convex combination of these boundary conditions. However, the techniques required to deal with this are very different from the one developed throughout this paper and we therefore did not looked into it much further.

7.1. **Positivity of solutions.** This subsection is dedicated to proving the following positivity property.

Proposition 7.1. *Let $m = e^{\kappa_1 |v|^{\kappa_2}}$ with $\kappa_1 > 0$ and κ_2 in $(0,2)$ or $m = \langle v \rangle^k$ with $k > k_\infty$. Let f_0 be in $L^\infty_{x,v}(m)$ with $\Pi_G(f_0) = 0$ and*

$$\|f_0\|_{L^\infty_{x,v}(m)} \leqslant \eta,$$

where $\eta > 0$ is chosen such that Theorems 6.1 and 6.9 hold and denote f the unique solution of the perturbed equation associated to f_0.
Suppose that $F_0 = \mu + f_0 \geqslant 0$ then $F = \mu + f \geqslant 0$.

Proof of Proposition 7.1. Since we are working with hard potential kernels we can decompose the nonlinear operator into

$$Q(F,F) = -Q^-(F,F) + Q^+(F,F)$$

where

$$\begin{aligned} Q^-(F,F)(v) &= \left(\int_{\mathbb{R}^3 \times \mathbb{S}^2} B\left(|v - v_*|, \cos\theta\right) F_* \, dv_* d\sigma \right) F(v) = q(F)(v) F(v), \\ Q^+(F,F) &= \int_{\mathbb{R}^3 \times \mathbb{S}^2} B\left(|v - v_*|, \cos\theta\right) F' F'_* \, dv_* d\sigma. \end{aligned}$$

Following the idea of [7][6], we obtain an equivalent definition of being a solution to (7.1) by applying the Duhamel formula along backward characteristics that is stopped right after the first collision against the boundary. If F is solution to the Boltzmann equation then for almost all (x,v) in $\Omega \times \mathbb{R}^3$,

$$\begin{aligned} F(t,x,v) =& F_0(x - vt, v) \exp\left(- \int_0^t q[F(s, x - (t-s)v, \cdot)](v) \, ds \right) \\ &+ \int_0^t \exp\left(- \int_s^t q[F(s', x - (s-s')v, \cdot)](v) \, ds' \right) \\ &\qquad \times Q^+\left[F(s, x - (t-s)v, \cdot), F(s, x - (t-s)v, \cdot) \right](v) \, ds \end{aligned} \tag{7.2}$$

if $t \leqslant t_{min}(x,v) := t_0$ or else

$$\begin{aligned} F(t,x,v) =& F_\Lambda(t_0, x - t_0 v, v) \exp\left(- \int_{t_0}^t q[F(s, x - (t-s)v, \cdot)](v) \, ds \right) \\ &+ \int_{t_0}^t \exp\left(- \int_s^t q[F(s', x - (s-s')v, \cdot)](v) \, ds' \right) \\ &\qquad \times Q^+\left[F(s, x - (t-s)v, \cdot), F(s, x - (t-s)v, \cdot) \right](v) \, ds. \end{aligned} \tag{7.3}$$

We denoted by F_Λ the Maxwell boundary condition for (t', x', v) in $\mathbb{R}^+ \times \Lambda$

$$F_\Lambda(t', x', v) = (1 - \alpha) F(t', x', \mathcal{R}_{x'}(v)) + \alpha P_\Lambda\left(F(t', x', \cdot)\right)(v).$$

We construct an iterative scheme $(F^{(n)})_{n\in\mathbb{N}}$ with $F^{(0)}=\mu$ and $F^{(n+1)}(t,x,v)$ being defined by (7.2) and (7.3) with all the F on the right-hand side being replaced by $F^{(n)}$ except in the definition of F_Λ where we keep $F^{(n+1)}$ instead. In other terms, $F^{(n+1)}$ is solution to

$$\left[\partial_t + v\cdot\nabla_x + q(F^{(n)})\right] F^{(n+1)} = Q(F^{(n)},F^{(n)})$$

with the Maxwell boundary condition; which is an approximative scheme to the Boltzmann equation (7.1).

