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# Notes on the geometric Satake equivalence

Pierre Baumann \*

Simon Riche †

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# 1 Introduction

## 1.1 Description

These notes are devoted to a detailed exposition of the proof of the Geometric Satake Equivalence by Mirković–Vilonen [MV2]. This celebrated result provides, for  $G$  a complex connected reductive group and  $\mathbf{k}$  a Noetherian commutative ring of finite global dimension, an equivalence of categories between the category  $\mathrm{P}_{G_{\mathcal{O}}}(\mathrm{Gr}_G, \mathbf{k})$  of  $G_{\mathcal{O}}$ -equivariant perverse sheaves on the affine Grassmannian  $\mathrm{Gr}_G$  of  $G$  (where  $G_{\mathcal{O}}$  is the loop group of  $G$ ) and the category  $\mathrm{Rep}_{\mathbf{k}}(G_{\mathbf{k}}^{\vee})$  of representations of the Langlands dual split reductive group over  $\mathbf{k}$  on finitely generated  $\mathbf{k}$ -modules. Under this equivalence, the tensor product of  $G_{\mathbf{k}}^{\vee}$ -modules corresponds to a geometric construction on perverse sheaves called *convolution*.

This result can be considered on the one hand as giving a geometric description of the category of representations of  $G_{\mathbf{k}}^{\vee}$ , and on the other hand as giving a “concrete” construction of the dual reductive group  $G_{\mathbf{k}}^{\vee}$  out of the original (complex) reductive group  $G$ .

## 1.2 History and idea of proof

The first evidence of a close relationship between perverse sheaves on  $\mathrm{Gr}_G$  and representations of  $G_{\mathbf{k}}^{\vee}$  was found in work of Lusztig [Lu], where a “combinatorial shadow” of the equivalence was proved (for  $\mathbf{k}$  a field of characteristic 0). The equivalence itself was first proved, in the case when  $\mathbf{k}$  is a field of characteristic 0, by Ginzburg [Gi]. (In this case, the existence of an equivalence of abelian categories  $\mathrm{P}_{G_{\mathcal{O}}}(\mathrm{Gr}_G, \mathbf{k}) \cong \mathrm{Rep}_{\mathbf{k}}(G_{\mathbf{k}}^{\vee})$  is obvious, since both categories are semisimple with isomorphism classes of simple objects parametrized by the same set. The content of the theorem is thus only the description of the tensor product in geometric terms.) A new proof, valid for general coefficients, was later given by Mirković–Vilonen [MV2]. This is the proof that we consider here; the main new ingredient of their approach is the definition of the *weight functors*, which give a geometric construction of the decomposition of  $G_{\mathbf{k}}^{\vee}$ -representations into weight spaces for a maximal torus. A later proof in the case of characteristic-0 fields (which applies for  $\ell$ -adic sheaves, when  $G$  is defined over a more general field) was given by Richarz [Rc]. (The main difference with the approaches of [Gi, MV2] lies in the identification of the group scheme, which relies on work of Kazhdan–Larsen–Varshavsky [KLV].)

All proofs are based on ideas from *Tannakian formalism*. The strategy is to construct enough structure on the category  $\mathrm{P}_{G_{\mathcal{O}}}(\mathrm{Gr}_G, \mathbf{k})$  so as to guarantee that this category is equivalent to the category of representations of a  $\mathbf{k}$ -group scheme. In the case when  $\mathbf{k}$  is a field, one can apply general results due to Saavedra Rivano [SR] and Deligne–Milne [DM] to prove this; for general coefficients no such theory is available, and Mirković–Vilonen construct the group scheme “by hand” using their weight functors. The next step is to identify this group scheme with  $G_{\mathbf{k}}^{\vee}$ . The case of fields of characteristic 0 is relatively easy. Then, in [MV2], the general case is deduced from this one using a detailed analysis of the group scheme in the case  $\mathbf{k}$  is an algebraic closure of a finite field, and a general result on reductive group schemes due to Prasad–Yu [PY].

### 1.3 Applications

The geometric Satake equivalence has found numerous applications in Representation Theory, Algebraic Geometry and Number Theory. For the latter applications (see in particular [La]; see also [Z4, §5.5] for other examples and references), it is important to have a version of this equivalence where the affine Grassmannian is defined not over  $\mathbf{C}$  (as we do here) but rather over an algebraically closed field of positive characteristic (and where the sheaves for the classical topology are replaced by étale sheaves). We will not consider this variant, but will only mention that the analogues in this setting of all results that we use on the geometry of the affine Grassmannian are known; see [Z4] for details and references.<sup>1</sup> With these results at hand, our considerations adapt in a straightforward way to this setting to prove the desired equivalence of categories. (Here of course the coefficients of sheaves cannot be arbitrary, and the role played by  $\mathbf{Z}$  in Section 14 should be played by  $\mathbf{Z}_\ell$ , where  $\ell$  is a prime number different from the characteristic of the field of definition of the affine Grassmannian.)

The applications to Number Theory have also motivated a number of generalizations of the geometric Satake equivalence (so far mainly in the case of characteristic-0 coefficients) which will not be reviewed here; see in particular [Rc, Z2, RZ, Z3].

### 1.4 Contents

The notes consist of two parts with different purposes. The first one is a gentle introduction to the proof of Mirković–Vilonen in the special case where  $\mathbf{k}$  is a field of characteristic 0. This case allows for important simplifications, but at the same time plays a crucial role in the proof for general coefficients. It is well understood, but (to our knowledge) has not been treated in detail in the literature from the point of view of Mirković–Vilonen (except of course in their paper). We follow their arguments closely, adding only a few details where their proofs might be considered a little bit sketchy. We also treat certain prerequisites (e.g. Tannakian formalism) in detail. On the other hand, most “standard” results on the affine Grassmannian are stated without proof; for details and references we refer e.g. to [Z4].

Part II is devoted to the proof for general coefficients. Some people have expressed doubts about the proof in this generality, so we have tried to make all the arguments explicit, and to clarify the proofs as much as possible. In this process, Geordie Williamson suggested a direct proof of the fact that the group scheme constructed by Mirković–Vilonen is of finite type in the case of field coefficients. This proof is reproduced in Lemma 14.2, and allows to simplify the arguments a little bit.

Finally, Appendix A provides proofs of some “well-known” results on equivariant perverse sheaves.

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<sup>1</sup>There is an additional subtlety in this setting if the characteristic of the base field is “small,” namely that the neutral connected component of the Grassmannian might not be isomorphic to the affine Grassmannian of the simply-connected cover of the derived subgroup; see [PR, Remark 6.4] for an example. However, as was explained to us by X. Zhu, in any case the natural morphism from the latter to the former is a universal homeomorphism (again, see [PR, Remark 6.4] for a special case) and hence is as good as an isomorphism, as far as étale sheaves are concerned.

## 1.5 Acknowledgements

These notes grew out of a  $2\frac{1}{2}$ -days mini-course given during the workshop “Geometric methods and Langlands functoriality in positive characteristic” held in Luminy in January 2016. This mini-course (which only covered the contents of Part I) also comprised reminders on constructible sheaves and equivariant derived categories (by D. Fratila), and on perverse sheaves (by V. Heiermann), which are not reproduced in the notes.

We thank Dragos Fratila for many discussions which helped clarify various constructions and proofs in Part I. We thank Volker Heiermann, Vincent Lafforgue and an anonymous referee for insisting that we should treat the case of general coefficients, which led to the work in Part II. We also thank Geordie Williamson for very helpful discussions on Part II, and for allowing us to reproduce his proof of Lemma 14.2. Finally, we thank Brian Conrad and Gopal Prasad for kindly answering some questions related to the application of the results of [PY], Julien Bichon for helpful discussions and providing some references, Xinwen Zhu for answering various questions, and Vincent Lafforgue and an (other?) anonymous referee for their comments on a previous version of these notes.

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## Part I

# The case of characteristic-0 coefficients

## 2 Tannakian reconstruction

In this section (where we follow closely [DM, §II]),  $\mathbf{k}$  is an arbitrary field, and we denote by  $\text{Vect}_{\mathbf{k}}$  the category of finite-dimensional  $\mathbf{k}$ -vector spaces. All categories are tacitly assumed to be essentially small. By a commutative diagram of functors we will mean a diagram commutative *up to isomorphism*.

Some important ideas of Tannakian reconstruction are already contained in the following easy exercise.

*Exercise 2.1.* Let  $A$  be a  $\mathbf{k}$ -algebra,  $X$  be an  $A$ -module which is finite-dimensional over  $\mathbf{k}$ , and  $\alpha \in \text{End}_{\mathbf{k}}(X)$ . Show that

$$\alpha \in \text{im}(A \rightarrow \text{End}_{\mathbf{k}}(X)) \iff \forall n \geq 0, \forall Y \subset X^{\oplus n} \text{ } A\text{-submodule, } \alpha^{\oplus n}(Y) \subset Y.$$

(*Hint* : Of course, the implication  $\Rightarrow$  is obvious. To prove the reverse direction, assume the condition in the right-hand side holds. Pick a  $\mathbf{k}$ -basis  $(e_1, \dots, e_n)$  of  $X$ , take for  $Y$  the  $A$ -submodule generated by  $(e_1, \dots, e_n) \in X^{\oplus n}$ , and write that  $Y$  contains  $\alpha^{\oplus n}(e_1, \dots, e_n) = (\alpha(e_1), \dots, \alpha(e_n))$ .)

Tannakian reconstruction actually amounts to veneer this exercise first with the language of categories and then with the language of Hopf algebras (i.e. affine group schemes).

### 2.1 A first reconstruction theorem

Let us denote the category of finite-dimensional  $\mathbf{k}$ -vector spaces by  $\text{Vect}_{\mathbf{k}}$ . Given a  $\mathbf{k}$ -algebra  $A$ , we denote the category of finite-dimensional left  $A$ -modules by  $\text{Mod}_A$ .

Recall that a category  $\mathcal{C}$  is called *additive* if

- each set  $\text{Hom}_{\mathcal{C}}(X, Y)$  is an abelian group;
- the composition of morphisms is a bilinear operation;
- $\mathcal{C}$  has a zero object;
- finite products and coproducts exist in  $\mathcal{C}$ .

Such a category is called  $\mathbf{k}$ -linear if each  $\text{Hom}_{\mathcal{C}}(X, Y)$  is a  $\mathbf{k}$ -vector space, and if the composition is  $\mathbf{k}$ -bilinear. An additive category  $\mathcal{C}$  is called *abelian* if

- each morphism has a kernel and a cokernel;

- for any morphism  $f$ , the natural morphism from the cokernel of the kernel (a.k.a. the coimage) of  $f$  to the kernel of the cokernel (a.k.a. the image) of  $f$  is an isomorphism.

Given an object  $X$  in an abelian category  $\mathcal{C}$ , we will denote by  $\langle X \rangle$  the full subcategory of  $\mathcal{C}$  formed by all objects that are isomorphic to a subquotient of a direct sum  $X^{\oplus n}$  for some  $n \in \mathbf{Z}_{\geq 0}$ .

**Proposition 2.2.** *Let  $\mathcal{C}$  be an abelian  $\mathbf{k}$ -linear category and let  $\omega : \mathcal{C} \rightarrow \mathbf{Vect}_{\mathbf{k}}$  be a  $\mathbf{k}$ -linear exact faithful functor. Fix an object  $X$  in  $\mathcal{C}$  and introduce the finite-dimensional  $\mathbf{k}$ -algebra*

$$A_X := \{ \alpha \in \text{End}_{\mathbf{k}}(\omega(X)) \mid \forall n \geq 0, \forall Y \subset X^{\oplus n} \text{ subobject, } \alpha^{\oplus n}(\omega(Y)) \subset \omega(Y) \}.$$

Then  $\omega|_{\langle X \rangle}$  admits a canonical factorization

$$\begin{array}{ccc} \langle X \rangle & \xrightarrow{\bar{\omega}_X} & \text{Mod}_{A_X} \\ & \searrow \omega & \swarrow \text{forget} \\ & & \mathbf{Vect}_{\mathbf{k}}, \end{array}$$

and  $\bar{\omega}_X$  is an equivalence of categories. In addition  $A_X$  is the endomorphism algebra of the functor  $\omega|_{\langle X \rangle}$ .

If  $A$  is a  $\mathbf{k}$ -algebra, and if we apply this proposition to the category  $\mathcal{C} = \text{Mod}_A$  of finite-dimensional  $A$ -modules with  $\omega$  the forgetful functor (which keeps the  $\mathbf{k}$ -vector space structure but forgets the structure of  $A$ -module), then Exercise 2.1 shows that the algebra  $A_X$  is precisely the image of  $A$  in  $\text{End}_{\mathbf{k}}(X)$ . The proposition is thus mainly saying that the exercise can be stated within the language of abelian categories.

For the proof of Proposition 2.2 we will need the following standard facts from Category Theory.

- Lemma 2.3.**
1. *An exact additive functor  $F : \mathcal{A} \rightarrow \mathcal{B}$  between two abelian categories preserves kernels and cokernels. It thus preserves finite intersections and finite sums (in an ambient object).*
  2. *A faithful functor  $F : \mathcal{A} \rightarrow \mathcal{B}$  between two abelian categories does not kill any nonzero object.*
  3. *Let  $F : \mathcal{A} \rightarrow \mathcal{B}$  be an exact faithful additive functor between two abelian categories and let  $u : X \rightarrow Y$  be a morphism in  $\mathcal{A}$ . Then  $u$  is an monomorphism (respectively, epimorphism) if and only if  $F(u)$  is so.*
  4. *Let  $F : \mathcal{A} \rightarrow \mathcal{B}$  be an exact faithful additive functor between two abelian categories. Assume that  $\mathcal{B}$  is Artinian and Noetherian: any monotone sequence of subobjects becomes eventually constant. Then arbitrary intersections and arbitrary sums (in an ambient object) exist in both  $\mathcal{A}$  and  $\mathcal{B}$ , and  $F$  preserves intersections and sums.*



*Proof.* (1) Any morphism  $u : X \rightarrow Y$  in  $\mathcal{A}$  gives rise to two short exact sequences

$$\begin{array}{ccccc}
 & & X & \xrightarrow{u} & Y \\
 & \nearrow & & & \searrow \\
 & \ker u & & \text{im } u & & \text{coker } u \\
 & \nearrow & & \nearrow & & \searrow \\
 0 & & 0 & & 0 & & 0.
 \end{array}$$

Applying  $F$  to this diagram and using the exactness assumption, we see that this functor preserves kernels and cokernels. The last assertion comes from the fact that the intersection (respectively, sum) of two subobjects can be expressed by a pull-back (respectively, push-forward) diagram, that is, as a kernel (respectively, cokernel).

(2) Assume that  $X$  is a nonzero object in  $\mathcal{A}$ . Then  $\text{id}_X \neq 0$  in  $\text{End}_{\mathcal{A}}(X)$ . The faithfulness assumption then implies that  $\text{id}_{F(X)} = F(\text{id}_X) \neq 0$  in  $\text{End}_{\mathcal{B}}(F(X))$ , whence  $F(X) \neq 0$ .

(3) It suffices to note that

$$\ker u = 0 \iff F(\ker u) = 0 \iff \ker F(u) = 0$$

and

$$\text{coker } u = 0 \iff F(\text{coker } u) = 0 \iff \text{coker } F(u) = 0.$$

(4) We first claim that  $\mathcal{A}$  is Artinian and Noetherian. Indeed given a monotone sequence of subobjects in  $\mathcal{A}$ , its image by  $F$  is a monotone sequence of subobjects in  $\mathcal{B}$ , so becomes eventually constant; (3) then implies that the sequence in  $\mathcal{A}$  also becomes eventually constant. Thus arbitrary intersections and sums exist in  $\mathcal{A}$  as well as in  $\mathcal{B}$  and are in fact finite intersections or sums (by the Artinian or Noetherian property, respectively). We conclude with the help of (1).  $\square$

We can now give the proof of Proposition 2.2.

*Proof.* By definition, for any  $\alpha \in A_X$ , the endomorphism  $\alpha^{\oplus n}$  of  $\omega(X)^{\oplus n}$  leaves stable  $\omega(Y)$  for all subobjects  $Y \subset X^{\oplus n}$ , and thus induces an endomorphism of  $\omega(Z)$  for all subquotients  $Z$  of  $X^{\oplus n}$ . In this way, for each object  $Z$  in  $\langle X \rangle$ , the  $\mathbf{k}$ -vector space  $\omega(Z)$  becomes an  $A_X$ -module. If  $Z$  is a subquotient of  $X^{\oplus n}$  and  $Z'$  is a subquotient of  $X^{\oplus m}$ , and if  $f : Z \rightarrow Z'$  is a morphism in  $\mathcal{C}$ , then  $Z \oplus Z'$  is a subquotient of  $X^{\oplus(n+m)}$ , and the image  $\text{gr}(f)$  of the morphism  $(\text{id}, f) : Z \rightarrow Z \oplus Z'$  (in other words the graph of  $f$ ) is a subobject of  $Z \oplus Z'$ , hence also a subquotient of  $X^{\oplus(n+m)}$ . The fact that  $A_X$  stabilizes  $\omega(\text{gr}(f))$  means that  $\omega(f)$  is a morphism of  $A_X$ -modules. In summary, we have proved that  $\omega|_{\langle X \rangle}$  factorizes through the category of finite-dimensional  $A_X$ -modules, as stated.

By definition, an endomorphism  $\alpha$  of the functor  $\omega|_{\langle X \rangle}$  is the datum of an endomorphism  $\alpha_Z \in \text{End}_{\mathbf{k}}(\omega(Z))$  for each  $Z \in \langle X \rangle$ , such that the diagram

$$\begin{array}{ccc}
 \omega(Z) & \xrightarrow{\alpha_Z} & \omega(Z) \\
 \omega(f) \downarrow & & \downarrow \omega(f) \\
 \omega(Z') & \xrightarrow{\alpha_{Z'}} & \omega(Z')
 \end{array}$$

commutes for any morphism  $f : Z \rightarrow Z'$  in  $\langle X \rangle$ . This compatibility condition and the definition of  $\langle X \rangle$  forces  $\alpha$  to be determined by  $\alpha_X \in \text{End}_{\mathbf{k}}(\omega(X))$ , and forces  $\alpha_X$  to belong to  $A_X$ . Conversely, any element in  $A_X$  gives rise to an endomorphism of  $\omega|_{\langle X \rangle}$ . This discussion shows the last assertion in the proposition.

It remains to show that the functor  $\bar{\omega}_X$  is an equivalence of categories. We already know that it is faithful, so we must show that it is full and essentially surjective. We will do that by constructing an inverse functor.

We will denote by  $\mathcal{C}^{\text{fin}}$  the category opposite to the category of  $\mathbf{k}$ -linear functors from  $\mathcal{C}$  to  $\text{Vect}_{\mathbf{k}}$ . Yoneda's lemma says that the functor  $Z \mapsto \text{Hom}_{\mathcal{C}}(Z, -)$  from  $\mathcal{C}$  to  $\mathcal{C}^{\text{fin}}$  is fully faithful, so  $\mathcal{C}$  is a full subcategory of  $\mathcal{C}^{\text{fin}}$ . Given an object  $Y \in \mathcal{C}$  and a finite-dimensional  $\mathbf{k}$ -vector space  $V$ , we define two objects in  $\mathcal{C}^{\text{fin}}$  by

$$\underline{\text{Hom}}(V, Y) := \text{Hom}_{\mathcal{C}}(Y, -) \otimes_{\mathbf{k}} V \quad \text{and} \quad Y \otimes V := \text{Hom}_{\mathbf{k}}(V, \text{Hom}_{\mathcal{C}}(Y, -)).$$

These functors are representable: if  $V = \mathbf{k}^n$ , then both functors are represented by  $Y^{\oplus n}$ . So we will regard  $\underline{\text{Hom}}(V, Y)$  and  $Y \otimes V$  as being objects in  $\mathcal{C}$  and forget everything about  $\mathcal{C}^{\text{fin}}$ . Note however that we gained functoriality in  $V$  in the process: given two  $\mathbf{k}$ -vector spaces  $V$  and  $W$  and an object  $Y \in \mathcal{C}$ , there is a linear map

$$\text{Hom}_{\mathbf{k}}(W, V) \rightarrow \text{Hom}_{\mathcal{C}}(\underline{\text{Hom}}(V, Y), \underline{\text{Hom}}(W, Y)) \quad (2.1)$$

that sends an element  $f \in \text{Hom}_{\mathbf{k}}(W, V)$  to the image of the identity by the map

$$\begin{aligned} \text{End}_{\mathcal{C}}(\underline{\text{Hom}}(W, Y)) &= \text{Hom}_{\mathcal{C}}(Y, \underline{\text{Hom}}(W, Y)) \otimes_{\mathbf{k}} W \\ &\quad \downarrow \text{id} \otimes f \\ \text{Hom}_{\mathcal{C}}(Y, \underline{\text{Hom}}(W, Y)) \otimes_{\mathbf{k}} V &= \text{Hom}_{\mathcal{C}}(\underline{\text{Hom}}(V, Y), \underline{\text{Hom}}(W, Y)). \end{aligned}$$

For two  $\mathbf{k}$ -vector spaces  $W \subset V$  and two objects  $Z \subset Y$  in  $\mathcal{C}$ , we define the transporter of  $W$  into  $Z$  as the subobject

$$(Z : W) := \ker(\underline{\text{Hom}}(V, Y) \rightarrow \underline{\text{Hom}}(W, Y/Z))$$

of  $\underline{\text{Hom}}(V, Y)$ , where the morphism  $\underline{\text{Hom}}(V, Y) \rightarrow \underline{\text{Hom}}(W, Y/Z)$  is the obvious one.

Now we define

$$P_X = \bigcap_{\substack{n \geq 0 \\ Y \subset X^{\oplus n}}} \left( \underline{\text{Hom}}(\omega(X), X) \cap (Y : \omega(Y)) \right).$$

Here the small intersection is computed in the ambient object  $\underline{\text{Hom}}(\omega(X)^{\oplus n}, X^{\oplus n})$ , the space  $\underline{\text{Hom}}(\omega(X), X)$  being embedded diagonally, and the large intersection, taken over all  $n \geq 0$  and all subobjects  $Y \subset X^{\oplus n}$ , is computed in the ambient object  $\underline{\text{Hom}}(\omega(X), X)$ . The existence of this intersection is guaranteed by Lemma 2.3(4); moreover, as a subobject of  $\underline{\text{Hom}}(\omega(X), X) \cong X^{\oplus \dim(\omega(X))}$ , the object  $P_X$  belongs to  $\langle X \rangle$ .

Equation (2.1) provides us with an algebra map

$$\text{End}_{\mathbf{k}}(\omega(X)) \rightarrow \text{End}_{\mathcal{C}}(\underline{\text{Hom}}(\omega(X), X)),$$

which induces an algebra map  $A_X \rightarrow \text{End}_{\mathcal{C}}(P_X)$ . This map can be seen as a morphism  $P_X \underline{\otimes} A_X \rightarrow P_X$  in  $\mathcal{C}$ , and we can thus define the coequalizer

$$P_X \underline{\otimes}_{A_X} V := \text{coeq}(P_X \underline{\otimes} (A_X \otimes_{\mathbf{k}} V) \rightrightarrows P_X \underline{\otimes} V)$$

for each  $A_X$ -module  $V$ . (Here, one of the maps is induced by the  $A_X$ -action on  $V$  via (2.1), and the other one by the map  $P_X \underline{\otimes} A_X \rightarrow P_X$  we have just constructed.) We will prove that the functor

$$P_X \underline{\otimes}_{A_X} - : \text{Mod}_{A_X} \rightarrow \langle X \rangle$$

is an inverse to  $\bar{\omega}_X$ .

First, we remark that for any  $\mathbf{k}$ -vector space  $V$  and any object  $Y \in \mathcal{C}$  there exists a canonical identification

$$\omega(Y \underline{\otimes} V) = \omega(Y) \otimes_{\mathbf{k}} V.$$

Indeed  $\text{id}_Y \otimes V$  defines an element in

$$\text{Hom}_{\mathcal{C}}(Y \underline{\otimes} V, Y \underline{\otimes} V) = \text{Hom}_{\mathbf{k}}(V, \text{Hom}_{\mathcal{C}}(Y, Y \underline{\otimes} V)).$$

The image of this element under the map

$$\text{Hom}_{\mathbf{k}}(V, \text{Hom}_{\mathcal{C}}(Y, Y \underline{\otimes} V)) \rightarrow \text{Hom}_{\mathbf{k}}(V, \text{Hom}_{\mathbf{k}}(\omega(Y), \omega(Y \underline{\otimes} V)))$$

induced by  $\omega$  provides a canonical element in

$$\text{Hom}_{\mathbf{k}}(V, \text{Hom}_{\mathbf{k}}(\omega(Y), \omega(Y \underline{\otimes} V))) \cong \text{Hom}_{\mathbf{k}}(\omega(Y) \otimes_{\mathbf{k}} V, \omega(Y \underline{\otimes} V)),$$

or in other words a canonical morphism  $\omega(Y) \otimes_{\mathbf{k}} V \rightarrow \omega(Y \underline{\otimes} V)$ . To check that this morphism is invertible one can assume that  $V = \mathbf{k}^n$ , in which case the claim is obvious. Likewise, we have an identification  $\omega(\underline{\text{Hom}}(V, Y)) = \text{Hom}_{\mathbf{k}}(V, \omega(Y))$ .

Using these identifications, the exactness of  $\omega$  implies that given two  $\mathbf{k}$ -vector spaces  $W \subset V$  and two objects  $Z \subset Y$  in  $\mathcal{C}$ ,

$$\omega((Z : W)) = \{\alpha \in \text{Hom}_{\mathbf{k}}(V, \omega(Y)) \mid \alpha(W) \subset \omega(Z)\}.$$

From Lemma 2.3(4), it then follows that  $\omega(P_X) = A_X$  (as a right  $A_X$ -module), and therefore, that for each  $V \in \text{Mod}_{A_X}$  we have

$$\bar{\omega}_X(P_X \underline{\otimes}_{A_X} V) = \bar{\omega}_X(P_X) \otimes_{A_X} V \cong V.$$

Hence  $\bar{\omega}_X(P_X \underline{\otimes}_{A_X} -)$  is naturally isomorphic to the identity functor of  $\text{Mod}_{A_X}$ .

For the other direction, we start by checking that

$$\text{Hom}_{\mathbf{k}}(V, \text{End}_{\mathcal{C}}(Y) \otimes_{\mathbf{k}} V) = \text{Hom}_{\mathbf{k}}(V, \text{Hom}_{\mathcal{C}}(\underline{\text{Hom}}(V, Y), Y)) = \text{Hom}_{\mathcal{C}}(\underline{\text{Hom}}(V, Y) \underline{\otimes} V, Y).$$

To the canonical element in the left-hand side (defined by  $v \mapsto \text{id}_Y \otimes v$ ) corresponds a canonical morphism  $\underline{\text{Hom}}(V, Y) \underline{\otimes} V \rightarrow Y$  in  $\mathcal{C}$ . Considering the latter for  $V = \omega(X)$  and  $Y = X^{\oplus n}$ , we obtain a canonical map

$$\underline{\text{Hom}}(\omega(X), X) \underline{\otimes} \omega(X^{\oplus n}) \rightarrow X^{\oplus n},$$

whence by restriction

$$P_X \otimes_{A_X} \omega(Y) \rightarrow Y$$

for any subobject  $Y \subset X^{\oplus n}$ . The latter map is an isomorphism because, as we saw above, its image by  $\omega$  is an isomorphism (see Lemma 2.3(3)). The right exactness of  $\omega$  and of  $P_X \otimes_{A_X} -$  then imply that  $P_X \otimes_{A_X} \omega(Z) \xrightarrow{\sim} Z$  for each subquotient  $Z$  of  $X^{\oplus n}$ , and we conclude that  $P_X \otimes_{A_X} \bar{\omega}_X(-)$  is naturally isomorphic to the identity functor of  $\langle X \rangle$ .  $\square$

In the setup of the proposition, if  $X$  and  $X'$  are two objects of  $\mathcal{C}$  such that  $\langle X \rangle \subset \langle X' \rangle$  (for instance if  $X'$  is of the form  $X \oplus Y$ ), then we have a restriction morphism

$$A_{X'} \cong \text{End}(\omega|_{\langle X' \rangle}) \rightarrow \text{End}(\omega|_{\langle X \rangle}) \cong A_X.$$

One would like to embrace the whole category  $\mathcal{C}$  by taking larger and larger subcategories  $\langle X \rangle$  and going to the limit, but the category of finite-dimensional modules over the inverse limit of a system of algebras is not the union of the categories of finite-dimensional modules over the algebras. Things work much better if one looks at comodules over coalgebras, mainly because tensor products commute with direct limits.

## 2.2 Algebras and coalgebras

Let us recall how the dictionary between finite-dimensional algebras and finite-dimensional coalgebras works (see for instance [Ka, Chap. III]):

|  |                       |   |
|--|-----------------------|---|
| A finite-dimensional $\mathbf{k}$ -algebra   | $\longleftrightarrow$ | its $\mathbf{k}$ -dual $B = A^\vee$ , a finite-dim. $\mathbf{k}$ -coalgebra;  |
| $m : A \otimes_{\mathbf{k}} A \rightarrow A$ multiplication<br>(associative)                                   | $\longleftrightarrow$ | $\Delta : B \rightarrow B \otimes_{\mathbf{k}} B$ comultiplication (coassociative)<br>$\varepsilon : B \rightarrow \mathbf{k}$ counit   |
| $\eta : \mathbf{k} \rightarrow A$ unit   |                       | (obtained by transposing $m$ and $\eta$ );  |
| left $A$ -module structure<br>on a space $M$ with action<br>map $\mu : A \otimes_{\mathbf{k}} M \rightarrow M$ | $\longleftrightarrow$ | right $B$ -comodule structure on $M$ with<br>coaction map $\delta : M \rightarrow M \otimes_{\mathbf{k}} B$<br>defined by $\mu(a \otimes m) = (\text{id}_M \otimes \text{ev}_a) \circ \delta(m)$ , where<br>$\text{ev}_a : B \rightarrow \mathbf{k}$ is the evaluation at $a$ . |

In the context of this dictionary, one can identify the category  $\text{Mod}_A$  of finite-dimensional left  $A$ -modules with the category  $\text{Comod}_B$  of finite-dimensional right  $B$ -comodules.

## 2.3 A second reconstruction theorem

Going back to the setting of §2.1, we see that whenever  $\langle X \rangle \subset \langle X' \rangle$ , we get a morphism of coalgebras

$$B_X := A_X^\vee \rightarrow A_{X'}^\vee =: B_{X'}.$$

Now we can choose  $X$  with more and more direct summands, so that  $\langle X \rangle$  grows larger and larger. Our running assumption that all categories are essentially small allows us to take the direct limit of the coalgebras  $B_X$  over the set of isomorphism classes of objects of  $\mathcal{C}$ , for the order determined by the inclusions  $\langle X \rangle \subset \langle X' \rangle$ . We then obtain the following statement.

**Theorem 2.4.** *Let  $\mathcal{C}$  and  $\omega$  be as in Proposition 2.2. Set*

$$B := \varinjlim_X B_X.$$

*Then  $\omega$  admits a canonical factorization*

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{\bar{\omega}} & \text{Comod}_B \\ & \searrow \omega & \swarrow \text{forget} \\ & \text{Vect}_{\mathbf{k}} & \end{array}$$

*where  $\bar{\omega}$  is an equivalence of categories.*

Here, the fact that  $\omega(X)$  admits a structure of  $B$ -comodule (for  $X$  in  $\mathcal{C}$ ) means that there exists a canonical morphism  $\omega(X) \rightarrow \omega(X) \otimes_{\mathbf{k}} B$  satisfying the appropriate axioms. In other words, we have obtained a canonical morphism of functors  $\omega \rightarrow \omega \otimes_{\mathbf{k}} B$ , where the right-hand side means the functor  $X \mapsto \omega(X) \otimes_{\mathbf{k}} B$  (and where we omit the natural functor from  $\text{Vect}_{\mathbf{k}}$  to the category of *all*  $\mathbf{k}$ -vector spaces).

*Example 2.5.* 1. Let  $V$  be a finite-dimensional  $\mathbf{k}$ -vector space, and take  $\mathcal{C} = \text{Vect}_{\mathbf{k}}$  and  $\omega = V \otimes_{\mathbf{k}} -$ . Then  $B = \text{End}_{\mathbf{k}}(V)^{\vee}$ . Indeed, the category of finite-dimensional left  $\text{End}_{\mathbf{k}}(V)$ -modules is semisimple, with just one simple object up to isomorphism, namely  $V$ .

2. Let  $M$  be a set, let  $\mathcal{C} = \text{Vect}_{\mathbf{k}}(M)$  be the category of finite-dimensional  $M$ -graded  $\mathbf{k}$ -vector spaces, and  $\omega : \mathcal{C} \rightarrow \text{Vect}_{\mathbf{k}}$  be the functor that forgets the  $M$ -grading. Then  $B = \mathbf{k}M$ , the  $\mathbf{k}$ -vector space with basis  $M$ , with the coalgebra structure given by

$$\Delta(m) = m \otimes m, \quad \varepsilon(m) = 1$$

for all  $m \in M$ . For each  $X \in \mathcal{C}$ , the coaction of  $B$  on  $\omega(X) = X$  is the map

$$X \rightarrow X \otimes_{\mathbf{k}} B, \quad x \mapsto \sum_{m \in M} x_m \otimes m,$$

where  $x = \sum_{m \in M} x_m$  is the decomposition of  $x$  into its homogeneous components.

3. Let  $C$  be a coalgebra, and take  $\mathcal{C} = \text{Comod}_C$ , with  $\omega$  the forgetful functor. Then there exists a canonical isomorphism  $C \cong B$ . Indeed, if  $X \in \text{Comod}_C$ , there exists a finite-dimensional subcoalgebra  $C' \subset C$  such that the  $C$ -comodule  $X$  is actually a  $C'$ -comodule.<sup>2</sup> Then the coaction morphism  $X \rightarrow X \otimes_{\mathbf{k}} C'$  defines an algebra morphism  $(C')^{\vee} \rightarrow A_X$ , hence a coalgebra morphism  $B_X \rightarrow C'$ . Composing with the embedding  $C' \hookrightarrow C$  and passing to the limit we deduce a coalgebra morphism  $B \rightarrow C$ . In the reverse direction, if  $C' \subset C$  is a finite-dimensional subcoalgebra, then  $C'$  is an object in  $\text{Comod}_C$ , hence it acquires a canonical  $B$ -comodule structure, i.e. a coalgebra map  $C' \rightarrow C' \otimes_{\mathbf{k}} B$ . Composing with the map induced by  $\varepsilon$  we deduce a coalgebra morphism  $C' \rightarrow B$ . Since  $C$  is the direct limit of its finite-dimensional subcoalgebras, we deduce a coalgebra morphism  $C \rightarrow B$ . It is easily seen that the morphisms we constructed are inverse to each other, proving our claim.

<sup>2</sup>In view of its importance, let us briefly recall the proof of this classical fact. Let  $\delta : X \rightarrow X \otimes_{\mathbf{k}} C$  be the structure map of the  $C$ -comodule  $X$  and let  $(e_1, \dots, e_n)$  be a  $\mathbf{k}$ -basis of  $X$ . Write  $\delta(e_j) = \sum_i e_i \otimes c_{i,j}$ . Then  $\Delta(c_{i,j}) = \sum_k c_{i,k} \otimes c_{k,j}$  and  $\varepsilon(c_{i,j}) = \delta_{i,j}$  (Kronecker's symbol), so  $C'$  can be chosen as the  $\mathbf{k}$ -span in  $C$  of the elements  $c_{i,j}$ .

An homomorphism of  $\mathbf{k}$ -coalgebras  $f : B \rightarrow C$  induces a functor  $f_* : \text{Comod}_B \rightarrow \text{Comod}_C$ . Specifically, given a  $B$ -comodule  $M$  with structure map  $\delta : M \rightarrow M \otimes_{\mathbf{k}} B$ , the  $C$ -comodule  $f_*M$  has the same underlying  $\mathbf{k}$ -vector space as  $M$  and has structure map  $(\text{id}_M \otimes f) \circ \delta : M \rightarrow M \otimes_{\mathbf{k}} C$ .

**Proposition 2.6.** 1. Let  $\mathcal{C}$  be an abelian  $\mathbf{k}$ -linear category, let  $C$  be a  $\mathbf{k}$ -coalgebra, and let  $F : \mathcal{C} \rightarrow \text{Comod}_C$  be a  $\mathbf{k}$ -linear exact faithful functor. If  $B$  is the coalgebra provided by Theorem 2.4 (for the functor given by the composition of  $F$  with the forgetful functor  $\text{Comod}_C \rightarrow \text{Vect}_{\mathbf{k}}$ ) and  $\overline{F} : \mathcal{C} \xrightarrow{\sim} \text{Comod}_B$  the corresponding equivalence, then there exists a unique morphism of  $\mathbf{k}$ -coalgebras  $f : B \rightarrow C$  such that the following diagram commutes:

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{\overline{F}} & \text{Comod}_B \\ & \searrow F & \swarrow f_* \\ & & \text{Comod}_C \end{array}$$

2. Let  $B$  and  $C$  be two  $\mathbf{k}$ -coalgebras. Any  $\mathbf{k}$ -linear functor  $F : \text{Comod}_B \rightarrow \text{Comod}_C$  such that

$$\begin{array}{ccc} \text{Comod}_B & \xrightarrow{F} & \text{Comod}_C \\ & \searrow \text{forget} & \swarrow \text{forget} \\ & & \text{Vect}_{\mathbf{k}} \end{array}$$

commutes is of the form  $F = f_*$  for a unique morphism of coalgebras  $f : B \rightarrow C$ .

*Proof.* (1) Let  $X$  be an object in  $\mathcal{C}$ . As seen in Example 2.5(3), there exists a finite-dimensional subcoalgebra  $C' \subset C$  such that the  $C$ -comodule  $F(X)$  is actually a  $C'$ -comodule. The restriction to the category  $\langle X \rangle$  of the functor  $F$  then factorizes through  $\text{Comod}_{C'} = \text{Mod}_{(C')^\vee}$ . Let  $\omega : \text{Mod}_{(C')^\vee} \rightarrow \text{Vect}_{\mathbf{k}}$  be the forgetful functor and let  $A_X$  be the endomorphism algebra of the functor  $\omega \circ F|_{\langle X \rangle}$ .

Consider the diagram

$$\begin{array}{ccc} \langle X \rangle & \xrightarrow{\overline{F}_X} & \text{Mod}_{A_X} \\ & \searrow F & \swarrow \text{dashed} \\ & & \text{Mod}_{(C')^\vee} \\ & & \downarrow \omega \\ & & \text{Vect}_{\mathbf{k}} \end{array}$$

Any  $\alpha \in (C')^\vee$  can be seen as an endomorphism of the functor  $\omega$ , so induces by restriction an endomorphism of  $\omega \circ F|_{\langle X \rangle}$ , or in other words an element of  $A_X$ . Our situation thus gives us a morphism of algebras  $(C')^\vee \rightarrow A_X$ , that is, a morphism of coalgebras  $A_X^\vee \rightarrow C'$ . Further,  $\overline{F}_X$  is an equivalence of categories, because the  $\mathbf{k}$ -linear functor  $\omega \circ F|_{\langle X \rangle}$  is exact and faithful. Taking as before the limit over  $\langle X \rangle$  yields the desired coalgebra  $B = \varinjlim A_X^\vee$ , the morphism of coalgebras  $f : B \rightarrow C$ , and the equivalence of categories  $\overline{F}$ .

(2) Statement (2) is essentially the special case of (1) in the context of Example 2.5(3). More concretely, the coalgebra  $B$  is a right comodule over itself. The functor  $F$  maps it to a  $C$ -comodule with the same underlying vector space. We thus get a structure map  $\delta : B \rightarrow B \otimes C$ . Composing with the augmentation  $\varepsilon : B \rightarrow \mathbf{k}$ , we get a map  $f = (\varepsilon \otimes \text{id}_C) \circ \delta$  from  $B$  to  $C$ . Abstract nonsense arguments (the functoriality of  $F$  and the axioms of comodules) imply that  $f : B \rightarrow C$  is a coalgebra map and that  $F = f_*$ .  $\square$

## 2.4 Tannakian reconstruction

As we saw in §2.3, a  $\mathbf{k}$ -linear abelian category equipped with an exact faithful  $\mathbf{k}$ -linear functor to  $\text{Vect}_{\mathbf{k}}$  is equivalent to the category of right comodules over a  $\mathbf{k}$ -coalgebra  $B$  equipped with the forgetful functor. On the other hand, an affine group scheme  $G$  over  $\mathbf{k}$  is a scheme represented by a commutative  $\mathbf{k}$ -Hopf algebra  $H = \mathbf{k}[G]$ , and representations of  $G$  are the same as right  $H$ -comodules (see e.g. [Wa, §1.4 and §3.2] or [Ja, §I.2]). A commutative Hopf algebra is a coalgebra with the extra datum of an associative and commutative multiplication with unit, plus the existence of the antipode. Striving to translate this setup into the language of categories, we look for the extra structures on an abelian  $\mathbf{k}$ -linear category that characterize categories of representations of affine group schemes.

The adequate notion is called rigid abelian tensor categories. Rather than studying this notion in the greatest possible generality, which would take too much space for the expected benefit, we will state a theorem tailored to our goal of understanding the geometric Satake equivalence. For a more thorough (and formal) treatment, the reader is referred to [SR] and [De], or to [Ka] for a more leisurely walk.

A last word before stating the main theorem of this subsection: a multiplication map

$$\text{mult} : B \otimes_{\mathbf{k}} B \rightarrow B$$

on a coalgebra  $B$  which is a coalgebra morphism allows to define a structure of  $B$ -comodule on the tensor product over  $\mathbf{k}$  of two  $B$ -comodules. Specifically, if  $M$  and  $M'$  are  $B$ -comodules with structure maps  $\delta_M : M \rightarrow M \otimes_{\mathbf{k}} B$  and  $\delta_{M'} : M' \rightarrow M' \otimes_{\mathbf{k}} B$ , then the structure map on  $M \otimes_{\mathbf{k}} M'$  is defined by the composition depicted on the diagram

$$\begin{array}{ccc} M \otimes_{\mathbf{k}} M' & \xrightarrow{\delta_{M \otimes M'}} & M \otimes_{\mathbf{k}} M' \otimes_{\mathbf{k}} B \\ \delta_M \otimes \delta_{M'} \downarrow & & \uparrow \text{id}_{M \otimes M'} \otimes \text{mult} \\ M \otimes_{\mathbf{k}} B \otimes_{\mathbf{k}} M' \otimes_{\mathbf{k}} B & \longrightarrow & M \otimes_{\mathbf{k}} M' \otimes_{\mathbf{k}} B \otimes_{\mathbf{k}} B \end{array}$$

where the bottom arrow is the usual commutativity constraint for tensor products of  $\mathbf{k}$ -vector spaces that swaps the second and third factors.

Given an affine group scheme  $G$  over  $\mathbf{k}$ , we denote the category of finite-dimensional representations of  $G$  (or equivalently finite dimensional right  $\mathbf{k}[G]$ -comodules) by  $\text{Rep}_{\mathbf{k}}(G)$ .

**Theorem 2.7.** *Let  $\mathcal{C}$  be an abelian  $\mathbf{k}$ -linear category equipped with the following data:*

- an exact  $\mathbf{k}$ -linear faithful functor  $\omega : \mathcal{C} \rightarrow \text{Vect}_{\mathbf{k}}$  (called the fiber functor);

- a  $\mathbf{k}$ -bilinear functor  $\otimes : \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$  (the tensor product);
- an object  $U \in \mathcal{C}$  (the tensor unit);
- an isomorphism  $\phi_{X,Y,Z} : X \otimes (Y \otimes Z) \xrightarrow{\sim} (X \otimes Y) \otimes Z$ , natural in  $X, Y$  and  $Z$  (the associativity constraint);
- isomorphisms  $U \otimes X \xrightarrow[\sim]{\lambda_X} X \xleftarrow[\sim]{\rho_X} X \otimes U$ , both natural in  $X$  (the unit constraints);
- an isomorphism  $\psi_{X,Y} : X \otimes Y \xrightarrow{\sim} Y \otimes X$  natural in  $X$  and  $Y$  (the commutativity constraint).

We also assume we are given isomorphisms  $v : \mathbf{k} \xrightarrow{\sim} \omega(U)$  and

$$\tau_{X,Y} : \omega(X) \otimes_{\mathbf{k}} \omega(Y) \xrightarrow{\sim} \omega(X \otimes Y) \quad (2.2)$$

in  $\mathbf{Vect}_{\mathbf{k}}$ , with  $\tau_{X,Y}$  natural in  $X, Y \in \mathcal{C}$ . Finally, we assume the following conditions hold:

1. Taking into account the identifications provided by  $\tau$  and  $v$ , the isomorphisms  $\omega(\phi_{X,Y,Z})$ ,  $\omega(\lambda_X)$ ,  $\omega(\rho_X)$  and  $\omega(\psi_{X,Y})$  are the usual associativity, unit and commutativity constraints in  $\mathbf{Vect}_{\mathbf{k}}$ .
2. If  $\dim_{\mathbf{k}}(\omega(X)) = 1$ , then there exists  $X^* \in \mathcal{C}$  such that  $X \otimes X^* \cong U$ .

Under these assumptions, there exists an affine group scheme  $G$  such that  $\omega$  admits a canonical factorization

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{\bar{\omega}} & \mathbf{Rep}_{\mathbf{k}}(G) \\ & \searrow \omega & \swarrow \text{forget} \\ & & \mathbf{Vect}_{\mathbf{k}} \end{array}$$

where  $\bar{\omega}$  is an equivalence of categories that respects the tensor product and the unit in the sense of the compatibility condition (1).

*Remark 2.8.* 1. It will be clear from the proof below that the group scheme  $G$  is the “automorphism group of the fiber functor.” This sentence means that for any commutative  $\mathbf{k}$ -algebra  $R$ , an element  $\alpha \in G(R)$  is a collection of elements  $\alpha_X \in \text{End}_R(\omega(X) \otimes_{\mathbf{k}} R)$ , natural in  $X \in \mathcal{C}$ , and compatible with  $\otimes$  and  $U$  via the isomorphisms  $\tau$  and  $v$ . There is no need to specifically ask for invertibility: this will automatically follow from the compatibility condition (2).

2. The datum of isomorphisms (2.2) satisfying condition (1) are usually worded as: “the functor  $\omega$  is a tensor functor.”
3. The faithfulness of  $\omega$  and the compatibility condition (1) imply that the associativity constraint  $\phi$  (respectively, the unit constraints  $\lambda$  and  $\rho$ , the commutativity constraint  $\psi$ ) of  $\mathcal{C}$  satisfies MacLane’s pentagon axiom (respectively, the triangle axiom, the hexagon axiom). Together, these coherence axioms imply that any diagram built from the constraints commutes. This makes multiple tensor products in  $\mathcal{C}$  non-ambiguous, see [McL, §VII.2].



4. Our formulation dropped completely the “rigidity condition” in the usual formulation of the Tannakian reconstruction theorem. This condition demands that each object  $X$  has a dual  $X^\vee$  characterized by an evaluation map  $X^\vee \otimes X \rightarrow U$  and a coevaluation map  $U \rightarrow X \otimes X^\vee$ . Its purpose is to guarantee the existence of inverses in  $G$ —without it,  $G$  would only be an affine monoid scheme. In Theorem 2.7, it has been replaced by condition (2), which is easier to check in the case we have in mind. See [DM, Proposition 1.20 and Remark 2.18] for a more precise study of the relationship between these conditions.
5. As in Example 2.5(3), if we start with the category  $\mathcal{C} = \text{Rep}_{\mathbf{k}}(G)$  for some  $\mathbf{k}$ -group scheme  $G$ , with  $\omega$  being the natural forgetful functor, then the group scheme reconstructed in Theorem 2.7 identifies canonically with  $G$ .

*Proof.* We first remark that the bifunctor  $\otimes : \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$  is exact in each variable: this follows from the analogous fact in the category  $\text{Vect}_{\mathbf{k}}$  together with Lemma 2.3(3).

We reuse the notation  $\langle X \rangle$ ,  $A_X$  and  $B_X$  from §§2.1–2.3. The direct limit of the coalgebras  $B_X$  is the coalgebra  $B$ , with comultiplication  $\Delta$  and counit  $\varepsilon$ .

Let  $X$  and  $X'$  two objects in  $\mathcal{C}$ . The isomorphism  $\tau_{X,X'} : \omega(X) \otimes_{\mathbf{k}} \omega(X') \xrightarrow{\sim} \omega(X \otimes X')$  induces an isomorphism of algebras

$$\text{End}_{\mathbf{k}}(\omega(X \otimes X')) \cong \text{End}_{\mathbf{k}}(\omega(X)) \otimes_{\mathbf{k}} \text{End}_{\mathbf{k}}(\omega(X')). \quad (2.3)$$

The opening remark in this proof implies that given subobjects  $Y \subset X^{\oplus n}$  and  $Y' \subset (X')^{\oplus n'}$ , the tensor products  $Y \otimes X'$  and  $X \otimes Y'$  are subobjects of respectively  $(X \otimes X')^{\oplus n}$  and  $(X \otimes X')^{\oplus n'}$ . It follows that the isomorphism (2.3) takes  $A_{X \otimes X'}$  into  $A_X \otimes_{\mathbf{k}} A_{X'}$ . Taking the duals, we get a morphism of coalgebras  $B_{X'} \otimes_{\mathbf{k}} B_X \rightarrow B_{X \otimes X'}$ , and taking the direct limit over  $\langle X \rangle$  and  $\langle X' \rangle$ , we obtain a morphism of coalgebras  $m : B \otimes_{\mathbf{k}} B \rightarrow B$ .

On the other hand, the  $\mathbf{k}$ -vector space  $\omega(U)$  has dimension 1, so the algebra  $A_U$  is reduced to  $\text{End}_{\mathbf{k}}(\omega(U)) = \mathbf{k}$ . Thus  $B_U$  is the trivial one-dimensional  $\mathbf{k}$ -coalgebra, and the definition of  $B$  as a direct limit of the  $B_X$  (including  $B_U$ ) leads to a morphism of coalgebras  $\eta : \mathbf{k} \rightarrow B$ .

Our coalgebra  $B$  is thus equipped with a multiplication  $m : B \otimes_{\mathbf{k}} B \rightarrow B$  and a unit  $\eta : \mathbf{k} \rightarrow B$ . The compatibility condition (1) implies that  $(B, m, \eta)$  is an associative and commutative  $\mathbf{k}$ -algebra with unit.

Let us call  $G$  the spectrum of the commutative  $\mathbf{k}$ -algebra  $(B, m, \eta)$ ; this is an affine scheme over  $\mathbf{k}$ . The commutative diagrams that express the fact that  $m$  and  $\eta$  are morphisms of coalgebras also say that  $\Delta$  and  $\varepsilon$  are morphisms of algebras. The latter thus define morphisms of schemes

$$\Delta^* : G \times_{\text{Spec}(\mathbf{k})} G \rightarrow G \quad \text{and} \quad \varepsilon^* : \text{Spec}(\mathbf{k}) \rightarrow G.$$

The coassociativity of  $\Delta$  and the counit property then imply that  $(G, \Delta^*, \varepsilon^*)$  is an affine monoid scheme. It thus only remains to show that the elements of  $G$  are invertible.

Unwinding the construction that led to the definition of  $G$ , we see that for any commutative  $\mathbf{k}$ -algebra  $R$ , an element  $\alpha \in G(R)$  is a collection of elements  $\alpha_X \in \text{End}_R(\omega(X) \otimes_{\mathbf{k}} R)$ , natural in  $X$ , and compatible with  $\otimes$  and  $U$ . We want to show that  $\alpha_X$  is invertible for all objects  $X$ .

First, if  $X$  is such that  $\dim \omega(X) = 1$ , then by condition (2) there exists  $X^*$  such that  $X \otimes X^* \cong U$ , and therefore  $\alpha_X \otimes \alpha_{X^*}$  is conjugate to  $\alpha_U = \text{id}_{\omega(U) \otimes_{\mathbf{k}} R}$ , so  $\alpha_X$  (an endomorphism of a free  $R$ -module of rank 1) is invertible.

To deal with the general case, one constructs the exterior power  $\bigwedge^d X$  as the quotient of the  $d$ -th tensor power of  $X$  by the appropriate relations (defined with the help of the commutativity constraint of  $\mathcal{C}$ ), for  $d = \dim \omega(X)$ . Since  $\omega$  is compatible with the commutativity constraint,  $\omega(\bigwedge^d X) \cong \bigwedge^d \omega(X)$  is 1-dimensional. As we saw in our particular case, this implies that  $\alpha_{\bigwedge^d X}$  is invertible. But this is  $\bigwedge^d \alpha_X$  (in other words the determinant of  $\alpha_X$ ), so we eventually obtain that  $\alpha_X$  (an endomorphism of a free  $R$ -module of rank  $d$ ) is invertible.  $\square$

*Example 2.9.* 1. Continue with Example 2.5(2), and suppose now that our category  $\mathcal{C}$  of finite-dimensional  $M$ -graded  $\mathbf{k}$ -vector spaces is endowed with a tensor product  $\otimes$ . There is then a law  $*$  on  $M$  such that

$$\mathbf{k}[m] \otimes \mathbf{k}[n] = \mathbf{k}[m * n]$$

for all  $m, n \in M$  (where  $\mathbf{k}[p]$  means  $\mathbf{k}$  placed in degree  $p$ ). The constraints (1) in the theorem impose that  $M$  is a commutative monoid, and then  $B = \mathbf{k}M$  is the associated monoid algebra. The condition (2), if verified, implies that  $M$  is indeed a group. The affine group scheme  $G = \text{Spec}(B)$  given by the theorem is then the Cartier dual of  $M$  (see [Wa, §2.4]).

2. Let  $X$  be a connected topological manifold, let  $\mathcal{C}$  be the category of local systems on  $X$  with coefficients in  $\mathbf{k}$ , let  $x \in X$ , and let  $\omega$  be the functor  $\mathcal{L} \mapsto \mathcal{L}_x$ , the fiber at point  $x$ . Then  $G$  is the constant group scheme equal to the fundamental group  $\pi_1(X, x)$ . On this example, we see how the choice of a fiber functor subtly changes the group.
3. We define the category  $\text{SVect}_{\mathbf{k}}$  of supersymmetric  $\mathbf{k}$ -vector spaces as the category of  $\mathbf{Z}/2\mathbf{Z}$ -graded vector spaces, equipped with the usual tensor product, with the usual associativity and unit constraints, but with the supersymmetric commutativity constraint: for  $V = V_0 \oplus V_1$  and  $W = W_0 \oplus W_1$ , the isomorphism  $\psi_{V,W} : V \otimes_{\mathbf{k}} W \rightarrow W \otimes_{\mathbf{k}} V$  is defined as

$$\psi_{V,W}(v \otimes w) = \begin{cases} w \otimes v & \text{if } v \in V_0 \text{ or } w \in W_0, \\ -(w \otimes v) & \text{if } v \in V_1 \text{ and } w \in W_1. \end{cases}$$

Then the forgetful functor from  $\text{SVect}_{\mathbf{k}}$  to  $\text{Vect}_{\mathbf{k}}$  does not respect the commutativity constraints, so one cannot apply the theorem to this situation.

Proposition 2.6 also has a tensor analog. We state without proof the assertion that is needed in the proof of the geometric Satake equivalence. Observe that a homomorphism of  $\mathbf{k}$ -group schemes  $f : H \rightarrow G$  induces a restriction functor  $f^* : \text{Rep}_{\mathbf{k}}(G) \rightarrow \text{Rep}_{\mathbf{k}}(H)$ .

**Proposition 2.10.** 1. *Let  $\mathcal{C}$  be an abelian  $\mathbf{k}$ -linear category with tensor product, tensor unit, and associativity, commutativity and unit constraints. Let  $H$  be an affine group scheme over  $\mathbf{k}$ . Let  $F : \mathcal{C} \rightarrow \text{Rep}_{\mathbf{k}}(H)$  be a  $\mathbf{k}$ -linear exact faithful functor, compatible with the monoidal structure and the various constraints (in the same sense as in Theorem 2.7). Let  $G$  be the affine  $\mathbf{k}$ -group scheme provided by Theorem 2.7 (for the composition of  $F$  with the forgetful functor  $\text{Rep}_{\mathbf{k}}(H) \rightarrow \text{Vect}_{\mathbf{k}}$ ) and  $\bar{F} : \mathcal{C} \xrightarrow{\sim} \text{Rep}_{\mathbf{k}}(G)$*

be the corresponding equivalence. Then there exists a unique morphism of group schemes  $f : H \rightarrow G$  such that the following diagram commutes:

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{\overline{F}} & \mathrm{Rep}_{\mathbf{k}}(G) \\ & \searrow F & \swarrow f^* \\ & & \mathrm{Rep}_{\mathbf{k}}(H). \end{array}$$

2. If  $G$  and  $H$  are affine  $\mathbf{k}$ -group schemes, any  $\mathbf{k}$ -linear functor  $F : \mathrm{Rep}_{\mathbf{k}}(G) \rightarrow \mathrm{Rep}_{\mathbf{k}}(H)$  compatible with tensor products, the forgetful functors, and the various constraints is of the form  $f^*$  for a unique  $\mathbf{k}$ -group scheme morphism  $f : H \rightarrow G$ .

## 2.5 Properties of $G$ visible on $\mathrm{Rep}_{\mathbf{k}}(G)$

Recall that an affine  $\mathbf{k}$ -group scheme  $G$  is called *algebraic* if the  $\mathbf{k}$ -algebra of regular functions on  $G$  is finitely generated. In this case, the construction of the group of connected components  $\pi_0(G)$  of  $G$  is recalled in [Mi, §XIII.3] or [Wa, §6.7]; this is an affine étale  $\mathbf{k}$ -group scheme endowed with a canonical morphism  $G \rightarrow \pi_0(G)$ , and  $G$  is connected iff  $\pi_0(G)$  is trivial.

Recall also that an affine algebraic group scheme  $G$  over  $\mathbf{k}$  is called *reductive*<sup>3</sup> if it is smooth<sup>4</sup> (hence in particular algebraic) and connected and, for an algebraic closure  $\overline{\mathbf{k}}$  of  $\mathbf{k}$ , the group  $\mathrm{Spec}(\overline{\mathbf{k}}) \times_{\mathrm{Spec}(\mathbf{k})} G$  is reductive in the usual sense, i.e. does not contain any nontrivial smooth connected normal unipotent subgroup; see [Mi, Definition XVII.2.1].

**Proposition 2.11.** 1. Let  $G$  be an affine group scheme over  $\mathbf{k}$ . Then  $G$  is algebraic if and only if there exists  $X \in \mathrm{Rep}_{\mathbf{k}}(G)$  such that  $X$  generates  $\mathrm{Rep}_{\mathbf{k}}(G)$  by taking direct sums, tensor products, duals, and subquotients.

2. Let  $G$  be an algebraic affine group scheme over  $\mathbf{k}$ . If  $G$  is not connected, then there exists a nontrivial representation  $X \in \mathrm{Rep}_{\mathbf{k}}(G)$  such that the subcategory  $\langle X \rangle$  (with the notation of §2.1) is stable under  $\otimes$ .

3. Let  $G$  be a connected algebraic affine group scheme over  $\mathbf{k}$ . Assume that  $\mathbf{k}$  has characteristic 0, and that  $\mathrm{Rep}_{\overline{\mathbf{k}}}(G_{\overline{\mathbf{k}}})$  is semisimple, where  $\overline{\mathbf{k}}$  is an algebraic closure of  $\mathbf{k}$  and  $G_{\overline{\mathbf{k}}} = \mathrm{Spec}(\overline{\mathbf{k}}) \times_{\mathrm{Spec}(\mathbf{k})} G$ . Then  $G$  is reductive.

*Proof.* (1) Suppose  $G$  is algebraic. Then  $G$  admits a faithful representation (see [Wa, §3.4]), i.e.  $G$  can be viewed as a closed subgroup of some  $\mathbf{GL}_n$ . It is then a classical result that any finite dimensional representation of  $G$  can be obtained from the representation on  $\mathbf{k}^n$  by the processes of forming tensor products, direct sums, subrepresentations, quotients and duals (see [Wa, §3.5]).

<sup>3</sup>Sometimes, the definition of reductive groups allows disconnected groups. All the reductive groups we will consider in these notes will be (sometimes tacitly) assumed to be connected. Note that connectedness can be checked after base change to an algebraic closure of the base field; see [Mi, Proposition XIII.3.8].

<sup>4</sup>Recall that this condition is automatic if  $G$  is algebraic and  $\mathrm{char}(\mathbf{k}) = 0$ ; see [Mi, Theorem VI.9.3].

Conversely, suppose the existence of a representation  $X$  that generates  $\text{Rep}_{\mathbf{k}}(G)$  in the sense explained in the statement of the proposition. Then  $X$  is necessarily a faithful representation of  $G$ , so  $G$  embeds as a closed subgroup in  $\mathbf{GL}(X)$  (see [Wa, §15.3]), and is therefore algebraic.

(2) The quotient  $G \rightarrow \pi_0(G)$  induces a fully faithful functor  $\text{Rep}_{\mathbf{k}}(\pi_0(G)) \rightarrow \text{Rep}_{\mathbf{k}}(G)$ . Taking for  $X$  the image of the regular representation of  $\pi_0(G)$ , we see that  $\langle X \rangle$  (which coincides with the essential image of  $\text{Rep}_{\mathbf{k}}(\pi_0(G))$ ) is stable under tensor products. If  $G$  is not connected, then  $X$  is not trivial.

(3) As explained in Footnote 4,  $G$  is automatically smooth since  $\text{char}(\mathbf{k}) = 0$ . Hence the only thing we have to check is that the unipotent radical  $R_u(G_{\overline{\mathbf{k}}})$  is trivial. In a simple representation  $X$  of  $G_{\overline{\mathbf{k}}}$ ,  $R_u(G_{\overline{\mathbf{k}}})$  acts trivially; indeed the subspace of points fixed by  $R_u(G_{\overline{\mathbf{k}}})$  is nontrivial by Kolchin's fixed point theorem (see [Wa, §8.2]) and is  $G_{\overline{\mathbf{k}}}$ -stable because  $R_u(G_{\overline{\mathbf{k}}})$  is a normal subgroup. This result immediately extends to semisimple representations of  $G_{\overline{\mathbf{k}}}$ . Now if  $\text{Rep}_{\overline{\mathbf{k}}}(G_{\overline{\mathbf{k}}})$  is semisimple, then  $G_{\overline{\mathbf{k}}}$  admits a semisimple faithful representation. On this representation,  $R_u(G_{\overline{\mathbf{k}}})$  acts trivially and faithfully. Therefore  $R_u(G_{\overline{\mathbf{k}}})$  is trivial.  $\square$

*Remark 2.12.* 1. An object which satisfies the conditions in Proposition 2.11(1) will be called a *tensor generator* of the category  $\text{Rep}_{\mathbf{k}}(G)$ .

