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Notes on the geometric Satake equivalence

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1 Introduction

1.1 Description

These notes are devoted to a detailed exposition of the proof of the Geometric Satake Equivalence in the case the coefficients are a field of characteristic 0, following Mirković–Vilonen [MV]. We follow their arguments closely, adding only a few details where this might be useful. (On the other hand, some of the proofs are only sketched.)

These notes grew out of a $2\frac{1}{2}$ -days mini-course given during the workshop “Geometric methods and Langlands functoriality in positive characteristic” held in Luminy in January 2016. This mini-course also comprised reminders on constructible sheaves and equivariant derived categories (by D. Fratila), and on perverse sheaves (by V. Heiermann), which are not reproduced in the notes.

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2 Tannakian reconstruction

In this section (where we follow closely [DM, §II]), \mathbf{k} is an arbitrary field, and all categories are essentially small.

Some important ideas of Tannakian reconstruction are already contained in the following easy exercise.

Exercise 2.1. Let A be a \mathbf{k} -algebra, X be a finite-dimensional A -module, and $\alpha \in \text{End}_{\mathbf{k}}(X)$. Show that

$$\alpha \in \text{im}(A \rightarrow \text{End}_{\mathbf{k}}(X)) \iff \forall n \geq 0, \forall Y \subset X^{\oplus n} \text{ } A\text{-submodule, } \alpha^{\oplus n}(Y) \subset Y.$$

(*Hint* : Of course, the implication \Rightarrow is obvious. To prove the reverse direction, assume the condition in the right-hand side holds. Pick a \mathbf{k} -basis (e_1, \dots, e_n) of X , take for Y the A -submodule generated by $(e_1, \dots, e_n) \in X^{\oplus n}$, and write that Y contains $\alpha^{\oplus n}(e_1, \dots, e_n) = (\alpha(e_1), \dots, \alpha(e_n))$.)

Tannakian reconstruction actually amounts to veneer this exercise first with the language of categories and then with the language of Hopf algebras (i.e. affine group schemes).

2.1 A first reconstruction theorem

Let us denote the category of finite-dimensional \mathbf{k} -vector spaces by $\text{Vect}_{\mathbf{k}}$. Given a \mathbf{k} -algebra A , we denote the category of finite-dimensional left A -modules by Mod_A .

Recall that a category \mathcal{C} is called additive (respectively, \mathbf{k} -linear) if each set $\text{Hom}_{\mathcal{C}}(X, Y)$ is an abelian group (respectively, a \mathbf{k} -vector space), if the composition of morphisms is a bilinear operation, and if one can always do finite products in \mathcal{C} . An additive category \mathcal{C} is abelian if each morphism has a kernel and a cokernel, and if for any morphism f , the natural morphism from the cokernel of the kernel (a.k.a. the coimage) of f to the kernel of the cokernel (a.k.a. the image) of f is an isomorphism.

Given an object X in an abelian category \mathcal{C} , we will denote by $\langle X \rangle$ the full subcategory of \mathcal{C} formed by all objects that are isomorphic to a subquotient of a direct sum $X^{\oplus n}$.

Proposition 2.2. *Let \mathcal{C} be an abelian \mathbf{k} -linear category and let $\omega : \mathcal{C} \rightarrow \text{Vect}_{\mathbf{k}}$ be a \mathbf{k} -linear exact faithful functor. Fix an object X in \mathcal{C} and introduce the finite-dimensional \mathbf{k} -algebra*

$$A_X = \{ \alpha \in \text{End}_{\mathbf{k}}(\omega(X)) \mid \forall n \geq 0, \forall Y \subset X^{\oplus n} \text{ subobject, } \alpha^{\oplus n}(\omega(Y)) \subset \omega(Y) \}.$$

Then $\omega|_{\langle X \rangle}$ admits a canonical factorization

$$\begin{array}{ccc} \langle X \rangle & \xrightarrow{\bar{\omega}_X} & \text{Mod}_{A_X} \\ & \searrow \omega & \swarrow \text{forget} \\ & \text{Vect}_{\mathbf{k}} & \end{array}$$

and $\bar{\omega}_X$ is an equivalence of categories. In addition A_X is the endomorphism algebra of the functor $\omega|_{\langle X \rangle}$.

If A is a \mathbf{k} -algebra, and if we apply this proposition to the category $\mathcal{C} = \text{Mod}_A$ with ω the forgetful functor (which keeps the \mathbf{k} -vector space structure but forgets the structure of A -module), then Exercise 2.1 shows that the algebra A_X is precisely the image of A in $\text{End}_{\mathbf{k}}(X)$. The proposition is thus mainly saying that the exercise can be stated within the language of abelian categories.

For the proof of Proposition 2.2 we will need the following standard facts from Category Theory.

- Lemma 2.3.**
1. *An exact additive functor $F : \mathcal{A} \rightarrow \mathcal{B}$ between two abelian categories preserves kernels and cokernels. It thus preserves finite intersections and finite sums (in an ambient object).*
 2. *A faithful functor $F : \mathcal{A} \rightarrow \mathcal{B}$ between two abelian categories does not kill any nonzero object.*
 3. *Let $F : \mathcal{A} \rightarrow \mathcal{B}$ be an exact faithful additive functor between two abelian categories and let $u : X \rightarrow Y$ be a morphism in \mathcal{A} . Then u is an monomorphism (respectively, epimorphism) if and only if $F(u)$ is so.*

4. Let $F : \mathcal{A} \rightarrow \mathcal{B}$ be an exact faithful additive functor between two abelian categories. Assume that \mathcal{B} is Artinian and Noetherian: any monotone sequence of subobjects becomes eventually constant. Then arbitrary intersections and arbitrary sums (in an ambient object) exist in both \mathcal{A} and \mathcal{B} , and F preserves intersections and sums.

Proof. (1) Any morphism $u : X \rightarrow Y$ in \mathcal{A} gives rise to two short exact sequences

$$\begin{array}{ccccc}
 & & X & \xrightarrow{u} & Y \\
 & \nearrow & & & \searrow \\
 & \text{ker } u & & \text{im } u & & \text{coker } u \\
 & \nearrow & & \nearrow & \searrow & \\
 0 & & 0 & & 0 & & 0
 \end{array}$$

Applying F to this diagram and using the exactness assumption, we see that it preserves kernels and cokernels. The last assertion comes from the fact that the intersection (respectively, sum) of two subobjects can be expressed by a pull-back (respectively, push-forward) diagram, that is, as a kernel (respectively, cokernel).

(2) Assume that X is a nonzero object in \mathcal{A} . Then $\text{id}_X \neq 0$ in $\text{End}_{\mathcal{A}}(X)$. The faithfulness assumption then implies that $\text{id}_{F(X)} = F(\text{id}_X) \neq 0$ in $\text{End}_{\mathcal{B}}(F(X))$, whence $F(X) \neq 0$.

(3) It suffices to note that

$$\text{ker } u = 0 \iff F(\text{ker } u) = 0 \iff \text{ker } F(u) = 0$$

and

$$\text{coker } u = 0 \iff F(\text{coker } u) = 0 \iff \text{coker } F(u) = 0.$$

(4) We first claim that \mathcal{A} is Artinian and Noetherian. Indeed given a monotone sequence of subobjects in \mathcal{A} , its image by F is a monotone sequence of subobjects in \mathcal{B} , so becomes eventually constant; (3) then implies that the sequence in \mathcal{A} also becomes eventually constant. Thus arbitrary intersections and sums exist in \mathcal{A} as well as in \mathcal{B} and are in fact finite intersections or sums. We conclude with the help of (1). \square

We can now give the proof of Proposition 2.2.

Proof. Certainly, for any $\alpha \in A_X$, the endomorphism $\alpha^{\oplus n}$ of $\omega(X)^{\oplus n}$ leaves stable $\omega(Y)$ for all subobjects $Y \subset X^{\oplus n}$, and thus induces an endomorphism of $\omega(Z)$ for all subquotients Z of $X^{\oplus n}$. In this way, for each object Z in $\langle X \rangle$, the \mathbf{k} -vector space $\omega(Z)$ becomes an A_X -module. If Z is a subquotient of $X^{\oplus n}$ and Z' is a subquotient of $X^{\oplus m}$, and if $f : Z \rightarrow Z'$ is a morphism in \mathcal{C} , then $Z \oplus Z'$ is a subquotient of $X^{\oplus(n+m)}$, and the image $\text{gr}(f)$ of the morphism $(\text{id}, f) : Z \rightarrow Z \oplus Z'$ (in other words the graph of f) is a subobject of $Z \oplus Z'$, hence also a subquotient of $X^{\oplus(n+m)}$. The fact that A_X stabilizes $\omega(\text{gr}(f))$ means that $\omega(f)$ is a morphism of A_X -modules. In summary, we have proved that $\omega|_{\langle X \rangle}$ factorizes through the category of finite-dimensional A_X -modules, as stated.

By definition, an endomorphism α of the functor $\omega|_{\langle X \rangle}$ is the datum of an endomorphism $\alpha_Z \in \text{End}_{\mathbf{k}}(\omega(Z))$ for each $Z \in \langle X \rangle$, such that the diagram

$$\begin{array}{ccc} \omega(Z) & \xrightarrow{\alpha_Z} & \omega(Z) \\ \omega(f) \downarrow & & \downarrow \omega(f) \\ \omega(Z') & \xrightarrow{\alpha_{Z'}} & \omega(Z') \end{array}$$

commutes for any morphism $f : Z \rightarrow Z'$ in $\langle X \rangle$. This compatibility condition and the definition of $\langle X \rangle$ forces α to be determined by $\alpha_X \in \text{End}_{\mathbf{k}}(\omega(X))$, and forces α_X to belong to A_X . Conversely, any element in A_X gives rise to an endomorphism of $\omega|_{\langle X \rangle}$. This discussion shows the last assertion in the proposition.

It remains to show that the functor $\bar{\omega}_X$ is an equivalence of categories. We already know that it is faithful, so we must show that it is full and essentially surjective. We will do that by constructing an inverse functor.

We will denote by \mathcal{C}^{fin} the category opposite to the category of \mathbf{k} -linear functors from \mathcal{C} to $\text{Vect}_{\mathbf{k}}$. Yoneda's lemma says that the functor $Z \mapsto \text{Hom}_{\mathcal{C}}(Z, -)$ from \mathcal{C} to \mathcal{C}^{fin} is fully faithful, so \mathcal{C} is a full subcategory of \mathcal{C}^{fin} . Given an object $Y \in \mathcal{C}$ and a finite-dimensional \mathbf{k} -vector space V , we define two objects in \mathcal{C}^{fin} by

$$\underline{\text{Hom}}(V, Y) : T \mapsto \text{Hom}_{\mathcal{C}}(Y, T) \otimes_{\mathbf{k}} V \quad \text{and} \quad Y \otimes V : T \mapsto \text{Hom}_{\mathbf{k}}(V, \text{Hom}_{\mathcal{C}}(Y, T)).$$

These functors are representable: if $V = \mathbf{k}^n$, then both functors are represented by $Y^{\oplus n}$. So we will regard $\underline{\text{Hom}}(V, Y)$ and $Y \otimes V$ as being objects in \mathcal{C} and forget everything about \mathcal{C}^{fin} .

Note however that we gained functoriality in V in the process: given two \mathbf{k} -vector spaces V and W and an object $Y \in \mathcal{C}$, there is a linear map

$$\text{Hom}_{\mathbf{k}}(W, V) \rightarrow \text{Hom}_{\mathcal{C}}(\underline{\text{Hom}}(V, Y), \underline{\text{Hom}}(W, Y)) \tag{2.1}$$

that sends an element $f \in \text{Hom}_{\mathbf{k}}(W, V)$ to the image of the identity by the map

$$\begin{array}{ccc} \text{End}_{\mathcal{C}}(\underline{\text{Hom}}(W, Y)) = \text{Hom}_{\mathbf{k}}(Y, \underline{\text{Hom}}(W, Y)) \otimes_{\mathbf{k}} W & & \\ \downarrow \text{id} \otimes f & & \\ \text{Hom}_{\mathbf{k}}(Y, \underline{\text{Hom}}(W, Y)) \otimes_{\mathbf{k}} V = \text{Hom}_{\mathcal{C}}(\underline{\text{Hom}}(V, Y), \underline{\text{Hom}}(W, Y)). & & \end{array}$$

For two \mathbf{k} -vector spaces $W \subset V$ and two objects $Z \subset Y$ in \mathcal{C} , we define the transporter of W into Z as the subobject

$$(Z : W) := \ker(\underline{\text{Hom}}(V, Y) \rightarrow \underline{\text{Hom}}(W, Y/Z))$$

of $\underline{\text{Hom}}(V, Y)$, where the morphism $\underline{\text{Hom}}(V, Y) \rightarrow \underline{\text{Hom}}(W, Y/Z)$ is the obvious one.

Now we define

$$P_X = \bigcap_{\substack{n \geq 0 \\ Y \subset X^{\oplus n}}} \left(\underline{\text{Hom}}(\omega(X), X) \cap (Y : \omega(Y)) \right).$$

Here the small intersection is computed in the ambient object $\underline{\mathbf{Hom}}(\omega(X)^{\oplus n}, X^{\oplus n})$, the space $\underline{\mathbf{Hom}}(\omega(X), X)$ being embedded diagonally, and the large intersection, taken over all subobjects $Y \subset X^{\oplus n}$, is computed in the ambient object $\underline{\mathbf{Hom}}(\omega(X), X)$. As a subobject of $\underline{\mathbf{Hom}}(\omega(X), X) \cong X^{\oplus \dim(\omega(X))}$, the object P_X belongs to $\langle X \rangle$.

Equation (2.1) provides us with an algebra map

$$\mathrm{End}_{\mathbf{k}}(\omega(X)) \rightarrow \mathrm{End}_{\mathcal{C}}(\underline{\mathbf{Hom}}(\omega(X), X)),$$

which induces an algebra map $A_X \rightarrow \mathrm{End}_{\mathcal{C}}(P_X)$. This map can be seen as a morphism $P_X \underline{\otimes}_{A_X} P_X \rightarrow P_X$ in \mathcal{C} , and we can thus define the coequalizer

$$P_X \underline{\otimes}_{A_X} V := \mathrm{coeq}(P_X \underline{\otimes} (A_X \otimes_{\mathbf{k}} V) \rightrightarrows P_X \underline{\otimes} V)$$

for each A_X -module V . We will prove that the functor

$$P_X \underline{\otimes}_{A_X} - : \mathrm{Mod}_{A_X} \rightarrow \langle X \rangle$$

is an inverse to $\bar{\omega}_X$.

First, we remark that for any \mathbf{k} -vector space V and any object $Y \in \mathcal{C}$ there exists a canonical identification

$$\omega(Y \underline{\otimes} V) = \omega(Y) \otimes_{\mathbf{k}} V.$$

Indeed $\mathrm{id}_{Y \underline{\otimes} V}$ defines an element in

$$\mathrm{Hom}_{\mathcal{C}}(Y \underline{\otimes} V, Y \underline{\otimes} V) = \mathrm{Hom}_{\mathbf{k}}(V, \mathrm{Hom}_{\mathcal{C}}(Y, Y \underline{\otimes} V)).$$

The image of this element under the map

$$\mathrm{Hom}_{\mathbf{k}}(V, \mathrm{Hom}_{\mathcal{C}}(Y, Y \underline{\otimes} V)) \rightarrow \mathrm{Hom}_{\mathbf{k}}(V, \mathrm{Hom}_{\mathbf{k}}(\omega(Y), \omega(Y \underline{\otimes} V)))$$

induced by ω provides a canonical element in

$$\mathrm{Hom}_{\mathbf{k}}(V, \mathrm{Hom}_{\mathbf{k}}(\omega(Y), \omega(Y \underline{\otimes} V))) \cong \mathrm{Hom}_{\mathbf{k}}(\omega(Y) \otimes_{\mathbf{k}} V, \omega(Y \underline{\otimes} V)),$$

or in other words a canonical morphism $\omega(Y) \otimes_{\mathbf{k}} V \rightarrow \omega(Y \underline{\otimes} V)$. To check that this morphism is invertible one can assume that $V = \mathbf{k}^n$, in which case the claim is obvious. Likewise, we have an identification $\omega(\underline{\mathbf{Hom}}(V, Y)) = \mathrm{Hom}_{\mathbf{k}}(V, \omega(Y))$.

Using these identifications, the exactness of ω implies that given two \mathbf{k} -vector spaces $W \subset V$ and two objects $Z \subset Y$ in \mathcal{C} ,

$$\omega((Z : W)) = \{\alpha \in \mathrm{Hom}_{\mathbf{k}}(V, \omega(Y)) \mid \alpha(W) \subset \omega(Z)\}.$$

From Lemma 2.3(4), it then follows that $\omega(P_X) = A_X$ (as a right A_X -module), and therefore, that for each $V \in \mathrm{Mod}_{A_X}$ we have

$$\bar{\omega}_X(P_X \underline{\otimes}_{A_X} V) = \bar{\omega}_X(P_X) \otimes_{A_X} V \cong V.$$

Hence $\bar{\omega}_X(P_X \underline{\otimes}_{A_X} -)$ is naturally isomorphic to the identity functor of Mod_{A_X} .

For the other direction, we start by checking that

$$\mathrm{Hom}_{\mathbf{k}}(V, \mathrm{End}_{\mathcal{C}}(Y) \otimes_{\mathbf{k}} V) = \mathrm{Hom}_{\mathbf{k}}(V, \mathrm{Hom}_{\mathcal{C}}(\underline{\mathrm{Hom}}(V, Y), Y)) = \mathrm{Hom}_{\mathcal{C}}(\underline{\mathrm{Hom}}(V, Y) \otimes V, Y).$$

To the canonical element in the left-hand side (defined by $v \mapsto \mathrm{id}_Y \otimes v$) corresponds a canonical morphism $\underline{\mathrm{Hom}}(V, Y) \otimes V \rightarrow Y$ in \mathcal{C} . Considering the latter for $V = \omega(X)$ and $Y = X^{\oplus n}$, we obtain a canonical map

$$\underline{\mathrm{Hom}}(\omega(X), X) \otimes \omega(X^{\oplus n}) \rightarrow X^{\oplus n},$$

whence by restriction

$$P_X \otimes_{A_X} \omega(Y) \rightarrow Y$$

for any subobject $Y \subset X^{\oplus n}$. The latter map is an isomorphism because, as we saw above, its image by ω is an isomorphism (see Lemma 2.3(3)). The right exactness of ω and of $P_X \otimes_{A_X} -$ then imply that $P_X \otimes_{A_X} \omega(Z) \xrightarrow{\sim} Z$ for each subquotient Z of $X^{\oplus n}$, and we conclude that $P_X \otimes_{A_X} \overline{\omega}_X(-)$ is naturally isomorphic to the identity functor of $\langle X \rangle$. \square

In the setup of the proposition, if X and X' are two objects of \mathcal{C} such that $\langle X \rangle \subset \langle X' \rangle$ (for instance if X' is of the form $X \oplus Y$), then we have a restriction morphism

$$A_{X'} \cong \mathrm{End}(\omega|_{\langle X' \rangle}) \rightarrow \mathrm{End}(\omega|_{\langle X \rangle}) \cong A_X.$$

One would like to embrace the whole category \mathcal{C} by taking larger and larger subcategories $\langle X \rangle$ and going to the limit, but the category of finite-dimensional modules over the inverse limit of a system of algebras is not the union of the categories of finite-dimensional modules over the algebras. Things work much better if one looks at comodules over coalgebras, mainly because tensor products commute with direct limits.

2.2 Algebras and coalgebras

Let us recall how the dictionary between finite-dimensional algebras and finite-dimensional coalgebras works (see for instance [Ka], chap. III):

A finite-dimensional \mathbf{k} -algebra	\longleftrightarrow	its \mathbf{k} -dual $B = A^\vee$, a finite-dim. \mathbf{k} -coalgebra
$m : A \otimes_{\mathbf{k}} A \rightarrow A$ multiplication	\longleftrightarrow	$\Delta : B \rightarrow B \otimes_{\mathbf{k}} B$ comultiplication (coassociative)
$\eta : \mathbf{k} \rightarrow A$ unit		$\varepsilon : B \rightarrow \mathbf{k}$ counit
		(obtained by transposing m and η)
left A -module structure on a space M with action map $\mu : A \otimes_{\mathbf{k}} M \rightarrow M$	\longleftrightarrow	right B -comodule structure on M with coaction map $\delta : M \rightarrow M \otimes_{\mathbf{k}} B$ defined by $\mu(a \otimes m) = (\mathrm{id}_M \otimes \mathrm{ev}_a) \circ \delta(m)$, where $\mathrm{ev}_a : B \rightarrow \mathbf{k}$ is the evaluation at a

In the context of this dictionary, one can identify the category Mod_A of finite-dimensional left A -modules with the category Comod_B of finite-dimensional right B -comodules.

2.3 A second reconstruction theorem

Going back to the setting of §2.1, we see that whenever $\langle X \rangle \subset \langle X' \rangle$, we get a morphism of coalgebras

$$B_X = A_X^\vee \rightarrow A_{X'}^\vee = B_{X'}.$$

Now we can choose X with more and more direct summands, so that $\langle X \rangle$ grows larger and larger. Our running assumption that all categories are essentially small allows us to take the direct limit of the algebras B_X over the set of isomorphism classes of objects of \mathcal{C} , for the order determined by the inclusions $\langle X \rangle \subset \langle X' \rangle$. We then obtain the following statement.

Theorem 2.4. *Let \mathcal{C} and ω be as in Proposition 2.2. Set $B = \varinjlim B_X$. Then ω admits a canonical factorization*

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{\bar{\omega}} & \text{Comod}_B \\ & \searrow \omega & \swarrow \text{forget} \\ & \text{Vect}_{\mathbf{k}} & \end{array}$$

where $\bar{\omega}$ is an equivalence of categories.

Example 2.5. 1. Let V be a finite-dimensional \mathbf{k} -vector space, and take $\mathcal{C} = \text{Vect}_{\mathbf{k}}$ and $\omega = V \otimes_{\mathbf{k}} -$. Then $B = \text{End}_{\mathbf{k}}(V)^\vee$. Indeed, the category of finite-dimensional left $\text{End}_{\mathbf{k}}(V)$ -modules is semisimple, with just one simple object up to isomorphism, namely V .