Defining $f^{(n)}=F^{(n)}-\mu$ we have the following differential iterative scheme

$$\partial_t f^{(n+1)} + v\cdot\nabla_x f^{(n+1)} = -\nu(v)\left(f^{(n+1)}\right) + K\left(f^{(n)}\right) + Q^+\left(f^{(n)}\right) - q\left(f^{(n)}\right) f^{(n+1)}.$$

As before, we prove that $\left(f^{(n)}\right)_{n\in\mathbb{N}}$ is well-defined and converges in $L^\infty_{t,x,v}(m)$ towards f, the unique solution of the perturbed Boltzmann equation. Therefore, the same holds for $\left(F^{(n)}\right)_{n\in\mathbb{N}}$ converging towards F the unique perturbed solution of the original Boltzmann equation (7.1).

From the positivity of q and Q^+ and F_0, a straightforward induction from (7.2) shows that $F^{(n)}(t,x,v)\geqslant 0$ for all n when $t\leqslant t_0$. This implies that for all n and all (x,v), $F^{(n+1)}_\Lambda(t_0,x-t_0v,v)\geqslant 0$ and therefore (7.3) gives $F^{(n+1)}(t,x,v)\geqslant 0$ for all (t,x,v) and all n. The positivity of F follows by taking the limit as n tends to infinity. □

7.2. **Continuity of solutions.** The last issue tackled in the present article is the continuity of the solutions described in Section 6. More precisely, we prove the following proposition.

Proposition 7.2. *Let $m=e^{\kappa_1|v|^{\kappa_2}}$ with $\kappa_1>0$ and κ_2 in $(0,2)$ or $m=\langle v\rangle^k$ with $k>k_\infty$. Let f_0 be in $L^\infty_{x,v}(m)$ with $\Pi_G(f_0)=0$ and*

$$\|f_0\|_{L^\infty_{x,v}(m)}\leqslant\eta,$$

where $\eta>0$ is chosen such that Theorems 6.1 and 6.9 hold and denote f the unique solution of the perturbed equation associated to f_0.
Suppose that $F_0=\mu+f_0$ is continuous on $\Omega\times\mathbb{R}^3\cup\left\{\Lambda^+\cup\mathfrak{C}^-_\Lambda\right\}$ and satisfies the Maxwell boundary condition (1.2) then $F=\mu+f$ is continuous on the continuity set $\mathfrak{C}$.

We recall the definition of inward inflection grazing boundary

$$\Lambda_0^{(I-)}=\Lambda_0\cap\left\{t_{min}(x,v)=0,\ t_{min}(x,-v)\neq 0 \text{ and } \exists\delta>0,\ \forall\tau\in[0,\delta],\ x-\tau v\in\overline{\Omega}^c\right\}.$$

We also rewrite the boundary continuity set

$$\mathfrak{C}^-_\Lambda=\Lambda^-\cup\Lambda_0^{(I-)}$$

and the continuity set

$$\begin{aligned}\mathfrak{C}=&\Big\{\{0\}\times\left\{\Omega\times\mathbb{R}^3\cup\left(\Lambda^+\cup\mathfrak{C}^-_\Lambda\right)\right\}\Big\}\cup\Big\{(0,+\infty)\times\mathfrak{C}^-_\Lambda\Big\}\\&\cup\Big\{(t,x,v)\in(0,+\infty)\times\left(\Omega\times\mathbb{R}^3\cup\Lambda^+\right):\\&\qquad\forall 1\leqslant k\leqslant N(t,x,v)\in\mathbb{N},(X_{k+1}(x,v),V_k(x,v))\in\mathfrak{C}^-_\Lambda\Big\}.\end{aligned}$$

The sequence $(T_k(x,v), X_k(x,v), V_k(x,v))_{k\in\mathbb{N}}$ is the sequence of footprints of the backward characteristic trajectory starting at (x,v); $N(t,x,v)$ is almost always finite and such that $T_{N(t,x,v)} \leqslant t < T_{N(t,x,v)+1}(x,v)$. We refer to Subsection 4.1 for more details.