2. An algebraic affine group scheme is called *strongly connected* if it admits no nontrivial finite quotient. (This property is in general stronger than being connected—which is equivalent to having no nontrivial finite *étale* quotient—but these notions are equivalent if  $\text{char}(\mathbf{k}) = 0$ .) If  $G$  is an algebraic affine group scheme over  $\mathbf{k}$ , the condition appearing in Proposition 2.11(2) is equivalent to  $G$  being strongly connected, see [Mi, §XVII.7].

3. A more precise version of Proposition 2.11(3) is proved in [DM, Proposition 2.23]. (But the simpler version we stated will be sufficient for our purposes.)

### 3 The affine Grassmannian

In this section we provide a brief introduction to the affine Grassmannian of a complex connected reductive algebraic group. For more details, examples and references, the reader can e.g. consult [Gö, §2] or [Z4]. (All of these properties are often considered as “well known,” and we have not tried to give the original references for them, but rather the most convenient one.)

#### 3.1 Definition

We set  $\mathcal{O} := \mathbf{C}[[t]]$  and  $\mathcal{K} := \mathbf{C}((t))$ , where  $t$  is an indeterminate. If  $H$  is a linear complex algebraic group, we denote by  $H_{\mathcal{O}}$ , resp.  $H_{\mathcal{K}}$ , the functor from  $\mathbf{C}$ -algebras to groups defined by

$$R \mapsto H(R[[t]]), \quad \text{resp.} \quad R \mapsto H(R((t))).$$

It is not difficult to check that  $H_{\mathcal{O}}$  is represented by a  $\mathbf{C}$ -group scheme (not of finite type in most cases), and that  $H_{\mathcal{K}}$  is represented by an ind-group scheme (i.e. an inductive limit

of schemes parametrized by  $\mathbf{Z}_{\geq 0}$ , with closed embeddings as transition maps). We will still denote these (ind-)group schemes by  $H_{\mathcal{O}}$  and  $H_{\mathcal{K}}$ .

We now fix a standard triple  $G \supset B \supset T$  of a connected complex reductive algebraic group, a Borel subgroup, and a maximal torus. We will denote by  $N$  the unipotent radical of  $B$ . We will denote by  $\Delta(G, T)$  the root system of  $(G, T)$ , by  $\Delta_+(G, B, T) \subset \Delta(G, T)$  the subset of positive roots (consisting of the  $T$ -weights in the Lie algebra of  $B$ ), and by  $\Delta_s(G, B, T)$  the corresponding subset of simple roots. For  $\alpha$  a root, we will denote the corresponding coroot by  $\alpha^\vee$ .

Let  $X_*(T)$  be the lattice of cocharacters of  $T$ ; it contains the coroot lattice  $Q^\vee$  and is endowed with the standard order  $\leq$  (such that nonnegative elements are nonnegative integral combinations of positive coroots). We will denote by  $X_*(T)^+ \subset X_*(T)$  the cone of dominant cocharacters. We define  $\rho$  as the halfsum of the positive roots and regard it as a linear form  $X_*(T) \rightarrow \frac{1}{2}\mathbf{Z}$ .

If  $L^{\leq 0}G$  denotes the ind-group scheme which represents the functor

$$R \mapsto G(R[t^{-1}])$$

and if  $L^{< 0}G$  is the kernel of the natural morphism  $L^{\leq 0}G \rightarrow G$  (sending  $t^{-1}$  to 0), then  $L^{< 0}G$  is a subgroup of  $G_{\mathcal{K}}$  in a natural way, and the multiplication morphism

$$L^{< 0}G \times G_{\mathcal{O}} \rightarrow G_{\mathcal{K}}$$

is an open embedding by [Fa, Lemma 3] (see also [NP, Lemme 2.1] or [Z4, Lemma 2.3.5]). In view of this property, the quotient

$$\widetilde{\mathrm{Gr}}_G := G_{\mathcal{K}}/G_{\mathcal{O}}$$

has a natural structure of ind-scheme. In fact, one can check that this ind-scheme is ind-proper, and of ind-finite type.

*Remark 3.1.* 1. In many references (but not [Fa]),  $\widetilde{\mathrm{Gr}}_G$  is rather defined as the object representing a certain presheaf on the category of affine  $\mathbf{C}$ -schemes (see e.g. [Z4, Theorem 1.2.2]) and then identified with a fpqc quotient  $G_{\mathcal{K}}/G_{\mathcal{O}}$ , see [Z4, Proposition 1.3.6]. Finally, it is realized that the quotient map  $G_{\mathcal{K}} \rightarrow \widetilde{\mathrm{Gr}}_G$  is Zariski locally trivial. In these notes the “moduli interpretation” of  $\widetilde{\mathrm{Gr}}_G$  will be introduced in Section 7 below.

2. Consider the group scheme  $\mathfrak{G} := G \times_{\mathrm{Spec}(\mathbf{C})} \mathrm{Spec}(\mathcal{O})$  over  $\mathrm{Spec}(\mathcal{O})$ . Then  $G_{\mathcal{O}}$  is the *arc space* of  $\mathfrak{G}$ , and  $G_{\mathcal{K}}$  is the *loop space* of  $\mathfrak{G} \times_{\mathrm{Spec}(\mathcal{O})} \mathrm{Spec}(\mathcal{K})$  in the sense of [Z4, Definition 1.3.1]. From this point of view one can consider the “affine Grassmannian” of more general (smooth, affine) group schemes over  $\mathrm{Spec}(\mathcal{O})$ . In particular, replacing  $\mathfrak{G}$  by the Iwahori group scheme constructed in Bruhat–Tits theory, then we obtain an ind-scheme  $\mathrm{Fl}_G$  which is often called the *affine flag variety* of  $G$ .

3. See also [Z4, §1.6] for a description of  $\widetilde{\mathrm{Gr}}_G$  in terms of the loop group of a maximal compact subgroup (which only makes sense for the case of *complex* reductive groups, unlike the other descriptions considered above). This approach is crucial in the proof of Ginzburg [Gi]; it also shows that the torsor  $G_{\mathcal{K}} \rightarrow \widetilde{\mathrm{Gr}}_G$  is topologically trivial.

In general, the quotient  $G_{\mathcal{K}}/G_{\mathcal{O}}$  is not reduced. (This can already be seen when  $G$  is the multiplicative group  $\mathbb{G}_m$ .) Since we will only consider constructible sheaves on this quotient, this non-reduced structure can be forgotten, and we will denote by  $\mathrm{Gr}_G$  the (reduced) ind-variety associated with the ind-scheme  $\widetilde{\mathrm{Gr}}_G$ .

Any cocharacter  $\nu \in X_*(T)$  defines a morphism  $\mathcal{K}^\times \rightarrow T_{\mathcal{K}}$ . The image of  $t$  under this morphism will be denoted by  $t^\nu$ . The coset  $t^\nu G_{\mathcal{O}}$  is a point in  $\mathrm{Gr}_G$ , which will be denoted by  $L_\nu$ .

The Cartan decomposition describes the  $G_{\mathcal{O}}$ -orbits in  $\mathrm{Gr}_G$ , in the following way (see [Z4, §2.1] for more details and references).

**Proposition 3.2.** *We have a decomposition*

$$\mathrm{Gr}_G = \bigsqcup_{\lambda \in X_*(T)^+} \mathrm{Gr}_G^\lambda, \quad \text{where } \mathrm{Gr}_G^\lambda := G_{\mathcal{O}} \cdot L_\lambda. \quad (3.1)$$

Moreover, this decomposition is a stratification of  $\mathrm{Gr}_G$  and, for any  $\lambda \in X_*(T)$ ,  $\mathrm{Gr}_G^\lambda$  is an affine bundle over the partial flag variety  $G/P_\lambda$  where  $P_\lambda$  is the parabolic subgroup of  $G$  containing  $B$  and associated with the subset of simple roots  $\{\alpha \in \Delta_s(G, B, T) \mid \langle \lambda, \alpha \rangle = 0\}$ . We also have

$$\dim(\mathrm{Gr}_G^\lambda) = \langle 2\rho, \lambda \rangle$$

and

$$\overline{\mathrm{Gr}_G^\lambda} = \bigsqcup_{\substack{\eta \in X_*(T)^+ \\ \eta \leq \lambda}} \mathrm{Gr}_G^\eta. \quad (3.2)$$

The stratification of  $\mathrm{Gr}_G$  by  $G_{\mathcal{O}}$ -orbits will be denoted by  $\mathcal{S}$ .

Finally, we will need a description of the connected components of  $\mathrm{Gr}_G$ . For any  $c \in X_*(T)/Q^\vee$ , let us set

$$\mathrm{Gr}_G^c := \bigsqcup_{\substack{\lambda \in X_*(T)^+ \\ \lambda + Q^\vee = c}} \mathrm{Gr}_G^\lambda.$$

Then the connected components of  $\mathrm{Gr}_G$  are exactly the subvarieties  $\mathrm{Gr}_G^c$  for  $c \in X_*(T)/Q^\vee$  (see [Z4, Comments after Theorem 1.3.11] for references). In particular, since  $\langle \rho, \lambda \rangle \in \mathbf{Z}$  for any  $\lambda \in Q^\vee$ , the parity of the dimensions of the Schubert varieties  $\mathrm{Gr}_G^\lambda$  is constant on each connected component. A connected component will be called even, resp. odd, if these dimensions are even, resp. odd.

### 3.2 Semi-infinite orbits

The Iwasawa decomposition describes (set-theoretically) the  $N_{\mathcal{K}}$ -orbits in  $\mathrm{Gr}_G$  as follows:

$$\mathrm{Gr}_G = \bigsqcup_{\mu \in X_*(T)} S_\mu, \quad \text{where } S_\mu := N_{\mathcal{K}} \cdot L_\mu. \quad (3.3)$$

Each orbit  $S_\mu$  is “infinite dimensional,” i.e. not contained in any finite type subscheme of  $\mathrm{Gr}_G$ .

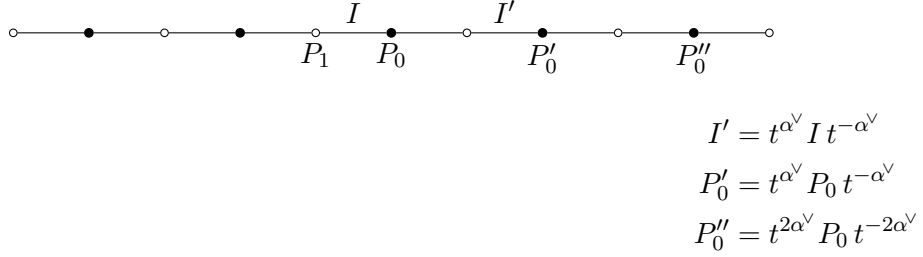


Figure 1: A small part of the Serre tree

The closure of these orbits for the inductive limit topology on  $\text{Gr}_G$  can be described in the following way:

$$\overline{S_\mu} = \bigsqcup_{\substack{\nu \in X_*(T) \\ \nu \leq \mu}} S_\nu.$$

We will soon sketch a formal proof of this equality (see Proposition 3.4 below), but let us first try to make this result intuitive, at least in the case  $G = \mathbf{SL}_2$ . For that, we denote by  $\alpha$  the unique positive root, and consider the standard Iwahori subgroup

$$I = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbf{SL}_{2,\mathcal{K}} \mid a, b, t^{-1}c, d \in \mathcal{O} \right\}$$

and the two maximal parahoric subgroups

$$P_0 = \mathbf{SL}_{2,\mathcal{O}}, \quad P_1 = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbf{SL}_{2,\mathcal{K}} \mid a, tb, t^{-1}c, d \in \mathcal{O} \right\}$$

that contain  $I$ .

A parahoric subgroup of  $\mathbf{SL}_{2,\mathcal{K}}$  is a subgroup conjugated to one of these three standard subgroups  $I$ ,  $P_0$  or  $P_1$ . Parahoric subgroups form a poset for the inclusion. The Serre tree is the simplicial realization of the opposite poset. Figure 1 shows a small part of this tree; namely we just pictured the parahoric subgroups that are conjugated to the standard ones by elements of the affine Weyl group (see §4.2). Note here that the inclusions  $P_1 \supset I \subset P_0$  translate to the fact that the vertices corresponding to  $P_1$  and  $P_0$  are incident to the edge corresponding to  $I$ .

Since Iwahori subgroups are conjugated in  $\mathbf{SL}_{2,\mathcal{K}}$  and since  $I$  is its own normalizer, the set of edges in the tree incident to a given edge is in bijection with the so-called affine flag variety  $\mathbf{SL}_{2,\mathcal{K}}/I \cong \mathbb{P}_{\mathcal{C}}^1$ . Likewise, the set of parahoric subgroups conjugated to  $P_0$ , depicted as black dots on the tree, is in bijection with the affine Grassmannian  $\text{Gr}_{\mathbf{SL}_2} = \mathbf{SL}_{2,\mathcal{K}}/P_0$ . One can rephrase this by saying that the group  $\mathbf{SL}_{2,\mathcal{K}}$  acts on the tree (transitively on the edges, on the black vertices, and on the white vertices) and that the stabilizer of the simplex associated to a parahoric subgroup is the subgroup itself. (The white dots form the second connected component in the affine Grassmannian  $\text{Gr}_{\mathbf{PGL}_2}$ .) A slightly more accurate picture of the Serre tree is given in Figure 2. (But here again we only draw a finite number of edges incident to each vertex, while as explained above such edges are in fact in bijection with  $\mathbb{P}_{\mathcal{C}}^1$ .)

Likewise, the Iwahori subgroups contained in  $P_0$  can be obtained by letting the normalizer of  $P_0$  act on  $I$ ; in other words, the set of edges incident to the vertex  $P_0$  is in bijection with

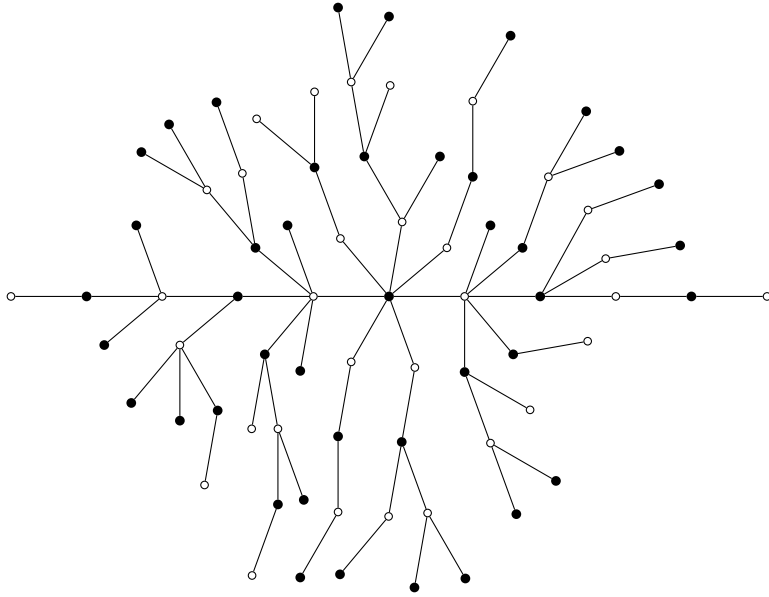


Figure 2: Intuitive picture for  $\text{Gr}_{\mathbf{SL}_2}$

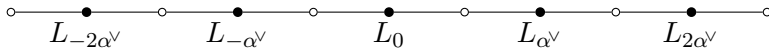


Figure 3: More points in  $\text{Gr}_{\mathbf{SL}_2}$

$P_0/I$ . This is a complex projective line. So the set of edges incident to a given black vertex is a complex projective line. (The same thing holds also for white vertices.) Our drawings are thus quite incomplete, because a lot of edges were omitted.

Again, our affine Grassmannian is the set of all black vertices. Here it is worth noting that the tree metric is related to the description of the ind-structure of  $\text{Gr}_{\mathbf{SL}_2}$ : one can take for the  $n$ -th finite-dimensional piece of  $\text{Gr}_{\mathbf{SL}_2}$  the set  $\text{Gr}_n$  of all vertices at distance  $\leq 2n$  from  $P_0$ . Further, the analytic (respectively, Zariski) topology of the variety  $\text{Gr}_n$  can also be seen on the tree: it comes from the analytic (respectively, Zariski) topology on all the projective lines mentioned in the previous paragraph. Thus, we can for instance see that  $\text{Gr}_n$  is dominated by a tower of  $2n$  projective lines, because each point at distance  $\leq 2n$  from the origin can be reached by choosing first an edge around the origin, then another edge around the white vertex at the end of this edge, and so on  $2n$  times.<sup>5</sup>

Now let us see how our orbits  $\text{Gr}_G^\lambda$  and  $S_\mu$  are depicted in this model. The point  $L_\nu$  corresponds to the Iwahori subgroup  $t^\nu P_0 t^{-\nu}$ , see Figure 3.

The Schubert cell  $\text{Gr}_G^\lambda$  is the orbit of  $L_\lambda$  under the stabilizer of the base point  $L_0$ ; it therefore looks like the sphere with center  $L_0$  going through  $L_\lambda$ . On Figure 4, the diamonds are points in  $\text{Gr}_G^{\alpha^\vee}$ .

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<sup>5</sup>From the algebro-geometric point of view, this process is a particular case of a Bott–Samelson resolution, as explained in [GL].



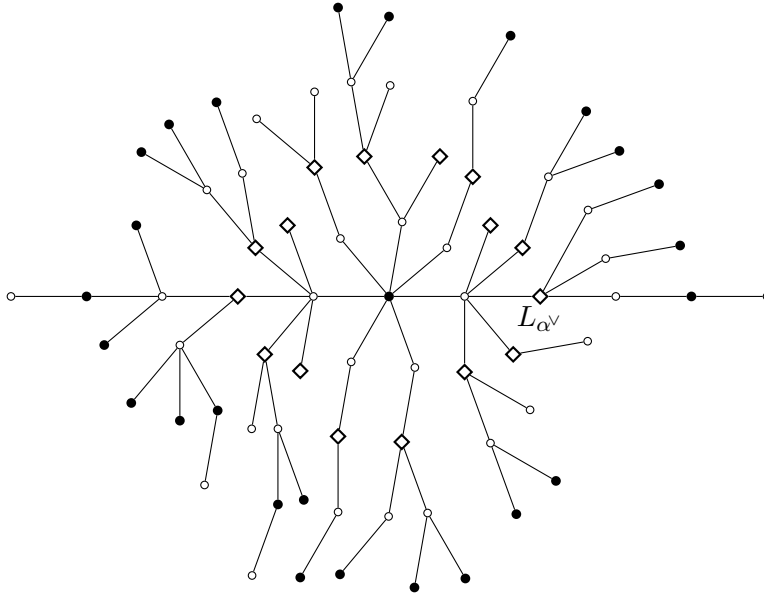


Figure 4: Illustration of a Schubert cell in  $\text{Gr}_{\text{SL}_2}$

Now look at the white vertex between the origin and  $L_{\alpha^\vee}$ : edges starting from this vertex form a projective line, and the other vertices of these edges belong to  $\text{Gr}_G^{\alpha^\vee}$ , with one exception, namely  $L_0$ . This point  $L_0$  appears thus as a limit (on the projective line) of points in  $\text{Gr}_G^{\alpha^\vee}$ , that is, belongs to the closure of  $\text{Gr}_G^{\alpha^\vee}$ . This provides an intuitive interpretation of the inclusion  $\text{Gr}_G^0 \subset \overline{\text{Gr}_G^{\alpha^\vee}}$ .

In the same line of ideas, the semi-infinite orbit  $S_\mu$  can be depicted as the sphere centered at  $-\infty$  (also called “horosphere”) and going through  $L_\mu$ . In Figure 5, the diamonds are points in  $S_{\alpha^\vee}$ .

For the same reason as before, we see that  $L_0$  belongs to the closure of  $S_{\alpha^\vee}$ . The reader can however feel cheated here, since we relied on geometrical intuition. For a more formal proof, one computes<sup>6</sup>

$$\begin{pmatrix} 1 & at^{-1} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} t^{-1} & 0 \\ a^{-1} & t \end{pmatrix} \begin{pmatrix} t & a \\ -a^{-1} & 0 \end{pmatrix} \quad \text{for } a \in \mathbf{C}^\times,$$

that is

$$\begin{pmatrix} 1 & at^{-1} \\ 0 & 1 \end{pmatrix} G_{\mathcal{O}} = \begin{pmatrix} t^{-1} & 0 \\ a^{-1} & t \end{pmatrix} G_{\mathcal{O}},$$

and therefore

$$\underbrace{\begin{pmatrix} 1 & at^{-1} \\ 0 & 1 \end{pmatrix} G_{\mathcal{O}}}_{\in S_0} \rightarrow \underbrace{\begin{pmatrix} t^{-1} & 0 \\ 0 & t \end{pmatrix} G_{\mathcal{O}}}_{=L_{-\alpha^\vee}} \quad \text{when } a \rightarrow \infty.$$

Multiplying on the left by  $t^\mu$  leads to  $L_{\mu-\alpha^\vee} \in \overline{S_\mu}$ , whence  $S_{\mu-\alpha^\vee} \subset \overline{S_\mu}$ . This justifies (at least

<sup>6</sup>In fact, this computation is precisely our observation on the tree.

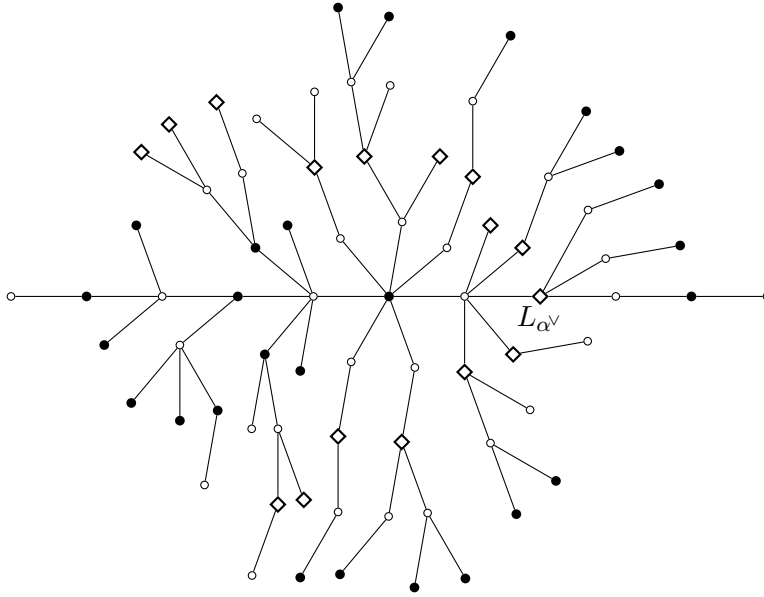


Figure 5: Illustration of a semi-infinite orbit in  $\text{GrSL}_2$

in the case of  $\mathbf{SL}_2$ , but the general case can be deduced from this special case) the inclusion

$$\overline{S_\mu} \supset \bigsqcup_{\substack{\nu \in X_*(T) \\ \nu \leq \mu}} S_\nu. \quad (3.4)$$

The proof of the reverse inclusion requires another tool, which is the subject of the next section.

### 3.3 Projective embeddings

We now want to embed the affine Grassmannian  $\text{Gr}_G$  in an (infinite dimensional) projective space  $\mathbf{P}(V)$  in order to get more control over its geometry. Replacing  $G$  by a simply connected cover of its derived subgroup may kill connected components, but has the advantage that the resulting group is a product of simple groups. Therefore in this subsection we assume that  $G$  is quasi-simple (i.e. that it is semisimple and that the quotient by its center is simple) and simply connected. (For the applications we consider, the general case will be reduced to this one.)

The character lattice  $X^*(T)$  of  $T$  is the  $\mathbf{Z}$ -dual of  $X_*(T)$ . Let  $W$  be the Weyl group of  $(G, T)$ , and let  $\tau : X_*(T) \rightarrow \mathbf{Z}$  be the  $W$ -invariant quadratic form that takes the value 1 at each short coroot. The polar form of  $\tau$  defines a map  $\iota : X_*(T) \rightarrow X^*(T)$ ; from the  $W$ -invariance of  $\tau$ , one deduces that

$$\iota(\alpha^\vee) = \tau(\alpha^\vee)\alpha \quad \text{for each coroot } \alpha^\vee. \quad (3.5)$$

Let  $\mathfrak{g}$  be the Lie algebra of  $G$ . The Lie algebra of  $T$  is a Cartan subalgebra  $\mathfrak{h}$  of  $\mathfrak{g}$ . Then  $\tau$

can be seen as the restriction to  $\mathfrak{h}$  of the Killing form of  $\mathfrak{g}$  (suitably rescaled), and  $X^*(T)$  is a lattice in the dual space  $\mathfrak{h}^\vee$ .

With the help of the Killing form of  $\mathfrak{g}$ , one defines a 2-cocycle of the Lie algebra  $\mathfrak{g} \otimes_{\mathbf{C}} \mathbf{C}[t, t^{-1}]$ , and thus a central extension

$$0 \rightarrow \mathbf{C}K \rightarrow \tilde{\mathfrak{g}} \xrightarrow{p} \mathfrak{g} \otimes_{\mathbf{C}} \mathbf{C}[t, t^{-1}] \rightarrow 0$$

of this algebra by a one-dimensional Lie algebra  $\mathbf{C}K$  (see [Kac, pp. 97–98]). The affine Kac–Moody Lie algebra is then obtained by taking a semidirect product

$$\hat{\mathfrak{g}} = \tilde{\mathfrak{g}} \rtimes \mathbf{C}d$$

with a one-dimensional Lie algebra  $\mathbf{C}d$ , where  $d$  acts as  $t \frac{d}{dt}$  on  $\mathfrak{g} \otimes_{\mathbf{C}} \mathbf{C}[t, t^{-1}]$ .

Further,  $\mathfrak{h} \subset \mathfrak{g} \otimes_{\mathbf{C}} \mathbf{C}[t, t^{-1}]$  can be canonically lifted in  $\tilde{\mathfrak{g}}$ . Then  $\hat{\mathfrak{h}} = \mathfrak{h} \oplus \mathbf{C}K \oplus \mathbf{C}d$  is a Cartan subalgebra of  $\hat{\mathfrak{g}}$ . Let  $\Lambda_0 \in (\hat{\mathfrak{h}})^\vee$  be the linear form that vanishes on  $\mathfrak{h} \oplus \mathbf{C}d$  and that maps  $K$  to 1. Let  $V(\Lambda_0)$  be the irreducible integrable highest weight representation of  $\hat{\mathfrak{g}}$  with highest weight  $\Lambda_0$ . It is generated by a highest weight vector  $v_{\Lambda_0}$ , and the stabilizer of the line  $[v_{\Lambda_0}]$  in  $\mathbf{P}(V(\Lambda_0))$  is the parabolic subalgebra  $p^{-1}(\mathfrak{g}[t]) \rtimes \mathbf{C}d$ .

Thanks to Garland’s work [Gar], we know that the representation  $V(\Lambda_0)$  can be integrated to the Kac–Moody group  $\hat{G}$  that corresponds to the Lie algebra  $\hat{\mathfrak{g}}$ . This group is the semi-direct product of a central extension

$$1 \rightarrow \mathbf{C}^\times \rightarrow \tilde{G} \rightarrow G(\mathbf{C}[t, t^{-1}]) \rightarrow 1$$

by another copy of  $\mathbf{C}^\times$ , acting by loop rotations. The central  $\mathbf{C}^\times$  in  $\tilde{G}$  acts by scalar multiplication on  $V(\Lambda_0)$ , so  $G(\mathbf{C}[t, t^{-1}])$  acts on  $\mathbf{P}(V(\Lambda_0))$ . Since the stabilizer of the line  $[v_{\Lambda_0}]$  for this action is the subgroup  $G(\mathbf{C}[t])$ , the map  $g \mapsto g \cdot [v_{\Lambda_0}]$  defines an embedding

$$\Psi : G(\mathbf{C}[t, t^{-1}])/G(\mathbf{C}[t]) \hookrightarrow \mathbf{P}(V(\Lambda_0)).$$

Further, using for instance the Iwasawa decomposition, one can show that on the level of  $\mathbf{C}$ -points, the obvious map

$$G(\mathbf{C}[t, t^{-1}])/G(\mathbf{C}[t]) \rightarrow G_{\mathcal{K}}/G_{\mathcal{O}} = \text{Gr}_G$$

is bijective. We eventually obtain a closed embedding

$$\Psi : \text{Gr}_G \rightarrow \mathbf{P}(V),$$

where (here and below) we write  $V$  instead of  $V(\Lambda_0)$  to shorten the notation.

Certainly,  $\mathbf{P}(V)$  has the structure of an ind-variety: the finite-dimensional pieces are all the finite-dimensional projective subspaces  $\mathbf{P}(W)$  inside  $\mathbf{P}(V)$ . Then  $\Psi$  is a morphism of ind-varieties. Even better: thanks to the work of Kumar (see [Ku, Chap. 7]), we know that the ind-variety structure of  $\text{Gr}_G$  is induced via  $\Psi$  by that of  $\mathbf{P}(V)$ .

Lastly, [Kac, (6.5.4)] implies that

$$\Psi(L_\nu) \in \mathbf{P}(V_{-\iota(\nu)}), \tag{3.6}$$

where  $V_{-\iota(\nu)}$  is the subspace of  $V$  of weight  $-\iota(\nu)$  for the action of  $\mathfrak{h} \subset \hat{\mathfrak{g}}$ .

*Remark 3.3.* See [PR, Remark 10.2(ii)] for a comparison between the group  $\tilde{G}$  considered above and a central extension of  $G_{\mathcal{K}}$  considered by Faltings in [Fa].

### 3.4 Consequences

After these lengthy preliminaries, we can go back to our problem. We drop our assumption that  $G$  is quasi-simple and simply connected.

**Proposition 3.4.** *Let  $\mu \in X_*(T)$ . Then*

$$\overline{S_\mu} = \bigsqcup_{\nu \leq \mu} S_\nu.$$

Moreover, there exists a  $\mathbf{C}$ -vector space  $V$  and a closed embedding  $\Psi : \mathrm{Gr}_G^{\mu+Q^\vee} \rightarrow \mathbf{P}(V)$  such that the boundary of  $S_\mu$  is the set-theoretic intersection of  $\overline{S_\mu}$  with a hyperplane  $H_\mu$  of  $\mathbf{P}(V)$ :

$$\partial S_\mu = \overline{S_\mu} \cap \Psi^{-1}(H_\mu).$$

*Proof.* First we assume that  $G$  is quasi-simple and simply-connected, and choose  $V$  and  $\Psi$  as in §3.3.

Let  $\lambda \in X_*(T)$ . Writing  $\Psi(L_\lambda) = \mathbf{C}v$ , the vector  $v$  belongs to the weight subspace  $V_{-\iota(\lambda)}$  of  $V$  by (3.6). The action on  $v$  of an element  $u \in N_{\mathcal{K}}$  can only increase weights,<sup>7</sup> hence  $uv - v \in \sum_{\chi > -\iota(\lambda)} V_\chi$ . (The order  $\geq$  on  $\mathfrak{h}^\vee$  used here is the dominance order: nonnegative elements in  $\mathfrak{h}^\vee$  are nonnegative integral combinations of positive roots.) It follows that

$$\Psi(u \cdot L_\lambda) \in \mathbf{P} \left( \sum_{\chi \geq -\iota(\lambda)} V_\chi \right) \setminus \mathbf{P} \left( \sum_{\chi > -\iota(\lambda)} V_\chi \right).$$

Letting  $u$  run over  $N_{\mathcal{K}}$ , we deduce

$$\Psi(S_\lambda) \subset \mathbf{P} \left( \sum_{\chi \geq -\iota(\lambda)} V_\chi \right) \setminus \mathbf{P} \left( \sum_{\chi > -\iota(\lambda)} V_\chi \right).$$

Writing these inclusions for all possible  $\lambda$ , we conclude that<sup>8</sup>

$$\bigsqcup_{\nu \leq \mu} S_\nu = \Psi^{-1} \left( \mathbf{P} \left( \sum_{\chi \geq -\iota(\mu)} V_\chi \right) \right).$$

This implies that  $\bigsqcup_{\nu \leq \mu} S_\nu$  is closed in  $\mathrm{Gr}_G$ , whence (in view of (3.4)) the first equality in the statement. For the second one, one chooses a linear form  $h \in V^\vee$  that vanishes on  $\sum_{\chi > -\iota(\mu)} V_\chi$  but does not vanish on  $\Psi(L_\mu)$  and takes  $H_\mu = \mathbf{P}(\ker h)$ .

<sup>7</sup>Certainly, above we have described  $V$  only as a projective representation of  $G(\mathbf{C}[t, t^{-1}])$ , so it seems hazardous to let  $N_{\mathcal{K}}$  act on  $V$ . To be more precise, we observe that the action of  $\hat{\mathfrak{g}}$  on  $V$  can be extended to its completion considered e.g. in [Ku, §13.1] (because the part one needs to complete acts in a locally nilpotent way). Then, using [Ku, Theorem 6.2.3 & Theorem 13.2.8] one sees that this action integrates to an action of a central extension of  $G_{\mathcal{K}}$ . Finally, one observes that the cocycle that defines the central extension is trivial on  $N_{\mathcal{K}}$ , so that this subgroup can be lifted to the central extension.

<sup>8</sup>Specifically, one must here observe that  $\nu \leq \mu$  in  $X_*(T)$  implies  $-\iota(\nu) \geq -\iota(\mu)$  in  $\mathfrak{h}^\vee$ . This follows from the equality (3.5).

The case of simply connected semisimple (but not necessarily quasi-simple) groups reduces to the preceding case since such a group is a product of simply connected quasi-simple groups. Finally, for  $G$  general, the action of  $t^{-\mu}$  identifies  $\mathrm{Gr}_G^{\mu+Q^\vee}$  with  $\mathrm{Gr}_G^{Q^\vee}$  (which itself identifies with the affine Grassmannian of a simply connected cover of the derived subgroup of  $G$ ) and sends  $S_\mu$  to  $S_0$ ; this reduces the proof to the case of simply connected semisimple groups, and allows to conclude.  $\square$

*Remark 3.5.* See [Z4, Corollary 5.3.8] for a different proof of Proposition 3.4, which avoids the use of Kac–Moody groups.

For symmetry reasons, one should also consider the Borel subgroup  $B^-$  opposite to  $B$  with respect to  $T$  and its unipotent radical  $N^-$ . One then has an Iwasawa decomposition

$$\mathrm{Gr}_G = \bigsqcup_{\mu \in X_*(T)} T_\mu, \quad \text{where } T_\mu = N_{\mathcal{K}}^- \cdot L_\mu,$$

and the closure of these orbits is given by

$$\overline{T}_\mu = \bigsqcup_{\substack{\nu \in X_*(T) \\ \nu \geq \mu}} T_\nu. \quad (3.7)$$

On the Serre tree (see §3.2),  $T_\mu$  is seen as the horosphere centered at  $+\infty$  going through  $L_\mu$ . This makes the following lemma quite intuitive.

**Lemma 3.6.** *Let  $\mu, \nu \in X_*(T)$ . Then  $\overline{S}_\mu \cap \overline{T}_\nu = \emptyset$  except if  $\nu \leq \mu$ , and  $\overline{S}_\mu \cap \overline{T}_\mu = \{L_\mu\}$ .*

(For a formal proof in the general case, one uses the projective embedding and weights arguments, as in the proof of Proposition 3.4.)

## 4 Semisimplicity of the Satake category

From now on in this part, we fix a field  $\mathbf{k}$  of characteristic 0. Our goal in this section is to show that the category  $\mathrm{P}_{\mathcal{S}}(\mathrm{Gr}_G, \mathbf{k})$  of perverse sheaves on  $\mathrm{Gr}_G$  with coefficients in  $\mathbf{k}$  and with  $\mathcal{S}$ -constructible cohomology is semisimple. Since every object of this abelian category has finite length, this result means that there are no non-trivial extensions between simple objects.

### 4.1 The Satake category

Recall the notion of t-structure introduced in [BBD].

**Definition 4.1.** Let  $\mathcal{D}$  be a triangulated category. A *t-structure* on  $\mathcal{D}$  is a pair  $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$  of strictly full subcategories of  $\mathcal{D}$  which satisfy the following properties:

1. If  $X \in \mathcal{D}^{\leq 0}$  and  $Y \in \mathcal{D}^{\geq 0}$ , then  $\mathrm{Hom}_{\mathcal{D}}(X, Y[-1]) = 0$ .
2. We have  $\mathcal{D}^{\leq 0} \subset \mathcal{D}^{\leq 0}[-1]$  and  $\mathcal{D}^{\geq 0} \supset \mathcal{D}^{\geq 0}[-1]$ .

3. For all  $X \in \mathcal{D}$ , there exists a distinguished triangle

$$A \rightarrow X \rightarrow B \xrightarrow{[1]}$$

in  $\mathcal{D}$  with  $A \in \mathcal{D}^{\leq 0}$  and  $B \in \mathcal{D}^{\geq 0}[-1]$ .

We will say that an object in  $\mathcal{D}^{\leq 0}$  (respectively,  $\mathcal{D}^{\geq 0}$ ) is concentrated in nonpositive (respectively, nonnegative) degrees with respect to the t-structure. By axiom (2) in Definition 4.1, these notions are compatible with the cohomological shift, so we may as well consider for instance the subcategory  $\mathcal{D}^{\geq 1} = (\mathcal{D}^{\geq 0})[-1]$  of objects concentrated in positive degrees. We also recall that the *heart* of the t-structure is the full subcategory  $\mathcal{A} := \mathcal{D}^{\leq 0} \cap \mathcal{D}^{\geq 0}$  of  $\mathcal{D}$ ; this is an abelian category, whose exact sequences are the distinguished triangles

$$X \rightarrow Y \rightarrow Z \xrightarrow{[1]}$$

in  $\mathcal{D}$  where  $X, Y$  and  $Z$  belong to  $\mathcal{A}$ . In particular, this means that for any  $X, Y$  in  $\mathcal{A}$  we have a canonical identification

$$\mathrm{Ext}_{\mathcal{A}}^1(X, Y) \cong \mathrm{Hom}_{\mathcal{D}}(X, Y[1]). \quad (4.1)$$

For instance, the bounded derived category  $D^b(\mathcal{A})$  of an abelian category  $\mathcal{A}$  has a natural t-structure, called the ordinary t-structure, whose heart is  $\mathcal{A}$ .

Let now  $X$  be a topological space and  $\mathcal{S}$  be a stratification which satisfies certain technical conditions; see [BBD, §2.1.3]. (These conditions will always tacitly be assumed to be satisfied when we consider perverse sheaves. They are obvious in the concrete cases we study.) Given  $S \in \mathcal{S}$ , we denote by  $i_S : S \hookrightarrow X$  the inclusion map. We denote by  $D_{\mathcal{S}}^b(X, \mathbf{k})$  the bounded derived category of sheaves of  $\mathbf{k}$ -vector spaces on  $X$  which are constructible with respect to  $\mathcal{S}$ . Thus, a complex  $\mathcal{F}$  of  $\mathbf{k}$ -sheaves on  $X$  belongs to  $D_{\mathcal{S}}^b(X, \mathbf{k})$  if the cohomology sheaves  $\mathcal{H}^n \mathcal{F}$  vanish for  $|n| \gg 0$  and if each restriction  $i_S^* \mathcal{H}^n \mathcal{F}$  is a local system (i.e. a locally free sheaf of finite rank).

In this setting, we define

$$\begin{aligned} {}^pD^{\leq 0} &= \{ \mathcal{F} \in D_{\mathcal{S}}^b(X, \mathbf{k}) \mid \forall S \in \mathcal{S}, \forall n > -\dim S, \mathcal{H}^n((i_S)^* \mathcal{F}) = 0 \}, \\ {}^pD^{\geq 0} &= \{ \mathcal{F} \in D_{\mathcal{S}}^b(X, \mathbf{k}) \mid \forall S \in \mathcal{S}, \forall n < -\dim S, \mathcal{H}^n((i_S)^! \mathcal{F}) = 0 \}. \end{aligned}$$

It is known (see [BBD, §2.1.13]) that  $({}^pD^{\leq 0}, {}^pD^{\geq 0})$  is a t-structure on  $D_{\mathcal{S}}^b(X, \mathbf{k})$ , called the *perverse t-structure*. The simplest example is the case where  $\mathcal{S}$  contains only one stratum (which requires that  $X$  is smooth); then the perverse t-structure is just the ordinary t-structure (restricted to  $D_{\mathcal{S}}^b(X, \mathbf{k})$ ), shifted to the left by  $\dim X$ . Objects in the heart  $\mathrm{P}_{\mathcal{S}}(X, \mathbf{k}) := {}^pD^{\leq 0} \cap {}^pD^{\geq 0}$  of this t-structure are called *perverse sheaves*. The truncation functors for this t-structure will be denoted  ${}^p\tau_{\leq i}$  and  ${}^p\tau_{\geq i}$ , and the corresponding cohomology functors will be denoted  ${}^p\mathcal{H}^i = {}^p\tau_{\leq i} \circ {}^p\tau_{\geq i} = {}^p\tau_{\geq i} \circ {}^p\tau_{\leq i}$ .

It is known that every object in  $\mathrm{P}_{\mathcal{S}}(X, \mathbf{k})$  has finite length, see [BBD, Théorème 4.3.1]. Moreover, the simple objects in this category are classified by pairs  $(S, \mathcal{L})$ , with  $S \in \mathcal{S}$  and  $\mathcal{L}$  a simple local system on  $S$ . Specifically, to  $(S, \mathcal{L})$  corresponds a unique object  $\mathcal{F} \in D_{\mathcal{S}}^b(X, \mathbf{k})$  characterized by the conditions

$$\mathcal{F}|_{X \setminus \overline{S}} = 0, \quad \mathcal{F}|_S = \mathcal{L}[\dim S], \quad i^* \mathcal{F} \in {}^pD^{\leq -1}(\overline{S} \setminus S, \mathbf{k}), \quad i^! \mathcal{F} \in {}^pD^{\geq 1}(\overline{S} \setminus S, \mathbf{k}), \quad (4.2)$$

where  $i : \overline{S} \setminus S \hookrightarrow X$  is the inclusion map. This  $\mathcal{F}$  is a simple perverse sheaf and is usually called the intersection cohomology sheaf on  $\overline{S}$  with coefficients in  $\mathcal{L}$ , and denoted  $\mathbf{IC}(S, \mathcal{L})$ . Then the assignment  $(S, \mathcal{L}) \mapsto \mathbf{IC}(S, \mathcal{L})$  induces a bijection between equivalence classes of pairs  $(S, \mathcal{L})$  as above (where  $(S, \mathcal{L}) \sim (S, \mathcal{L}')$  if  $\mathcal{L} \cong \mathcal{L}'$ ) and isomorphism classes of simple objects in  $\mathbf{P}_{\mathcal{F}}(X, \mathbf{k})$ .

We can finally define the main object of study of these notes. Consider the affine Grassmannian  $\mathrm{Gr}_G$ , and its stratification  $\mathcal{S}$  by  $G_O$ -orbits, see §3.1. Then we can consider the constructible derived category  $D_{\mathcal{F}}^b(\mathrm{Gr}_G, \mathbf{k})$ , and its full subcategory  $\mathbf{P}_{\mathcal{F}}(\mathrm{Gr}_G, \mathbf{k})$  of perverse sheaves. The main result of this section is the following.

**Theorem 4.2.** *The category  $\mathbf{P}_{\mathcal{F}}(\mathrm{Gr}_G, \mathbf{k})$  is semisimple.*

*Remark 4.3.* Note that the assumption that  $\mathrm{char}(\mathbf{k}) = 0$  is crucial here. The category  $\mathbf{P}_{\mathcal{F}}(\mathrm{Gr}_G, \mathbf{k})$  with  $\mathbf{k}$  a field of positive characteristic is *not* semisimple.

The strata  $\mathrm{Gr}_G^\lambda$  of  $\mathcal{S}$  are simply connected, because they are affine bundles over partial flag varieties (see Proposition 3.2). Thus, the only simple local system on  $\mathrm{Gr}_G^\lambda$  is the trivial local system  $\underline{\mathbf{k}}$ . We denote by  $\mathbf{IC}_\lambda$  the corresponding intersection cohomology sheaf. Then, the simple objects in  $\mathbf{P}_{\mathcal{F}}(\mathrm{Gr}_G, \mathbf{k})$  are (up to isomorphism) these complexes  $\mathbf{IC}_\lambda$ , for  $\lambda \in X_*(T)^+$ . Since every object in  $\mathbf{P}_{\mathcal{F}}(\mathrm{Gr}_G, \mathbf{k})$  has finite length, and in view of (4.1), Theorem 4.2 follows from the following claim.

**Proposition 4.4.** *For any  $\lambda, \mu \in X_*(T)^+$ , we have*

$$\mathrm{Hom}_{D_{\mathcal{F}}^b(\mathrm{Gr}_G, \mathbf{k})}(\mathbf{IC}_\lambda, \mathbf{IC}_\mu[1]) = 0.$$

The main ingredients in the proof of Proposition 4.4 are the following facts:

- the cohomology sheaves  $\mathcal{H}^k(\mathbf{IC}_\lambda)$  vanish unless  $k$  and  $\dim(\mathrm{Gr}_G^\lambda)$  have the same parity (see Lemma 4.5 below);
- if  $\mathrm{Gr}_G^\mu \subset \overline{\mathrm{Gr}_G^\lambda}$ , then  $\mathrm{codim}_{\overline{\mathrm{Gr}_G^\lambda}}(\mathrm{Gr}_G^\mu)$  is even (see §3.1).

## 4.2 Parity vanishing

As explained above, a key point in the proof of Proposition 4.4 is the following result.

**Lemma 4.5.** *For any  $\lambda \in X_*(T)^+$ , we have*

$$\mathcal{H}^n(\mathbf{IC}_\lambda) = 0 \quad \text{unless } n \equiv \dim(\mathrm{Gr}_G^\lambda) \pmod{2}.$$

A similar property in fact holds for Iwahori-constructible perverse sheaves on the affine flag variety. In this section, we argue that this property can be deduced from the existence of resolutions of closures of Iwahori orbits whose fibers are paved by affine spaces. (These arguments are sketched in [Ga, §A.7]. A different proof of this property can be given by imitating the case of the finite flag variety treated in [Sp].)

As in Section 3, let  $W$  be the Weyl group of  $(G, T)$  and let  $Q^\vee \subset X_*(T)$  be the coroot lattice. The affine Weyl group and the extended affine Weyl group are defined as

$$W_{\text{aff}} = W \ltimes Q^\vee \quad \text{and} \quad \widetilde{W}_{\text{aff}} = W \ltimes X_*(T)$$

respectively. As is well known,  $W_{\text{aff}}$  is generated by a set  $S_{\text{aff}}$  of simple reflections, and  $(W_{\text{aff}}, S_{\text{aff}})$  is a Coxeter system with length function  $\ell$  which satisfies

$$\ell(w \cdot \lambda) = \sum_{\substack{\alpha \in \Delta_+(G, B, T) \\ w(\alpha) \in \Delta_+(G, B, T)}} |\langle \lambda, \alpha \rangle| + \sum_{\substack{\alpha \in \Delta_+(G, B, T) \\ w(-\alpha) \in \Delta_+(G, B, T)}} |1 + \langle \lambda, \alpha \rangle|$$

for  $w \in W$  and  $\lambda \in Q^\vee$ . This formula makes sense more generally for  $\lambda \in X_*(T)$ , which allows to extend  $\ell$  to  $\widetilde{W}_{\text{aff}}$ . Then if  $\Omega = \{w \in \widetilde{W}_{\text{aff}} \mid \ell(w) = 0\}$ , the conjugation action of the subgroup  $\Omega$  on  $\widetilde{W}_{\text{aff}}$  preserves  $S_{\text{aff}}$ , hence also  $W_{\text{aff}}$ , and we have  $\widetilde{W}_{\text{aff}} = W_{\text{aff}} \rtimes \Omega$ .

As in §3.3, let  $B^- \subset G$  be the Borel subgroup opposite to  $B$  with respect to  $T$ , and let  $I \subset G_{\mathcal{O}}$  be the corresponding Iwahori subgroup, defined as the preimage of  $B^-$  under the evaluation map  $G_{\mathcal{O}} \rightarrow G$  given by  $t \mapsto 0$  (i.e. the arc space of the group scheme considered in Remark 3.1(2), for the Borel subgroup  $B^-$  instead of  $B$ ). The Bruhat decomposition then yields

$$\text{Gr}_G = \bigsqcup_{w \in \widetilde{W}_{\text{aff}}/W} IwG_{\mathcal{O}}/G_{\mathcal{O}},$$

and  $IwG_{\mathcal{O}}/G_{\mathcal{O}}$  is an affine space of dimension  $\ell(w)$  if  $w$  is of minimal length in the coset  $wW$ . (Here, if  $w = v \cdot \lambda$  with  $v \in W$  and  $\lambda \in X_*(T)$ , by  $IwG_{\mathcal{O}}/G_{\mathcal{O}}$  we mean the  $I$ -orbit of  $\dot{v}t^\lambda G_{\mathcal{O}}/G_{\mathcal{O}}$ , where  $\dot{v}$  is any lift of  $v$  in  $N_G(T) \subset G$ .)

Let  $\lambda \in X_*(T)^+$ . Then

$$\text{Gr}_G^\lambda = \bigsqcup_{w \in Wt_\lambda W/W} IwG_{\mathcal{O}}/G_{\mathcal{O}}$$

is a union of Schubert cells. One of these cells is open dense in  $\text{Gr}_G^\lambda$ ; we denote by  $w_\lambda$  the unique element in  $Wt_\lambda W$  which is minimal in  $w_\lambda W$  and such that  $Iw_\lambda G_{\mathcal{O}}/G_{\mathcal{O}}$  is open in  $\text{Gr}_G^\lambda$ . Certainly then we have

$$\mathbf{IC}_\lambda = \mathbf{IC}(Iw_\lambda G_{\mathcal{O}}/G_{\mathcal{O}}, \mathbf{k}).$$

Hence Lemma 4.5 follows from the claim that for any  $w \in \widetilde{W}_{\text{aff}}$  which is minimal in  $wW$  we have

$$\mathcal{H}^n(\mathbf{IC}(IwG_{\mathcal{O}}/G_{\mathcal{O}}, \mathbf{k})) \neq 0 \implies n \equiv \ell(w) \pmod{2}. \quad (4.3)$$

To prove (4.3) we introduce the affine flag variety

$$\text{Fl}_G := G_{\mathcal{K}}/I$$

(see also Remark 3.1(2)). As for  $\text{Gr}_G$ , this variety has a natural complex ind-variety structure, and a Bruhat decomposition

$$\text{Fl}_G = \bigsqcup_{w \in \widetilde{W}_{\text{aff}}} IwI/I,$$



see [Gö] for details and references. This decomposition provides a stratification of  $\mathrm{Fl}_G$ , which we denote by  $\mathcal{S}$ . Then we can consider the constructible derived category  $D_{\mathcal{S}}^b(\mathrm{Fl}_G, \mathbf{k})$  and the corresponding category  $\mathrm{P}_{\mathcal{S}}(\mathrm{Fl}_G, \mathbf{k})$  of perverse sheaves.

Let  $\pi : \mathrm{Fl}_G \rightarrow \mathrm{Gr}_G$  be the natural projection. This morphism is smooth; in fact it is a locally trivial fibration<sup>9</sup> with fiber  $G/B^-$ . From this property and the characterization of the intersection cohomology complex given in (4.2), it is not difficult to check that for any  $w \in \widetilde{W}_{\mathrm{aff}}$  which is minimal in  $wW$ , we have

$$\pi^* \mathbf{IC}(IwG_{\mathcal{O}}/G_{\mathcal{O}}, \mathbf{k})[\ell(w_0)] \cong \mathbf{IC}(Iww_0I/I, \mathbf{k}),$$

where  $w_0 \in W$  is the longest element (so that  $\ell(w_0) = \dim(G/B^-)$ ). This shows that (4.3) (hence also Lemma 4.5) follows from the following claim.

**Lemma 4.6.** *For any  $w \in \widetilde{W}_{\mathrm{aff}}$  we have*

$$\mathcal{H}^n(\mathbf{IC}(IwI/I, \mathbf{k})) \neq 0 \implies n \equiv \ell(w) \pmod{2}.$$

*Proof.* For any  $s \in S_{\mathrm{aff}}$ , denote by  $J_s = IsI \cup I$  the minimal parahoric subgroup of  $G_{\mathcal{K}}$  associated with  $s$ . Fix  $w \in \widetilde{W}_{\mathrm{aff}}$ , and choose a reduced expression  $\underline{w} = (s_1, \dots, s_r, \omega)$  for  $w$  (with  $s_j \in S_{\mathrm{aff}}$  and  $\ell(\omega) = 0$ ). We can then consider the Bott-Samelson resolution

$$\pi_{\underline{w}} : J_{s_1} \times^I \cdots \times^I J_{s_r} \times^I \underbrace{(I\omega I/I)}_{\text{(a point)}} \rightarrow \overline{IwI/I}$$

induced by multiplication in  $G_{\mathcal{K}}$ . It is known that  $\pi_{\underline{w}}$  is proper and is an isomorphism over  $IwI/I$ . It is known also that each fiber  $\pi_{\underline{w}}^{-1}(x)$  is paved by affine spaces. (For this claim in the case of finite flag varieties, see [Gau]. See also [Ha] for a different proof, which works *mutatis mutandis* in the affine setting.) Therefore

$$\mathrm{H}_c^{n+\ell(w)}(\pi_{\underline{w}}^{-1}(x); \mathbf{k})$$

is nonzero only if  $n + \ell(w)$  is even. By proper base change, this cohomology group is the stalk at  $x$  of the cohomology sheaf  $\mathcal{H}^n((\pi_{\underline{w}})_! \mathbf{k}[\ell(w)])$ , so that

$$\mathcal{H}^n((\pi_{\underline{w}})_! \mathbf{k}[\ell(w)]) \neq 0 \implies n \equiv \ell(w) \pmod{2}.$$

Our desired parity vanishing property then follows from the celebrated Decomposition Theorem (see [BBD, Theorem 6.2.5]), which here implies that  $\mathbf{IC}(IwI/I; \mathbf{k})$  is a direct summand of the complex  $(\pi_{\underline{w}})_! \mathbf{k}[\ell(w)]$ .  $\square$

### 4.3 Proof of Proposition 4.4

We follow the arguments in [Ga, Proof of Proposition 1] (but adding more details). We distinguish 3 cases (of which only the third one will use Lemma 4.5).

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<sup>9</sup>This fibration is in fact topologically trivial, as follows from the realization of  $\mathrm{Gr}_G$  as a topological group, see [Gi, §1.2].

**First case:**  $\lambda = \mu$ .

Consider the diagram

$$\begin{array}{ccc} \mathrm{Gr}_G^\lambda & \xrightarrow{j} & \overline{\mathrm{Gr}}_G^\lambda \leftarrow i \overline{\mathrm{Gr}}_G^\lambda \setminus \mathrm{Gr}_G^\lambda \\ & \searrow j_\lambda & \downarrow i_\lambda \\ & & \mathrm{Gr}_G \end{array}$$

where all maps are the obvious embeddings. Set  $\mathcal{F} = (i_\lambda i)^* \mathbf{IC}_\lambda$ ; by (4.2), this complex of sheaves is concentrated in negative perverse degrees. Likewise, the complex of sheaves  $(i_\lambda i)^! \mathbf{IC}_\lambda$  is concentrated in positive perverse degrees. It follows that

$$\mathrm{Hom}_{D_{\mathcal{F}}^b(\overline{\mathrm{Gr}}_G^\lambda \setminus \mathrm{Gr}_G^\lambda, \mathbf{k})}(\mathcal{F}, (i_\lambda i)^! \mathbf{IC}_\lambda[1]) = 0. \quad (4.4)$$

Applying the cohomological functor  $\mathrm{Hom}_{D_{\mathcal{F}}^b(\mathrm{Gr}_G, \mathbf{k})}^\bullet((i_\lambda i)^! \mathbf{IC}_\lambda[1])$  to the distinguished triangle

$$j! j^* \left( \mathbf{IC}_\lambda |_{\overline{\mathrm{Gr}}_G^\lambda} \right) \rightarrow \left( \mathbf{IC}_\lambda |_{\overline{\mathrm{Gr}}_G^\lambda} \right) \rightarrow i! i^* \left( \mathbf{IC}_\lambda |_{\overline{\mathrm{Gr}}_G^\lambda} \right) \xrightarrow{[1]},$$

we get an exact sequence

$$\begin{aligned} \mathrm{Hom}_{D_{\mathcal{F}}^b(\mathrm{Gr}_G, \mathbf{k})}((i_\lambda i)^! \mathcal{F}, \mathbf{IC}_\lambda[1]) &\rightarrow \mathrm{Hom}_{D_{\mathcal{F}}^b(\mathrm{Gr}_G, \mathbf{k})}(\mathbf{IC}_\lambda, \mathbf{IC}_\lambda[1]) \\ &\rightarrow \mathrm{Hom}_{D_{\mathcal{F}}^b(\mathrm{Gr}_G, \mathbf{k})}((j_\lambda)! \mathbf{k}_{\mathrm{Gr}_G^\lambda}[\dim \mathrm{Gr}_G^\lambda], \mathbf{IC}_\lambda[1]). \end{aligned} \quad (4.5)$$

The first space in (4.5) is zero, thanks to (4.4) and because by adjunction we have

$$\mathrm{Hom}_{D_{\mathcal{F}}^b(\mathrm{Gr}_G, \mathbf{k})}((i_\lambda i)^! \mathcal{F}, \mathbf{IC}_\lambda[1]) \cong \mathrm{Hom}_{D_{\mathcal{F}}^b(\overline{\mathrm{Gr}}_G^\lambda \setminus \mathrm{Gr}_G^\lambda, \mathbf{k})}(\mathcal{F}, (i_\lambda i)^! \mathbf{IC}_\lambda[1]).$$

By adjunction again, the third space in (4.5) is

$$\begin{aligned} \mathrm{Hom}_{D_{\mathcal{F}}^b(\mathrm{Gr}_G, \mathbf{k})}((j_\lambda)! \mathbf{k}_{\mathrm{Gr}_G^\lambda}[\dim \mathrm{Gr}_G^\lambda], \mathbf{IC}_\lambda[1]) &\cong \mathrm{Hom}_{D_{\mathcal{F}}^b(\mathrm{Gr}_G^\lambda, \mathbf{k})}(\mathbf{k}_{\mathrm{Gr}_G^\lambda}[\dim \mathrm{Gr}_G^\lambda], (j_\lambda)^! \mathbf{IC}_\lambda[1]) \\ &\cong \mathrm{Hom}_{D_{\mathcal{F}}^b(\mathrm{Gr}_G^\lambda, \mathbf{k})}(\mathbf{k}_{\mathrm{Gr}_G^\lambda}, \mathbf{k}_{\mathrm{Gr}_G^\lambda}[1]) \\ &= \mathrm{H}^1(\mathrm{Gr}_G^\lambda; \mathbf{k}). \end{aligned}$$

This last space is again zero since  $\mathrm{Gr}_G^\lambda$  is an affine bundle over a partial flag variety (see Proposition 3.2), so has only cohomology in even degrees.

We conclude that  $\mathrm{Hom}_{D_{\mathcal{F}}^b(\mathrm{Gr}_G, \mathbf{k})}(\mathbf{IC}_\lambda, \mathbf{IC}_\lambda[1]) = 0$ .

**Second case:** Neither  $\mathrm{Gr}_G^\lambda \subset \overline{\mathrm{Gr}}_G^\mu$  nor  $\mathrm{Gr}_G^\mu \subset \overline{\mathrm{Gr}}_G^\lambda$ .

Consider the inclusion  $i_\mu : \overline{\mathrm{Gr}}_G^\mu \hookrightarrow \mathrm{Gr}_G$ . Since  $\mathbf{IC}_\mu$  is supported on  $\overline{\mathrm{Gr}}_G^\mu$ , we have  $\mathbf{IC}_\mu = (i_\mu)_* (i_\mu)^* \mathbf{IC}_\mu$  and therefore by adjunction

$$\mathrm{Hom}_{D_{\mathcal{F}}^b(\mathrm{Gr}_G, \mathbf{k})}(\mathbf{IC}_\lambda, \mathbf{IC}_\mu[1]) \cong \mathrm{Hom}_{D_{\mathcal{F}}^b(\overline{\mathrm{Gr}}_G^\mu, \mathbf{k})}((i_\mu)^* \mathbf{IC}_\lambda, (i_\mu)^* \mathbf{IC}_\mu[1]).$$

Now set  $Z = \overline{\text{Gr}_G^\lambda} \cap \overline{\text{Gr}_G^\mu}$  and consider the inclusion  $f : Z \hookrightarrow \overline{\text{Gr}_G^\mu}$ . Since  $(i_\mu)^* \mathbf{IC}_\lambda$  is supported on  $Z$ , it is of the form  $f_! \mathcal{F}$  for some complex of sheaves  $\mathcal{F} \in D_{\mathcal{S}}^b(Z, \mathbf{k})$ . Arguing as in the first case, we see that  $\mathcal{F}$  is concentrated in negative perverse degrees and that  $f^!(i_\mu)^* \mathbf{IC}_\mu \cong (i_\mu f)^! \mathbf{IC}_\mu$  is concentrated in positive perverse degrees. Therefore

$$\begin{aligned} \text{Hom}_{D_{\mathcal{S}}^b(\text{Gr}_G, \mathbf{k})}(\mathbf{IC}_\lambda, \mathbf{IC}_\mu[1]) &\cong \text{Hom}_{D_{\mathcal{S}}^b(\overline{\text{Gr}_G^\mu}, \mathbf{k})}(f_! \mathcal{F}, (i_\mu)^* \mathbf{IC}_\mu[1]) \\ &\cong \text{Hom}_{D_{\mathcal{S}}^b(Z, \mathbf{k})}(\mathcal{F}, f^!(i_\mu)^* \mathbf{IC}_\mu[1]) = 0, \end{aligned}$$

as desired.

**Third case:**  $\lambda \neq \mu$  and either  $\text{Gr}_G^\lambda \subset \overline{\text{Gr}_G^\mu}$  or  $\text{Gr}_G^\mu \subset \overline{\text{Gr}_G^\lambda}$ .