2. Let M be a set, let \mathcal{C} be the category of finite-dimensional M -graded \mathbf{k} -vector spaces, and $\omega : \mathcal{C} \rightarrow \text{Vect}_{\mathbf{k}}$ be the functor that forgets the M -grading. Then $B = \mathbf{k}M$, the \mathbf{k} -vector space with basis M , with the coalgebra structure given by

$$\Delta(m) = m \otimes m, \quad \varepsilon(m) = 1$$

for all $m \in M$. For each $X \in \mathcal{C}$, the coaction of B on $\omega(X) = X$ is the map

$$X \rightarrow X \otimes_{\mathbf{k}} B, \quad x \mapsto \sum_{m \in M} x_m \otimes m,$$

where $x = \sum_{m \in M} x_m$ is the decomposition of x into its homogeneous components.

3. Let C be a coalgebra, and take $\mathcal{C} = \text{Comod}_C$, with ω the forgetful functor. Then there exists a canonical isomorphism $C \cong B$. Indeed, if $X \in \text{Comod}_C$, there exists a finite-dimensional subcoalgebra $C' \subset C$ such that the C -comodule X is actually a C' -comodule¹. Then the coaction morphism $X \rightarrow X \otimes C'$ defines an algebra morphism $(C')^\vee \rightarrow A_X$, hence a coalgebra morphism $B_X \rightarrow C'$. Composing with the embedding $C' \hookrightarrow C$ and passing to the limit we deduce a coalgebra morphism $B \rightarrow C$. In the reverse direction, if $C' \subset C$ is a finite-dimensional subcoalgebra, then C' is an object in Comod_C , hence it acquires a canonical B -comodule structure, i.e. a coalgebra map $C' \rightarrow C' \otimes B$. Composing with the map induced by ε we deduce a coalgebra morphism $C' \rightarrow B$. Since C is the direct limit of its finite-dimensional subcoalgebras, we deduce a coalgebra morphism $C \rightarrow B$. It is easily seen that the morphisms constructed are inverse to each other, proving our claim.

¹Let $\delta : X \rightarrow X \otimes_{\mathbf{k}} C$ be the structure map of the C -comodule X and let (e_1, \dots, e_n) be a \mathbf{k} -basis of X . Write $\delta(e_j) = \sum_i e_i \otimes c_{i,j}$. Then $\Delta(c_{i,j}) = \sum_k c_{i,k} \otimes c_{k,j}$ and $\varepsilon(c_{i,j}) = \delta_{i,j}$ (Kronecker's symbol), so C' can be chosen as the \mathbf{k} -span in C of the elements $c_{i,j}$.

An homomorphism of \mathbf{k} -coalgebras $f : B \rightarrow C$ induces a functor $f_* : \text{Comod}_B \rightarrow \text{Comod}_C$. Specifically, given a B -comodule M with structure map $\delta : M \rightarrow M \otimes_{\mathbf{k}} B$, the C -comodule f_*M has the same underlying \mathbf{k} -vector space as M and has structure map $(\text{id}_M \otimes f) \circ \delta : M \rightarrow M \otimes_{\mathbf{k}} C$.

Proposition 2.6. 1. Let \mathcal{C} be an abelian \mathbf{k} -linear category, let C be a \mathbf{k} -coalgebra, and let $F : \mathcal{C} \rightarrow \text{Comod}_C$ be a \mathbf{k} -linear exact faithful functor. If B is the coalgebra provided by Theorem 2.4 (for the functor given by the composition of F with the forgetful functor $\text{Comod}_C \rightarrow \text{Vect}_{\mathbf{k}}$) and $\overline{F} : \mathcal{C} \xrightarrow{\sim} \text{Comod}_B$ the corresponding equivalence, then there exists a morphism of coalgebras $f : B \rightarrow C$ such that the following diagram commutes:

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{\overline{F}} & \text{Comod}_B \\ & \searrow F & \swarrow f_* \\ & \text{Comod}_C & \end{array}$$

2. Let B and C be two \mathbf{k} -coalgebras. Any functor $F : \text{Comod}_B \rightarrow \text{Comod}_C$ such that

$$\begin{array}{ccc} \text{Comod}_B & \xrightarrow{F} & \text{Comod}_C \\ & \searrow \text{forget} & \swarrow \text{forget} \\ & \text{Vect}_{\mathbf{k}} & \end{array}$$

commutes is of the form $F = f_*$ for a unique morphism of coalgebras $f : B \rightarrow C$.

Proof. (1) Let X be an object in \mathcal{C} . As seen in Example 2.5(3), there exists a finite-dimensional subcoalgebra $C' \subset C$ such that the C -comodule $F(X)$ is actually a C' -comodule. The restriction to the category $\langle X \rangle$ of the functor F then factorizes through $\text{Comod}_{C'} = \text{Mod}_{(C')^\vee}$. Let $\omega : \text{Mod}_{(C')^\vee} \rightarrow \text{Vect}_{\mathbf{k}}$ be the forgetful functor and let A_X be the endomorphism algebra of the functor $\omega \circ F|_{\langle X \rangle}$.

Consider the diagram

$$\begin{array}{ccc} \langle X \rangle & \xrightarrow{\overline{F}_X} & \text{Mod}_{A_X} \\ & \searrow F & \swarrow \text{dashed} \\ & \text{Mod}_{(C')^\vee} & \\ & \omega \downarrow & \swarrow \\ & \text{Vect}_{\mathbf{k}} & \end{array}$$

Any $\alpha \in (C')^\vee$ can be seen as an endomorphism of the functor ω , so induces by restriction an endomorphism of $\omega \circ F|_{\langle X \rangle}$, in other words, an element of A_X . Our situation thus gives us a morphism of algebras $(C')^\vee \rightarrow A_X$, that is, a morphism of coalgebras $A_X^\vee \rightarrow C'$. Further, \overline{F}_X is an equivalence of categories, because the \mathbf{k} -linear functor $\omega \circ F|_{\langle X \rangle}$ is exact and faithful. (This was the key ingredient in the proof of Proposition 2.2.) Taking as before the limit over $\langle X \rangle$ yields the desired coalgebra $B = \varinjlim A_X^\vee$, the morphism of coalgebras $f : B \rightarrow C$, and the equivalence of categories \overline{F} .

Statement (2) follows from (1) and Example 2.5(3). \square

2.4 Tannakian reconstruction

As we saw in §2.3, a \mathbf{k} -linear abelian category equipped with an exact faithful \mathbf{k} -linear functor to $\mathbf{Vect}_{\mathbf{k}}$ is equivalent to the category of right comodules over a \mathbf{k} -coalgebra B equipped with the forgetful functor. On the other hand, an affine group scheme G over \mathbf{k} is a scheme represented by a commutative \mathbf{k} -Hopf algebra H , and representations of G are the same as right H -comodules (see e.g. [Wa, §1.4 and §3.2]). A commutative Hopf algebra is a coalgebra with the extra datum of an associative and commutative multiplication with unit, plus the existence of the antipode. Striving to translate this setup into the language of categories, we look for the extra structures on an abelian \mathbf{k} -linear category that characterize categories of representations of affine group schemes.

The adequate notion is called tensor (or monoidal) symmetric rigid categories. Rather than studying this notion in the greatest possible generality, which would take too much space for the expected benefit, we will state a theorem tailored to our goal of understanding the geometric Satake correspondence. For a more thorough (and formal) treatment, the reader is referred to [SR] and [De], or to [Ka] for a more leisurely walk.

A last word before stating the main theorem of this subsection: a multiplication map $\text{mult} : B \otimes_{\mathbf{k}} B \rightarrow B$ on a coalgebra B which is a coalgebra morphism allows to define a structure of B -comodule on the tensor product over \mathbf{k} of two B -comodules. Specifically, if M and M' are B -comodules with structure maps $\delta_M : M \rightarrow M \otimes_{\mathbf{k}} B$ and $\delta_{M'} : M' \rightarrow M' \otimes_{\mathbf{k}} B$, then the structure map on $M \otimes_{\mathbf{k}} M'$ is defined by the composition depicted on the diagram

$$\begin{array}{ccc}
 M \otimes_{\mathbf{k}} M' & \xrightarrow{\delta_{M \otimes M'}} & M \otimes_{\mathbf{k}} M' \otimes_{\mathbf{k}} B \\
 \delta_M \otimes \delta_{M'} \downarrow & & \uparrow \text{id}_{M \otimes M'} \otimes \text{mult} \\
 M \otimes_{\mathbf{k}} B \otimes_{\mathbf{k}} M' \otimes_{\mathbf{k}} B & \longrightarrow & M \otimes_{\mathbf{k}} M' \otimes_{\mathbf{k}} B \otimes_{\mathbf{k}} B
 \end{array}$$

where the bottom arrow is the usual commutativity constraint for tensor products of \mathbf{k} -vector spaces that swaps the second and third factors.

Given an affine group scheme G over \mathbf{k} , we denote the category of finite-dimensional representations of G by $\text{Rep}_{\mathbf{k}}(G)$.

Theorem 2.7. *Let \mathcal{C} be an abelian \mathbf{k} -linear category equipped with the following data:*

- an exact \mathbf{k} -linear faithful functor $\omega : \mathcal{C} \rightarrow \mathbf{Vect}_{\mathbf{k}}$ (called the fiber functor);
- a \mathbf{k} -bilinear functor $\otimes : \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$ (the tensor product);
- an object $U \in \mathcal{C}$ (the tensor unit);
- an isomorphism $\phi_{X,Y,Z} : X \otimes (Y \otimes Z) \rightarrow (X \otimes Y) \otimes Z$, natural in X, Y and Z (the associativity constraint);
- isomorphisms $U \otimes X \xrightarrow{\lambda_X} X \xleftarrow{\rho_X} X \otimes U$, both natural in X (the unit constraints);
- an isomorphism $\psi_{X,Y} : X \otimes Y \rightarrow Y \otimes X$ natural in X and Y (the commutativity constraint).

We assume the following compatibility conditions:

1. There are isomorphisms $\tau_{X,Y} : \omega(X) \otimes_{\mathbf{k}} \omega(Y) \rightarrow \omega(X \otimes Y)$ and $v : \mathbf{k} \rightarrow \omega(U)$ in $\text{Vect}_{\mathbf{k}}$, with $\tau_{X,Y}$ natural in X and Y .
2. Taking into account the identifications provided by τ and v , the isomorphisms $\omega(\phi_{X,Y,Z})$, $\omega(\lambda_X)$, $\omega(\rho_X)$ and $\omega(\psi_{X,Y})$ are the usual associativity, unit and commutativity constraints in $\text{Vect}_{\mathbf{k}}$.
3. If $\dim \omega(X) = 1$, then there exists $X^* \in \mathcal{C}$ such that $X \otimes X^* \cong U$.

Under these assumptions, there exists an affine group scheme G such that ω admits a canonical factorization

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{\bar{\omega}} & \text{Rep}_{\mathbf{k}}(G) \\ & \searrow \omega & \swarrow \text{forget} \\ & & \text{Vect}_{\mathbf{k}} \end{array}$$

where $\bar{\omega}$ is an equivalence of categories that respects the tensor product and the unit in the sense of the compatibility conditions (1) and (2).

- Remark 2.8.*
1. The group scheme G is the “automorphism group of the fiber functor”. This sentence means that for any commutative \mathbf{k} -algebra R , an element $\alpha \in G(R)$ is a collection of elements $\alpha_X \in \text{End}_R(\omega(X) \otimes_{\mathbf{k}} R)$, natural in $X \in \mathcal{C}$, and compatible with \otimes and U via the isomorphisms τ and v . There is no need to specifically ask for invertibility: this will automatically follow from the compatibility condition (3).
 2. The conditions (1) and (2) are usually worded as: “the functor ω is a tensor functor”.
 3. The faithfulness of ω and the compatibility condition (2) imply that the associativity constraint ϕ (respectively, the unit constraints λ and ρ , the commutativity constraint ψ) of \mathcal{C} satisfies MacLane’s pentagon axiom (respectively, the triangle axiom, the hexagon axiom). Together, these coherence axioms imply that any diagram built from the constraints commutes. This makes multiple tensor products in \mathcal{C} non-ambiguous, see [McL, §VII.2].
 4. Our formulation dropped completely the “rigidity condition” in the usual formulation of the Tannakian reconstruction theorem. This condition demands that each object X has a dual X^\vee characterized by an evaluation map $X^\vee \otimes X \rightarrow U$ and a coevaluation map $U \rightarrow X \otimes X^\vee$. Its purpose is to guarantee the existence of inverses in G — without it, G would only be an affine monoid scheme. In Theorem 2.7, it has been replaced by condition (3), which is sometimes easier to check.

Proof. We first remark that the bifunctor $\otimes : \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$ is exact in each variable: this follows from the analogous fact in the category $\text{Vect}_{\mathbf{k}}$ together with Lemma 2.3(3).

We reuse the notation $\langle X \rangle$, A_X and B_X from §§2.1–2.3. The direct limit of the coalgebras B_X is the coalgebra B , with comultiplication Δ and counit ε .

Let X and X' two objects in \mathcal{C} . The isomorphism $\tau_{X,X'} : \omega(X) \otimes_{\mathbf{k}} \omega(X') \rightarrow \omega(X \otimes X')$ induces an isomorphism of algebras

$$\mathrm{End}_{\mathbf{k}}(\omega(X \otimes X')) \cong \mathrm{End}_{\mathbf{k}}(\omega(X)) \otimes_{\mathbf{k}} \mathrm{End}_{\mathbf{k}}(\omega(X')). \quad (2.2)$$

The opening remark in this proof implies that given subobjects $Y \subset X^{\oplus n}$ and $Y' \subset (X')^{\oplus n'}$, the tensor products $Y \otimes X'$ and $X \otimes Y'$ are subobjects of respectively $(X \otimes X')^{\oplus n}$ and $(X \otimes X')^{\oplus n'}$. It follows that the isomorphism (2.2) takes $A_{X \otimes X'}$ into $A_X \otimes_{\mathbf{k}} A_{X'}$. Taking the duals, we get a morphism of coalgebras $B_{X'} \otimes_{\mathbf{k}} B_X \rightarrow B_{X \otimes X'}$, and taking the direct limit over $\langle X \rangle$ and $\langle X' \rangle$, we obtain a morphism of coalgebras $m : B \otimes_{\mathbf{k}} B \rightarrow B$.

On the other hand, the \mathbf{k} -vector space $\omega(U)$ has dimension 1, so the algebra A_U is reduced to $\mathrm{End}_{\mathbf{k}}(\omega(U)) = \mathbf{k}$. Thus B_U is the trivial one-dimensional \mathbf{k} -coalgebra, and the definition of B as a direct limit of the B_X (including B_U) leads to a morphism of coalgebras $\eta : \mathbf{k} \rightarrow B$.

Our coalgebra B is thus equipped with a multiplication $m : B \otimes_{\mathbf{k}} B \rightarrow B$ and a unit $\eta : \mathbf{k} \rightarrow B$. The compatibility condition (2) implies that (B, m, η) is an associative and commutative \mathbf{k} -algebra with unit.

Let us call G the spectrum of the commutative \mathbf{k} -algebra (B, m, η) ; this is an affine scheme over \mathbf{k} . The commutative diagrams that express the fact that m and η are morphisms of coalgebras also say that Δ and ε are morphisms of algebras. The latter thus define morphisms of schemes

$$\Delta^* : G \times_{\mathrm{Spec}(\mathbf{k})} G \rightarrow G \quad \text{and} \quad \varepsilon^* : \mathrm{Spec}(\mathbf{k}) \rightarrow G.$$

The coassociativity of Δ and the counit property then imply that $(G, \Delta^*, \varepsilon^*)$ is an affine monoid scheme. It thus remains to show that the elements of G are invertible.

Unwinding the construction that led to the definition of G , we see that for any commutative \mathbf{k} -algebra R , an element $\alpha \in G(R)$ is a collection of elements $\alpha_X \in \mathrm{End}_R(\omega(X) \otimes_{\mathbf{k}} R)$, natural in X , and compatible with \otimes and U . We want to show that α_X is invertible for all objects X .

First, if X is such that $\dim \omega(X) = 1$, then by condition (3) there exists X^* such that $X \otimes X^* \cong U$, and therefore $\alpha_X \otimes \alpha_{X^*}$ is conjugated to $\alpha_U = \mathrm{id}_{\omega(U) \otimes_{\mathbf{k}} R}$, so α_X (an endomorphism of a free R -module of rank 1) is invertible.

To deal with the general case, one constructs the exterior power $\bigwedge^d X$ with the help of the commutativity constraint of \mathcal{C} , for $d = \dim \omega(X)$. Since ω is compatible with this commutativity constraint, $\omega(\bigwedge^d X) \cong \bigwedge^d \omega(X)$ is one-dimensional. As we saw in our particular case, this implies that $\alpha_{\bigwedge^d X}$ is invertible. But this is $\bigwedge^d \alpha_X$, so we eventually obtain that α_X (an endomorphism of a free R -module of rank d) is invertible. \square

Example 2.9. 1. Continue with Example 2.5(2), and suppose now that our category \mathcal{C} of finite-dimensional M -graded \mathbf{k} -vector spaces is endowed with a tensor product \otimes . There is then a law $*$ on M such that

$$\mathbf{k}[m] \otimes \mathbf{k}[n] = \mathbf{k}[m * n]$$

for all $m, n \in M$ (where $\mathbf{k}[p]$ means \mathbf{k} placed in degree p). The constraints (2) in the theorem impose that M is a commutative monoid, and then $B = \mathbf{k}M$ is the monoid algebra. The condition (3), if verified, implies that M is indeed a group. The affine group scheme $G = \mathrm{Spec}(B)$ given by the theorem is then the Cartier dual of M (see [Wa, §2.4]).

2. Let X be a connected topological manifold, let \mathcal{C} be the category of local systems on X with coefficients in \mathbf{k} , let $x \in X$, and let ω be the functor $\mathcal{L} \mapsto \mathcal{L}_x$, the fiber at point x . Then G is the constant group scheme equal to the fundamental group $\pi_1(X, x)$. On this example, we see how the choice of a fiber functor subtly changes the group.
3. We define the category $\text{SVect}_{\mathbf{k}}$ of supersymmetric \mathbf{k} -vector spaces as the category of $\mathbf{Z}/2\mathbf{Z}$ -graded vector spaces, equipped with the usual tensor product, with the usual associativity and unit constraints, but with the supersymmetric commutativity constraint: for $V = V_0 \oplus V_1$ and $W = W_0 \oplus W_1$, the isomorphism $\psi_{V,W} : V \otimes_{\mathbf{k}} W \rightarrow W \otimes_{\mathbf{k}} V$ is defined as

$$\psi_{V,W}(v \otimes w) = \begin{cases} w \otimes v & \text{if } v \in V_0 \text{ or } w \in W_0, \\ -(w \otimes v) & \text{if } v \in V_1 \text{ and } w \in W_1. \end{cases}$$

Then the forgetful functor from $\text{SVect}_{\mathbf{k}}$ to $\text{Vect}_{\mathbf{k}}$ does not respect the commutativity constraints, so one cannot apply the theorem to this situation.

Proposition 2.6 also has a tensor analog. We state without proof the assertion that is needed in the proof of the geometric Satake equivalence. Observe that an homomorphism of \mathbf{k} -group schemes $f : H \rightarrow G$ induces a restriction functor $f^* : \text{Rep}_{\mathbf{k}}(G) \rightarrow \text{Rep}_{\mathbf{k}}(H)$.

Proposition 2.10. *Let \mathcal{C} be an abelian \mathbf{k} -linear category with tensor product, tensor unit, and the relevant constraints. Let H be an affine group scheme over \mathbf{k} . Let $F : \mathcal{C} \rightarrow \text{Rep}_{\mathbf{k}}(H)$ be a \mathbf{k} -linear exact faithful tensor functor, compatible with the various constraints. Let G be the affine \mathbf{k} -group scheme provided by Theorem 2.7 (for the composition of F with the forgetful functor $\text{Rep}_{\mathbf{k}}(H) \rightarrow \text{Vect}_{\mathbf{k}}$) and $\overline{F} : \mathcal{C} \xrightarrow{\sim} \text{Rep}_{\mathbf{k}}(G)$ be the corresponding equivalence. Then there exists a morphism of group schemes $f : H \rightarrow G$ such that the following diagram commutes:*

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{\overline{F}} & \text{Rep}_{\mathbf{k}}(G) \\ & \searrow F & \swarrow f^* \\ & & \text{Rep}_{\mathbf{k}}(H). \end{array}$$

2.5 Properties of G visible on $\text{Rep}_{\mathbf{k}}(G)$

Recall that an affine \mathbf{k} -group scheme G is called *algebraic* if the \mathbf{k} -algebra of regular functions on G is finitely generated.

Proposition 2.11. *1. Let G be an affine group scheme over \mathbf{k} . Then G is algebraic if and only if there exists $X \in \text{Rep}_{\mathbf{k}}(G)$ such that X generates $\text{Rep}_{\mathbf{k}}(G)$ by taking direct sums, tensor products, duals, and subquotients.*

2. Let G be an algebraic affine group scheme over \mathbf{k} . If G is not connected, then there exists a nontrivial representation $X \in \text{Rep}_{\mathbf{k}}(G)$ such that the subcategory $\langle X \rangle$ (with the notation of §2.1) is stable under \otimes .

3. Let G be an algebraic affine group scheme over \mathbf{k} . Assume that \mathbf{k} has characteristic 0. Then G is reductive if and only if the category $\text{Rep}_{\mathbf{k}}(G)$ is semisimple.

Proof. (1) Suppose G is algebraic. Then G admits a faithful representation ([Wa, §3.4]), i.e. G can be viewed as a closed subgroup of some \mathbf{GL}_n . It is then a classical result that any finite dimensional representation of G can be obtained from the representation on \mathbf{k}^n by the processes of forming tensor products, direct sums, subrepresentations, quotients and duals (see [Wa, §3.5]).

Conversely, suppose the existence of a representation X that generates $\mathrm{Rep}_{\mathbf{k}}(G)$ in the sense explained in the statement of the proposition. Then X is necessarily a faithful representation of G , so G embeds as a closed subgroup in $\mathbf{GL}(X)$ ([Wa, §15.3]), and is therefore algebraic.

(2) The quotient $G \rightarrow \pi_0(G)$ induces the inclusion $\mathrm{Rep}_{\mathbf{k}}(\pi_0(G)) \subset \mathrm{Rep}_{\mathbf{k}}(G)$. Taking for X the regular representation of $\pi_0(G)$, we see that $\langle X \rangle = \mathrm{Rep}_{\mathbf{k}}(\pi_0(G))$ is stable under tensor products. If G is not connected, then X is not trivial.

(3) In a simple representation X of G , the unipotent radical $R_u(G)$ of G acts trivially; indeed the subspace of points fixed by $R_u(G)$ is nontrivial by Kolchin's fixed point theorem ([Wa, §8.2]) and is G -stable because $R_u(G)$ is a normal subgroup. This result immediately extends to semisimple representations of G . Now suppose that $\mathrm{Rep}_{\mathbf{k}}(G)$ is semisimple. Then G admits a semisimple faithful representation. On this representation, $R_u(G)$ acts trivially and faithfully. Therefore $R_u(G)$ is trivial, that is, G is reductive.