As explained in Lemma 4.3, the set $\mathfrak{C}^-_\lambda$ describes the boundary points in the phase space that lead to continuous specular reflections.

The proof of Proposition 7.2 relies on a continuity result for the non-homogeneous transport equation with a mixed specular and in-flow boundary conditions when Ω is not necessarily convex.

Similar results have been obtained in [24, Lemma 12] or [21, Lemma 13] (when Ω is convex) for purely in-flow boundary condition as well as for purely bounce-back reflections [24, Lemma 15]. We recover their results when $\alpha = 1$ or by replacing $(T_k, X_k, V_k)_k$ by the sequence associated to bounce-back characteristics. The continuity for pure specular reflections has been tackled in [21, Lemma 21] but required strict convexity of Ω.

The following lemma therefore improves and extends the existing results.

Lemma 7.3. *Let Ω be a C^1 bounded domain of $\mathbb{R}^3$ and let $f_0(x,v)$ be continuous on $\Omega\times\mathbb{R}^3 \cup \{\Lambda^+ \cup \mathfrak{C}^-_\Lambda\}$ and $g(t,x,v)$ be a boundary datum continuous on $[0,T]\times\mathfrak{C}^-_\Lambda$. At last, let $q_1(t,x,v)$ and $q_2(t,x,v)$ be two continuous function in the interior of $[0,T]\times\Omega\times\mathbb{R}^3$ satisfying*

$$\sup_{t\in[0,T]} \|q_1(t,x,v)\|_{L^\infty_{x,v}(m)} < \infty \quad \text{and} \quad \sup_{t\in[0,T]} \|q_2(t,x,v)\|_{L^\infty_{x,v}(m)} < \infty.$$

Assume f_0 satisfies the mixed specular and in-flow boundary condition:

$$\forall (x,v)\in\mathfrak{C}^-_\Lambda, \quad f_0(x,v) = (1-\alpha) f_0(x,\mathcal{R}_x(v)) + g(0,x,v)$$

and suppose $f(t,x,v)$ is the solution to

$$\begin{cases} [\partial_t + v\cdot\nabla_x + q_1(t,x,v)]\, f(t,x,v) = q_2(t,x,v) \\ \forall (t,x,v)\in[0,T]\times\Lambda^-,\ f(t,x,v) = (1-\alpha) f(t,x,\mathcal{R}_x(v)) + g(t,x,v) \end{cases}$$

associated to the initial datum f_0. Then $f(t,x,v)$ is continuous on the continuity set $\mathfrak{C}$.

Proof of Lemma 7.3. As now standard, in the homogeneous case $q_2 = 0$, we can use a Duhamel formula along the backward characteristics because q_1 belongs to $L^\infty_{t,x,v}$. More precisely, as in Subsection 4.3.1 with $q_1(t,x,v)$ replacing $\nu(v)$ we obtain that if $h(t,x,v)$ is solution to

$$[\partial_t + v\cdot\nabla_x + q_1(t,x,v)]\, h(t,x,v) = 0$$

with the mixed specular and in-flow boundary conditions then h takes the form

- if $t \leqslant t_{min}(x,v) = T_1$,

$$h(t,x,v) = h_0(x-tv,v) e^{-\int_0^t q_1(s,x-(t-s)v,v)ds};$$

- if $t > T_1$

$$h(t,x,v) = [(1-\alpha)h(t-T_1,X_1,V_1) + g(t-T_1,X_1,V_1)]\, e^{-\int_{t-T_1}^t q_1(s,x-(t-s)v,v)ds}.$$