Since Verdier duality is an anti-autoequivalence of  $D_{\mathcal{S}}^b(\text{Gr}_G, \mathbf{k})$  which fixes  $\mathbf{IC}_\lambda$  and  $\mathbf{IC}_\mu$ , we can assume that  $\text{Gr}_G^\mu \subset \overline{\text{Gr}_G^\lambda}$ . Let  $j_\mu : \text{Gr}_G^\mu \hookrightarrow \text{Gr}_G$  be the inclusion, and let  $\mathcal{G} \in D_{\mathcal{S}}^b(\text{Gr}_G, \mathbf{k})$  be the cone of the adjunction map  $\mathbf{IC}_\mu \rightarrow (j_\mu)_*(j_\mu)^* \mathbf{IC}_\mu \cong (j_\mu)_* \underline{\mathbf{k}}_{\text{Gr}_G^\mu}[\dim \text{Gr}_G^\mu]$ . It follows from the definition of the perverse t-structure that  $(j_\mu)_* \underline{\mathbf{k}}_{\text{Gr}_G^\mu}[\dim \text{Gr}_G^\mu]$  is concentrated in nonnegative perverse degrees, and it is a classical fact that the morphism  $\mathbf{IC}_\mu \rightarrow {}^p \mathcal{H}^0((j_\mu)_* \underline{\mathbf{k}}_{\text{Gr}_G^\mu}[\dim \text{Gr}_G^\mu])$  induced by the adjunction map considered above (where  ${}^p \mathcal{H}^0(?)$  means the degree-0 perverse cohomology) is injective, see e.g. [BBD, (1.4.22.1)]. Therefore,  $\mathcal{G}$  is concentrated in nonnegative perverse degrees.

From the triangle

$$\mathbf{IC}_\mu \rightarrow (j_\mu)_* \underline{\mathbf{k}}_{\text{Gr}_G^\mu}[\dim \text{Gr}_G^\mu] \rightarrow \mathcal{G} \xrightarrow{[1]}$$

we get an exact sequence

$$\begin{aligned} \text{Hom}_{D_{\mathcal{S}}^b(\text{Gr}_G, \mathbf{k})}(\mathbf{IC}_\lambda, \mathcal{G}) &\rightarrow \text{Hom}_{D_{\mathcal{S}}^b(\text{Gr}_G, \mathbf{k})}(\mathbf{IC}_\lambda, \mathbf{IC}_\mu[1]) \\ &\rightarrow \text{Hom}_{D_{\mathcal{S}}^b(\text{Gr}_G, \mathbf{k})}(\mathbf{IC}_\lambda, (j_\mu)_* \underline{\mathbf{k}}_{\text{Gr}_G^\mu}[\dim \text{Gr}_G^\mu + 1]). \end{aligned} \quad (4.6)$$

As in the second case (but now using the  $((-)^*, (-)_*)$  adjunction), using the fact that  $\mathcal{G}$  is concentrated in nonnegative perverse degrees and supported on  $\overline{\text{Gr}_G^\lambda}$ , which is included in  $\overline{\text{Gr}_G^\lambda} \setminus \text{Gr}_G^\lambda$ , one checks that the left Hom space is zero.

By (4.2),  $(j_\mu)^* \mathbf{IC}_\lambda$  is concentrated in degrees  $< -\dim \text{Gr}_G^\mu$ . On the other hand, by Lemma 4.5, this complex has cohomology only in degrees of the same parity as  $\dim(\text{Gr}_G^\lambda)$ . Noting that  $\dim(\text{Gr}_G^\lambda) \equiv \dim(\text{Gr}_G^\mu) \pmod{2}$  (because these orbits belong to the same connected component of  $\text{Gr}_G$ ), this implies that in fact  $(j_\mu)^* \mathbf{IC}_\lambda$  is concentrated in degrees  $\leq -\dim \text{Gr}_G^\mu - 2$ . It follows that

$$\text{Hom}_{D_{\mathcal{S}}^b(\text{Gr}_G, \mathbf{k})}(\mathbf{IC}_\lambda, (j_\mu)_* \underline{\mathbf{k}}_{\text{Gr}_G^\mu}[\dim \text{Gr}_G^\mu + 1]) = \text{Hom}_{D_{\mathcal{S}}^b(\text{Gr}_G^\mu, \mathbf{k})}((j_\mu)^* \mathbf{IC}_\lambda, \underline{\mathbf{k}}_{\text{Gr}_G^\mu}[\dim \text{Gr}_G^\mu + 1])$$

vanishes.

Our exact sequence (4.6) then yields the desired equality  $\text{Hom}_{D_{\mathcal{S}}^b(\text{Gr}_G, \mathbf{k})}(\mathbf{IC}_\lambda, \mathbf{IC}_\mu[1]) = 0$ .

*Remark 4.7.* One can give a slightly shorter proof of Proposition 4.4 as follows. Lemma 4.5 and the Verdier self-duality of the objects  $\mathbf{IC}_\lambda$  show that these objects are *parity complexes*

in the sense of [JMW, Definition 2.4] (for the constant pariversity). More precisely,  $\mathbf{IC}_\lambda$  is even if  $\dim(\mathrm{Gr}_G^\lambda)$  is even, and odd if  $\dim(\mathrm{Gr}_G^\lambda)$  is odd. Now Proposition 4.4 is obvious if  $\dim(\mathrm{Gr}_G^\lambda)$  and  $\dim(\mathrm{Gr}_G^\mu)$  do not have the same parity (because then  $\mathbf{IC}_\lambda$  and  $\mathbf{IC}_\mu$  live on different connected components of  $\mathrm{Gr}_G$ ) and follows from [JMW, Corollary 2.8] if they do have the same parity.

#### 4.4 Consequence on equivariance

Consider the category

$$\mathrm{P}_{G_{\mathcal{O}}}(\mathrm{Gr}_G, \mathbf{k})$$

of  $G_{\mathcal{O}}$ -equivariant perverse sheaves on  $\mathrm{Gr}_G$ ; see §A.1. (Here the stratification we consider is  $\mathcal{S}$ .) We have a forgetful functor

$$\mathrm{P}_{G_{\mathcal{O}}}(\mathrm{Gr}_G, \mathbf{k}) \rightarrow \mathrm{P}_{\mathcal{S}}(\mathrm{Gr}_G, \mathbf{k}),$$

which is fully faithful by construction. As a consequence of Theorem 4.2, each object in  $\mathrm{P}_{\mathcal{S}}(\mathrm{Gr}_G, \mathbf{k})$  is isomorphic to a direct sum of the simple objects  $\mathbf{IC}_\lambda$ , hence belongs to the essential image of this functor. We deduce the following.

**Corollary 4.8.** *The forgetful functor*

$$\mathrm{P}_{G_{\mathcal{O}}}(\mathrm{Gr}_G, \mathbf{k}) \rightarrow \mathrm{P}_{\mathcal{S}}(\mathrm{Gr}_G, \mathbf{k})$$

*is an equivalence of categories.*

*Remark 4.9.* See §10.2 below for a different proof of Corollary 4.8 which does not use the semisimplicity of  $\mathrm{P}_{\mathcal{S}}(\mathrm{Gr}_G, \mathbf{k})$  (but requires much more sophisticated tools).

## 5 Dimension estimates and the weight functors

### 5.1 Overview

Recall that if  $\mathbf{F}$  is a field, the split<sup>10</sup> reductive groups over  $\mathbf{F}$  are classified, up to isomorphism, by their root datum<sup>11</sup> (see e.g. [SGA3, Exposé XXIII, Corollaire 5.4 and Exposé XXII, Proposition 2.2]). In particular, we can consider the reductive  $\mathbf{k}$ -group  $G_{\mathbf{k}}^\vee$  which is Langlands dual to  $G$ , i.e. whose root datum is dual to that of  $G$  (which means that it is obtained from that of  $G$  by exchanging weights and coweights and roots and coroots); this group is defined up to isomorphism.

<sup>10</sup>A reductive group is called *split* if it admits a maximal torus which is split, i.e. isomorphic to a product of copies of the multiplicative group over  $\mathbf{F}$ . Here a maximal torus of a reductive group  $H$  is a closed subgroup which is a torus and whose base change to an algebraic closure  $\overline{\mathbf{F}}$  of  $\mathbf{F}$  is a maximal torus of  $\mathrm{Spec}(\overline{\mathbf{F}}) \times_{\mathrm{Spec}(\mathbf{F})} H$  in the “traditional” sense, see e.g. [Hu].

<sup>11</sup>If  $H$  is a split reductive group and  $K \subset H$  is a maximal torus, then the root datum of  $H$  with respect to  $K$  is the quadruple  $(X^*(K_{\overline{\mathbf{F}}}), X_*(K_{\overline{\mathbf{F}}}), \Delta(H_{\overline{\mathbf{F}}}, K_{\overline{\mathbf{F}}}), \Delta^\vee(H_{\overline{\mathbf{F}}}, K_{\overline{\mathbf{F}}}))$  where  $\overline{\mathbf{F}}$  is an algebraic closure of  $\mathbf{F}$ ,  $H_{\overline{\mathbf{F}}} := \mathrm{Spec}(\overline{\mathbf{F}}) \times_{\mathrm{Spec}(\mathbf{F})} H$ ,  $K_{\overline{\mathbf{F}}} := \mathrm{Spec}(\overline{\mathbf{F}}) \times_{\mathrm{Spec}(\mathbf{F})} K$ ,  $\Delta(H_{\overline{\mathbf{F}}}, K_{\overline{\mathbf{F}}})$ , resp.  $\Delta^\vee(H_{\overline{\mathbf{F}}}, K_{\overline{\mathbf{F}}})$ , is the root system, resp. coroot system, of  $H_{\overline{\mathbf{F}}}$  with respect to  $K_{\overline{\mathbf{F}}}$ , together with the bijection  $\Delta(H_{\overline{\mathbf{F}}}, K_{\overline{\mathbf{F}}}) \xrightarrow{\sim} \Delta^\vee(H_{\overline{\mathbf{F}}}, K_{\overline{\mathbf{F}}})$  given by  $\alpha \mapsto \alpha^\vee$ .

The *geometric Satake equivalence* is the statement that the category  $P_{G_{\mathcal{O}}}(\mathrm{Gr}_G, \mathbf{k})$  is equivalent to the category of finite-dimensional representations of  $G_{\mathbf{k}}^{\vee}$ , in such a way that the tensor product of representations corresponds to a natural operation in  $P_{G_{\mathcal{O}}}(\mathrm{Gr}_G, \mathbf{k})$  called *convolution*.<sup>12</sup> In fact, we will even explain how to construct a *canonical* group scheme  $G_{\mathbf{k}}^{\vee}$  which is split reductive (with a canonical maximal torus) and whose root datum is dual to that of  $G$ , and a canonical equivalence of monoidal categories  $P_{G_{\mathcal{O}}}(\mathrm{Gr}_G, \mathbf{k}) \xrightarrow{\sim} \mathrm{Rep}_{\mathbf{k}}(G_{\mathbf{k}}^{\vee})$ .

To achieve this goal, the method is to define a convolution product on the abelian category  $P_{G_{\mathcal{O}}}(\mathrm{Gr}_G, \mathbf{k})$  so that this category satisfies the conditions of Theorem 2.7 with respect to the functor

$$H^{\bullet}(\mathrm{Gr}_G, ?) : P_{G_{\mathcal{O}}}(\mathrm{Gr}_G, \mathbf{k}) \rightarrow \mathrm{Vect}_{\mathbf{k}}.$$

We will then need to identify the affine group scheme provided by Theorem 2.7. The construction of a (split) maximal torus in this group scheme, which is the first step in this direction, is based on Mirković and Vilonen's *weight functors*, which we introduce in this section.

Recall that we have chosen a maximal torus and a Borel subgroup  $T \subset B \subset G$ . Then  $T \subset G_{\mathcal{K}}$  acts on  $\mathrm{Gr}_G = G_{\mathcal{K}}/G_{\mathcal{O}}$  with fixed points

$$(\mathrm{Gr}_G)^T = \{L_{\mu} : \mu \in X_*(T)\}.$$

The choice of a dominant regular cocharacter  $\eta \in X_*(T)$  provides a one-parameter subgroup  $\mathbb{G}_{\mathbf{m}} \subset T$ , whence a  $\mathbf{C}^{\times}$ -action on  $\mathrm{Gr}_G$  with fixed points  $(\mathrm{Gr}_G)^T$ . The attractive and repulsive varieties relative to the fixed point  $L_{\mu}$  coincide with the semi-infinite orbits  $S_{\mu}$  and  $T_{\mu}$  defined in Section 3:

$$S_{\mu} = \{x \in \mathrm{Gr}_G \mid \eta(a) \cdot x \rightarrow L_{\mu} \text{ when } a \rightarrow 0\}$$

and

$$T_{\mu} = \{x \in \mathrm{Gr}_G \mid \eta(a) \cdot x \rightarrow L_{\mu} \text{ when } a \rightarrow \infty\}$$

(see §5.2 for details). With these notations, the weight functor  $F_{\mu}$  is defined either as the cohomology with compact support of the restriction to  $S_{\mu}$ , or as the cohomology with support in  $T_{\mu}$ . These two definitions are equivalent, thanks to Braden's theorem on hyperbolic localization.

*Remark 5.1.* In Ginzburg's approach to the geometric Satake equivalence [Gi], the maximal torus in  $G_{\mathbf{k}}^{\vee}$  is instead constructed using equivariant cohomology and the functors of co-restriction to points  $L_{\lambda}$ . For a comparison between these points of view, the reader may consult [GR].

## 5.2 Dimension estimates

Recall from §3.1 that  $\rho$  denotes the halfsum of the positive roots, considered as a linear form  $X_*(T) \rightarrow \frac{1}{2}\mathbf{Z}$ , and that  $Q^{\vee} \subset X_*(T)$  denotes the coroot lattice.

**Theorem 5.2.** *Let  $\lambda, \mu \in X_*(T)$  with  $\lambda$  dominant.*

---

<sup>12</sup>Note that if we drop this requirement, the statement becomes vacuous, because the categories  $P_{G_{\mathcal{O}}}(\mathrm{Gr}_G, \mathbf{k})$  and  $\mathrm{Rep}_{\mathbf{k}}(G_{\mathbf{k}}^{\vee})$  are both semisimple with simple objects parametrized by  $X_*(T)^+$ . This weaker statement might be already nontrivial, however, for more general coefficients (see Part II).

1. We have

$$\overline{\text{Gr}}_G^\lambda \cap S_\mu \neq \emptyset \iff L_\mu \in \overline{\text{Gr}}_G^\lambda \iff \mu \in \text{Conv}(W\lambda) \cap (\lambda + Q^\vee),$$

where  $\text{Conv}$  denotes the convex hull.

2. If  $\mu$  satisfies the condition in (1), the intersection  $\overline{\text{Gr}}_G^\lambda \cap S_\mu$  has pure dimension<sup>13</sup>  $\langle \rho, \lambda + \mu \rangle$ .
3. If  $\mu$  satisfies the condition in (1), then  $\overline{\text{Gr}}_G^\lambda \cap S_\mu$  is open dense in  $\overline{\text{Gr}}_G^\lambda \cap S_\mu$ ; in particular, the irreducible components of  $\overline{\text{Gr}}_G^\lambda \cap S_\mu$  and  $\text{Gr}_G^\lambda \cap S_\mu$  are in a canonical bijection.

*Proof.* (1) Let  $\eta \in X_*(T)$  be regular dominant. If  $g \in N_K$ , then  $\eta(a)g\eta(a)^{-1} \rightarrow 1$  when  $a \rightarrow 0$ . Therefore, looking at the induced action of  $\mathbf{C}^\times$  on  $\text{Gr}_G$ , we obtain that for any  $\mu \in X_*(T)$ ,

$$S_\mu \subset \{x \in \text{Gr}_G \mid \eta(a) \cdot x \rightarrow L_\mu \text{ when } a \rightarrow 0\}.$$

In view of the Iwasawa decomposition (3.3), this inclusion is in fact an equality. Then the stability of  $\overline{\text{Gr}}_G^\lambda$  by the action of  $T$  implies the first equivalence.

On the other hand, we have

$$W\lambda \subset \{\mu \in X_*(T) \mid L_\mu \in \text{Gr}_G^\lambda\},$$

and using the Cartan decomposition (3.1), we see that this inclusion is in fact an equality. The description of  $\overline{\text{Gr}}_G^\lambda$  recalled in (3.2) then implies that  $L_\mu \in \overline{\text{Gr}}_G^\lambda$  if and only if the dominant  $W$ -conjugate  $\mu^+$  of  $\mu$  satisfies  $\mu^+ \leq \lambda$ , that is, if and only if

$$W\mu \subset \{\nu \in X_*(T) \mid \nu \leq \lambda\}.$$

Using [Bou, chap. VIII, §7, exerc. 1], we see that this condition is equivalent to

$$\mu \in \text{Conv}(W\lambda) \cap (\lambda + Q^\vee).$$

(2) We start with the following remarks. From (1), we deduce that  $\overline{\text{Gr}}_G^\lambda$  meets only those  $S_\mu$  such that  $\mu \leq \lambda$ , therefore  $\overline{\text{Gr}}_G^\lambda \subset \overline{S_\lambda}$ .<sup>14</sup> If  $w_0 \in W$  is the longest element (so that  $w_0\lambda$  is the unique antidominant element in  $W\lambda$ ), then conjugating by a lift of  $w_0$  we deduce that  $\overline{\text{Gr}}_G^\lambda \subset \overline{T_{w_0\lambda}}$ .

Note that if  $\mu$  satisfies the condition in (1), then  $w_0\lambda \leq \mu \leq \lambda$ . We will now prove, by induction on  $\langle \rho, \mu - w_0\lambda \rangle$ , that

$$\dim(\overline{\text{Gr}}_G^\lambda \cap \overline{S_\mu}) \leq \langle \rho, \lambda + \mu \rangle. \quad (5.1)$$

If  $\mu = w_0\lambda$ , then from the remarks above we have  $\overline{\text{Gr}}_G^\lambda \cap \overline{S_{w_0\lambda}} \subset \overline{S_{w_0\lambda}} \cap \overline{T_{w_0\lambda}} = \{L_{w_0\lambda}\}$  (see Lemma 3.6), so that the claim holds in this case.

<sup>13</sup>By this, we mean that all the irreducible components of this variety have dimension  $\langle \rho, \lambda + \mu \rangle$ .

<sup>14</sup>To prove the inclusion  $\overline{\text{Gr}}_G^\lambda \subset \overline{S_\lambda}$ , one can also argue as follows. The open cell  $N_{\mathcal{O}}B_{\mathcal{O}}^-$  is dense in  $G_{\mathcal{O}}$  and  $B_{\mathcal{O}}^-$  stabilizes  $L_\lambda$ , therefore  $\overline{\text{Gr}}_G^\lambda = G_{\mathcal{O}} \cdot L_\lambda$  contains  $N_{\mathcal{O}} \cdot L_\lambda$  as a dense subset, whence  $\overline{\text{Gr}}_G^\lambda \subset \overline{N_{\mathcal{O}} \cdot L_\lambda} \subset \overline{S_\lambda}$ .

Assume now that  $\mu > w_0\lambda$ , and choose a hyperplane  $H_\mu$  as in Proposition 3.4. Let  $C$  be an irreducible component of  $\overline{\text{Gr}_G^\lambda} \cap \overline{S_\mu}$  and let  $D$  be an irreducible component of  $C \cap \Psi^{-1}(H_\mu)$ . Then  $\dim(D) \geq \dim(C) - 1$ , and  $D$  contained in

$$\Psi^{-1}(H_\mu) \cap \overline{\text{Gr}_G^\lambda} \cap \overline{S_\mu} = \partial S_\mu \cap \overline{\text{Gr}_G^\lambda} = \bigcup_{\nu < \mu} \overline{S_\nu} \cap \overline{\text{Gr}_G^\lambda},$$

so by induction  $\dim D \leq \max_{\nu < \mu} \langle \rho, \lambda + \nu \rangle = \langle \rho, \lambda + \mu \rangle - 1$ . We deduce that  $\dim C \leq \dim D + 1 \leq \langle \rho, \lambda + \mu \rangle$ , which finishes the proof of (5.1).

The inequality (5.1) implies that each irreducible component  $C$  of  $\overline{\text{Gr}_G^\lambda} \cap \overline{S_\mu}$  has dimension at most  $\langle \rho, \lambda + \mu \rangle$ . We will now prove that this dimension is always exactly  $\langle \rho, \lambda + \mu \rangle$ . First, if  $\mu = \lambda$  then as observed above we have  $\overline{\text{Gr}_G^\lambda} \cap \overline{S_\lambda} = \overline{\text{Gr}_G^\lambda}$ . Since this variety is irreducible of dimension  $\langle 2\rho, \lambda \rangle = \langle \rho, \lambda + \lambda \rangle$  by Proposition 3.2, this implies that its (nonempty) open subset  $\overline{\text{Gr}_G^\lambda} \cap \overline{S_\lambda}$  has the same properties. Now, assume that  $\mu < \lambda$ , and fix an irreducible component  $C$  of  $\overline{\text{Gr}_G^\lambda} \cap \overline{S_\mu}$ . Set  $d := \langle \rho, 2\lambda \rangle - \dim(C)$ , and let  $H_\lambda$  be as in Proposition 3.4. Then we have

$$\overline{C} \subset \overline{\text{Gr}_G^\lambda} \cap \partial S_\lambda = \overline{\text{Gr}_G^\lambda} \cap \Psi^{-1}(H_\lambda).$$

Hence there exists an irreducible component  $D_1$  of the right-hand side containing  $\overline{C}$ . Then  $\dim(D_1) = \langle \rho, 2\lambda \rangle - 1$ , and  $D_1$  is the disjoint union of its locally closed intersections with the orbits  $S_\nu$  with  $w_0\lambda \leq \nu < \lambda$ ; hence there exists such a  $\nu_1$  such that  $C_1 := D_1 \cap \overline{S_{\nu_1}}$  is open dense in  $D_1$ . We necessarily have  $\nu_1 \geq \mu$  since otherwise  $\overline{C}$  would be contained in  $\overline{\text{Gr}_G^\lambda} \cap \partial S_\mu$ , which is not the case. Now  $C_1$  is an irreducible component of  $\overline{\text{Gr}_G^\lambda} \cap \overline{S_{\nu_1}}$  of dimension  $\langle \rho, 2\lambda \rangle - 1$  such that  $\overline{C_1}$  contains  $\overline{C}$ . If  $d > 1$  we must have  $\mu < \nu_1$ ; in fact, otherwise from the facts that  $\overline{C} \subset \overline{C_1}$  and that

$$C_1 = \overline{C_1} \cap S_{\nu_1} \quad \text{and} \quad C = \overline{C} \cap S_\mu$$

we would deduce that  $C \subset C_1$ , so that  $C = C_1$  (which is impossible for reasons of dimension) since both of these varieties are irreducible components of  $\overline{\text{Gr}_G^\lambda} \cap \overline{S_\mu}$ .

Repeating this argument we find coweights  $\nu_1, \dots, \nu_d$  which satisfy

$$\mu \leq \nu_d < \nu_{d-1} < \dots < \nu_1 < \lambda \tag{5.2}$$

and irreducible components  $C_i$  of  $\overline{\text{Gr}_G^\lambda} \cap \overline{S_{\nu_i}}$  such that  $\overline{C} \subset \overline{C_i}$  and  $\dim(C_i) = \langle \rho, 2\lambda \rangle - i$ . Then (5.2) implies that  $\langle \rho, \mu \rangle \leq \langle \rho, \lambda \rangle - d$ , or in other words that  $d \leq \langle \rho, \lambda \rangle - \langle \rho, \mu \rangle$ ; this implies that

$$\dim(C) \geq \langle \rho, \lambda + \mu \rangle,$$

as expected.

(3) Let  $Z$  be an irreducible component of  $\overline{\text{Gr}_G^\lambda} \cap \overline{S_\mu}$ . Then  $Z$  must meet  $\text{Gr}_G^\lambda$ , otherwise by (3.2) it would be contained in some  $\overline{\text{Gr}_G^\eta}$  with  $\eta < \lambda$ , and the inequality  $\dim Z = \langle \rho, \lambda + \mu \rangle > \langle \rho, \eta + \mu \rangle$  would contradict (2). Therefore  $Z \cap \text{Gr}_G^\lambda$  is open dense in  $Z$ .  $\square$

*Remark 5.3.* The irreducible components of the intersections  $\overline{\text{Gr}_G^\lambda} \cap \overline{S_\mu}$ , or sometimes those of the intersections  $\text{Gr}_G^\lambda \cap S_\mu$ , are called *Mirković–Vilonen cycles*; they have been studied and used extensively in various fields since their introduction in [MV2], see e.g. [BrG, GL, BaG, Kam].

The following corollary will prove to be useful.

**Corollary 5.4.** *Let  $\lambda \in X_*(T)^+$ , and let  $X \subset \overline{\text{Gr}_G^\lambda}$  be a closed  $T$ -invariant subvariety. Then*

$$\dim(X) \leq \max_{\substack{\mu \in X_*(T) \\ L_\mu \in X}} \langle \rho, \lambda + \mu \rangle.$$

*Proof.* Let  $\eta \in X_*(T)$  be regular dominant. We saw during the proof of Theorem 5.2(1) that

$$S_\mu = \{x \in \text{Gr}_G \mid \eta(a) \cdot x \rightarrow L_\mu \text{ when } a \rightarrow 0\}.$$

Therefore  $X$  meets  $S_\mu$  if and only if  $L_\mu \in X$ , whence

$$X \subset \bigcup_{\substack{\mu \in X_*(T) \\ L_\mu \in X}} S_\mu,$$

and therefore

$$X \subset \bigcup_{\substack{\mu \in X_*(T) \\ L_\mu \in X}} (\overline{\text{Gr}_G^\lambda} \cap S_\mu).$$

The corollary now follows from Theorem 5.2(2).  $\square$

The following theorem is the analogue of Theorem 5.2 for the Borel subgroup  $B^-$  in place of  $B$ .

**Theorem 5.5.** *Let  $\lambda, \mu \in X_*(T)$  with  $\lambda$  dominant.*

1. *We have*

$$\overline{\text{Gr}_G^\lambda} \cap T_\mu \neq \emptyset \iff L_\mu \in \overline{\text{Gr}_G^\lambda} \iff \mu \in \text{Conv}(W\lambda) \cap (\lambda + Q^\vee).$$

2. *If  $\mu$  satisfies the condition in (1), the intersection  $\overline{\text{Gr}_G^\lambda} \cap T_\mu$  has pure dimension  $\langle \rho, \lambda - \mu \rangle$ .*

3. *If  $\mu$  satisfies the condition in (1), then  $\overline{\text{Gr}_G^\lambda} \cap T_\mu$  is open dense in  $\overline{\text{Gr}_G^\lambda} \cap T_\mu$ ; in particular, the irreducible components of  $\overline{\text{Gr}_G^\lambda} \cap T_\mu$  and  $\text{Gr}_G^\lambda \cap T_\mu$  are in a canonical bijection.*

### 5.3 Weight functors

Recall that if  $X$  is a topological space,  $i : Y \rightarrow X$  is the inclusion of a locally closed subspace and  $\mathcal{F} \in D_c^b(X, \mathbf{k})$ , then the local cohomology groups  $H_Y^k(X, \mathcal{F})$  are defined as  $H^k(Y, i^! \mathcal{F})$ .

**Proposition 5.6.** *For each  $\mathcal{A} \in P_{G_O}(\text{Gr}_G, \mathbf{k})$ ,  $\mu \in X_*(T)$  and  $k \in \mathbf{Z}$ , there exists a canonical isomorphism*

$$H_{T_\mu}^k(\text{Gr}_G, \mathcal{A}) \xrightarrow{\sim} H_c^k(S_\mu, \mathcal{A}),$$

*and both terms vanish if  $k \neq \langle 2\rho, \mu \rangle$ .*



*Proof.* For all  $\lambda \in X_*(T)^+$ , we have  $\mathcal{A}|_{\mathrm{Gr}_G^\lambda} \in D^{\leq -(2\rho, \lambda)}(\mathrm{Gr}_G^\lambda, \mathbf{k})$  by the perversity conditions (see §4.1). Further, the dimension estimates from Theorem 5.2(2) imply that  $\mathrm{H}_c^k(\mathrm{Gr}_G^\lambda \cap S_\mu; \mathbf{k}) = 0$  for  $k > \langle 2\rho, \lambda + \mu \rangle$ , see [Iv, Proposition X.1.4]. Using an easy dévissage argument, we deduce that

$$\mathrm{H}_c^k(\mathrm{Gr}_G^\lambda \cap S_\mu, \mathcal{A}) = 0 \quad \text{for } k > \langle 2\rho, \mu \rangle.$$

Filtering the support of  $\mathcal{A}$  by the closed subsets  $\overline{\mathrm{Gr}_G^\lambda}$ , we deduce that

$$\mathrm{H}_c^k(S_\mu, \mathcal{A}) = 0 \quad \text{for } k > \langle 2\rho, \mu \rangle.$$

(To prove this formally, one can either use a spectral sequence or write down distinguished triangles associated to inclusions of an open subset and its closed complement. With both methods, in order to deal with a sequence of closed subsets, it is convenient to enumerate the dominant weights as  $(\lambda_n)_{n \geq 0}$  in such a way that  $(\lambda_i \leq \lambda_j) \Rightarrow (i \leq j)$ .)

An analogous (dual) argument, using [Iv, Theorem X.2.1], shows that

$$\mathrm{H}_{T_\mu}^k(\mathrm{Gr}_G, \mathcal{A}) = 0 \quad \text{for } k < \langle 2\rho, \mu \rangle.$$

Lastly, Braden's hyperbolic localization theorem [Br, Theorem 1] provides a canonical isomorphism

$$\mathrm{H}_{T_\mu}^k(\mathrm{Gr}_G, \mathcal{A}) \cong \mathrm{H}_c^k(S_\mu, \mathcal{A})$$

for any  $k \in \mathbf{Z}$ . The claim follows.  $\square$

*Remark 5.7.* 1. See [Xu, §1.8.1] for a discussion of the validity of the normality assumption needed to apply Braden's theorem, and for an alternative proof using [DrG] instead on [Br] (and which therefore avoids this normality question).

2. Explicitly, the isomorphism in Proposition 5.6 is constructed as follows. Let

$$t_\mu : T_\mu \rightarrow \mathrm{Gr}_G \quad \text{and} \quad s_\mu : S_\mu \rightarrow \mathrm{Gr}_G$$

be the embeddings, and consider also the natural maps

$$\pi_\mu^T : T_\mu \rightarrow \{L_\mu\}, \quad \pi_\mu^S : S_\mu \rightarrow \{L_\mu\}, \quad i_\mu^T : \{L_\mu\} \rightarrow T_\mu, \quad i_\mu^S : \{L_\mu\} \rightarrow S_\mu.$$

By adjunction and the base change theorem, there exist canonical isomorphisms

$$\begin{aligned} \mathrm{Hom}((i_\mu^T)^*(t_\mu)^!(-), (i_\mu^S)^!(s_\mu)^*(-)) &\cong \mathrm{Hom}((t_\mu)^!(-), (i_\mu^T)_*(i_\mu^S)^!(s_\mu)^*(-)) \\ &\cong \mathrm{Hom}((t_\mu)^!(-), (t_\mu)^!(s_\mu)_*(s_\mu)^*(-)); \end{aligned}$$

hence the adjunction morphism  $\mathrm{id} \rightarrow (s_\mu)_*(s_\mu)^*$  induces a morphism of functors

$$(i_\mu^T)^*(t_\mu)^! \rightarrow (i_\mu^S)^!(s_\mu)^*.$$

Finally, one identifies the functors  $(i_\mu^T)^*$  and  $(\pi_\mu^T)_*$ , resp.  $(i_\mu^S)^!$  and  $(\pi_\mu^S)!$ , when applied to “weakly equivariant” objects; see [Br, Equation (1)].

In view of this proposition, for any  $\mu \in X_*(T)$  we consider the functor

$$F_\mu : P_{G_{\mathcal{O}}}(\mathrm{Gr}_G, \mathbf{k}) \rightarrow \mathrm{Vect}_{\mathbf{k}}$$

defined by

$$F_\mu(\mathcal{A}) = H_{T_\mu}^{(2\rho, \mu)}(\mathrm{Gr}_G, \mathcal{A}) \cong H_c^{(2\rho, \mu)}(S_\mu, \mathcal{A}).$$

Since the category  $P_{G_{\mathcal{O}}}(\mathrm{Gr}_G, \mathbf{k})$  is semisimple (see Theorem 4.2 and Corollary 4.8), this functor is automatically exact.

*Remark 5.8.* 1. The  $G_{\mathcal{O}}$ -invariance is not used in the proof of Proposition 5.6 (only the constructibility with respect to  $G_{\mathcal{O}}$ -orbits matters).

2. The same arguments show more generally that if  $\mathcal{F}$  is in  $P_{\mathcal{S}}(Z, \mathbf{k})$ , where  $Z \subset \mathrm{Gr}_G$  is a locally closed union of  $G_{\mathcal{O}}$ -orbits (and where by abuse we still denote by  $\mathcal{S}$  the restriction of this stratification to  $Z$ ), then for any  $\lambda \in X_*(T)^+$  such that  $L_\lambda \in Z$  and any  $k \in \mathbf{Z}$  there exists a canonical isomorphism

$$H_{T_\lambda \cap Z}^k(Z, \mathcal{F}) \xrightarrow{\sim} H_c^k(S_\lambda \cap Z, \mathcal{F}),$$

and that these spaces vanish unless  $k = \langle 2\rho, \lambda \rangle$ . (Note that if  $Z$  is not closed, the condition  $L_\lambda \in Z$  is *not* equivalent to the condition  $S_\lambda \cap Z \neq \emptyset$ . In particular,  $Z$  might not be covered by the intersections  $Z \cap S_\lambda$  where  $\lambda \in X_*(T)$  is such that  $L_\lambda \in Z$ .)

## 5.4 Total cohomology and weight functors

We now consider the functor

$$F : P_{G_{\mathcal{O}}}(\mathrm{Gr}_G, \mathbf{k}) \rightarrow \mathrm{Vect}_{\mathbf{k}}$$

defined by

$$F(\mathcal{A}) = H^\bullet(\mathrm{Gr}_G, \mathcal{A}).$$

**Theorem 5.9.** 1. *There exists a canonical isomorphism of functors*

$$F \cong \bigoplus_{\mu \in X_*(T)} F_\mu : P_{G_{\mathcal{O}}}(\mathrm{Gr}_G, \mathbf{k}) \rightarrow \mathrm{Vect}_{\mathbf{k}}.$$

2. *The functor  $F$  is exact and faithful.*

*Proof.* (1) Let  $\mathcal{A} \in P_{G_{\mathcal{O}}}(\mathrm{Gr}_G, \mathbf{k})$ . Our aim is to construct a canonical isomorphism

$$H^\bullet(\mathrm{Gr}_G, \mathcal{A}) \cong \bigoplus_{\mu \in X_*(T)} F_\mu(\mathcal{A}),$$

and more precisely to construct a canonical isomorphism

$$H^k(\mathrm{Gr}_G, \mathcal{A}) \cong \bigoplus_{\substack{\mu \in X_*(T) \\ \langle 2\rho, \mu \rangle = k}} F_\mu(\mathcal{A})$$

for each  $k \in \mathbf{Z}$ .

Without loss of generality, we may assume that  $\mathcal{A}$  is indecomposable, and in particular that the support of  $\mathcal{A}$  is connected.

For  $n \in \frac{1}{2}\mathbf{Z}$ , set

$$Z_n = \bigsqcup_{\substack{\mu \in X_*(T) \\ \langle \rho, \mu \rangle = n}} T_\mu.$$

Then both

$$\bigcup_{n \in \mathbf{Z}} Z_n \quad \text{and} \quad \bigcup_{n \in \frac{1}{2} + \mathbf{Z}} Z_n$$

are unions of connected components of  $\text{Gr}_G$ . As  $\text{supp } \mathcal{A}$  was assumed to be connected, it is contained in one of these subsets. Let us assume that it is contained in the first one, the reasoning in the other case being entirely similar.

We endow  $Z_n$  with the topology induced from that of  $\text{Gr}_G$ . Then  $Z_n$  is the topological disjoint union of the  $T_\mu$  contained in it, and it follows that

$$\mathbf{H}_{Z_n}^k(\text{Gr}_G, \mathcal{A}) = \begin{cases} 0 & \text{if } k \neq 2n; \\ \bigoplus_{\langle \rho, \mu \rangle = n} F_\mu(\mathcal{A}) & \text{if } k = 2n. \end{cases} \quad (5.3)$$

By (3.7), the closure of  $Z_n$  is

$$\overline{Z_n} = Z_n \sqcup Z_{n+1} \sqcup Z_{n+2} \sqcup \cdots = Z_n \sqcup \overline{Z_{n+1}},$$

so there is a diagram of complementary open and closed inclusions

$$\overline{Z_{n+1}} \xrightarrow{i} \overline{Z_n} \xleftarrow{j} Z_n.$$

Applying the cohomological functor  $\mathbf{H}^\bullet(\overline{Z_n}, ?)$  to the distinguished triangle

$$i_* i^! \mathcal{A}_n \rightarrow \mathcal{A}_n \rightarrow j_* j^! \mathcal{A}_n \xrightarrow{[1]}$$

where  $\mathcal{A}_n$  is the corestriction of  $\mathcal{A}$  to  $\overline{Z_n}$ , we obtain a long exact sequence

$$\cdots \rightarrow \mathbf{H}_{Z_{n+1}}^k(\text{Gr}_G, \mathcal{A}) \rightarrow \mathbf{H}_{Z_n}^k(\text{Gr}_G, \mathcal{A}) \rightarrow \mathbf{H}_{Z_n}^k(\text{Gr}_G, \mathcal{A}) \rightarrow \mathbf{H}_{Z_{n+1}}^{k+1}(\text{Gr}_G, \mathcal{A}) \rightarrow \cdots$$

For  $n$  large enough,  $\text{supp } \mathcal{A}$  is disjoint from  $\overline{Z_n}$ , because  $\text{supp } \mathcal{A}$  is compact and  $\overline{Z_n}$  is far away from the origin of  $\text{Gr}_G$ .<sup>15</sup> Consequently  $\mathbf{H}_{Z_n}^\bullet(\text{Gr}_G, \mathcal{A}) = 0$  for  $n$  large enough. Using the long exact sequence above and (5.3), a decreasing induction on  $n$  leads to

$$\begin{array}{ll} \mathbf{H}_{Z_n}^k(\text{Gr}_G, \mathcal{A}) = 0 & \text{if } k \text{ is odd or if } n > \frac{k}{2}, \\ \mathbf{H}_{Z_{k/2}}^k(\text{Gr}_G, \mathcal{A}) \xrightarrow{\sim} \mathbf{H}_{Z_n}^k(\text{Gr}_G, \mathcal{A}) & \text{if } k \text{ is even and } n \leq \frac{k}{2}. \\ \downarrow \wr & \\ \mathbf{H}_{Z_{k/2}}^k(\text{Gr}_G, \mathcal{A}) & \end{array}$$

<sup>15</sup>The reader may here have in mind the Serre tree considered in §3.2:  $\overline{Z_n}$  is a union of horospheres centered at  $+\infty$  and going through  $L_{n\alpha^\vee}$ ; for  $n$  large enough, this is located far away on the right.

One concludes by taking  $n$  small enough so that  $\text{supp } \mathcal{A} \subset \overline{Z_n}$ .

(2) The exactness is automatic since the category  $\mathbf{P}_{G_O}(\text{Gr}_G, \mathbf{k})$  is semisimple (see the comments before the theorem). Given the exactness, the faithfulness means that  $\mathbf{F}$  does not kill any nonzero object in  $\mathbf{P}_{G_O}(\text{Gr}_G, \mathbf{k})$ . So let us take a nonzero perverse sheaf  $\mathcal{A}$  in our category. Then  $\text{supp } \mathcal{A}$  is a finite union of Schubert cells  $\text{Gr}_G^\lambda$ . Let us choose  $\lambda$  maximal for this property. Then  $\mathcal{A}|_{\text{Gr}_G^\lambda} \cong \underline{\mathbf{k}}[\dim \text{Gr}_G^\lambda]$  and as in the proof of Theorem 5.2(2) we have

$$((\text{supp } \mathcal{A}) \setminus \text{Gr}_G^\lambda) \cap T_\lambda = \emptyset \quad \text{and} \quad \text{Gr}_G^\lambda \cap T_\lambda = \{L_\lambda\},$$

and therefore  $\mathbf{F}_\lambda(\mathcal{A}) \neq 0$ , which implies that  $\mathbf{F}(\mathcal{A}) \neq 0$ .  $\square$

*Remark 5.10.* The proof of Theorem 5.9 has broken the symmetry between the two sides of hyperbolic localization, so let us try to restore it. Given  $\mu \in X_*(T)$ , let us define the inclusion maps

$$T_\mu \begin{array}{c} \xrightarrow{t'_\mu} \overline{T}_\mu \xrightarrow{t''_\mu} \\ \searrow \quad \quad \quad \nearrow \\ \text{Gr}_G \end{array} \quad \text{and} \quad S_\mu \begin{array}{c} \xrightarrow{s'_\mu} \overline{S}_\mu \xrightarrow{s''_\mu} \\ \searrow \quad \quad \quad \nearrow \\ \text{Gr}_G \end{array}.$$

Then for each  $\mathcal{A} \in \mathbf{P}_{G_O}(\text{Gr}_G, \mathbf{k})$ , we have

$$\begin{aligned} \mathbf{H}_{T_\mu}^k(\text{Gr}_G, \mathcal{A}) &= \mathbf{H}^k(\text{Gr}_G, (t''_\mu)! (t'_\mu)^! \mathcal{A}), \\ \mathbf{H}^k(\overline{S}_\mu, \mathcal{A}) &= \mathbf{H}^k(\text{Gr}_G, (s''_\mu)_* (s'_\mu)^* \mathcal{A}), \\ \mathbf{H}_{T_\mu}^k(\text{Gr}_G, \mathcal{A}) &= \mathbf{H}^k(\text{Gr}_G, \underbrace{(t''_\mu)! (t'_\mu)_* (t'_\mu)^* (t''_\mu)^!}_{\cong t_{\mu*} t'_\mu^!} \mathcal{A}), \\ \mathbf{H}_c^k(S_\mu, \mathcal{A}) &= \mathbf{H}^k(\text{Gr}_G, \underbrace{(s''_\mu)_* (s'_\mu)! (s'_\mu)^! (s''_\mu)^*}_{\cong s_{\mu!} s_\mu^*} \mathcal{A}). \end{aligned}$$

One can check that the adjunction maps and hyperbolic localization give rise to a commutative diagram

$$\begin{array}{ccc} & \mathbf{H}^k(\text{Gr}_G, \mathcal{A}) & \\ & \nearrow \scriptstyle (t''_\mu)! (t'_\mu)^! \rightarrow \text{id} & \searrow \scriptstyle \text{id} \rightarrow (s''_\mu)_* (s'_\mu)^* \\ \mathbf{H}_{T_\mu}^k(\text{Gr}_G, \mathcal{A}) & & \mathbf{H}^k(\overline{S}_\mu, \mathcal{A}) \\ & \searrow \scriptstyle \text{id} \rightarrow (t'_\mu)_* (t'_\mu)^* & \nearrow \scriptstyle (s'_\mu)! (s'_\mu)^! \rightarrow \text{id} \\ & \mathbf{H}_{T_\mu}^k(\text{Gr}_G, \mathcal{A}) & \xrightarrow[\sim]{\text{hyperbolic loc.}} \mathbf{H}_c^k(S_\mu, \mathcal{A}) \end{array}$$

If  $k = \langle 2\rho, \mu \rangle$ , then the three bottom arrows are isomorphisms, so the four bottom spaces can be identified: they define the functor  $\mathbf{F}_\mu$ . At this point, let us write

$$\mathbf{F}_\mu(\mathcal{A}) \xrightarrow{i_\mu} \mathbf{H}^k(\text{Gr}_G, \mathcal{A}) \xrightarrow{p_\mu} \mathbf{F}_\mu(\mathcal{A})$$

for the two top arrows of the diagram above. Theorem 5.9 shows that for each  $k \in \mathbf{Z}$ ,

$$\mathbf{H}^k(\text{Gr}_G, \mathcal{A}) = \bigoplus_{\substack{\mu \in X_*(T) \\ \langle 2\rho, \mu \rangle = k}} \text{im}(i_\mu), \quad (5.4)$$

and likewise

$$\mathrm{H}^k(\mathrm{Gr}_G, \mathcal{A}) \xrightarrow{\sim} \bigoplus_{\substack{\nu \in X_*(T) \\ \langle 2\rho, \nu \rangle = k}} \mathrm{coim}(p_\nu), \quad (5.5)$$

where  $\mathrm{coim}(p_\nu) = \mathrm{H}^k(\mathrm{Gr}_G, \mathcal{A}) / \ker(p_\nu)$  is the coimage of  $p_\nu$ . Further, if  $\mu \neq \nu$  and  $\langle 2\rho, \mu \rangle = \langle 2\rho, \nu \rangle$ , then  $\mu \not\leq \nu$ , so  $\overline{S}_\nu \cap \overline{T}_\mu = \emptyset$  by Lemma 3.6, and therefore  $p_\nu \circ i_\mu = 0$ . This implies that the decompositions (5.4) and (5.5) of  $\mathrm{H}^k(\mathrm{Gr}_G, \mathcal{A})$  coincide. The decomposition

$$\mathrm{H}^k(\mathrm{Gr}_G, \mathcal{A}) = \bigoplus_{\substack{\mu \in X_*(T) \\ \langle 2\rho, \mu \rangle = k}} \mathrm{F}_\mu(\mathcal{A})$$

is therefore defined without ambiguity.

## 5.5 Independence of the choice of Torel

To define the functors  $\mathrm{F}_\mu$  we started by choosing a Torel (or Borus)  $T \subset B$ . In this subsection we show that these functors are in fact independent of this choice, in the following way. If we fix a Torel  $T \subset B$ , then any other Torel will be of the form  $gTg^{-1} \subset gBg^{-1}$  for some  $g \in G$ , whose class  $gT \in G/T$  is uniquely determined. Then there exists a canonical isomorphism  $X_*(T) \xrightarrow{\sim} X_*(gTg^{-1})$  sending  $\lambda : \mathbf{C}^\times \rightarrow T$  to the cocharacter  $z \mapsto g\lambda(z)g^{-1}$ . We use this operation to identify  $X_*(T)$  and  $X_*(gTg^{-1})$ .<sup>16</sup> Then for  $\lambda \in X_*(T)$  we can consider both the  $\lambda$ -weight functor  $\mathrm{F}_\lambda$  constructed out of the Torel  $T \subset B$ , and the  $\lambda$ -weight functor  $\mathrm{F}_\lambda^{gT}$  constructed out of the Torel  $gTg^{-1} \subset gBg^{-1}$ .

**Lemma 5.11.** *In the setting considered above, for any  $gT \in G/T$  there exists a canonical isomorphism of functor  $\mathrm{F}_\lambda \xrightarrow{\sim} \mathrm{F}_\lambda^{gT}$ .*

*Proof.* Set  $X_\lambda := \{(x, gT) \in \mathrm{Gr}_G \times G/T \mid x \in g \cdot S_\lambda\}$ , and consider the diagram

$$\begin{array}{ccccc} \mathrm{Gr}_G & \xleftarrow{a} & \mathrm{Gr}_G \times G/T & \xleftarrow{c} & X_\lambda \\ & & \searrow b & & \searrow d \\ & & & & G/T, \end{array}$$

where  $a, b, d$  are the natural projections, and  $c$  is the embedding. Let  $\mathcal{F}$  in  $\mathrm{P}_{\mathcal{S}}(\mathrm{Gr}_G, \mathbf{k})$ , and consider the complex of sheaves  $b_*c_!c^*a^*\mathcal{F} \cong d_!(ac)^*\mathcal{F}$ . By the base change theorem, the fiber of this complex over  $gT$  is  $R\Gamma_c(g \cdot S_\lambda, \mathcal{F})$ . Applying Proposition 5.6 for the choice of Torel  $gTg^{-1} \subset gBg^{-1}$ , we see that this fiber is concentrated in degree  $\langle \lambda, 2\rho \rangle$ . Hence the complex  $b_*c_!c^*a^*\mathcal{F}$  itself is concentrated in degree  $\langle \lambda, 2\rho \rangle$ .

Next, the proof of Theorem 5.9 (and the comments in Remark 5.10) can also be written “in family” over  $G/T$ ; this shows that  $\mathcal{H}^{\langle 2\rho, \lambda \rangle}(b_*c_!c^*a^*\mathcal{F})$  is a direct factor of

$$\mathcal{H}^{\langle 2\rho, \lambda \rangle}(b_*a^*\mathcal{F}) \cong \mathrm{H}^{\langle 2\rho, \lambda \rangle}(\mathrm{Gr}_G, \mathcal{F}) \otimes_{\mathbf{k}} \underline{\mathbf{k}}_{G/T}$$

<sup>16</sup>Note that this construction depends on the choice of Borel subgroup: for two general maximal tori in  $G$ , there is no canonical identification of their cocharacter lattices!

(where the isomorphism follows from the projection formula). Since this sheaf is a constant local system, we deduce that  $\mathcal{H}^{\langle 2\rho, \lambda \rangle}(b_*c_1c^*a^*\mathcal{F})$  is also a constant local system. Hence its fibers over any two points can be identified canonically (because they both identify with global sections); in particular we deduce a canonical isomorphism  $F_\lambda \xrightarrow{\sim} F_\lambda^{gT}$  for any  $gT \in G/T$ .  $\square$

*Remark 5.12.* Note that the proof of Lemma 5.11 only relies on the  $\mathcal{S}$ -constructibility of  $\mathcal{F}$ , and not on its  $G_{\mathcal{O}}$ -equivariance; in particular, this proof is independent of Corollary 4.8. (This fact does not play any role in the present case when  $\mathbf{k}$  is a field of characteristic 0, but will be important in the case of general coefficients considered in Part II.)

## 5.6 Weight spaces of simple objects

**Proposition 5.13.** *Let  $\lambda, \mu \in X_*(T)$  with  $\lambda$  dominant. Then  $\dim F_\mu(\mathbf{IC}_\lambda)$  is the number of irreducible components of  $\mathrm{Gr}_G^\lambda \cap S_\mu$ . In particular, it is nonzero if and only if  $\mu \in \mathrm{Conv}(W\lambda) \cap (\lambda + Q^\vee)$ .*

*Proof.* For each  $\eta \in X_*(T)^+$ , one of the following three possibilities hold:

- $\mathrm{Gr}_G^\eta$  does not meet  $\mathrm{supp} \mathbf{IC}_\lambda$ , and  $\mathbf{IC}_\lambda|_{\mathrm{Gr}_G^\eta} = 0$ ;
- $\eta = \lambda$  and  $\mathbf{IC}_\lambda|_{\mathrm{Gr}_G^\eta} \in D^{\leq -\langle 2\rho, \eta \rangle}(\mathrm{Gr}_G^\eta, \mathbf{k})$ ;
- $\eta < \lambda$  and  $\mathbf{IC}_\lambda|_{\mathrm{Gr}_G^\eta} \in D^{\leq -\langle 2\rho, \eta \rangle - 1}(\mathrm{Gr}_G^\eta, \mathbf{k})$

(see (4.2)). In the last case, we can in fact replace  $-\langle 2\rho, \eta \rangle - 1$  by  $-\langle 2\rho, \eta \rangle - 2$  because of Lemma 4.5 (and the fact that  $\eta < \lambda \Rightarrow \langle 2\rho, \lambda \rangle \equiv \langle 2\rho, \eta \rangle \pmod{2}$ ).

When we gather these facts to reconstruct  $H_c^{\langle 2\rho, \mu \rangle}(S_\mu, \mathbf{IC}_\lambda)$  using the same method as in the proof of Proposition 5.6, only the stratum  $\mathrm{Gr}_G^\lambda$  contributes, and we obtain an isomorphism

$$H_c^{\langle 2\rho, \mu \rangle}(S_\mu, \mathbf{IC}_\lambda) \cong H_c^{\langle 2\rho, \mu \rangle}(\mathrm{Gr}_G^\lambda \cap S_\mu, \mathbf{IC}_\lambda|_{\mathrm{Gr}_G^\lambda}).$$

Therefore

$$F_\mu(\mathbf{IC}_\lambda) \cong H_c^{\langle 2\rho, \mu \rangle}(\mathrm{Gr}_G^\lambda \cap S_\mu, \mathbf{IC}_\lambda|_{\mathrm{Gr}_G^\lambda}) = H_c^{\langle 2\rho, \lambda + \mu \rangle}(\mathrm{Gr}_G^\lambda \cap S_\mu; \mathbf{k}).$$

The right-hand side is the top cohomology group with compact support of  $\mathrm{Gr}_G^\lambda \cap S_\mu$  by Theorem 5.2; it therefore has a natural basis indexed by the irreducible components of top dimension of this intersection.<sup>17</sup>

The last claim then follows from Theorem 5.2.  $\square$

*Remark 5.14.* 1. See Proposition 11.1 below for a proof, based on slightly different ideas, of a statement which reduces to Proposition 5.13 in the case  $\mathbf{k}$  is a field of characteristic 0.

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<sup>17</sup>This property is a classical fact about the top cohomology with compact supports of algebraic varieties, which follows e.g. from the considerations in [Iv, §X.1].

2. In a similar vein, one can describe the multiplicity space of a simple object  $\mathbf{IC}_\nu$  as a direct summand of a product  $\mathbf{IC}_\lambda \star \mathbf{IC}_\mu$  (where  $\star$  is the convolution product introduced in §6.2 below) in terms of cohomology of a certain variety, see [Z4, Corollary 5.1.5] for details.

## 6 Convolution product: “classical” point of view

Our goal in Sections 6–7 is to endow the category  $\mathbf{P}_{G_{\mathcal{O}}}(\mathrm{Gr}_G, \mathbf{k})$  of  $G_{\mathcal{O}}$ -equivariant perverse sheaves on  $\mathrm{Gr}_G$  with the structure of a symmetric monoidal category. We first define the convolution product of two equivariant perverse sheaves, and with the help of the notion of stratified semismall map, we show that the result of the operation is a perverse sheaf. We also define an associativity constraint. To proceed further, we will need a different point of view on convolution, which uses an important auxiliary construction, known as the Beilinson–Drinfeld Grassmannian. This is considered in Section 7.

### 6.1 Stratified semismall maps

We first consider a general result, which guarantees that the direct image of a perverse sheaf under a *stratified semismall morphism*<sup>18</sup> is a perverse sheaf.

Let  $(X, \mathcal{T})$  and  $(Y, \mathcal{U})$  be two stratified algebraic varieties, and let  $f : Y \rightarrow X$  be a proper map such that for each  $U \in \mathcal{U}$ , the set  $f(U)$  is a union of strata. We say that  $f$  is *stratified semismall* if for any stratum  $T \subset f(U)$  and any  $x \in T$ , we have

$$\dim(f^{-1}(x) \cap U) \leq \frac{1}{2}(\dim U - \dim T).$$

We say that  $f$  is *locally trivial* if for any  $(T, U) \in \mathcal{T} \times \mathcal{U}$  such that  $T \subset f(U)$ , the map  $U \cap f^{-1}(T) \rightarrow T$  induced by  $f$  is a Zariski locally trivial fibration.

**Proposition 6.1.** *If  $f$  is stratified semismall and locally trivial and if  $\mathcal{F}$  is a perverse sheaf on  $Y$  constructible with respect to  $\mathcal{U}$ , then  $f_*\mathcal{F}$  is a perverse sheaf on  $X$  constructible with respect to  $\mathcal{T}$ .*

*Proof.* For any stratum  $T \in \mathcal{T}$ , we can consider the restriction

$$\begin{array}{ccc} f^{-1}(T) & \xrightarrow{f_T} & T \\ \parallel & & \\ \bigsqcup_{U \in \mathcal{U}} f^{-1}(T) \cap U & & \end{array}$$

We denote by  $f_{T,U} : f^{-1}(T) \cap U \rightarrow T$  the restriction of  $f$  (which is a Zariski locally trivial fibration by assumption if  $T \subset f(U)$ ). Note here that since  $f(U)$  is a union of strata in  $\mathcal{T}$ , the assertions that  $T \subset f(U)$  and that  $f^{-1}(T) \cap U \neq \emptyset$  are equivalent.

<sup>18</sup>This notion is a refinement of the more familiar notion of *semismall morphism* (see e.g. [GM, BM]) which takes into account the stratifications on the varieties under consideration.

First, let us prove that for any  $\mathcal{F}$  in the  $\mathcal{U}$ -constructible derived category  $D_{\mathcal{U}}^b(Y, \mathbf{k})$ , the complex  $f_*\mathcal{F} = f_!\mathcal{F}$  belongs to the  $\mathcal{T}$ -constructible derived category  $D_{\mathcal{T}}^b(X, \mathbf{k})$ . We proceed by induction on the smallest number of strata whose union is a closed subvariety  $Z$  of  $Y$  such that  $\mathcal{F}|_{Y \setminus Z} = 0$ . So, let us consider such a closed union of strata, and choose some  $U \in \mathcal{U}$  which is open in  $Z$ . We can consider  $\mathcal{F}$  as a complex in  $D_{\mathcal{U}}^b(Z, \mathbf{k})$ . Then, if we denote by  $j : U \rightarrow Z$  and  $i : Z \setminus U \rightarrow Z$  the embeddings, we have a standard distinguished triangle

$$j_!j^*\mathcal{F} \rightarrow \mathcal{F} \rightarrow i_*i^*\mathcal{F} \xrightarrow{[1]}.$$

Applying  $f_!$ , we deduce a distinguished triangle

$$(f \circ j)_!j^*\mathcal{F} \rightarrow \mathcal{F} \rightarrow (f \circ i)_!i^*\mathcal{F} \xrightarrow{[1]}.$$

By induction, the third term in this triangle belongs to  $D_{\mathcal{T}}^b(X, \mathbf{k})$ . Since  $D_{\mathcal{T}}^b(X, \mathbf{k})$  is a triangulated subcategory of the derived category of  $\mathbf{k}$ -sheaves on  $X$ , we are reduced to prove that  $(f \circ j)_!j^*\mathcal{F}$  belongs to  $D_{\mathcal{T}}^b(X, \mathbf{k})$ . Using truncation triangles, for this it suffices to prove that for each  $n \in \mathbf{Z}$ ,  $(f \circ j)_!\mathcal{H}^n(j^*\mathcal{F})$  belongs to  $D_{\mathcal{T}}^b(X, \mathbf{k})$ . Let  $T \in \mathcal{T}$  such that  $T \subset f(U)$ , and let  $g : T \rightarrow X$  be the embedding. By the base change theorem, we have

$$g^*(f \circ j)_!\mathcal{H}^n(j^*\mathcal{F}) \cong (f_{T,U})_!\mathcal{H}^n(\mathcal{F})|_{f^{-1}(T) \cap U}. \quad (6.1)$$

Now since  $\mathcal{F}$  is  $\mathcal{U}$ -constructible,  $\mathcal{H}^n(\mathcal{F})|_{f^{-1}(T) \cap U}$  is a local system; since  $f_{T,U}$  is a locally trivial fibration we deduce that the cohomology sheaves of  $g^*(f \circ j)_!\mathcal{H}^n(j^*\mathcal{F})$  are local systems on  $T$ , and finally that  $\mathcal{F}$  belongs to  $D_{\mathcal{T}}^b(X, \mathbf{k})$ .<sup>19</sup>

Next, we prove that if  $\mathcal{F}$  is in nonpositive perverse degrees, then  $f_!\mathcal{F}$  is in nonpositive perverse degrees. Let as above  $T \in \mathcal{T}$  be a stratum, and consider the Cartesian diagram

$$\begin{array}{ccc} f^{-1}(T) & \xrightarrow{h} & Y \\ f_T \downarrow & & \downarrow f \\ T & \xrightarrow{g} & X, \end{array}$$

where  $g$  and  $h$  are the embeddings. Then we need to prove that

$$g^*f_!\mathcal{F} \cong (f_T)_!h^*\mathcal{F}$$

is concentrated in degrees  $\leq -\dim T$ . By the same arguments as above, it suffices to prove that for any  $U \in \mathcal{U}$  such that  $U \cap f^{-1}(T) \neq \emptyset$ , the complex  $(f_{T,U})_!\mathcal{F}|_{U \cap f^{-1}(T)}$  satisfies this property. This follows from a classical vanishing result for cohomology with compact supports already used in the proof of Proposition 5.6, see [Iv, Proposition X.1.4].

Finally, we need to prove that if  $\mathcal{F}$  is in nonnegative perverse degrees, then  $f_*\mathcal{F}$  is in nonnegative perverse degrees. This can be deduced from what we proved above using Verdier duality, or alternatively by an argument “dual” to the preceding one: for  $T, h, g$  as above we need to prove that

$$g^!f_*\mathcal{F} \cong (f_T)_*h^!\mathcal{F}$$

<sup>19</sup>In this argument we use the compatibility of external products with  $!$ -pushforwards; see [Ly, Proposition 2.9.II] for a precise statement.



is concentrated in degrees  $\geq -\dim T$ . Again, this can be reduced to proving that for any  $U \in \mathcal{U}$  the complex

$$(f_{T,U})_* k^! \mathcal{F}$$

is concentrated in degrees  $\geq -\dim T$ , where  $k : f^{-1}(T) \cap U \rightarrow Y$  is the embedding. If  $x \in T$  is any point and  $i_x : \{x\} \rightarrow T$  is the embedding, for this it suffices to prove that  $(i_x)^!(f_{T,U})_* k^! \mathcal{F}$  is concentrated in degrees  $\geq \dim(T)$ . In turn, this follows from a classical result for cohomology with support, see e.g. [AHR, Lemma 4.12].<sup>20</sup>  $\square$

## 6.2 Definition of convolution on $\mathrm{Gr}_G$

To define the convolution operation on  $\mathrm{P}_{G_{\mathcal{O}}}(\mathrm{Gr}_G, \mathbf{k})$ , we will identify this category with the heart of the perverse t-structure on the *constructible equivariant derived category*

$$D_{c,G_{\mathcal{O}}}^b(\mathrm{Gr}_G, \mathbf{k})$$

in the sense of Bernstein–Lunts [BL], see §A.1. (See also §§A.3–A.4 for details on the definition of  $D_{c,G_{\mathcal{O}}}^b(\mathrm{Gr}_G, \mathbf{k})$  and of convolution in a more general context.)

We denote by  $[h] \in \mathrm{Gr}_G$  the coset  $hG_{\mathcal{O}}$  of an element  $h \in G_{\mathcal{K}}$ . Likewise, letting the group  $G_{\mathcal{O}}$  act on  $G_{\mathcal{K}} \times \mathrm{Gr}_G$  by  $k \cdot (g, [h]) = (gk^{-1}, [kh])$ , we denote by  $[g, h]$  the orbit of  $(g, [h])$ . We form the diagram

$$\mathrm{Gr}_G \times \mathrm{Gr}_G \xleftarrow{p} G_{\mathcal{K}} \times \mathrm{Gr}_G \xrightarrow{q} G_{\mathcal{K}} \times^{G_{\mathcal{O}}} \mathrm{Gr}_G \xrightarrow{m} \mathrm{Gr}_G, \quad (6.2)$$

where  $p$  is the map  $(g, [h]) \mapsto ([g], [h])$ ,  $q$  is the map  $(g, [h]) \mapsto [g, h]$ , and  $m$  is the map  $[g, h] \mapsto [gh]$ .