The converse (which will not be needed in these notes) is more difficult, and essentially amounts to Weyl's unitary trick ([Wa, §12.5]). \square

3 The affine Grassmannian

In this section we provide a brief introduction to the affine Grassmannian of a complex reductive algebraic group. For more details, examples and references, the reader can e.g. consult [Gö, §2].

3.1 Definition

We set $\mathcal{O} := \mathbf{C}[[t]]$ and $\mathcal{K} := \mathbf{C}((t))$, where t is an indeterminate. If H is a linear complex algebraic group, we denote by $H_{\mathcal{O}}$, resp. $H_{\mathcal{K}}$, the functor from \mathbf{C} -algebras to groups defined by

$$R \mapsto H(R[[t]]), \quad \text{resp.} \quad R \mapsto H(R((t))).$$

It is not difficult to check that $H_{\mathcal{O}}$ is represented by a \mathbf{C} -group scheme (not of finite type in most cases), and that $H_{\mathcal{K}}$ is represented by an ind-group scheme (i.e. an inductive limit of schemes parametrized by $\mathbf{Z}_{\geq 0}$, with closed embeddings as transition maps). We will still denote these (ind-)group schemes by $H_{\mathcal{O}}$ and $H_{\mathcal{K}}$.

We now fix a standard triple $G \supset B \supset T$ of a connected complex reductive algebraic group, a Borel subgroup, and a maximal torus. We will denote by N the unipotent radical of B . We will denote by Φ the root system of (G, T) , by $\Phi^+ \subset \Phi$ the subset of positive roots (consisting of the T -weights in the Lie algebra of B), and by Φ_s the corresponding subset of simple roots.

Let $X_*(T)$ be the lattice of cocharacters of T ; it contains the coroot lattice Q^\vee and is endowed with the dominance order \leq (such that nonnegative elements are nonnegative integral combination of positive coroots). We will denote by $X_*(T)^+ \subset X_*(T)$ the cone of dominant cocharacters. We define ρ as half the sum of the positive roots and regard it as a linear form $X_*(T) \rightarrow \frac{1}{2}\mathbf{Z}$.

If $L^{\leq 0}G$ denotes the ind-group scheme which represents the functor

$$R \mapsto G(R[t^{-1}])$$

and if $L^{< 0}G$ is the kernel of the natural morphism $L^{\leq 0}G \rightarrow G$ (sending t^{-1} to 0), then $L^{< 0}G$ is a subgroup of $G_{\mathcal{K}}$ in a natural way, and the multiplication morphism

$$L^{< 0}G \times G_{\mathcal{O}} \rightarrow G_{\mathcal{K}}$$

is an open embedding by [NP, Lemme 2.1]. In view of this property, the quotient

$$G_{\mathcal{K}}/G_{\mathcal{O}}$$

has a natural structure of ind-scheme. In fact, one can check that this ind-scheme is ind-proper, and of ind-finite type.

In general, the quotient $G_{\mathcal{K}}/G_{\mathcal{O}}$ is not reduced. Since we will only consider constructible sheaves on this quotient, this non-reduced structure can be forgotten, and we will denote by Gr_G the (reduced) ind-variety associated with the ind-scheme $G_{\mathcal{K}}/G_{\mathcal{O}}$.

Any cocharacter $\nu \in X_*(T)$ defines a morphism $(\mathcal{K})^\times \rightarrow T_{\mathcal{K}}$. The image of t under this morphism will be denoted by t^ν . The coset $t^\nu G_{\mathcal{O}}$ is a point in Gr_G , which will be denoted by L_ν .

The Cartan decomposition describes the $G_{\mathcal{O}}$ -orbits in Gr_G , in the following way.

Proposition 3.1. *We have*

$$\text{Gr}_G = \bigsqcup_{\lambda \in X_*(T)^+} \text{Gr}_G^\lambda, \quad \text{where } \text{Gr}_G^\lambda := G_{\mathcal{O}} \cdot L_\lambda. \quad (3.1)$$

Moreover, for any $\lambda \in X_*(T)$, Gr_G^λ is an affine bundle over the partial flag variety G/P_λ where P_λ is the parabolic subgroup of G containing B and associated with the subset of simple roots $\{\alpha \in \Phi_s \mid \langle \lambda, \alpha \rangle = 0\}$, and we have

$$\dim(\text{Gr}_G^\lambda) = \langle 2\rho, \lambda \rangle.$$

Finally, we have

$$\overline{\text{Gr}_G^\lambda} = \bigsqcup_{\substack{\eta \in X_*(T)^+ \\ \eta \leq \lambda}} \text{Gr}_G^\eta. \quad (3.2)$$

The collection of $G_{\mathcal{O}}$ -orbits in Gr_G is a Whitney stratification, which will be denoted by \mathcal{S} .

Finally, we will need a description of the connected components of Gr_G . For any $c \in X_*(T)/Q^\vee$, let us set

$$\mathrm{Gr}_G^c := \bigsqcup_{\substack{\lambda \in X_*(T)^+ \\ \lambda + Q^\vee = c}} \mathrm{Gr}_G^\lambda.$$

Then the connected components of Gr_G are exactly the subvarieties Gr_G^c for $c \in X_*(T)/Q^\vee$. In particular, note that the parity of the dimensions of the Schubert varieties Gr_G^λ is constant on each connected component. A connected component will be called even, resp. odd, if these dimensions are even, resp. odd.

3.2 Semi-infinite orbits

The Iwasawa decomposition describes the $N_{\mathcal{K}}$ -orbits in Gr_G as follows:

$$\mathrm{Gr}_G = \bigsqcup_{\mu \in X_*(T)} S_\mu, \quad \text{where } S_\mu := N_{\mathcal{K}} \cdot L_\mu. \quad (3.3)$$

Each orbit S_μ is infinite dimensional and therefore is not contained in any stratum of the ind-structure of Gr_G .

The closure of these orbits for the inductive limit topology on Gr_G can be described in the following way:

$$\overline{S_\mu} = \bigsqcup_{\substack{\nu \in X_*(T) \\ \nu \leq \mu}} S_\nu. \quad (3.4)$$

We will soon provide a formal proof of this equality (see Proposition 3.2 below), but let us first try to make this result intuitive, at least in the case $G = \mathbf{SL}_2$. For that, we denote by α the unique positive root, and consider the standard Iwahori subgroup

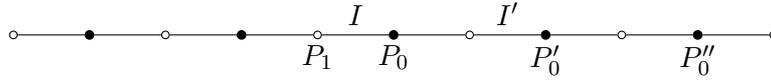
$$I = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbf{SL}_2(\mathcal{K}) \mid a, b, t^{-1}c, d \in \mathcal{O} \right\}$$

and the two maximal parahoric subgroups

$$P_0 = \mathbf{SL}_2(\mathcal{O}), \quad P_1 = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbf{SL}_2(\mathcal{K}) \mid a, tb, t^{-1}c, d \in \mathcal{O} \right\}$$

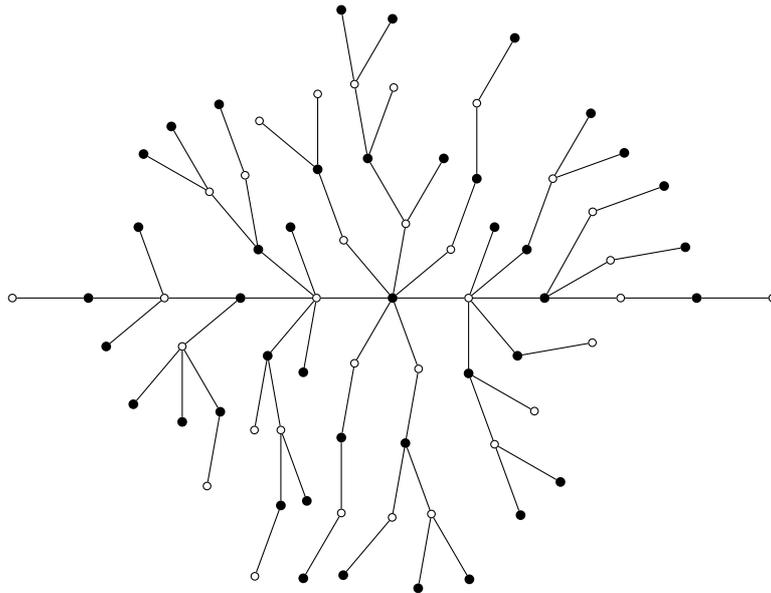
that contain I .

A parahoric subgroup of $\mathbf{SL}_2(\mathcal{K})$ is a subgroup conjugated to one of these three standard subgroups I , P_0 or P_1 . Parahoric subgroups form a poset for the inclusion. The Serre tree is the simplicial realization of the opposite poset. Here is a picture of a small part of the tree, namely we just pictured the parahoric subgroups that are conjugated to the standard ones by elements of the affine Weyl group. Note here that the inclusions $P_1 \supset I \subset P_0$ translate to the fact that the vertices corresponding to P_1 and P_0 are incident to the edge corresponding to I .



$$\begin{aligned}
 I' &= t^{\alpha^\vee} I t^{-\alpha^\vee} \\
 P'_0 &= t^{\alpha^\vee} P_0 t^{-\alpha^\vee} \\
 P''_0 &= t^{2\alpha^\vee} P_0 t^{-2\alpha^\vee}
 \end{aligned}$$

Since Iwahori subgroups are conjugated in $\mathbf{SL}_2(\mathcal{K})$ and since I is its own normalizer, the set of edges in the tree is in bijection with the so-called affine flag variety $\mathbf{SL}_2(\mathcal{K})/I$. Likewise, the set of parahoric subgroups conjugated to P_0 , depicted as black dots on the tree, is in bijection with the affine Grassmannian $\mathrm{Gr}_{\mathbf{SL}_2} = \mathbf{SL}_2(\mathcal{K})/P_0$. One can rephrase this by saying that the group $\mathbf{SL}_2(\mathcal{K})$ acts on the tree (transitively on the edges, on the black vertices, and on the white vertices) and that the stabilizer of the simplex associated to a parahoric subgroup is the subgroup itself. (The white dots form the second connected component in the affine Grassmannian $\mathrm{Gr}_{\mathbf{PGL}_2}$.)



Likewise, the Iwahori subgroups contained in P_0 can be obtained by letting the normalizer of P_0 act on I ; in other words, the set of edges incident to the vertex P_0 is in bijection with P_0/I . This is a complex projective line. So the set of edges incident to a given black vertex is a complex projective line. (The same thing holds also for white vertices.) Our drawings are thus quite incomplete, because a lot of edges were omitted.

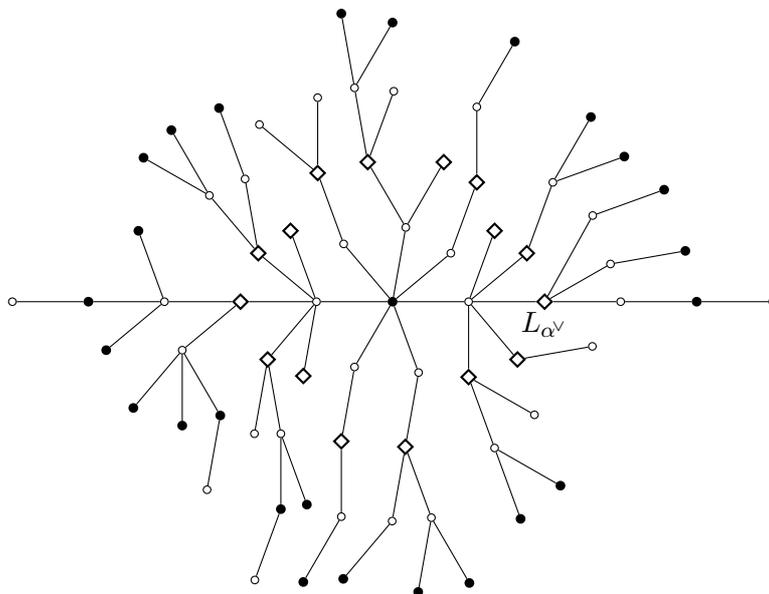
Again, our affine Grassmannian is the set of all black vertices. Here it is worth noting that the tree metric is related to the description of the ind-structure of $\mathrm{Gr}_{\mathbf{SL}_2}$: one can take for the n -th finite-dimensional piece of $\mathrm{Gr}_{\mathbf{SL}_2}$ the set Gr_n of all vertices at distance $\leq 2n$ from P_0 . Further, the analytic (respectively, Zariski) topology of the variety Gr_n can also be seen on the tree: it comes from the analytic (respectively, Zariski) topology on all the projective lines

mentioned in the previous paragraph. Thus, we can for instance see that Gr_n is dominated by a tower of $2n$ projective lines, because each point at distance $\leq 2n$ from the origin can be reached by choosing first an edge around the origin, then another edge around the white vertex at the end of this edge, and so on $2n$ times².

Now let us see how our orbits Gr_G^λ and S_μ are depicted in this model. The point L_ν corresponds to the Iwahori subgroup $t^\nu P_0 t^{-\nu}$:



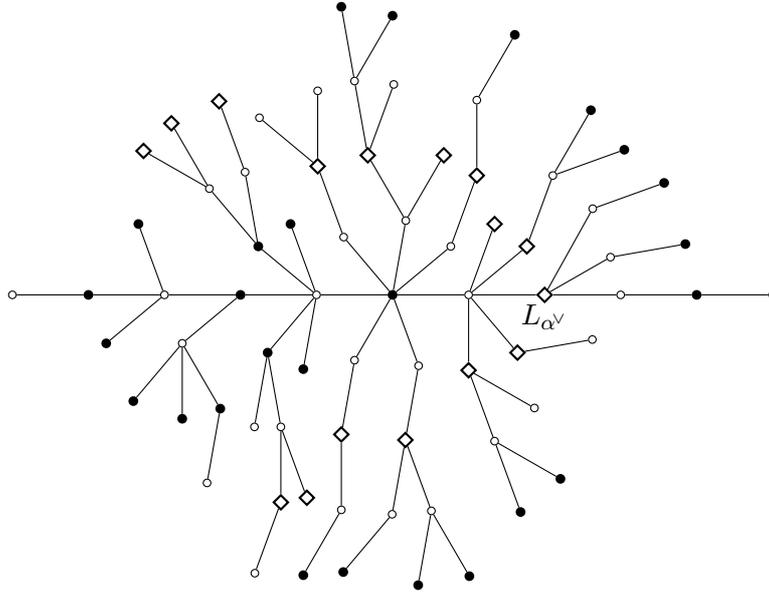
The Schubert cell Gr_G^λ is the orbit of L_λ under the stabilizer of the base point L_0 ; it therefore looks like the sphere with center L_0 going through L_λ . On our next figure, the diamonds are points in $\text{Gr}_G^{\alpha^\vee}$:



Now look at the white vertex between the origin and L_{α^\vee} : edges starting from this vertex form a projective line, and the black vertices of these edges belong to $\text{Gr}_G^{\alpha^\vee}$, with one exception, namely L_0 . This point L_0 appears thus as a limit (on the projective line) of points in $\text{Gr}_G^{\alpha^\vee}$, that is, belongs to the closure of $\text{Gr}_G^{\alpha^\vee}$. This provides an intuitive interpretation of the inclusion $\text{Gr}_G^0 \subset \overline{\text{Gr}_G^{\alpha^\vee}}$.

In the same line of ideas, the semi-infinite orbit S_μ can be depicted as the sphere centered at $-\infty$ (also called “horosphere”) and going through L_μ . In the figure below, the diamonds are points in S_{α^\vee} :

²Algebraically, this process is a particular case of a Bott–Samelson resolution, as explained in [GL].



For the same reason as before, we see that L_0 belongs to the closure of S_{α^\vee} . The reader can however feel cheated here, since we relied on geometrical intuition. For a more formal proof, one computes³

$$\begin{pmatrix} 1 & at^{-1} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} t^{-1} & 0 \\ a^{-1} & t \end{pmatrix} \begin{pmatrix} t & a \\ -a^{-1} & 0 \end{pmatrix} \quad \text{for } a \in \mathbb{C}^\times,$$

that is

$$\begin{pmatrix} 1 & at^{-1} \\ 0 & 1 \end{pmatrix} G_{\mathcal{O}} = \begin{pmatrix} t^{-1} & 0 \\ a^{-1} & t \end{pmatrix} G_{\mathcal{O}},$$

and therefore

$$\underbrace{\begin{pmatrix} 1 & at^{-1} \\ 0 & 1 \end{pmatrix} G_{\mathcal{O}}}_{\in S_0} \rightarrow \underbrace{\begin{pmatrix} t^{-1} & 0 \\ 0 & t \end{pmatrix} G_{\mathcal{O}}}_{=L_{-\alpha^\vee}} \quad \text{when } a \rightarrow \infty.$$

Multiplying on the left by t^μ leads to $L_{\mu-\alpha^\vee} \in \overline{S_\mu}$, whence $S_{\mu-\alpha^\vee} \subset \overline{S_\mu}$. This justifies (at least in the case of \mathbf{SL}_2 , but the general case can be deduced from this special case) the inclusion

$$\overline{S_\mu} \supset \bigsqcup_{\substack{\nu \in X_*(T) \\ \nu \leq \mu}} S_\nu.$$

The proof of the reverse inclusion requires another tool, which is the subject of the next section.

3.3 Projective embeddings

We want to embed the affine Grassmannian Gr_G in an (infinite dimensional) projective space $\mathbf{P}(V)$ in order to get some control. Replacing G by its simply connected cover may kill

³In fact, this computation is precisely our observation on the tree.

connected components, but has the advantage that the resulting group is a product of simple groups. Therefore in this subsection we assume that G is simple and simply connected. (The general case can be reduced to this one.)

The character lattice $X^*(T)$ of T is the \mathbf{Z} -dual of $X_*(T)$. Let W be the Weyl group, and let $\tau : X_*(T) \rightarrow \mathbf{Z}$ be the W -invariant quadratic form that takes the value 1 at each short coroot. The polar form of τ defines a map $\iota : X_*(T) \rightarrow X^*(T)$; from the W -invariance of τ , one deduces that

$$\iota(\alpha^\vee) = \tau(\alpha^\vee)\alpha \quad \text{for each coroot } \alpha^\vee. \quad (3.5)$$

Let \mathfrak{g} be the Lie algebra of G . The Lie algebra of T is a Cartan subalgebra \mathfrak{h} of \mathfrak{g} . Then τ can be seen as the restriction to \mathfrak{h} of the Killing form of \mathfrak{g} (suitably rescaled), and $X^*(T)$ is a lattice in the dual space \mathfrak{h}^\vee .

With the help of the Killing form of \mathfrak{g} , one defines a 2-cocycle of the Lie algebra $\mathfrak{g} \otimes_{\mathbf{C}} \mathbf{C}[t, t^{-1}]$, and thus a central extension

$$0 \rightarrow \mathbf{C}K \rightarrow \tilde{\mathfrak{g}} \xrightarrow{p} \mathfrak{g} \otimes_{\mathbf{C}} \mathbf{C}[t, t^{-1}] \rightarrow 0$$

of this algebra by a one-dimensional Lie algebra $\mathbf{C}K$ (see [Kac, pp. 97–98]). The affine Kac–Moody Lie algebra is then obtained by taking a semidirect product

$$\hat{\mathfrak{g}} = \tilde{\mathfrak{g}} \rtimes \mathbf{C}d$$

with a one-dimensional Lie algebra $\mathbf{C}d$, where d acts as $t \frac{d}{dt}$ on $\mathfrak{g} \otimes_{\mathbf{C}} \mathbf{C}[t, t^{-1}]$.

Further, $\mathfrak{h} \subset \mathfrak{g} \otimes_{\mathbf{C}} \mathbf{C}[t, t^{-1}]$ can be canonically lifted in $\tilde{\mathfrak{g}}$. Then $\hat{\mathfrak{h}} = \mathfrak{h} \oplus \mathbf{C}K \oplus \mathbf{C}d$ is a Cartan subalgebra of $\hat{\mathfrak{g}}$. Let $\Lambda_0 \in (\hat{\mathfrak{h}})^\vee$ be the linear form that vanishes on $\mathfrak{h} \oplus \mathbf{C}d$ and that maps K to 1. Let $V(\Lambda_0)$ be the irreducible integrable highest weight representation of $\hat{\mathfrak{g}}$ with highest weight Λ_0 . It is generated by a highest weight vector v_{Λ_0} , and the stabilizer of the line $[v_{\Lambda_0}]$ in $\mathbf{P}(V(\Lambda_0))$ is the parabolic subalgebra $p^{-1}(\mathfrak{g}[t]) \rtimes \mathbf{C}d$.

Thanks to Garland’s work [Gar], the representation $L(\Lambda_0)$ can be integrated to the Kac–Moody group \hat{G} that corresponds to the Lie algebra $\hat{\mathfrak{g}}$. This group is the semi-direct product of a central extension

$$1 \rightarrow \mathbf{C}^\times \rightarrow \tilde{G} \rightarrow G(\mathbf{C}[t, t^{-1}]) \rightarrow 1$$

by another copy of \mathbf{C}^\times , acting by loop rotations. The central \mathbf{C}^\times in \tilde{G} acts by scalar multiplication on $V(\Lambda_0)$, so $G(\mathbf{C}[t, t^{-1}])$ acts on $\mathbf{P}(V(\Lambda_0))$. Since the stabilizer of the line $[v_{\Lambda_0}]$ for this action is the subgroup $G(\mathbf{C}[t])$, the map $g \mapsto g[v_{\Lambda_0}]$ defines an embedding

$$\Psi : G(\mathbf{C}[t, t^{-1}])/G(\mathbf{C}[t]) \hookrightarrow \mathbf{P}(V(\Lambda_0)).$$

Further, using for instance the Iwasawa decomposition, one shows that on the level of \mathbf{C} -points, the obvious map

$$G(\mathbf{C}[t, t^{-1}])/G(\mathbf{C}[t]) \rightarrow G(\mathbf{C}((t)))/G(\mathbf{C}[[t]]) = \text{Gr}_G$$

is bijective. We eventually obtain an embedding

$$\Psi : \text{Gr}_G \rightarrow \mathbf{P}(V),$$

where we wrote V instead of $V(\Lambda_0)$ to shorten the notation.

Certainly, $\mathbf{P}(V)$ has the structure of an ind-variety: the final-dimensional pieces are all the finite-dimensional projective subspaces $\mathbf{P}(W)$ inside $\mathbf{P}(V)$. Then Ψ is a morphism of ind-varieties. Even better: thanks to the work of Kumar (see [Ku, Chap. 7]), we know that the ind-variety structure of Gr_G is induced via Ψ by that of $\mathbf{P}(V)$.