Unlike the case of Maxwell boundary condition, we see that in the case of mixed specular and in-flow boundary condition we can always reach the initial plane $\{t = 0\}$. We obtain an explicit form for $h(t, x, v)$ for almost every (t, x, v) by iterating the property above (see [21, lemma 20] for more details or [24, Lemma 15] replacing the bounce-back trajectories by the specular ones). It reads with $N = N(t, x, v))$ and the usual notation $t_k = t - T_k(x, v)$

$$\begin{aligned} h(t,x,v) =&(1-\alpha)^N h_0(X_N - t_N V_N, V_N)\, e^{-\sum_{k=0}^{N}\int_{\min\{0,t_{k+1}\}}^{t_k} q_1(s,X_k-(t_k-s)V_k,V_k)\,ds} \\ &+\sum_{k=0}^{N-1} g(t_{k+1},X_{k+1},V_{k+1})\, e^{-\int_{\min\{0,t_{k+1}\}}^{t_k} q_1(s,X_k-(t_k-s)V_k,V_k)\,ds} \end{aligned} \tag{7.4}$$

for almost every (t, x, v). Note that this expression is indeed well-defined since $N(t, x, v)$ is finite almost everywhere and q_1 belongs to $L^\infty_{t,x,v}$. We also emphasize that $\min\{0, t_{k+1}\}$ only plays a role when $k = N(t, x, v)$; it encodes the fact that we integrate all the complete lines between t_k and t_{k+1} and only the remaining part $[t - T_N, t]$ of the last line.

Since the source term q_2 also belongs to $L^\infty_{t,x,v}$, we obtain an explicit formula for $f(t, x, v)$ from (7.4). It reads, for almost every (t, x, v),

$$\begin{aligned} f(t,x,v) =&(1-\alpha)^N f_0(X_N - t_N V_N, V_N)\, e^{-\sum_{k=0}^{N}\int_{\min\{0,t_{k+1}\}}^{t_k} q_1(s,X_k-(t_k-s)V_k,V_k)\,ds} \\ &+\sum_{k=0}^{N-1} g(t_{k+1},X_{k+1},V_{k+1})\, e^{-\int_{\min\{0,t_{k+1}\}}^{t_k} q_1(s,X_k-(t_k-s)V_k,V_k)\,ds} \\ &+\sum_{k=0}^{N-1}\int_{\min\{0,t_{k+1}\}}^{t_k} \exp\left[-\sum_{l=0}^{k}\int_{\max\{s,t_{l+1}\}}^{t_l} q_1(s_1, X_l-(t_l-s_1)V_l, V_l)ds_1\right] \\ &\times q_2(s, X_k-(t_k-s)V_k, V_k)\,ds. \end{aligned} \tag{7.5}$$

Note that in the expression above we used the change of variable $s_1 \mapsto t - s_1$ to recover exactly the sequence (t_l, X_l, V_l) associated to (t, x, v) instead of $(\widetilde{t_l}, \widetilde{X_l}, \widetilde{V_l})$ associated with $(t - s, x, v)$.

By assumptions on f_0 and g, we deduce that f is continuous on

$$\Big\{\{0\}\times\big\{\Omega\times\mathbb{R}^3\cup\big(\Lambda^+\cup\mathfrak{C}^-_\Lambda\big)\big\}\Big\}\cup\Big\{(0,+\infty)\times\mathfrak{C}^-_\Lambda\Big\}.$$

Now if (t, x, v) belongs to

$$\begin{aligned}\Big\{&(t,x,v)\in(0,+\infty)\times\big(\Omega\times\mathbb{R}^3\cup\Lambda^+\big):\\ &\forall 1\leqslant k\leqslant N(t,x,v)\in\mathbb{N}, (X_{k+1}(x,v),V_k(x,v))\in\mathfrak{C}^-_\Lambda\Big\}\end{aligned}$$

we have by iterating Lemma 4.3 that the finite sequence $(T_k, X_k, V_k)_{0\leqslant k\leqslant N(t,x,v)}$ is continuous around (x, v).