Let  $\mathcal{F}$  and  $\mathcal{G}$  be two complexes of sheaves in the equivariant derived category  $D_{c,G_{\mathcal{O}}}^b(\mathrm{Gr}_G, \mathbf{k})$ . Since the  $G_{\mathcal{O}}$ -action on  $G_{\mathcal{K}} \times \mathrm{Gr}_G$  considered above is free, the functor  $q^*$  induces an equivalence of categories

$$D_{c,G_{\mathcal{O}}}^b(G_{\mathcal{K}} \times^{G_{\mathcal{O}}} \mathrm{Gr}_G, \mathbf{k}) \xrightarrow{\sim} D_{c,G_{\mathcal{O}} \times G_{\mathcal{O}}}^b(G_{\mathcal{K}} \times \mathrm{Gr}_G, \mathbf{k}),$$

see [BL, Theorem 2.6.2]. (Here,  $G_{\mathcal{O}}$  acts on  $G_{\mathcal{K}} \times^{G_{\mathcal{O}}} \mathrm{Gr}_G$  via multiplication on the left on  $G_{\mathcal{K}}$ ; for the action of  $G_{\mathcal{O}} \times G_{\mathcal{O}}$  on  $G_{\mathcal{K}} \times \mathrm{Gr}_G$ , the first copy of  $G_{\mathcal{O}}$  acts via left multiplication on  $G_{\mathcal{K}}$  and the second copy acts as above.) The complex  $p^*(\mathcal{F} \boxtimes \mathcal{G})$  defines an object of  $D_{c,G_{\mathcal{O}} \times G_{\mathcal{O}}}^b(G_{\mathcal{K}} \times \mathrm{Gr}_G)$ . Therefore, we can consider the unique object  $\mathcal{F} \boxtimes \tilde{\mathcal{G}} \in D_{c,G_{\mathcal{O}}}^b(G_{\mathcal{K}} \times^{G_{\mathcal{O}}} \mathrm{Gr}_G, \mathbf{k})$  such that

$$q^*(\mathcal{F} \boxtimes \tilde{\mathcal{G}}) = p^*(\mathcal{F} \boxtimes \mathcal{G}).$$

We then set

$$\mathcal{F} \star \mathcal{G} := m_*(\mathcal{F} \boxtimes \tilde{\mathcal{G}}) \in D_{c,G_{\mathcal{O}}}^b(\mathrm{Gr}_G, \mathbf{k}).$$

*Remark 6.2.* When stating this construction in these terms we cheat a little bit; see §A.4.

<sup>20</sup>In the cases of interest to us here, the local system appearing in [AHR, Lemma 4.12] will be constant; then the claim we need is the statement [Iv, Theorem X.2.1] already used in the proof of Proposition 5.6.

### 6.3 Exactness of convolution

The first important property of the convolution product  $\star$  on  $D_{c,G\mathcal{O}}^b(\mathrm{Gr}_G, \mathbf{k})$  is the following.

**Proposition 6.3.** *Assume that  $\mathcal{F}$  and  $\mathcal{G}$  belong to  $P_{G\mathcal{O}}(\mathrm{Gr}_G, \mathbf{k})$ . Then  $\mathcal{F} \star \mathcal{G}$  also belongs to  $P_{G\mathcal{O}}(\mathrm{Gr}_G, \mathbf{k})$ .*

To prove this result we will need an auxiliary lemma. Here, for  $\lambda, \mu \in X_*(T)^+$  we set

$$\widetilde{\mathrm{Gr}}_G^{\lambda, \mu} := q(p^{-1}(\mathrm{Gr}_G^\lambda \times \mathrm{Gr}_G^\mu)).$$

**Lemma 6.4.** *For any  $\lambda, \mu \in X_*(T)^+$  and  $\nu \in -X_*(T)^+$ , we have*

$$\dim(\widetilde{\mathrm{Gr}}_G^{\lambda, \mu} \cap m^{-1}(L_\nu)) \leq \langle \rho, \lambda + \mu + \nu \rangle.$$

*Proof.* We consider the  $T$ -action on  $G_{\mathcal{K}} \times^{G\mathcal{O}} \mathrm{Gr}_G$  induced by left multiplication on  $G_{\mathcal{K}}$ , and the diagonal  $T$ -action on  $\mathrm{Gr}_G \times \mathrm{Gr}_G$ . Then the map

$$\phi : G_{\mathcal{K}} \times^{G\mathcal{O}} \mathrm{Gr}_G \rightarrow \mathrm{Gr}_G \times \mathrm{Gr}_G$$

that sends  $[g, h]$  to  $([g], [gh])$  is a  $T$ -equivariant isomorphism. We deduce that the  $T$ -fixed points in  $G_{\mathcal{K}} \times^{G\mathcal{O}} \mathrm{Gr}_G$  are of the form  $[t^\alpha, t^\beta]$ , with  $\alpha, \beta \in X_*(T)$ ; indeed  $\phi([t^\alpha, t^\beta]) = (L_\alpha, L_{\alpha+\beta})$ . Further,  $[t^\alpha, t^\beta]$  belongs to

$$X_{\lambda, \mu} := \overline{\widetilde{\mathrm{Gr}}_G^{\lambda, \mu}} = q(p^{-1}(\overline{\mathrm{Gr}}_G^\lambda \times \overline{\mathrm{Gr}}_G^\mu))$$

if and only if the dominant  $W$ -conjugate  $\alpha^+$  of  $\alpha \in X_*(T)$  is  $\leq \lambda$  and the dominant  $W$ -conjugate  $\beta^+$  of  $\beta$  is  $\leq \mu$  with respect to the dominance order.

The morphism  $\phi$  maps  $m^{-1}(L_\nu)$  to  $\mathrm{Gr}_G \times \{L_\nu\}$ . This allows (by projecting onto the first factor) to regard  $X_{\lambda, \mu} \cap m^{-1}(L_\nu)$  as a closed subvariety of  $\overline{\mathrm{Gr}}_G^\lambda$ . Now by Corollary 5.4 we have

$$\dim(X_{\lambda, \mu} \cap m^{-1}(L_\nu)) \leq \max_{\substack{\alpha, \beta \in X_*(T) \\ [t^\alpha, t^\beta] \in X_{\lambda, \mu} \cap m^{-1}(L_\nu)}} \langle \rho, \lambda + \alpha \rangle.$$

The pairs  $(\alpha, \beta)$  occurring here satisfy  $\alpha + \beta = \nu$  and

$$\langle \rho, \mu + \beta \rangle = \langle \rho, \mu - w_0(\beta) \rangle \geq 0$$

since  $w_0(\beta) \leq \beta^+ \leq \mu$ ; hence they satisfy

$$\langle \rho, \lambda + \alpha \rangle \leq \langle \rho, \lambda + \alpha \rangle + \langle \rho, \mu + \beta \rangle = \langle \rho, \lambda + \mu + \nu \rangle,$$

which entails the desired result.  $\square$

We can now give the proof of Proposition 6.3.

*Proof.* We consider the situation

$$\begin{array}{ccc} G_{\mathcal{K}} \times^{G_{\mathcal{O}}} \mathrm{Gr}_G & \xrightarrow{m} & \mathrm{Gr}_G \\ \parallel & & \parallel \\ \bigsqcup_{\lambda, \mu \in X_*(T)^+} \widetilde{\mathrm{Gr}}_G^{\lambda, \mu} & & \bigsqcup_{\nu \in X_*(T)^+} \mathrm{Gr}_G^{\nu} \end{array}$$

Here certainly  $m$  is ind-proper. It is locally trivial, because the whole situation is  $G_{\mathcal{O}}$ -equivariant. Also, it follows from the definitions that the complex  $\mathcal{F} \boxtimes \mathcal{G} \in D_{c, G_{\mathcal{O}}}^b(G_{\mathcal{K}} \times^{G_{\mathcal{O}}} \mathrm{Gr}_G, \mathbf{k})$  defined in §6.2 is perverse and is constructible with respect to the stratification given by the subsets  $\widetilde{\mathrm{Gr}}_G^{\lambda, \mu}$ . To show that  $\mathcal{F} \star \mathcal{G}$  is perverse, using Proposition 6.1 it thus suffices to prove that  $m$  is stratified semismall. This is exactly the content of Lemma 6.4 (since  $\dim(\mathrm{Gr}_G^{w_0(\nu)}) = \langle 2\rho, w_0(\nu) \rangle = -\langle 2\rho, \nu \rangle$  if  $\nu \in -X_*(T)$ ).  $\square$

*Remark 6.5.* A different proof of Proposition 6.3 is due to Gaitsgory. In fact, the convolution  $\mathcal{F} \star \mathcal{G}$  makes sense for any  $\mathcal{F}$  in  $D_c^b(\mathrm{Gr}_G, \mathbf{k})$  and  $\mathcal{G}$  in  $D_{c, G_{\mathcal{O}}}^b(\mathrm{Gr}_G, \mathbf{k})$ . It follows from [Ga, Proposition 6] that, in this generality,  $\mathcal{F} \star \mathcal{G}$  is perverse as soon as  $\mathcal{F}$  and  $\mathcal{G}$  are perverse. This approach uses an interpretation of convolution in terms of nearby cycles. (See also [Z4, §5.4] for an exposition of closely related ideas, based on the notion of universal local acyclicity.)

## 6.4 Associativity of convolution

For  $\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3$  in  $\mathrm{P}_{G_{\mathcal{O}}}(\mathrm{Gr}_G, \mathbf{k})$ , one can define

$$\mathrm{Conv}_3(\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3) = (m_3)_* (\mathcal{F}_1 \widetilde{\boxtimes} \mathcal{F}_2 \widetilde{\boxtimes} \mathcal{F}_3),$$

where  $m_3 : G_{\mathcal{K}} \times^{G_{\mathcal{O}}} G_{\mathcal{K}} \times^{G_{\mathcal{O}}} \mathrm{Gr}_G \rightarrow \mathrm{Gr}_G$  is the map  $[g_1, g_2, g_3] \mapsto [g_1 g_2 g_3]$ , with an obvious notation, and the twisted product  $\mathcal{F}_1 \widetilde{\boxtimes} \mathcal{F}_2 \widetilde{\boxtimes} \mathcal{F}_3$  is defined in the obvious way. Then base change yields natural isomorphisms

$$(\mathcal{F}_1 \star \mathcal{F}_2) \star \mathcal{F}_3 \xleftarrow{\sim} \mathrm{Conv}_3(\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3) \xrightarrow{\sim} \mathcal{F}_1 \star (\mathcal{F}_2 \star \mathcal{F}_3).$$

The composition of these isomorphisms provides an associativity constraint that turns the pair  $(\mathrm{P}_{G_{\mathcal{O}}}(\mathrm{Gr}_G, \mathbf{k}), \star)$  into a monoidal category.

## 7 Convolution and fusion

In this section we describe a different construction of the convolution product on  $\mathrm{P}_{G_{\mathcal{O}}}(\mathrm{Gr}_G, \mathbf{k})$ . This construction uses the Beilinson–Drinfeld Grassmannian, hence ultimately the moduli interpretation of  $\mathrm{Gr}_G$ . It plays a crucial role in the definition of the commutativity constraint for  $\star$ . (The ideas behind all of this go back to work of Beilinson–Drinfeld [BD]. For more details and references on this point of view, the reader might consult [Z4].)

## 7.1 A moduli interpretation of the affine Grassmannian

In this section, we adopt the following setup. We consider a smooth curve  $X$  over  $\mathbf{C}$ , and for any point  $x \in X$ , we denote by  $\mathcal{O}_x$  the completion of the local ring of  $X$  at  $x$  and by  $\mathcal{K}_x$  the fraction field of  $\mathcal{O}_x$ ; the choice of a local coordinate  $t$  on  $X$  around  $x$  leads to isomorphisms  $\mathcal{O}_x \cong \mathbf{C}[[t]]$  and  $\mathcal{K}_x \cong \mathbf{C}((t))$ . Using these data we can define a “local” version of  $\mathrm{Gr}_G$  at  $x$  by  $\mathrm{Gr}_{G,x} := (G_{\mathcal{K}_x}/G_{\mathcal{O}_x})_{\mathrm{red}}$ , where  $G_{\mathcal{K}_x}$  and  $G_{\mathcal{O}_x}$  are defined in the obvious way.

*Remark 7.1.* Below, to lighten notation (and since this does not play any role for us) we will not distinguish between the ind-scheme  $G_{\mathcal{K}_x}/G_{\mathcal{O}_x}$  and the associated ind-variety  $\mathrm{Gr}_{G,x}$ . We leave it to the attentive (and interested) reader to check which version is more appropriate in each statement.

We define

$$\mathcal{D}_x = \mathrm{Spec}(\mathcal{O}_x) \quad \text{and} \quad \mathcal{D}_x^\times = \mathrm{Spec}(\mathcal{K}_x).$$

For a  $\mathbf{C}$ -algebra  $R$ , we consider the completed tensor products  $R \widehat{\otimes} \mathcal{O}_x$  and  $R \widehat{\otimes} \mathcal{K}_x$ , so that

$$R \widehat{\otimes} \mathcal{O}_x \cong R[[t]] \quad \text{and} \quad R \widehat{\otimes} \mathcal{K}_x \cong R((t)).$$

We set

$$\mathcal{D}_{x,R} = \mathrm{Spec}(R \widehat{\otimes} \mathcal{O}_x) \quad \text{and} \quad \mathcal{D}_{x,R}^\times = \mathrm{Spec}(R \widehat{\otimes} \mathcal{K}_x).$$

For a  $\mathbf{C}$ -algebra  $R$ , we set

$$X_R = X \times_{\mathrm{Spec}(\mathbf{C})} \mathrm{Spec}(R) \quad \text{and} \quad X_R^\times = (X \setminus \{x\}) \times_{\mathrm{Spec}(\mathbf{C})} \mathrm{Spec}(R).$$

*Remark 7.2.* Note that the subscript “ $R$ ” does not have the same meaning in the notation “ $\mathcal{D}_{x,R}$ ” and “ $X_R$ ,” in that it is *not* true that  $\mathcal{D}_{x,R} \cong \mathcal{D}_x \otimes_{\mathrm{Spec}(\mathbf{C})} \mathrm{Spec}(R)$ .

The following proposition gives a first description of  $\mathrm{Gr}_{G,x}$  in terms of moduli of bundles on  $X$ .

**Proposition 7.3.** *1. The ind-scheme  $G_{\mathcal{K}_x}$  represents the functor*

$$R \mapsto \left\{ (\mathcal{F}, \nu, \mu) \left| \begin{array}{l} \mathcal{F} \text{ } G\text{-bundle on } X_R \\ \nu : G \times X_R^\times \xrightarrow{\sim} \mathcal{F}|_{X_R^\times} \text{ trivialization on } X_R^\times \\ \mu : G \times \mathcal{D}_{x,R} \xrightarrow{\sim} \mathcal{F}|_{\mathcal{D}_{x,R}} \text{ trivialization on } \mathcal{D}_{x,R} \end{array} \right. \right\} / \text{isomorphism}.$$

*2. The ind-scheme  $\mathrm{Gr}_{G,x}$  represents the functor*

$$R \mapsto \left\{ (\mathcal{F}, \nu) \left| \begin{array}{l} \mathcal{F} \text{ } G\text{-bundle on } X_R \\ \nu : G \times X_R^\times \xrightarrow{\sim} \mathcal{F}|_{X_R^\times} \text{ trivialization on } X_R^\times \end{array} \right. \right\} / \text{isomorphism}.$$

Here, a  $G$ -bundle on a scheme  $Z$  is a scheme  $\mathcal{F} \rightarrow Z$  equipped with a right  $G$ -action and which, locally in the fpqc topology, is isomorphic to the product  $G \times Z$  as a  $G$ -scheme. (In fact, since  $G$  is smooth here, a  $G$ -bundle will also be locally trivial in the étale topology; see [So, Remark 2.1.2] for more comments and references.) The proof of this proposition is given in [LS, Propositions 3.8 and 3.10]. The main ingredients are:

1. The Beauville–Laszlo theorem [BL], which says that the datum of a  $G$ -bundle on  $X_R$  is equivalent to the datum of a  $G$ -bundle on  $X_R^\times$ , of a  $G$ -bundle on  $\mathcal{D}_{x,R}$ , and of a gluing datum on  $\mathcal{D}_{x,R}^\times = \mathcal{D}_{x,R} \cap X_R^\times$ .
2. The fact that any  $G$ -bundle on  $\mathcal{D}_{x,R}$  becomes trivial when pulled back to  $\mathcal{D}_{x,R'}$  for some faithfully flat extension  $R \rightarrow R'$ .<sup>21</sup>

The Beauville–Laszlo theorem also shows that restriction induces an isomorphism

$$\left\{ (\mathcal{F}, \nu) \left| \begin{array}{l} \mathcal{F} \text{ } G\text{-bundle on } X_R \\ \nu \text{ trivialization on } X_R^\times \end{array} \right. \right\} / \text{isom.} \xrightarrow{\sim} \left\{ (\mathcal{F}, \nu) \left| \begin{array}{l} \mathcal{F} \text{ } G\text{-bundle on } \mathcal{D}_{x,R} \\ \nu \text{ trivialization on } \mathcal{D}_{x,R}^\times \end{array} \right. \right\} / \text{isom.}$$

In particular, we deduce that  $\text{Gr}_{G,x}$  also represents the functor

$$R \mapsto \left\{ (\mathcal{F}, \nu) \left| \begin{array}{l} \mathcal{F} \text{ } G\text{-bundle on } \mathcal{D}_{x,R} \\ \nu : G \times \mathcal{D}_{x,R}^\times \xrightarrow{\sim} \mathcal{F}|_{\mathcal{D}_{x,R}^\times} \text{ trivialization on } \mathcal{D}_{x,R}^\times \end{array} \right. \right\} / \text{isomorphism.} \quad (7.1)$$

*Remark 7.4.* The description of  $\text{Gr}_G$  (or in fact more precisely  $\widetilde{\text{Gr}}_G$ ) in terms of  $G$ -bundles on  $\text{Spec}(\mathbf{C}[[t]])$  as in (7.1) is in fact often taken as the definition of this ind-scheme, see e.g. [Z4, §1.2]. The identification with the quotient  $G_{\mathcal{K}}/G_{\mathcal{O}}$  is “purely local” and does not require the Beauville–Laszlo theorem.

## 7.2 Moduli interpretation of the convolution diagram

We now give a similar geometric interpretation of the diagram

$$\text{Gr}_{G,x} \times \text{Gr}_{G,x} \xleftarrow{p} G_{\mathcal{K}_x} \times \text{Gr}_{G,x} \xrightarrow{q} G_{\mathcal{K}_x} \times^{G_{\mathcal{O}_x}} \text{Gr}_{G,x} \xrightarrow{m} \text{Gr}_{G,x}, \quad (7.2)$$

which is the “local version at  $x$ ” of the diagram (6.2). We first remark that  $G_{\mathcal{K}_x} \times^{G_{\mathcal{O}_x}} \text{Gr}_{G,x}$  represents the functor

$$R \mapsto \left\{ (\mathcal{F}_1, \mathcal{F}, \nu_1, \eta) \left| \begin{array}{l} \mathcal{F}_1, \mathcal{F} \text{ } G\text{-bundles on } X_R \\ \nu_1 \text{ trivialization of } \mathcal{F}_1 \text{ on } X_R^\times \\ \eta : \mathcal{F}_1|_{X_R^\times} \xrightarrow{\sim} \mathcal{F}|_{X_R^\times} \text{ isomorphism} \end{array} \right. \right\} / \text{isom.}$$

To check this, one observes that the datum of  $(\mathcal{F}_1, \mathcal{F}, \nu_1, \eta)$  is equivalent to the datum of  $((\mathcal{F}_1, \nu_1), (\mathcal{F}, \eta \circ \nu_1))$ , and one notes that this transformation is completely similar to the isomorphism  $G_{\mathcal{K}} \times^{G_{\mathcal{O}}} \text{Gr}_G \xrightarrow{\sim} \text{Gr}_G \times \text{Gr}_G$  used in the proof of Lemma 6.4.

Likewise,  $G_{\mathcal{K}_x} \times \text{Gr}_{G,x}$  represents the functor

$$R \mapsto \left\{ (\mathcal{F}_1, \mathcal{F}_2, \nu_1, \nu_2, \mu_1) \left| \begin{array}{l} \mathcal{F}_1, \mathcal{F}_2 \text{ } G\text{-bundles on } X_R \\ \nu_1, \nu_2 \text{ trivializations of } \mathcal{F}_1, \mathcal{F}_2 \text{ on } X_R^\times \\ \mu_1 \text{ trivialization of } \mathcal{F}_1 \text{ on } \mathcal{D}_{x,R} \end{array} \right. \right\} / \text{isom.}$$

---

<sup>21</sup>See also [Z4, Lemma 1.3.7] for a slightly different statement in the same vein.

With these identifications, the maps  $m$  and  $p$  in diagram (7.2) are given by

$$\begin{aligned} m(\mathcal{F}_1, \mathcal{F}, \nu_1, \eta) &= (\mathcal{F}, \eta \circ \nu_1), \\ p(\mathcal{F}_1, \mathcal{F}_2, \nu_1, \nu_2, \mu_1) &= ((\mathcal{F}_1, \nu_1), (\mathcal{F}_2, \nu_2)), \end{aligned}$$

and the map  $q$  associates to  $(\mathcal{F}_1, \mathcal{F}_2, \nu_1, \nu_2, \mu_1)$  the quadruple  $(\mathcal{F}_1, \mathcal{F}, \nu_1, \eta)$ , where  $\mathcal{F}$  is obtained by gluing  $\mathcal{F}_1|_{X_R^\times}$  and  $\mathcal{F}_2|_{\mathcal{D}_{x,R}}$  along the isomorphism

$$\mathcal{F}_1|_{\mathcal{D}_{x,R}^\times} \xleftarrow[\mu_1]{\sim} G \times \mathcal{D}_{x,R}^\times \xrightarrow[\nu_2]{\sim} \mathcal{F}_2|_{\mathcal{D}_{x,R}^\times}$$

and  $\eta$  is the natural isomorphism obtained in the process. (This gluing datum indeed defines a  $G$ -bundle on  $X_R$  thanks to the Beauville–Laszlo theorem, see §7.1.)

### 7.3 The Beilinson–Drinfeld Grassmannian

The idea behind the fusion procedure is to regard the geometric situation described in §§7.1–7.2 as the degeneration of a simpler situation. This involves the Beilinson–Drinfeld Grassmannian.

Specifically, we define  $\mathrm{Gr}_{G,X}$  as the ind-scheme over  $X$  that represents the functor

$$R \mapsto \left\{ (\mathcal{F}, \nu, x) \left| \begin{array}{l} x \in X(R) \\ \mathcal{F} \text{ } G\text{-bundle on } X_R \\ \nu \text{ trivialization of } \mathcal{F} \text{ on } X_R \setminus x \end{array} \right. \right\} / \text{isom.},$$

where the symbol  $X_R \setminus x$  indicates the complement in  $X_R$  of the graph of  $x : \mathrm{Spec}(R) \rightarrow X$  (a closed subscheme of  $X_R = X \times \mathrm{Spec}(R)$ ).

In the same way, we define  $\mathrm{Gr}_{G,X^2}$  as the ind-scheme over  $X^2$  that represents the functor

$$R \mapsto \left\{ (\mathcal{F}, \nu, x_1, x_2) \left| \begin{array}{l} (x_1, x_2) \in X^2(R) \\ \mathcal{F} \text{ } G\text{-bundle on } X_R \\ \nu \text{ trivialization of } \mathcal{F} \text{ on } X_R \setminus (x_1 \cup x_2) \end{array} \right. \right\} / \text{isom.}$$

By definition there is an obvious morphism  $\mathrm{Gr}_{G,X^2} \rightarrow X^2$ . Plainly, the restriction of  $\mathrm{Gr}_{G,X^2}$  to the diagonal  $\Delta_X$  of  $X^2$ , namely  $\mathrm{Gr}_{G,X^2} \times_{X^2} \Delta_X$ , is isomorphic to  $\mathrm{Gr}_{G,X}$ . Away from the diagonal, we have an isomorphism

$$\mathrm{Gr}_{G,X^2}|_{X^2 \setminus \Delta_X} \cong (\mathrm{Gr}_{G,X} \times \mathrm{Gr}_{G,X})|_{X^2 \setminus \Delta_X} \quad (7.3)$$

given by  $(\mathcal{F}, \nu, x_1, x_2) \mapsto ((\mathcal{F}_1, \nu_1, x_1), (\mathcal{F}_2, \nu_2, x_2))$ , with  $\mathcal{F}_i$  obtained by gluing the trivial  $G$ -bundle on  $X_R \setminus x_i$  and the bundle  $\mathcal{F}|_{X_R \setminus x_j}$  along the map  $\nu$  (where  $\{i, j\} = \{1, 2\}$ ). Under the converse isomorphism  $((\mathcal{F}_1, \nu_1, x_1), (\mathcal{F}_2, \nu_2, x_2)) \mapsto (\mathcal{F}, \nu, x_1, x_2)$ , the  $G$ -bundle  $\mathcal{F}$  is obtained by gluing  $\mathcal{F}_1|_{X_R \setminus x_2}$  and  $\mathcal{F}_2|_{X_R \setminus x_1}$  along the isomorphism

$$\mathcal{F}_1|_{X_R \setminus (x_1 \cup x_2)} \xleftarrow[\nu_1]{\sim} G \times (X_R \setminus (x_1 \cup x_2)) \xrightarrow[\nu_2]{\sim} \mathcal{F}_2|_{X_R \setminus (x_1 \cup x_2)}.$$

- Remark 7.5.* 1. Of course one can define more generally Beilinson–Drinfeld Grassmannians over arbitrary powers of  $X$ , which satisfy appropriate analogues of the isomorphism (7.3). More formally this collection satisfies the “factorization” properties spelled out e.g. in [Z4, Theorem 3.2.1]; see also [BD, §§5.3.10–16] and [Rc, §3].
2. One can also consider Beilinson–Drinfeld Grassmannians associated with more general affine smooth group schemes over  $X$ ; see [Z1] for references and applications.

## 7.4 Global version of the convolution diagram

We can also define global analogues of  $G_{\mathcal{K}_x} \times \mathrm{Gr}_{G,x}$  and  $G_{\mathcal{K}_x} \times^{G_{\mathcal{O}_x}} \mathrm{Gr}_{G,x}$ . For that, we define  $\widetilde{\mathrm{Gr}}_{G,X} \times \mathrm{Gr}_{G,X}$  as the ind-scheme that represents the functor

$$R \mapsto \left\{ (\mathcal{F}_1, \nu_1, \mu_1, \mathcal{F}_2, \nu_2, x_1, x_2) \left| \begin{array}{l} (x_1, x_2) \in X^2(R) \\ \mathcal{F}_1, \mathcal{F}_2 \text{ } G\text{-bundles on } X_R \\ \nu_i \text{ trivialization of } \mathcal{F}_i \text{ on } X_R \setminus x_i \\ \mu_1 \text{ trivialization of } \mathcal{F}_1 \text{ on } \mathcal{D}_{x_2,R} \end{array} \right. \right\} / \text{isomorphism.}$$

(Here and below,  $\mathcal{D}_{x_2,R}$  means the formal neighborhood of the graph of  $x_2$  in  $X_R$ , considered as a scheme.) We also define  $\widetilde{\mathrm{Gr}}_{G,X} \times \widetilde{\mathrm{Gr}}_{G,X}$  as the ind-scheme that represents the functor

$$R \mapsto \left\{ (\mathcal{F}_1, \mathcal{F}, \nu_1, \eta, x_1, x_2) \left| \begin{array}{l} (x_1, x_2) \in X^2(R) \\ \mathcal{F}_1, \mathcal{F} \text{ } G\text{-bundles on } X_R \\ \nu_1 \text{ trivialization of } \mathcal{F}_1 \text{ on } X_R \setminus x_1 \\ \eta : \mathcal{F}_1|_{X_R \setminus x_2} \xrightarrow{\sim} \mathcal{F}|_{X_R \setminus x_2} \text{ isomorphism} \end{array} \right. \right\} / \text{isomorphism.}$$

We then get a diagram

$$\mathrm{Gr}_{G,X} \times \mathrm{Gr}_{G,X} \xleftarrow{p} \widetilde{\mathrm{Gr}}_{G,X} \times \mathrm{Gr}_{G,X} \xrightarrow{q} \widetilde{\mathrm{Gr}}_{G,X} \times \widetilde{\mathrm{Gr}}_{G,X} \xrightarrow{m} \mathrm{Gr}_{G,X^2} \quad (7.4)$$

over  $X^2$  by setting

$$\begin{aligned} m(\mathcal{F}_1, \mathcal{F}, \nu_1, \eta, x_1, x_2) &= (\mathcal{F}, \eta \circ \nu_1, x_1, x_2), \\ p(\mathcal{F}_1, \nu_1, \mu_1, \mathcal{F}_2, \nu_2, x_1, x_2) &= ((\mathcal{F}_1, \nu_1, x_1), (\mathcal{F}_2, \nu_2, x_2)), \end{aligned}$$

and by defining  $q$  as the map  $(\mathcal{F}_1, \nu_1, \mu_1, \mathcal{F}_2, \nu_2, x_1, x_2) \mapsto (\mathcal{F}_1, \mathcal{F}, \nu_1, \eta, x_1, x_2)$ , where  $\mathcal{F}$  is obtained by gluing  $\mathcal{F}_1|_{X_R \setminus x_2}$  and  $\mathcal{F}_2|_{\mathcal{D}_{x_2,R}}$  along the isomorphism

$$\mathcal{F}_1|_{\mathcal{D}_{x_2,R}^\times} \xleftarrow[\mu_1]{\sim} G \times \mathcal{D}_{x_2,R}^\times \xrightarrow[\nu_2]{\sim} \mathcal{F}_2|_{\mathcal{D}_{x_2,R}^\times}.$$

*Remark 7.6.* To justify the gluing procedure used here, one cannot simply quote the Beauville–Laszlo theorem, since the point  $x_2$  might not be constant. The more general result that we need is discussed in [BD, Remark 2.3.7 and §2.12].

We now explain that  $p$  and  $q$  are principal bundles for a group scheme over  $X^2$ . For that, we define  $G_{X,\mathcal{O}}$  as the group scheme over  $X$  that represents the functor

$$R \mapsto \left\{ (x, \mu) \left| \begin{array}{l} x \in X(R) \\ \mu \text{ trivialization of } G \times X_R \text{ on } \mathcal{D}_{x,R} \end{array} \right. \right\}.$$

In the description of the functor that  $\mathrm{Gr}_{G,X}$  represents, as in §7.1 one can replace  $(\mathcal{F}, \nu)$  by a pair  $(\mathcal{F}', \nu')$  where  $\mathcal{F}'$  is a  $G$ -bundle on  $\mathcal{D}_{x,R}$  and  $\nu'$  is a trivialization of  $\mathcal{F}'$  on  $\mathcal{D}_{x,R}^\times$ ; thus  $G_{X,\mathcal{O}}$  acts on  $\mathrm{Gr}_{G,X}$  by twisting the trivialization; specifically,  $\nu'$  gets replaced by  $\nu' \circ \mu^{-1}$ . (See also [Z4, §3.1] for more details about these groups schemes—over arbitrary powers of  $X$ —and their relation with the Beilinson–Drinfeld Grassmannians.)

We consider the second projection  $X^2 \rightarrow X$ , and the pullback  $G_{X,\mathcal{O}} \times_X X^2$  of the group scheme  $G_{X,\mathcal{O}}$ . The result acts on  $\mathrm{Gr}_{G,X} \times \mathrm{Gr}_{G,X}$  by twisting  $\mu_1$ , which defines  $p$  as a bundle.

In the definition of  $\mathrm{Gr}_{G,X} \times \mathrm{Gr}_{G,X}$ , as above one can replace  $(\mathcal{F}_2, \nu_2)$  by a pair  $(\mathcal{F}'_2, \nu'_2)$  where  $\mathcal{F}'_2$  is a  $G$ -bundle on  $\mathcal{D}_{x_2,R}$  and  $\nu'_2$  is a trivialization of  $\mathcal{F}'_2$  on  $\mathcal{D}_{x_2,R}^\times$ . The group scheme  $G_{X,\mathcal{O}} \times_X X^2$  then acts on  $\mathrm{Gr}_{G,X} \times \mathrm{Gr}_{G,X}$  by simultaneously twisting both  $\mu_1$  and  $\nu'_2$ . This action defines  $q$  as a principal bundle.

## 7.5 Convolution product and fusion

We go back to our convolution problem, starting this time with diagram (7.4). Since  $p$  and  $q$  are principal bundles, we can define a convolution product  $\star_X$  on  $\mathrm{P}_{G_{X,\mathcal{O}}}(\mathrm{Gr}_{G,X}, \mathbf{k})$  by setting

$$\mathcal{M} \star_X \mathcal{N} := m_*(\mathcal{M} \tilde{\boxtimes} \mathcal{N}),$$

where again  $\mathcal{M} \tilde{\boxtimes} \mathcal{N}$  is defined by the condition that

$$q^*(\mathcal{M} \tilde{\boxtimes} \mathcal{N}) = p^*(\mathcal{M} \boxtimes \mathcal{N}).$$

Here  $\mathcal{M}$  and  $\mathcal{N}$  are perverse sheaves on  $\mathrm{Gr}_{G,X}$ , and the result  $\mathcal{M} \star_X \mathcal{N}$  is in  $D_c^b(\mathrm{Gr}_{G,X^2}, \mathbf{k})$ .

*Remark 7.7.* 1. To define the category  $\mathrm{P}_{G_{X,\mathcal{O}}}(\mathrm{Gr}_{G,X}, \mathbf{k})$  we use a slight variant of the constructions of Appendix A, where algebraic groups are replaced by group schemes over  $X$ . This does not require any new ingredient: one simply replaces products by fiber products over  $X$  everywhere. The same remarks as in §A.4 are also in order here: we must consider perverse sheaves supported on a closed finite union of  $G_{X,\mathcal{O}}$ -orbits, and equivariant under some quotient  $(G/H_n)_{X,\mathcal{O}}$ . (A more sensible definition of a category of perverse sheaves on  $\mathrm{Gr}_{G,X}$  is due to Reich; see [Z4, §5.4]. These more sophisticated considerations will not be needed here.)

2. It will follow from Lemma 7.10 below that in fact  $\mathcal{M} \star_X \mathcal{N}$  is a perverse sheaf. This perverse sheaf is clearly  $G_{X,\mathcal{O}}$ -equivariant, so that this operation indeed defines a functor from  $\mathrm{P}_{G_{X,\mathcal{O}}}(\mathrm{Gr}_{G,X}, \mathbf{k}) \times \mathrm{P}_{G_{X,\mathcal{O}}}(\mathrm{Gr}_{G,X}, \mathbf{k})$  to  $\mathrm{P}_{G_{X,\mathcal{O}}}(\mathrm{Gr}_{G,X}, \mathbf{k})$ .

For the sake of simplicity,<sup>22</sup> from now on we restrict to the special case  $X = \mathbb{A}^1$ . We can then use a global coordinate on  $X$ , which yields a local coordinate at any point  $x \in X$ , and therefore allows to identify  $\mathrm{Gr}_{G,x}$  with the affine Grassmannian  $\mathrm{Gr}_G$  as we originally defined it. This also leads to an identification  $\mathrm{Gr}_{G,X} = \mathrm{Gr}_G \times X$ . We let  $\tau : \mathrm{Gr}_{G,X} \rightarrow \mathrm{Gr}_G$  be the projection and define  $\tau^\circ := \tau^*[1] \cong \tau^![-1]$ ; the shift is introduced so that  $\tau^\circ$  takes a perverse sheaf on  $\mathrm{Gr}_G$  to a perverse sheaf on  $\mathrm{Gr}_{G,X}$ .

<sup>22</sup>The general situation can be dealt with by putting the torsor of change of coordinates into the picture; see e.g. [Ga, §2.1.2] or [Z4, Discussion surrounding (3.1.10)] for details.



We explained in §7.3 that the restriction of  $\mathrm{Gr}_{G,X^2}$  to the diagonal  $\Delta_X$  in  $X^2$  is isomorphic to  $\mathrm{Gr}_{G,X}$ ; we may then denote by  $i : \mathrm{Gr}_{G,X} = \mathrm{Gr}_{G,X^2}|_{\Delta_X} \hookrightarrow \mathrm{Gr}_{G,X^2}$  the closed embedding, and consider the functors  $i^\circ := i^*[-1]$  and  $i^\bullet := i^![1]$ .

**Lemma 7.8.** *For  $\mathcal{F}_1$  and  $\mathcal{F}_2$  in  $\mathrm{P}_{G_{\mathcal{O}}}(\mathrm{Gr}_G, \mathbf{k})$ , we have canonical isomorphisms*

$$i^\circ(\tau^\circ(\mathcal{F}_1) \star_X \tau^\circ(\mathcal{F}_2)) \cong \tau^\circ(\mathcal{F}_1 \star \mathcal{F}_2) \cong i^\bullet(\tau^\circ(\mathcal{F}_1) \star_X \tau^\circ(\mathcal{F}_2)).$$

*Proof.* Since the map  $m$  in (7.4) is proper, and restricts over the diagonal  $\Delta_X$  to the product of the map denoted  $m$  in (6.2) by  $\mathrm{id}_{\Delta_X}$ , using the base change theorem it suffices to provide canonical isomorphisms

$$(i')^*(\tau^\circ \mathcal{F}_1 \tilde{\boxtimes} \tau^\circ \mathcal{F}_2) \cong (\tau')^\circ(\mathcal{F}_1 \tilde{\boxtimes} \mathcal{F}_2)[1], \quad (i')^!(\tau^\circ \mathcal{F}_1 \tilde{\boxtimes} \tau^\circ \mathcal{F}_2) \cong (\tau')^\circ(\mathcal{F}_1 \tilde{\boxtimes} \mathcal{F}_2)[-1]$$

where  $i' : (G_{\mathcal{K}} \times^{G_{\mathcal{O}}} \mathrm{Gr}_G) \times \Delta_X \rightarrow \mathrm{Gr}_{G,X} \tilde{\times} \mathrm{Gr}_{G,X}$  is the embedding,  $\tau' : (G_{\mathcal{K}} \times^{G_{\mathcal{O}}} \mathrm{Gr}_G) \times \Delta_X \rightarrow G_{\mathcal{K}} \times^{G_{\mathcal{O}}} \mathrm{Gr}_G$  is the projection, and  $(\tau')^\circ = (\tau')^*[-1] \cong (\tau')^![-1]$ . The first isomorphism is immediate from the definitions. The proof of the second one is similar, using Remark A.4.  $\square$

*Remark 7.9.* The isomorphism  $i^\circ(\tau^\circ(\mathcal{F}_1) \star_X \tau^\circ(\mathcal{F}_2)) \cong i^\bullet(\tau^\circ(\mathcal{F}_1) \star_X \tau^\circ(\mathcal{F}_2))$  observed in Lemma 7.8 can also be deduced from more general considerations related to universal local acyclicity; see [Z4, Theorem A.2.6 and proof of Proposition 5.4.2].

We now analyze the convolution diagram over  $U = X^2 \setminus \Delta_X$ :

$$\begin{array}{ccc} (\mathrm{Gr}_{G,X} \times \mathrm{Gr}_{G,X})|_U \xleftarrow{p} (\widetilde{\mathrm{Gr}_{G,X} \times \mathrm{Gr}_{G,X}})|_U \xrightarrow{q} (\mathrm{Gr}_{G,X} \tilde{\times} \mathrm{Gr}_{G,X})|_U & \xrightarrow{\sim} & \mathrm{Gr}_{G,X^2}|_U \\ & \searrow \pi \circ m & \downarrow \pi \\ & & (\mathrm{Gr}_{G,X} \times \mathrm{Gr}_{G,X})|_U. \end{array} \quad (7.5)$$

Here  $\pi$  is the isomorphism of (7.3), defined by

$$(\mathcal{F}, \nu, x_1, x_2) \mapsto ((\mathcal{F}_1, \nu_1, x_1), (\mathcal{F}_2, \nu_2, x_2)),$$

where  $\mathcal{F}_i$  is obtained by gluing the trivial bundle on  $X_R \setminus x_i$  and the bundle  $\mathcal{F}$  on  $\mathcal{D}_{x_i,R}$  using  $\nu$ . We note that there exists an isomorphism

$$(\widetilde{\mathrm{Gr}_{G,X} \times \mathrm{Gr}_{G,X}})|_U \xrightarrow{\sim} \left( (\mathrm{Gr}_{G,X} \times \mathrm{Gr}_{G,X}) \times_{X^2} (X \times G_{X,\mathcal{O}}) \right)|_U$$

defined by

$$(\mathcal{F}_1, \nu_1, \mu_1, \mathcal{F}_2, \nu_2, x_1, x_2) \mapsto \left( ((\mathcal{F}_1, \nu_1, x_1), (\mathcal{F}_2, \nu_2, x_2)), (x_1, (x_2, \mu_1^{-1} \circ \nu_1|_{\mathcal{D}_{x_2,R}})) \right).$$

Under this identification, the maps  $p$  and  $\pi \circ m \circ q$  identify with

$$(\mathrm{Gr}_{G,X} \times \mathrm{Gr}_{G,X})|_U \xleftarrow{p_1} \left[ (\mathrm{Gr}_{G,X} \times \mathrm{Gr}_{G,X}) \times_{X^2} (X \times G_{X,\mathcal{O}}) \right]|_U \xrightarrow{a} (\mathrm{Gr}_{G,X} \times \mathrm{Gr}_{G,X})|_U,$$

where  $p_1$  is the projection on the first factor and  $a$  is the action of  $G_{X,\mathcal{O}}$  on the second copy of  $\mathrm{Gr}_{G,X}$ .

It follows that if we identify the three spaces on the right-hand side of the convolution diagram (7.5), then for any  $\mathcal{M}_1, \mathcal{M}_2$  in  $\text{Perv}_{G_X, \mathcal{O}}(\text{Gr}_{G, X}, \mathbf{k})$ , the equivariant structure of  $\mathcal{M}_2$  leads to canonical identifications

$$\begin{array}{ccc} (\mathcal{M}_1 \tilde{\boxtimes} \mathcal{M}_2)|_U & \xlongequal{\quad} & (\mathcal{M}_1 \star_X \mathcal{M}_2)|_U \\ & \searrow & \parallel \\ & & (\mathcal{M}_1 \boxtimes \mathcal{M}_2)|_U. \end{array} \quad (7.6)$$

Consider now the open embedding  $j : (\text{Gr}_{G, X} \times \text{Gr}_{G, X})|_U \xrightarrow{(7.3)} \text{Gr}_{G, X^2}|_U \hookrightarrow \text{Gr}_{G, X^2}$ .

**Lemma 7.10.** *For any  $\mathcal{F}_1, \mathcal{F}_2 \in \text{P}_{G_{\mathcal{O}}}(\text{Gr}_G, \mathbf{k})$ , we have*

$$j!_*((\tau^\circ \mathcal{F}_1 \boxtimes \tau^\circ \mathcal{F}_2)|_U) \cong (\tau^\circ \mathcal{F}_1) \star_X (\tau^\circ \mathcal{F}_2).$$

*Proof.* We will use the characterization of the left-hand side given by [BBD, Corollaire 1.4.24]. In fact, in (7.6) we have already obtained the desired description of  $(\tau^\circ \mathcal{F}_1) \star_X (\tau^\circ \mathcal{F}_2)$  over  $U$ . Hence to conclude it suffices to prove that

$$i^*((\tau^\circ \mathcal{F}_1) \star_X (\tau^\circ \mathcal{F}_2)) \in {}^pD^{\leq -1} \quad \text{and} \quad i^!((\tau^\circ \mathcal{F}_1) \star_X (\tau^\circ \mathcal{F}_2)) \in {}^pD^{\geq 1}. \quad (7.7)$$

However, it follows from Lemma 7.8 that

$$i^*((\tau^\circ \mathcal{F}_1) \star_X (\tau^\circ \mathcal{F}_2)) \cong \tau^\circ(\mathcal{F}_1 \star \mathcal{F}_2)[1].$$

By Proposition 6.3 the right-hand side is concentrated in perverse degree  $-1$ , proving the first condition in (7.7). The second condition can be checked similarly, using the second isomorphism in Lemma 7.8.  $\square$

*Remark 7.11.* Once again, Lemma 7.10 can also be deduced from more general considerations related to universal local acyclicity; see [Z4, Theorem A.2.6 and proof of Proposition 5.4.2].

## 7.6 Construction of the commutativity constraint

Combining Lemma 7.8 and Lemma 7.10, we obtain a canonical isomorphism

$$\tau^\circ(\mathcal{F}_1 \star \mathcal{F}_2) \cong i^\circ j!_*((\tau^\circ \mathcal{F}_1 \boxtimes \tau^\circ \mathcal{F}_2)|_U), \quad (7.8)$$

valid for any  $\mathcal{F}_1, \mathcal{F}_2 \in \text{P}_{G_{\mathcal{O}}}(\text{Gr}_G, \mathbf{k})$ . In other words, the convolution product  $\mathcal{F}_1 \star \mathcal{F}_2$  can also be obtained by a procedure based on the Beilinson–Drinfeld Grassmannians  $\text{Gr}_{G, X}$  and  $\text{Gr}_{G, X^2}$ , called the *fusion product*.

Let  $\text{swap} : \text{Gr}_{G, X^2} \rightarrow \text{Gr}_{G, X^2}$  be the automorphism that swaps  $x_1$  and  $x_2$ . Then we have  $(\text{swap} \circ i) = i$ . Moreover,  $\text{swap}$  stabilizes  $\text{Gr}_{G, X^2}|_U$ , and under the identification (7.3) the

induced automorphism of  $(\mathrm{Gr}_{G,X} \times \mathrm{Gr}_{G,X})|_U$  (which we will denote  $\mathrm{swap}_U$ ) swaps the two factors  $\mathrm{Gr}_{G,X}$ . Therefore we obtain canonical isomorphisms

$$\begin{aligned} \tau^\circ(\mathcal{F}_1 \star \mathcal{F}_2) &\cong i^\circ j_{!*}((\tau^\circ \mathcal{F}_1 \boxtimes \tau^\circ \mathcal{F}_2)|_U) \\ &\cong i^\circ \mathrm{swap}^* j_{!*}((\tau^\circ \mathcal{F}_1 \boxtimes \tau^\circ \mathcal{F}_2)|_U) \\ &\cong i^\circ j_{!*}(\mathrm{swap}_U)^*((\tau^\circ \mathcal{F}_1 \boxtimes \tau^\circ \mathcal{F}_2)|_U) \\ &\cong i^\circ j_{!*}((\tau^\circ \mathcal{F}_2 \boxtimes \tau^\circ \mathcal{F}_1)|_U) \\ &= \tau^\circ(\mathcal{F}_2 \star \mathcal{F}_1). \end{aligned}$$

Restricting to a point of  $X$ , we deduce a canonical isomorphism

$$\mathcal{F}_1 \star \mathcal{F}_2 \xrightarrow{\sim} \mathcal{F}_2 \star \mathcal{F}_1,$$

which provides a commutativity constraint for the category  $\mathrm{P}_{G\mathcal{O}}(\mathrm{Gr}_G, \mathbf{k})$ .

*Remark 7.12.* 1. Later we will modify this commutativity constraint by a sign to make sure that the functor  $F$  sends it to the standard commutativity constraint on vector spaces; see §8.2.

2. One may note here that the twisted product  $\mathrm{Gr}_{G,X} \widetilde{\times} \mathrm{Gr}_{G,X}$ , while playing a key role in the proof, is not involved in the definition of the fusion product, since the maps  $i$  and  $j$  only deal with the Beilinson–Drinfeld Grassmannians  $\mathrm{Gr}_{G,X}$  and  $\mathrm{Gr}_{G,X^2}$ . The two points  $x_1$  and  $x_2$ , which are not interchangeable in the definition of  $\mathrm{Gr}_{G,X} \times \mathrm{Gr}_{G,X}$ , play the same role in  $\mathrm{Gr}_{G,X^2}$ . This property is the basis for the construction of the commutativity constraint.
3. One can describe the associativity constraint considered in §6.4 also in terms of the fusion procedure, using the Beilinson–Drinfeld Grassmannian  $\mathrm{Gr}_{G,X^3}$  over  $X^3$ .

## 8 Further study of the fiber functor

### 8.1 Compatibility of $F$ with the convolution product

In Sections 6–7 we have endowed our category  $\mathrm{P}_{G\mathcal{O}}(\mathrm{Gr}_G, \mathbf{k})$  with a convolution product  $\star$ , defined either in the “easy” way with the convolution diagram (6.2) or with the fusion procedure. The latter even allows to define a commutativity constraint. We now want to show that the functor

$$F = H^\bullet(\mathrm{Gr}_G, ?) : \mathrm{P}_{G\mathcal{O}}(\mathrm{Gr}_G, \mathbf{k}) \rightarrow \mathrm{Vect}_{\mathbf{k}}$$

is a fiber functor in the sense of Remark 2.8(2); in other words that this is an exact and faithful functor that maps the convolution product of sheaves to the tensor product of vector spaces while respecting the associativity, the unit, and the commutativity constraints of these categories.

The exactness and the faithfulness of  $F$  have already been proved in Theorem 5.9(2). The goal of this subsection is to prove the following.

**Proposition 8.1.** *For any  $\mathcal{A}_1, \mathcal{A}_2$  in  $\mathbf{P}_{G_{\mathcal{O}}}(\mathrm{Gr}_G, \mathbf{k})$ , there exists a canonical identification*

$$\mathbf{F}(\mathcal{A}_1 \star \mathcal{A}_2) = \mathbf{F}(\mathcal{A}_1) \otimes_{\mathbf{k}} \mathbf{F}(\mathcal{A}_2).$$

*Proof.* The proof will use the fusion procedure. Recall the setup of Section 7 (in the special case  $X = \mathbb{A}^1$ ), and in particular diagram (7.4).

Let  $\mathcal{A}_1, \mathcal{A}_2$  in  $\mathbf{P}_{G_{\mathcal{O}}}(\mathrm{Gr}_G, \mathbf{k})$ , and set  $\mathcal{B} := (\tau^\circ \mathcal{A}_1) \star_X (\tau^\circ \mathcal{A}_2)$ . Then if  $f : \mathrm{Gr}_{G, X^2} \rightarrow X^2$  is the natural map, Lemma 7.8 and (7.6) translate to the following properties: for each  $k \in \mathbf{Z}$ ,

- the  $k$ -th cohomology sheaf of the complex  $(f_* \mathcal{B})|_{\Delta_X}[-2]$  is locally constant on  $\Delta_X$ , with stalk  $\mathbf{H}^k(\mathrm{Gr}_G, \mathcal{A}_1 \star \mathcal{A}_2)$ ;
- the  $k$ -th cohomology sheaf of the complex  $(f_* \mathcal{B})|_U[-2]$  is locally constant on  $U$ , with stalk  $\mathbf{H}^k(\mathrm{Gr}_G \times \mathrm{Gr}_G, \mathcal{A}_1 \boxtimes \mathcal{A}_2)$ , which identifies with  $\bigoplus_{i+j=k} \mathbf{H}^i(\mathrm{Gr}_G, \mathcal{A}_1) \otimes \mathbf{H}^j(\mathrm{Gr}_G, \mathcal{A}_2)$  by the Künneth formula.

From there, we will be able to deduce the desired identification

$$\mathbf{H}^k(\mathrm{Gr}_G, \mathcal{A}_1 \star \mathcal{A}_2) \cong \bigoplus_{i+j=k} \mathbf{H}^i(\mathrm{Gr}_G, \mathcal{A}_1) \otimes \mathbf{H}^j(\mathrm{Gr}_G, \mathcal{A}_2)$$

as soon as we know that  $\mathcal{H}^{k-2}(f_* \mathcal{B})$  is locally constant on the whole space  $X^2$ . (Indeed, then this sheaf will be constant, so that we will be able to identify any of its fibers with its global sections canonically.)

We now prove this fact. Set  $\tilde{\mathcal{B}} := (\tau^\circ \mathcal{A}_1) \tilde{\boxtimes} (\tau^\circ \mathcal{A}_2)$ , so that  $\mathcal{B} = m_* \tilde{\mathcal{B}}$ . If we set  $\tilde{f} = f \circ m$ , we have  $f_* \mathcal{B} = \tilde{f}_* \tilde{\mathcal{B}}$ . For  $\lambda, \mu \in X_*(T)^+$ , set

$$\mathrm{Gr}_{G, X}^\lambda = \tau^{-1}(\mathrm{Gr}_G^\lambda), \quad \mathrm{Gr}_{G, X}^\mu = \tau^{-1}(\mathrm{Gr}_G^\mu),$$

and define  $\mathrm{Gr}_{G, X}^\lambda \tilde{\times} \mathrm{Gr}_{G, X}^\mu \subset \mathrm{Gr}_{G, X} \tilde{\times} \mathrm{Gr}_{G, X}$  by the requirement

$$q^{-1}(\mathrm{Gr}_{G, X}^\lambda \tilde{\times} \mathrm{Gr}_{G, X}^\mu) = p^{-1}(\mathrm{Gr}_{G, X}^\lambda \times \mathrm{Gr}_{G, X}^\mu).$$

(This definition makes sense, since  $\mathrm{Gr}_{G, X}^\mu$  is stable under the left action of  $G_{X, \mathcal{O}}$ ). Then

$$\tilde{\mathcal{F}} = \{\mathrm{Gr}_{G, X}^\lambda \tilde{\times} \mathrm{Gr}_{G, X}^\mu \mid \lambda, \mu \in X_*(T)^+\}$$

is a stratification of  $\mathrm{Gr}_{G, X} \tilde{\times} \mathrm{Gr}_{G, X}$ , and  $\tilde{\mathcal{B}}$  is  $\tilde{\mathcal{F}}$ -constructible. To show that the cohomology sheaves of  $\tilde{f}_* \tilde{\mathcal{B}}$  are locally constant, it suffices by dévissage<sup>23</sup> to check that for each  $k \in \mathbf{Z}$  and each stratum  $S \in \tilde{\mathcal{F}}$ , the sheaf  $\mathcal{H}^k \tilde{f}_* \tilde{\mathbf{k}}_S$  is locally constant.

<sup>23</sup>More precisely, one uses the following claim: the complexes  $\mathcal{M}$  such that the cohomology sheaves  $\mathcal{H}^k \tilde{f}_* \mathcal{M}$  are local systems form a full *triangulated* subcategory of  $D_c^b(\mathrm{Gr}_{G, X} \tilde{\times} \mathrm{Gr}_{G, X}, \mathbf{k})$ . To prove this claim, consider a distinguished triangle  $\mathcal{M}' \rightarrow \mathcal{M} \rightarrow \mathcal{M}'' \xrightarrow{[1]}$  with  $\mathcal{M}'$  and  $\mathcal{M}''$  in the subcategory. The long exact sequence in cohomology expresses  $\mathcal{H}^k \tilde{f}_* \mathcal{M}$  as an extension of  $\ker(\mathcal{H}^k \tilde{f}_* \mathcal{M}'' \rightarrow \mathcal{H}^{k+1} \tilde{f}_* \mathcal{M}')$  by  $\mathrm{coker}(\mathcal{H}^{k-1} \tilde{f}_* \mathcal{M}'' \rightarrow \mathcal{H}^k \tilde{f}_* \mathcal{M}')$ , hence as an extension of two local systems. Therefore  $\mathcal{H}^k \tilde{f}_* \mathcal{M}$  is a local system for each  $k$ , which means that  $\mathcal{M}$  belongs to our subcategory.

Let  $\widehat{\text{Gr}}_{G,X^2}$  be the ind-scheme representing the functor

$$R \mapsto \left\{ (\mathcal{F}_1, \nu_1, \mu_1, x_1, x_2) \left| \begin{array}{l} (x_1, x_2) \in X^2(R) \\ \mathcal{F}_1 \text{ } G\text{-bundle on } X_R \\ \nu_1 \text{ trivialization of } \mathcal{F}_1 \text{ on } X_R \setminus x_1 \\ \mu_1 \text{ trivialization of } \mathcal{F}_1 \text{ on } \mathcal{D}_{x_2,R} \end{array} \right. \right\} / \text{isomorphism.}$$

There is a natural map  $q' : \widehat{\text{Gr}}_{G,X^2} \rightarrow \text{Gr}_{G,X} \times X$ , that simply forgets  $\mu_1$ . The group scheme  $X \times G_{X,\mathcal{O}}$  acts on  $\widehat{\text{Gr}}_{G,X^2}$  by twisting  $\mu_1$ , and  $q'$  is a bundle for this action. (To justify this, we need to check that a trivialization  $\mu_1$  exists for any  $(\mathcal{F}_1, \nu_1, x_1)$  in  $\text{Gr}_{G,X}(R)$ , possibly after base change associated with a faithfully flat extension  $R \rightarrow R'$ . This fact is clear if  $x_2 \neq x_1$ , and follows from the results recalled in §7.1 when  $x_1 = x_2$ .)

On the other hand,  $\text{Gr}_{G,X}$  classifies data  $(\mathcal{F}_2, \nu_2, x_2)$  with  $\mathcal{F}_2$  a  $G$ -bundle on  $\mathcal{D}_{x_2}$  and  $\nu_2$  a trivialization on  $\mathcal{D}_{x_2}^\times$  (see §7.1), so the group scheme  $G_{X,\mathcal{O}}$  acts on  $\text{Gr}_{G,X}$  by twisting  $\nu_2$ . We claim that we have an identification

$$\text{Gr}_{G,X} \widetilde{\times} \text{Gr}_{G,X} = \widehat{\text{Gr}}_{G,X^2} \times^{(X \times G_{X,\mathcal{O}})} (X \times \text{Gr}_{G,X})$$

such that the map induced by  $q'$  identifies with the map  $\text{Gr}_{G,X} \widetilde{\times} \text{Gr}_{G,X} \rightarrow \text{Gr}_{G,X} \times X$  induced by the projection  $\text{Gr}_{G,X} \rightarrow X$  in the second factor. In fact, this identification sends an element  $(\mathcal{F}_1, \mathcal{F}, \nu_1, \eta, x_1, x_2)$  in  $(\text{Gr}_{G,X} \widetilde{\times} \text{Gr}_{G,X})(R)$  to the class of the pair

$$((\mathcal{F}_1, \nu_1, \mu_1, x_1, x_2), (\mathcal{F}_2, \nu_2, x_2))$$

where  $\mu_1$  is a choice of trivialization of  $\mathcal{F}_1$  on  $\mathcal{D}_{x_2,R}$ ,  $\mathcal{F}_2$  is the restriction of  $\mathcal{F}$  to  $\mathcal{D}_{x_2,R}$ , and  $\nu_1$  is the composition of the isomorphism  $\mathcal{F}_1|_{\mathcal{D}_{x_2,R}^\times} \xrightarrow{\sim} \mathcal{F}|_{\mathcal{D}_{x_2,R}^\times}$  induced by  $\eta$  with the trivialization of  $\mathcal{F}_1|_{\mathcal{D}_{x_2,R}^\times}$  induced by  $\mu_1$ . The inverse map sends the class of  $((\mathcal{F}_1, \nu_1, \mu_1, x_1, x_2), (\mathcal{F}_2, \nu_2, x_2))$  to  $(\mathcal{F}_1, \mathcal{F}, \nu_1, \eta, x_1, x_2)$ , where  $\mathcal{F}$  is the  $G$ -bundle obtained by gluing  $\mathcal{F}_1|_{X \setminus x_2}$  and  $\mathcal{F}_2$  using the gluing datum provided by the trivializations  $\mu_1$  and  $\nu_2$ .

These considerations show that the morphism  $\text{Gr}_{G,X} \widetilde{\times} \text{Gr}_{G,X} \rightarrow \text{Gr}_{G,X} \times X$  is a locally trivial fibration. Now, take  $\lambda, \mu \in X_*(T)^+$ , and set  $S = \text{Gr}_{G,X}^\lambda \widetilde{\times} \text{Gr}_{G,X}^\mu$ . The base change corresponding to the inclusion  $\text{Gr}_{G,X}^\lambda \hookrightarrow \text{Gr}_{G,X}$  and the fiber change corresponding to the inclusion  $\text{Gr}_{G,X}^\mu \hookrightarrow \text{Gr}_{G,X}$  show that the natural map

$$S = \text{Gr}_{G,X}^\lambda \widetilde{\times} \text{Gr}_{G,X}^\mu \rightarrow \text{Gr}_{G,X}^\lambda \times X$$

is a locally trivial fibration with fiber  $\text{Gr}_G^\mu$ . It follows that the cohomology sheaves of the pushforward of  $\underline{\mathbf{k}}_S$  along this map are locally constant on  $\text{Gr}_{G,X}^\lambda \times X$ . Further, the projection  $\text{Gr}_{G,X}^\lambda \rightarrow X$  is also a locally trivial fibration, and by a last dévissage argument, we conclude that  $f_* \underline{\mathbf{k}}_S$  has locally constant cohomology sheaves, as desired.  $\square$

*Remark 8.2.* See [Z4, §5.2] for a sketch of a different proof of Proposition 8.1, based on the use of equivariant cohomology. (This proof does not extend to more general coefficients, since [Z4, Theorem A.1.10] has no analogue for general coefficients.)

## 8.2 Compatibility with the commutativity constraint

It should be clear (see in particular Remark 7.12(3)) that the identification provided by Proposition 8.1 sends the associativity constraint on  $P_{G_{\mathcal{O}}}(\mathrm{Gr}_G, \mathbf{k})$  to the natural associativity constraint on  $\mathrm{Vect}_{\mathbf{k}}$ . The situation is slightly more subtle for the commutativity constraint.

By construction, the fiber functor  $F$  factors through a functor from  $P_{G_{\mathcal{O}}}(\mathrm{Gr}_G, \mathbf{k})$  to the category  $\mathrm{Vect}_{\mathbf{k}}(\mathbf{Z})$  of  $\mathbf{Z}$ -graded  $\mathbf{k}$ -vector spaces. We can endow the latter with the usual structure of tensor category or with the supersymmetric structure; the difference between the two structures is the definition of the commutativity constraint, which involves a sign in the super case (see in particular Example 2.9(3)).