Lastly, [Kac, (6.5.4)] implies that

$$\Psi(L_\nu) \subset \mathbf{P}(V_{-\iota(\nu)}), \quad (3.6)$$

where $V_{-\iota(\nu)}$ is the subspace of V of weight $-\iota(\nu)$ for the action of $\mathfrak{h} \subset \hat{\mathfrak{g}}$.

After these lengthy preliminaries, we can go back to our problem.

Proposition 3.2. *Let $\mu \in X_*(T)$. Then $\overline{S_\mu} = \bigsqcup_{\nu \leq \mu} S_\nu$ and the boundary of S_μ is the set-theoretic intersection of $\overline{S_\mu}$ with a hyperplane H_μ of $\mathbf{P}(V)$:*

$$\partial S_\mu = \overline{S_\mu} \cap \Psi^{-1}(H_\mu).$$

Proof. Let $\lambda \in X_*(T)$. Writing $\Psi(L_\lambda) = \mathbf{C}v$, the vector v belongs to the weight subspace $V_{-\iota(\lambda)}$ of V by (3.6). The action on v of an element $u \in N_{\mathcal{K}}$ can only increase weights⁴, hence $uv - v \in \sum_{\chi \geq -\iota(\lambda)} V_\chi$. (The order \geq on \mathfrak{h}^\vee used here is the dominance order: nonnegative elements in \mathfrak{h}^\vee are nonnegative integral combinations of positive roots.) It follows that

$$\Psi(u \cdot L_\lambda) \in \mathbf{P} \left(\sum_{\chi \geq -\iota(\lambda)} V_\chi \right) \setminus \mathbf{P} \left(\sum_{\chi > -\iota(\lambda)} V_\chi \right).$$

Letting u run over $N_{\mathcal{K}}$, we deduce

$$\Psi(S_\lambda) \subset \mathbf{P} \left(\sum_{\chi \geq -\iota(\lambda)} V_\chi \right) \setminus \mathbf{P} \left(\sum_{\chi > -\iota(\lambda)} V_\chi \right).$$

Writing these inclusions for all possible λ , we conclude that⁵

$$\bigsqcup_{\nu \leq \mu} S_\nu = \Psi^{-1} \left(\mathbf{P} \left(\sum_{\chi \geq -\iota(\mu)} V_\chi \right) \right).$$

This implies that $\bigsqcup_{\nu \leq \mu} S_\nu$ is closed in Gr_G , whence (in view of (3.4)) the first equality in the statement. For the second one, one chooses a linear form $h \in V^\vee$ that vanishes on $\sum_{\chi > -\iota(\mu)} V_\chi$ but does not vanish on $\Psi(L_\mu)$ and takes $H_\mu = \ker h$. \square

⁴Certainly, V is only a projective representation of $G_{\mathcal{K}}$, so it seems hazardous to let $N_{\mathcal{K}}$ act on V . One may here argue that the cocycle that defines the central extension of $G_{\mathcal{K}}$ is trivial, so $N_{\mathcal{K}}$ can be lifted to a subgroup of \tilde{G} . This argument is however not needed on our reasoning, for we only need the result of the computation up to a nonzero scalar.

⁵Specifically, one must here observe that $\nu \leq \mu$ in $X_*(T)$ implies $-\iota(\nu) \geq -\iota(\mu)$ in \mathfrak{h}^\vee . This comes from the equality (3.5).

For symmetry reasons, one should also consider the Borel subgroup B^- opposite to B with respect to T and its unipotent radical N^- . One then has an Iwasawa decomposition

$$\mathrm{Gr}_G = \bigsqcup_{\mu \in X_*(T)} T_\mu, \quad \text{where } T_\mu = N_{\mathcal{K}}^- \cdot L_\mu,$$

and the closure of these orbits is given by

$$\overline{T}_\mu = \bigsqcup_{\substack{\nu \in X_*(T) \\ \nu \geq \mu}} T_\nu. \quad (3.7)$$

On the Serre tree, T_μ is seen as the horosphere centered at $+\infty$ going through L_μ . This makes the following lemma quite intuitive.

Lemma 3.3. *Let $\mu, \nu \in X_*(T)$. Then $\overline{S}_\mu \cap \overline{T}_\nu = \emptyset$ except if $\nu \leq \mu$, and $\overline{S}_\mu \cap \overline{T}_\mu = \{L_\mu\}$.*

(For a formal proof in the general case, one uses the projective embedding and weights arguments, as in the proof of Proposition 3.2.)

4 Semisimplicity of the Satake category

Let \mathbf{k} be a field of characteristic 0. Our goal in this section is to show that the category $\mathrm{P}_{\mathcal{S}}(\mathrm{Gr}_G, \mathbf{k})$ of perverse sheaves on Gr_G with coefficients in \mathbf{k} and with \mathcal{S} -constructible cohomology is semisimple. Since every object of this abelian category has finite length, this result means that there are no non-trivial extensions between simple objects.

4.1 The Satake category

Recall the notion of t -structure introduced in [BBD].

Definition 4.1. Let \mathcal{D} be a triangulated category. A t -structure on \mathcal{D} is a pair $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ of strictly full subcategories of \mathcal{D} which satisfy the following properties:

1. If $X \in \mathcal{D}^{\leq 0}$ and $Y \in \mathcal{D}^{\geq 0}$, then $\mathrm{Hom}_{\mathcal{D}}(X, Y[-1]) = \{0\}$.
2. We have $\mathcal{D}^{\leq 0} \subset \mathcal{D}^{\leq 0}[-1]$ and $\mathcal{D}^{\geq 0} \supset \mathcal{D}^{\geq 0}[-1]$.
3. For all $X \in \mathcal{D}$, there exists a distinguished triangle

$$A \rightarrow X \rightarrow B \xrightarrow{[1]}$$

in \mathcal{D} with $A \in \mathcal{D}^{\leq 0}$ and $B \in \mathcal{D}^{\geq 0}[-1]$.

We will say that an object in $\mathcal{D}^{\leq 0}$ (respectively, $\mathcal{D}^{\geq 0}$) is concentrated in nonpositive (respectively, nonnegative) degrees with respect to the t -structure. By axiom (2) in Definition 4.1, these notions are compatible with the cohomological shift, so we may as well consider for instance the subcategory $\mathcal{D}^{\geq 1} = (\mathcal{D}^{\geq 0})[-1]$ of objects concentrated in positive degrees. We also recall that the *heart* of the t -structure is the full subcategory $\mathcal{A} := \mathcal{D}^{\leq 0} \cap \mathcal{D}^{\geq 0}$ of \mathcal{D} ; this is an abelian category, whose exact sequences are the distinguished triangles

$$X \rightarrow Y \rightarrow Z \xrightarrow{[1]}$$

in \mathcal{D} where X , Y and Z belong to \mathcal{A} . In particular, this means that for any X, Y in \mathcal{A} we have a canonical identification

$$\mathrm{Ext}_{\mathcal{A}}^1(X, Y) \cong \mathrm{Hom}_{\mathcal{D}}(X, Y[1]). \quad (4.1)$$

For instance, the bounded derived category $D^b(\mathcal{A})$ of an abelian category \mathcal{A} has a natural t -structure, called the usual t -structure, whose heart is \mathcal{A} .

Let now X be a topological space endowed with a Whitney stratification \mathcal{S} . Given $S \in \mathcal{S}$, we denote by $i_S : S \hookrightarrow X$ the inclusion map. We denote by $D_{\mathcal{S}}^b(X, \mathbf{k})$ the bounded derived category of sheaves on X with coefficients in \mathbf{k} which are constructible with respect to \mathcal{S} . Thus, a complex \mathcal{F} of \mathbf{k} -sheaves on X belongs to $D_{\mathcal{S}}^b(X, \mathbf{k})$ if the cohomology sheaves $\mathcal{H}^n \mathcal{F}$ vanish for $|n| \gg 0$ and if each restriction $i_S^* \mathcal{H}^n \mathcal{F}$ is a local system (i.e. a locally free sheaf of finite rank).

In this setting, we define

$$\begin{aligned} {}^pD^{\leq 0} &= \{ \mathcal{F} \in D_{\mathcal{S}}^b(X, \mathbf{k}) \mid \forall S \in \mathcal{S}, \forall n > -\dim S, \mathcal{H}^n((i_S)^* \mathcal{F}) = 0 \}, \\ {}^pD^{\geq 0} &= \{ \mathcal{F} \in D_{\mathcal{S}}^b(X, \mathbf{k}) \mid \forall S \in \mathcal{S}, \forall n > -\dim S, \mathcal{H}^n((i_S)^! \mathcal{F}) = 0 \}. \end{aligned}$$

It is known (see [BBD, §2.1]) that $({}^pD^{\leq 0}, {}^pD^{\geq 0})$ is a t -structure on $D_{\mathcal{S}}^b(X, \mathbf{k})$, called the *perverse t -structure*. The simplest example is the case where \mathcal{S} contains only one stratum (which requires that X is smooth); then the perverse t -structure is just the ordinary t -structure, shifted to the left by $\dim X$. Objects in the heart $\mathrm{P}_{\mathcal{S}}(X, \mathbf{k}) := {}^pD^{\leq 0} \cap {}^pD^{\geq 0}$ of this t -structure are called *perverse sheaves*.

It is known that every object in $\mathrm{P}_{\mathcal{S}}(X, \mathbf{k})$ has finite length, see [BBD, Théorème 4.3.1]. Moreover, the simple objects in this category are classified by pairs (S, \mathcal{L}) , with $S \in \mathcal{S}$ and \mathcal{L} a simple local system on S . Specifically, to (S, \mathcal{L}) corresponds a unique object $\mathcal{F} \in D_{\mathcal{S}}^b(X, \mathbf{k})$ characterized by the conditions

$$\mathcal{F}|_{X \setminus \overline{S}} = 0, \quad \mathcal{F}|_S = \mathcal{L}[\dim S], \quad i^* \mathcal{F} \in {}^pD^{\leq -1}(\overline{S} \setminus S, \mathbf{k}), \quad i^! \mathcal{F} \in {}^pD^{\geq 1}(\overline{S} \setminus S, \mathbf{k}), \quad (4.2)$$

where $i : \overline{S} \setminus S \hookrightarrow X$ is the inclusion map. This \mathcal{F} is a perverse sheaf and is usually called the intersection cohomology sheaf on \overline{S} with coefficients in \mathcal{L} , and denoted $\mathbf{IC}(S, \mathcal{L})$. Then the assignment $(S, \mathcal{L}) \mapsto \mathbf{IC}(S, \mathcal{L})$ induces a bijection between equivalence classes of pairs (S, \mathcal{L}) as above (where $(S, \mathcal{L}) \sim (S, \mathcal{L}')$ if $\mathcal{L} \cong \mathcal{L}'$) and isomorphism classes of simple objects in $\mathrm{P}_{\mathcal{S}}(X, \mathbf{k})$.

We can finally define the main object of study of these notes. Consider the affine Grassmannian Gr_G , and its (Whitney) stratification \mathcal{S} by $G_{\mathcal{O}}$ -orbits, see §3.1. Then we can consider the constructible derived category $D_{\mathcal{S}}^b(\mathrm{Gr}_G, \mathbf{k})$, and its full subcategory $\mathrm{P}_{\mathcal{S}}(\mathrm{Gr}_G, \mathbf{k})$ of perverse sheaves. The main result of this section is the following.

Theorem 4.2. *The category $P_{\mathcal{S}}(\mathrm{Gr}_G, \mathbf{k})$ is semisimple.*

Remark 4.3. Note that the assumption that $\mathrm{char}(\mathbf{k}) = 0$ is crucial here. The category $P_{\mathcal{S}}(\mathrm{Gr}_G, \mathbf{k})$ with \mathbf{k} a field of positive characteristic is *not* semisimple.

Recall that the strata Gr_G^λ of \mathcal{S} are simply connected. Thus, the only simple local system on Gr_G^λ is the trivial one. We denote by \mathbf{IC}_λ the corresponding intersection cohomology sheaf. Then, the simple objects in $P_{\mathcal{S}}(\mathrm{Gr}_G, \mathbf{k})$ are (up to isomorphism) these complexes \mathbf{IC}_λ , for $\lambda \in X_*(T)^+$. Since every object in $P_{\mathcal{S}}(\mathrm{Gr}_G, \mathbf{k})$ has finite length, and in view of (4.1), Theorem 4.2 follows from the following claim.

Proposition 4.4. *For any $\lambda, \mu \in X_*(T)^+$, we have*

$$\mathrm{Hom}_{D_{\mathcal{S}}^b(\mathrm{Gr}_G, \mathbf{k})}(\mathbf{IC}_\lambda, \mathbf{IC}_\mu[1]) = 0.$$

With these notations, the main ingredient in the proof of the semisimplicity property can be stated as follows: the homology sheaves $\mathcal{H}^k(\mathbf{IC}_\lambda)$ vanish unless k and $\dim(\mathrm{Gr}_G^\lambda)$ have the same parity. (We will also use the fact that an orbit Gr_G^μ contained in the closure $\overline{\mathrm{Gr}_G^\lambda}$ has even codimension.)

4.2 Parity vanishing

As explained above, a key point in the proof of Proposition 4.4 is the following result.

Lemma 4.5. *For any $\lambda \in X_*(T)^+$, we have*

$$\mathcal{H}^n(\mathbf{IC}_\lambda) = 0 \quad \text{unless } n \equiv \dim(\mathrm{Gr}_G^\lambda) \pmod{2}.$$

A similar property in fact holds for Iwahori-constructible perverse sheaves on the affine flag variety. In this section, we argue that this property can be deduced from the existence of resolutions of closures of Iwahori orbits whose fibers are paved by affine spaces. (A different proof of this property can be given by imitating the case of the finite flag variety treated in [Sp].)

Let W be the Weyl group of (G, T) and let $Q^\vee \subset X_*(T)$ be the coroot lattice. The affine Weyl group and the extended affine Weyl group are defined as

$$W_{\mathrm{aff}} = W \ltimes Q^\vee \quad \text{and} \quad \widetilde{W}_{\mathrm{aff}} = W \ltimes X_*(T).$$

As is well known, W_{aff} is generated by a set S_{aff} of simple reflections, and $(W_{\mathrm{aff}}, S_{\mathrm{aff}})$ is a Coxeter system with length function ℓ which satisfies

$$\ell(w \cdot t_\lambda) = \sum_{\alpha \in \Phi^+ \cap w^{-1}(\Phi^+)} |\langle \lambda, \alpha \rangle| + \sum_{\alpha \in \Phi^+ \cap w^{-1}(\Phi^-)} |1 + \langle \lambda, \alpha \rangle|$$

for $w \in W$ and $\lambda \in Q^\vee$. This formula makes sense more generally for $\lambda \in X_*(T)$, which allows to extend ℓ to $\widetilde{W}_{\mathrm{aff}}$. Then if $\Omega = \{w \in \widetilde{W}_{\mathrm{aff}} \mid \ell(w) = 0\}$, then the conjugation action of the subgroup Ω on $\widetilde{W}_{\mathrm{aff}}$ preserves S_{aff} , and we have $\widetilde{W}_{\mathrm{aff}} = W_{\mathrm{aff}} \rtimes \Omega$.

As in §3.3, let $B^- \subset G$ be the Borel subgroup opposite to B with respect to T , and let $I \subset G_{\mathcal{O}}$ be the corresponding Iwahori subgroup, defined as the preimage of B^- under the evaluation map $G_{\mathcal{O}} \rightarrow G$ given by $t \mapsto 0$. The Bruhat decomposition then yields

$$\mathrm{Gr}_G = \bigsqcup_{w \in \widetilde{W}_{\mathrm{aff}}/W} IwG_{\mathcal{O}}/G_{\mathcal{O}},$$

and $IwG_{\mathcal{O}}/G_{\mathcal{O}}$ is an affine space of dimension $\ell(w)$ if w is of minimal length in the coset wW .

Let $\lambda \in X_*(T)^+$. Then

$$\mathrm{Gr}_G^\lambda = \bigsqcup_{w \in Wt_\lambda W/W} IwG_{\mathcal{O}}/G_{\mathcal{O}}$$

is a union of Schubert cells. One of these cells is open dense in Gr_G^λ ; we denote by w_λ the unique element in $Wt_\lambda W$ which is minimal in $w_\lambda W$ and such that $Iw_\lambda G_{\mathcal{O}}/G_{\mathcal{O}}$ is open in Gr_G^λ . Certainly then we have

$$\mathbf{IC}_\lambda = \mathbf{IC}(Iw_\lambda G_{\mathcal{O}}/G_{\mathcal{O}}, \underline{\mathbf{k}}).$$

Hence Lemma 4.5 follows from the claim that for any $w \in \widetilde{W}_{\mathrm{aff}}$ which is minimal in wW we have

$$\mathcal{H}^n(\mathbf{IC}(IwG_{\mathcal{O}}/G_{\mathcal{O}}, \underline{\mathbf{k}})) \neq 0 \implies n \equiv \ell(w) \pmod{2}. \quad (4.3)$$

To prove (4.3) we introduce the affine flag variety

$$\mathrm{Fl}_G := G_{\mathcal{K}}/I.$$

As for Gr_G , this variety has a natural complex ind-variety structure, and a Bruhat decomposition

$$\mathrm{Fl}_G = \bigsqcup_{w \in \widetilde{W}_{\mathrm{aff}}} IwI/I,$$

see [Gö] for details and references. This decomposition provides a Whitney stratification of Fl_G , which we denote by \mathcal{S} . Then we can consider the constructible derived category $D_{\mathcal{S}}^b(\mathrm{Fl}_G, \underline{\mathbf{k}})$ and the corresponding category $\mathrm{P}_{\mathcal{S}}(\mathrm{Fl}_G, \underline{\mathbf{k}})$ of perverse sheaves.

Let $\pi : \mathrm{Fl}_G \rightarrow \mathrm{Gr}_G$ be the natural projection. This morphism is smooth; in fact it is a locally trivial fibration with fiber G/B^- . From this property and the characterization of the intersection cohomology complex given in (4.2), it is not difficult to check that for any $w \in \widetilde{W}_{\mathrm{aff}}$ which is minimal in wW , we have

$$\pi^* \mathbf{IC}(IwG_{\mathcal{O}}/G_{\mathcal{O}}, \underline{\mathbf{k}})[\ell(w_0)] \cong \mathbf{IC}(Iww_0I/I, \underline{\mathbf{k}}),$$

where $w_0 \in W$ is the longest element (so that $\ell(w_0) = \dim(G/B^-)$). This shows that (4.3) (hence also Lemma 4.5) follows from the following claim.

Lemma 4.6. *For any $w \in \widetilde{W}_{\mathrm{aff}}$ we have*

$$\mathcal{H}^n(\mathbf{IC}(IwI/I, \underline{\mathbf{k}})) \neq 0 \implies n \equiv \ell(w) \pmod{2}.$$

Proof. For any $s \in S_{\text{aff}}$, denote by $J_s = IsI \cup I$ the minimal parahoric subgroup of $G_{\mathcal{K}}$ associated with s . Fix $w \in \widetilde{W}_{\text{aff}}$, and choose a reduced expression $\underline{w} = (s_1, \dots, s_r, \omega)$ for w (with $s_j \in S_{\text{aff}}$ and $\ell(\omega) = 0$). We can then consider the Bott-Samelson resolution

$$\pi_{\underline{w}} : J_{s_1} \times^I \dots \times^I J_{s_r} \times^I \underbrace{(I\omega I/I)}_{\text{(a point)}} \rightarrow \overline{IwI/I}$$

induced by multiplication in $G_{\mathcal{K}}$. It is known that $\pi_{\underline{w}}$ is proper and is an isomorphism over IwI/I . It is known also that each fiber $\pi_{\underline{w}}^{-1}(x)$ is paved by affine spaces. (For this claim in the case of finite flag varieties, see [Gau]. See also [Ha] for a different proof, which works mutatis mutandis in the affine setting.) Therefore

$$H_c^{n+\ell(w)}(\pi_{\underline{w}}^{-1}(x); \mathbf{k})$$

is nonzero only if $n + \ell(w)$ is even. By proper base change, this cohomology group is the stalk at x of the cohomology sheaf $\mathcal{H}^n((\pi_{\underline{w}})_! \underline{\mathbf{k}}[\ell(w)])$, so that

$$\mathcal{H}^n((\pi_{\underline{w}})_! \underline{\mathbf{k}}[\ell(w)]) \neq 0 \implies n \equiv \ell(w) \pmod{2}.$$

Our desired parity vanishing property then follows from the decomposition theorem, which here says that $\mathbf{IC}(IwI/I; \underline{\mathbf{k}})$ is a direct summand of the complex $(\pi_{\underline{w}})_! \underline{\mathbf{k}}[\ell(w)]$. \square

4.3 Proof of Proposition 4.4

We distinguish 3 cases.

First case: $\lambda = \mu$.