Let (t', x', v') be in the same set as (t, x, v). In the case $T_{N(t,x,v)} \leqslant t \leqslant t' < T_{N(t,x,v)+1}$ or $T_{N(t,x,v)} \leqslant t' \leqslant t < T_{N(t,x,v)+1}$, by continuity of the $t - T_k(x, v)$ we have that for (t', x', v') sufficiently close to t, $N(t', x', v') = N(t, x, v)$ and the continuity

of $(T_k, X_k, V_k)_{0\leqslant k\leqslant N(t,x,v)}$, g, q_1 and q_2 implies $f(t', x', v') \to f(t, x, v)$. It remains to deal with the case $t' \leqslant t = T_{N(t,x,v)}$ where $N(t', x, v) = N(t, x, v) - 1$. Exactly as proved in [21, Lemma 21], in that case $t_{N(t,x,v)} = 0$ and the integrals from 0 to t_N are null in formula (7.5). Moreover, $(X_{N(t,x,v)-1} - (t' - T_{N(t,x,v)-1})V_{N(t,x,v)-1})$ converges to $X_{N(t,x,v)}$ as t' tends to t. Finally, since f_0 satisfies the boundary condition, we obtain here again that $f(t', x', v') \to f(t, x, v)$. Which concludes the proof. □

We now prove the continuity of the solutions constructed in Section 6.

Proof of Proposition 7.2. We use a sequence to approximate the solution of the full Boltzmann equation with initial datum $F_0 = \mu + f_0$. We start from $F^{(0)} = \mu$ and define by induction $F^{(n+1)} = \mu + f^{(n+1)}$ such that

$$\left[\partial_t + v \cdot \nabla_x + q(F^{(n)})\right] F^{(n+1)} = Q(F^{(n)}, F^{(n)})$$

with the mixed specular and diffusive boundary conditions

$$\forall (x, v) \in \Lambda^-, \quad F^{(n+1)}(x, v) = (1 - \alpha)F^{(n+1)}(x, \mathcal{R}_x(v)) + \alpha P_\Lambda(F^{(n)}(x, \cdot))(v).$$

Since we impose a specular part in the boundary condition, similar computations as in Section 4 show that $\left(f^{(n)}\right)_{n\in\mathbb{N}}$ is well-defined in $L^\infty_{t,x,v}(m)$. Moreover, similar computations as Subsection 6.1.2 prove that $\left(f^{(n)}\right)_{n\in\mathbb{N}}$ is a Cauchy sequence, at least on $[0, T]$ for T sufficiently small, as well as $\left(F^{(n)}\right)_{n\in\mathbb{N}}$. Therefore $\left(f^{(n)}\right)_{n\in\mathbb{N}}$ converges towards f the unique solution of the perturbed Boltzmann equation with initial datum f_0 and $\left(F^{(n)}\right)_{n\in\mathbb{N}}$ converges to F the unique solution of the full Boltzmann equation with initial datum $F_0 = \mu + f_0$.

We apply Lemma 7.3 inductively on $\nu(v)^{-1}F^{(n+1)}$. Indeed, [24, Theorem 4 and Corollary 5] showed that $q_1 = \nu(v)^{-1}q(F^{(n)})$ and $q_2 = \nu(v)^{-1}Q(F^{(n)}, F^{(n)})$ are continuous in the interior of $[0, T] \times \overline{\Omega} \times \mathbb{R}^3$ if $F^{(n)}$ is continous on $\mathfrak{C}$ (see also Lemma 6.6). And [24, Proof of 2 of Theorem 3, Step 1] proved that $P_\Lambda(F^{(n)})$ is continuous on $[0, T] \times \mathfrak{C}^-_\Lambda$ even if $F^{(n)}$ is only continuous on

$$[0, T] \times \overline{\Omega} \times \mathbb{R}^3 - \left\{(x, v) \in \overline{\Omega} \times \mathbb{R}^3, \quad n(X_1(x, v)) \cdot v = 0\right\}$$

which is included in $\mathfrak{C}$.

Hence, by induction $F^{(n)}$ is continuous on $\mathfrak{C}$ for all n and is a Cauchy sequence. Therefore its limit F is continuous as well. □

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