Recall the notion of even and odd components of  $\mathrm{Gr}_G$  from §3.1. It follows in particular from Theorem 5.9(1) that if  $\mathcal{A}$  is supported on an even (resp. odd) component of  $\mathrm{Gr}_G$  then  $\mathbf{H}^{\bullet}(\mathrm{Gr}_G, \mathcal{A})$  is concentrated in even (resp. odd) degrees. Looking closely at the constructions in §8.1, one can check that the functor  $F$  maps the commutativity constraint on  $P_{G_{\mathcal{O}}}(\mathrm{Gr}_G, \mathbf{k})$  defined in §7.6 to the supersymmetric commutativity constraint on  $\mathrm{Vect}_{\mathbf{k}}(\mathbf{Z})$ . (This is related to the fact that the canonical isomorphism  $(\mathrm{swap}_U)^*((\tau^{\circ} \mathcal{F}_1 \boxtimes \tau^{\circ} \mathcal{F}_2)|_U) \cong (\tau^{\circ} \mathcal{F}_2 \boxtimes \tau^{\circ} \mathcal{F}_1)|_U$  involves some signs, since it requires to swap the order in a tensor product of complexes.)

However, to be in a position to apply the Tannakian reconstruction theorem from Section 2, we need to make sure that  $F$  maps the commutativity constraint on  $P_{G_{\mathcal{O}}}(\mathrm{Gr}_G, \mathbf{k})$  to the usual (unsigned) commutativity constraint on  $\mathrm{Vect}_{\mathbf{k}}(\mathbf{Z})$ . One solution consists in altering the commutativity constraint on  $P_{G_{\mathcal{O}}}(\mathrm{Gr}_G, \mathbf{k})$  by an appropriate sign. In fact, due to the change of parity introduced in the functor  $\tau^{\circ}$ , one must multiply the isomorphism of §7.6 by  $-1$  for the summands of the perverse sheaves  $\mathcal{F}_1$  and  $\mathcal{F}_2$  supported on even components of  $\mathrm{Gr}_G$ . This is the commutativity constraint that we will consider below.

## 8.3 Compatibility with the weight functors

We have noticed in §8.2 that the fiber functor  $F : P_{G_{\mathcal{O}}}(\mathrm{Gr}_G, \mathbf{k}) \rightarrow \mathrm{Vect}_{\mathbf{k}}$  in fact factors through the category  $\mathrm{Vect}_{\mathbf{k}}(\mathbf{Z})$  of  $\mathbf{Z}$ -graded  $\mathbf{k}$ -vector spaces. We can enhance this result using the weight functors of §5.3. In fact by Theorem 5.9(1) we have a commutative diagram

$$\begin{array}{ccc}
 & & \mathrm{Vect}_{\mathbf{k}}(X_*(T)) \\
 & \nearrow \bigoplus_{\mu} F_{\mu} & \downarrow \text{forget} \\
 P_{G_{\mathcal{O}}}(\mathrm{Gr}_G, \mathbf{k}) & & \mathrm{Vect}_{\mathbf{k}} \\
 & \searrow F & 
 \end{array}$$

where  $\mathrm{Vect}_{\mathbf{k}}(X_*(T))$  is the category of  $X_*(T)$ -graded  $\mathbf{k}$ -vector spaces. Recall from Example 2.9(1) that the category  $\mathrm{Vect}_{\mathbf{k}}(X_*(T))$  admits a natural tensor product, with commutativity and associativity constraints.

**Proposition 8.3.** *The functor  $\bigoplus_{\mu} F_{\mu}$  sends the convolution product  $\star$  to the tensor product of  $X_*(T)$ -graded  $\mathbf{k}$ -vector spaces, in a way compatible with the associativity and commutativity constraints.*

*Proof.* We need to provide an identification

$$F_\mu(\mathcal{A}_1 \star \mathcal{A}_2) = \bigoplus_{\mu_1 + \mu_2 = \mu} F_{\mu_1}(\mathcal{A}_1) \otimes_{\mathbf{k}} F_{\mu_2}(\mathcal{A}_2) \quad (8.1)$$

for each  $\mu \in X_*(T)$  and all  $\mathcal{A}_1, \mathcal{A}_2 \in P_{G_{\mathcal{O}}}(\mathrm{Gr}_G, \mathbf{k})$ .

Recall how the weight functors  $F_\mu$  are defined (see Remark 5.10). We have chosen a maximal torus and a Borel subgroup  $T \subset B \subset G$ . Then  $T \subset G_{\mathcal{K}}$  acts on  $\mathrm{Gr}_G = G_{\mathcal{K}}/G_{\mathcal{O}}$  with fixed points

$$(\mathrm{Gr}_G)^T = \{L_\mu : \mu \in X_*(T)\}.$$

We picked a dominant regular cocharacter  $\eta \in X_*(T)$ , which provides a one-parameter subgroup  $\mathbb{G}_m \subset T$  and a  $\mathbf{C}^\times$ -action on  $\mathrm{Gr}_G$  with fixed points  $(\mathrm{Gr}_G)^T$ . For  $\mu \in X_*(T)$ , the attractive variety relative to the fixed point  $L_\mu$  is

$$S_\mu = \{x \in \mathrm{Gr}_G \mid \eta(a) \cdot x \rightarrow L_\mu \text{ when } a \rightarrow 0\}$$

(see the proof of Theorem 5.2), and for each  $\mathcal{A} \in P_{G_{\mathcal{O}}}(\mathrm{Gr}_G, \mathbf{k})$ ,

$$H_c^k(S_\mu, \mathcal{A}) \neq 0 \implies k = \langle 2\rho, \mu \rangle.$$

For  $\mu \in X_*(T)$  and  $\mathcal{A} \in P_{G_{\mathcal{O}}}(\mathrm{Gr}_G, \mathbf{k})$ , we have  $F_\mu(\mathcal{A}) := H_c^{\langle 2\rho, \mu \rangle}(\overline{S}_\mu, \mathcal{A})$ . We get adjunction maps (see Remark 5.10)

$$\begin{array}{ccc} & H_c^{\langle 2\rho, \mu \rangle}(S_\mu, \mathcal{A}) & \\ & \downarrow \wr & \\ H_c^{\langle 2\rho, \mu \rangle}(\mathrm{Gr}_G, \mathcal{A}) & \longrightarrow & H_c^{\langle 2\rho, \mu \rangle}(\overline{S}_\mu, \mathcal{A}); \end{array}$$

moreover for each  $k \in \mathbf{Z}$ , there is a decomposition

$$H^k(\mathrm{Gr}_G, \mathcal{A}) = \bigoplus_{\substack{\mu \in X_*(T) \\ \langle 2\rho, \mu \rangle = k}} F_\mu(\mathcal{A}).$$

We need to insert this construction in the reasoning in §8.1. The various spaces considered in §7.4 carry an action of  $T$ . Specifically, this action twists  $\nu$  in  $\mathrm{Gr}_{G,X}$  and  $\mathrm{Gr}_{G,X^2}$ , and twists  $\nu_1$  on  $\widetilde{\mathrm{Gr}}_{G,X} \times \mathrm{Gr}_{G,X}$  and  $\mathrm{Gr}_{G,X} \widetilde{\times} \mathrm{Gr}_{G,X}$ . The maps  $q$  and  $m$  in diagram (7.4) and the isomorphism  $\pi$  in diagram (7.5) are  $T$ -equivariant.

To each pair  $(\mu_1, \mu_2) \in X_*(T)^2$  corresponds a connected component  $\widetilde{C}_{(\mu_1, \mu_2)}$  of the set of  $T$ -fixed points in  $\mathrm{Gr}_{G,X} \widetilde{\times} \mathrm{Gr}_{G,X}$ , namely

$$\widetilde{C}_{(\mu_1, \mu_2)} = \{([t^{\mu_1}, L_{\mu_2}], x, x) \mid x \in X\} \cup \{(L_{\mu_1}, L_{\mu_2}, x_1, x_2) \mid (x_1, x_2) \in U\}$$

where  $[t^{\mu_1}, L_{\mu_2}]$  is seen as a point in  $G_{\mathcal{K}_x} \times^{G_{\mathcal{O}_x}} \mathrm{Gr}_{G,x}$ , identified with the fiber of the twisted product  $\mathrm{Gr}_{G,X} \widetilde{\times} \mathrm{Gr}_{G,X}$  over a point  $(x, x) \in \Delta_X$ , and  $(L_{\mu_1}, L_{\mu_2})$  is likewise seen as a point in  $\mathrm{Gr}_{G,x_1} \times \mathrm{Gr}_{G,x_2}$ , identified with the fiber of  $\mathrm{Gr}_{G,X} \widetilde{\times} \mathrm{Gr}_{G,X}$  over  $(x_1, x_2) \in U$  thanks to the

map  $\pi \circ m \circ q$  in (7.5). Moreover the projection  $\tilde{C}_{\mu_1, \mu_2} \rightarrow X^2$  is an isomorphism. (Recall that we have chosen  $X = \mathbb{A}^1$ , so that we have a canonical identification  $\mathrm{Gr}_{G,x} \cong \mathrm{Gr}_G$  for any  $x$ .)

The map  $m : \mathrm{Gr}_{G,X} \tilde{\times} \mathrm{Gr}_{G,X} \rightarrow \mathrm{Gr}_{G,X^2}$  glues together along the diagonal  $\Delta_X$  the various connected components  $\tilde{C}_{(\mu_1, \mu_2)}$  for which  $\mu_1 + \mu_2$  is the same. Therefore, to each  $\mu \in X_*(T)$  corresponds a connected component

$$C_\mu := \bigsqcup_{\mu_1 + \mu_2 = \mu} m(\tilde{C}_{(\mu_1, \mu_2)})$$

of the set of  $T$ -fixed points in  $\mathrm{Gr}_{G,X^2}$ .

Our dominant regular cocharacter  $\eta \in X_*(T)$  defines a  $\mathbf{C}^\times$ -action on  $\mathrm{Gr}_{G,X^2}$ . Denote the attractive variety<sup>24</sup> around  $C_\mu$  by  $S_\mu(X^2)$ ; one can check<sup>25</sup> that  $S_\mu(X^2)$  is a locally closed subscheme of  $\mathrm{Gr}_{G,X^2}$ . Over a point  $(x, x) \in \Delta_X$ , the fiber of the map  $S_\mu(X^2) \rightarrow X^2$  is the semi-infinite orbit  $S_\mu$ , viewed as a subvariety of  $\mathrm{Gr}_{G,x}$  thanks to the isomorphism  $\mathrm{Gr}_{G,X^2}|_{\Delta_X} = \mathrm{Gr}_{G,X}$ ; over a point  $(x_1, x_2) \in U$ , the fiber of  $S_\mu(X^2) \rightarrow X^2$  is the union

$$\bigsqcup_{\mu_1 + \mu_2 = \mu} S_{\mu_1} \times S_{\mu_2} \subset \mathrm{Gr}_{G,x_1} \times \mathrm{Gr}_{G,x_2},$$

where we use the isomorphism  $\pi : \mathrm{Gr}_{G,X^2}|_U \xrightarrow{\sim} (\mathrm{Gr}_{G,X} \times \mathrm{Gr}_{G,X})|_U$  of §7.5 (see [DrG, Lemma 1.4.9]).

Consider again  $\mathcal{B} := (\tau^\circ \mathcal{A}_1) \star_X (\tau^\circ \mathcal{A}_2)$  and consider the natural maps depicted in the following diagram:

$$\begin{array}{ccc} S_\mu(X^2) & \xrightarrow{\tilde{s}'_\mu} \overline{S_\mu(X^2)} & \xrightarrow{\tilde{s}''_\mu} \mathrm{Gr}_{G,X^2} \\ & \searrow \tilde{s}_\mu & \downarrow f \\ & & X^2. \end{array}$$

The stalks of the complex of sheaves  $(f\tilde{s}_\mu)_!(\tilde{s}_\mu)^* \mathcal{B}$  can be computed by base change. Using Lemma 7.8 and (7.6), and taking into account the shift in the definition of  $\tau^\circ$ , we obtain:

- The sheaf  $\mathcal{H}^{k-2}(f\tilde{s}_\mu)_!(\tilde{s}_\mu)^* \mathcal{B}$  is locally constant on  $\Delta_X$ , with stalk  $\mathrm{H}_c^k(S_\mu, \mathcal{A}_1 \star \mathcal{A}_2)$ , so is  $\mathrm{F}_\mu(\mathcal{A}_1 \star \mathcal{A}_2)$  if  $k = \langle 2\rho, \mu \rangle$  and is zero otherwise.
- The sheaf  $\mathcal{H}^{k-2}(f\tilde{s}_\mu)_!(\tilde{s}_\mu)^* \mathcal{B}$  is locally constant on  $U$ , with stalk isomorphic to

$$\bigoplus_{\mu_1 + \mu_2 = \mu} \mathrm{H}_c^k(S_{\mu_1} \times S_{\mu_2}, \mathcal{A}_1 \boxtimes \mathcal{A}_2),$$

so is isomorphic to

$$\bigoplus_{\mu_1 + \mu_2 = \mu} \mathrm{F}_{\mu_1}(\mathcal{A}_1) \otimes \mathrm{F}_{\mu_2}(\mathcal{A}_2)$$

if  $k = \langle 2\rho, \mu \rangle$  and is zero otherwise.

<sup>24</sup>See [DrG, Definition 1.4.2 and Corollary 1.5.3] for the general construction of the attractor for a  $\mathbf{C}^\times$ -action on a scheme.

<sup>25</sup>This fact is not automatic (i.e. it does not follow from the general result [DrG, Theorem 1.6.8]) because the finite-dimensional pieces of  $\mathrm{Gr}_{G,X^2}$  might not be normal. One way to prove this is to first check that  $\bigsqcup_{\nu \leq \mu} S_\nu(X^2)$  is closed in  $\mathrm{Gr}_{G,X^2}$ ; then  $S_\mu(X^2)$  is the complement of  $\bigsqcup_{\nu < \mu} S_\nu(X^2)$  in  $\bigsqcup_{\nu \leq \mu} S_\nu(X^2)$ .



In particular,

$$\mathcal{H}^{k-2}(f\tilde{s}_\mu)_!(\tilde{s}_\mu)^*\mathcal{B} \neq 0 \implies k = \langle 2\rho, \mu \rangle.$$

Given  $\mu \in X_*(T)$ , denote the sheaf  $\mathcal{H}^{\langle 2\rho, \mu \rangle - 2}(f\tilde{s}_\mu)_!(\tilde{s}_\mu)^*\mathcal{B}$  by  $\mathcal{L}_\mu(\mathcal{B})$ . Adjunction yields maps

$$\begin{array}{ccc} & \mathcal{H}^{\langle 2\rho, \mu \rangle - 2}(f\tilde{s}_\mu)_!(\tilde{s}_\mu)^*\mathcal{B} & \\ & \downarrow & \\ \mathcal{H}^{\langle 2\rho, \mu \rangle - 2}f_*\mathcal{B} & \longrightarrow & \mathcal{L}_\mu(\mathcal{B}). \end{array}$$

Since this is true over any point of  $X^2$ , the vertical arrow is an isomorphism and the horizontal arrow is an epimorphism; moreover for each  $k \in \mathbf{Z}$  the projections provide an isomorphism

$$\mathcal{H}^{k-2}f_*\mathcal{B} \cong \bigoplus_{\substack{\mu \in X_*(T) \\ \langle 2\rho, \mu \rangle = k}} \mathcal{L}_\mu(\mathcal{B}).$$

We see here that  $\mathcal{L}_\mu(\mathcal{B})$  is a direct summand of the local system  $\mathcal{H}^{k-2}f_*\mathcal{B}$  (see §8.1), so it is a local system itself. As we saw, its stalk over a point in  $\Delta_X$  is  $F_\mu(\mathcal{A}_1 \star \mathcal{A}_2)$  and its stalk over a point in  $U$  is  $\bigoplus_{\mu_1 + \mu_2 = \mu} F_{\mu_1}(\mathcal{A}_1) \otimes F_{\mu_2}(\mathcal{A}_2)$ . We thus obtain the desired identification (8.1), as in the proof of Proposition 8.1.  $\square$

*Remark 8.4.* 1. See Proposition 15.2 below for a different proof of the compatibility of  $\bigoplus_\mu F_\mu$  with convolution, in a more general context.

2. Once again, Proposition 8.3 can be proved in a more elementary way using equivariant cohomology, see [Z4, Proposition 5.3.14] (but this proof is specific to the characteristic-0 setting).

## 9 Identification of the dual group

At this point, we have constructed the convolution product  $\star$  on  $P_{G_{\mathcal{O}}}(\mathrm{Gr}_G, \mathbf{k})$ , a  $\mathbf{k}$ -linear faithful exact functor  $F : P_{G_{\mathcal{O}}}(\mathrm{Gr}_G, \mathbf{k}) \rightarrow \mathrm{Vect}_{\mathbf{k}}$ , an associativity constraint, a commutativity constraint, and a unit object  $U = \mathbf{IC}_0$  such that:

1.  $F \circ \star = \otimes \circ (F \otimes F)$  and  $F(U) = \mathbf{k}$ ;
2.  $F$  maps the associativity constraint, the commutativity constraint and the unit constraints of  $P_{G_{\mathcal{O}}}(\mathrm{Gr}_G, \mathbf{k})$  to the corresponding constraints of  $\mathrm{Vect}_{\mathbf{k}}$ ;
3. If  $F(L)$  has dimension 1, then there exists  $L^{-1}$  such that  $L \star L^{-1} \cong U$ .

(For (3), one observes that for  $L = \mathbf{IC}_\lambda$  to satisfy  $\dim F(L) = 1$ , by Proposition 5.13  $\lambda$  must be orthogonal to each root  $\alpha \in \Delta(G, T)$ , so  $\mathrm{Gr}_G^\lambda = \{L_\lambda\}$ , and we can take  $L^{-1} = \mathbf{IC}_{-\lambda}$  since  $-\lambda$  is dominant.)

Tannakian reconstruction (see Theorem 2.7) then gives us an affine group scheme  $\tilde{G}_{\mathbf{k}}$  over  $\mathbf{k}$  and an equivalence  $\mathbb{S}$  which fits in the following commutative diagram:

$$\begin{array}{ccc} P_{G_{\mathcal{O}}}(\mathrm{Gr}_G, \mathbf{k}) & \xrightarrow[\mathbb{S}]{\sim} & \mathrm{Rep}_{\mathbf{k}}(\tilde{G}_{\mathbf{k}}) \\ & \searrow \mathrm{F} & \swarrow \omega \\ & & \mathrm{Vect}_{\mathbf{k}}, \end{array}$$

where  $\omega$  is the forgetful functor. We now need to identify  $\tilde{G}_{\mathbf{k}}$ .

*Remark 9.1.* The group  $\tilde{G}_{\mathbf{k}}$  considered here should not be confused with the group  $\tilde{G}$  of §3.3.

## 9.1 First step: $\tilde{G}_{\mathbf{k}}$ is a split connected reductive algebraic group over $\mathbf{k}$

**Lemma 9.2.** *The affine group scheme  $\tilde{G}_{\mathbf{k}}$  is algebraic.*

*Proof.* Choose a finite set of generators  $\lambda_1, \dots, \lambda_n$  of the monoid  $X_*(T)^+$  of dominant cocharacters. Then for any nonnegative integral linear combination  $\lambda = k_1\lambda_1 + \dots + k_n\lambda_n$ , the sheaf  $\mathbf{IC}_{\lambda}$  appears as a direct summand of the convolution product

$$\underbrace{\mathbf{IC}_{\lambda_1} \star \dots \star \mathbf{IC}_{\lambda_1}}_{k_1 \text{ copies}} \star \dots \star \underbrace{\mathbf{IC}_{\lambda_n} \star \dots \star \mathbf{IC}_{\lambda_n}}_{k_n \text{ copies}}.$$

(In fact, this convolution product is a semisimple perverse sheaf by Proposition 4.2. Moreover it is easily seen to be supported on  $\overline{\mathrm{Gr}_G^{\lambda}}$ , with restriction to  $\mathrm{Gr}_G^{\lambda}$  isomorphic to  $\mathbf{k}_{\mathrm{Gr}_G^{\lambda}}[\dim(\mathrm{Gr}_G^{\lambda})]$ . Hence it must admit  $\mathbf{IC}_{\lambda}$  as a direct summand.) Therefore  $\mathcal{X} := \mathbf{IC}_{\lambda_1} \oplus \dots \oplus \mathbf{IC}_{\lambda_n}$  is a tensor generator of the category  $P_{G_{\mathcal{O}}}(\mathrm{Gr}_G, \mathbf{k})$ ; namely, any object of  $P_{G_{\mathcal{O}}}(\mathrm{Gr}_G, \mathbf{k})$  appears as a subquotient of a direct sum of tensor powers of  $\mathcal{X}$ . Thus  $\mathrm{Rep}_{\mathbf{k}}(\tilde{G}_{\mathbf{k}})$  admits a tensor generator, which implies that  $\tilde{G}_{\mathbf{k}}$  is algebraic by Proposition 2.11(1).  $\square$

**Lemma 9.3.** *The affine algebraic group scheme  $\tilde{G}_{\mathbf{k}}$  is connected.*

*Proof.* If  $\lambda$  is a nonzero dominant cocharacter of  $T$ , then the objects  $\mathbf{IC}_{m\lambda}$  are pairwise non isomorphic for  $m \in \mathbf{Z}_{\geq 0}$  (since they have different supports). It follows that for any nontrivial object  $\mathcal{X}$  in  $P_{G_{\mathcal{O}}}(\mathrm{Gr}_G, \mathbf{k})$ , the full subcategory formed by subquotients of direct sums  $\mathcal{X}^{\oplus n}$  cannot be stable under  $\star$ . The same property then also holds for the tensor category  $\mathrm{Rep}_{\mathbf{k}}(\tilde{G}_{\mathbf{k}})$ . This in turn implies that  $\tilde{G}_{\mathbf{k}}$  is connected by Proposition 2.11(2).  $\square$

**Lemma 9.4.** *The connected affine algebraic group scheme  $\tilde{G}_{\mathbf{k}}$  is reductive.*

*Proof.* If  $\bar{\mathbf{k}}$  is an algebraic closure of  $\mathbf{k}$ , it is clear that  $\tilde{G}_{\bar{\mathbf{k}}} := \mathrm{Spec}(\bar{\mathbf{k}}) \times_{\mathrm{Spec}(\mathbf{k})} \tilde{G}_{\mathbf{k}}$  is the group scheme provided by Tannakian formalism out of the category  $P_{G_{\mathcal{O}}}(\mathrm{Gr}_G, \bar{\mathbf{k}})$ . This category is semisimple by Theorem 4.2. We conclude using Proposition 2.11(3).  $\square$

We now explain the construction of a split maximal torus in  $\tilde{G}_{\mathbf{k}}$  (see §5.1).

As in §8.3, we denote by  $\text{Vect}_{\mathbf{k}}(X_*(T))$  the category of finite dimensional  $X_*(T)$ -graded  $\mathbf{k}$ -vector spaces. This is a monoidal category, and the weight functors provide us with a factorization of  $\mathbf{F}$  as

$$P_{G_{\mathcal{O}}}(\text{Gr}_G, \mathbf{k}) \xrightarrow{F'} \text{Vect}_{\mathbf{k}}(X_*(T)) \xrightarrow{\text{forget}} \text{Vect}_{\mathbf{k}},$$

see §8.3. Let  $T_{\mathbf{k}}^{\vee}$  be the unique split  $\mathbf{k}$ -torus such that  $X^*(T_{\mathbf{k}}^{\vee}) = X_*(T)$ ; then  $\text{Vect}_{\mathbf{k}}(X_*(T)) \cong \text{Rep}_{\mathbf{k}}(T_{\mathbf{k}}^{\vee})$  canonically (see e.g. [Ja, §I.2.11]), and  $F'$  induces a functor  $F_{T_{\mathbf{k}}^{\vee}} : \text{Rep}_{\mathbf{k}}(\tilde{G}_{\mathbf{k}}) \rightarrow \text{Rep}_{\mathbf{k}}(T_{\mathbf{k}}^{\vee})$  compatible with the monoidal structures. There is then a commutative diagram

$$\begin{array}{ccc} P_{G_{\mathcal{O}}}(\text{Gr}_G, \mathbf{k}) & \xrightarrow{F'} & \text{Vect}_{\mathbf{k}}(X_*(T)) , \\ \downarrow \wr & & \parallel \\ \text{Rep}_{\mathbf{k}}(\tilde{G}_{\mathbf{k}}) & \xrightarrow{F_{T_{\mathbf{k}}^{\vee}}} & \text{Rep}_{\mathbf{k}}(T_{\mathbf{k}}^{\vee}) \end{array}$$

and the functor  $F_{T_{\mathbf{k}}^{\vee}}$  commutes with the forgetful functors to  $\text{Vect}_{\mathbf{k}}$  and satisfies the conditions of Proposition 2.10. Hence this functor is induced by a unique morphism  $\varphi : T_{\mathbf{k}}^{\vee} \rightarrow \tilde{G}_{\mathbf{k}}$  of algebraic groups.

Each character  $\lambda \in X^*(T_{\mathbf{k}}^{\vee})$  appears in at least one  $F_{T_{\mathbf{k}}^{\vee}}(\mathbf{1C}_{\mu})$ . (One can here e.g. choose  $\mu$  as the dominant  $W$ -conjugate of  $\lambda$  and use Theorem 5.2 and Proposition 5.13.) It follows that  $\varphi$  is an embedding of a closed subgroup, see [DM, Proposition 2.21(b)]; so  $T_{\mathbf{k}}^{\vee}$  can be considered as a split torus in  $\tilde{G}_{\mathbf{k}}$ .

Now, for any reductive group  $H$  over a field  $\mathbf{F}$ , if  $\overline{\mathbf{F}}$  is an algebraic closure of  $\mathbf{F}$  and if we set  $H_{\overline{\mathbf{F}}} := \text{Spec}(\overline{\mathbf{F}}) \times_{\text{Spec}(\mathbf{F})} H$ , then

$$\text{rk}(H) = \dim \text{Spec}(\mathbf{Q} \otimes_{\mathbf{Z}} K^0(\text{Rep}_{\overline{\mathbf{F}}}(H_{\overline{\mathbf{F}}})))). \quad (9.1)$$

In fact, the right-hand side admits a basis consisting of classes of induced modules (i.e. the modules denoted  $H^0(\lambda)$  in [Ja, Chap. II.2]), whose characters are given by the Weyl character formula, see e.g. [Ja, Corollary II.5.11]. Therefore it identifies with  $K_{\mathbf{Q}}/W_H$ , where  $K_{\mathbf{Q}}$  is the split  $\mathbf{Q}$ -torus with character lattice the character lattice of any chosen maximal torus  $K_{\overline{\mathbf{F}}}$  in  $H_{\overline{\mathbf{F}}}$  and  $W_H$  is the Weyl group of  $H$  with respect to this torus. There is a finite morphism  $K_{\mathbf{Q}} \rightarrow K_{\mathbf{Q}}/W_H$ , so that this scheme has the same dimension as  $K_{\mathbf{Q}}$ , i.e. has dimension the rank of  $H_{\overline{\mathbf{F}}}$ , which by definition is the rank of  $H$ .

In our case, the functor  $F_{T_{\mathbf{k}}^{\vee}}$  provides a morphism of schemes

$$T_{\mathbf{Q}}^{\vee} \rightarrow \text{Spec}(\mathbf{Q} \otimes_{\mathbf{Z}} K^0(\text{Rep}_{\overline{\mathbf{k}}}(\tilde{G}_{\overline{\mathbf{k}}})))),$$

where  $\overline{\mathbf{k}}$  and  $\tilde{G}_{\overline{\mathbf{k}}}$  are as in the proof of Lemma 9.4, and  $T_{\mathbf{Q}}^{\vee}$  is the  $\mathbf{Q}$ -torus with characters  $X_*(T)$ . In view of the description of the simple objects in  $P_{G_{\mathcal{O}}}(\text{Gr}_G, \mathbf{k})$  (see Section 4), this morphism identifies the right-hand side with  $T_{\mathbf{Q}}^{\vee}/W$ . We deduce that the rank of  $\tilde{G}_{\mathbf{k}}$  is the dimension of  $T_{\mathbf{k}}^{\vee}$ , i.e. that  $T_{\mathbf{k}}^{\vee}$  is a maximal torus of  $\tilde{G}_{\mathbf{k}}$ .

## 9.2 Second step: identification of the root datum of $(\tilde{G}_{\mathbf{k}}, T_{\mathbf{k}}^{\vee})$

In view of the general results recalled in §5.1, to finish our determination of the group scheme  $\tilde{G}_{\mathbf{k}}$ , it only remains to identify the root datum of  $(\tilde{G}_{\mathbf{k}}, T_{\mathbf{k}}^{\vee})$ . By the remarks in the proof of Lemma 9.4 and the definitions recalled above, for this we can (and shall) assume that  $\mathbf{k}$  is algebraically closed.

We first determine a “canonical” Borel subgroup in  $\tilde{G}_{\mathbf{k}}$ . Consider the sum  $2\rho \in X^*(T)$  of the positive roots of  $G$ . Then there exists a (possibly non unique) Borel subgroup  $\tilde{B} \subset \tilde{G}_{\mathbf{k}}$  that contains  $T_{\mathbf{k}}^{\vee}$  and such that  $2\rho$  is a dominant coweight for the choice of positive roots of  $\tilde{G}_{\mathbf{k}}$  given by the  $T_{\mathbf{k}}^{\vee}$ -weights in the Lie algebra of  $\tilde{B}$ .

**Lemma 9.5.** *For such a choice of Borel subgroup  $\tilde{B}$ , hence of positive roots, the dominant weights for  $T_{\mathbf{k}}^{\vee}$  are exactly the dominant coweights  $X_*(T)^+$  of  $T$  (for the choice of the positive roots as the  $T$ -weights in the Lie algebra of  $B$ ).*

*Proof.* Given  $\lambda \in X_*(T)^+$  (that is, dominant for  $T \subset B \subset G$ ), let  $V = \mathbf{S}(\mathbf{IC}_{\lambda})$  be the simple  $\tilde{G}_{\mathbf{k}}$ -module corresponding to the simple object  $\mathbf{IC}_{\lambda}$  of  $\mathbf{P}_{G_{\mathcal{O}}}(\mathrm{Gr}_G, \mathbf{k})$ . By Proposition 5.13 the maximal value of  $\langle 2\rho, \mu \rangle$  for  $\mu$  a weight of  $V$  is obtained for  $\mu = \lambda$ , and only for this weight. Therefore  $\lambda$  is dominant for  $T_{\mathbf{k}}^{\vee} \subset \tilde{B} \subset \tilde{G}_{\mathbf{k}}$ , and is the highest weight of  $V$ .

Conversely, let  $\mu \in X^*(T_{\mathbf{k}}^{\vee})$  be dominant for  $T_{\mathbf{k}}^{\vee} \subset \tilde{B} \subset \tilde{G}_{\mathbf{k}}$ . Let  $V$  be the simple  $\tilde{G}_{\mathbf{k}}$ -module of highest weight  $\mu$ . Then  $V = \mathbf{S}(\mathbf{IC}_{\lambda})$  for a unique  $\lambda \in X_*(T)^+$ , and by the first step  $\lambda = \mu$ . Thus  $\mu$  is dominant for  $T \subset B \subset G$ .  $\square$

This claim implies in particular that  $\tilde{B}$  is uniquely determined; that is, no root of  $(\tilde{G}_{\mathbf{k}}, T_{\mathbf{k}}^{\vee})$  is orthogonal to  $2\rho$ . From now on we fix this choice of Borel subgroup in  $\tilde{G}_{\mathbf{k}}$ , and hence of positive roots of  $\tilde{G}_{\mathbf{k}}$  with respect to  $T_{\mathbf{k}}^{\vee}$ . We will denote by  $\Delta(\tilde{G}_{\mathbf{k}}, T_{\mathbf{k}}^{\vee})$  the root system of  $\tilde{G}_{\mathbf{k}}$  with respect to  $T_{\mathbf{k}}^{\vee}$ , by  $\Delta_+(\tilde{G}_{\mathbf{k}}, \tilde{B}, T_{\mathbf{k}}^{\vee})$  the subset of positive roots determined by  $\tilde{B}$ , and by  $\Delta_s(\tilde{G}_{\mathbf{k}}, \tilde{B}, T_{\mathbf{k}}^{\vee})$  the corresponding set of simple roots. We use similar notation (with a superscript “ $\vee$ ”) for coroots, and also for the roots and coroots of  $G$ . (This is of course consistent with the notation introduced in §3.1.)

*Remark 9.6.* 1. Recall (see §5.5) that the maximal torus  $T_{\mathbf{k}}^{\vee} \subset G_{\mathbf{k}}^{\vee}$  does not depend on any choice. Viewed as a coweight of  $T_{\mathbf{k}}^{\vee}$ , the element  $2\rho$  does not depend on any choice either: it is the only coweight such that the weights of restriction of the action of  $G_{\mathbf{k}}^{\vee}$  on  $\mathbf{H}^{\bullet}(\mathrm{Gr}_G, \mathbf{k})$  to  $\mathbf{k}^{\times}$  are given by the cohomological grading. Therefore,  $\tilde{B}$  is also canonical in the sense that it does not depend on any choice.

2. In various sources (e.g. [MV2, End of §7] or [Z4, Discussion after Lemma 5.3.17] the “canonical” Borel subgroup in  $\tilde{G}_{\mathbf{k}}$  is constructed using a “Plücker formalism.” We were not able to find references supporting this construction, hence decided to use a more elementary approach. In any case the two constructions have to produce the same subgroup, see e.g. [Z4, Corollary 5.3.20].
3. Using a construction involving the action of the first Chern class of line bundles on  $\mathrm{Gr}_G$ , viewed as elements of  $\mathbf{H}^{\bullet}(\mathrm{Gr}_G, \mathbf{k})$  (following ideas of Ginzburg [Gi]) one can “complete” the datum of  $\tilde{B}$  and  $T_{\mathbf{k}}^{\vee}$  to a canonical pinning on  $\tilde{G}_{\mathbf{k}}$ ; see in particular [Va], [Ba,

§3.4], [YZ, §5.3] or [Z4, Theorem 5.3.23]. More precisely, this construction provides a group morphism from the Picard group  $\text{Pic}(\text{Gr}_G)$  to the Lie algebra  $\text{Lie}(G_{\mathbf{k}}^{\vee})$ , which sends ample line bundles to regular nilpotent elements belonging to  $\text{Lie}(\tilde{B})$ . (This property provides another canonical description of  $\tilde{B}$ , as the unique Borel subgroup of  $G_{\mathbf{k}}^{\vee}$  containing  $T_{\mathbf{k}}^{\vee}$  and whose Lie algebra contains these regular nilpotent elements.) In particular, if the derived subgroup of  $G$  is quasi-simple then there exists a canonical ample line bundle on  $\text{Gr}$  characterized by the fact that its restriction to each connected component is a generator of the corresponding Picard group; decomposing the regular nilpotent element obtained from this line bundle on root spaces we obtain the desired pinning.

Lemma 9.5 implies that the simple root directions of  $T \subset B \subset G$  are the simple coroot directions of  $T_{\mathbf{k}}^{\vee} \subset \tilde{B} \subset \tilde{G}_{\mathbf{k}}$ :

$$\{\mathbf{Q}_+ \cdot \alpha : \alpha \in \Delta_s^{\vee}(\tilde{G}_{\mathbf{k}}, \tilde{B}, T_{\mathbf{k}}^{\vee})\} = \{\mathbf{Q}_+ \cdot \beta : \beta \in \Delta_s(G, B, T)\}. \quad (9.2)$$

(In fact, these sets are the extreme rays of the rational convex polyhedral cone determined by  $\{\lambda \in \mathbf{Q} \otimes_{\mathbf{Z}} X^*(T) \mid \forall \mu \in X_*(T)^+, \langle \lambda, \mu \rangle \geq 0\}$ .)

**Lemma 9.7.** *We have  $\Delta_s(\tilde{G}_{\mathbf{k}}, \tilde{B}, T_{\mathbf{k}}^{\vee}) = \Delta_s^{\vee}(G, B, T)$  as subsets of  $X_*(T) = X^*(T_{\mathbf{k}}^{\vee})$ .*

*Proof.* Let  $G_{\mathbf{k}}^{\vee}$  be the (connected, split) reductive  $\mathbf{k}$ -group which is Langlands dual to  $G$ , i.e. whose root datum is dual to that of  $(G, T)$ . Then  $T_{\mathbf{k}}^{\vee}$  is also a maximal torus in  $G_{\mathbf{k}}^{\vee}$ . Choose the positive roots of  $(G_{\mathbf{k}}^{\vee}, T_{\mathbf{k}}^{\vee})$  as the positive coroots of  $T \subset B \subset G$ , so that the dominant weights of  $(G_{\mathbf{k}}^{\vee}, T_{\mathbf{k}}^{\vee})$  are  $X_*(T)^+$ .

Given  $\lambda \in X_*(T)^+$ , we can consider the simple  $G_{\mathbf{k}}^{\vee}$ -module  $V_{\lambda}(G_{\mathbf{k}}^{\vee})$  with highest weight  $\lambda$ , and the simple  $\tilde{G}_{\mathbf{k}}$ -module  $V_{\lambda}(\tilde{G}_{\mathbf{k}}) = \mathbf{S}(\mathbf{IC}_{\lambda})$  with highest weight  $\lambda$ . The crucial observation is that these two  $T_{\mathbf{k}}^{\vee}$ -modules have the same weights; specifically, the set of weights of both of these modules is

$$\left\{ \mu \in X_*(T) \mid \begin{array}{l} \mu - \lambda \text{ is in the coroot lattice of } (G, T) \\ \text{and } \mu \text{ is in the convex hull of } W\lambda \end{array} \right\},$$

see again Proposition 5.13. (Note however that we do not yet know that these two  $T_{\mathbf{k}}^{\vee}$ -modules have the same character.)

We now observe that

$$\{\lambda - \mu \mid \lambda \in X_*(T)^+, \mu \text{ a weight of } V_{\lambda}(G_{\mathbf{k}}^{\vee})\}$$

is the  $\mathbf{N}$ -span of the positive roots of  $(G_{\mathbf{k}}^{\vee}, T_{\mathbf{k}}^{\vee})$ . The argument just above shows that this is also the  $\mathbf{N}$ -span of the positive roots of  $(\tilde{G}_{\mathbf{k}}, T_{\mathbf{k}}^{\vee})$ . Looking at the indecomposable elements of this monoid, we deduce that the simple roots of  $\tilde{G}_{\mathbf{k}}$  are the simple roots of  $G_{\mathbf{k}}^{\vee}$ , i.e. the simple coroots of  $G$ .  $\square$

We can finally conclude.

**Theorem 9.8.** *The group  $\tilde{G}_{\mathbf{k}}$  is Langlands dual to  $G$ ; more precisely the root datum of  $\tilde{G}_{\mathbf{k}}$  with respect to  $T_{\mathbf{k}}^{\vee}$  is dual to the root datum of  $G$  with respect to  $T$ .*

*Proof.* By construction  $X^*(T_{\mathbf{k}}^{\vee})$  is dual to  $X^*(T)$ . What remains to be proved is that the roots and coroots of  $\tilde{G}_{\mathbf{k}}$ , together with the canonical bijection between these two sets, coincide with the coroots and roots of  $G$ , together with their canonical bijection.

Let  $\alpha \in \Delta_s(G, B, T)$ . By Lemma 9.7, the corresponding coroot  $\alpha^{\vee}$  belongs to  $\Delta_s(\tilde{G}_{\mathbf{k}}, \tilde{B}, T_{\mathbf{k}}^{\vee})$ . The coroot  $\tilde{\alpha}$  of  $\tilde{G}_{\mathbf{k}}$  associated with this root is  $\mathbf{Q}_+$ -proportional to a simple root of  $T \subset B \subset G$  by (9.2). The conditions

$$\begin{cases} \langle \tilde{\alpha}, \alpha^{\vee} \rangle = 2, \\ \langle \tilde{\alpha}, \beta^{\vee} \rangle \leq 0 \quad \text{for } \beta^{\vee} \in \Delta_s(\tilde{G}_{\mathbf{k}}, \tilde{B}, T_{\mathbf{k}}^{\vee}) \setminus \{\alpha^{\vee}\} \end{cases}$$

then give  $\tilde{\alpha} = \alpha$ .

We thus have an identification

$$\Delta_s(G, B, T) = \Delta_s^{\vee}(\tilde{G}_{\mathbf{k}}, \tilde{B}, T_{\mathbf{k}}^{\vee}).$$

By Lemma 9.7 we also have

$$\Delta_s(\tilde{G}_{\mathbf{k}}, \tilde{B}, T_{\mathbf{k}}^{\vee}) = \Delta_s^{\vee}(G, B, T),$$

and the bijections between simple roots and simple coroots are the same. We may thus identify the Weyl groups of  $G$  and  $\tilde{G}_{\mathbf{k}}$  and extend the above equalities between simple roots/coroots of  $\tilde{G}_{\mathbf{k}}$  and coroots/roots of  $G$  to equalities between *all* roots and coroots. It is clear from this proof that the bijections between roots and coroots are the same for the two groups, and thus our proof is complete.  $\square$

### 9.3 Conclusion

We have finally constructed our canonical equivalence of monoidal categories  $\mathbf{S}$  which fits in the commutative diagram

$$\begin{array}{ccc} P_{G_{\mathcal{O}}}(\mathrm{Gr}_G, \mathbf{k}) & \xrightarrow[\mathbf{S}]{\sim} & \mathrm{Rep}_{\mathbf{k}}(G_{\mathbf{k}}^{\vee}) \\ & \searrow \mathbf{F} := \mathbf{H}^{\bullet}(\mathrm{Gr}_G, ?) & \swarrow \text{forget} \\ & & \mathrm{Vect}_{\mathbf{k}} \end{array}$$

## Part II

# The case of arbitrary coefficients

In this part,  $\mathbf{k}$  is an arbitrary commutative Noetherian ring of finite global dimension,<sup>26</sup> and we denote by  $\text{Mod}_{\mathbf{k}}$  the abelian category of finitely generated  $\mathbf{k}$ -modules. We continue with the geometric setting of Part I:  $G$  is a (connected) complex reductive algebraic group, and we consider the affine Grassmannian  $\text{Gr}_G$  of  $G$ . Our main object of study is now the category  $\text{P}_{G_{\mathcal{O}}}(\text{Gr}_G, \mathbf{k})$  of  $G_{\mathcal{O}}$ -equivariant  $\mathbf{k}$ -perverse sheaves<sup>27</sup> on  $\text{Gr}_G$ . We will see in §10.2 that this category is equivalent to the category  $\text{P}_{\mathcal{S}}(\text{Gr}_G, \mathbf{k})$  of  $\mathcal{S}$ -constructible perverse sheaves (as for fields of characteristic 0, see Corollary 4.8), but at first we need to distinguish these two categories.

## 10 Convolution and weight functors for general coefficients

In this section we explain how to modify the definition of the convolution bifunctor, and the proof of its main properties, to treat the case of general coefficients.

### 10.1 Weight functors

Proposition 5.6 still holds in this generality, with the same proof.

**Proposition 10.1.** *For  $\mathcal{A} \in \text{P}_{\mathcal{S}}(\text{Gr}_G, \mathbf{k})$ ,  $\mu \in X_*(T)$  and  $k \in \mathbf{Z}$ , there exists a canonical isomorphism*

$$\mathrm{H}_{T_{\mu}}^k(\text{Gr}_G, \mathcal{A}) \xrightarrow{\sim} \mathrm{H}_c^k(S_{\mu}, \mathcal{A}),$$

and both terms vanish if  $k \neq \langle 2\rho, \mu \rangle$ .

*Remark 10.2.* The same comments as in Remark 5.8 apply here also.

In view of this fact, as in Section 5, for any  $\mu \in X_*(T)$  we denote by

$$\mathrm{F}_{\mu} : \text{P}_{\mathcal{S}}(\text{Gr}_G, \mathbf{k}) \rightarrow \text{Mod}_{\mathbf{k}}$$

the functor defined by

$$\mathrm{F}_{\mu}(\mathcal{A}) = \mathrm{H}_{T_{\mu}}^{\langle 2\rho, \mu \rangle}(\text{Gr}_G, \mathcal{A}) \cong \mathrm{H}_c^{\langle 2\rho, \mu \rangle}(S_{\mu}, \mathcal{A}).$$

**Lemma 10.3.** *For any  $\mu \in X_*(T)$ , the functor  $\mathrm{F}_{\mu}$  is exact.*

<sup>26</sup>These assumptions on  $\mathbf{k}$  are needed to have a “good” six-functors formalism for derived categories of sheaves of  $\mathbf{k}$ -modules, hence to apply the theory of perverse sheaves; see [KS].

<sup>27</sup>The definition of perverse sheaves in this generality is literally the same as that recalled in §4.1. The main difference with the case of fields is that now this subcategory is not stable under Verdier duality in general.

*Proof.* Any exact sequence  $\mathcal{F}_1 \hookrightarrow \mathcal{F}_2 \twoheadrightarrow \mathcal{F}_3$  in  $\mathcal{P}_{\mathcal{S}}(\mathrm{Gr}_G, \mathbf{k})$  is defined by a distinguished triangle

$$\mathcal{F}_1 \rightarrow \mathcal{F}_2 \rightarrow \mathcal{F}_3 \xrightarrow{[1]}$$

in  $D_{\mathcal{S}}^b(\mathrm{Gr}_G, \mathbf{k})$ . Such a triangle induces a long exact sequence

$$\cdots \rightarrow H_c^{k-1}(S_\mu, \mathcal{F}_3) \rightarrow H_c^k(S_\mu, \mathcal{F}_1) \rightarrow H_c^k(S_\mu, \mathcal{F}_2) \rightarrow H_c^k(S_\mu, \mathcal{F}_3) \rightarrow H_c^{k+1}(S_\mu, \mathcal{F}_1) \rightarrow \cdots$$

in  $\mathrm{Mod}_{\mathbf{k}}$ . Using the vanishing claim in Proposition 10.1 we deduce an exact sequence of  $\mathbf{k}$ -modules

$$H_c^{\langle 2\rho, \mu \rangle}(S_\mu, \mathcal{F}_1) \hookrightarrow H_c^{\langle 2\rho, \mu \rangle}(S_\mu, \mathcal{F}_2) \twoheadrightarrow H_c^{\langle 2\rho, \mu \rangle}(S_\mu, \mathcal{F}_3),$$

which finishes the proof.  $\square$

Then we define the functor

$$F : \mathcal{P}_{\mathcal{S}}(\mathrm{Gr}_G, \mathbf{k}) \rightarrow \mathrm{Mod}_{\mathbf{k}}$$

by

$$F(\mathcal{A}) = H^\bullet(\mathrm{Gr}_G, \mathcal{A}).$$

The same proof as that of Theorem 5.9, together with Lemma 10.3, gives the following result.

**Theorem 10.4.** *There exists a canonical isomorphism of functors*

$$F \cong \bigoplus_{\mu \in X_*(T)} F_\mu : \mathcal{P}_{\mathcal{S}}(\mathrm{Gr}_G, \mathbf{k}) \rightarrow \mathrm{Mod}_{\mathbf{k}}.$$

Moreover,  $F$  is exact and faithful.  $\square$

*Remark 10.5.* Using Theorem 10.4 one can also generalize the proof of Lemma 5.11: the functors  $F_\lambda$  do not depend on the choice of Torel  $T \subset B$ , up to canonical isomorphism.

Below we will also need the following claim (where, as in §6.2, we denote by  $D_{c, G_{\mathcal{O}}}^b(\mathrm{Gr}_G, \mathbf{k})$  the constructible  $G_{\mathcal{O}}$ -equivariant derived category).

**Lemma 10.6.** *For any  $\mathcal{F}$  in  $D_{c, G_{\mathcal{O}}}^b(\mathrm{Gr}_G, \mathbf{k})$ , the following conditions are equivalent:*

1.  $\mathcal{F}$  is a perverse sheaf
2. for any  $\mu \in X_*(T)$  and  $k \in \mathbf{Z}$  we have

$$H_c^k(S_\mu, \mathcal{F}) \neq 0 \quad \Rightarrow \quad k = \langle 2\rho, \mu \rangle.$$

3. for any  $\mu \in X_*(T)$  and  $k \in \mathbf{Z}$  we have

$$H_{S_\mu}^k(\mathrm{Gr}_G, \mathcal{F}) \neq 0 \quad \Rightarrow \quad k = -\langle 2\rho, \mu \rangle.$$



*Proof.* If  $\mathcal{F}$  is perverse, then the conditions (2) and (3) hold by Proposition 10.1 together with the facts that  $\mathcal{F}$  is  $G$ -equivariant and that  $T_{w_0\mu} = \dot{w}_0 \cdot S_\mu$ , where  $\dot{w}_0$  is any lift of the longest element  $w_0$  of  $W$  in  $G$ .

Now, let us assume that (2) holds, and prove that  $\mathcal{F}$  is perverse. Of course we can assume that  $\mathcal{F} \neq 0$ . Let  $n$  be the highest degree for which  ${}^p\mathcal{H}^n(\mathcal{F}) \neq 0$  (where  ${}^p\mathcal{H}^n(?)$  is the  $n$ -th perverse cohomology functor). Then we have a “truncation triangle”

$$\mathcal{F}' \rightarrow \mathcal{F} \rightarrow {}^p\mathcal{H}^n(\mathcal{F})[-n] \xrightarrow{[1]}$$

where  $\mathcal{F}'$  is concentrated in perverse degrees  $\leq n - 1$ . By Theorem 10.4, there exists  $\mu \in X_*(T)$  such that  $F_\mu({}^p\mathcal{H}^n(\mathcal{F})) \neq 0$ . Then Proposition 10.1 implies that  $H_c^k(S_\mu, \mathcal{F}') = 0$  if  $k \geq n + \langle 2\rho, \mu \rangle$ , so that the natural morphism

$$H_c^{n+\langle 2\rho, \mu \rangle}(S_\mu, \mathcal{F}) \rightarrow H_c^{(2\rho, \mu)}(S_\mu, {}^p\mathcal{H}^n(\mathcal{F}))$$

is an isomorphism. Since the right-hand side is nonzero by our choice of  $\mu$ , so is the left-hand side, and then our assumption implies that  $n = 0$ .

If now  $m$  is the lowest degree such that  ${}^p\mathcal{H}^m(\mathcal{F}) \neq 0$ , then similar arguments using the truncation triangle

$${}^p\mathcal{H}^n(\mathcal{F})[-m] \rightarrow \mathcal{F} \rightarrow \mathcal{F}'' \xrightarrow{[1]}$$

(where  $\mathcal{F}''$  is concentrated in perverse degrees  $\geq m + 1$ ) show that  $m = 0$ , which finally proves that  $\mathcal{F}$  is perverse.

The fact that (3) implies that  $\mathcal{F}$  is perverse can be proved similarly using the other description of the functor  $F_\mu$  and the relation between  $S_\mu$  and  $T_{w_0\mu}$  noticed at the beginning of the proof.  $\square$

More generally, using arguments similar to those in the proof of Lemma 10.6 one can show the following claim by induction on  $\#\{m \in \mathbf{Z} \mid {}^p\mathcal{H}^m(\mathcal{F}) \neq 0\}$ .

**Lemma 10.7.** *For any  $\mathcal{F}$  in  $D_{\mathcal{S}}^b(\mathrm{Gr}_G, \mathbf{k})$  and any  $n \in \mathbf{Z}$  we have*

$$H_c^{n+\langle 2\rho, \mu \rangle}(S_\mu, \mathcal{F}) \cong F_\mu({}^p\mathcal{H}^n(\mathcal{F})).$$

## 10.2 Equivariant and constructible perverse sheaves

Now we can prove that Corollary 4.8 is still true in this context (but for more serious reasons). By definition, the forgetful functor  $D_{c, G_{\mathcal{O}}}^b(\mathrm{Gr}_G, \mathbf{k}) \rightarrow D_{\mathcal{S}}^b(\mathrm{Gr}_G, \mathbf{k})$  is t-exact for the perverse t-structures. In the following proposition we consider the restriction of this functor to perverse sheaves.

**Proposition 10.8.** *The forgetful functor*

$$P_{G_{\mathcal{O}}}(\mathrm{Gr}_G, \mathbf{k}) \rightarrow P_{\mathcal{S}}(\mathrm{Gr}_G, \mathbf{k})$$

*is an equivalence of categories.*

In view of this result, below we will not distinguish the categories  $\mathbf{P}_{G_{\mathcal{O}}}(\mathrm{Gr}_G, \mathbf{k})$  and  $\mathbf{P}_{\mathcal{S}}(\mathrm{Gr}_G, \mathbf{k})$  anymore. In particular, we will now consider  $F$  and  $F_{\mu}$  as functors from  $\mathbf{P}_{G_{\mathcal{O}}}(\mathrm{Gr}_G, \mathbf{k})$  to  $\mathrm{Mod}_{\mathbf{k}}$ .

To explain the proof of Proposition 10.8 we need to recall a construction from [Vi]. Consider some categories  $\mathcal{A}$  and  $\mathcal{B}$ , two functors  $F, G : \mathcal{A} \rightarrow \mathcal{B}$ , and a morphism of functor  $\vartheta : F \rightarrow G$ . Then we define a new category  $\mathcal{C}(F, G; \vartheta)$  with:

- objects: quadruples  $(A, B, m, n)$  with  $A$  in  $\mathcal{A}$ ,  $B$  in  $\mathcal{B}$ , and  $m : F(A) \rightarrow B$ ,  $n : B \rightarrow G(A)$  morphisms in  $\mathcal{B}$  such that  $\vartheta(A) = n \circ m$ ;
- morphisms from  $(A, B, m, n)$  to  $(A', B', m', n')$ : pairs  $(f, g)$  where  $f : A \rightarrow A'$  and  $g : B \rightarrow B'$  are morphisms in  $\mathcal{A}$  and  $\mathcal{B}$  respectively, such that both squares in the following diagram commute:

$$\begin{array}{ccccc} F(A) & \xrightarrow{m} & B & \xrightarrow{n} & G(A) \\ F(f) \downarrow & & g \downarrow & & \downarrow G(f) \\ F(A') & \xrightarrow{m'} & B' & \xrightarrow{n'} & G(A'). \end{array}$$

If  $\mathcal{A}$ ,  $\mathcal{B}$  are abelian,  $F$  is right exact and  $G$  is left exact then  $\mathcal{C}(F, G; \vartheta)$  is abelian (see [Vi, Proposition 1.1]). In practice we will only consider this situation, but this fact will not play any role in our arguments.

*Proof of Proposition 10.8.* First, we claim that the forgetful functor

$$\mathbf{P}_{G_{\mathcal{O}}}(\mathrm{Gr}_G, \mathbf{k}) \rightarrow \mathbf{P}_{\mathcal{S}, G}(\mathrm{Gr}_G, \mathbf{k}) \tag{10.1}$$

is an equivalence of categories. In fact, since  $G_{\mathcal{O}}$  is the semi-direct product of  $G$  with a pro-unipotent subgroup (namely the kernel of the natural morphism  $G_{\mathcal{O}} \rightarrow G$ ), [BL, Theorem 3.7.3] shows that the forgetful functor  $D_{c, G_{\mathcal{O}}}^b(\mathrm{Gr}_G, \mathbf{k}) \rightarrow D_{\mathcal{S}, G}^b(\mathrm{Gr}_G, \mathbf{k})$  is fully-faithful. Since the codomain of this functor is generated (as a triangulated category) by the objects of the form  $(j_{\lambda})_! \underline{\mathbf{k}}_{\mathrm{Gr}_G^{\lambda}}$ , which belong to its essential image, this functor is also essentially surjective, hence an equivalence. Restricting to perverse sheaves we deduce that (10.1) is an equivalence as well.

On the other hand, the forgetful functor  $\mathbf{P}_{\mathcal{S}, G}(\mathrm{Gr}_G, \mathbf{k}) \rightarrow \mathbf{P}_{\mathcal{S}}(\mathrm{Gr}_G, \mathbf{k})$  is fully faithful, see §A.1; hence what we have to prove is the following claim: for any finite closed union of  $G_{\mathcal{O}}$ -orbits  $Z$  and any  $\mathcal{S}$ -constructible<sup>28</sup> perverse sheaf  $\mathcal{F}$  on  $Z$ , there exists an isomorphism  $(p_Z)^* \mathcal{F} \xrightarrow{\sim} (a_Z)^* \mathcal{F}$ , where  $a_Z, p_Z : G \times Z \rightarrow Z$  are the action map and the projection, respectively. In fact, we will prove this property for any *locally closed* union of strata, by induction of the number of strata in  $Z$ .

We note that the claim is obvious if  $Z$  contains only one  $G_{\mathcal{O}}$ -orbit. (In fact, in this case the category  $\mathbf{P}_{\mathcal{S}}(Z, \mathbf{k})$  is equivalent to the category  $\mathrm{Mod}_{\mathbf{k}}$  via  $V \mapsto \underline{V}_Z[\dim Z]$ .) Now we consider a general  $Z$ , choose  $\lambda \in X_*(T)^+$  such that  $\mathrm{Gr}_G^{\lambda} \subset Z$  is closed in  $Z$ , and set  $U := Z \setminus \mathrm{Gr}_G^{\lambda}$ . We denote by  $i : \mathrm{Gr}_G^{\lambda} \rightarrow Z$  and  $j : U \rightarrow Z$  the closed and open embeddings, respectively. We also

<sup>28</sup>Here (and below), by abuse, we still denote by  $\mathcal{S}$  the restriction of the stratification  $\mathcal{S}$  to  $Z$  (or to any locally closed union of strata in  $\mathrm{Gr}_G$ ).

consider the varieties  $\tilde{Z} := G \times Z$ ,  $\tilde{U} := G \times U$ , and denote by  $\tilde{i} : G \times \text{Gr}_G^\lambda \rightarrow \tilde{Z}$  and  $\tilde{j} : \tilde{U} \rightarrow \tilde{Z}$  the closed and open embeddings, respectively. Finally, we denote by  $\mathcal{F}$  the stratification of  $\tilde{Z}$  whose strata are the products  $G \times \text{Gr}_G^\mu$  with  $\text{Gr}_G^\mu \subset Z$ , and also the restriction of this stratification to  $\tilde{U}$ .

By induction, we know that the forgetful functor  $\text{P}_{\mathcal{F}, G}(U, \mathbf{k}) \rightarrow \text{P}_{\mathcal{F}}(U, \mathbf{k})$  is an equivalence of categories. Now we take  $\mathcal{F}$  in  $\text{P}_{\mathcal{F}}(Z, \mathbf{k})$ , and need to show that there exists an isomorphism  $(p_Z)^* \mathcal{F} \xrightarrow{\sim} (a_Z)^* \mathcal{F}$ . To check this, we set  $\mathcal{A} := \text{P}_{\mathcal{F}}(U, \mathbf{k})$ , and denote by  $\mathcal{B}$  the category of  $\mathbf{k}$ -local systems on  $G$ . We consider the functor

$$E := \mathcal{H}^{(\lambda, 2\rho) + \dim(G)}((\tilde{\sigma}_\lambda)_!(\tilde{s}_\lambda)^*?) : \text{P}_{\mathcal{F}}(\tilde{Z}, \mathbf{k}) \rightarrow \mathcal{B},$$

where  $\tilde{\sigma}_\lambda : G \times (S_\lambda \cap Z) \rightarrow G$  is the projection and  $\tilde{s}_\lambda : G \times (S_\lambda \cap Z) \rightarrow \tilde{Z}$  is the embedding. (Here, the fact that  $E$  takes values in local systems rather than more general sheaves follows from the observation that the simple objects in  $\text{P}_{\mathcal{F}}(\tilde{Z}, \mathbf{k})$  are actually  $G$ -equivariant, so that their images under  $E$  are also  $G$ -equivariant, hence are local systems.) For  $\mathcal{G}$  in  $\text{P}_{\mathcal{F}}(\tilde{Z}, \mathbf{k})$ , if  $g \in G$  the fiber of the complex  $(\tilde{\sigma}_\lambda)_!(\tilde{s}_\lambda)^* \mathcal{G}$  at  $g$  is  $R\Gamma_c(g \cdot S_\lambda \cap Z, \mathcal{G}_{\{g\} \times Z})$ ; hence this fiber is concentrated in degree  $\langle \lambda, 2\rho \rangle$  by Remark 10.2 (for the choice of Torel  $gTg^{-1} \subset gBg^{-1}$ ). This implies (as in the proof of Lemma 10.3) that  $E$  is an exact functor.