Consider the diagram

$$\begin{array}{ccc} \text{Gr}_G^\lambda & \xrightarrow{j} & \overline{\text{Gr}_G^\lambda} \leftarrow i \overline{\text{Gr}_G^\lambda} \setminus \text{Gr}_G^\lambda \\ & \searrow j_\lambda & \downarrow i_\lambda \\ & & \text{Gr}_G \end{array}$$

where all maps are the obvious embeddings. Set $\mathcal{F} = (i_\lambda i)^* \mathbf{IC}_\lambda$; by (4.2), this complex of sheaves is concentrated in negative perverse degrees. Likewise, the complex of sheaves $(i_\lambda i)^! \mathbf{IC}_\lambda$ is concentrated in positive perverse degrees. It follows that

$$\text{Hom}_{D_{\mathcal{F}}^b(\overline{\text{Gr}_G^\lambda} \setminus \text{Gr}_G^\lambda, \mathbf{k})}(\mathcal{F}, (i_\lambda i)^! \mathbf{IC}_\lambda[1]) = 0. \quad (4.4)$$

Applying the cohomological functor $\text{Hom}_{D_{\mathcal{F}}^b(\text{Gr}_G, \mathbf{k})}^\bullet((i_\lambda i)^? \mathbf{IC}_\lambda[1])$ to the distinguished triangle

$$j! j^* \left(\mathbf{IC}_\lambda|_{\overline{\text{Gr}_G^\lambda}} \right) \rightarrow \left(\mathbf{IC}_\lambda|_{\overline{\text{Gr}_G^\lambda}} \right) \rightarrow i! i^* \left(\mathbf{IC}_\lambda|_{\overline{\text{Gr}_G^\lambda}} \right) \xrightarrow{[1]},$$

we get an exact sequence

$$\begin{aligned} \text{Hom}_{D_{\mathcal{F}}^b(\text{Gr}_G, \mathbf{k})}((i_\lambda i)^! \mathcal{F}, \mathbf{IC}_\lambda[1]) &\rightarrow \text{Hom}_{D_{\mathcal{F}}^b(\text{Gr}_G, \mathbf{k})}(\mathbf{IC}_\lambda, \mathbf{IC}_\lambda[1]) \\ &\rightarrow \text{Hom}_{D_{\mathcal{F}}^b(\text{Gr}_G, \mathbf{k})}((j_\lambda)_! \underline{\mathbf{k}}_{\text{Gr}_G^\lambda}[\dim \text{Gr}_G^\lambda], \mathbf{IC}_\lambda[1]). \end{aligned} \quad (4.5)$$

The first space in (4.5) is zero, thanks to (4.4) and because by adjunction we have

$$\mathrm{Hom}_{D_{\mathcal{F}}^b(\mathrm{Gr}_G, \mathbf{k})}((i_\lambda i)^\dagger \mathcal{F}, \mathbf{IC}_\lambda[1]) \cong \mathrm{Hom}_{D_{\mathcal{F}}^b(\overline{\mathrm{Gr}_G^\lambda}, \mathrm{Gr}_G^\lambda, \mathbf{k})}(\mathcal{F}, (i_\lambda i)^\dagger \mathbf{IC}_\lambda[1]).$$

By adjunction again, the third space in (4.5) is

$$\begin{aligned} \mathrm{Hom}_{D_{\mathcal{F}}^b(\mathrm{Gr}_G, \mathbf{k})}((j_\lambda)^\dagger \underline{\mathbf{k}}_{\mathrm{Gr}_G^\lambda}[\dim \mathrm{Gr}_G^\lambda], \mathbf{IC}_\lambda[1]) &\cong \mathrm{Hom}_{D_{\mathcal{F}}^b(\mathrm{Gr}_G^\lambda, \mathbf{k})}(\underline{\mathbf{k}}_{\mathrm{Gr}_G^\lambda}[\dim \mathrm{Gr}_G^\lambda], (j_\lambda)^\dagger \mathbf{IC}_\lambda[1]) \\ &\cong \mathrm{Hom}_{D_{\mathcal{F}}^b(\mathrm{Gr}_G^\lambda, \mathbf{k})}(\underline{\mathbf{k}}_{\mathrm{Gr}_G^\lambda}, \underline{\mathbf{k}}_{\mathrm{Gr}_G^\lambda}[1]) \\ &= H^1(\mathrm{Gr}_G^\lambda; \mathbf{k}). \end{aligned}$$

This last space is again zero since Gr_G^λ is an affine bundle over a partial flag variety (see Proposition 3.1), so has only cohomology in even degrees.

We conclude that $\mathrm{Hom}_{D_{\mathcal{F}}^b(\mathrm{Gr}_G, \mathbf{k})}(\mathbf{IC}_\lambda, \mathbf{IC}_\lambda[1]) = 0$.

Second case: Neither $\mathrm{Gr}_G^\lambda \subset \overline{\mathrm{Gr}_G^\mu}$ nor $\mathrm{Gr}_G^\mu \subset \overline{\mathrm{Gr}_G^\lambda}$.

Consider the inclusion $i_\mu : \overline{\mathrm{Gr}_G^\mu} \hookrightarrow \mathrm{Gr}_G$. Since \mathbf{IC}_μ is supported on $\overline{\mathrm{Gr}_G^\mu}$, we have $\mathbf{IC}_\mu = (i_\mu)_*(i_\mu)^* \mathbf{IC}_\mu$ and therefore

$$\mathrm{Hom}_{D_{\mathcal{F}}^b(\mathrm{Gr}_G, \mathbf{k})}(\mathbf{IC}_\lambda, \mathbf{IC}_\mu[1]) \cong \mathrm{Hom}_{D_{\mathcal{F}}^b(\overline{\mathrm{Gr}_G^\mu}, \mathbf{k})}((i_\mu)^* \mathbf{IC}_\lambda, (i_\mu)^* \mathbf{IC}_\mu[1]).$$

Now set $Z = \overline{\mathrm{Gr}_G^\lambda} \cap \overline{\mathrm{Gr}_G^\mu}$ and consider the inclusion $f : Z \hookrightarrow \overline{\mathrm{Gr}_G^\mu}$. Since $(i_\mu)^* \mathbf{IC}_\lambda$ is supported on Z , it is of the form $f_! \mathcal{F}$ for some complex of sheaves $\mathcal{F} \in D_{\mathcal{F}}^b(Z, \mathbf{k})$. Arguing as in the first case, we see that \mathcal{F} is concentrated in negative perverse degrees and that $f^!(i_\mu)^* \mathbf{IC}_\mu \cong (i_\mu f)^\dagger \mathbf{IC}_\mu$ is concentrated in positive perverse degrees. Therefore

$$\begin{aligned} \mathrm{Hom}_{D_{\mathcal{F}}^b(\mathrm{Gr}_G, \mathbf{k})}(\mathbf{IC}_\lambda, \mathbf{IC}_\mu[1]) &\cong \mathrm{Hom}_{D_{\mathcal{F}}^b(\overline{\mathrm{Gr}_G^\mu}, \mathbf{k})}(f_! \mathcal{F}, (i_\mu)^* \mathbf{IC}_\mu[1]) \\ &\cong \mathrm{Hom}_{D_{\mathcal{F}}^b(Z, \mathbf{k})}(\mathcal{F}, f^!(i_\mu)^* \mathbf{IC}_\mu[1]) = 0, \end{aligned}$$

as desired.

Third case: $\lambda \neq \mu$ and either $\mathrm{Gr}_G^\lambda \subset \overline{\mathrm{Gr}_G^\mu}$ or $\mathrm{Gr}_G^\mu \subset \overline{\mathrm{Gr}_G^\lambda}$.

Since Verdier duality is an anti-autoequivalence of $D_{\mathcal{F}}^b(\mathrm{Gr}_G, \mathbf{k})$ which fixes \mathbf{IC}_λ and \mathbf{IC}_μ , we can assume that $\mathrm{Gr}_G^\mu \subset \overline{\mathrm{Gr}_G^\lambda}$. Let $j_\mu : \mathrm{Gr}_G^\mu \hookrightarrow \mathrm{Gr}_G$ be the inclusion, and let $\mathcal{G} \in D_{\mathcal{F}}^b(\mathrm{Gr}_G, \mathbf{k})$ be the cone of the adjunction map $\mathbf{IC}_\mu \rightarrow (j_\mu)_*(j_\mu)^* \mathbf{IC}_\mu \cong (j_\mu)_* \underline{\mathbf{k}}_{\mathrm{Gr}_G^\mu}[\dim \mathrm{Gr}_G^\mu]$. It follows from the definition of the perverse t-structure that $(j_\mu)_* \underline{\mathbf{k}}_{\mathrm{Gr}_G^\mu}[\dim \mathrm{Gr}_G^\mu]$ is concentrated in nonnegative perverse degrees, and it is a classical fact that the morphism $\mathbf{IC}_\mu \rightarrow {}^p \mathcal{H}^0((j_\mu)_* \underline{\mathbf{k}}_{\mathrm{Gr}_G^\mu}[\dim \mathrm{Gr}_G^\mu])$ induced by the adjunction map considered above (where ${}^p \mathcal{H}^0(?)$ means the degree-0 perverse cohomology) is injective, see e.g. [BBD, (1.4.22.1)]. Therefore, \mathcal{G} is concentrated in nonnegative perverse degrees.

From the triangle

$$\mathbf{IC}_\mu \rightarrow (j_\mu)_* \underline{\mathbf{k}}_{\mathrm{Gr}_G^\mu}[\dim \mathrm{Gr}_G^\mu] \rightarrow \mathcal{G} \xrightarrow{[1]}$$

we get the exact sequence

$$\begin{aligned} \mathrm{Hom}_{D_{\mathcal{S}}^b(\mathrm{Gr}_G, \mathbf{k})}(\mathbf{IC}_\lambda, \mathcal{G}) &\rightarrow \mathrm{Hom}_{D_{\mathcal{S}}^b(\mathrm{Gr}_G, \mathbf{k})}(\mathbf{IC}_\lambda, \mathbf{IC}_\mu[1]) \\ &\rightarrow \mathrm{Hom}_{D_{\mathcal{S}}^b(\mathrm{Gr}_G, \mathbf{k})}(\mathbf{IC}_\lambda, (j_\mu)_* \underline{\mathbf{k}}_{\mathrm{Gr}_G^\mu}[\dim \mathrm{Gr}_G^\mu + 1]). \end{aligned}$$

As in the second case, using the fact that \mathcal{G} is concentrated in nonnegative perverse degrees and supported on $\overline{\mathrm{Gr}_G^\lambda}$, which is included in $\overline{\mathrm{Gr}_G^\lambda} \setminus \mathrm{Gr}_G^\lambda$, one checks that the left Hom is zero.

By (4.2) $(j_\mu)^* \mathbf{IC}_\lambda$ is concentrated in degrees $< -\dim \mathrm{Gr}_G^\mu$. On the other hand, by Lemma 4.5, this complex has cohomology only in degrees of the same parity as $\dim(\mathrm{Gr}_G^\lambda)$. Noting that $\dim(\mathrm{Gr}_G^\lambda) \equiv \dim(\mathrm{Gr}_G^\mu) \pmod{2}$ (because these orbits belong to the same connected component of Gr_G), this implies that in fact $(j_\mu)^* \mathbf{IC}_\lambda$ is concentrated in degrees $\leq -\dim \mathrm{Gr}_G^\mu - 2$. It follows that

$$\mathrm{Hom}_{D_{\mathcal{S}}^b(\mathrm{Gr}_G, \mathbf{k})}(\mathbf{IC}_\lambda, (j_\mu)_* \underline{\mathbf{k}}_{\mathrm{Gr}_G^\mu}[\dim \mathrm{Gr}_G^\mu + 1]) = \mathrm{Hom}_{D_{\mathcal{S}}^b(\mathrm{Gr}_G^\mu, \mathbf{k})}((j_\mu)^* \mathbf{IC}_\lambda, \underline{\mathbf{k}}_{\mathrm{Gr}_G^\mu}[\dim \mathrm{Gr}_G^\mu + 1])$$

vanishes.

Our exact sequence then yields the desired equality $\mathrm{Hom}_{D_{\mathcal{S}}^b(\mathrm{Gr}_G, \mathbf{k})}(\mathbf{IC}_\lambda, \mathbf{IC}_\mu[1]) = 0$.

4.4 Consequence on equivariance

Consider the category

$$\mathrm{P}_{G_{\mathcal{O}}}(\mathrm{Gr}_G, \mathbf{k})$$

of $G_{\mathcal{O}}$ -equivariant perverse sheaves; see §A.1. (Here the stratification we consider is \mathcal{S} .) We have a forgetful functor

$$\mathrm{P}_{G_{\mathcal{O}}}(\mathrm{Gr}_G, \mathbf{k}) \rightarrow \mathrm{P}_{\mathcal{S}}(\mathrm{Gr}_G, \mathbf{k}),$$

which is fully faithful by construction. As a consequence of Theorem 4.2, each object in $\mathrm{P}_{\mathcal{S}}(\mathrm{Gr}_G, \mathbf{k})$ is isomorphic to a direct sum of the simple objects \mathbf{IC}_λ , hence belongs to the essential images of this functor. We deduce the following.

Corollary 4.7. *The forgetful functor*

$$\mathrm{P}_{G_{\mathcal{O}}}(\mathrm{Gr}_G, \mathbf{k}) \rightarrow \mathrm{P}_{\mathcal{S}}(\mathrm{Gr}_G, \mathbf{k})$$

is an equivalence of categories.

Remark 4.8. This statement remains true for more general coefficients, as proved in [MV]. However, the proof in this generality is much more involved.

5 Dimension estimates and the weight functors

The geometric Satake equivalence is the statement that for any connected reductive algebraic group G , the category $\mathrm{P}_{G_{\mathcal{O}}}(\mathrm{Gr}_G, \mathbf{k})$ is equivalent to the category of representations of the Langlands dual \check{G} of G . The method is to define a convolution product on $\mathrm{P}_{G_{\mathcal{O}}}(\mathrm{Gr}_G, \mathbf{k})$ so that this category becomes Tannakian; the fiber functor here is the global cohomology functor

$H(\mathrm{Gr}_G, ?)$. One then has to identify the affine group scheme provided by Theorem 2.7 with \check{G} . A key construction for this last step is Mirković and Vilonen's weight functors, which define an action of the maximal torus of \check{G} on $H(\mathrm{Gr}_G, \mathcal{A})$, for each $\mathcal{A} \in \mathcal{P}_{G_{\mathcal{O}}}(\mathrm{Gr}_G, \mathbf{k})$.

We here explain the construction of these weight functors. Recall that we have chosen a maximal torus and a Borel subgroup $T \subset B \subset G$. Then $T \subset G_{\mathcal{K}}$ acts on $\mathrm{Gr}_G = G_{\mathcal{K}}/G_{\mathcal{O}}$ with fixed points

$$(\mathrm{Gr}_G)^T = \{L_{\mu} \mid \mu \in X_*(T)\}.$$

The choice of a dominant regular cocharacter $\eta \in X_*(T)$ provides a one-parameter subgroup $\mathbf{G}_m \subset T$, whence a \mathbf{C}^{\times} -action on Gr_G with fixed points $(\mathrm{Gr}_G)^T$. The attractive and repulsive varieties relative to the fixed point L_{μ} coincide with the semi-infinite orbits S_{μ} and T_{μ} defined in section 3:

$$S_{\mu} = \{x \in \mathrm{Gr}_G \mid \eta(a) \cdot x \rightarrow L_{\mu} \text{ when } a \rightarrow 0\}$$

and

$$T_{\mu} = \{x \in \mathrm{Gr}_G \mid \eta(a) \cdot x \rightarrow L_{\mu} \text{ when } a \rightarrow \infty\}.$$

With these notations, the weight functor F_{μ} is defined as either the cohomology with compact support of the restriction to S_{μ} , or as the cohomology with support in T_{μ} . These two definitions are equivalent, thanks to Braden's theorem on hyperbolic localization.

5.1 Dimension estimates

Recall from §3.1 that ρ denotes the half-sum of the positive roots, considered as a linear form $X_*(T) \rightarrow \frac{1}{2}\mathbf{Z}$, and that $Q^{\vee} \subset X_*(T)$ denotes the coroot lattice.

Theorem 5.1. *Let $\lambda, \mu \in X_*(T)$ with λ dominant.*

1. *We have*

$$\overline{\mathrm{Gr}_G^{\lambda}} \cap S_{\mu} \neq \emptyset \iff L_{\mu} \in \overline{\mathrm{Gr}_G^{\lambda}} \iff \mu \in \mathrm{Conv}(W\lambda) \cap (\lambda + Q^{\vee}),$$

where Conv denotes the convex hull.

2. *If μ satisfies the condition in (1), the intersection $\overline{\mathrm{Gr}_G^{\lambda}} \cap \overline{S_{\mu}}$ has pure dimension $\langle \rho, \lambda + \mu \rangle$.*

3. *If μ satisfies the condition in (1), we have $\overline{\mathrm{Gr}_G^{\lambda}} \cap \overline{S_{\mu}} = \overline{\mathrm{Gr}_G^{\lambda} \cap S_{\mu}}$.*

Proof. (1) Let $\eta \in X_*(T)$ be regular dominant. If $g \in N_{\mathcal{K}}$, then $\eta(a)g\eta(a)^{-1} \rightarrow 1$ when $a \rightarrow 0$. Therefore, looking at the induced action of \mathbf{C}^{\times} on Gr_G , we obtain that for any $\mu \in X_*(T)$,

$$S_{\mu} \subset \{x \in \mathrm{Gr}_G \mid \eta(a) \cdot x \rightarrow L_{\mu} \text{ when } a \rightarrow 0\}.$$

In view of the Iwasawa decomposition (3.3), this inclusion is in fact an equality. Then the stability of $\overline{\mathrm{Gr}_G^{\lambda}}$ by the action of T implies the first equivalence.

On the other hand, we have

$$W\lambda \subset \{\mu \in X_*(T) \mid L_{\mu} \in \overline{\mathrm{Gr}_G^{\lambda}}\},$$

and using the Cartan decomposition (3.1), we see that this inclusion is in fact an equality. The description of $\overline{\text{Gr}_G^\lambda}$ recalled in (3.2) then implies that $L_\mu \in \overline{\text{Gr}_G^\lambda}$ if and only if the W -conjugate μ^+ of μ satisfies $\mu^+ \leq \lambda$. Then comparing e.g. [Hu, Proposition 21.3] and [HL, Theorem 7.41] we see that this condition is equivalent to $\mu \in \text{Conv}(W\lambda) \cap (\lambda + Q^\vee)$.

(2) From (1), we deduce that $\overline{\text{Gr}_G^\lambda}$ meets only those S_μ such that $\mu \leq \lambda$, therefore $\overline{\text{Gr}_G^\lambda} \subset \overline{S_\lambda}$.⁶ The claim for $\mu = \lambda$ then follows from the fact that Gr_G^λ is irreducible of dimension $\langle 2\rho, \lambda \rangle$.

Let $w_0 \in W$ be the longest element, so that $w_0\lambda$ is the unique antidominant element in $W\lambda$. Then we have similarly $\overline{\text{Gr}_G^\lambda} = \overline{\text{Gr}_G^{w_0\lambda}} \subset \overline{T_{w_0\lambda}}$, and using Lemma 3.3, we get $\overline{\text{Gr}_G^\lambda} \cap \overline{S_{w_0\lambda}} = \{L_{w_0\lambda}\}$. The claim therefore holds also for $\mu = w_0\lambda$.

To prove the general case, we navigate between these two extreme cases by cutting by hyperplane sections, as in Proposition 3.2, using equation (3.4); see [MV] for details.

(3) Let Z be an irreducible component of $\overline{\text{Gr}_G^\lambda} \cap \overline{S_\mu}$. Then Z must meet Gr_G^λ , otherwise by (3.2) it would be contained in some $\overline{\text{Gr}_G^\eta}$ with $\eta < \lambda$, and the inequality $\dim Z = \langle \rho, \lambda + \mu \rangle > \langle \rho, \eta + \mu \rangle$ would contradict (2). Therefore $Z \cap \text{Gr}_G^\lambda$ is open dense in Z . Likewise, using (3.4), we see that $Z \cap S_\mu$ is open dense in Z . We conclude that $Z \cap \text{Gr}_G^\lambda \cap S_\mu$ is dense in Z . \square

The following corollary will be useful later.

Corollary 5.2. *Let $\lambda \in X_*(T)^+$, and let $X \subset \overline{\text{Gr}_G^\lambda}$ be a closed T -invariant subvariety. Then*

$$\dim(X) \leq \max\{\langle \rho, \lambda + \mu \rangle \mid L_\mu \in X\}.$$

Proof. Let $\eta \in X_*(T)$ be regular dominant. We saw during the proof of Theorem 5.1(1) that

$$S_\mu = \{x \in \text{Gr}_G \mid \eta(a) \cdot x \rightarrow L_\mu \text{ when } a \rightarrow 0\}.$$

Therefore X meets S_μ if and only if $L_\mu \in X$, whence

$$X \subset \bigcup_{\substack{\mu \in X_*(T) \\ L_\mu \in X}} S_\mu,$$

and therefore

$$X \subset \bigcup_{\substack{\mu \in X_*(T) \\ L_\mu \in X}} (\overline{\text{Gr}_G^\lambda} \cap S_\mu).$$

The corollary now follows from Theorem 5.1(2). \square

The following theorem is the analogue of Theorem 5.1 for the Borel subgroup B^- in place of B .

Theorem 5.3. *Let $\lambda, \mu \in X_*(T)$ with λ dominant. Then $\overline{\text{Gr}_G^\lambda} \cap \overline{T_\mu} = \overline{\text{Gr}_G^\lambda \cap T_\mu}$, and this intersection has pure dimension $\langle \rho, \lambda - \mu \rangle$.*

⁶To prove the inclusion $\overline{\text{Gr}_G^\lambda} \subset \overline{S_\lambda}$, one can also argue as follows. The open cell $N_{\mathcal{O}}B_{\mathcal{O}}^-$ is dense in $G_{\mathcal{O}}$ and $B_{\mathcal{O}}^-$ stabilizes L_λ , therefore $\text{Gr}_G^\lambda = G_{\mathcal{O}} \cdot L_\lambda$ contains $N_{\mathcal{O}} \cdot L_\lambda$ as a dense subset, whence $\text{Gr}_G^\lambda \subset \overline{N_{\mathcal{O}} \cdot L_\lambda} \subset \overline{S_\lambda}$.

5.2 Weight functors

Recall that if X is a topological space, $i : Y \rightarrow X$ is the inclusion of a locally closed subspace and $\mathcal{F} \in D^b(X, \mathbf{k})$, then the local cohomology groups $H_Y^k(X, \mathcal{F})$ are defined as $H^k(Y, i^! \mathcal{F})$.

Proposition 5.4. *For each $\mathcal{A} \in P_{G_{\mathcal{O}}}(\mathrm{Gr}_G, \mathbf{k})$, $\mu \in X_*(T)$ and $k \in \mathbf{Z}$, there exists a canonical isomorphism*

$$H_{T_\mu}^k(\mathrm{Gr}_G, \mathcal{A}) \xrightarrow{\sim} H_c^k(S_\mu, \mathcal{A}),$$

and both terms vanish if $k \neq \langle 2\rho, \mu \rangle$.

Proof. For all $\lambda \in X_*(T)^+$, we have $\mathcal{A}|_{\mathrm{Gr}_G^\lambda} \in D^{\leq -\langle 2\rho, \lambda \rangle}(\mathrm{Gr}_G^\lambda, \mathbf{k})$ by the perversity conditions (see §4.1). Further, the dimension estimates from Theorem 5.1(2) imply that $H_c^k(\mathrm{Gr}_G^\lambda \cap S_\mu; \mathbf{k}) = 0$ for $k > \langle 2\rho, \lambda + \mu \rangle$. Using an easy dévissage argument, we deduce that

$$H_c^k(\mathrm{Gr}_G^\lambda \cap S_\mu, \mathcal{A}) = 0 \quad \text{for } k > \langle 2\rho, \mu \rangle.$$

Filtering the support of \mathcal{A} by the closed subsets $\overline{\mathrm{Gr}_G^\lambda}$, we deduce that

$$H_c^k(S_\mu, \mathcal{A}) = 0 \quad \text{for } k > \langle 2\rho, \mu \rangle.$$

(To prove this formally, one can either use a spectral sequence or write down distinguished triangles associated to inclusions of an open subset and its closed complement. With both methods, in order to deal with a sequence of closed subsets, it is convenient to enumerate the dominant weights as $(\lambda_n)_{n \geq 0}$ in such a way that $(\lambda_i \leq \lambda_j) \Rightarrow (i \leq j)$.)