We then set

$$\hat{F} := {}^{p\tilde{j}}_!(?), \quad \hat{G} := {}^{p\tilde{j}*}?(?) : \text{P}_{\mathcal{F}}(\tilde{U}, \mathbf{k}) \rightarrow \text{P}_{\mathcal{F}}(\tilde{Z}, \mathbf{k}).$$

We also denote by  $\theta : \hat{F} \rightarrow \hat{G}$  the natural morphism of functors (provided by adjunction and the fact that  $j^* \circ \hat{F} \cong \text{id}$ , or equivalently the fact that  $j^* \hat{G} \cong \text{id}$ ). Finally, we set  $\tilde{\mathcal{B}} := \text{P}_{\mathcal{F}}(G \times \text{Gr}_G^\lambda, \mathbf{k})$ , which we consider as a full subcategory of  $\text{P}_{\mathcal{F}}(\tilde{Z}, \mathbf{k})$  via the functor  $\tilde{i}_*$ . Then we are exactly in the setting of [Vi, Proposition 1.2], which claims that the functor

$$\tilde{E} : \text{P}_{\mathcal{F}}(Z, \mathbf{k}) \rightarrow \mathcal{C}(E \circ \hat{F}, E \circ \hat{G}; E \circ \theta),$$

sending  $\mathcal{G}$  to the quadruple  $(\tilde{j}^* \mathcal{G}, E(\mathcal{G}), m, n)$  where  $m : E \circ \hat{F}(\tilde{j}^* \mathcal{G}) \rightarrow E(\mathcal{G})$  and  $n : E(\mathcal{G}) \rightarrow E \circ \hat{G}(\tilde{j}^* \mathcal{G})$  are the images under  $E$  of the adjunction morphisms, is fully faithful.<sup>29</sup>

Now, recall our object  $\mathcal{F}$  of  $\text{P}_{\mathcal{F}}(Z, \mathbf{k})$ . The induction hypothesis provides a canonical isomorphism

$$\tilde{j}^*(p_Z)^* \mathcal{F}[\dim G] \cong (p_U)^* j^* \mathcal{F}[\dim G] \xrightarrow{\sim} (a_U)^* j^* \mathcal{F}[\dim G] \cong \tilde{j}^*(a_Z)^* \mathcal{F}[\dim G].$$

On the other hand, for  $g \in G$ , the fiber of  $E((p_Z)^* \mathcal{F}[\dim G])$ , resp. of  $E((a_Z)^* \mathcal{F}[\dim G])$ , at  $g$  is  $\text{H}_c^{(\lambda, 2\rho)}(S_\lambda \cap Z, \mathcal{F})$ , resp.  $\text{H}_c^{(\lambda, 2\rho)}((g \cdot S_\lambda) \cap Z, \mathcal{F})$ . If  $k : Z \rightarrow \text{Gr}_G$  is the embedding, then we have

$$\text{H}_c^{(\lambda, 2\rho)}(S_\lambda \cap Z, \mathcal{F}) \cong \text{H}_c^{(\lambda, 2\rho)}(S_\lambda, k_! \mathcal{F}) \cong \text{F}_\lambda({}^p \mathcal{H}^0(k_! \mathcal{F}))$$

(by the base change theorem and then Lemma 10.7) and similarly

$$\text{H}_c^{(\lambda, 2\rho)}((g \cdot S_\lambda) \cap Z, \mathcal{F}) \cong \text{H}_c^{(\lambda, 2\rho)}(g \cdot S_\lambda, k_! \mathcal{F}) \cong \text{F}_\lambda^{gT}({}^p \mathcal{H}^0(k_! \mathcal{F})),$$

<sup>29</sup>In [MV2, Proof of Proposition A.1], the authors claim (without proof) that this functor is in fact an equivalence of categories. Since this fact is not necessary for the proof of Proposition 10.8, we will not consider this problem here.

where  $F_\lambda^{gT}$  denotes the  $\lambda$ -weight functor constructed using the Torel  $gTg^{-1} \subset gBg^{-1}$  (as in §5.5). The independence of the functor  $F_\lambda$  on the choice of Torel (see Remark 10.5) provides a canonical identification between these spaces, and then an isomorphism of local systems  $E((p_Z)^* \mathcal{F}[\dim G]) \xrightarrow{\sim} E((a_Z)^* \mathcal{F}[\dim G])$ . The pair of isomorphisms we have constructed provides an isomorphism  $\tilde{E}((p_Z)^* \mathcal{F}[\dim G]) \xrightarrow{\sim} \tilde{E}((a_Z)^* \mathcal{F}[\dim G])$ . Since  $\tilde{E}$  is fully faithful, we deduce that  $(p_Z)^* \mathcal{F}$  and  $(a_Z)^* \mathcal{F}$  are (canonically) isomorphic, which shows that  $\mathcal{F}$  is  $G$ -equivariant.  $\square$

### 10.3 The convolution bifunctor

Recall the setting of §6.2. If  $\mathcal{F}$  and  $\mathcal{G}$  are in  $P_{G_O}(\mathrm{Gr}_G, \mathbf{k})$ , the convolution product  $\mathcal{F} \star \mathcal{G}$  is again defined by

$$\mathcal{F} \star \mathcal{G} := m_* (\mathcal{F} \tilde{\boxtimes} \mathcal{G}),$$

but where now  $\mathcal{F} \tilde{\boxtimes} \mathcal{G}$  is defined by the property that

$$q^* (\mathcal{F} \tilde{\boxtimes} \mathcal{G}) = p^* ({}^p \mathcal{H}^0(\mathcal{F} \overset{L}{\boxtimes}_{\mathbf{k}} \mathcal{G})),$$

where  $\overset{L}{\boxtimes}_{\mathbf{k}}$  is now the *derived* external tensor product over  $\mathbf{k}$ . The same considerations as in §6.3 (based on the use of stratified semismall maps) show that  $\mathcal{F} \star \mathcal{G}$  is a perverse sheaf.

An associativity constraint for this bifunctor can be constructed as in §6.4, using the observation that

$${}^p \mathcal{H}^0(\mathcal{F}_1 \overset{L}{\boxtimes}_{\mathbf{k}} {}^p \mathcal{H}^0(\mathcal{F}_2 \overset{L}{\boxtimes}_{\mathbf{k}} \mathcal{F}_3)) \cong {}^p \mathcal{H}^0(\mathcal{F}_1 \overset{L}{\boxtimes}_{\mathbf{k}} \mathcal{F}_2 \overset{L}{\boxtimes}_{\mathbf{k}} \mathcal{F}_3) \cong {}^p \mathcal{H}^0({}^p \mathcal{H}^0(\mathcal{F}_1 \overset{L}{\boxtimes}_{\mathbf{k}} \mathcal{F}_2) \overset{L}{\boxtimes}_{\mathbf{k}} \mathcal{F}_3)$$

for  $\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3$  in  $P_{G_O}(\mathrm{Gr}_G, \mathbf{k})$ .

Finally, the same considerations as in Section 7 apply in this generality, and lead to a description of this convolution bifunctor in terms of fusion and to the construction of a commutativity constraint (which we then modify as in §8.2). In fact, the only change that is required is the replacement of the formula (7.8) by an isomorphism

$$\tau^\circ(\mathcal{F}_1 \star \mathcal{F}_2) \cong i^\circ j_{!*} ({}^p \mathcal{H}^0(\tau^\circ \mathcal{F}_1 \overset{L}{\boxtimes}_{\mathbf{k}} \tau^\circ \mathcal{F}_2)|_U). \quad (10.2)$$

### 10.4 Compatibility with the fiber functor

In this subsection we study the compatibility of convolution with the functor  $F$  (considered either with values in finitely generated  $\mathbf{k}$ -modules, or in  $X_*(T)$ -graded finitely generated  $\mathbf{k}$ -modules). The proof will use the following lemma.

**Lemma 10.9.** *If  $F(\mathcal{F})$  or  $F(\mathcal{G})$  is flat over  $\mathbf{k}$ , then  $\mathcal{F} \overset{L}{\boxtimes}_{\mathbf{k}} \mathcal{G}$  is perverse.*

*Proof.* By Lemma 10.6 (applied to the group  $G \times G$  instead of  $G$ ), it suffices to prove that

$$H_c^k(S_{\nu_1} \times S_{\nu_2}, \mathcal{F} \overset{L}{\boxtimes}_{\mathbf{k}} \mathcal{G}) = 0$$

unless  $k = \langle 2\rho, \nu_1 + \nu_2 \rangle$ , for any  $\nu_1, \nu_2 \in X_*(T)$ . However, if for  $\mu \in X_*(T)$  we denote by  $s_\mu : S_\mu \rightarrow \text{Gr}_G$  the embedding, and by  $\sigma_\mu : S_\mu \rightarrow \text{pt}$  the projection, we have

$$(\sigma_{\nu_1} \times \sigma_{\nu_2})!(s_{\nu_1} \times s_{\nu_2})^*(\mathcal{F} \boxtimes_{\mathbf{k}}^L \mathcal{G}) \cong ((\sigma_{\nu_1})!(s_{\nu_1})^* \mathcal{F}) \boxtimes_{\mathbf{k}}^L ((\sigma_{\nu_2})!(s_{\nu_2})^* \mathcal{G}).$$

(Here we use the compatibility of external products with \*-pullback functors, which is easy, and with derived global sections with compact supports, which is proved in [Bor, Theorem V.10.19].) By Proposition 10.1  $(\sigma_{\nu_1})!(s_{\nu_1})^* \mathcal{F}$  is isomorphic to a  $\mathbf{k}$ -module shifted by  $[-\langle 2\rho, \nu_1 \rangle]$  and  $(\sigma_{\nu_2})!(s_{\nu_2})^* \mathcal{G}$  is isomorphic to a  $\mathbf{k}$ -module shifted by  $[-\langle 2\rho, \nu_2 \rangle]$ . Moreover, our assumption and Theorem 10.4 imply that one of these  $\mathbf{k}$ -modules is flat; hence  $(\sigma_{\nu_1} \times \sigma_{\nu_2})!(s_{\nu_1} \times s_{\nu_2})^*(\mathcal{F} \boxtimes_{\mathbf{k}}^L \mathcal{G})$  is concentrated in degree  $\langle 2\rho, \nu_1 + \nu_2 \rangle$ , which finishes the proof.  $\square$

Our next task is to define a canonical isomorphism

$$F(\mathcal{A}_1 \star \mathcal{A}_2) \cong F(\mathcal{A}_1) \otimes_{\mathbf{k}} F(\mathcal{A}_2)$$

for  $\mathcal{A}_1, \mathcal{A}_2$  in  $\text{P}_{G_O}(\text{Gr}_G, \mathbf{k})$ . The proof is similar to the one explained in §8.1, using the following lemma.

**Lemma 10.10.** *For  $\mathcal{A}_1, \mathcal{A}_2$  in  $\text{P}_{G_O}(\text{Gr}_G, \mathbf{k})$ , there exists a canonical isomorphism*

$$\mathbf{H}^\bullet(\text{Gr}_G \times \text{Gr}_G, {}^p\mathcal{H}^0(\mathcal{A}_1 \boxtimes_{\mathbf{k}}^L \mathcal{A}_2)) \cong \mathbf{H}^\bullet(\text{Gr}_G, \mathcal{A}_1) \otimes_{\mathbf{k}} \mathbf{H}^\bullet(\text{Gr}_G, \mathcal{A}_2).$$

*Proof.* First, we construct a natural morphism from the right-hand side to the left-hand side. For this we consider  $f \in \mathbf{H}^n(\text{Gr}_G, \mathcal{A}_1)$ , considered as a morphism  $\underline{\mathbf{k}}_{\text{Gr}_G} \rightarrow \mathcal{A}_1[n]$ , and  $g \in \mathbf{H}^m(\text{Gr}_G, \mathcal{A}_2)$ , considered as a morphism  $\underline{\mathbf{k}}_{\text{Gr}_G} \rightarrow \mathcal{A}_2[m]$ . Then we can consider

$$f \boxtimes_{\mathbf{k}}^L g : \underline{\mathbf{k}}_{\text{Gr}_G \times \text{Gr}_G} \rightarrow \mathcal{A}_1 \boxtimes_{\mathbf{k}}^L \mathcal{A}_2[n+m].$$

Now, since  $\mathcal{A}_1 \boxtimes_{\mathbf{k}}^L \mathcal{A}_2$  is concentrated in nonpositive perverse degrees, we have a canonical (truncation) morphism  $\mathcal{A}_1 \boxtimes_{\mathbf{k}}^L \mathcal{A}_2 \rightarrow {}^p\mathcal{H}^0(\mathcal{A}_1 \boxtimes_{\mathbf{k}}^L \mathcal{A}_2)$ . Composing  $f \boxtimes_{\mathbf{k}}^L g$  with the shift of this morphism by  $n+m$  provides the desired element of  $\mathbf{H}^{n+m}(\text{Gr}_G \times \text{Gr}_G, {}^p\mathcal{H}^0(\mathcal{A}_1 \boxtimes_{\mathbf{k}}^L \mathcal{A}_2))$ .

We next prove that our morphism is an isomorphism. If  $F(\mathcal{A}_1)$  is projective over  $\mathbf{k}$ , then by Lemma 10.9 the left-hand side identifies with  $\mathbf{H}^\bullet(\text{Gr}_G \times \text{Gr}_G, \mathcal{A}_1 \boxtimes_{\mathbf{k}}^L \mathcal{A}_2)$ . By the formula [Bor, Theorem V.10.19] (already used in the proof of this lemma), we have

$$R\Gamma(\text{Gr}_G \times \text{Gr}_G, \mathcal{A}_1 \boxtimes_{\mathbf{k}}^L \mathcal{A}_2) \cong R\Gamma(\text{Gr}_G, \mathcal{A}_1) \otimes_{\mathbf{k}}^L R\Gamma(\text{Gr}_G, \mathcal{A}_2).$$

The cohomology of the left-hand side is  $\mathbf{H}^\bullet(\text{Gr}_G \times \text{Gr}_G, \mathcal{A}_1 \boxtimes_{\mathbf{k}}^L \mathcal{A}_2)$ . Now since  $F(\mathcal{A}_1)$  is projective,  $R\Gamma(\text{Gr}_G, \mathcal{A}_1)$  is isomorphic, in the derived category of  $\mathbf{k}$ -modules, to its cohomology; it follows that the cohomology of the right-hand side is  $F(\mathcal{A}_1) \otimes_{\mathbf{k}} F(\mathcal{A}_2)$ , and then that our morphism is an isomorphism.

To deduce the general case, we observe that by the results of Section 12 below (see in particular the remarks at the end of §12.1 and Proposition 12.3(3)) there exists an exact sequence

$$\mathcal{F} \rightarrow \mathcal{G} \rightarrow \mathcal{A}_1 \rightarrow 0$$

in  $P_{G_{\mathcal{O}}}(\mathrm{Gr}_G, \mathbf{k})$  where  $F(\mathcal{F})$  and  $F(\mathcal{G})$  are free over  $\mathbf{k}$ . By right-exactness of the functor  ${}^p\mathcal{H}^0(? \boxtimes_{\mathbf{k}}^L \mathcal{A}_2)$ , we deduce an exact sequence of perverse sheaves

$${}^p\mathcal{H}^0(\mathcal{F} \boxtimes_{\mathbf{k}}^L \mathcal{A}_2) \rightarrow {}^p\mathcal{H}^0(\mathcal{G} \boxtimes_{\mathbf{k}}^L \mathcal{A}_2) \rightarrow {}^p\mathcal{H}^0(\mathcal{A}_1 \boxtimes_{\mathbf{k}}^L \mathcal{A}_2) \rightarrow 0.$$

Since the functor  $\mathbf{H}^\bullet(\mathrm{Gr}_G \times \mathrm{Gr}_G, ?)$  is exact on  $G_{\mathcal{O}}$ -equivariant perverse sheaves (by Theorem 10.4 applied to the group  $G \times G$ ), and the case already proven, we deduce an exact sequence

$$\begin{aligned} \mathbf{H}^\bullet(\mathrm{Gr}_G, \mathcal{F}) \otimes_{\mathbf{k}} \mathbf{H}^\bullet(\mathrm{Gr}_G, \mathcal{A}_2) &\rightarrow \mathbf{H}^\bullet(\mathrm{Gr}_G, \mathcal{G}) \otimes_{\mathbf{k}} \mathbf{H}^\bullet(\mathrm{Gr}_G, \mathcal{A}_2) \\ &\rightarrow \mathbf{H}^\bullet(\mathrm{Gr}_G \times \mathrm{Gr}_G, {}^p\mathcal{H}^0(\mathcal{A}_1 \boxtimes_{\mathbf{k}}^L \mathcal{A}_2)) \rightarrow 0. \end{aligned}$$

Comparing with the exact sequence obtained by applying the functor  $? \otimes_{\mathbf{k}} \mathbf{H}^\bullet(\mathrm{Gr}_G, \mathcal{A}_2)$  to the exact sequence

$$\mathbf{H}^\bullet(\mathrm{Gr}_G, \mathcal{F}) \rightarrow \mathbf{H}^\bullet(\mathrm{Gr}_G, \mathcal{G}) \rightarrow \mathbf{H}^\bullet(\mathrm{Gr}_G, \mathcal{A}_1) \rightarrow 0,$$

we finally deduce that our morphism is an isomorphism in general.  $\square$

We also have the following generalization of Proposition 8.3, where we denote by  $\mathrm{Mod}_{\mathbf{k}}(X_*(T))$  the category of finitely generated  $X_*(T)$ -graded  $\mathbf{k}$ -modules.

**Proposition 10.11.** *The functor*

$$\bigoplus_{\mu \in X_*(T)} F_{\mu} : P_{G_{\mathcal{O}}}(\mathrm{Gr}_G, \mathbf{k}) \rightarrow \mathrm{Mod}_{\mathbf{k}}(X_*(T))$$

*sends the convolution product  $\star$  to the tensor product of  $X_*(T)$ -graded  $\mathbf{k}$ -modules, in a way compatible with the associativity and commutativity constraints.*

Here again, the proof is similar to that of Proposition 8.3, except that now we have to provide a canonical isomorphism

$$\mathbf{H}_c^\bullet(S_{\mu_1} \times S_{\mu_2}, {}^p\mathcal{H}^0(\mathcal{A}_1 \boxtimes_{\mathbf{k}}^L \mathcal{A}_2)) \cong \mathbf{H}_c^\bullet(S_{\mu_1}, \mathcal{A}_1) \otimes_{\mathbf{k}} \mathbf{H}_c^\bullet(S_{\mu_2}, \mathcal{A}_2)$$

for  $\mathcal{A}_1, \mathcal{A}_2$  in  $P_{G_{\mathcal{O}}}(\mathrm{Gr}_G, \mathbf{k})$  and  $\mu_1, \mu_2 \in X_*(T)$ . The proof is completely similar to that of Lemma 10.10.

## 11 Study of standard and costandard sheaves

### 11.1 Definitions

Recall that for any  $\lambda \in X_*(T)^+$  we denote by  $j_{\lambda} : \mathrm{Gr}_G^{\lambda} \rightarrow \mathrm{Gr}_G$  the embedding. We set

$$\mathcal{I}_!(\lambda, \mathbf{k}) := {}^p\mathcal{H}^0((j_{\lambda})_! \mathbf{k}_{\mathrm{Gr}_G^{\lambda}}[\dim \mathrm{Gr}_G^{\lambda}]), \quad \mathcal{I}_*(\lambda, \mathbf{k}) := {}^p\mathcal{H}^0((j_{\lambda})_* \mathbf{k}_{\mathrm{Gr}_G^{\lambda}}[\dim \mathrm{Gr}_G^{\lambda}]).$$

By adjunction there exists a canonical morphism of complexes

$$(j_\lambda)! \underline{\mathbf{k}}_{\mathrm{Gr}_G^\lambda}[\dim \mathrm{Gr}_G^\lambda] \rightarrow (j_\lambda)_* \underline{\mathbf{k}}_{\mathrm{Gr}_G^\lambda}[\dim \mathrm{Gr}_G^\lambda],$$

hence a canonical morphism of perverse sheaves

$$\mathcal{J}_!(\lambda, \mathbf{k}) \rightarrow \mathcal{J}_*(\lambda, \mathbf{k}),$$

and we denote its image (in the abelian category of perverse sheaves) by  $\mathcal{J}_{!*}(\lambda, \mathbf{k})$ . It follows from the definition of the perverse t-structure that  $(j_\lambda)! \underline{\mathbf{k}}_{\mathrm{Gr}_G^\lambda}[\dim \mathrm{Gr}_G^\lambda]$  is concentrated in perverse degrees  $\leq 0$ , hence that for any perverse sheaf  $\mathcal{F}$  we have

$$\mathrm{Hom}(\mathcal{J}_!(\lambda, \mathbf{k}), \mathcal{F}) \cong \mathrm{Hom}((j_\lambda)! \underline{\mathbf{k}}_{\mathrm{Gr}_G^\lambda}[\dim \mathrm{Gr}_G^\lambda], \mathcal{F}).$$

In particular, using adjunction we see that  $\mathcal{J}_!(\lambda, \mathbf{k})$  has no nonzero morphism to a perverse sheaf supported on  $\overline{\mathrm{Gr}_G^\lambda} \setminus \mathrm{Gr}_G^\lambda$ . Similar arguments show that  $\mathcal{J}_*(\lambda, \mathbf{k})$  has no nonzero morphism from a perverse sheaf supported on  $\overline{\mathrm{Gr}_G^\lambda} \setminus \mathrm{Gr}_G^\lambda$ .

If  $\mathbf{k}$  is a field then  $\mathcal{J}_{!*}(\lambda, \mathbf{k})$  is simple, and coincides with the object denoted  $\mathbf{IC}_\lambda$  in Section 4. If moreover  $\mathbf{k}$  has characteristic 0, then the category  $\mathrm{P}_{G_O}(\mathrm{Gr}_G, \mathbf{k})$  is semisimple by Theorem 4.2. In view of the properties of  $\mathcal{J}_!(\lambda, \mathbf{k})$  and  $\mathcal{J}_*(\lambda, \mathbf{k})$  recalled above, it follows that the canonical morphisms

$$\mathcal{J}_!(\lambda, \mathbf{k}) \twoheadrightarrow \mathcal{J}_{!*}(\lambda, \mathbf{k}) \hookrightarrow \mathcal{J}_*(\lambda, \mathbf{k})$$

are isomorphisms in this case.

Now we come back to the case of a general Noetherian commutative ring  $\mathbf{k}$  of finite global dimension. In view of the remarks above, the following result is a generalization of Proposition 5.13.

**Proposition 11.1.** *Let  $\lambda, \mu \in X_*(T)$  with  $\lambda$  dominant. Then the  $\mathbf{k}$ -module  $F_\mu(\mathcal{J}_!(\lambda, \mathbf{k}))$ , resp.  $F_\mu(\mathcal{J}_*(\lambda, \mathbf{k}))$ , is free, with a canonical basis parametrized by the irreducible components of  $\mathrm{Gr}_G^\lambda \cap S_\mu$ , resp. of  $\mathrm{Gr}_G^\lambda \cap T_\mu$ .*

*Proof.* By Lemma 10.7, we have

$$F_\mu(\mathcal{J}_!(\lambda, \mathbf{k})) \cong H_c^{(2\rho, \mu)}(S_\mu, (j_\lambda)! \underline{\mathbf{k}}_{\mathrm{Gr}_G^\lambda}[\langle 2\rho, \lambda \rangle]).$$

Using the base change theorem, it is not difficult to check that there exists a canonical isomorphism

$$H_c^{(2\rho, \mu)}(S_\mu, (j_\lambda)! \underline{\mathbf{k}}_{\mathrm{Gr}_G^\lambda}[\langle 2\rho, \lambda \rangle]) \cong H_c^{(2\rho, \lambda + \mu)}(\mathrm{Gr}_G^\lambda \cap S_\mu; \mathbf{k}).$$

Since  $\langle 2\rho, \lambda + \mu \rangle = 2 \dim(\mathrm{Gr}_G^\lambda \cap S_\mu)$  (see Theorem 5.2), the right-hand side is free, with a canonical basis parametrized by irreducible components of  $\mathrm{Gr}_G^\lambda \cap S_\mu$ .

The case of  $\mathcal{J}_*(\lambda, \mathbf{k})$  is similar, using the description of  $F_\mu$  as  $H_{T_\mu}^{(2\rho, \mu)}(\mathrm{Gr}_G, ?)$ , and Borel–Moore homology instead of cohomology with compact supports.  $\square$

*Remark 11.2.* More generally, if  $M$  is a finitely generated  $\mathbf{k}$ -module, we can consider the perverse sheaves

$$\mathcal{J}_!(\lambda, M) := {}^p\mathcal{H}^0((j_\lambda)_! \underline{M}_{\mathrm{Gr}_G^\lambda}[\dim \mathrm{Gr}_G^\lambda]), \quad \mathcal{J}_*(\lambda, M) := {}^p\mathcal{H}^0((j_\lambda)_* \underline{M}_{\mathrm{Gr}_G^\lambda}[\dim \mathrm{Gr}_G^\lambda]).$$

Considerations similar to those in the proof of Proposition 11.1 show that there exist canonical isomorphisms

$$F_\mu(\mathcal{J}_!(\lambda, M)) \cong F_\mu(\mathcal{J}_!(\lambda, \mathbf{k})) \otimes_{\mathbf{k}} M, \quad F_\mu(\mathcal{J}_*(\lambda, M)) \cong F_\mu(\mathcal{J}_*(\lambda, \mathbf{k})) \otimes_{\mathbf{k}} M \quad (11.1)$$

for any  $\mu \in X_*(T)$ .

## 11.2 Extension of scalars

In the following proposition, we denote by

$$\mathbf{k} \otimes_{\mathbf{Z}}^L (?) : D_{c, G_{\mathcal{O}}}^b(\mathrm{Gr}_G, \mathbf{Z}) \rightarrow D_{c, G_{\mathcal{O}}}^b(\mathrm{Gr}_G, \mathbf{k})$$

the (derived) extension-of-scalars functor. (Note that this functor does not send perverse sheaves to perverse sheaves.) Below we will use this notation also for varieties other than  $\mathrm{Gr}_G$ .

**Proposition 11.3.** *For any  $\lambda \in X_*(T)^+$ , we have a canonical isomorphism*

$$\mathcal{J}_!(\lambda, \mathbf{k}) \cong \mathbf{k} \otimes_{\mathbf{Z}}^L \mathcal{J}_!(\lambda, \mathbf{Z}), \quad \mathcal{J}_*(\lambda, \mathbf{k}) \cong \mathbf{k} \otimes_{\mathbf{Z}}^L \mathcal{J}_*(\lambda, \mathbf{Z}).$$

*Proof.* As in the proof of Lemma 10.9, for  $\mu \in X_*(T)$  we denote by  $s_\mu : S_\mu \rightarrow \mathrm{Gr}_G$  the embedding, and by  $\sigma_\mu : S_\mu \rightarrow \mathrm{pt}$  the projection. Then by definition we have

$$H_c^\bullet(S_\mu, ?) \cong H^\bullet((\sigma_\mu)_!(s_\mu)^*(?)),$$

where we identify the derived category of  $\mathbf{k}$ -sheaves on  $\mathrm{pt}$  with the derived category of  $\mathbf{k}$ -modules.

By general considerations, we have

$$\mathbf{k} \otimes_{\mathbf{Z}}^L (\sigma_\mu)_!(s_\mu)^*(?) \cong (\sigma_\mu)_!(s_\mu)^*(\mathbf{k} \otimes_{\mathbf{Z}}^L (?)).$$

We apply this isomorphism to  $\mathcal{J}_!(\lambda, \mathbf{Z})$ . In this case, by Proposition 10.1 and Proposition 11.1 the complex  $(\sigma_\mu)_!(s_\mu)^*(\mathcal{J}_!(\lambda, \mathbf{Z}))$  is isomorphic to the shift by  $[-\langle 2\rho, \mu \rangle]$  of a free  $\mathbf{Z}$ -module. Hence  $\mathbf{k} \otimes_{\mathbf{Z}}^L (\sigma_\mu)_!(s_\mu)^*(\mathcal{J}_!(\lambda, \mathbf{Z}))$  is concentrated in degree  $\langle 2\rho, \mu \rangle$  which, by Lemma 10.6, shows that  $\mathbf{k} \otimes_{\mathbf{Z}}^L \mathcal{J}_!(\lambda, \mathbf{Z})$  is a perverse sheaf.

Now, we clearly have

$$(j_\lambda)_!(\mathbf{k} \otimes_{\mathbf{Z}}^L \mathcal{J}_!(\lambda, \mathbf{Z})) \cong \underline{\mathbf{k}}_{\mathrm{Gr}_G^\lambda}[\dim \mathrm{Gr}_G^\lambda].$$

By adjunction we deduce a canonical morphism  $(j_\lambda)_! \underline{\mathbf{k}}_{\mathrm{Gr}_G^\lambda}[\dim \mathrm{Gr}_G^\lambda] \rightarrow \mathbf{k} \otimes_{\mathbf{Z}}^L \mathcal{J}_!(\lambda, \mathbf{Z})$ , and then taking the 0-th perverse cohomology we deduce a canonical morphism

$$\mathcal{J}_!(\lambda, \mathbf{k}) \rightarrow \mathbf{k} \otimes_{\mathbf{Z}}^L \mathcal{J}_!(\lambda, \mathbf{Z}). \quad (11.2)$$



Using Proposition 11.1 and the same kind of considerations as above, we see that this morphism induces an isomorphism

$$F_\mu(\mathcal{J}_!(\lambda, \mathbf{k})) \xrightarrow{\sim} F_\mu\left(\mathbf{k} \otimes_{\mathbf{Z}}^L \mathcal{J}_!(\lambda, \mathbf{Z})\right)$$

for any  $\mu \in X_*(T)$ . By the faithfulness claim in Theorem 10.4, this implies that (11.2) is an isomorphism, and concludes the proof of the first isomorphism.

The proof of the isomorphism  $\mathcal{J}_*(\lambda, \mathbf{k}) \cong \mathbf{k} \otimes_{\mathbf{Z}}^L \mathcal{J}_*(\lambda, \mathbf{Z})$  is similar.  $\square$

*Remark 11.4.* These results imply that there exists a canonical isomorphism  $\mathbb{D}(\mathcal{J}_*(\lambda, \mathbf{k})) \cong \mathcal{J}_!(\lambda, \mathbf{k})$ , where  $\mathbb{D}$  is the Verdier duality functor on the category  $D_{c, G_O}^b(\mathrm{Gr}_G, \mathbf{k})$ . In fact, by general considerations  $\mathbb{D}(\mathcal{J}_*(\lambda, \mathbf{k}))$  is the 0-th cohomology of  $\mathbb{D}((j_\lambda)_* \underline{\mathbf{Z}}_{\mathrm{Gr}_G^\lambda}[\dim \mathrm{Gr}_G^\lambda]) \cong (j_\lambda)_! \underline{\mathbf{Z}}_{\mathrm{Gr}_G^\lambda}[\dim \mathrm{Gr}_G^\lambda]$  for the t-structure  $p^+$  of [BBD, §3.3]. Now consider the truncation triangle for the *usual* perverse t-structure:

$${}^p\tau_{<0}((j_\lambda)_! \underline{\mathbf{Z}}_{\mathrm{Gr}_G^\lambda}[\dim \mathrm{Gr}_G^\lambda]) \rightarrow (j_\lambda)_! \underline{\mathbf{Z}}_{\mathrm{Gr}_G^\lambda}[\dim \mathrm{Gr}_G^\lambda] \rightarrow \mathcal{J}_!(\lambda, \mathbf{Z}) \xrightarrow{[1]}.$$

By definition the left-hand side belongs to  ${}^pD_{\mathcal{S}}^b(\mathrm{Gr}_G, \mathbf{Z})^{<0}$ , hence to  ${}^pD_{\mathcal{S}}^b(\mathrm{Gr}_G, \mathbf{Z})^{<0}$ . On the other hand, the fact that  $\mathbf{k} \otimes_{\mathbf{Z}}^L \mathcal{J}_!(\lambda, \mathbf{Z})$  is a perverse sheaf (see Proposition 11.3) shows that  $\mathcal{J}_!(\lambda, \mathbf{Z})$  is torsion-free; in view of [BBD, §3.3.4] this implies that this object belongs to  ${}^pD_{\mathcal{S}}^b(\mathrm{Gr}_G, \mathbf{Z})^{\geq 0}$ . Hence the triangle above is also the truncation triangle for the t-structure  $p^+$ ; in other words we have

$$\mathcal{J}_!(\lambda, \mathbf{Z}) \cong {}^{p^+}\mathcal{H}^0((j_\lambda)_! \underline{\mathbf{Z}}_{\mathrm{Gr}_G^\lambda}[\dim \mathrm{Gr}_G^\lambda]) \cong \mathbb{D}(\mathcal{J}_*(\lambda, \mathbf{k})).$$

(See [MV2, Proposition 8.1(c)] for a proof of this isomorphism which does not refer to the t-structure  $p^+$ .)

### 11.3 Relation between integral standard and IC-sheaves

**Lemma 11.5.** *For any  $\lambda \in X_*(T)^+$ , the canonical surjection*

$$\mathcal{J}_!(\lambda, \mathbf{Z}) \rightarrow \mathcal{J}_{!*}(\lambda, \mathbf{Z})$$

*is an isomorphism.*

*Proof.* The claim amounts to saying that the canonical morphism

$$\mathcal{J}_!(\lambda, \mathbf{Z}) \rightarrow \mathcal{J}_*(\lambda, \mathbf{Z})$$

(see §11.1) is injective or, in view of Theorem 10.4, that for any  $\mu \in X_*(T)$  it induces an embedding

$$F_\mu(\mathcal{J}_!(\lambda, \mathbf{Z})) \rightarrow F_\mu(\mathcal{J}_*(\lambda, \mathbf{Z})).$$

However, since the left-hand side is free over  $\mathbf{Z}$  by Proposition 11.1, it suffices to prove that the induced morphism

$$\mathbf{Q} \otimes_{\mathbf{Z}} F_\mu(\mathcal{J}_!(\lambda, \mathbf{Z})) \rightarrow \mathbf{Q} \otimes_{\mathbf{Z}} F_\mu(\mathcal{J}_*(\lambda, \mathbf{Z}))$$

is an embedding. By Proposition 11.3 and its proof, this morphism identifies with the morphism

$$F_\mu(\mathcal{J}_!(\lambda, \mathbf{Q})) \rightarrow F_\mu(\mathcal{J}_*(\lambda, \mathbf{Q}))$$

induced by the canonical morphism  $\mathcal{J}_!(\lambda, \mathbf{Q}) \rightarrow \mathcal{J}_*(\lambda, \mathbf{Q})$ . The latter morphism is an isomorphism (see §11.1), which concludes the proof.  $\square$

## 12 Representability of the weight functors

In Section 2, we presented Deligne and Milne's proof of (a version of) Tannakian reconstruction for rigid tensor  $\mathbf{k}$ -linear abelian categories, where  $\mathbf{k}$  is a field. One of the key steps is Proposition 2.2, which established an equivalence, for each object  $X$  in an abelian  $\mathbf{k}$ -linear category  $\mathcal{C}$  endowed with a  $\mathbf{k}$ -linear exact faithful functor  $\omega : \mathcal{C} \rightarrow \text{Vect}_{\mathbf{k}}$ , between the abelian subcategory  $\langle X \rangle$  generated by  $X$  and the category  $\text{Mod}_{A_X}$  of modules over an algebra  $A_X \subset \text{End}_{\mathbf{k}}(\omega(X))$ .

The condition that  $\mathbf{k}$  is a field, needed for the proof of Proposition 2.2, is too restrictive for our current setup. Mirković and Vilonen choose therefore another approach. Rather than an equivalence  $\langle X \rangle \cong \text{Mod}_{A_X}$  for each object  $X \in \mathcal{C}$ , they produce a Morita equivalence  $\text{P}_{G_{\mathcal{O}}}(Z, \mathbf{k}) \cong \text{Mod}_{A_Z(\mathbf{k})}$  for each closed subset  $Z \subset \text{Gr}_G$  union of finitely many  $G_{\mathcal{O}}$ -orbits. Here  $\text{P}_{G_{\mathcal{O}}}(Z, \mathbf{k})$  is the subcategory of  $\text{P}_{G_{\mathcal{O}}}(\text{Gr}_G, \mathbf{k})$  consisting of  $G_{\mathcal{O}}$ -equivariant perverse sheaves supported on  $Z$  and  $A_Z(\mathbf{k})$  is the (opposite algebra of the) endomorphism algebra of a projective generator  $P_Z(\mathbf{k})$  of  $\text{P}_{G_{\mathcal{O}}}(Z, \mathbf{k})$ .

The aim of this section is to construct and study the objects  $P_Z(\mathbf{k})$ . Since the diagram

$$\begin{array}{ccc} \text{P}_{G_{\mathcal{O}}}(Z, \mathbf{k}) & \xrightarrow{\text{Hom}(P_Z(\mathbf{k}), ?)} & \text{Mod}_{A_Z(\mathbf{k})} \\ & \searrow \text{F} & \swarrow \text{forget} \\ & \text{Mod}_{\mathbf{k}} & \end{array}$$

has to commute, we will choose  $P_Z(\mathbf{k})$  so that it represents  $\text{F}$ .

### 12.1 Construction of projective objects

Let  $Z$  be a closed subset of  $\text{Gr}_G$ , union of finitely many  $G_{\mathcal{O}}$ -orbits. For  $n \geq 0$ , we set  $\mathcal{O}_n = \mathcal{O}/t^{n+1}\mathcal{O}$  and let  $G_{\mathcal{O}_n}$  be the complex algebraic group which represents the functor  $R \mapsto G(R \otimes_{\mathbf{C}} \mathcal{O}_n)$ . We choose (as we may)  $n \in \mathbf{Z}_{\geq 0}$  large enough so that the  $G_{\mathcal{O}}$ -action on  $Z$  factors through  $G_{\mathcal{O}_n}$ ; then by definition (see §A.4) we have  $D_{c, G_{\mathcal{O}}}^b(Z, \mathbf{k}) = D_{c, G_{\mathcal{O}_n}}^b(Z, \mathbf{k})$ .

Let  $\nu \in X_*(T)$ . For any  $\mathcal{A} \in \text{P}_{G_{\mathcal{O}}}(Z, \mathbf{k})$ , we have

$$F_\nu(\mathcal{A}) = H_{T_\nu}^{(2\rho, \nu)}(Z, \mathcal{A}) = \text{Hom}_{D_c^b(Z, \mathbf{k})}(i_! \mathbf{k}_{T_\nu \cap Z}[-\langle 2\rho, \nu \rangle], \mathcal{A}),$$

where  $i : T_\nu \cap Z \rightarrow Z$  is the embedding. To represent the functor  $F_\nu$  on the category  $\text{P}_{G_{\mathcal{O}}}(Z, \mathbf{k})$ , we need to transform the nonequivariant object  $i_! \mathbf{k}_{T_\nu \cap Z}[-\langle 2\rho, \nu \rangle]$  of  $D_c^b(Z, \mathbf{k})$  into an object of  $\text{P}_{G_{\mathcal{O}}}(Z, \mathbf{k})$ . We do this using the (!)-induction functor (whose construction is recalled in §A.2).

Concretely, we consider the commutative diagram

$$\begin{array}{ccccc}
T_\nu \cap Z & \longleftarrow & G_{\mathcal{O}_n} \times (T_\nu \cap Z) & \xrightarrow{\tilde{a}} & Z \\
\downarrow i & & \downarrow & & \parallel \\
Z & \xleftarrow{p} & G_{\mathcal{O}_n} \times Z & \xrightarrow{a} & Z,
\end{array} \tag{12.1}$$

where  $a$  is the action map and  $p$  is the projection, and we define

$$\begin{aligned}
P_Z(\nu, \mathbf{k}) &:= {}^p\mathcal{H}^0(a_! p^! i_! \underline{\mathbf{k}}_{T_\nu \cap Z}[-\langle 2\rho, \nu \rangle]) \\
&\cong {}^p\mathcal{H}^0(a_! p^* i_! \underline{\mathbf{k}}_{T_\nu \cap Z}[2 \dim(G_{\mathcal{O}_n}) - \langle 2\rho, \nu \rangle]) \\
&\cong {}^p\mathcal{H}^0(\tilde{a}_! \underline{\mathbf{k}}_{G_{\mathcal{O}_n} \times (T_\nu \cap Z)}[2 \dim(G_{\mathcal{O}_n}) - \langle 2\rho, \nu \rangle]),
\end{aligned}$$

the last equality being given by base change along the left (Cartesian) square in (12.1).

**Proposition 12.1.** *The perverse sheaf  $P_Z(\nu, \mathbf{k})$  is a projective object of  $\mathrm{P}_{G_{\mathcal{O}}}(Z, \mathbf{k})$  that represents the weight functor  $\mathrm{F}_\nu$ .*

*Proof.* We set

$$\mathcal{F} := a_! p^! i_! \underline{\mathbf{k}}_{T_\nu \cap Z}[-\langle 2\rho, \nu \rangle].$$

For any  $\mathcal{A} \in \mathrm{P}_{G_{\mathcal{O}}}(Z, \mathbf{k})$ , we have by Lemma A.3

$$\begin{aligned}
\mathrm{F}_\nu(\mathcal{A}) &= \mathrm{H}_{T_\nu}^{\langle 2\rho, \nu \rangle}(Z, \mathcal{A}) = \mathrm{Hom}_{D_c^b(Z, \mathbf{k})}(i_! \underline{\mathbf{k}}_{T_\nu \cap Z}[-\langle 2\rho, \nu \rangle], \mathcal{A}) \\
&\cong \mathrm{Hom}_{D_{c, G_{\mathcal{O}_n}}^b(Z, \mathbf{k})}(a_! p^! i_! \underline{\mathbf{k}}_{T_\nu \cap Z}[-\langle 2\rho, \nu \rangle], \mathcal{A}) = \mathrm{Hom}_{D_{c, G_{\mathcal{O}_n}}^b(Z, \mathbf{k})}(\mathcal{F}, \mathcal{A}).
\end{aligned}$$

We claim that  $\mathcal{F}$  is concentrated in nonpositive perverse degrees. Indeed, let  $n$  be the largest integer such that  ${}^p\mathcal{H}^n(\mathcal{F}) \neq 0$ . The second arrow in the truncation triangle

$${}^p\tau_{<n}\mathcal{F} \rightarrow \mathcal{F} \rightarrow {}^p\mathcal{H}^n(\mathcal{F})[-n] \xrightarrow{[1]}$$

is nonzero, so that

$$0 \neq \mathrm{Hom}_{D_{c, G_{\mathcal{O}_n}}^b(Z, \mathbf{k})}(\mathcal{F}, {}^p\mathcal{H}^n(\mathcal{F})[-n]) = \mathrm{F}_\nu({}^p\mathcal{H}^n(\mathcal{F})[-n]) = \mathrm{H}_{T_\nu}^{\langle 2\rho, \nu \rangle - n}(Z, {}^p\mathcal{H}^n(\mathcal{F}));$$

applying Lemma 10.6, we deduce that  $n = 0$ , proving our claim.

Our truncation triangle now reads

$${}^p\tau_{<0}\mathcal{F} \rightarrow \mathcal{F} \rightarrow P_Z(\nu, \mathbf{k}) \xrightarrow{[1]}.$$

For any  $\mathcal{A} \in \mathrm{P}_{G_{\mathcal{O}}}(Z, \mathbf{k})$ , we have a long exact sequence

$$\begin{aligned}
\mathrm{Hom}_{D_{c, G_{\mathcal{O}_n}}^b(Z, \mathbf{k})}({}^p\tau_{<0}\mathcal{F}, \mathcal{A}[-1]) &\rightarrow \mathrm{Hom}_{D_{c, G_{\mathcal{O}_n}}^b(Z, \mathbf{k})}(P_Z(\nu, \mathbf{k}), \mathcal{A}) \\
&\rightarrow \mathrm{Hom}_{D_{c, G_{\mathcal{O}_n}}^b(Z, \mathbf{k})}(\mathcal{F}, \mathcal{A}) \rightarrow \mathrm{Hom}_{D_{c, G_{\mathcal{O}_n}}^b(Z, \mathbf{k})}({}^p\tau_{<0}\mathcal{F}, \mathcal{A}).
\end{aligned}$$

By perverse degrees considerations, the first and the last spaces above are zero; we conclude that we have a canonical isomorphism

$$F_\nu(\mathcal{A}) = \mathrm{Hom}_{D_{c, G_{\mathcal{O}}_n}^b(Z, \mathbf{k})}(P_Z(\nu, \mathbf{k}), \mathcal{A}).$$

Thus  $P_Z(\nu, \mathbf{k})$  represents the functor  $F_\nu$  on  $\mathrm{P}_{G_{\mathcal{O}}}(Z, \mathbf{k})$ . Since the latter is exact (see Lemma 10.3),  $P_Z(\nu, \mathbf{k})$  is projective.  $\square$

For a fixed  $Z$ , there are only finitely many  $\nu \in X_*(T)$  such that  $T_\nu \cap Z \neq \emptyset$  (see Theorem 5.5(1)), so that the sum

$$\bigoplus_{\nu \in X_*(T)} P_Z(\nu, \mathbf{k})$$

involves finitely many nonzero terms; it therefore defines an object  $P_Z(\mathbf{k})$  of  $\mathrm{P}_{G_{\mathcal{O}}}(Z, \mathbf{k})$ . Theorem 10.4 and Proposition 12.1 imply that  $P_Z(\mathbf{k})$  represents the functor  $F$ . Since  $F$  is exact,  $P_Z(\mathbf{k})$  is projective. Since  $F$  is faithful,  $P_Z(\mathbf{k})$  is a generator of the category  $\mathrm{P}_{G_{\mathcal{O}}}(Z, \mathbf{k})$  (see e.g. [Bas, chap. II, §1]). Specifically, for each object  $\mathcal{A} \in \mathrm{P}_{G_{\mathcal{O}}}(Z, \mathbf{k})$ , there exists an epimorphism  $P_Z(\mathbf{k})^n \twoheadrightarrow \mathcal{A}$  for some  $n \geq 0$  (because the  $\mathbf{k}$ -module  $\mathrm{Hom}_{\mathrm{P}_{G_{\mathcal{O}}}(Z, \mathbf{k})}(P_Z(\mathbf{k}), \mathcal{A})$  is finitely generated).

## 12.2 Structure of the projective objects

Let  $Y \subset Z$  be closed subsets of  $\mathrm{Gr}_G$ , unions of finitely many  $G_{\mathcal{O}}$ -orbits. Let  $i : Y \rightarrow Z$  be the inclusion. The functor  ${}^p i^* := {}^p \mathcal{H}^0(i^*(?))$  maps  $\mathrm{P}_{G_{\mathcal{O}}}(Z, \mathbf{k})$  to  $\mathrm{P}_{G_{\mathcal{O}}}(Y, \mathbf{k})$  and is the left adjoint to the inclusion  $i_* : \mathrm{P}_{G_{\mathcal{O}}}(Y, \mathbf{k}) \rightarrow \mathrm{P}_{G_{\mathcal{O}}}(Z, \mathbf{k})$ .

**Proposition 12.2.** *There exists a canonical isomorphism  $P_Y(\mathbf{k}) \cong {}^p i^* P_Z(\mathbf{k})$  and a canonical surjection  $P_Z(\mathbf{k}) \twoheadrightarrow P_Y(\mathbf{k})$ .*

*Proof.* Since  $P_Z(\mathbf{k})$  represents  $F$  on  $\mathrm{P}_{G_{\mathcal{O}}}(Z, \mathbf{k})$ , its restriction  ${}^p i^* P_Z(\mathbf{k})$  represents  $F$  on the subcategory  $\mathrm{P}_{G_{\mathcal{O}}}(Y, \mathbf{k})$ . Since  $P_Y(\mathbf{k})$  also represents  $F$  on  $\mathrm{P}_{G_{\mathcal{O}}}(Y, \mathbf{k})$ , we get a canonical isomorphism  ${}^p i^* P_Z(\mathbf{k}) \xrightarrow{\sim} P_Y(\mathbf{k})$ .

Composing with the adjunction morphism  $P_Z(\mathbf{k}) \rightarrow i_* {}^p i^* P_Z(\mathbf{k})$ , we get a canonical map  $u : P_Z(\mathbf{k}) \rightarrow i_* P_Y(\mathbf{k})$ . Let  $f : i_* P_Y(\mathbf{k}) \rightarrow C$  be the cokernel of  $u$ . As a quotient of  $i_* P_Y(\mathbf{k})$ , the sheaf  $C$  is supported on  $Y$ , and since  $i_*$  is full,  $f$  is of the form  $i_* g$  for some map  $g : P_Y(\mathbf{k}) \rightarrow C'$  with  $C = i_* C'$ . Under the adjunction isomorphism

$$\mathrm{Hom}_{\mathrm{P}_{G_{\mathcal{O}}}(Y, \mathbf{k})}(P_Y(\mathbf{k}), C') \cong \mathrm{Hom}_{\mathrm{P}_{G_{\mathcal{O}}}(Y, \mathbf{k})}({}^p i^* P_Z(\mathbf{k}), C') \cong \mathrm{Hom}_{\mathrm{P}_{G_{\mathcal{O}}}(Z, \mathbf{k})}(P_Z(\mathbf{k}), i_* C'),$$

$g$  goes to  $(i_* g) \circ u = 0$ , hence  $g = 0$ , and we conclude that  $u$  is surjective.  $\square$

**Proposition 12.3.** *Let  $Z$  be a closed subset of  $\mathrm{Gr}_G$ , union of finitely many  $G_{\mathcal{O}}$ -orbits.*

1. *The object  $P_Z(\mathbf{k})$  admits a filtration in the abelian category  $\mathrm{P}_{G_{\mathcal{O}}}(Z, \mathbf{k})$  parametrized by  $\{\lambda \in X_*(T)^+ \mid \mathrm{Gr}_G^\lambda \subset Z\}$  (endowed with any total order refining  $\leq$ ) and with subquotients isomorphic to*

$$F(\mathcal{J}_*(\lambda, \mathbf{k})) \otimes_{\mathbf{k}} \mathcal{J}_!(\lambda, \mathbf{k}).$$

2. There exists a canonical isomorphism  $P_Z(\mathbf{k}) \cong \mathbf{k} \otimes_{\mathbf{Z}}^L P_Z(\mathbf{Z})$ .

3.  $F(P_Z(\mathbf{Z}))$  is a finitely generated free  $\mathbf{Z}$ -module and we have  $F(P_Z(\mathbf{k})) = \mathbf{k} \otimes_{\mathbf{Z}} F(P_Z(\mathbf{Z}))$ .

*Proof.* The proof proceeds by induction on the number of  $G_{\mathcal{O}}$ -orbits in  $Z$ .

Let us pick an orbit  $\text{Gr}_G^\lambda$  which is open in  $Z$ , let  $j : \text{Gr}_G^\lambda \rightarrow Z$  be the inclusion, and set  $Y = Z \setminus \text{Gr}_G^\lambda$ . Our goal is to analyze the kernel  $K(\mathbf{k})$  of the surjection constructed in Proposition 12.2:

$$0 \rightarrow K(\mathbf{k}) \rightarrow P_Z(\mathbf{k}) \rightarrow P_Y(\mathbf{k}) \rightarrow 0. \quad (12.2)$$

Let  $M$  be a finitely generated  $\mathbf{k}$ -module and let  $\mathcal{M} := \underline{M}_{\text{Gr}_G^\lambda}[\langle 2\rho, \lambda \rangle]$  be the shifted constant sheaf with stalk  $M$  on  $\text{Gr}_G^\lambda$ . From the truncation triangle

$$\mathcal{J}_*(\lambda, M) \rightarrow j_*\mathcal{M} \rightarrow {}^p\tau_{>0}(j_*\mathcal{M}) \xrightarrow{[1]}$$

we get an embedding

$$\text{Ext}_{D_{c, G_{\mathcal{O}}}^b(Z, \mathbf{k})}^i(P_Y(\mathbf{k}), \mathcal{J}_*(\lambda, M)) \hookrightarrow \text{Ext}_{D_{c, G_{\mathcal{O}}}^b(Z, \mathbf{k})}^i(P_Y(\mathbf{k}), j_*\mathcal{M})$$

for  $i \in \{0, 1\}$ , because  $\text{Ext}_{D_{c, G_{\mathcal{O}}}^b(Z, \mathbf{k})}^{i-1}(P_Y(\mathbf{k}), {}^p\tau_{>0}(j_*\mathcal{M})) = 0$  for (perverse) degree reasons. Since, by adjunction,

$$\text{Ext}_{D_{c, G_{\mathcal{O}}}^b(Z, \mathbf{k})}^i(P_Y(\mathbf{k}), j_*\mathcal{M}) \cong \text{Ext}_{D_{c, G_{\mathcal{O}}}^b(\text{Gr}_G^\lambda, \mathbf{k})}^i(j^*(P_Y(\mathbf{k})), \mathcal{M}) = 0,$$

we deduce that  $\text{Ext}_{D_{c, G_{\mathcal{O}}}^b(Z, \mathbf{k})}^i(P_Y(\mathbf{k}), \mathcal{J}_*(\lambda, M)) = 0$  for  $i \in \{0, 1\}$ . Applying the functor

$$\text{Hom}_{P_{G_{\mathcal{O}}}(Z, \mathbf{k})}(\cdot, \mathcal{J}_*(\lambda, M))$$

to the exact sequence (12.2) and using this vanishing, we get an isomorphism

$$\text{Hom}_{P_{G_{\mathcal{O}}}(Z, \mathbf{k})}(P_Z(\mathbf{k}), \mathcal{J}_*(\lambda, M)) \cong \text{Hom}_{P_{G_{\mathcal{O}}}(Z, \mathbf{k})}(K(\mathbf{k}), \mathcal{J}_*(\lambda, M)),$$

and thus

$$\begin{aligned} \text{Hom}_{D_c^b(\text{Gr}_G^\lambda, \mathbf{k})}(j^*K(\mathbf{k}), \mathcal{M}) &\cong \text{Hom}_{P_{G_{\mathcal{O}}}(Z, \mathbf{k})}(K(\mathbf{k}), {}^p\tau_{\leq 0}(j_*\mathcal{M})) \\ &= \text{Hom}_{P_{G_{\mathcal{O}}}(Z, \mathbf{k})}(K(\mathbf{k}), \mathcal{J}_*(\lambda, M)) \\ &\cong \text{Hom}_{P_{G_{\mathcal{O}}}(Z, \mathbf{k})}(P_Z(\mathbf{k}), \mathcal{J}_*(\lambda, M)) \\ &\cong F(\mathcal{J}_*(\lambda, M)) \\ &\cong F(\mathcal{J}_*(\lambda, \mathbf{k})) \otimes_{\mathbf{k}} M && \text{by (11.1)} \\ &\cong \text{Hom}_{\mathbf{k}}(F(\mathcal{J}_*(\lambda, \mathbf{k}))^*, M) && \text{by Proposition 11.1.} \end{aligned}$$

Since  $K(\mathbf{k})$  is an object of  $P_{G_{\mathcal{O}}}(Z, \mathbf{k})$  and  $\text{Gr}_G^\lambda$  is open in  $Z$ , the restriction  $j^*K(\mathbf{k})$  is a shifted local system on  $\text{Gr}_G^\lambda$ . From the isomorphism

$$\text{Hom}_{D_c^b(\text{Gr}_G^\lambda, \mathbf{k})}(j^*K(\mathbf{k}), \mathcal{M}) \cong \text{Hom}_{\mathbf{k}}(F(\mathcal{J}_*(\lambda, \mathbf{k}))^*, M)$$

proved above for any  $M$ , we deduce that

$$j^* K(\mathbf{k}) \cong F(\mathcal{J}_*(\lambda, \mathbf{k}))^* \otimes_{\mathbf{k}} \underline{\mathbf{k}}_{\text{Gr}^\lambda}[\langle 2\rho, \lambda \rangle].$$

The adjunction map  $j_! j^* K(\mathbf{k}) \rightarrow K(\mathbf{k})$  then gives, after truncation in nonnegative perverse degrees, a map

$$\alpha : F(\mathcal{J}_*(\lambda, \mathbf{k}))^* \otimes_{\mathbf{k}} \mathcal{J}_!(\lambda, \mathbf{k}) \rightarrow K(\mathbf{k})$$

in  $\mathbf{P}_{G_{\mathcal{O}}}(Z, \mathbf{k})$  (see again (11.1)).

Since  $j^*(\alpha)$  is an isomorphism, the cokernel  $C$  of  $\alpha$  is supported on  $Y$ . Applying the functor  $\text{Hom}_{\mathbf{P}_{G_{\mathcal{O}}}(Z, \mathbf{k})}(\mathcal{?}, C)$  to the exact sequence (12.2), we get an exact sequence

$$\begin{aligned} 0 \rightarrow \text{Hom}_{\mathbf{P}_{G_{\mathcal{O}}}(Z, \mathbf{k})}(P_Y(\mathbf{k}), C) &\xrightarrow{\beta} \text{Hom}_{\mathbf{P}_{G_{\mathcal{O}}}(Z, \mathbf{k})}(P_Z(\mathbf{k}), C) \\ &\rightarrow \text{Hom}_{\mathbf{P}_{G_{\mathcal{O}}}(Z, \mathbf{k})}(K(\mathbf{k}), C) \rightarrow \text{Ext}_{\mathbf{P}_{G_{\mathcal{O}}}(Z, \mathbf{k})}^1(P_Y(\mathbf{k}), C). \end{aligned}$$

Since  $C$  belongs to  $\mathbf{P}_{G_{\mathcal{O}}}(Y, \mathbf{k})$ , the map  $\beta$  is an isomorphism between two copies of  $F(C)$ . Moreover, using (4.1) we have

$$\begin{aligned} \text{Ext}_{\mathbf{P}_{G_{\mathcal{O}}}(Z, \mathbf{k})}^1(P_Y(\mathbf{k}), C) &\cong \text{Ext}_{D_{c, G_{\mathcal{O}}}^b(Z, \mathbf{k})}^1(P_Y(\mathbf{k}), C) \\ &\cong \text{Ext}_{D_{c, G_{\mathcal{O}}}^b(Y, \mathbf{k})}^1(P_Y(\mathbf{k}), C) \cong \text{Ext}_{\mathbf{P}_{G_{\mathcal{O}}}(Y, \mathbf{k})}^1(P_Y(\mathbf{k}), C) = 0 \end{aligned}$$

since  $P_Y(\mathbf{k})$  is projective in  $\mathbf{P}_{G_{\mathcal{O}}}(Y, \mathbf{k})$ . It follows that  $\text{Hom}_{\mathbf{P}_{G_{\mathcal{O}}}(Z, \mathbf{k})}(K(\mathbf{k}), C) = 0$ , and therefore that  $C = 0$ . This shows that  $\alpha$  is an epimorphism. We will see shortly that it is in fact an isomorphism.

Let  $K'(\mathbf{k})$  be the kernel of  $\alpha$ , so that we have an exact sequence

$$0 \rightarrow K'(\mathbf{k}) \rightarrow F(\mathcal{J}_*(\lambda, \mathbf{k}))^* \otimes_{\mathbf{k}} \mathcal{J}_!(\lambda, \mathbf{k}) \xrightarrow{\alpha} K(\mathbf{k}) \rightarrow 0 \quad (12.3)$$

in  $\mathbf{P}_{G_{\mathcal{O}}}(Z, \mathbf{k})$ . As for  $C$  above, since  $j^*(\alpha)$  is an isomorphism,  $K'(\mathbf{k})$  is supported on  $Y$ .

Now we consider the case  $\mathbf{k} = \mathbf{Z}$ . As a consequence of Lemma 11.5 (see also the remarks in §11.1), the perverse sheaf  $\mathcal{J}_!(\lambda, \mathbf{Z})$  does not have any subobject supported on  $Y$ , and therefore  $K'(\mathbf{Z}) = 0$ . Thus

$$K(\mathbf{Z}) \cong F(\mathcal{J}_*(\lambda, \mathbf{Z}))^* \otimes_{\mathbf{Z}} \mathcal{J}_!(\lambda, \mathbf{Z}),$$

and from (12.2), we easily get statement (1) by induction in this case.

We come back to the general case. Since  $\mathcal{J}_!(\lambda, \mathbf{k}) \cong \mathbf{k} \otimes_{\mathbf{Z}}^L \mathcal{J}_!(\lambda, \mathbf{Z})$  (see Proposition 11.3), each object

$$\mathbf{k} \otimes_{\mathbf{Z}}^L \left( F(\mathcal{J}_*(\lambda, \mathbf{Z})) \otimes_{\mathbf{Z}} \mathcal{J}_!(\lambda, \mathbf{Z}) \right) \cong F(\mathcal{J}_*(\lambda, \mathbf{k})) \otimes_{\mathbf{k}} \mathcal{J}_!(\lambda, \mathbf{k})$$

is a perverse sheaf. The complex  $\mathbf{k} \otimes_{\mathbf{Z}}^L P_Z(\mathbf{Z})$  is thus an iterated extension (in the sense of triangulated categories) of perverse sheaves, and is therefore perverse. On the other hand, for each  $\mathcal{A} \in \mathbf{P}_{G_{\mathcal{O}}}(Z, \mathbf{k})$ , we have by [KS, First formula in (2.6.8)]

$$\begin{aligned} \text{Hom}_{\mathbf{P}_{G_{\mathcal{O}}}(Z, \mathbf{k})}(\mathbf{k} \otimes_{\mathbf{Z}}^L P_Z(\mathbf{Z}), \mathcal{A}) &= \text{Hom}_{D_{c, G_{\mathcal{O}}}^b(Z, \mathbf{Z})}(P_Z(\mathbf{Z}), R\mathcal{H}om_{\mathbf{k}}(\underline{\mathbf{k}}_{\mathbf{Z}}, \mathcal{A})) \\ &= \text{Hom}_{\mathbf{P}_{G_{\mathcal{O}}}(Z, \mathbf{Z})}(P_Z(\mathbf{Z}), \mathcal{A}) = F(\mathcal{A}), \end{aligned}$$

naturally in  $\mathcal{A}$ . Thus  $P_Z(\mathbf{k})$  and  $\mathbf{k} \otimes_{\mathbf{Z}}^L P_Z(\mathbf{Z})$  both represent  $\mathbf{F}$  on  $\mathbf{P}_{G_{\mathcal{O}}}(Z, \mathbf{k})$ , and therefore  $P_Z(\mathbf{k}) = \mathbf{k} \otimes_{\mathbf{Z}}^L P_Z(\mathbf{Z})$ , as claimed in statement (2).

Using this description for  $P_Z(\mathbf{k})$  and  $P_Y(\mathbf{k})$  in (12.2), we get  $K(\mathbf{k}) = \mathbf{k} \otimes_{\mathbf{Z}}^L K(\mathbf{Z})$ . Turning to (12.3), we see that  $K'(\mathbf{k}) = \mathbf{k} \otimes_{\mathbf{Z}}^L K'(\mathbf{Z})$ ; and since  $K'(\mathbf{Z}) = 0$ , we eventually get that  $K'(\mathbf{k}) = 0$ , or in other words that

$$K(\mathbf{k}) \cong \mathbf{F}(\mathcal{J}_*(\lambda, \mathbf{k}))^* \otimes_{\mathbf{k}} \mathcal{J}_!(\lambda, \mathbf{k}).$$

This information leads to statement (1) by induction.

Finally, statement (3) follows from the discussion above and Proposition 11.1 (since an extension of free  $\mathbf{Z}$ -modules is free).  $\square$

### 12.3 Consequence: highest weight structure

In this subsection we assume that  $\mathbf{k}$  is a field. Recall the notion of highest weight category, whose definition is spelled out e.g. in [Ri, Definition 7.1]. (These conditions are obvious extensions of those considered in [BGS, §3.2], which are inspired by earlier work of Cline–Parshall–Scott [CPS].) Our goal in this subsection is to prove the following proposition.

**Proposition 12.4.** *The category  $\mathbf{P}_{G_{\mathcal{O}}}(\mathrm{Gr}_G, \mathbf{k})$ , together with the “weight poset”  $(X_*(T)^+, \leq)$ , the “standard objects”  $\{\mathcal{J}_!(\lambda, \mathbf{k}) : \lambda \in X_*(T)^+\}$  and the “costandard objects”  $\{\mathcal{J}_*(\lambda, \mathbf{k}) : \lambda \in X_*(T)^+\}$ , is a highest weight category.*

*Proof.* Condition (1) in [Ri, Definition 7.1] is obvious, and conditions (2)–(4) are easily checked using adjunction and the general theory of perverse sheaves. Hence to conclude it suffices to prove that for any  $\lambda, \mu \in X_*(T)^+$  we have  $\mathrm{Ext}_{\mathbf{P}_{G_{\mathcal{O}}}(\mathrm{Gr}_G, \mathbf{k})}^2(\mathcal{J}_!(\lambda, \mathbf{k}), \mathcal{J}_*(\mu, \mathbf{k})) = 0$ . And for this it suffices to prove that for any finite closed union of  $G_{\mathcal{O}}$ -orbits  $Z \subset \mathrm{Gr}_G$  containing  $\mathrm{Gr}_G^\lambda$  and  $\mathrm{Gr}_G^\mu$  we have  $\mathrm{Ext}_{\mathbf{P}_{G_{\mathcal{O}}}(Z, \mathbf{k})}^2(\mathcal{J}_!(\lambda, \mathbf{k}), \mathcal{J}_*(\mu, \mathbf{k})) = 0$ . Before that, let us note that we have

$$\mathrm{Ext}_{\mathbf{P}_{G_{\mathcal{O}}}(Z, \mathbf{k})}^1(\mathcal{J}_!(\lambda, \mathbf{k}), \mathcal{J}_*(\mu, \mathbf{k})) = 0. \tag{12.4}$$

In fact, using (4.1) we can assume that  $Z = \overline{\mathrm{Gr}_G^\lambda} \cup \overline{\mathrm{Gr}_G^\mu}$ . Then the vanishing follows from the fact that either  $\mathcal{J}_!(\lambda, \mathbf{k})$  is projective (if  $\mu \not\prec \lambda$ ) or  $\mathcal{J}_*(\mu, \mathbf{k})$  is injective (if  $\lambda \not\prec \mu$ ) in  $\mathbf{P}_{G_{\mathcal{O}}}(\overline{\mathrm{Gr}_G^\lambda} \cup \overline{\mathrm{Gr}_G^\mu}, \mathbf{k})$ .