An analogous (dual) argument shows that

$$H_{T_\mu}^k(\mathrm{Gr}_G, \mathcal{A}) = 0 \quad \text{for } k < 2\langle \rho, \mu \rangle.$$

Lastly, Braden's hyperbolic localization theorem [Br] provides a canonical isomorphism

$$H_{T_\mu}^k(\mathrm{Gr}_G, \mathcal{A}) \cong H_c^k(S_\mu, \mathcal{A})$$

for any $k \in \mathbf{Z}$. The claim follows. \square

In view of this proposition, for any $\mu \in X_*(T)$ we consider the functor

$$F_\mu : P_{G_{\mathcal{O}}}(\mathrm{Gr}_G, \mathbf{k}) \rightarrow \mathrm{Vect}_{\mathbf{k}}$$

defined by

$$F_\mu(\mathcal{A}) = H_{T_\mu}^{\langle 2\rho, \mu \rangle}(\mathrm{Gr}_G, \mathcal{A}) \cong H_c^{\langle 2\rho, \mu \rangle}(S_\mu, \mathcal{A}).$$

Since the category $P_{G_{\mathcal{O}}}(\mathrm{Gr}_G, \mathbf{k})$ is semisimple (see Theorem 4.2 and Corollary 4.7), this functor is automatically exact.

Remark 5.5. 1. The running assumption that $\mathrm{char}(\mathbf{k}) = 0$ was not used in the proof of Proposition 5.4, and this claim in fact holds for very general coefficients. However in this setting, the exactness of the functors F_μ requires some justification: it follows from the vanishing of $H_{T_\mu}^{\langle 2\rho, \mu \rangle \pm 1}(\mathrm{Gr}_G, \mathcal{A})$ and a long exact sequence argument.

2. The $G_{\mathcal{O}}$ -invariance is not used in the proof of Proposition 5.4 (only the constructibility with respect to $G_{\mathcal{O}}$ -orbits matters), but has however the following important consequence: the functors F_{μ} do not depend on the choice of the Torel (or Borus) $T \subset B \subset G$. Specifically, the action of an element $g \in G$ on a pair (T, B) moves also the cocharacter lattice $X_*(T)$ and the dominance order; the weight functor F_{μ} then defines a G -equivariant local system on the set of all Torels; since G acts transitively on this set (in fact it is isomorphic to G/T for any choice of maximal torus T), this local system is trivial.

We now consider the functor

$$F : P_{G_{\mathcal{O}}}(\mathrm{Gr}_G, \mathbf{k}) \rightarrow \mathrm{Vect}_{\mathbf{k}}$$

defined by

$$F(\mathcal{A}) = H^{\bullet}(\mathrm{Gr}_G, \mathcal{A}).$$

Theorem 5.6. 1. *There exists a canonical isomorphism of functors*

$$F \cong \bigoplus_{\mu \in X_*(T)} F_{\mu} : P_{G_{\mathcal{O}}}(\mathrm{Gr}_G, \mathbf{k}) \rightarrow \mathrm{Vect}_{\mathbf{k}}.$$

2. *The functor F is exact and faithful.*

Proof. (1) Let $\mathcal{A} \in P_{G_{\mathcal{O}}}(\mathrm{Gr}_G, \mathbf{k})$. Our aim is to construct a canonical isomorphism

$$H^{\bullet}(\mathrm{Gr}_G, \mathcal{A}) \cong \bigoplus_{\mu \in X_*(T)} F_{\mu}(\mathcal{A}),$$

and more precisely to construct a canonical isomorphism

$$H^k(\mathrm{Gr}_G, \mathcal{A}) \cong \bigoplus_{\substack{\mu \in X_*(T) \\ \langle 2\rho, \mu \rangle = k}} F_{\mu}(\mathcal{A})$$

for each $k \in \mathbf{Z}$.

Without loss of generality, we may assume that \mathcal{A} is indecomposable, and in particular that the support of \mathcal{A} is connected.

For $n \in \frac{1}{2}\mathbf{Z}$, set

$$Z_n = \bigsqcup_{\substack{\mu \in X_*(T) \\ \langle \rho, \mu \rangle = n}} T_{\mu}.$$

Then both

$$\bigcup_{n \in \mathbf{Z}} Z_n \quad \text{and} \quad \bigcup_{n \in \frac{1}{2} + \mathbf{Z}} Z_n$$

are union of connected components of Gr_G . As $\mathrm{supp} \mathcal{A}$ was assumed to be connected, it is contained in one of these subsets. Let us say that it is contained in the first one, the reasoning in the other case being entirely similar.

We endow Z_n with the topology induced from that of Gr_G . Then Z_n is the topological disjoint union of the T_μ contained in it, and it follows that

$$H_{Z_n}^k(\mathrm{Gr}_G, \mathcal{A}) = \begin{cases} 0 & \text{if } k \neq 2n; \\ \bigoplus_{\langle \rho, \mu \rangle = n} F_\mu(\mathcal{A}) & \text{if } k = 2n. \end{cases} \quad (5.1)$$

By (3.7), the closure of Z_n is

$$\overline{Z_n} = Z_n \sqcup Z_{n+1} \sqcup Z_{n+2} \sqcup \cdots = Z_n \sqcup \overline{Z_{n+1}},$$

so there is a diagram of complementary open and closed inclusions

$$\overline{Z_{n+1}} \xrightarrow{i} \overline{Z_n} \xleftarrow{j} Z_n.$$

Applying then the cohomological functor $H^\bullet(\mathrm{Gr}_G, ?)$ to the distinguished triangle

$$i_* i^! \mathcal{A} \rightarrow \mathcal{A} \rightarrow j_* j^! \mathcal{A} \xrightarrow{[1]}$$

in $D^b(\mathrm{Gr}_G, \mathbf{k})$, we obtain the long exact sequence

$$\cdots \rightarrow H_{Z_{n+1}}^k(\mathrm{Gr}_G, \mathcal{A}) \rightarrow H_{Z_n}^k(\mathrm{Gr}_G, \mathcal{A}) \rightarrow H_{Z_n}^k(\mathrm{Gr}_G, \mathcal{A}) \rightarrow H_{Z_{n+1}}^{k+1}(\mathrm{Gr}_G, \mathcal{A}) \rightarrow \cdots$$

For n large enough, $\mathrm{supp} \mathcal{A}$ is disjoint from $\overline{Z_n}$, because $\mathrm{supp} \mathcal{A}$ is compact and $\overline{Z_n}$ is far away from the origin of Gr_G .⁷ Consequently $H_{Z_n}^\bullet(\mathrm{Gr}_G, \mathcal{A}) = 0$ for n large enough. Using the long exact sequence above and (5.1), a decreasing induction on n leads to

$$\begin{aligned} H_{Z_n}^k(\mathrm{Gr}_G, \mathcal{A}) &= 0 && \text{if } k \text{ is odd or if } k < 2n, \\ H_{Z_{k/2}}^k(\mathrm{Gr}_G, \mathcal{A}) &\xrightarrow{\sim} H_{Z_n}^k(\mathrm{Gr}_G, \mathcal{A}) && \text{if } k \text{ is even and } \geq 2n. \\ &\downarrow \wr && \\ H_{Z_{k/2}}^k(\mathrm{Gr}_G, \mathcal{A}) &&& \end{aligned}$$

One concludes by taking n small enough so that $\mathrm{supp} \mathcal{A} \subset \overline{Z_n}$.

(2) The exactness is automatic since the category $\mathrm{P}_{G_{\mathcal{O}}}(\mathrm{Gr}_G, \mathbf{k})$ is semisimple (see the comments before the theorem). Given the exactness, the faithfulness means that $H^\bullet(\mathrm{Gr}_G, -)$ does not kill any object in $\mathrm{P}_{G_{\mathcal{O}}}(\mathrm{Gr}_G, \mathbf{k})$. So let us take a perverse sheaf \mathcal{A} in our category. Then $\mathrm{supp} \mathcal{A}$ is a finite union of Schubert cells Gr_G^λ . Let us choose λ maximal for this property. Then $\mathcal{A}|_{\mathrm{Gr}_G^\lambda} \cong \mathbf{k}[\dim \mathrm{Gr}_G^\lambda]$ and as in the proof of Theorem 5.1(2) we have $\mathrm{Gr}^\lambda \cap T_\lambda = \{L_\lambda\}$, and therefore $F_\lambda(\mathcal{A}) \neq 0$. \square

⁷You may here have in mind the Serre tree considered in §3.2: $\overline{Z_n}$ is a union of horospheres centered at $+\infty$ and going through $L_{n\alpha^\vee}$; for n large enough, this is located far away on the right.

Remark 5.7. The proof of Theorem 5.6 has broken the symmetry between the two sides of hyperbolic localization, so let us try to restore it. Given $\mu \in X_*(T)$, let us define the inclusion maps

$$T_\mu \begin{array}{c} \xrightarrow{t'_\mu} \overline{T}_\mu \xrightarrow{t''_\mu} \\ \searrow \quad \quad \quad \nearrow \\ \text{Gr}_G \end{array} \quad \text{and} \quad S_\mu \begin{array}{c} \xrightarrow{s'_\mu} \overline{S}_\mu \xrightarrow{s''_\mu} \\ \searrow \quad \quad \quad \nearrow \\ \text{Gr}_G \end{array}.$$

Then for each $\mathcal{A} \in \text{P}_{G\mathcal{O}}(\text{Gr}_G, \mathbf{k})$, we have

$$\begin{aligned} H_{\overline{T}_\mu}^k(\text{Gr}_G, \mathcal{A}) &= H^k(\text{Gr}_G, (t''_\mu)!(t'_\mu)^!\mathcal{A}), \\ H^k(\overline{S}_\mu, \mathcal{A}) &= H^k(\text{Gr}_G, (s''_\mu)_*(s'_\mu)^*\mathcal{A}), \\ H_{T_\mu}^k(\text{Gr}_G, \mathcal{A}) &= H^k(\text{Gr}_G, \underbrace{(t''_\mu)!(t'_\mu)_*(t'_\mu)^*(t''_\mu)^!}_{\cong t_{\mu*}t'_\mu} \mathcal{A}), \\ H_c^k(S_\mu, \mathcal{A}) &= H^k(\text{Gr}_G, \underbrace{(s''_\mu)_*(s'_\mu)!(s'_\mu)^!(s''_\mu)^*}_{\cong s_{\mu!}s_\mu^*} \mathcal{A}). \end{aligned}$$

One can check that the adjunction maps and hyperbolic localization give rise to a commutative diagram

$$\begin{array}{ccccc} & & H^k(\text{Gr}_G, \mathcal{A}) & & \\ & \nearrow^{(t''_\mu)!(t'_\mu)^! \rightarrow \text{id}} & & \searrow^{\text{id} \rightarrow (s''_\mu)_*(s'_\mu)^*} & \\ H_{\overline{T}_\mu}^k(\text{Gr}_G, \mathcal{A}) & & & & H^k(\overline{S}_\mu, \mathcal{A}). \\ & \searrow_{\text{id} \rightarrow (t'_\mu)_*(t'_\mu)^*} & & \nearrow_{(s'_\mu)!(s'_\mu)^! \rightarrow \text{id}} & \\ & & H_{T_\mu}^k(\text{Gr}_G, \mathcal{A}) & \xrightarrow{\text{hyperbolic loc.}} & H_c^k(S_\mu, \mathcal{A}) \end{array}$$

If $k = \langle 2\rho, \mu \rangle$, then the three bottom arrows are isomorphisms, so the four bottom spaces can be identified: they define the functor $\mathcal{A} \mapsto F_\mu(\mathcal{A})$. At this point, let us write

$$F_\mu(\mathcal{A}) \xrightarrow{i_\mu} H^k(\text{Gr}_G, \mathcal{A}) \xrightarrow{p_\mu} F_\mu(\mathcal{A})$$

for the two top arrows of the diagram above. Theorem 5.6 shows that for each $k \in \mathbf{Z}$,

$$H^k(\text{Gr}_G, \mathcal{A}) = \bigoplus_{\substack{\mu \in X_*(T) \\ \langle 2\rho, \mu \rangle = k}} \text{im}(i_\mu), \quad (5.2)$$

and likewise

$$H^k(\text{Gr}_G, \mathcal{A}) \xrightarrow{\cong} \bigoplus_{\substack{\nu \in X_*(T) \\ \langle 2\rho, \nu \rangle = k}} \text{coim}(p_\nu), \quad (5.3)$$

where $\text{coim}(p_\nu) = H^k(\text{Gr}_G, \mathcal{A}) / \ker(p_\nu)$ is the coimage of p_ν . Further, if $\mu \neq \nu$ and $\langle 2\rho, \mu \rangle = \langle 2\rho, \nu \rangle$, then $\mu \not\leq \nu$, so $\overline{S}_\nu \cap \overline{T}_\mu = \emptyset$ by Lemma 3.3, and therefore $p_\nu \circ i_\mu = 0$. This implies that the decompositions (5.2) and (5.3) of $H^k(\text{Gr}_G, \mathcal{A})$ coincide. The decomposition

$$H^k(\text{Gr}_G, \mathcal{A}) = \bigoplus_{\substack{\mu \in X_*(T) \\ \langle 2\rho, \mu \rangle = k}} F_\mu(\mathcal{A})$$

is therefore defined without ambiguity.

5.3 Weight spaces of simple objects

Proposition 5.8. *Let $\lambda, \mu \in X_*(T)$ with λ dominant. Then $\dim F_\mu(\mathbf{IC}_\lambda)$ is the number of irreducible components of $\mathrm{Gr}_G^\lambda \cap S_\mu$. In particular, it is nonzero if and only if $\mu \in \mathrm{Conv}(W\lambda) \cap (\lambda + Q^\vee)$.*

Proof. For each $\eta \in X_*(T)^+$, one of the following three possibilities hold:

- Gr_G^η does not meet $\mathrm{supp} \mathbf{IC}_\lambda$, and $\mathbf{IC}_\lambda|_{\mathrm{Gr}_G^\eta} = 0$;
- $\eta = \lambda$ and $\mathbf{IC}_\lambda|_{\mathrm{Gr}_G^\eta} \in D^{\leq -\langle 2\rho, \eta \rangle}(\mathrm{Gr}_G^\eta, \mathbf{k})$;
- $\eta < \lambda$ and $\mathbf{IC}_\lambda|_{\mathrm{Gr}_G^\eta} \in D^{\leq -2\rho(\eta) - 1}(\mathrm{Gr}_G^\eta, \mathbf{k})$

(see (4.2)). In the last case, we can in fact replace $-\langle 2\rho, \eta \rangle - 1$ by $-\langle 2\rho, \eta \rangle - 2$ because of Lemma 4.5 (and the fact that $\eta < \lambda \Rightarrow \langle 2\rho, \lambda \rangle \equiv \langle 2\rho, \eta \rangle \pmod{2}$).

When we gather these facts to reconstruct $H_c^{(2\rho, \mu)}(S_\mu, \mathbf{IC}_\lambda)$ using the same method as in the proof of Proposition 5.4, only the stratum Gr_G^λ contributes, and we obtain an isomorphism

$$H_c^{(2\rho, \mu)}(S_\mu, \mathbf{IC}_\lambda) \cong H_c^{(2\rho, \mu)}(\mathrm{Gr}_G^\lambda \cap S_\mu, \mathbf{IC}_\lambda|_{\mathrm{Gr}_G^\lambda}).$$

Therefore

$$F_\mu(\mathbf{IC}_\lambda) \cong H_c^{(2\rho, \mu)}(\mathrm{Gr}_G^\lambda \cap S_\mu, \mathbf{IC}_\lambda|_{\mathrm{Gr}_G^\lambda}) = H_c^{(2\rho, \lambda + \mu)}(\mathrm{Gr}_G^\lambda \cap S_\mu; \mathbf{k}).$$

The right-hand side is the top cohomology group with compact support of $\mathrm{Gr}_G^\lambda \cap S_\mu$; it therefore has a natural basis indexed by the irreducible components of top dimension of this intersection.⁸

The last claim then follows from Theorem 5.1. □

Remark 5.9. Proposition 5.8 is only valid under the assumption $\mathrm{char}(\mathbf{k}) = 0$, because we use Lemma 4.5. Without this assumption, one has to replace the intersection cohomology sheaf \mathbf{IC}_λ by the sheaf denoted by $\mathcal{I}_1(\lambda, \mathbf{k})$ in [MV]. Indeed this sheaf always has the property that $\mathcal{I}_1(\lambda, \mathbf{k})|_{\mathrm{Gr}_G^\eta} \in D^{\leq -\langle 2\rho, \eta \rangle - 2}(\mathrm{Gr}_G^\eta, \mathbf{k})$ for each $\eta < \lambda$.

6 Convolution product: “classical” point of view

Our goal in Sections 6–7 is to endow the category $P_{G_\mathcal{O}}(\mathrm{Gr}_G, \mathbf{k})$ of $G_\mathcal{O}$ -equivariant perverse sheaves on Gr_G with the structure of a symmetric monoidal category. We first define the convolution product of two equivariant perverse sheaves, and with the help of the notion of

⁸This property is a classical fact about the top cohomology with compact supports of algebraic varieties, which follows e.g. from the considerations in [Iv, §X.1].

stratified semismall map, we show that the result of the operation is a perverse sheaf. We also define an associativity constraint. To proceed further, we will need a different point of view on convolution, which uses an important auxiliary construction, known as the Beilinson–Drinfeld Grassmannian. This is considered in Section 7.

6.1 Stratified semismall maps

We first consider a general result, which guarantees that the direct image of a perverse sheaf under a *stratified semismall morphism* is a perverse sheaf.

Let (X, \mathcal{T}) and (Y, \mathcal{U}) be two stratified algebraic varieties (with finite Whitney stratifications), and let $f : Y \rightarrow X$ be a proper map such that for each $U \in \mathcal{U}$, the set $f(U)$ is a union of strata. We say that f is *stratified semismall* if for any stratum $T \subset f(U)$ and any $x \in T$, we have

$$\dim(f^{-1}(x) \cap U) \leq \frac{1}{2}(\dim U - \dim T).$$

We say that f is *locally trivial* if for any $(T, U) \in \mathcal{T} \times \mathcal{U}$ such that $T \subset f(U)$, the map $U \cap f^{-1}(T) \rightarrow T$ induced by f is a Zariski locally trivial fibration.

Proposition 6.1. *If f is stratified semismall and locally trivial and if \mathcal{F} is a perverse sheaf on Y constructible with respect to \mathcal{U} , then $f_*\mathcal{F}$ is a perverse sheaf on X constructible with respect to \mathcal{T} .*

Proof. For any stratum $T \in \mathcal{T}$, we can consider the restriction

$$\begin{array}{ccc} f^{-1}(T) & \xrightarrow{f_T} & T \\ \parallel & & \\ \bigsqcup_{U \in \mathcal{U}} f^{-1}(T) \cap U & & \end{array}$$

We denote by $f_{T,U} : f^{-1}(T) \cap U \rightarrow T$ the restriction of f (which is a Zariski locally trivial fibration by assumption if $T \subset f(U)$). Note here that since $f(U)$ is a union of strata in \mathcal{T} , the assertions that $T \subset f(U)$ and that $f^{-1}(T) \cap U \neq \emptyset$ are equivalent.

First, let us prove that for any \mathcal{F} in the \mathcal{U} -constructible derived category $D_{\mathcal{U}}^b(Y, \mathbf{k})$, the complex $f_*\mathcal{F} = f_!\mathcal{F}$ belongs to the \mathcal{T} -constructible derived category $D_{\mathcal{T}}^b(X, \mathbf{k})$. We proceed by induction on the smallest number of strata whose union is a closed subvariety Z of Y such that $\mathcal{F}|_{Y \setminus Z} = 0$. So, let us consider such a closed union of strata, and choose some $U \in \mathcal{U}$ which is open in Z . We can consider \mathcal{F} as a complex in $D_{\mathcal{U}}^b(Z, \mathbf{k})$. Then, if we denote by $j : U \rightarrow Z$ and $i : Z \setminus U \rightarrow Z$ the embeddings, we have a standard distinguished triangle

$$j_!j^*\mathcal{F} \rightarrow \mathcal{F} \rightarrow i_*i^*\mathcal{F} \xrightarrow{[1]}.$$

Applying $f_!$, we deduce a distinguished triangle

$$(f \circ j)_!j^*\mathcal{F} \rightarrow \mathcal{F} \rightarrow (f \circ i)_!i^*\mathcal{F} \xrightarrow{[1]}.$$

By induction, the third term in this triangle belongs to $D_{\mathcal{T}}^b(X, \mathbf{k})$. Since $D_{\mathcal{T}}^b(X, \mathbf{k})$ is a triangulated subcategory of the derived category of \mathbf{k} -sheaves on X , we are reduced to prove

that $(f \circ j)_! j^* \mathcal{F}$ belongs to $D_{\mathcal{T}}^b(X, \mathbf{k})$. Using truncation triangles, for this it suffices to prove that for each $n \in \mathbf{Z}$, $(f \circ j)_! \mathcal{H}^n(j^* \mathcal{F})$ belongs to $D_{\mathcal{T}}^b(X, \mathbf{k})$. So, let $T \in \mathcal{T}$ such that $T \subset f(U)$, and let $g: T \rightarrow X$ be the embedding. By the base change theorem, we have

$$g^*(f \circ j)_! \mathcal{H}^n(j^* \mathcal{F}) \cong (f_{T,U})_! \mathcal{H}^n(\mathcal{F})|_{f^{-1}(T) \cap U}. \quad (6.1)$$

Now since \mathcal{F} is \mathcal{U} -constructible, $\mathcal{H}^n(\mathcal{F})|_{f^{-1}(T) \cap U}$ is a local system; since $f_{T,U}$ is a locally trivial fibration we deduce that the cohomology sheaves of $g^*(f \circ j)_! \mathcal{H}^n(j^* \mathcal{F})$ are local systems on T , and finally that \mathcal{F} belongs to $D_{\mathcal{T}}^b(X, \mathbf{k})$.