We denote by  $Q_{Z, \lambda}$  the projective cover of the simple object  $\mathcal{J}_{!*}(\lambda, \mathbf{k})$  in the abelian category  $\mathbf{P}_{G_{\mathcal{O}}}(Z, \mathbf{k})$ . (This category is equivalent to the category of finite-dimensional modules over a finite-dimensional  $\mathbf{k}$ -algebra, see §13.1 below for details; in particular we can indeed consider projective covers.) We claim that  $Q_{Z, \lambda}$  has a filtration with  $\mathcal{J}_!(\lambda, \mathbf{k})$  at the top and with subquotients of the form  $\mathcal{J}_!(\nu, \mathbf{k})$  for some  $\nu$ 's in  $X_*(T)^+$ .

This property is true if  $\mathrm{Gr}_G^\lambda$  is open in  $Z$ , since then  $Q_{Z, \lambda} = \mathcal{J}_!(\lambda, \mathbf{k})$  by condition (3) in the definition of a highest weight category. When  $\mathrm{Gr}_G^\lambda$  is not open in  $Z$ , we proceed along the lines of the proof of Proposition 12.3. We note that  $Q_{Z, \lambda}$  is a direct summand of  $P_Z(\mathbf{k})$ , for

the latter is a projective generator of  $\mathbf{P}_{G_{\mathcal{O}}}(Z, \mathbf{k})$ . Let  $\mathrm{Gr}'_G \subset Z$  be an open  $G_{\mathcal{O}}$ -orbit and set  $Y := Z \setminus \mathrm{Gr}'_G$ . The short exact sequence (12.2) then induces a short exact sequence

$$0 \rightarrow K' \rightarrow Q_{Z,\lambda} \rightarrow Q'_{Y,\lambda} \rightarrow 0.$$

Here  $Q'_{Y,\lambda} := i_*^p \mathcal{H}^0(i^* Q_{Z,\lambda})$ , and  $K'$  is a direct summand of the sheaf  $K(\mathbf{k})$  in (12.2), which is a direct sum of copies of  $\mathcal{J}_!(\nu, \mathbf{k})$ . Since the perverse sheaf  $\mathcal{J}_!(\nu, \mathbf{k})$  is indecomposable,  $K'$  must also be a direct sum of copies of  $\mathcal{J}_!(\nu, \mathbf{k})$ . Further, there is no nonzero map  $\mathcal{J}_!(\nu, \mathbf{k}) \rightarrow \mathcal{J}_{!*}(\lambda, \mathbf{k})$  (see §11.1), so the kernel of the covering map

$$Q_{Z,\lambda} \twoheadrightarrow \mathcal{J}_{!*}(\lambda, \mathbf{k})$$

contains  $K'$ , whence a surjective map

$$Q'_{Y,\lambda} \twoheadrightarrow \mathcal{J}_{!*}(\lambda, \mathbf{k}).$$

Moreover  $Q'_{Y,\lambda}$  is a direct summand of the term  $P_Y$  appearing in (12.2), so is a projective object of  $\mathbf{P}_{G_{\mathcal{O}}}(Y, \mathbf{k})$ . Lastly, the projectivity of  $Q_{Z,\lambda}$  gives a surjective map

$$\mathrm{Hom}_{\mathbf{P}_{G_{\mathcal{O}}}(Z, \mathbf{k})}(Q_{Z,\lambda}, Q_{Z,\lambda}) \twoheadrightarrow \mathrm{Hom}_{\mathbf{P}_{G_{\mathcal{O}}}(Z, \mathbf{k})}(Q_{Z,\lambda}, Q'_{Y,\lambda}),$$

and since by adjunction we can identify

$$\begin{aligned} \mathrm{Hom}_{\mathbf{P}_{G_{\mathcal{O}}}(Z, \mathbf{k})}(Q_{Z,\lambda}, Q'_{Y,\lambda}) &= \mathrm{Hom}_{\mathbf{P}_{G_{\mathcal{O}}}(Z, \mathbf{k})}(Q_{Z,\lambda}, i_*^p \mathcal{H}^0(i^* Q_{Z,\lambda})) \\ &\cong \mathrm{Hom}_{\mathbf{P}_{G_{\mathcal{O}}}(Y, \mathbf{k})}(i^* Q_{Z,\lambda}, \mathcal{H}^0(i^* Q_{Z,\lambda})) \\ &= \mathrm{Hom}_{\mathbf{P}_{G_{\mathcal{O}}}(Y, \mathbf{k})}(Q'_{Y,\lambda}, Q'_{Y,\lambda}), \end{aligned}$$

we obtain the existence of a surjective ring homomorphism

$$\mathrm{End}_{\mathbf{P}_{G_{\mathcal{O}}}(Z, \mathbf{k})}(Q_{Z,\lambda}) \twoheadrightarrow \mathrm{End}_{\mathbf{P}_{G_{\mathcal{O}}}(Y, \mathbf{k})}(Q'_{Y,\lambda}).$$

Therefore  $Q'_{Y,\lambda}$  has a local endomorphism ring, so is indecomposable. We finally conclude that  $Q'_{Y,\lambda}$  can be identified with the projective cover  $Q_{Y,\lambda}$  of  $\mathcal{J}_{!*}(\lambda, \mathbf{k})$  in  $\mathbf{P}_{G_{\mathcal{O}}}(Y, \mathbf{k})$ . To sum up, we have a short exact sequence

$$0 \rightarrow K' \rightarrow Q_{Z,\lambda} \rightarrow Q_{Y,\lambda} \rightarrow 0,$$

where  $K'$  is a direct sum of copies of  $\mathcal{J}_!(\nu, \mathbf{k})$ . Our claim now easily follows by induction on the number of  $G_{\mathcal{O}}$ -orbits in  $Z$ .

At this point, we have shown the existence of a short exact sequence

$$0 \rightarrow R_{Z,\lambda} \rightarrow Q_{Z,\lambda} \rightarrow \mathcal{J}_!(\lambda, \mathbf{k}) \rightarrow 0$$

such that  $R_{Z,\lambda}$  admits a filtration with subquotients of the form  $\mathcal{J}_!(\nu, \mathbf{k})$  for some  $\nu$ 's in  $X_*(T)^+$ . We then consider the exact sequence

$$\mathrm{Ext}_{\mathbf{P}_{G_{\mathcal{O}}}(Z, \mathbf{k})}^1(R_{Z,\lambda}, \mathcal{J}_*(\mu, \mathbf{k})) \rightarrow \mathrm{Ext}_{\mathbf{P}_{G_{\mathcal{O}}}(Z, \mathbf{k})}^2(\mathcal{J}_!(\lambda, \mathbf{k}), \mathcal{J}_*(\mu, \mathbf{k})) \rightarrow \mathrm{Ext}_{\mathbf{P}_{G_{\mathcal{O}}}(Z, \mathbf{k})}^2(Q_{Z,\lambda}, \mathcal{J}_*(\mu, \mathbf{k}))$$

obtained by applying the functor  $\mathrm{Hom}(\mathcal{J}_*(\mu, \mathbf{k}), ?)$  to this exact sequence. Here the first term vanishes because  $\mathrm{Ext}_{\mathbf{P}_{G_{\mathcal{O}}}(Z, \mathbf{k})}^1(\mathcal{J}_!(\nu, \mathbf{k}), \mathcal{J}_*(\mu, \mathbf{k})) = 0$  for any  $\nu$ , and the third term vanishes because  $Q_{Z,\lambda}$  is projective. We deduce the desired vanishing, and finally the proposition.  $\square$



## 13 Construction of the group scheme

In this section, we construct an affine  $\mathbf{k}$ -group scheme  $\tilde{G}_{\mathbf{k}}$  and an equivalence of monoidal categories  $\mathbf{S}$  from  $P_{G_{\mathcal{O}}}(\mathrm{Gr}_G, \mathbf{k})$  to the category  $\mathrm{Rep}_{\mathbf{k}}(\tilde{G}_{\mathbf{k}})$  of representations of this group scheme on finitely generated  $\mathbf{k}$ -modules. Along the way, we will show that the function algebra  $\mathbf{Z}[\tilde{G}_{\mathbf{Z}}]$  is a free  $\mathbf{Z}$ -module and that  $\mathbf{k}[\tilde{G}_{\mathbf{k}}] \cong \mathbf{k} \otimes_{\mathbf{Z}} \mathbf{Z}[\tilde{G}_{\mathbf{Z}}]$ . (These facts will play a key role in Section 14 below.)

### 13.1 Abelian reconstruction

Let us recall the following variant of Gabriel and Mitchell's theorem. Here we will denote by  $\mathrm{mod}_{\mathbf{k}}$  the category of *all* (i.e. not necessarily finitely generated)  $\mathbf{k}$ -modules.

**Proposition 13.1.** *Let  $\mathcal{C}$  be a  $\mathbf{k}$ -linear abelian category. Let  $P$  be a projective object and let  $A = \mathrm{End}_{\mathcal{C}}(P)$ . Let  $\mathrm{Modfp}_P$  be the full subcategory of  $\mathcal{C}$  consisting of those objects that admit a presentation of the form  $P_1 \rightarrow P_0 \rightarrow M \rightarrow 0$ , where  $P_1$  and  $P_0$  are direct sums of finitely many copies of  $P$ . Let also  $\mathrm{Modfp}_A^r$  be the category of finitely presented right  $A$ -modules.*

1. *The functor  $G = \mathrm{Hom}_{\mathcal{C}}(P, ?)$  defines an equivalence of categories from  $\mathrm{Modfp}_P$  to  $\mathrm{Modfp}_A^r$ .*
2. *The endomorphism ring of the functor  $G : \mathrm{Modfp}_P \rightarrow \mathrm{mod}_{\mathbf{k}}$  is canonically isomorphic to  $A^{\mathrm{op}}$ .*

*Proof.* Statement (1) is proved as in [ARS, Proposition II.2.5]. The proof of (2) is similar to that of the corresponding claim in Proposition 2.2.  $\square$

Let  $Z$  be a closed subset of  $\mathrm{Gr}_G$ , union of finitely many  $G_{\mathcal{O}}$ -orbits. As we mentioned at the end of §12.1, each object in  $P_{G_{\mathcal{O}}}(Z, \mathbf{k})$  is a quotient of a module  $P_Z(\mathbf{k})^n$ , so each object  $\mathcal{A} \in P_{G_{\mathcal{O}}}(Z, \mathbf{k})$  admits a presentation of the form  $P_1 \rightarrow P_0 \rightarrow \mathcal{A} \rightarrow 0$  with  $P_1$  and  $P_0$  isomorphic to direct sums of finitely many copies of  $P_Z(\mathbf{k})$ . Moreover, the ring

$$A_Z(\mathbf{k}) := \mathrm{End}_{P_{G_{\mathcal{O}}}(Z, \mathbf{k})}(P_Z(\mathbf{k}))^{\mathrm{op}}$$

is a finitely generated  $\mathbf{k}$ -module, hence is left Noetherian, so that each finitely generated left  $A_Z(\mathbf{k})$ -module is finitely presented. In the present situation, Proposition 13.1 thus states that the functor  $F = \mathrm{Hom}_{P_{G_{\mathcal{O}}}(Z, \mathbf{k})}(P_Z(\mathbf{k}), ?)$  induces an equivalence of categories  $\mathbf{S}_Z$ , as depicted on the following diagram:

$$\begin{array}{ccc}
 P_{G_{\mathcal{O}}}(Z, \mathbf{k}) & \xrightarrow[\sim]{\mathbf{S}_Z} & \mathrm{Mod}_{A_Z(\mathbf{k})} \\
 & \searrow F & \swarrow \text{forget} \\
 & & \mathrm{Mod}_{\mathbf{k}}
 \end{array}$$

Let  $i : Y \hookrightarrow Z$  be the inclusion of a closed subset, union of (finitely many)  $G_{\mathcal{O}}$ -orbits. The perverse restriction functor

$${}^p i^* = {}^p \mathcal{H}^0(i^*(?)) : P_{G_{\mathcal{O}}}(Z, \mathbf{k}) \rightarrow P_{G_{\mathcal{O}}}(Y, \mathbf{k})$$

is left adjoint to the extension-by-zero functor  $i_* : P_{G_{\mathcal{O}}}(Y, \mathbf{k}) \rightarrow P_{G_{\mathcal{O}}}(Z, \mathbf{k})$ . Further, this functor sends  $P_Z(\mathbf{k})$  to  $P_Y(\mathbf{k})$  (see Proposition 12.2) and thus induces a morphism of algebras  $f_Y^Z$  from  $A_Z(\mathbf{k}) = \text{End}_{P_{G_{\mathcal{O}}}(Z, \mathbf{k})}(P_Z(\mathbf{k}))^{\text{op}}$  to  $A_Y(\mathbf{k}) = \text{End}_{P_{G_{\mathcal{O}}}(Y, \mathbf{k})}(P_Y(\mathbf{k}))^{\text{op}}$ . By functoriality and adjointness, for each  $\mathcal{A} \in P_{G_{\mathcal{O}}}(Y, \mathbf{k})$ , the action of an element  $a \in A_Z(\mathbf{k})$  on

$$S_Z(\mathcal{A}) = \text{Hom}_{P_{G_{\mathcal{O}}}(Z, \mathbf{k})}(P_Z(\mathbf{k}), i_* \mathcal{A})$$

coincides with the action of  $f_Y^Z(a) \in A_Y(\mathbf{k})$  on

$$S_Y(\mathcal{A}) = \text{Hom}_{P_{G_{\mathcal{O}}}(Y, \mathbf{k})}(P_Y(\mathbf{k}), \mathcal{A}).$$

As a consequence, the diagram

$$\begin{array}{ccc} P_{G_{\mathcal{O}}}(Y, \mathbf{k}) & \xrightarrow{S_Y} & \text{Mod}_{A_Y(\mathbf{k})} \\ i_* \downarrow & & \downarrow (f_Y^Z)^* \\ P_{G_{\mathcal{O}}}(Z, \mathbf{k}) & \xrightarrow{S_Z} & \text{Mod}_{A_Z(\mathbf{k})} \\ & \searrow F & \swarrow \text{forget} \\ & \text{Mod}_{\mathbf{k}} & \end{array}$$

commutes, where  $(f_Y^Z)^*$  is the restriction-of-scalars functor associated with  $f_Y^Z$ .

Since  $A_Z(\mathbf{k}) \cong F(P_Z(\mathbf{k}))$  is a finitely generated free  $\mathbf{k}$ -module (see Proposition 12.3(3)), the same dictionary as the one set up in §2.2 can be used in the present context. Namely, we may endow the dual  $\mathbf{k}$ -module

$$B_Z(\mathbf{k}) := \text{Hom}_{\mathbf{k}}(A_Z(\mathbf{k}), \mathbf{k})$$

with the structure of a  $\mathbf{k}$ -coalgebra and identify the category  $\text{Mod}_{A_Z(\mathbf{k})}$  with the category  $\text{Comod}_{B_Z(\mathbf{k})}$  of right  $B_Z(\mathbf{k})$ -comodules that are finitely generated over  $\mathbf{k}$ . The dual of the algebra map  $f_Y^Z : A_Z(\mathbf{k}) \rightarrow A_Y(\mathbf{k})$  is a coalgebra map  $B_Y(\mathbf{k}) \rightarrow B_Z(\mathbf{k})$ , and we can consider the limit  $B(\mathbf{k})$  of the directed system of coalgebras thus constructed (over the poset of closed finite unions of  $G_{\mathcal{O}}$ -orbits under inclusion).

**Proposition 13.2.** *The  $\mathbf{Z}$ -module  $B(\mathbf{Z})$  is free, and we have a canonical isomorphism of  $\mathbf{k}$ -coalgebras  $B(\mathbf{k}) \cong \mathbf{k} \otimes_{\mathbf{Z}} B(\mathbf{Z})$ .*

*Proof.* The freeness assertion follows from Proposition 12.3(1) and its proof. The second assertion follows directly from Proposition 12.3(3).  $\square$

We eventually get an equivalence of abelian categories  $\mathcal{S}$  and a commutative diagram

$$\begin{array}{ccc} P_{G_{\mathcal{O}}}(\text{Gr}_G, \mathbf{k}) & \xrightarrow[\sim]{\mathcal{S}} & \text{Comod}_{B(\mathbf{k})} \\ & \searrow F & \swarrow \text{forget} \\ & \text{Mod}_{\mathbf{k}} & \end{array}$$

### 13.2 Tannakian reconstruction

We now want to endow  $B(\mathbf{k})$  with the structure of a Hopf algebra, and upgrade  $\mathbf{S}$  to an equivalence of monoidal categories.

For  $\lambda \in X_*(T)^+$ , we set  $Z_\lambda := \overline{\text{Gr}_G^\lambda}$  and we shorten the notation  $P_{Z_\lambda}(\mathbf{k})$ ,  $A_{Z_\lambda}(\mathbf{k})$  and  $B_{Z_\lambda}(\mathbf{k})$  to respectively  $P_\lambda(\mathbf{k})$ ,  $A_\lambda(\mathbf{k})$  and  $B_\lambda(\mathbf{k})$ . We note that for  $\lambda, \mu \in X_*(T)^+$ , the perverse sheaf  $\mathcal{A} \star \mathcal{B}$  belongs to  $\text{P}_{G_{\mathcal{O}}}(Z_{\lambda+\mu}, \mathbf{k})$  whenever  $\mathcal{A} \in \text{P}_{G_{\mathcal{O}}}(Z_\lambda, \mathbf{k})$  and  $\mathcal{B} \in \text{P}_{G_{\mathcal{O}}}(Z_\mu, \mathbf{k})$ .

An element  $a \in A_{\lambda+\mu}(\mathbf{k})$  defines an endomorphism of the bifunctor

$$\text{Hom}_{\text{P}_{G_{\mathcal{O}}}(Z_{\lambda+\mu}, \mathbf{k})}(P_{\lambda+\mu}(\mathbf{k}), ?\star?) : \text{P}_{G_{\mathcal{O}}}(Z_\lambda, \mathbf{k}) \times \text{P}_{G_{\mathcal{O}}}(Z_\mu, \mathbf{k}) \rightarrow \text{Mod}_{\mathbf{k}}.$$

Now since  $\mathbf{F}$  is a tensor functor, we have a canonical isomorphism of bifunctors

$$\begin{aligned} \text{Hom}_{\text{P}_{G_{\mathcal{O}}}(Z_{\lambda+\mu}, \mathbf{k})}(P_{\lambda+\mu}(\mathbf{k}), ?\star?) &\cong \mathbf{F}(?\star?) \\ &\cong \mathbf{F}(?) \otimes_{\mathbf{k}} \mathbf{F}(?) \\ &\cong \text{Hom}_{\text{P}_{G_{\mathcal{O}}}(Z_\lambda, \mathbf{k})}(P_\lambda(\mathbf{k}), ?) \otimes_{\mathbf{k}} \text{Hom}_{\text{P}_{G_{\mathcal{O}}}(Z_\mu, \mathbf{k})}(P_\mu(\mathbf{k}), ?). \end{aligned}$$

By an immediate generalization of Proposition 13.1(2), our element  $a$  thus defines an element of the ring  $A_\lambda(\mathbf{k}) \otimes_{\mathbf{k}} A_\mu(\mathbf{k})$ . This leads to a ring homomorphism

$$A_{\lambda+\mu}(\mathbf{k}) \rightarrow A_\lambda(\mathbf{k}) \otimes_{\mathbf{k}} A_\mu(\mathbf{k}).$$

Dualizing, we get a coalgebra map

$$B_\lambda(\mathbf{k}) \otimes_{\mathbf{k}} B_\mu(\mathbf{k}) \rightarrow B_{\lambda+\mu}(\mathbf{k}).$$

Taking the limit of these maps over  $\lambda$  and  $\mu$ , this construction provides a multiplication map on  $B(\mathbf{k})$ , which can be seen to be associative and commutative.

On the other hand, it is clear that  $B_0(\mathbf{k}) = \mathbf{k}$ , so that the natural morphism  $B_0(\mathbf{k}) \rightarrow B(\mathbf{k})$  defines a canonical element in  $B(\mathbf{k})$  which is easily seen to be a unit. Altogether, we have thus constructed a bialgebra structure on  $B(\mathbf{k})$ . Since our construction is based on natural transformations of functors, the functor  $\mathbf{S}$  is easily seen to be compatible with the monoidal structures.

If we set

$$\tilde{G}_{\mathbf{k}} := \text{Spec}(B(\mathbf{k})),$$

then the bialgebra structure on  $B(\mathbf{k})$  translates to a structure of monoid scheme on  $\tilde{G}_{\mathbf{k}}$ . To conclude, what remains to show is that  $B(\mathbf{k})$  admits an antipode, or in other words that  $\tilde{G}_{\mathbf{k}}$  is a group scheme. Since, by Proposition 13.2, we have

$$\tilde{G}_{\mathbf{k}} \cong \text{Spec}(\mathbf{k}) \times_{\text{Spec}(\mathbf{Z})} \tilde{G}_{\mathbf{Z}}, \tag{13.1}$$

it suffices to prove this when  $\mathbf{k} = \mathbf{Z}$ . This will be done in Proposition 13.4 below.

**Lemma 13.3.** *Assume that  $\mathbf{k} = \mathbf{Z}$ . If  $M$  is an object of  $\text{P}_{G_{\mathcal{O}}}(\text{Gr}_G, \mathbf{Z})$  such that  $\mathbf{F}(M)$  is free of rank 1, then there exists  $M^*$  in  $\text{P}_{G_{\mathcal{O}}}(\text{Gr}_G, \mathbf{Z})$  such that  $M \star M^*$  is the unit object.*

*Proof.* Consider the object  $\mathbf{Q} \otimes_{\mathbf{Z}}^L M \in \mathcal{P}_{G_{\mathcal{O}}}(\mathrm{Gr}_G, \mathbf{Q})$ . This object is such that  $F(\mathbf{Q} \otimes_{\mathbf{Z}}^L M)$  has dimension 1, where here  $F$  means the tensor functor for coefficients  $\mathbf{Q}$ ; as noticed at the beginning of Section 9, this implies that  $\mathbf{Q} \otimes_{\mathbf{Z}}^L M \cong \mathcal{J}_{!*}(\lambda, \mathbf{Q})$  for some  $\lambda \in X_*(T)^+$  orthogonal to all the roots of  $G$ , i.e. such that  $\mathrm{Gr}_G^\lambda = \{L_\lambda\}$ .

By the results in §13.1, we have an embedding

$$f : \mathrm{Hom}_{\mathcal{P}_{G_{\mathcal{O}}}(\mathrm{Gr}_G, \mathbf{Z})}(M, \mathcal{J}_{!*}(\lambda, \mathbf{Z})) \hookrightarrow \mathrm{Hom}_{\mathbf{Z}}(F(M), F(\mathcal{J}_{!*}(\lambda, \mathbf{Z})))$$

whose image is the set of all the  $B(\mathbf{Z})$ -comodule maps. Since  $F(M) \cong \mathbf{Z} \cong F(\mathcal{J}_{!*}(\lambda, \mathbf{Z}))$ , the codomain of  $f$  is a free  $\mathbf{Z}$ -module of rank 1. Therefore  $\mathrm{Hom}_{\mathcal{P}_{G_{\mathcal{O}}}(\mathrm{Gr}_G, \mathbf{Z})}(M, \mathcal{J}_{!*}(\lambda, \mathbf{Z}))$  is either 0 or a free  $\mathbf{Z}$ -module of rank 1; since

$$\mathbf{Q} \otimes_{\mathbf{Z}} \mathrm{Hom}_{\mathcal{P}_{G_{\mathcal{O}}}(\mathrm{Gr}_G, \mathbf{Z})}(M, \mathcal{J}_{!*}(\lambda, \mathbf{Z})) \cong \mathrm{Hom}_{\mathcal{P}_{G_{\mathcal{O}}}(\mathrm{Gr}_G, \mathbf{Q})}(\mathbf{Q} \otimes_{\mathbf{Z}}^L M, \mathcal{J}_{!*}(\lambda, \mathbf{Q})) = \mathbf{Q},$$

it is in fact free of rank 1. We see moreover that the cokernel of  $f$  is either 0 or a cyclic group. Now if a nonzero multiple of a  $\mathbf{Z}$ -linear map  $f : F(M) \rightarrow F(\mathcal{J}_{!*}(\lambda, \mathbf{Z}))$  is a morphism of  $B(\mathbf{Z})$ -comodules, then the map  $f$  itself is a morphism of comodules, because  $F(\mathcal{J}_{!*}(\lambda, \mathbf{Z})) \otimes_{\mathbf{Z}} B(\mathbf{Z})$  is torsion-free. The cokernel of  $f$  is therefore torsion-free, hence is zero. In other words,  $f$  is an isomorphism, and any map in  $\mathrm{Hom}_{\mathbf{Z}}(F(M), F(\mathcal{J}_{!*}(\lambda, \mathbf{Z})))$  is a  $B(\mathbf{Z})$ -comodule map.

The image by  $f^{-1}$  of an isomorphism of  $\mathbf{Z}$ -modules  $F(M) \xrightarrow{\sim} F(\mathcal{J}_{!*}(\lambda, \mathbf{Z}))$  is thus an isomorphism  $M \xrightarrow{\sim} \mathcal{J}_{!*}(\lambda, \mathbf{Z})$ . One can then take  $M^* := \mathcal{J}_{!*}(-\lambda, \mathbf{Z})$ .  $\square$

**Proposition 13.4.** *The monoid scheme  $\tilde{G}_{\mathbf{Z}}$  is a group scheme.*

*Proof.* First, we remark that if  $M$  is a right  $B(\mathbf{Z})$ -comodule which is free of rank 1 over  $\mathbf{Z}$ , then Lemma 13.3 implies that  $M$  is invertible in the monoidal category of  $\tilde{G}_{\mathbf{Z}}$ -modules, hence that  $\tilde{G}_{\mathbf{Z}}(R)$  acts by invertible endomorphisms on  $R \otimes_{\mathbf{Z}} M$ , for any  $\mathbf{Z}$ -algebra  $R$ . As in the case of fields (see the proof of Theorem 2.7), this implies the same claim for any right  $B(\mathbf{Z})$ -comodule which is free of finite rank. Then, consider an arbitrary object  $M$  in  $\mathrm{Comod}_{B(\mathbf{Z})}$ . By [S2, Proposition 3], there exist right  $B(\mathbf{Z})$ -comodules  $M'$  and  $M''$  which are free of finite rank over  $\mathbf{Z}$  and an exact sequence of  $B(\mathbf{Z})$ -comodules

$$M'' \rightarrow M' \rightarrow M \rightarrow 0.$$

Then for any  $\mathbf{Z}$ -algebra  $R$  we have an exact sequence

$$R \otimes_{\mathbf{Z}} M'' \rightarrow R \otimes_{\mathbf{Z}} M' \rightarrow R \otimes_{\mathbf{Z}} M \rightarrow 0.$$

Any element of  $\tilde{G}_{\mathbf{Z}}(R)$  acts on  $R \otimes_{\mathbf{Z}} M''$  and  $R \otimes_{\mathbf{Z}} M'$  by invertible endomorphisms by the case treated above; the 5-lemma implies that the same claim holds also for  $M$ . This implies the proposition since the statement in Remark 2.8(1) holds in our present setting, see [SR, Chap. II, Scholie 3.1.1(3)].  $\square$

## 14 Identification of the group scheme

### 14.1 Statement and overview of the proof

In Section 13 we have constructed an affine  $\mathbf{k}$ -group scheme  $\tilde{G}_{\mathbf{k}}$  and an equivalence of monoidal categories

$$P_{G_{\mathcal{O}}}(\mathrm{Gr}_G, \mathbf{k}) \xrightarrow{\sim} \mathrm{Rep}_{\mathbf{k}}(\tilde{G}_{\mathbf{k}}).$$

Our goal now is to identify  $\tilde{G}_{\mathbf{k}}$ . To state this result we need some terminology. Recall that:

- a *reductive group* over a scheme  $S$  is a smooth affine group scheme over  $S$  all of whose geometric fibers are connected reductive algebraic groups; see [SGA3, Exposé XIX, Définition 2.7];
- a *split torus* over  $S$  is a group scheme which is isomorphic to a finite product of copies of the multiplicative group  $\mathbb{G}_{\mathbf{m}, S}$ ;
- a *split maximal torus* of a group scheme  $H$  over  $S$  is a closed subgroup scheme  $K$  of  $H$  which is a split torus and such that for any geometric fiber  $\bar{s}$ , the morphism  $K_{\bar{s}} \rightarrow H_{\bar{s}}$  identifies  $K_{\bar{s}}$  with a maximal torus of  $H_{\bar{s}}$ ; see [SGA3, Exposé XIX, p. 10].

When  $S = \mathrm{Spec}(\mathbf{Z})$ , it is known that a reductive group  $H$  over  $\mathrm{Spec}(\mathbf{Z})$  which admits a split maximal torus is determined, up to isomorphism, by the root datum of  $\mathrm{Spec}(\mathbf{C}) \times_{\mathrm{Spec}(\mathbf{Z})} H$ ; see [SGA3, Exposé XXIII, Corollaire 5.4]. For such a group, if  $\mathbf{k}$  is an algebraically closed field, the root datum of  $\mathrm{Spec}(\mathbf{k}) \times_{\mathrm{Spec}(\mathbf{Z})} H$  does not depend on  $\mathbf{k}$ , and will be called the root datum of  $H$ .

When  $\mathbf{k} = \mathbf{Z}$ , the answer to our question is provided by the following theorem.

**Theorem 14.1.** *The group scheme  $\tilde{G}_{\mathbf{Z}}$  is the unique reductive group over  $\mathbf{Z}$  which admits a split torus and whose root datum is dual to that of  $G$ .*

In fact, below we will prove a slightly more precise result: we will construct a maximal torus of  $\tilde{G}_{\mathbf{Z}}$  whose group of characters identifies with  $X_*(T)$ , and show that the root datum of  $\tilde{G}_{\mathbf{Z}}$  with respect to this maximal torus is dual to the root datum of  $(G, T)$ . For a general  $\mathbf{k}$ , since  $\tilde{G}_{\mathbf{k}} \cong \mathrm{Spec}(\mathbf{k}) \times_{\mathrm{Spec}(\mathbf{Z})} \tilde{G}_{\mathbf{Z}}$  (see (13.1)), Theorem 14.1 determines  $\tilde{G}_{\mathbf{k}}$  also up to isomorphism.

When  $\mathbf{k}$  is a field of characteristic 0, this description<sup>30</sup> has already been proved in Theorem 9.8; this special case will play an important role in the proof below. In fact, a result of Prasad–Yu [PY, Proposition 1.5] ensures that a flat affine group scheme  $H$  over  $\mathbf{Z}$  such that  $\mathrm{Spec}(\mathbf{k}) \times_{\mathrm{Spec}(\mathbf{Z})} H$  is a connected reductive group for any algebraically closed<sup>31</sup> field  $\mathbf{k}$ , whose dimension is independent of  $\mathbf{k}$ , is necessarily reductive. Hence what remains to be done is:

<sup>30</sup>Note that in this setting there are two different groups that we have denoted  $\tilde{G}_{\mathbf{k}}$ : the one constructed in Section 9 using Tannakian reconstruction, and the one constructed “by hand” in Section 13. These two groups are canonically identified thanks to [Mi, Theorem X.1.2].

<sup>31</sup>As stated in [PY], the claim requires this property rather when  $\mathbf{k}$  is either  $\mathbf{Q}$  or a finite field  $\mathbf{F}_p$ . But an affine group scheme over a field is reductive iff its base change to an algebraic closure of the field is reductive; this follows from the fact that smoothness can be checked on this base change, see [GW, Remark 6.30(2)], and similarly for connectedness, see Footnote 3.

1. construct a subgroup scheme of  $\tilde{G}_{\mathbf{Z}}$  which is a split torus;
2. check that for an algebraic closure  $\mathbf{k}$  of a finite field, the group scheme  $\tilde{G}_{\mathbf{k}}$  is reductive;
3. show that the base change to  $\mathbf{k}$  of our  $\mathbf{Z}$ -torus is a maximal torus of  $\tilde{G}_{\mathbf{k}}$ ;
4. and finally, show that  $\tilde{G}_{\mathbf{k}}$  has the appropriate root datum with respect to this maximal torus.

Here (1) will be easy, and based on the same arguments as for fields of characteristic 0, see §14.2. The proof of (2)–(4) will rely on another result of Prasad–Yu [PY, Theorem 1.2] which, in our setting, characterizes reductive group schemes over  $\mathbf{Z}_p$  in terms of properties of their base change to  $\mathbf{Q}_p$  and to an algebraic closure of  $\mathbf{F}_p$ . (More precisely, this result will be needed to show that  $\tilde{G}_{\mathbf{k}}$  is reduced; the other properties will be checked directly.)

## 14.2 First properties

For any  $\mathbf{k}$ , by construction  $\tilde{G}_{\mathbf{k}}$  is an affine group scheme over  $\mathbf{k}$ . Moreover, this group scheme is flat over  $\mathbf{k}$  by Proposition 13.2.

**Lemma 14.2.** *If  $\mathbf{k}$  is a field, the group scheme  $\tilde{G}_{\mathbf{k}}$  is algebraic and connected.*

*Proof.*<sup>32</sup> By Proposition 2.11(1), to prove that  $\tilde{G}_{\mathbf{k}}$  is algebraic we need to exhibit a tensor generator of the category  $\text{Rep}_{\mathbf{k}}(\tilde{G}_{\mathbf{k}}) \cong \text{P}_{G_{\mathcal{O}}}(\text{Gr}_G, \mathbf{k})$ . By Proposition 12.4, the category  $\text{P}_{G_{\mathcal{O}}}(\text{Gr}_G, \mathbf{k})$  has a natural highest weight structure. Hence we can consider the *tilting* objects in this category, namely those which admit both a filtration with subquotients of the form  $\mathcal{J}_1(\lambda, \mathbf{k})$ , and a filtration with subquotients of the form  $\mathcal{J}_*(\lambda, \mathbf{k})$ ; see e.g. [Ri, §7.5]. If we denote by  $\text{Tilt}_{G_{\mathcal{O}}}(\text{Gr}_G, \mathbf{k})$  the full subcategory of  $\text{P}_{G_{\mathcal{O}}}(\text{Gr}_G, \mathbf{k})$  consisting of the tilting objects, then the indecomposable objects in  $\text{Tilt}_{G_{\mathcal{O}}}(\text{Gr}_G, \mathbf{k})$  are parametrized by  $X_*(T)^+$  (see e.g. [Ri, Theorem 7.14]), and the natural functor

$$K^b\text{Tilt}_{G_{\mathcal{O}}}(\text{Gr}_G, \mathbf{k}) \rightarrow D^b\text{P}_{G_{\mathcal{O}}}(\text{Gr}_G, \mathbf{k})$$

is an equivalence of categories (see [Ri, Proposition 7.17]). In particular, any object of  $\text{P}_{G_{\mathcal{O}}}(\text{Gr}_G, \mathbf{k})$  is a subquotient of a tilting object.

Now, it is known that the subcategory  $\text{Tilt}_{G_{\mathcal{O}}}(\text{Gr}_G, \mathbf{k})$  is stable under the convolution bifunctor  $\star$ . In fact, consider the “parity sheaves”  $\{\mathcal{E}_{\lambda} : \lambda \in X_*(T)^+\}$  in  $D_{\mathcal{S}}^b(\text{Gr}_G, \mathbf{k})$  in the sense of Juteau–Mautner–Williamson [JMW] (for the constant pariversity). It follows from [JMW2, Proposition 3.3] that if these objects are perverse, then they coincide with the tilting objects in  $\text{P}_{\mathcal{S}}(\text{Gr}_G, \mathbf{k}) \cong \text{P}_{G_{\mathcal{O}}}(\text{Gr}_G, \mathbf{k})$ . The fact that they are indeed perverse is proved in [JMW2] under certain technical conditions on  $\text{char}(\mathbf{k})$ , and in [MR, Corollary 1.6] under the assumption that  $\text{char}(\mathbf{k})$  is good for  $G$ .<sup>33</sup> This settles the question in this case, since convolution preserves

<sup>32</sup>This proof was suggested to us by G. Williamson.

<sup>33</sup>Recall that a prime number  $p$  is called *bad* for  $G$  if  $p = 2$  and  $\Delta(G, T)$  has a component not of type  $\mathbf{A}$ , or if  $p = 3$  and  $\Delta(G, T)$  has a component of type  $\mathbf{E}$ ,  $\mathbf{F}$  or  $\mathbf{G}$ , or finally if  $p = 5$  and  $\Delta(G, T)$  has a component of type  $\mathbf{E}_8$ . A prime number is called *good* for  $G$  if it is not bad for  $G$ .

parity complexes; see [JMW2, Theorem 1.5]. The proof that  $\text{Tilt}_{G_{\mathcal{O}}}(\text{Gr}_G, \mathbf{k})$  is stable under convolution for a general field  $\mathbf{k}$  will appear in [BGMRR].

Finally we can conclude: if  $(\lambda_1, \dots, \lambda_n)$  is a finite generating subset of the monoid  $X_*(T)^+$ , and if  $\mathcal{T}_i$  is the indecomposable tilting object attached to  $\lambda_i$  for any  $i \in \{1, \dots, n\}$ , then by support considerations we see that any indecomposable tilting object in  $\text{P}_{G_{\mathcal{O}}}(\text{Gr}_G, \mathbf{k})$  is a direct summand of a tensor power of  $\mathcal{T}_1 \oplus \dots \oplus \mathcal{T}_n$ , and therefore that  $\mathcal{T}_1 \oplus \dots \oplus \mathcal{T}_n$  is a tensor generator of the category  $\text{P}_{G_{\mathcal{O}}}(\text{Gr}_G, \mathbf{k})$ .

Once we know that  $\tilde{G}_{\mathbf{k}}$  is algebraic, the fact that it is connected follows from Proposition 2.11(2), using the same considerations as in the proof of Lemma 9.3.  $\square$

*Remark 14.3.* 1. The algebraicity claim in Lemma 14.2 is not proved in this way in [MV2]. In fact, in order to apply the results of [PY] we only need to know that the *reduced* subgroup  $(\tilde{G}_{\mathbf{k}})_{\text{red}}$  is of finite type, when  $\mathbf{k}$  is an algebraic closure of a finite field. The proof of this claim in the published version of [MV2] is incomplete, but the authors have recently added in the arXiv version of their paper an appendix explaining how to fill this gap. In any case, the prior knowledge of the fact that  $\tilde{G}_{\mathbf{k}}$  is algebraic will allow us to simplify some later steps of the proof.

2. The fact that  $\tilde{G}_{\mathbf{k}}$  is connected implies that  $(\tilde{G}_{\mathbf{k}})_{\text{red}}$  is connected; see [Wa, §6.6].

**Lemma 14.4.** *If  $\mathbf{k}$  is an algebraic closure of a finite field, then the dimension of  $\tilde{G}_{\mathbf{k}}$  is at most the dimension of the reductive  $\mathbf{k}$ -group with root datum dual to that of  $G$  (i.e.  $\dim(G)$ ).*

*Proof.* This property follows from the general fact that the dimension of fibers of a flat morphism of finite presentation is a lower semicontinuous function (on the target), see [SP, Tag 0D4H]. In more “down to earth” terms, one can argue as follows. Let  $p$  be the characteristic of  $\mathbf{k}$ , and set  $d := \dim(\tilde{G}_{\mathbf{k}})$ . Then there exist  $d$  algebraically independent functions  $f_1, \dots, f_d$  in  $\mathbf{k}[\tilde{G}_{\mathbf{k}}]$  (see e.g. [GW, Theorem 5.22]). Since

$$\mathbf{k}[\tilde{G}_{\mathbf{k}}] = \mathbf{k} \otimes_{\mathbf{F}_p} \mathbf{F}_p[\tilde{G}_{\mathbf{F}_p}],$$

there exists a finite field  $\mathbf{F} \subset \mathbf{k}$  such that each  $f_i$  belongs to

$$\mathbf{F} \otimes_{\mathbf{F}_p} \mathbf{F}_p[\tilde{G}_{\mathbf{F}_p}] \cong \mathbf{F}[\tilde{G}_{\mathbf{F}}].$$

Let  $\mathbf{O}$  be a finite extension of  $\mathbf{Z}_p$  with residue field  $\mathbf{F}$ . Then since

$$\mathbf{F}[\tilde{G}_{\mathbf{F}}] = \mathbf{F} \otimes_{\mathbf{O}} \mathbf{O}[\tilde{G}_{\mathbf{O}}],$$

each  $f_i$  can be lifted to a function  $\tilde{f}_i \in \mathbf{O}[\tilde{G}_{\mathbf{O}}]$ . Since  $\mathbf{O}[\tilde{G}_{\mathbf{O}}]$  is torsion-free, the collection  $\tilde{f}_1, \dots, \tilde{f}_d$  does not satisfy any algebraic equation with coefficients in  $\mathbf{O}$ . Finally, since  $\mathbf{O}[\tilde{G}_{\mathbf{O}}]$  is free over  $\mathbf{O}$ , if  $\mathbf{K}$  is the fraction field of  $\mathbf{O}$  this collection is algebraically independent in

$$\mathbf{K} \otimes_{\mathbf{O}} \mathbf{O}[\tilde{G}_{\mathbf{O}}] \cong \mathbf{K}[\tilde{G}_{\mathbf{K}}].$$

Hence  $d$  is at most  $\dim(\tilde{G}_{\mathbf{K}})$ . We conclude using the fact that  $\dim(\tilde{G}_{\mathbf{K}})$  is the dimension of the split reductive  $\mathbf{k}$ -group with root datum dual to that of  $G$ , see Theorem 9.8.  $\square$

We finish this subsection with the following remark, valid for any ring  $\mathbf{k}$ . We denote by  $T_{\mathbf{k}}^{\vee}$  the split  $\mathbf{k}$ -torus whose group of characters is  $X_*(T)$ . Then, as in the case of fields of characteristic 0 (see §9.1), the weight functors define a canonical functor  $P_{G_{\mathcal{O}}}(\mathrm{Gr}_G, \mathbf{k}) \rightarrow \mathrm{Rep}(T_{\mathbf{k}}^{\vee})$  sending convolution to tensor product and the functor  $F$  to the natural forgetful functor. In view of [Mi, Theorem X.1.2] (compare with Proposition 2.6 and Proposition 2.10), this defines for any  $\mathbf{k}$ -algebra  $\mathbf{k}'$  a group morphism  $T_{\mathbf{k}}^{\vee}(\mathbf{k}') \rightarrow \tilde{G}_{\mathbf{k}}(\mathbf{k}')$ , or in other words a  $\mathbf{k}$ -group scheme morphism  $T_{\mathbf{k}}^{\vee} \rightarrow \tilde{G}_{\mathbf{k}}$ . Again as in the characteristic-0 case, for any  $\lambda \in X_*(T)$  the free rank-1  $T_{\mathbf{k}}^{\vee}$ -module defined by  $\lambda$  appears as a direct summand of the image of an object of  $P_{G_{\mathcal{O}}}(\mathrm{Gr}_G, \mathbf{k})$ ; considering matrix coefficients we deduce that  $\lambda$  belongs to the image of the associated morphism  $\mathbf{k}[\tilde{G}_{\mathbf{k}}] \rightarrow \mathbf{k}[T_{\mathbf{k}}^{\vee}]$ . This shows that this morphism is surjective, i.e. that the morphism  $T_{\mathbf{k}}^{\vee} \rightarrow \tilde{G}_{\mathbf{k}}$  is a closed embedding.

### 14.3 Study of the group $(\tilde{G}_{\mathbf{k}})_{\mathrm{red}}$ for $\mathbf{k}$ an algebraic closure of a finite field

In this subsection we fix a prime number  $p$  and assume that  $\mathbf{k}$  is an algebraic closure of  $\mathbf{F}_p$ . We study in detail the algebraic  $\mathbf{k}$ -group scheme<sup>34</sup>  $(\tilde{G}_{\mathbf{k}})_{\mathrm{red}}$ . Recall that this group is connected; see Remark 14.3(2). We also remark that the embedding  $T_{\mathbf{k}}^{\vee} \rightarrow \tilde{G}_{\mathbf{k}}$  factors through an embedding  $T_{\mathbf{k}}^{\vee} \rightarrow (\tilde{G}_{\mathbf{k}})_{\mathrm{red}}$  since  $T_{\mathbf{k}}^{\vee}$  is reduced. The goal of this subsection is to prove the following proposition.

**Proposition 14.5.** *The group scheme  $(\tilde{G}_{\mathbf{k}})_{\mathrm{red}}$  is a connected reductive group,  $T_{\mathbf{k}}^{\vee}$  is a maximal torus of this group, and the root datum of  $(\tilde{G}_{\mathbf{k}})_{\mathrm{red}}$  with respect to  $T_{\mathbf{k}}^{\vee}$  is dual to that of  $(G, T)$ .*

Note that Proposition 14.5 is sufficient to complete the program outlined in §14.1. Indeed, once this result is proved, we will know that the group scheme  $\tilde{G}_{\mathbf{Z}_p}$  over  $\mathbf{Z}_p$  satisfies the following conditions:

- $\tilde{G}_{\mathbf{Z}_p}$  is affine and flat over  $\mathbf{Z}_p$  (see §14.2);
- the generic fiber  $\tilde{G}_{\mathbf{Q}_p} = \mathrm{Spec}(\mathbf{Q}_p) \times_{\mathrm{Spec}(\mathbf{Z}_p)} \tilde{G}_{\mathbf{Z}_p}$  is connected and smooth over  $\mathbf{Q}_p$  (see Theorem 9.8);
- the reduced geometric special fiber  $(\tilde{G}_{\mathbf{k}})_{\mathrm{red}} = (\mathrm{Spec}(\mathbf{k}) \times_{\mathrm{Spec}(\mathbf{Z}_p)} \tilde{G}_{\mathbf{Z}_p})_{\mathrm{red}}$  is of finite type over  $\mathbf{k}$  (see Lemma 14.2) and its identity component  $(\tilde{G}_{\mathbf{k}})_{\mathrm{red}}^{\circ}$  is a reductive group of the same dimension as  $\tilde{G}_{\mathbf{Q}_p}$  (see Proposition 14.5).

In the terminology of [PY], this means that  $\tilde{G}_{\mathbf{Z}_p}$  is quasi-reductive. We will also know that

- the root data of  $\tilde{G}_{\mathbf{Q}_p}$  and  $(\tilde{G}_{\mathbf{k}})_{\mathrm{red}}^{\circ}$  coincide.

By [PY, Theorem 1.2], it will follow that  $\tilde{G}_{\mathbf{Z}_p}$  is a reductive group over  $\mathbf{Z}_p$ . This will imply in particular that  $\tilde{G}_{\mathbf{k}}$  is reduced, hence that in Proposition 14.5 we can omit the subscript “red,” and thus will finally prove the properties (2)–(4) of §14.1.

<sup>34</sup>Recall that if  $H$  is a group scheme over a field  $\mathbf{F}$ , the associated reduced scheme  $H_{\mathrm{red}}$  is not necessarily a closed subgroup. But this is true if  $\mathbf{F}$  is perfect, which is the case here; see [Mi, §VI.6].



The proof of Proposition 14.5 will be based on the same ideas as in Section 9, but with many additional difficulties. We need some preparatory lemmas. We denote by  $R$  the quotient of  $(\tilde{G}_{\mathbf{k}})_{\text{red}}$  by its unipotent radical. Then the composition  $T_{\mathbf{k}}^{\vee} \rightarrow (\tilde{G}_{\mathbf{k}})_{\text{red}} \rightarrow R$  is injective, so that we can also consider  $T_{\mathbf{k}}^{\vee}$  as a closed subgroup of  $R$ .

**Lemma 14.6.**  $T_{\mathbf{k}}^{\vee}$  is a maximal torus of  $R$ .

*Proof.* First, by [DG, Corollary III.3.6.4],  $\tilde{G}_{\mathbf{k}}$  is isomorphic, as a scheme, to the product (over  $\text{Spec}(\mathbf{k})$ ) of  $(\tilde{G}_{\mathbf{k}})_{\text{red}}$  with a scheme of the form  $\text{Spec}(\mathbf{k}[X_1, \dots, X_r]/(X_1^{p_1}, \dots, X_r^{p_r}))$  for some positive integers  $n_1, \dots, n_r$ . It follows<sup>35</sup> that for some  $n$ , the  $n$ -th Frobenius morphism  $\text{Fr}_{\tilde{G}_{\mathbf{k}}}^n : \tilde{G}_{\mathbf{k}} \rightarrow (\tilde{G}_{\mathbf{k}})^{(n)}$  (see e.g. [Ja, §I.9.2]) factors through  $((\tilde{G}_{\mathbf{k}})_{\text{red}})^{(n)}$ . Hence we can consider the diagram

$$\begin{array}{ccc}
\tilde{G}_{\mathbf{k}} & \xrightarrow{\text{Fr}_{\tilde{G}_{\mathbf{k}}}^n} & (\tilde{G}_{\mathbf{k}})^{(n)} \\
\uparrow & \dashrightarrow & \uparrow \\
(\tilde{G}_{\mathbf{k}})_{\text{red}} & \xrightarrow{\text{Fr}_{(\tilde{G}_{\mathbf{k}})_{\text{red}}}^n} & ((\tilde{G}_{\mathbf{k}})_{\text{red}})^{(n)} \\
\downarrow & & \downarrow \\
R & \xrightarrow{\text{Fr}_R^n} & R^{(n)}.
\end{array}$$

Now, consider a simple representation  $V$  of  $R^{(n)}$ , seen as a (simple) representation of the group  $((\tilde{G}_{\mathbf{k}})_{\text{red}})^{(n)}$ . Our factorization above allows to see  $V$  as a representation of  $\tilde{G}_{\mathbf{k}}$ . This representation is simple: in fact, its restriction to  $(\tilde{G}_{\mathbf{k}})_{\text{red}}$  is simply the twist of  $V$  by  $\text{Fr}_{(\tilde{G}_{\mathbf{k}})_{\text{red}}}^n$ , hence it is simple by [Ja, Proposition I.9.5]. In this way we obtain an injective ring morphism

$$\mathbf{Q} \otimes_{\mathbf{Z}} K^0(\text{Rep}_{\mathbf{k}}(R^{(n)})) \hookrightarrow \mathbf{Q} \otimes_{\mathbf{Z}} K^0(\text{Rep}_{\mathbf{k}}(\tilde{G}_{\mathbf{k}})).$$

By [GW, Theorem 5.22(3)], this shows that

$$\dim \text{Spec}(\mathbf{Q} \otimes_{\mathbf{Z}} K^0(\text{Rep}_{\mathbf{k}}(\tilde{G}_{\mathbf{k}}))) \geq \dim \text{Spec}(\mathbf{Q} \otimes_{\mathbf{Z}} K^0(\text{Rep}_{\mathbf{k}}(R^{(n)}))).$$

Here, by (9.1) the right-hand side is equal to  $\text{rk}(R^{(n)}) = \text{rk}(R)$ , and the left-hand side is equal to  $\dim(T)$  (by the same considerations as in the characteristic-0 case, see §9.1). Hence this inequality means that  $\text{rk}(R) \leq \dim(T_{\mathbf{k}}^{\vee})$ , hence that  $T_{\mathbf{k}}^{\vee}$  is a maximal torus in  $R$ .  $\square$

Now we choose a Borel subgroup  $\tilde{B}$  of  $R$  containing  $T_{\mathbf{k}}^{\vee}$  for which the sum  $2\rho$  of the positive roots of  $G$  is a dominant cocharacter (for the choice of positive roots given by the  $T_{\mathbf{k}}^{\vee}$ -weights in the Lie algebra of  $\tilde{B}$ ). We then use the same notation as in §9.2 for roots and coroots of  $G$  and  $R$ .

**Lemma 14.7.** *The set of dominant weights of  $(R, T_{\mathbf{k}}^{\vee})$  relative to the system of positive roots  $\Delta_+(R, \tilde{B}, T_{\mathbf{k}}^{\vee})$  is  $X_*(T)^+ \subset X_*(T) = X^*(T_{\mathbf{k}}^{\vee})$ .*

<sup>35</sup>See also [Mi, Corollary VI.10.2] for a direct proof of this fact.

*Proof.* For  $\lambda \in X_*(T)$  a dominant weight relative to the system of positive roots  $\Delta_+(R, \tilde{B}, T_{\mathbf{k}}^\vee)$ , we denote by  $L^R(\lambda)$  the corresponding simple  $R$ -module. Let  $n$  be as in the proof of Lemma 14.6. Then for  $\lambda$  as above, the action of  $R$  on  $L^R(p^n\lambda)$  factors through an action of  $R^{(n)}$  by Steinberg's theorem (see [Ja, Proposition II.3.16]), hence this module determines a simple  $\tilde{G}_{\mathbf{k}}$ -module (see the proof of Lemma 14.6). The action of  $T_{\mathbf{k}}^\vee$  on this module is then determined by the character of the  $R$ -module  $L^R(p^n\lambda)$ .

On the other hand, let  $\mu \in X_*(T)^+$  be the dominant coweight of  $G$  such that the simple perverse sheaf corresponding to  $L^R(p^n\lambda)$  is  $\mathcal{J}_{!*}(\mu, \mathbf{k})$ . Then we can write in the Grothendieck group of  $\mathrm{P}_{G_{\mathcal{O}}}(\mathrm{Gr}_G, \mathbf{k})$

$$[\mathcal{J}_{!*}(\mu, \mathbf{k})] = [\mathcal{J}_!(\mu, \mathbf{k})] + \sum_{\substack{\nu \in X_*(T)^+ \\ \nu < \mu}} c_{\mu, \nu} \cdot [\mathcal{J}_!(\nu, \mathbf{k})]$$

for some coefficients  $c_{\mu, \nu} \in \mathbf{Z}$ . This gives rise to a second way of expressing the action of  $T_{\mathbf{k}}^\vee$  on this  $\tilde{G}_{\mathbf{k}}$ -module using Proposition 11.1. In particular, since the highest weight of  $L^R(p^n\lambda)$  (considered as an  $R$ -module, with the choice of positive roots determined by  $\tilde{B}$ ) is a weight for which the function  $\langle 2\rho, ? \rangle$  attains its maximum, we must have  $\mu = p^n\lambda$ , so that  $p^n\lambda$  belongs to  $X_*(T)^+$ , and finally  $\lambda \in X_*(T)^+$ .

On the other hand, let  $\lambda \in X_*(T)^+$ . Consider the simple  $\tilde{G}_{\mathbf{k}}$ -module  $L^{\tilde{G}_{\mathbf{k}}}(\lambda)$  corresponding to the simple perverse sheaf  $\mathcal{J}_{!*}(\lambda, \mathbf{k})$ . The  $T_{\mathbf{k}}^\vee$ -weights of this module, or equivalently of its restriction to  $(\tilde{G}_{\mathbf{k}})_{\mathrm{red}}$ , can be estimated as above using Proposition 11.1; in particular  $\lambda$  is a weight of this module. Hence there exists a composition factor  $M$  of the  $(\tilde{G}_{\mathbf{k}})_{\mathrm{red}}$ -module  $L^{\tilde{G}_{\mathbf{k}}}(\lambda)$  which admits  $\lambda$  as a  $T_{\mathbf{k}}^\vee$ -weight. Since  $M$  is simple, the  $(\tilde{G}_{\mathbf{k}})_{\mathrm{red}}$ -action factors through an  $R$ -action. Considering once again the values of the function  $\langle 2\rho, ? \rangle$ , we see that  $\lambda$  must be the highest weight of  $M$ , and thus that  $\lambda$  is dominant with respect to the system of positive roots  $\Delta_+(R, \tilde{B}, T_{\mathbf{k}}^\vee)$ .  $\square$

As for (9.2), Lemma 14.7 implies that

$$\{\mathbf{Q}_+ \cdot \alpha : \alpha \in \Delta_s^\vee(R, \tilde{B}, T_{\mathbf{k}}^\vee)\} = \{\mathbf{Q}_+ \cdot \beta : \beta \in \Delta_s(G, B, T)\}. \quad (14.1)$$

**Lemma 14.8.** *We have  $\mathbf{Z} \cdot \Delta(R, T_{\mathbf{k}}^\vee) \subset \mathbf{Z} \cdot \Delta^\vee(G, T)$  (in  $X_*(T) = X^*(T_{\mathbf{k}}^\vee)$ ).*

*Proof.* Recall that the connected components of  $\mathrm{Gr}_G$  are in a natural bijection with the quotient  $X_*(T)/\mathbf{Z}\Delta^\vee(G, T)$ , see §3.1. Let  $Z \subset T_{\mathbf{k}}^\vee$  be the (scheme-theoretic) intersection of the kernels of all the elements in  $\mathbf{Z}\Delta^\vee(G, T)$ , so that  $Z$  is a diagonalisable group scheme with  $X^*(Z) \cong X_*(T)/\mathbf{Z}\Delta^\vee(G, T)$ . Then any object of  $\mathrm{P}_{G_{\mathcal{O}}}(\mathrm{Gr}_G, \mathbf{k})$  is naturally graded by the group of characters of  $Z$ , in a way compatible with the functor

$$\mathrm{P}_{G_{\mathcal{O}}}(\mathrm{Gr}_G, \mathbf{k}) \cong \mathrm{Rep}_{\mathbf{k}}(\tilde{G}_{\mathbf{k}}) \rightarrow \mathrm{Rep}_{\mathbf{k}}(Z)$$

(where the second arrow is the forgetful functor). In particular, for any  $\chi \in X^*(Z)$ , the subspace of the left regular representation  $\mathbf{k}[\tilde{G}_{\mathbf{k}}]$  consisting of the functions  $f$  satisfying  $f(z^{-1}g) = \chi(z)f(g)$  is stable under the action of  $\tilde{G}_{\mathbf{k}}$ ; hence  $Z$  is a central subgroup of  $\tilde{G}_{\mathbf{k}}$ , and then its image in  $R$  is central also. We deduce that all the roots of  $(R, T_{\mathbf{k}}^\vee)$  restrict trivially to  $Z$ , i.e. that the morphism  $X_*(T) \rightarrow X_*(T)/\mathbf{Z}\Delta^\vee(G, T)$  factors through  $X_*(T)/\mathbf{Z} \cdot \Delta(R, T_{\mathbf{k}}^\vee)$ , whence the claim.  $\square$

**Lemma 14.9.** *The Weyl groups of  $(G, T)$  and of  $(R, T_{\mathbf{k}}^{\vee})$ , seen as groups of automorphisms of  $X_*(T)$ , together with their subsets of simple reflections, coincide.*

*Proof.* Recall that the Weyl group of  $(G, T)$  is denoted by  $W$ . We also denote by  $S \subset W$  the subset of simple reflections (i.e. the reflections associated with simple roots). We will denote by  $W'$  the Weyl group of  $(R, T_{\mathbf{k}}^{\vee})$ , and by  $S' \subset W'$  the subset of simple reflections. We fix  $n$  as in the proof of Lemma 14.6.

For  $\lambda \in X_*(T)^+$ , we can recover the orbit  $W' \cdot (p^n \lambda)$  as the set of extremal points of the convex polytope consisting of the convex hull of the weights of the simple  $R$ -module  $L^R(p^n \lambda)$ . Using the same considerations as in the proof of Lemma 14.6 we see that this set coincides with the orbit  $W \cdot (p^n \lambda)$ , so that  $W' \cdot \lambda = W \cdot \lambda$ .

Now, we define an element of  $X_*(T)^+$  to be *regular* if its orbit under  $W'$  (or equivalently under  $W$ ) has the maximal possible cardinality, or equivalently if it is not orthogonal to any simple root of  $(G, T)$ , or equivalently if it is not orthogonal to any simple coroot of  $(R, T_{\mathbf{k}}^{\vee})$ . Then for  $\lambda$  regular, we can recover the subset  $\{s \cdot \lambda : s \in S\} \subset W \cdot \lambda$  as the subset consisting of elements  $\mu$  such that the segment joining  $\lambda$  to  $\mu$  is also extremal in the convex hull of  $W \cdot \lambda$ . A similar description applies for  $\{s' \cdot \lambda : s' \in S'\}$ , from which we deduce that

$$\{s \cdot \lambda : s \in S\} = \{s' \cdot \lambda : s' \in S'\}.$$

This implies that  $S = S'$ : in fact if  $s \in S$ , then for any  $\lambda \in X_*(T)^+$  regular there exists  $s' \in S'$  such that  $s \cdot \lambda = s' \cdot \lambda$ , and  $s'$  does not depend on  $\lambda$  because the direction of  $\lambda - s' \cdot \lambda$  is the line generated by the coroot of  $G$  associated with  $s$  and also the line generated by the root of  $R$  associated with  $s'$ ; then we have  $s = s'$ .

Finally, once we know that  $S = S'$  we deduce that  $W = W'$ , since  $W$ , resp.  $W'$ , is generated by  $S$ , resp.  $S'$ .  $\square$

**Lemma 14.10.** *We have  $\mathbf{Z}\Delta(G, T) \subset \mathbf{Z}\Delta^{\vee}(R, T_{\mathbf{k}}^{\vee})$  in  $X^*(T) = X_*(T_{\mathbf{k}}^{\vee})$ . Moreover, if this inclusion is an equality the root datum of  $(R, T_{\mathbf{k}}^{\vee})$  is dual to that of  $(G, T)$ .*

*Proof.* Let  $\alpha \in \Delta_s(G, B, T)$ . By (14.1), we know that there exists  $a \in \mathbf{Q}_+ \setminus \{0\}$  such that  $a\alpha \in \Delta_s^{\vee}(R, \tilde{B}, T_{\mathbf{k}}^{\vee})$ . We can also consider the coroot  $\alpha^{\vee}$  of  $(G, T)$  associated with the root  $\alpha$ , and the root  $(a\alpha)^{\wedge}$  of  $(R, T_{\mathbf{k}}^{\vee})$  associated with the coroot  $a\alpha$ . By Lemma 14.9, we have

$$\text{id} - \langle \alpha^{\vee}, ? \rangle \alpha = \text{id} - \langle (a\alpha)^{\wedge}, ? \rangle (a\alpha)$$

as automorphisms of  $X^*(T) = X_*(T_{\mathbf{k}}^{\vee})$ ; it follows that  $(a\alpha)^{\wedge} = \frac{1}{a}\alpha^{\vee}$ . On the other hand, Lemma 14.8 shows that  $(a\alpha)^{\wedge} \in \mathbf{Z}\Delta^{\vee}(G, T)$ ; hence  $\frac{1}{a} \in \mathbf{Z}$ , and  $\alpha = \frac{1}{a}(a\alpha) \in \mathbf{Z}\Delta^{\vee}(R, T_{\mathbf{k}}^{\vee})$ .

If the inclusion  $\mathbf{Z}\Delta(G, T) \subset \mathbf{Z}\Delta^{\vee}(R, T_{\mathbf{k}}^{\vee})$  is an equality, then with the notation used above we must have  $a = 1$  for any  $\alpha$ ; then  $\Delta_s(R, \tilde{B}, T_{\mathbf{k}}^{\vee}) = \Delta_s^{\vee}(G, B, T)$  and  $\Delta_s^{\vee}(R, \tilde{B}, T_{\mathbf{k}}^{\vee}) = \Delta_s(G, B, T)$ , and the canonical bijections between simple roots and coroots of  $R$  and of  $G$  coincide. Taking orbits under the Weyl groups, it follows that  $\Delta(R, T_{\mathbf{k}}^{\vee}) = \Delta^{\vee}(G, T)$  and  $\Delta^{\vee}(R, T_{\mathbf{k}}^{\vee}) = \Delta(G, T)$ , in a way compatible with the bijections between roots and coroots.  $\square$

**Lemma 14.11.** *If  $G$  is semisimple of adjoint type, then Proposition 14.5 holds.*

*Proof.* If  $G$  is semisimple of adjoint type, then  $\mathbf{Z}\Delta(G, T) = X^*(T)$ . It follows that the inclusion in Lemma 14.10 is an equality, and then that the root datum of  $R$  with respect to  $T_{\mathbf{k}}^{\vee}$  is dual to that of  $(G, T)$ .

Then we conclude as follows: of course we have  $\dim(R) \leq \dim((\widetilde{G}_{\mathbf{k}})_{\text{red}}) = \dim(\widetilde{G}_{\mathbf{k}})$ . Lemma 14.4 and our determination of  $\Delta(R, T_{\mathbf{k}}^{\vee})$  imply that this inequality is in fact an equality, so that  $(\widetilde{G}_{\mathbf{k}})_{\text{red}} = R$ , and then the claim follows from our identification of the root datum of  $R$ .  $\square$

**Lemma 14.12.** *If  $G$  is semisimple, then Proposition 14.5 holds.*

*Proof.* We assume that  $G$  is semisimple. Now that the claim is known if  $G$  is of adjoint type (see Lemma 14.11), we will in fact prove directly that  $\widetilde{G}_{\mathbf{k}}$  is a semisimple group with maximal torus  $T_{\mathbf{k}}^{\vee}$  and root datum dual to that of  $(G, T)$ .

Let  $G_{\text{ad}}$  be the adjoint quotient of  $G$ , and let  $T_{\text{ad}}$  be the image of  $T$  in  $G_{\text{ad}}$ . Then we can consider the group scheme  $(\widetilde{G}_{\text{ad}})_{\mathbf{k}}$  constructed in Section 13 starting from the group  $G_{\text{ad}}$ . By Lemma 14.11 and the remarks following Proposition 14.5, we know that  $(\widetilde{G}_{\text{ad}})_{\mathbf{k}}$  is semisimple with root datum dual to that of  $(G_{\text{ad}}, T_{\text{ad}})$ . The morphism  $G \rightarrow G_{\text{ad}}$  induces a closed embedding  $\text{Gr}_G \hookrightarrow \text{Gr}_{G_{\text{ad}}}$ , which then defines a group scheme morphism  $(\widetilde{G}_{\text{ad}})_{\mathbf{k}} \rightarrow \widetilde{G}_{\mathbf{k}}$  via Tannakian formalism.

The connected components of  $\text{Gr}_{G_{\text{ad}}}$  are parametrized by  $X_*(T_{\text{ad}})/\mathbf{Z}\Delta^{\vee}(G_{\text{ad}}, T_{\text{ad}})$ , and  $\text{Gr}_G$  is the union of those corresponding to elements in the subset  $X_*(T)/\mathbf{Z}\Delta^{\vee}(G_{\text{ad}}, T_{\text{ad}})$ . (Here  $\Delta^{\vee}(G_{\text{ad}}, T_{\text{ad}})$  is included in  $X_*(T)$ , and identifies with  $\Delta^{\vee}(G, T)$ .) Hence if  $Z \subset (T_{\text{ad}})_{\mathbf{k}}^{\vee}$  is the (scheme-theoretic) intersection of the kernels of the elements of  $X_*(T)$ , so that  $Z$  is a diagonalisable  $\mathbf{k}$ -group scheme with  $X^*(Z) \cong X_*(T_{\text{ad}})/X_*(T)$ , then any object  $\mathcal{F}$  of  $\text{P}_{G_{\text{ad}}, \mathcal{O}}(\text{Gr}_{G_{\text{ad}}}, \mathbf{k})$  admits a canonical grading  $\mathcal{F} = \bigoplus_{\chi \in X^*(Z)} \mathcal{F}_{\chi}$ , and using the equivalence of Proposition 10.8 we see that  $\text{P}_{G_{\mathcal{O}}}(\text{Gr}_G, \mathbf{k})$  identifies with the full subcategory of objects  $\mathcal{F}$  such that  $\mathcal{F}_{\chi} = 0$  for  $\chi \neq 1$ . This means that  $\widetilde{G}_{\mathbf{k}}$  is the quotient of  $(\widetilde{G}_{\text{ad}})_{\mathbf{k}}$  by the finite central subgroup scheme  $Z$ . Hence  $\widetilde{G}_{\mathbf{k}}$  is semisimple, and its root datum is dual to that of  $(G, T)$ .  $\square$

Finally, we conclude the proof of Proposition 14.5 with the following lemma.

**Lemma 14.13.** *Proposition 14.5 holds for a general reductive group  $G$ .*

We will give two proofs of this lemma: the first one is a slightly expanded version of the proof given in [MV2], and the second one is new (to the best of our knowledge).

*First proof of Lemma 14.13.* Here also, we will prove directly that  $\widetilde{G}_{\mathbf{k}}$  is reduced and reductive, and compute its root datum.

Let  $Z(G)$  be the center of  $G$ , and set  $H := Z(G)^{\circ}$ . Then  $H$  is a torus and  $G/H$  is a semisimple group; in particular the group  $\widetilde{H}_{\mathbf{k}}$  constructed as for  $G$  is the  $\mathbf{k}$ -torus dual to  $H$ , and  $\widetilde{G}/\widetilde{H}_{\mathbf{k}}$  is the semisimple group dual to  $G/H$ .

The natural maps  $H \hookrightarrow G$  and  $G \twoheadrightarrow G/H$  induce morphisms

$$\mathrm{Gr}_H \xrightarrow{i} \mathrm{Gr}_G \xrightarrow{\pi} \mathrm{Gr}_{G/H},$$

which exhibit  $\mathrm{Gr}_G$  as a trivial cover of  $\mathrm{Gr}_{G/H}$  with fiber  $\mathrm{Gr}_H$ . (In fact, if we choose a lattice  $Y \subset X_*(T)$  such that the composition  $Y \hookrightarrow X_*(T) \twoheadrightarrow X_*(T/H)$  is an isomorphism, then  $\mathrm{Gr}_{G/H}$  identifies with the union of the connected components of  $\mathrm{Gr}_G$  corresponding to elements in  $Y/\mathbf{Z}\Delta^\vee(G, T) \subset X_*(T)/\mathbf{Z}\Delta^\vee(G, T)$ . Note that there exists an isomorphism of varieties  $\mathrm{Gr}_G \cong \mathrm{Gr}_H \times \mathrm{Gr}_{G/H}$ , but that in general such an isomorphism cannot be chosen to be compatible—in any reasonable sense—with the construction of the convolution product.) We have associated exact functors

$$\mathrm{P}_{H_{\mathcal{O}}}(\mathrm{Gr}_H, \mathbf{k}) \xrightarrow{i_*} \mathrm{P}_{G_{\mathcal{O}}}(\mathrm{Gr}_G, \mathbf{k}) \xrightarrow{\pi_*} \mathrm{P}_{(G/H)_{\mathcal{O}}}(\mathrm{Gr}_{G/H}, \mathbf{k}), \quad (14.2)$$

where  $i_*$  is fully faithful and  $\pi_*$  is essentially surjective. (Here we use Proposition 10.8 to make sense of the functor  $i_*$  as a functor between the categories of *equivariant* perverse sheaves.) These functors are compatible with the monoidal structures and forgetful functors, hence induce group scheme morphisms

$$\widetilde{G}/\widetilde{H}_{\mathbf{k}} \rightarrow \widetilde{G}_{\mathbf{k}} \rightarrow \widetilde{H}_{\mathbf{k}} \quad (14.3)$$

via Tannakian formalism. If  $\mathcal{F}$  is in  $\mathrm{P}_{G_{\mathcal{O}}}(\mathrm{Gr}_G, \mathbf{k})$  and if we set  $\mathcal{F}_0 := i_*^p \mathcal{H}^0(i^* \mathcal{F})$ , then  $\mathcal{F}_0$  is a subobject of  $\mathcal{F}$  and  $\pi_*(\mathcal{F}_0) \subset \pi_* \mathcal{F}$  is the largest subobject isomorphic to a direct sum of copies of the unit object. This shows that (14.2) is an exact sequence of tensor categories in the sense of [BN, Definition 3.7]; in view of [BN, Remark 3.13] we deduce that (14.3) is an exact sequence of  $\mathbf{k}$ -group schemes. (Here the fact that the first morphism is a closed embedding can be seen using [DM, Proposition 2.21(b)], and the fact that the second morphism is a quotient morphism in the sense of [Wa, §15.1] or [Mi, §VII.7] follows from [DM, Proposition 2.21(a)]; however exactness at the middle term is less obvious, in particular since it is not clear a priori that  $\widetilde{G}/\widetilde{H}_{\mathbf{k}}$  is a normal subgroup. In fact, the property stated right after (14.3) essentially guarantees this.)

We have just proved that  $\widetilde{G}_{\mathbf{k}}$  is an extension of  $\widetilde{H}_{\mathbf{k}}$  by  $\widetilde{G}/\widetilde{H}_{\mathbf{k}}$ . Since both of these group schemes are smooth, by [Mi, Proposition VII.10.1] this implies that  $\widetilde{G}_{\mathbf{k}}$  is also a smooth group, i.e. that (14.3) is an extension of  $\mathbf{k}$ -algebraic groups in the “traditional” sense of e.g. [Hu]. The unipotent radical of  $\widetilde{G}_{\mathbf{k}}$  has trivial image in the torus  $\widetilde{H}_{\mathbf{k}}$ , hence is included in  $\widetilde{G}/\widetilde{H}_{\mathbf{k}}$ ; since the latter group is semisimple it follows that this unipotent radical is trivial, i.e. that  $\widetilde{G}_{\mathbf{k}}$  is reductive.

Since  $\widetilde{H}_{\mathbf{k}}$  is commutative,  $\widetilde{G}/\widetilde{H}_{\mathbf{k}}$  contains the derived subgroup of  $\widetilde{G}_{\mathbf{k}}$ ; and since  $\widetilde{G}/\widetilde{H}_{\mathbf{k}}$  is semisimple it coincides with the derived subgroup of  $\widetilde{G}_{\mathbf{k}}$ . The torus  $(T/H)_{\mathbf{k}}^\vee$  dual to  $T/H$  embeds naturally in  $T_{\mathbf{k}}^\vee$ , and identifies with a maximal torus in  $\widetilde{G}/\widetilde{H}_{\mathbf{k}}$ ; hence the associated embedding  $X_*((T/H)_{\mathbf{k}}^\vee) \hookrightarrow X_*(T_{\mathbf{k}}^\vee)$  induces an isomorphism

$$\mathbf{Z}\Delta^\vee(\widetilde{G}/\widetilde{H}_{\mathbf{k}}, (T/H)_{\mathbf{k}}^\vee) \xrightarrow{\sim} \mathbf{Z}\Delta^\vee(\widetilde{G}_{\mathbf{k}}, T_{\mathbf{k}}^\vee).$$

On the other hand, in terms of  $G$  this embedding identifies with the morphism  $X^*(T/H) \rightarrow X^*(T)$  induced by the quotient morphism  $T \rightarrow T/H$ ; hence it induces an isomorphism

$$\mathbf{Z}\Delta(G/H, T/H) \xrightarrow{\sim} \mathbf{Z}\Delta(G, T).$$

Since the embedding  $\mathbf{Z}\Delta(G/H, T/H) \subset \mathbf{Z}\Delta^\vee(\widetilde{G/H}_{\mathbf{k}}, (T/H)_{\mathbf{k}}^\vee)$  of Lemma 14.10 is known to be an equality, we deduce that the embedding  $\mathbf{Z}\Delta(G, T) \subset \mathbf{Z}\Delta^\vee(\widetilde{G}_{\mathbf{k}}, T_{\mathbf{k}}^\vee)$  is an equality also, hence by Lemma 14.10 that the root datum of  $(\widetilde{G}_{\mathbf{k}}, T_{\mathbf{k}}^\vee)$  is dual to that of  $(G, T)$ .  $\square$

*Second proof of Lemma 14.13.* We again set  $H = Z(G)^\circ$ , and consider the quotient  $G/H$  and the closed embedding  $\widetilde{G/H}_{\mathbf{k}} \hookrightarrow \widetilde{G}_{\mathbf{k}}$ . Since  $\widetilde{G/H}_{\mathbf{k}}$  is known to be reduced this embedding factors through  $(\widetilde{G}_{\mathbf{k}})_{\text{red}}$ , and since  $\widetilde{G/H}_{\mathbf{k}}$  is semisimple the composition with the quotient morphism  $(\widetilde{G}_{\mathbf{k}})_{\text{red}} \rightarrow R$  is injective; hence  $\widetilde{G/H}_{\mathbf{k}}$  can (and will) be considered as a closed subgroup of  $R$ . Consider the subspaces

$$\text{Lie}(\widetilde{G/H}_{\mathbf{k}}), \text{Lie}(T_{\mathbf{k}}^\vee) \subset \text{Lie}(R),$$

where  $\text{Lie}(\?)$  means the Lie algebra. We have

$$\begin{aligned} \text{Lie}(\widetilde{G/H}_{\mathbf{k}}) \cap \text{Lie}(T_{\mathbf{k}}^\vee) &= \{x \in \text{Lie}(\widetilde{G/H}_{\mathbf{k}}) \mid \forall t \in T_{\mathbf{k}}^\vee, t \cdot x = x\} \\ &\subset \{x \in \text{Lie}(\widetilde{G/H}_{\mathbf{k}}) \mid \forall t \in (T/H)_{\mathbf{k}}^\vee, t \cdot x = x\} = \text{Lie}((T/H)_{\mathbf{k}}^\vee) \end{aligned}$$

(where the  $\mathbf{k}$ -torus  $(T/H)_{\mathbf{k}}^\vee$  dual to  $T/H$  is seen as a closed subgroup of  $T_{\mathbf{k}}^\vee$ , and as the maximal torus of  $\widetilde{G/H}_{\mathbf{k}}$ ). We deduce that

$$\dim(\text{Lie}(R)) \geq \dim(\text{Lie}(\widetilde{G/H}_{\mathbf{k}}) + \text{Lie}(T_{\mathbf{k}}^\vee)) \geq \dim(G/H) + \dim(H) = \dim(G).$$

Since the left-hand side coincides with  $\dim(R)$  (see [Wa, §12.2]), which is at most  $\dim((\widetilde{G}_{\mathbf{k}})_{\text{red}})$ , using Lemma 14.4 we deduce that all the inequalities above are equalities. In particular,  $(\widetilde{G}_{\mathbf{k}})_{\text{red}} = R$  is reductive, and we have

$$\#\Delta(R, T_{\mathbf{k}}^\vee) = \#\Delta(\widetilde{G/H}_{\mathbf{k}}, (T/H)_{\mathbf{k}}^\vee) = \#\Delta(G, T).$$

This formula, together with Lemma 14.9, implies that if  $\mathcal{D}(R)$  is the derived subgroup of  $R$  we have

$$\dim(\mathcal{D}(R)) = \#\Delta(R, T_{\mathbf{k}}^\vee) + \#\Delta_s(R, T_{\mathbf{k}}^\vee) = \dim(\widetilde{G/H}_{\mathbf{k}}).$$

Since  $\widetilde{G/H}_{\mathbf{k}}$  is semisimple it is included in  $\mathcal{D}(R)$ , which is connected (see [Wa, Theorem 10.2]); hence this equality implies that  $\widetilde{G/H}_{\mathbf{k}} = \mathcal{D}(R)$ .

Once this equality is known, we can conclude essentially as in the last part of the first proof: the embedding  $X_*((T/H)_{\mathbf{k}}^\vee) \hookrightarrow X_*(T_{\mathbf{k}}^\vee)$  induces an isomorphism  $\mathbf{Z}\Delta^\vee(\widetilde{G/H}_{\mathbf{k}}, (T/H)_{\mathbf{k}}^\vee) \xrightarrow{\sim} \mathbf{Z}\Delta^\vee(R, T_{\mathbf{k}}^\vee)$  and an isomorphism  $\mathbf{Z}\Delta(G/H, T/H) \xrightarrow{\sim} \mathbf{Z}\Delta(G, T)$ , which shows that the embedding  $\mathbf{Z}\Delta(G, T) \subset \mathbf{Z}\Delta^\vee(R, T_{\mathbf{k}}^\vee)$  of Lemma 14.10 is an equality, and then that the root datum of  $(R, T_{\mathbf{k}}^\vee)$  is dual to that of  $(G, T)$ . Since  $R = (\widetilde{G}_{\mathbf{k}})_{\text{red}}$ , this concludes the proof of Lemma 14.13.  $\square$

*Remark 14.14.* From the point of view of Geometric Representation Theory, the most interesting case of the geometric Satake equivalence is when  $\mathbf{k}$  is an algebraically closed field. As explained above, for this special case the results of [PY] are required only to justify that the group scheme  $\widetilde{G}_{\mathbf{k}}$  is reduced. It would be desirable to find a direct justification for this fact (but we were not able to do so).

## 15 Complement: restriction to a Levi subgroup

In this subsection we construct a geometric counterpart of the functor of restriction to a Levi subgroup, following [BD, §§5.3.27–31]. This construction plays a key role in various applications of the geometric Satake equivalence, see e.g. [BrG, AHR].

### 15.1 The geometric restriction functor

Let  $P \subset G$  be a parabolic subgroup containing  $B$ , and let  $L \subset P$  be the Levi factor containing  $T$ . If  $B_L = B \cap L$ , then  $B_L$  is a Borel subgroup of  $L$ , and  $P$  is determined by the subset  $\Delta_s(L, B_L, T) \subset \Delta_s(G, B, T)$ .

The embedding  $L \hookrightarrow G$  induces a closed embedding  $\mathrm{Gr}_L \hookrightarrow \mathrm{Gr}_G$ , whose image identifies with the fixed points  $(\mathrm{Gr}_G)^{Z(L)^\circ}$  (where  $Z(L) \subset L$  is the center of  $L$ , and  $Z(L)^\circ$  is the identity component of  $Z(L)$ ). In fact, choose a dominant cocharacter  $\eta \in X_*(T)$  which is orthogonal to the simple roots in  $\Delta_s(L, B_L, T)$ , but not to any other simple root. Then (the image of)  $\mathrm{Gr}_L$  identifies with  $(\mathrm{Gr}_G)^{\eta(\mathbf{C}^\times)}$ . We will denote by  $\mathcal{S}_L$  the stratification of  $\mathrm{Gr}_L$  by  $L_{\mathcal{O}}$ -orbits.

The connected components of the affine Grassmannian  $\mathrm{Gr}_L$  are in a canonical bijection with the quotient  $X_*(T)/\mathbf{Z}\Delta^\vee(L, T)$ ; see §3.1. If  $c$  belongs to this quotient, then we denote by  $\mathrm{Gr}_L^c$  the corresponding connected component of  $\mathrm{Gr}_L$  and we set

$$S_c := \left\{ x \in \mathrm{Gr}_G \mid \lim_{a \rightarrow 0} (\eta(a) \cdot x) \in \mathrm{Gr}_L^c \right\};$$

$$T_c := \left\{ x \in \mathrm{Gr}_G \mid \lim_{a \rightarrow \infty} (\eta(a) \cdot x) \in \mathrm{Gr}_L^c \right\}.$$

If  $N_P \subset P$  is the unipotent radical and  $N_{\bar{P}} \subset G$  is the unipotent radical of the parabolic subgroup of  $G$  which is opposite to  $P$  with respect to  $T$ , then we have

$$S_c = (N_P)_{\mathcal{K}} \cdot \mathrm{Gr}_L^c, \quad T_c = (N_{\bar{P}})_{\mathcal{K}} \cdot \mathrm{Gr}_L^c.$$

We will denote by

$$\mathrm{Gr}_G \xleftarrow{s_c} S_c \xrightarrow{\sigma_c} \mathrm{Gr}_L^c, \quad \mathrm{Gr}_G \xleftarrow{t_c} T_c \xrightarrow{\tau_c} \mathrm{Gr}_L^c$$

the natural maps.

If  $\rho_L$  is the half sum of the positive roots of  $L$  determined by  $B_L$ , then for any  $\lambda \in \Delta^\vee(L, T)$  we have  $\langle 2\rho - 2\rho_L, \lambda \rangle = 0$ . It follows that the pairing  $\langle 2\rho - 2\rho_L, c \rangle$  makes sense for  $c \in X_*(T)/\mathbf{Z}\Delta^\vee(L, T)$ .

**Lemma 15.1.** *For any  $c \in X_*(T)/\mathbf{Z}\Delta^\vee(L, T)$  and any  $\mathcal{F}$  in  $\mathrm{P}_{G_{\mathcal{O}}}(\mathrm{Gr}_G, \mathbf{k})$ , there exists a canonical isomorphism*

$$(\tau_c)_*(t_c)! \mathcal{F} \xrightarrow{\sim} (\sigma_c)_!(s_c)^* \mathcal{F}$$

in  $D_{\mathcal{S}_L}^b(\mathrm{Gr}_L, \mathbf{k})$ . Moreover, this complex is concentrated in perverse degree  $\langle 2\rho - 2\rho_L, c \rangle$ .

*Proof.* As in the case  $L = T$  (see Proposition 10.1), the isomorphism follows from Braden's hyperbolic localization theorem [Br, Theorem 1]. If, for  $\lambda \in X_*(T)$ , we denote by  $S_\lambda^L, T_\lambda^L \subset$

$\mathrm{Gr}_L$  the semi-infinite orbits for the group  $L$ , then for any  $\lambda \in c$  the base change isomorphism provides a canonical isomorphism

$$\mathbf{H}_c^\bullet(S_\lambda^L, (\sigma_c)_!(s_c)^*\mathcal{F}) \cong \mathbf{H}_c^\bullet(S_\lambda, \mathcal{F}).$$

By Lemma 10.6, this implies that  $(\sigma_c)_!(s_c)^*\mathcal{F}[-\langle 2\rho - 2\rho_L, c \rangle]$  is a perverse sheaf, and finishes the proof.  $\square$

In view of this lemma, for  $c \in X_*(T)/\mathbf{Z}\Delta^\vee(L, T)$  we consider the functor

$$F_c := (\sigma_c)_!(s_c)^*(?)[-\langle 2\rho - 2\rho_L, c \rangle] : P_{G_\mathcal{O}}(\mathrm{Gr}_G, \mathbf{k}) \rightarrow P_{L_\mathcal{O}}(\mathrm{Gr}_L, \mathbf{k}).$$

We also set

$$R_L^G := \bigoplus_{c \in X_*(T)/\mathbf{Z}\Delta^\vee(L, T)} F_c : P_{G_\mathcal{O}}(\mathrm{Gr}_G, \mathbf{k}) \rightarrow P_{L_\mathcal{O}}(\mathrm{Gr}_L, \mathbf{k}).$$

The arguments of Lemma 15.1 provide, for any  $\lambda \in X_*(T)$ , a canonical isomorphism

$$F_\lambda^L \circ R_L^G \xrightarrow{\sim} F_\lambda \quad (15.1)$$

(where  $F_\lambda^L$  is the  $\lambda$ -weight functor for the group  $L$ ). In particular, summing over  $\lambda$  and using Theorem 10.4 we deduce a canonical isomorphism of functors.

$$F^L \circ R_L^G \cong F$$

where  $F^L := \mathbf{H}^\bullet(\mathrm{Gr}_L, ?)$ .

**Proposition 15.2.** *The functor  $R_L^G$  sends the convolution product on  $P_{G_\mathcal{O}}(\mathrm{Gr}_G, \mathbf{k})$  to the convolution product on  $P_{L_\mathcal{O}}(\mathrm{Gr}_L, \mathbf{k})$ , in a way compatible with associativity and commutativity constraints.*

*Proof.* Recall the objects considered in Section 7. As in the proof of Proposition 8.3 (which was only concerned with the case  $L = T$ ) we can consider “relative” versions  $S_c(X) \subset \mathrm{Gr}_{G, X}$ ,  $S_c(X^2) \subset \mathrm{Gr}_{G, X^2}$  of the varieties  $S_c$ , and denote the corresponding embeddings and projections by

$$\begin{aligned} \tilde{s}_c : S_c(X) &\rightarrow \mathrm{Gr}_{G, X}, & \tilde{\sigma}_c : S_c(X) &\rightarrow \mathrm{Gr}_{L, X}^c, \\ \tilde{s}_c^2 : S_c(X^2) &\rightarrow \mathrm{Gr}_{G, X^2}, & \tilde{\sigma}_c^2 : S_c(X^2) &\rightarrow \mathrm{Gr}_{L, X^2}^c, \end{aligned}$$

where  $\mathrm{Gr}_{L, X}^c$  and  $\mathrm{Gr}_{L, X^2}^c$  are the connected components of  $\mathrm{Gr}_{L, X}$  and  $\mathrm{Gr}_{L, X^2}$  defined by  $c$ . Here, for  $x \in X$ , the fiber of  $S_c(X^2)$  over  $(x, x) \in X^2$  is canonically identified with  $S_c$ , and the fiber over  $(x_1, x_2)$  with  $x_1 \neq x_2$  is canonically identified with  $\bigsqcup_{c_1+c_2=c} S_{c_1} \times S_{c_2}$ .

Now, consider the diagram

$$\begin{array}{ccccc} (\mathrm{Gr}_{G, X} \times \mathrm{Gr}_{G, X})|_U & \xrightarrow{j} & \mathrm{Gr}_{G, X^2} & \xleftarrow{i} & \mathrm{Gr}_{G, X} \\ (\tilde{s}_c^2)|_U \uparrow & & \uparrow \tilde{s}_c^2 & & \uparrow \tilde{s}_c \\ \bigsqcup_{c_1+c_2=c} (S_{c_1}(X) \times S_{c_2}(X))|_U & \xrightarrow{j^c} & S_c(X^2) & \xleftarrow{i_c} & S_c(X) \\ (\tilde{\sigma}_c^2)|_U \downarrow & & \downarrow \tilde{\sigma}_c^2 & & \downarrow \tilde{\sigma}_c \\ \bigsqcup_{c_1+c_2=c} (\mathrm{Gr}_{L, X}^{c_1} \times \mathrm{Gr}_{L, X}^{c_2})|_U & \xrightarrow{j_L^c} & \mathrm{Gr}_{L, X^2}^c & \xleftarrow{i_L^c} & \mathrm{Gr}_{L, X}^c \end{array} \quad (15.2)$$



where  $i_c$  and  $j_c$  are the restrictions of  $i$  and  $j$ . All the squares in this diagram are Cartesian by [DrG, Lemma 1.4.9]. Moreover,  $(\tilde{s}_c^2)|_U$  identifies with the restriction to  $U$  of the disjoint union of inclusions  $\tilde{s}_{c_1} \times \tilde{s}_{c_2}$ , and similarly for  $\tilde{\sigma}_c^2$ .

We fix  $\mathcal{A}_1, \mathcal{A}_2$  in  $\mathbf{P}_{G_{\mathcal{O}}}(\mathrm{Gr}_G, \mathbf{k})$ . Then by (10.2) we have

$$\tau^\circ(\mathcal{A}_1 \star \mathcal{A}_2) \cong i^\circ j_{!*}(p\mathcal{H}^0(\tau^\circ \mathcal{A}_1 \boxtimes_{\mathbf{k}}^L \tau^\circ \mathcal{A}_2)|_U).$$

We set

$$\tilde{F}_c := (\tilde{\sigma}_c)_!(\tilde{s}_c)^*(?)[- \langle 2\rho - 2\rho_L, c \rangle], \quad \tilde{F}_c^2 := (\tilde{\sigma}_c^2)_!(\tilde{s}_c^2)^*(?)[- \langle 2\rho - 2\rho_L, c \rangle].$$

Then on the one hand we have

$$\tilde{F}_c(\tau^\circ(\mathcal{A}_1 \star \mathcal{A}_2)) \cong (\tau_L)^\circ(\mathbf{F}_c(\mathcal{A}_1 \star \mathcal{A}_2)), \quad (15.3)$$

and on the other hand we have

$$\tilde{F}_c\left(i^\circ j_{!*}(p\mathcal{H}^0(\tau^\circ \mathcal{A}_1 \boxtimes_{\mathbf{k}}^L \tau^\circ \mathcal{A}_2)|_U)\right) \cong (i_L^c)^\circ\left(\tilde{F}_c^2 \circ j_{!*}(p\mathcal{H}^0(\tau^\circ \mathcal{A}_1 \boxtimes_{\mathbf{k}}^L \tau^\circ \mathcal{A}_2)|_U)\right)$$

by the base change theorem. We claim that

$$\begin{aligned} & \tilde{F}_c^2 \circ j_{!*}(p\mathcal{H}^0(\tau^\circ \mathcal{A}_1 \boxtimes_{\mathbf{k}}^L \tau^\circ \mathcal{A}_2)|_U) \\ & \cong (j_L^c)_* \left( \bigoplus_{c_1+c_2=c} p\mathcal{H}^0((\tau_L)^\circ \mathbf{F}_{c_1}(\mathcal{A}_1) \boxtimes_{\mathbf{k}}^L (\tau_L)^\circ \mathbf{F}_{c_2}(\mathcal{A}_2))|_U \right). \end{aligned} \quad (15.4)$$

In fact, to check this it suffices to prove that the left-hand side satisfies the properties (4.2) which characterize the right-hand side. The isomorphism over  $U$  follows from the base change theorem applied in the left-hand side of diagram (15.2) and the description above of the maps  $(\tilde{s}_c^2)|_U$  and  $(\tilde{\sigma}_c^2)|_U$ . The restriction of our complex to the inverse image of  $X$  is computed in (15.3), and satisfies the required property. Finally, the co-restriction to the inverse image of  $X$  can be computed similarly, using the other description of the functors  $\tilde{F}_c$  and  $\tilde{F}_c^2$  provided by Braden's theorem.<sup>36</sup> Finally, comparing (15.3) and (15.4) and using the isomorphism (10.2) for  $L$ , we obtain a canonical isomorphism

$$(\tau_L)^\circ(\mathbf{F}_c(\mathcal{A}_1 \star \mathcal{A}_2)) \cong \bigoplus_{c_1+c_2=c} (\tau_L)^\circ(\mathbf{F}_{c_1}(\mathcal{A}_1) \star \mathbf{F}_{c_2}(\mathcal{A}_2)).$$

Restricting to a point in  $x$  and then summing over  $c$ , we deduce the wished-for isomorphism

$$\mathbf{R}_L^G(\mathcal{A}_1 \star \mathcal{A}_2) \cong \mathbf{R}_L^G(\mathcal{A}_1) \star \mathbf{R}_L^G(\mathcal{A}_2).$$

The proof of compatibility with the constraints is left to the reader.  $\square$

<sup>36</sup>Here we need to apply Braden's theorem on a finite-dimensional subvariety of  $\mathrm{Gr}_{G, X^2}$ . Since such a variety is not necessarily normal, the proof in [Br] does not apply in this context. The more general form of this result that we need is proved in [DrG].

## 15.2 Description of the induced morphism of group schemes

The results of Section 13 provide canonical equivalences of monoidal categories

$$P_{G_{\mathcal{O}}}(\mathrm{Gr}_G, \mathbf{k}) \cong \mathrm{Rep}_{\mathbf{k}}(\tilde{G}_{\mathbf{k}}), \quad P_{L_{\mathcal{O}}}(\mathrm{Gr}_L, \mathbf{k}) \cong \mathrm{Rep}_{\mathbf{k}}(\tilde{L}_{\mathbf{k}}).$$

In view of [Mi, Theorem X.1.2], the functor  $R_L^G$  defines a  $\mathbf{k}$ -group scheme morphism

$$\varphi_L^G : \tilde{L}_{\mathbf{k}} \rightarrow \tilde{G}_{\mathbf{k}}.$$

The isomorphisms (15.1) show that the composition of  $\varphi_L^G$  with the canonical embedding  $T_{\mathbf{k}}^{\vee} \rightarrow \tilde{L}_{\mathbf{k}}$  (see §14.2) is the canonical morphism  $T_{\mathbf{k}}^{\vee} \rightarrow \tilde{G}_{\mathbf{k}}$ .

**Proposition 15.3.** *The morphism  $\varphi_L^G$  is a closed embedding, which induces an isomorphism between  $\tilde{L}_{\mathbf{k}}$  and the Levi subgroup<sup>37</sup> of  $\tilde{G}_{\mathbf{k}}$  containing  $T_{\mathbf{k}}^{\vee}$  whose roots are the coroots of  $L$ .*

*Proof.* First, we assume that  $\mathbf{k}$  is a field. In this case, by [DM, Proposition 2.21(b)], to prove that  $\varphi_L^G$  is a closed embedding it suffices to prove that any object of  $P_{L_{\mathcal{O}}}(\mathrm{Gr}_L, \mathbf{k})$  is a subquotient of an object in the essential image of  $R_L^G$ . However, as in the proof of Lemma 14.2, any object of  $P_{L_{\mathcal{O}}}(\mathrm{Gr}_L, \mathbf{k})$  is a subquotient of a tilting object. Now the functor  $R_L^G$  sends tilting objects of  $P_{G_{\mathcal{O}}}(\mathrm{Gr}_G, \mathbf{k})$  to tilting objects of  $P_{L_{\mathcal{O}}}(\mathrm{Gr}_L, \mathbf{k})$ . (In the case  $\mathrm{char}(\mathbf{k})$  is good for  $G$ , this fact follows from [JMW2, Theorem 1.6] and the results of [MR, §1.5]; the general case is treated in [BGMRR].) Moreover, it is not difficult to check that if  $\lambda \in X_*(T)$  is dominant for  $L$ , then the indecomposable tilting object in  $P_{L_{\mathcal{O}}}(\mathrm{Gr}_L, \mathbf{k})$  labelled by  $\lambda$  is a direct summand of the image under  $R_L^G$  of the indecomposable tilting object in  $P_{G_{\mathcal{O}}}(\mathrm{Gr}_G, \mathbf{k})$  labelled by the unique  $W$ -conjugate of  $\lambda$  belonging to  $X_*(T)^+$ . It follows that any tilting object in  $P_{L_{\mathcal{O}}}(\mathrm{Gr}_L, \mathbf{k})$  is a direct summand of an object in the essential image of  $R_L^G$ , which finishes the proof of the fact that  $\varphi_L^G$  is a closed embedding.

Once this fact is established, we note that since  $\varphi_L^G$  intertwines the canonical morphisms  $T_{\mathbf{k}}^{\vee} \rightarrow \tilde{L}_{\mathbf{k}}$  and  $T_{\mathbf{k}}^{\vee} \rightarrow \tilde{G}_{\mathbf{k}}$ , it must induce, for any  $\alpha \in \Delta_s^{\vee}(L, B_L, T)$ , an isomorphism between the root subgroup of  $\tilde{L}_{\mathbf{k}}$  associated with  $\alpha$  and the root subgroup of  $\tilde{G}_{\mathbf{k}}$  associated with  $\alpha$ . Now the group  $\tilde{L}_{\mathbf{k}}$ , resp. the Levi subgroup  $\tilde{L}'_{\mathbf{k}}$  of  $\tilde{G}_{\mathbf{k}}$  containing  $T_{\mathbf{k}}^{\vee}$  whose roots are the coroots of  $L$ , is generated by  $T_{\mathbf{k}}^{\vee}$  and these subgroups. We deduce that the image of  $\varphi_L^G$  is  $\tilde{L}'_{\mathbf{k}}$ , or in other words that  $\varphi_L^G$  induces an isomorphism between  $\tilde{L}_{\mathbf{k}}$  and  $\tilde{L}'_{\mathbf{k}}$ .

Now we treat the case  $\mathbf{k} = \mathbf{Z}$ . Consider the morphism  $(\varphi_L^G)^* : \mathbf{Z}[\tilde{G}_{\mathbf{Z}}] \rightarrow \mathbf{Z}[\tilde{L}_{\mathbf{Z}}]$ . If  $C$  is the cokernel of this morphism, then  $C$  is a finitely generated  $\mathbf{Z}[\tilde{G}_{\mathbf{Z}}]$ -module which satisfies  $C \otimes_{\mathbf{Z}} \mathbf{F} = 0$  for any field  $\mathbf{F}$ . By [BR, Claim (\*) in the proof of Lemma 1.4.1], it follows that  $C = 0$ , i.e. that  $(\varphi_L^G)^*$  is surjective, and hence that  $\varphi_L^G$  is a closed embedding. It is easily checked, using similar arguments, that the image of  $\varphi_L^G$  satisfies condition (b) in [SGA3, Exposé XXVI, Proposition 1.6(ii)] (for the parabolic subgroup containing  $T_{\mathbf{Z}}^{\vee}$  and whose roots are  $\Delta_+^{\vee}(L, B_L, T) \sqcup (-\Delta_+^{\vee}(G, B, T))$ ). By the unicity claim in this statement, it follows that this image is the Levi subgroup of  $\tilde{G}_{\mathbf{Z}}$  containing  $T_{\mathbf{Z}}^{\vee}$  whose roots are the coroots of  $L$ .

Finally, the general case follows from the case  $\mathbf{k} = \mathbf{Z}$  by base change.  $\square$

<sup>37</sup>See [SGA3, Exposé XXVI, §1.7] for the notion of Levi subgroup of a reductive group over a base scheme.

## A Equivariant perverse sheaves

### A.1 Equivariant perverse sheaves

Let  $X$  be a complex algebraic variety, let  $H$  be a connected<sup>38</sup> algebraic group acting on  $X$ , and consider a commutative Noetherian ring of finite global dimension  $\mathbf{k}$ . Let

$$a, p : H \times X \rightarrow X, \quad e : X \rightarrow H \times X$$

be the maps defined by

$$p(g, x) = x, \quad a(g, x) = g \cdot x, \quad e(x) = (1, x).$$

Let also  $p_{23} : H \times H \times X \rightarrow H \times X$  be the projection on the last two components, and  $m : H \times H \rightarrow H$  be the multiplication map.

Let  $\mathcal{T}$  be a stratification of  $X$  whose strata are stable under the  $H$ -action. Then there are at least 3 “reasonable” definitions of the category of  $\mathcal{T}$ -constructible  $H$ -equivariant perverse sheaves on  $X$ :

1. the heart  $\mathbf{P}_{\mathcal{T}, H}^{\#}(X, \mathbf{k})$  of the perverse t-structure on the  $\mathcal{T}$ -constructible equivariant derived category  $D_{\mathcal{T}, H}^b(X, \mathbf{k})$  in the sense of Bernstein–Lunts, see [BL, §5];
2. the category  $\mathbf{P}_{\mathcal{T}, H}^b(X, \mathbf{k})$  whose objects are pairs  $(\mathcal{F}, \vartheta)$  where  $\mathcal{F} \in \mathbf{P}_{\mathcal{T}}(X, \mathbf{k})$  and  $\vartheta : a^* \mathcal{F} \rightarrow p^* \mathcal{F}$  is an isomorphism such that

$$e^*(\vartheta) = \text{id}_{\mathcal{F}} \quad \text{and} \quad (m \times \text{id}_X)^*(\vartheta) = (p_{23})^*(\vartheta) \circ (\text{id}_H \times a)^*(\vartheta), \quad (\text{A.1})$$

and whose morphisms from  $(\mathcal{F}, \vartheta)$  to  $(\mathcal{F}', \vartheta')$  are morphisms  $f : \mathcal{F} \rightarrow \mathcal{F}'$  in  $\mathbf{P}_{\mathcal{T}}(X, \mathbf{k})$  such that the following diagram commutes:

$$\begin{array}{ccc} a^* \mathcal{F} & \xrightarrow{\vartheta} & p^* \mathcal{F} \\ a^*(f) \downarrow & & \downarrow p^*(f) \\ a^* \mathcal{F}' & \xrightarrow{\vartheta'} & p^* \mathcal{F}' \end{array};$$

3. the full subcategory  $\mathbf{P}_{\mathcal{T}, H}(X, \mathbf{k})$  of  $\mathbf{P}_{\mathcal{T}}(X, \mathbf{k})$  consisting of objects  $\mathcal{F}$  such that there exists an isomorphism  $p^* \mathcal{F} \cong a^* \mathcal{F}$ .

There exists an obvious forgetful functor  $\mathbf{P}_{\mathcal{T}, H}^b(X, \mathbf{k}) \rightarrow \mathbf{P}_{\mathcal{T}, H}(X, \mathbf{k})$ . Next, we will define a canonical functor

$$\mathbf{P}_{\mathcal{T}, H}^{\#}(X, \mathbf{k}) \rightarrow \mathbf{P}_{\mathcal{T}, H}^b(X, \mathbf{k}). \quad (\text{A.2})$$

For this we need the following observation. We denote by  $\text{For}_H : D_{\mathcal{T}, H}^b(X, \mathbf{k}) \rightarrow D_{\mathcal{T}}^b(X, \mathbf{k})$  the forgetful functor. The morphism  $p$  is a  $\phi$ -morphism of varieties in the sense of [BL, §0.1],

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<sup>38</sup>This assumption is crucial; in case  $H$  is disconnected, only the first definition of equivariant perverse sheaves has favorable properties.

where  $\phi$  is the unique morphism  $H \rightarrow \{1\}$  and where  $H$  acts on  $H \times X$  via left multiplication on the first factor. Therefore, this map defines a functor

$$p^* : D_{\mathcal{F}}^b(X, \mathbf{k}) \rightarrow D_{\widetilde{\mathcal{F}}, H}^b(H \times X, \mathbf{k}) \quad (\text{A.3})$$

(where  $\widetilde{\mathcal{F}}$  is the stratification of  $H \times X$  whose strata are the subvarieties  $H \times S$  with  $S \in \mathcal{F}$ ), see [BL, §6.5].

**Lemma A.1.** *For any  $\mathcal{F}$  in  $D_{\mathcal{F}, H}^b(X, \mathbf{k})$ , there exists a canonical isomorphism*

$$a^* \mathcal{F} \xrightarrow{\sim} p^* \text{For}_H(\mathcal{F})$$

in  $D_{\widetilde{\mathcal{F}}, H}^b(H \times X, \mathbf{k})$ .

*Proof.* In view of [BL, §6.6, Item 5], the functor (A.3) is an equivalence of categories, whose quasi-inverse is the composition  $e^* \circ \text{For}_H$  (where we also denote by  $\text{For}_H$  the forgetful functor  $D_{\widetilde{\mathcal{F}}, H}^b(H \times X, \mathbf{k}) \rightarrow D_{\mathcal{F}}^b(X, \mathbf{k})$ ). Therefore, to define an isomorphism as in the lemma it suffices to construct an isomorphism

$$e^* \circ \text{For}_H(a^* \mathcal{F}) \xrightarrow{\sim} \text{For}_H(\mathcal{F}).$$

In fact, such an isomorphism is clear from the facts that  $a^*$  commutes with forgetful functors in the obvious way and that  $a \circ e = \text{id}_X$ .  $\square$

If  $\mathcal{F}$  is in  $P_{\mathcal{F}, H}^\#(X, \mathbf{k})$ , applying the forgetful functor to the isomorphism of Lemma A.1 we obtain a canonical isomorphism  $\vartheta : a^* \text{For}_H(\mathcal{F}) \xrightarrow{\sim} p^* \text{For}_H(\mathcal{F})$  in  $D_{\widetilde{\mathcal{F}}, H}^b(H \times X, \mathbf{k})$ . We leave it to the reader to check that this isomorphism satisfies the conditions (A.1); then the pair  $(\text{For}_H(\mathcal{F}), \vartheta)$  defines an object of  $P_{\mathcal{F}, H}^b(X, \mathbf{k})$ . This construction provides the wished-for functor (A.2).

The following result is well known, but not explicitly proved in the literature to the best of our knowledge (except for a very brief treatment in [MV1, Appendix A]).

**Proposition A.2.** *The forgetful functors*

$$P_{\mathcal{F}, H}^\#(X, \mathbf{k}) \rightarrow P_{\mathcal{F}, H}^b(X, \mathbf{k}) \rightarrow P_{\mathcal{F}, H}(X, \mathbf{k})$$

are equivalences of categories.

In view of this proposition, in the body of these notes we identify the three categories above, and denote them by  $P_{\mathcal{F}, H}(X, \mathbf{k})$ .

In the proof of this proposition we will use the fact (see [BBD, Théorème 3.2.4]) that perverse sheaves form a stack for the smooth topology. In our particular case, if  $\pi : P \rightarrow X$  is a smooth resolution (in the sense of [BL]),  $\mathcal{U}$  denotes the stratification on  $P$  whose strata are the subsets  $\pi^{-1}(S)$  for  $S \in \mathcal{F}$ ,  $\mathcal{V}$  denotes the stratification on  $P/H$  whose strata are the subsets  $q(U)$  with  $U \in \mathcal{U}$  (where  $q : P \rightarrow P/H$  is the projection), and if

$$r_1, r_2 : P \times_{P/H} P \rightarrow P, \quad r_{12}, r_{23}, r_{13} : P \times_{P/H} P \times_{P/H} P \rightarrow P \times_{P/H} P$$

are the natural projections, this means that the category  $\mathbf{P}_{\mathcal{V}}(P/H, \mathbf{k})$  is equivalent, via the functor  $q^*$ , to the category whose objects are pairs  $(\mathcal{F}, \sigma)$  where  $\mathcal{F} \in \mathbf{P}_{\mathcal{U}}(P, \mathbf{k})[-\dim(H)]$  and  $\sigma : (r_1)^*\mathcal{F} \xrightarrow{\sim} (r_2)^*\mathcal{F}$  is an isomorphism such that  $(r_{23})^*(\sigma) \circ (r_{12})^*(\sigma) = (r_{13})^*(\sigma)$ , and whose morphisms  $(\mathcal{F}, \sigma) \rightarrow (\mathcal{F}', \sigma')$  are morphisms  $f \in \mathrm{Hom}_{D_{\mathcal{U}}^b(P, \mathbf{k})}(\mathcal{F}, \mathcal{F}')$  such that  $(r_2)^*(f) \circ \sigma = \sigma' \circ (r_1)^*(f)$ .

With this result at hand we can give the proof of Proposition A.2.

*Proof.* The second functor is an equivalence by [Le, §4.2.10]. Hence what remains to be proved is that the composition  $\mathbf{P}_{\mathcal{F}, H}^{\#}(X, \mathbf{k}) \rightarrow \mathbf{P}_{\mathcal{F}, H}(X, \mathbf{k})$  is an equivalence.

Fix a free  $H$ -space  $P$  and a smooth  $\dim(X)$ -acyclic map  $\pi : P \rightarrow X$  of relative dimension  $d$  (which exist thanks to the results of [BL, §3.1]), and let  $q : P \rightarrow P/H$  be the quotient morphism. Then  $\mathbf{P}_{\mathcal{F}, H}^{\#}(X, \mathbf{k})$  is (by definition, see [BL, §2.2.4]) equivalent to the category whose objects are the triples  $(\mathcal{F}_P, \mathcal{F}_X, \beta)$  where  $\mathcal{F}_P \in D_c^b(P/H, \mathbf{k})$ ,  $\mathcal{F}_X \in \mathbf{P}_{\mathcal{F}}(X, \mathbf{k})$  and  $\beta : q^*\mathcal{F}_P \xrightarrow{\sim} \pi^*\mathcal{F}_X$  is an isomorphism, and whose morphisms from  $(\mathcal{F}_P, \mathcal{F}_X, \beta)$  to  $(\mathcal{F}'_P, \mathcal{F}'_X, \beta')$  are the pairs  $(f_P, f_X)$  with  $f_P : \mathcal{F}_P \rightarrow \mathcal{F}'_P$  and  $f_X : \mathcal{F}_X \rightarrow \mathcal{F}'_X$  compatible (in the natural sense) with  $\beta$  and  $\beta'$ .

First we show that our functor is faithful. Let  $(f_P, f_X) : (\mathcal{F}_P, \mathcal{F}_X, \beta) \rightarrow (\mathcal{F}'_P, \mathcal{F}'_X, \beta')$  be a morphism in  $\mathbf{P}_{\mathcal{F}, H}^{\#}(X, \mathbf{k})$  such that  $f_X = 0$ . Then by the compatibility of  $(f_P, f_X)$  with  $\beta$  and  $\beta'$  we deduce that  $q^*(f_P) = 0$ . Now it is easily seen that  $\mathcal{F}_P$  belongs to  $\mathbf{P}_{\mathcal{V}}(P/H, \mathbf{k})[\dim(H) - d]$ . Since  $q$  is smooth with connected fibers, the functor  $q^*$  is fully faithful on perverse sheaves (see [BBD, Proposition 4.2.5]); we deduce that  $f_P = 0$ , finishing the proof of faithfulness.

Next we prove that our functor is full. Let  $(\mathcal{F}_P, \mathcal{F}_X, \beta)$  and  $(\mathcal{F}'_P, \mathcal{F}'_X, \beta')$  be in  $\mathbf{P}_{\mathcal{F}, H}^{\#}(X, \mathbf{k})$ , and let  $f : \mathcal{F}_X \rightarrow \mathcal{F}'_X$  be a morphism. To construct a morphism  $f_P : \mathcal{F}_P \rightarrow \mathcal{F}'_P$  such that  $\beta' \circ q^*(f_P) = \pi^*(f) \circ \beta$ , we use the stack property recalled above: we remark that the morphism  $(\beta')^{-1} \circ \pi^*(f) \circ \beta$  satisfies the descent condition, hence is of the form  $q^*(f_P)$  for a unique morphism  $f_P : \mathcal{F}_P \rightarrow \mathcal{F}'_P$ .

Finally, we prove that our functor is essentially surjective. Let  $\mathcal{F}$  be in  $\mathbf{P}_{\mathcal{F}, H}(X, \mathbf{k})$ . Then there exists a (unique) isomorphism  $\vartheta : a^*(\mathcal{F}) \rightarrow p^*(\mathcal{F})$  which satisfies the conditions (A.1). Identifying  $H \times P$  with  $P \times_{P/H} P$  via the morphism  $(a, p)$ ,  $(\mathrm{id}_H \times \pi)^*(\vartheta)$  defines an isomorphism  $\sigma : (r_1)^*(\pi^*\mathcal{F}) \rightarrow (r_2)^*(\pi^*\mathcal{F})$ . Identifying  $H \times H \times P$  with  $P \times_{P/H} P \times_{P/H} P$  via  $(g, h, x) \mapsto (ghx, hx, x)$ , we see that the second condition in (A.1) guarantees that  $\sigma$  satisfies the descent condition, so that the pair  $(\pi^*\mathcal{F}, \sigma)$  defines an object  $\mathcal{F}_P \in D_c^b(P/H, \mathbf{k})$  such that  $\pi^*\mathcal{F} \cong q^*\mathcal{F}_P$ . Fixing such an isomorphism, we obtain an object of  $\mathbf{P}_{\mathcal{F}, H}^{\#}(X, \mathbf{k})$  whose image in  $\mathbf{P}_{\mathcal{F}, H}(X, \mathbf{k})$  is  $\mathcal{F}$ .  $\square$

## A.2 Induction

Let  $X$ ,  $H$  and  $\mathbf{k}$  be as in §A.1. We consider the constructible derived category  $D_c^b(X, \mathbf{k})$  of  $\mathbf{k}$ -sheaves on  $X$ , and its  $H$ -equivariant version  $D_{c, H}^b(X, \mathbf{k})$ . We also denote by

$$\mathrm{For}_H : D_{c, H}^b(X, \mathbf{k}) \rightarrow D_c^b(X, \mathbf{k})$$

the forgetful functor. Recall that if  $H \times X$  is considered as an  $H$ -variety via left multiplication on the first factor, and if  $p : H \times X \rightarrow X$  is the projection, then the functor  $p^!$  induces an equivalence of categories  $D_c^b(X, \mathbf{k}) \rightarrow D_{c,H}^b(H \times X, \mathbf{k})$ , see [BL, Proposition 2.2.5]. We consider the functor

$$\mathrm{ind}_H : D_c^b(X, \mathbf{k}) \rightarrow D_{c,H}^b(X, \mathbf{k})$$

defined by

$$\mathrm{ind}_H(\mathcal{F}) = a_! p^!(\mathcal{F}).$$

**Lemma A.3.** *The functor  $\mathrm{ind}_H$  is left adjoint to  $\mathrm{For}_H$ .*

*Proof.* Let  $\mathcal{F}$  in  $D_c^b(X, \mathbf{k})$  and  $\mathcal{G}$  in  $D_{c,H}^b(X, \mathbf{k})$ . Using first the fact that  $p^!$  is an equivalence, then Lemma A.1, and finally adjunction, we obtain canonical isomorphisms

$$\begin{aligned} \mathrm{Hom}_{D_c^b(X, \mathbf{k})}(\mathcal{F}, \mathrm{For}_H(\mathcal{G})) &\cong \mathrm{Hom}_{D_{c,H}^b(H \times X, \mathbf{k})}(p^! \mathcal{F}, p^! \mathrm{For}_H(\mathcal{G})) \\ &\cong \mathrm{Hom}_{D_{c,H}^b(H \times X, \mathbf{k})}(p^! \mathcal{F}, a^! \mathcal{G}) \cong \mathrm{Hom}_{D_{c,H}^b(X, \mathbf{k})}(a_! p^! \mathcal{F}, \mathcal{G}). \end{aligned}$$

The claim follows.  $\square$

### A.3 Convolution

Let  $H$  be a complex algebraic group, and let  $K \subset H$  be a closed subgroup. Recall that the  $K$ -bundle given by the quotient morphism  $H \rightarrow H/K$  is locally trivial for the analytic topology, see [S1]. (In all the cases we will consider, this morphism is in fact locally trivial for the Zariski topology.) We consider the constructible equivariant derived category  $D_{c,K}^b(H/K, \mathbf{k})$ . This category admits a natural convolution bifunctor, constructed as follows. Consider the diagram

$$H/K \times H/K \xleftarrow{p} H \times H/K \xrightarrow{q} H \times^K H/K \xrightarrow{m} H/K, \quad (\text{A.4})$$

where  $H \times^K H/K$  is the quotient of  $H \times H/K$  by the action defined by  $k \cdot (g, hK) = (gk^{-1}, khK)$  for  $k \in K$  and  $g, h \in H$ ,  $q$  is the quotient morphism, and the maps  $p$  and  $m$  are defined by

$$p(g, hK) = (gK, hK), \quad m([g, hK]) = ghK.$$

Since  $K$  acts freely on  $H \times H/K$ , by [BL, Theorem 2.6.2] the functor  $q^*$  induces an equivalence

$$D_{c,K}^b(H \times^K H/K, \mathbf{k}) \xrightarrow{\sim} D_{c,K \times K}^b(H \times H/K, \mathbf{k})$$

(where  $K$  acts on  $H \times^K H/K$  via left multiplication on  $H$ , and  $K \times K$  acts on  $H \times H/K$  via  $(k_1, k_2) \cdot (g, hK) = (k_1 g k_2^{-1}, k_2 hK)$ ). Now, consider some objects  $\mathcal{F}_1, \mathcal{F}_2$  in  $D_{c,K}^b(H/K, \mathbf{k})$ . Then  $\mathcal{F}_1 \boxtimes_{\mathbf{k}}^L \mathcal{F}_2$  belongs to  $D_{c,K \times K}^b(H/K \times H/K, \mathbf{k})$ . Since  $p$  is a  $(K \times K)$ -equivariant morphism,  $p^*(\mathcal{F}_1 \boxtimes_{\mathbf{k}}^L \mathcal{F}_2)$  defines an object in  $D_{c,K \times K}^b(H \times H/K, \mathbf{k})$ . Hence there exists a unique object  $\mathcal{F}_1 \tilde{\boxtimes} \mathcal{F}_2$  in  $D_{c,K}^b(H \times^K H/K, \mathbf{k})$  such that

$$q^*(\mathcal{F}_1 \tilde{\boxtimes} \mathcal{F}_2) \cong p^*(\mathcal{F}_1 \boxtimes_{\mathbf{k}}^L \mathcal{F}_2). \quad (\text{A.5})$$

We then set

$$\mathcal{F}_1 \star \mathcal{F}_2 := m_*(\mathcal{F}_1 \tilde{\boxtimes} \mathcal{F}_2).$$

It is a classical fact that this construction defines a monoidal structure on the category  $D_{c,K}^b(H/K, \mathbf{k})$  (which does not, in general, restrict to a monoidal structure on  $\mathrm{P}_K(H/K, \mathbf{k})$ ).

*Remark A.4.* 1. Since the maps  $p$  and  $q$  are smooth of relative dimension  $\dim(K)$ , we have canonical isomorphisms  $p^! \cong p^*[\dim(K)]$  and  $q^! \cong q^*[\dim(K)]$ , so that the condition (A.5) can be replaced by  $q^!(\mathcal{F}_1 \boxtimes \mathcal{F}_2) \cong p^!(\mathcal{F}_1 \boxtimes_{\mathbf{k}}^L \mathcal{F}_2)$ .

2. In the special case considered for the geometric Satake equivalence, when  $\mathbf{k}$  is not a field one modifies this construction slightly so that it sends pairs of perverse sheaves to perverse sheaves; see §10.3.

#### A.4 The case of $\mathrm{Gr}_G$

The main object of study in these notes is the category  $\mathrm{P}_{G_{\mathcal{O}}}(\mathrm{Gr}_G, \mathbf{k})$ . This setting does not fit exactly in the framework of §§A.1–A.3 because  $G_{\mathcal{K}}$  and  $G_{\mathcal{O}}$  are not algebraic groups in the usual sense. But the category  $D_{c, G_{\mathcal{O}}}^b(\mathrm{Gr}_G, \mathbf{k})$  still makes sense, as follows.

For any  $n \in \mathbf{Z}_{\geq 1}$ , we denote by  $H_n \subset G_{\mathcal{O}}$  the kernel of the morphism

$$G_{\mathcal{O}} \rightarrow G_{\mathcal{O}/t^n \mathcal{O}}$$

induced by the quotient morphism  $\mathcal{O} \rightarrow \mathcal{O}/t^n \mathcal{O}$ . (Here the group scheme  $G_{\mathcal{O}/t^n \mathcal{O}}$  is defined in a way similar to  $G_{\mathcal{O}}$ .) Note that if  $m \geq n$ , then  $H_m$  is a normal subgroup in  $H_n$ , and the quotient  $H_n/H_m$  is a unipotent group. If  $X \subset \mathrm{Gr}_G$  is a closed finite union of  $G_{\mathcal{O}}$ -orbits, there exists  $n \in \mathbf{Z}_{\geq 1}$  such that  $H_n$  acts trivially on  $X$ . Then it makes sense to consider the equivariant derived category  $D_{c, G_{\mathcal{O}}/H_n}^b(X, \mathbf{k})$ . Since  $H_n/H_m$  is unipotent for any  $m \geq n$ , one can check using [BL, Theorem 3.7.3] that the functor

$$D_{c, G_{\mathcal{O}}/H_n}^b(X, \mathbf{k}) \rightarrow D_{c, G_{\mathcal{O}}/H_m}^b(X, \mathbf{k})$$

given by inverse image under the projection  $G/H_m \rightarrow G/H_n$  is an equivalence of categories. Hence one can define the category  $D_{c, G_{\mathcal{O}}}^b(X, \mathbf{k})$  to be  $D_{c, G_{\mathcal{O}}/H_n}^b(X, \mathbf{k})$  for any  $n$  such that  $H_n$  acts trivially on  $X$ .

If  $X \subset Y \subset \mathrm{Gr}_G$  are closed finite unions of  $G_{\mathcal{O}}$ -orbits, the direct image under the embedding  $X \hookrightarrow Y$  induces a fully-faithful functor  $D_{c, G_{\mathcal{O}}}^b(X, \mathbf{k}) \rightarrow D_{c, G_{\mathcal{O}}}^b(Y, \mathbf{k})$ . Hence we can finally define  $D_{c, G_{\mathcal{O}}}^b(\mathrm{Gr}_G, \mathbf{k})$  as the union of the categories  $D_{c, G_{\mathcal{O}}}^b(X, \mathbf{k})$  for all closed finite unions of  $G_{\mathcal{O}}$ -orbits  $X \subset \mathrm{Gr}_G$ .

A construction similar to that of §A.3 produces a convolution bifunctor  $\star$  on the category  $D_{c, G_{\mathcal{O}}}^b(\mathrm{Gr}_G, \mathbf{k})$ . More precisely, if  $\mathcal{F}_1$  and  $\mathcal{F}_2$  are in  $D_{c, G_{\mathcal{O}}}^b(\mathrm{Gr}_G, \mathbf{k})$ , one should choose a closed finite union of  $G_{\mathcal{O}}$ -orbits  $X \subset \mathrm{Gr}_G$  such that  $\mathcal{F}_2$  belongs to  $D_{c, G_{\mathcal{O}}}^b(X, \mathbf{k})$ , and  $n \in \mathbf{Z}_{\geq 1}$  such that  $H_n$  acts trivially on  $X$ , and replace diagram (A.4) by the similar diagram

$$\mathrm{Gr}_G \times X \leftarrow G_{\mathcal{K}}/H_n \times X \rightarrow (G_{\mathcal{K}}/H_n) \times^{(G_{\mathcal{O}}/H_n)} X \rightarrow \mathrm{Gr}_G,$$

and proceed as before. In the body of the paper, as in [MV2], to lighten the notation we neglect these technical subtleties.

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