Next, we prove that if \mathcal{F} is in nonpositive perverse degrees, then $f_! \mathcal{F}$ is in nonpositive perverse degrees. Let as above $T \in \mathcal{T}$ be a stratum, and $g: T \rightarrow X$ be the embedding. Then we need to prove that

$$g^* f_! \mathcal{F} \cong (f_T)_! \mathcal{F}|_{f^{-1}(T)}$$

is concentrated in degrees $\leq -\dim T$. By the same arguments as above, it suffices to prove that for any $U \in \mathcal{U}$ such that $U \cap f^{-1}(T) \neq \emptyset$, the complex $(f_{T,U})_! \mathcal{F}|_{U \cap f^{-1}(T)}$ satisfies this property. This follows from a classical vanishing result for cohomology with compact supports, see [Iv, Proposition X.1.4].

Since the functor $f_! = f_*$ commutes with Verdier duality, we now deduce that this functor also sends complexes concentrated in nonnegative perverse degrees to complexes with the same property. And finally that this functor sends perverse sheaves to perverse sheaves. \square

6.2 Definition of convolution on Gr_G

To define the convolution operation on $\mathrm{P}_{G_{\mathcal{O}}}(\mathrm{Gr}_G, \mathbf{k})$, we will identify this category with the heart of the perverse t-structure on the *equivariant derived category* $D_{G_{\mathcal{O}}}^b(\mathrm{Gr}_G, \mathbf{k})$ in the sense of Bernstein–Lunts [BL], see §A.1. (See also §A.3 for the definition of $D_{G_{\mathcal{O}}}^b(\mathrm{Gr}_G, \mathbf{k})$.)

We denote by $[h] \in \mathrm{Gr}_G$ the coset $hG_{\mathcal{O}}$ of an element $h \in G_{\mathcal{K}}$. Likewise, letting the group $G_{\mathcal{O}}$ act on $G_{\mathcal{K}} \times \mathrm{Gr}_G$ by $k \cdot (g, [h]) = (gk^{-1}, [kh])$, we denote by $[g, h]$ the orbit of $(g, [h])$. We form the diagram

$$\mathrm{Gr}_G \times \mathrm{Gr}_G \xleftarrow{p} G_{\mathcal{K}} \times \mathrm{Gr}_G \xrightarrow{q} G_{\mathcal{K}} \times^{G_{\mathcal{O}}} \mathrm{Gr}_G \xrightarrow{m} \mathrm{Gr}_G, \quad (6.2)$$

where p is the map $(g, [h]) \mapsto ([g], [h])$, q is the map $(g, [h]) \mapsto [g, h]$, and m is the map $[g, h] \mapsto [gh]$.

Let \mathcal{F} and \mathcal{G} be two complexes of sheaves in the equivariant category $D_{G_{\mathcal{O}}}^b(\mathrm{Gr}_G, \mathbf{k})$. Since the $G_{\mathcal{O}}$ action on $G_{\mathcal{K}} \times \mathrm{Gr}_G$ considered above is free, the functor q^* induces an equivalence of categories

$$D_{G_{\mathcal{O}}}^b(G_{\mathcal{K}} \times^{G_{\mathcal{O}}} \mathrm{Gr}_G) \xrightarrow{\sim} D_{G_{\mathcal{O}} \times G_{\mathcal{O}}}^b(G_{\mathcal{K}} \times \mathrm{Gr}_G),$$

see [BL, Theorem 2.6.2]. (Here, $G_{\mathcal{O}}$ acts on $G_{\mathcal{K}} \times^{G_{\mathcal{O}}} \mathrm{Gr}_G$ via multiplication on the left on $G_{\mathcal{K}}$; for the action of $G_{\mathcal{O}} \times G_{\mathcal{O}}$ on $G_{\mathcal{K}} \times \mathrm{Gr}_G$, the first copy of $G_{\mathcal{O}}$ acts via left multiplication on $G_{\mathcal{K}}$ and the second copy acts as above.) The complex $p^*(\mathcal{F} \boxtimes \mathcal{G})$ defines an object of $D_{G_{\mathcal{O}} \times G_{\mathcal{O}}}^b(G_{\mathcal{K}} \times \mathrm{Gr}_G)$. Therefore, we can consider the unique object $\mathcal{F} \tilde{\boxtimes} \mathcal{G} \in D_{G_{\mathcal{O}}}^b(G_{\mathcal{K}} \times^{G_{\mathcal{O}}} \mathrm{Gr}_G, \mathbf{k})$ such that

$$q^*(\mathcal{F} \tilde{\boxtimes} \mathcal{G}) = p^*(\mathcal{F} \boxtimes \mathcal{G}).$$

We then set

$$\mathcal{F} \star \mathcal{G} := m_*(\mathcal{F} \widetilde{\boxtimes} \mathcal{G}) \in D_{G^\circ}^b(\mathrm{Gr}_G, \mathbf{k}).$$

Remark 6.2. When stating this construction in these terms we cheat a little bit; see §A.3.

6.3 Exactness of convolution

The first important property of the convolution product \star on $D_{G^\circ}^b(\mathrm{Gr}_G, \mathbf{k})$ is the following.

Proposition 6.3. *Assume that \mathcal{F} and \mathcal{G} belong to $\mathrm{P}_{G^\circ}(\mathrm{Gr}_G, \mathbf{k})$. Then $\mathcal{F} \star \mathcal{G}$ also belongs to $\mathrm{P}_{G^\circ}(\mathrm{Gr}_G, \mathbf{k})$.*

To prove this result we will need an auxiliary lemma. Here, for $\lambda, \mu \in X_*(T)^+$ we set

$$\widetilde{\mathrm{Gr}}^{\lambda, \mu} := q(p^{-1}(\mathrm{Gr}_G^\lambda \times \mathrm{Gr}_G^\mu)).$$

Lemma 6.4. *For any $\lambda, \mu \in X_*(T)^+$ and $\nu \in -X_*(T)^+$, we have*

$$\dim(\widetilde{\mathrm{Gr}}^{\lambda, \mu} \cap m^{-1}(L_\nu)) \leq \langle \rho, \lambda + \mu + \nu \rangle.$$

Proof. We consider the T -action on $G_{\mathcal{K}} \times^{G^\circ} \mathrm{Gr}_G$ induced by left multiplication on $G_{\mathcal{K}}$, and the diagonal T -action on $\mathrm{Gr}_G \times \mathrm{Gr}_G$. Then the map

$$\phi : G_{\mathcal{K}} \times^{G^\circ} \mathrm{Gr}_G \rightarrow \mathrm{Gr}_G \times \mathrm{Gr}_G$$

that sends $[g, h]$ to $([g], [gh])$ is a T -equivariant isomorphism. We deduce that the T -fixed points in $G_{\mathcal{K}} \times^{G^\circ} \mathrm{Gr}_G$ are of the form $[t^\alpha, t^\beta]$, with $\alpha, \beta \in X_*(T)$; indeed $\phi([t^\alpha, t^\beta]) = (L_\alpha, L_{\alpha+\beta})$. Further, $[t^\alpha, t^\beta]$ belongs to

$$\overline{\widetilde{\mathrm{Gr}}^{\lambda, \mu}} = q(p^{-1}(\overline{\mathrm{Gr}_G^\lambda} \times \overline{\mathrm{Gr}_G^\mu}))$$

if and only if the dominant W -conjugate α^+ of $\alpha \in X_*(T)$ is $\leq \lambda$ and the dominant W -conjugate β^+ of β is $\leq \mu$ with respect to the dominance order.

The morphism ϕ maps $m^{-1}(L_\nu)$ to $\mathrm{Gr}_G \times \{L_\nu\}$. This allows (by projecting onto the first factor) to regard $\overline{\widetilde{\mathrm{Gr}}^{\lambda, \mu}} \cap m^{-1}(L_\nu)$ as a closed subvariety of $\overline{\mathrm{Gr}_G^\lambda}$. Now by Corollary 5.2 we have

$$\dim(\overline{\widetilde{\mathrm{Gr}}^{\lambda, \mu}} \cap m^{-1}(L_\nu)) \leq \max\left\{ \langle \rho, \lambda + \alpha \rangle \mid [t^\alpha, t^\beta] \in \overline{\widetilde{\mathrm{Gr}}^{\lambda, \mu}} \cap m^{-1}(L_\nu) \right\}.$$

The pairs (α, β) occurring here satisfy $\alpha + \beta = \nu$ and

$$\langle \rho, \mu + \beta \rangle = \langle \rho, \mu - w_0(\beta) \rangle \geq 0$$

since $w_0(\beta) \leq \beta^+ \leq \mu$; so that

$$\langle \rho, \lambda + \alpha \rangle \leq \langle \rho, \lambda + \alpha \rangle + \langle \rho, \mu + \beta \rangle = \langle \rho, \lambda + \mu + \nu \rangle,$$

which entails the desired result. \square

We can now give the proof of Proposition 6.3.

Proof. We consider the situation

$$\begin{array}{ccc} G_{\mathcal{K}} \times^{G_{\mathcal{O}}} \mathrm{Gr}_G & \xrightarrow{m} & \mathrm{Gr}_G \\ \parallel & & \parallel \\ \bigsqcup_{\lambda, \mu \in X_*(T)^+} \widetilde{\mathrm{Gr}}^{\lambda, \mu} & & \bigsqcup_{\nu \in X_*(T)^+} \mathrm{Gr}_G^{\nu} \end{array}$$

Here certainly m is ind-proper. It is locally trivial, because the whole situation is $G_{\mathcal{O}}$ -equivariant. Also, it follows from the definitions that the complex $\mathcal{F} \widetilde{\boxtimes} \mathcal{G} \in D_{G_{\mathcal{O}}}^b(G_{\mathcal{K}} \times^{G_{\mathcal{O}}} \mathrm{Gr}_G, \mathbf{k})$ defined in §6.2 is perverse and is constructible with respect to the stratification given by the subsets $\widetilde{\mathrm{Gr}}^{\lambda, \mu}$. To show that $\mathcal{F} \star \mathcal{G}$ is perverse, it thus suffices to prove that m is stratified semismall. This is exactly the content of Lemma 6.4 (since $\dim(\mathrm{Gr}_G^{w_0(\nu)}) = \langle 2\rho, w_0(\nu) \rangle = -\langle 2\rho, \nu \rangle$ if $\nu \in -X_*(T)$). \square

Remark 6.5. A different proof of Proposition 6.3 is due to Gaitsgory. In fact, the convolution $\mathcal{F} \star \mathcal{G}$ makes sense for any \mathcal{F} in $D_c^b(\mathrm{Gr}_G, \mathbf{k})$ and \mathcal{G} in $D_{G_{\mathcal{O}}}^b(\mathrm{Gr}_G, \mathbf{k})$. It follows from [Ga, Proposition 6] that, in this generality, $\mathcal{F} \star \mathcal{G}$ is perverse as soon as \mathcal{F} and \mathcal{G} are perverse. This approach uses an interpretation of convolution in terms of nearby cycles.

6.4 Associativity of convolution

For $\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3$ in $\mathrm{P}_{G_{\mathcal{O}}}(\mathrm{Gr}_G, \mathbf{k})$, one can define

$$\mathrm{Conv}_3(\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3) = (m_3)_* (\mathcal{F}_1 \widetilde{\boxtimes} \mathcal{F}_2 \widetilde{\boxtimes} \mathcal{F}_3),$$

where $m_3 : G_{\mathcal{K}} \times^{G_{\mathcal{O}}} G_{\mathcal{K}} \times^{G_{\mathcal{O}}} \mathrm{Gr}_G \rightarrow \mathrm{Gr}_G$ is the map $[g_1, g_2, g_3] \mapsto [g_1 g_2 g_3]$, with an obvious notation, and the twisted product $\mathcal{F}_1 \widetilde{\boxtimes} \mathcal{F}_2 \widetilde{\boxtimes} \mathcal{F}_3$ is defined in the obvious way. Then base change yields natural isomorphisms

$$(\mathcal{F}_1 \star \mathcal{F}_2) \star \mathcal{F}_3 \xleftarrow{\sim} \mathrm{Conv}_3(\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3) \xrightarrow{\sim} \mathcal{F}_1 \star (\mathcal{F}_2 \star \mathcal{F}_3).$$

The composition of these isomorphisms provides an associativity constraint that turns the pair $(\mathrm{P}_{G_{\mathcal{O}}}(\mathrm{Gr}_G, \mathbf{k}), \star)$ into a monoidal category.

7 Convolution and fusion

In this section we describe a different construction of the convolution product on $\mathrm{P}_{G_{\mathcal{O}}}(\mathrm{Gr}_G, \mathbf{k})$. This construction uses the Beilinson–Drinfeld Grassmannian, hence ultimately the moduli interpretation of Gr_G . It plays a crucial role in the definition of the commutativity constraint for \star .

7.1 A moduli interpretation of the affine Grassmannian

In this section, we adopt the following setup. We consider a smooth curve X over \mathbb{C} , and for any point $x \in X$, we denote by \mathcal{O}_x the completion of the local ring of X at x and by \mathcal{K}_x the fraction field of \mathcal{O}_x ; the choice of a local coordinate t on X around x leads to isomorphisms $\mathcal{O}_x \cong \mathbb{C}[[t]]$ and $\mathcal{K}_x \cong \mathbb{C}((t))$. Using these data we can define a “local” version of Gr_G at x by $\mathrm{Gr}_{G,x} := G_{\mathcal{K}_x}/G_{\mathcal{O}_x}$.

We define

$$\mathcal{D}_x = \mathrm{Spec}(\mathcal{O}_x) \quad \text{and} \quad \mathcal{D}_x^\times = \mathrm{Spec}(\mathcal{K}_x).$$

For a \mathbb{C} -algebra R , we consider the completed tensor products $R \widehat{\otimes} \mathcal{O}_x$ and $R \widehat{\otimes} \mathcal{K}_x$, so that

$$R \widehat{\otimes} \mathcal{O}_x \cong R[[t]] \quad \text{and} \quad R \widehat{\otimes} \mathcal{K}_x \cong R((t)).$$

We set

$$\mathcal{D}_{x,R} = \mathrm{Spec}(R \widehat{\otimes} \mathcal{O}_x) \quad \text{and} \quad \mathcal{D}_{x,R}^\times = \mathrm{Spec}(R \widehat{\otimes} \mathcal{K}_x).$$

For a \mathbb{C} -algebra R , we set

$$X_R = X \times_{\mathrm{Spec}(\mathbb{C})} \mathrm{Spec}(R) \quad \text{and} \quad X_R^\times = (X \setminus \{x\}) \times_{\mathrm{Spec}(\mathbb{C})} \mathrm{Spec}(R).$$

The following proposition gives a first description of $\mathrm{Gr}_{G,x}$ in terms of moduli of bundles on X .

Proposition 7.1. *1. The ind-scheme $G_{\mathcal{K}_x}$ represents the functor*

$$R \mapsto \left\{ (\mathcal{F}, \nu, \mu) \left| \begin{array}{l} \mathcal{F} \text{ } G\text{-bundle on } X_R \\ \nu : G \times X_R^\times \xrightarrow{\sim} \mathcal{F}|_{X_R^\times} \text{ trivialization on } X_R^\times \\ \mu : G \times \mathcal{D}_{x,R} \xrightarrow{\sim} \mathcal{F}|_{\mathcal{D}_{x,R}} \text{ trivialization on } \mathcal{D}_{x,R} \end{array} \right. \right\} / \text{isomorphism}.$$

2. The ind-scheme $\mathrm{Gr}_{G,x}$ represents the functor

$$R \mapsto \left\{ (\mathcal{F}, \nu) \left| \begin{array}{l} \mathcal{F} \text{ } G\text{-bundle on } X_R \\ \nu : G \times X_R^\times \xrightarrow{\sim} \mathcal{F}|_{X_R^\times} \text{ trivialization on } X_R^\times \end{array} \right. \right\} / \text{isomorphism}.$$

Here, a G -bundle on a scheme Z is a scheme $\mathcal{F} \rightarrow Z$ equipped with a right G -action and which, locally in the flat topology, is isomorphic to the product $G \times Z$ as a G -scheme. The proof of this proposition is given in [LS, Propositions 3.8 and 3.10]. The main ingredients are:

1. The Beauville–Laszlo theorem [BL], which says that the datum of a G -bundle on X_R is equivalent to the datum of a G -bundle on X_R^\times , of a G -bundle on $\mathcal{D}_{x,R}$, and of a gluing datum on $\mathcal{D}_{x,R}^\times$.
2. The fact that any G -bundle on $\mathcal{D}_{x,R}$ which is trivial on $\mathcal{D}_{x,R}^\times$ becomes trivial when pulled back to $\mathcal{D}_{x,R'}$ for some faithfully flat extension $R \rightarrow R'$.

The Beauville–Laszlo theorem also shows that restriction induces an isomorphism

$$\left\{ (\mathcal{F}, \nu) \left| \begin{array}{l} \mathcal{F} \text{ } G\text{-bundle on } X_R \\ \nu \text{ trivialization on } X_R^\times \end{array} \right. \right\} / \text{isom.} \xrightarrow{\sim} \left\{ (\mathcal{F}, \nu) \left| \begin{array}{l} \mathcal{F} \text{ } G\text{-bundle on } \mathcal{D}_{x,R} \\ \nu \text{ trivialization on } \mathcal{D}_{x,R}^\times \end{array} \right. \right\} / \text{isom.}$$

In particular, we deduce that $\text{Gr}_{G,x}$ also represents the functor

$$R \mapsto \left\{ (\mathcal{F}, \nu) \left| \begin{array}{l} \mathcal{F} \text{ } G\text{-bundle on } \mathcal{D}_{x,R} \\ \nu : G \times \mathcal{D}_{x,R}^\times \xrightarrow{\sim} \mathcal{F}|_{\mathcal{D}_{x,R}^\times} \text{ trivialization on } \mathcal{D}_{x,R}^\times \end{array} \right. \right\} / \text{isomorphism.}$$

7.2 Moduli interpretation of the convolution diagram

We now give a similar geometric interpretation of the diagram

$$\text{Gr}_{G,x} \times \text{Gr}_{G,x} \xleftarrow{p} G_{\mathcal{K}_x} \times \text{Gr}_{G,x} \xrightarrow{q} G_{\mathcal{K}_x} \times^{G_{\mathcal{O}_x}} \text{Gr}_{G,x} \xrightarrow{m} \text{Gr}_{G,x}, \quad (7.1)$$

which is the “local version at x ” of the diagram (6.2). We first remark that $G_{\mathcal{K}_x} \times^{G_{\mathcal{O}_x}} \text{Gr}_{G,x}$ represents the functor

$$R \mapsto \left\{ (\mathcal{F}_1, \mathcal{F}, \nu_1, \eta) \left| \begin{array}{l} \mathcal{F}_1, \mathcal{F} \text{ } G\text{-bundles on } X_R \\ \nu_1 \text{ trivialization of } \mathcal{F}_1 \text{ on } X_R^\times \\ \eta : \mathcal{F}_1|_{X_R^\times} \xrightarrow{\sim} \mathcal{F}|_{X_R^\times} \text{ isomorphism} \end{array} \right. \right\} / \text{isom.}$$

To check this, one observes that the datum of $(\mathcal{F}_1, \mathcal{F}, \nu_1, \eta)$ is equivalent to the datum of $((\mathcal{F}_1, \nu_1), (\mathcal{F}, \eta \circ \nu_1))$, and one notes that this transformation is completely similar to the isomorphism $G_{\mathcal{K}} \times^{G_{\mathcal{O}}} \text{Gr}_G \xrightarrow{\sim} \text{Gr}_G \times \text{Gr}_G$ used in the proof of Lemma 6.4.

Likewise, $G_{\mathcal{K}_x} \times \text{Gr}_x$ represents the functor

$$R \mapsto \left\{ (\mathcal{F}_1, \mathcal{F}_2, \nu_1, \nu_2, \mu_1) \left| \begin{array}{l} \mathcal{F}_1, \mathcal{F}_2 \text{ } G\text{-bundles on } X_R \\ \nu_1, \nu_2 \text{ trivializations of } \mathcal{F}_1, \mathcal{F}_2 \text{ on } X_R^\times \\ \mu_1 \text{ trivialization of } \mathcal{F}_1 \text{ on } \mathcal{D}_{x,R} \end{array} \right. \right\} / \text{isom.}$$

With these identifications, the maps m and p in the diagram (7.1) are given by

$$\begin{aligned} m(\mathcal{F}_1, \mathcal{F}, \nu_1, \eta) &= (\mathcal{F}, \eta \circ \nu_1), \\ p(\mathcal{F}_1, \mathcal{F}_2, \nu_1, \nu_2, \mu_1) &= ((\mathcal{F}_1, \nu_1), (\mathcal{F}_2, \nu_2)), \end{aligned}$$

and the map q associates to $(\mathcal{F}_1, \mathcal{F}_2, \nu_1, \nu_2, \mu_1)$ the quadruple $(\mathcal{F}_1, \mathcal{F}, \nu_1, \eta)$, where \mathcal{F} is obtained by gluing $\mathcal{F}_1|_{X_R^\times}$ and $\mathcal{F}_2|_{\mathcal{D}_{x,R}}$ along the isomorphism

$$\mathcal{F}_1|_{\mathcal{D}_{x,R}^\times} \xleftarrow[\mu_1]{\sim} G \times \mathcal{D}_{x,R}^\times \xrightarrow[\nu_2]{\sim} \mathcal{F}_2|_{\mathcal{D}_{x,R}^\times}$$

and η is the natural isomorphism obtained in the process. (This gluing datum indeed defines a G -bundle on X_R thanks to the Beauville–Laszlo theorem, see §7.1.)

7.3 The Beilinson–Drinfeld Grassmannian

The idea behind the fusion procedure is to regard the geometric situation described in §§7.1–7.2 as the degeneration of a simpler situation. This involves the Beilinson–Drinfeld Grassmannian.

Specifically, we define $\mathrm{Gr}_{G,X}$ as the ind-scheme that represents the functor

$$R \mapsto \left\{ (\mathcal{F}, \nu, x) \left| \begin{array}{l} x \in X(R) \\ \mathcal{F} \text{ } G\text{-bundle on } X_R \\ \nu \text{ trivialization of } \mathcal{F} \text{ on } X_R \setminus x \end{array} \right. \right\} / \text{isom.},$$

where the symbol $X_R \setminus x$ indicates the complement in X_R of the graph of $x : \mathrm{Spec}(R) \rightarrow X$ (a closed subscheme of $X_R = X \times \mathrm{Spec}(R)$).

In the same way, we define Gr_{G,X^2} as the ind-scheme that represents the functor

$$R \mapsto \left\{ (\mathcal{F}, \nu, x_1, x_2) \left| \begin{array}{l} (x_1, x_2) \in X^2(R) \\ \mathcal{F} \text{ } G\text{-bundle on } X_R \\ \nu \text{ trivialization of } \mathcal{F} \text{ on } X_R \setminus (x_1 \cup x_2) \end{array} \right. \right\} / \text{isom.}$$

By definition there is an obvious morphism $\mathrm{Gr}_{G,X^2} \rightarrow X^2$. Plainly, the restriction of Gr_{G,X^2} to the diagonal Δ_X of X^2 , namely $\mathrm{Gr}_{G,X^2} \times_{X^2} \Delta_X$, is isomorphic to $\mathrm{Gr}_{G,X}$. Away from the diagonal, we have an isomorphism

$$\mathrm{Gr}_{G,X^2} \Big|_{X^2 \setminus \Delta_X} \cong (\mathrm{Gr}_{G,X} \times \mathrm{Gr}_{G,X}) \Big|_{X^2 \setminus \Delta_X} \quad (7.2)$$

given by $(\mathcal{F}, \nu, x_1, x_2) \mapsto ((\mathcal{F}_1, \nu_1, x_1), (\mathcal{F}_2, \nu_2, x_2))$, with \mathcal{F}_i obtained by gluing the trivial G -bundle on $X_R \setminus x_i$ and the bundle $\mathcal{F}|_{\mathcal{D}_{x_i,R}}$ along the map ν . (Here and below, $\mathcal{D}_{x_i,R}$ means the formal neighborhood of the graph of x_i in X_R , considered as a scheme.) Under the converse isomorphism $((\mathcal{F}_1, \nu_1, x_1), (\mathcal{F}_2, \nu_2, x_2)) \mapsto (\mathcal{F}, \nu, x_1, x_2)$, the G -bundle \mathcal{F} is obtained by gluing $\mathcal{F}_1|_{X_R \setminus x_2}$ and $\mathcal{F}_2|_{X_R \setminus x_1}$ along the isomorphism

$$\mathcal{F}_1|_{X_R \setminus (x_1 \cup x_2)} \xleftarrow[\nu_1]{\sim} G \times (X_R \setminus (x_1 \cup x_2)) \xrightarrow[\nu_2]{\sim} \mathcal{F}_2|_{X_R \setminus (x_1 \cup x_2)}.$$

Remark 7.2. To justify the gluing procedures used to define the morphism $\mathrm{Gr}_{G,X^2} \Big|_{X^2 \setminus \Delta_X} \rightarrow (\mathrm{Gr}_{G,X} \times \mathrm{Gr}_{G,X}) \Big|_{X^2 \setminus \Delta_X}$, one cannot simply quote the Beauville–Laszlo theorem, since the points x_i might not be constant. The more general result that we need is discussed in [BD, Remark 2.3.7].

7.4 Global version of the convolution diagram

We can also define global analogues of $G_{\mathcal{K}_x} \times \mathrm{Gr}_{G,x}$ and $G_{\mathcal{K}_x} \times^{G \circ x} \mathrm{Gr}_{G,x}$. For that, we define $\widetilde{\mathrm{Gr}}_{G,X} \times \mathrm{Gr}_{G,X}$ as the ind-scheme that represents the functor

$$R \mapsto \left\{ (\mathcal{F}_1, \nu_1, \mu_1, \mathcal{F}_2, \nu_2, x_1, x_2) \left| \begin{array}{l} (x_1, x_2) \in X^2(R) \\ \mathcal{F}_1, \mathcal{F}_2 \text{ } G\text{-bundles on } X_R \\ \nu_i \text{ trivialization of } \mathcal{F}_i \text{ on } X_R \setminus x_i \\ \mu_1 \text{ trivialization of } \mathcal{F}_1 \text{ on } \mathcal{D}_{x_2,R} \end{array} \right. \right\} / \text{isom.}$$

and $\mathrm{Gr}_{G,X} \widetilde{\times} \mathrm{Gr}_{G,X}$ as the ind-scheme that represents the functor

$$R \mapsto \left\{ (\mathcal{F}_1, \mathcal{F}, \nu_1, \eta, x_1, x_2) \left| \begin{array}{l} (x_1, x_2) \in X^2(R) \\ \mathcal{F}_1, \mathcal{F} \text{ } G\text{-bundles on } X_R \\ \nu_1 \text{ trivialization of } \mathcal{F}_1 \text{ on } X_R \setminus x_1 \\ \eta : \mathcal{F}_1|_{X_R \setminus x_2} \xrightarrow{\sim} \mathcal{F}|_{X_R \setminus x_2} \text{ isomorphism} \end{array} \right. \right\} / \text{isom.}$$

We then get a diagram

$$\mathrm{Gr}_{G,X} \times \mathrm{Gr}_{G,X} \xleftarrow{p} \mathrm{Gr}_{G,X} \widetilde{\times} \mathrm{Gr}_{G,X} \xrightarrow{q} \mathrm{Gr}_{G,X} \widetilde{\times} \mathrm{Gr}_{G,X} \xrightarrow{m} \mathrm{Gr}_{G,X^2} \quad (7.3)$$

by setting

$$\begin{aligned} m(\mathcal{F}_1, \mathcal{F}, \nu_1, \eta, x_1, x_2) &= (\mathcal{F}, \eta \circ \nu_1, x_1, x_2), \\ p(\mathcal{F}_1, \nu_1, \mu_1, \mathcal{F}_2, \nu_2, x_1, x_2) &= ((\mathcal{F}_1, \nu_1, x_1), (\mathcal{F}_2, \nu_2, x_2)), \end{aligned}$$

and by defining q as the map $(\mathcal{F}_1, \nu_1, \mu_1, \mathcal{F}_2, \nu_2, x_1, x_2) \mapsto (\mathcal{F}_1, \mathcal{F}, \nu_1, \eta, x_1, x_2)$, where \mathcal{F} is obtained by gluing $\mathcal{F}_1|_{X_R \setminus x_2}$ and $\mathcal{F}_2|_{\mathcal{D}_{x_2,R}}$ along the isomorphism

$$\mathcal{F}_1|_{\mathcal{D}_{x_2,R}^\times} \xleftarrow[\mu_1]{\sim} G \times \mathcal{D}_{x_2,R}^\times \xrightarrow[\nu_2]{\sim} \mathcal{F}_2|_{\mathcal{D}_{x_2,R}^\times},$$

see Remark 7.2.

We now explain that p and q are principal bundles for a group. For that, we define $G_{X,\mathcal{O}}$ as the group scheme over X that represents the functor

$$R \mapsto \left\{ (x, \mu) \left| \begin{array}{l} x \in X(R) \\ \mu \text{ trivialization of } G \times X_R \text{ on } \mathcal{D}_{x,R} \end{array} \right. \right\}.$$

In the description of the functor that $\mathrm{Gr}_{G,X}$ represents, as in §7.1 one can replace (\mathcal{F}, ν) by a pair (\mathcal{F}', ν') where \mathcal{F}' is a G -bundle on $\mathcal{D}_{x,R}$ and ν' is a trivialization of \mathcal{F}' on $\mathcal{D}_{x,R}^\times$; thus $G_{X,\mathcal{O}}$ acts on $\mathrm{Gr}_{G,X}$ by twisting the trivialization (specifically, ν' gets replaced by $\nu' \circ \mu^{-1}$).

We can view $G_{X,\mathcal{O}}$ as a group scheme over X^2 by the base change $X^2 \rightarrow X$ given by the second projection. The result acts on $\mathrm{Gr}_{G,X} \widetilde{\times} \mathrm{Gr}_{G,X}$ by twisting μ_1 , which defines p as a bundle.

In the definition of $\mathrm{Gr}_{G,X} \widetilde{\times} \mathrm{Gr}_{G,X}$, as above one can replace (\mathcal{F}_2, ν_2) by a pair (\mathcal{F}'_2, ν'_2) where \mathcal{F}'_2 is a G -bundle on $\mathcal{D}_{x_2,R}$ and ν'_2 is a trivialization of \mathcal{F}'_2 on $\mathcal{D}_{x_2,R}^\times$. The group scheme $G_{X,\mathcal{O}} \times_X X^2$ then acts on $\mathrm{Gr}_{G,X} \widetilde{\times} \mathrm{Gr}_{G,X}$ by simultaneously twisting both μ_1 and ν'_2 . This action defines q as a principal bundle.

7.5 Convolution product and fusion

We go back to our convolution problem, starting this time with the diagram (7.3). Since p and q are principal bundles, we can define a convolution product \star_X on $\mathrm{P}_{G_{X,\mathcal{O}}}(\mathrm{Gr}_{G,X}, \mathbf{k})$ by setting

$$\mathcal{M} \star_X \mathcal{N} := m_*(\mathcal{M} \widetilde{\boxtimes} \mathcal{N}),$$

where again $\mathcal{M} \widetilde{\boxtimes} \mathcal{N}$ is defined by the condition that

$$q^*(\mathcal{M} \widetilde{\boxtimes} \mathcal{N}) = p^*(\mathcal{M} \boxtimes \mathcal{N}).$$

Here \mathcal{M} and \mathcal{N} are perverse sheaves on $\mathrm{Gr}_{G,X}$, and the result $\mathcal{M} \star_X \mathcal{N}$ is in $D^b(\mathrm{Gr}_{G,X^2}, \mathbf{k})$.

Remark 7.3. To define the category $\mathrm{P}_{G_X, \mathcal{O}}(\mathrm{Gr}_{G,X}, \mathbf{k})$ we use a slight variant of the constructions of Appendix A, where algebraic groups are replaced by group schemes over X . This does not require any new ingredient: one simply replace product by fiber products over X everywhere. The same remarks as in §A.3 are also in order here: we must consider perverse sheaves supported on a closed finite union of $G_{X, \mathcal{O}}$ -orbits, and equivariant under some quotient $(G/H_n)_{X, \mathcal{O}}$.

For the sake of simplicity⁹, we will use $X = \mathbb{A}^1$. We can then use a global coordinate on X , which yields a local coordinate at any point $x \in X$, and therefore allows to identify $\mathrm{Gr}_{G,x}$ with the affine Grassmannian Gr_G as we originally defined it. This also leads to an identification $\mathrm{Gr}_{G,X} = \mathrm{Gr}_G \times X$. We let $\tau : \mathrm{Gr}_{G,X} \rightarrow \mathrm{Gr}_G$ be the projection and define $\tau^\circ := \tau^*[1] \cong \tau^![-1]$; the shift is introduced so that τ° takes a perverse sheaf on Gr_G to a perverse sheaf on $\mathrm{Gr}_{G,X}$.

We explained in §7.3 that the restriction of Gr_{G,X^2} to the diagonal Δ_X in X^2 is isomorphic to $\mathrm{Gr}_{G,X}$; we may then denote by $i : \mathrm{Gr}_{G,X} = \mathrm{Gr}_{G,X^2}|_{\Delta_X} \hookrightarrow \mathrm{Gr}_{G,X^2}$ the closed embedding, and consider the functors $i^\circ := i^*[-1]$ and $i^\bullet := i^![1]$.

Lemma 7.4. *For \mathcal{F}_1 and \mathcal{F}_2 in $\mathrm{P}_{G_{\mathcal{O}}}(\mathrm{Gr}_G, \mathbf{k})$, we have canonical isomorphisms*

$$i^\circ(\tau^\circ(\mathcal{F}_1) \star_X \tau^\circ(\mathcal{F}_2)) \cong \tau^\circ(\mathcal{F}_1 \star \mathcal{F}_2) \cong i^\bullet(\tau^\circ(\mathcal{F}_1) \star_X \tau^\circ(\mathcal{F}_2)).$$

Proof. Since the map m in (7.3) is proper, and restricts over the diagonal Δ_X to the product of the map denoted m in (6.2) by id_{Δ_X} , using the base change theorem it suffices to provide canonical isomorphisms

$$(i')^*(\tau^\circ \mathcal{F}_1 \widetilde{\boxtimes} \tau^\circ \mathcal{F}_2) \cong (\tau')^\circ(\mathcal{F}_1 \widetilde{\boxtimes} \mathcal{F}_2)[1], \quad (i')^!(\tau^\circ \mathcal{F}_1 \widetilde{\boxtimes} \tau^\circ \mathcal{F}_2) \cong (\tau')^\circ(\mathcal{F}_1 \widetilde{\boxtimes} \mathcal{F}_2)[-1]$$

where $i' : (G_{\mathcal{K}} \times^{G_{\mathcal{O}}} \mathrm{Gr}_G) \times \Delta_X \rightarrow \mathrm{Gr}_{G,X} \widetilde{\times} \mathrm{Gr}_{G,X}$ is the embedding, $\tau' : (G_{\mathcal{K}} \times^{G_{\mathcal{O}}} \mathrm{Gr}_G) \times \Delta_X \rightarrow G_{\mathcal{K}} \times^{G_{\mathcal{O}}} \mathrm{Gr}_G$ is the projection, and $(\tau')^\circ = (\tau')^*[1] \cong (\tau')^![-1]$. The first isomorphism is immediate from the definitions. The proof of the second one is similar, using Remark A.2. \square

We now analyze the convolution diagram over $U = X^2 \setminus \Delta_X$:

$$\begin{array}{ccc} (\mathrm{Gr}_{G,X} \times \mathrm{Gr}_{G,X})|_U & \xleftarrow{p} & (\mathrm{Gr}_{G,X} \widetilde{\times} \mathrm{Gr}_{G,X})|_U & \xrightarrow{q} & (\mathrm{Gr}_{G,X} \widetilde{\times} \mathrm{Gr}_{G,X})|_U & \xrightarrow{\widetilde{m}} & \mathrm{Gr}_{G,X^2}|_U \\ & & & & \searrow^{\pi \circ m} & & \downarrow \pi \\ & & & & & & (\mathrm{Gr}_{G,X} \times \mathrm{Gr}_{G,X})|_U \end{array} \quad (7.4)$$

Here π is the isomorphism of (7.2), defined by

$$(\mathcal{F}, \nu, x_1, x_2) \mapsto ((\mathcal{F}_1, \nu_1, x_1), (\mathcal{F}_2, \nu_2, x_2)),$$

⁹The general situation can be dealt with by putting the torsor of change of coordinates into the picture; see e.g. [Ga, §2.1.2] for details.

where \mathcal{F}_i is obtained by gluing the trivial bundle on $X_R \setminus x_i$ and the bundle \mathcal{F} on $\mathcal{D}_{x_i, R}$ using ν . We note that there exists an isomorphism

$$(\mathrm{Gr}_{G, X} \widetilde{\times} \mathrm{Gr}_{G, X})|_U \xrightarrow{\sim} \left((\mathrm{Gr}_{G, X} \times \mathrm{Gr}_{G, X}) \times_{X^2} (X \times G_{X, \mathcal{O}}) \right)|_U$$

defined by

$$(\mathcal{F}_1, \nu_1, \mu_1, \mathcal{F}_2, \nu_2, x_1, x_2) \mapsto \left(((\mathcal{F}_1, \nu_1, x_1), (\mathcal{F}_2, \nu_2, x_2)), (x_1, (x_2, \mu_1^{-1} \circ \nu_1|_{\mathcal{D}_{x_2, R}})) \right).$$

Under this identification, the maps p and $\pi \circ m \circ q$ identify with

$$(\mathrm{Gr}_{G, X} \times \mathrm{Gr}_{G, X})|_U \xleftarrow{p_1} [(\mathrm{Gr}_{G, X} \times \mathrm{Gr}_{G, X}) \times_{X^2} (X \times G_{X, \mathcal{O}})]|_U \xrightarrow{a} (\mathrm{Gr}_{G, X} \times \mathrm{Gr}_{G, X})|_U,$$

where p_1 is the projection on the first factor and a is the action of $G_{X, \mathcal{O}}$ on the second copy of $\mathrm{Gr}_{G, X}$.

It follows that if we identify the three spaces on the right in the convolution diagram (7.4), then for any $\mathcal{M}_1, \mathcal{M}_2$ in $\mathrm{Perv}_{G_{X, \mathcal{O}}}(\mathrm{Gr}_{G, X}, \mathbf{k})$, the equivariant structure of \mathcal{M}_2 leads to canonical identifications

$$\begin{array}{ccc} (\mathcal{M}_1 \widetilde{\boxtimes} \mathcal{M}_2)|_U & \xlongequal{\quad} & (\mathcal{M}_1 \star_X \mathcal{M}_2)|_U \\ & \searrow & \parallel \\ & & (\mathcal{M}_1 \boxtimes \mathcal{M}_2)|_U. \end{array} \quad (7.5)$$

Consider now the open embedding $j : (\mathrm{Gr}_X \times \mathrm{Gr}_X)|_U \xrightarrow{(7.2)} \mathrm{Gr}_{X^2}|_U \hookrightarrow \mathrm{Gr}_{X^2}$.

Lemma 7.5. *For any $\mathcal{F}_1, \mathcal{F}_2 \in \mathrm{P}_{G_{\mathcal{O}}}(\mathrm{Gr}_G, \mathbf{k})$, we have*

$$j_{!*}((\tau^\circ \mathcal{F}_1 \boxtimes \tau^\circ \mathcal{F}_2)|_U) \cong (\tau^\circ \mathcal{F}_1) \star_X (\tau^\circ \mathcal{F}_2).$$

Proof. We will use the characterization of $j_{!*}((\tau^\circ \mathcal{F}_1 \boxtimes \tau^\circ \mathcal{F}_2)|_U)$ given by (4.2). In fact, in (7.5) we have already obtained the desired description over U . Hence to conclude it suffices to prove that

$$i^*((\tau^\circ \mathcal{F}_1) \star_X (\tau^\circ \mathcal{F}_2)) \in {}^p D^{\leq -1} \quad \text{and} \quad i^!((\tau^\circ \mathcal{F}_1) \star_X (\tau^\circ \mathcal{F}_2)) \in {}^p D^{\geq 1}. \quad (7.6)$$

However, it follows from Lemma 7.4 that

$$i^*((\tau^\circ \mathcal{F}_1) \star_X (\tau^\circ \mathcal{F}_2)) \cong \tau^\circ(\mathcal{F}_1 \star \mathcal{F}_2)[1].$$

By Proposition 6.3 the right-hand side is concentrated in perverse degree -1 , proving the first condition in (7.6). The second condition can be checked similarly, using the second isomorphism in Lemma 7.4. \square

7.6 Construction of the commutativity constraint

Combining Lemma 7.4 and Lemma 7.5, we obtain a canonical isomorphism

$$\tau^\circ(\mathcal{F}_1 \star \mathcal{F}_2) \cong i^\circ j_{!*}((\tau^\circ \mathcal{F}_1 \boxtimes \tau^\circ \mathcal{F}_2)|_U),$$

valid for any $\mathcal{F}_1, \mathcal{F}_2 \in \mathbf{P}_{G_\circ}(\mathrm{Gr}_G, \mathbf{k})$. In other words, the convolution product $\mathcal{F}_1 \star \mathcal{F}_2$ can also be obtained by a procedure based on the Beilinson–Drinfeld Grassmannians $\mathrm{Gr}_{G,X}$ and Gr_{G,X^2} , called the *fusion product*.

Let $\mathrm{swap} : \mathrm{Gr}_{G,X^2} \rightarrow \mathrm{Gr}_{G,X^2}$ be the automorphism that swaps x_1 and x_2 . Then we have $(\mathrm{swap} \circ i) = i$. Moreover, swap stabilizes $\mathrm{Gr}_{G,X^2}|_U$, and under the identification (7.2) the induced automorphism of $(\mathrm{Gr}_{G,X} \times \mathrm{Gr}_{G,X})|_U$ (which we will denote swap_U) swaps the two factors $\mathrm{Gr}_{G,X}$. Therefore we obtain canonical isomorphisms

$$\begin{aligned} \tau^\circ(\mathcal{F}_1 \star \mathcal{F}_2) &\cong i^\circ j_{!*}((\tau^\circ \mathcal{F}_1 \boxtimes \tau^\circ \mathcal{F}_2)|_U) \\ &\cong i^\circ \mathrm{swap}^* j_{!*}((\tau^\circ \mathcal{F}_1 \boxtimes \tau^\circ \mathcal{F}_2)|_U) \\ &\cong i^\circ j_{!*}(\mathrm{swap}_U)^*((\tau^\circ \mathcal{F}_1 \boxtimes \tau^\circ \mathcal{F}_2)|_U) \\ &\cong i^\circ j_{!*}((\tau^\circ \mathcal{F}_2 \boxtimes \tau^\circ \mathcal{F}_1)|_U) \\ &= \tau^\circ(\mathcal{F}_2 \star \mathcal{F}_1). \end{aligned}$$

Restricting to a point of X , we deduce a canonical isomorphism

$$\mathcal{F}_1 \star \mathcal{F}_2 \xrightarrow{\sim} \mathcal{F}_2 \star \mathcal{F}_1,$$

which provides a commutativity constraint for the category $\mathbf{P}_{G_\circ}(\mathrm{Gr}_G, \mathbf{k})$.

- Remark 7.6.*
1. Later we will modify this commutativity constraint by a sign to make sure that the functor $\mathbf{H}^\bullet(\mathrm{Gr}_G, ?)$ sends it to the standard commutativity constraint on vector spaces; see §8.2.
 2. One may note here that the twisted product $\mathrm{Gr}_{G,X} \widetilde{\times} \mathrm{Gr}_{G,X}$, while playing a key role in the proof, is not involved in the definition of the fusion product, since the maps i and j only deal with the Beilinson–Drinfeld Grassmannians $\mathrm{Gr}_{G,X}$ and Gr_{G,X^2} . The two points x_1 and x_2 , which are not interchangeable in the definition of $\mathrm{Gr}_{G,X} \widetilde{\times} \mathrm{Gr}_{G,X}$, play the same role in Gr_{G,X^2} . This property is the basis for the construction of the commutativity constraint.
 3. Note that one can describe the associativity constraint considered in §6.4 in a similar way, using Gr_{G,X^3} .

8 Study of the fiber functor

8.1 Compatibility of \mathbf{F} with the convolution product

In Sections 6–7 we have endowed our category $\mathbf{P}_{G_\circ}(\mathrm{Gr}_G, \mathbf{k})$ with a convolution product \star , defined either in the easy way with the convolution diagram (6.2) or with the fusion procedure.

The latter even allows to define a commutativity constraint. We now want to show that the functor

$$F = H^\bullet(\mathrm{Gr}_G, ?) : P_{G\mathcal{O}}(\mathrm{Gr}_G, \mathbf{k}) \rightarrow \mathrm{Vect}_{\mathbf{k}}$$

is a fiber functor in the sense of Remark 2.8(2); in other words that this is an exact and faithful functor that maps the convolution product of sheaves to the tensor product of vector spaces while respecting the associativity, the unit, and the commutativity constraints of these categories.

The exactness and the faithfulness of F have already been proved in Theorem 5.6(2). We will here show that

$$F(\mathcal{A}_1 \star \mathcal{A}_2) = F(\mathcal{A}_1) \otimes F(\mathcal{A}_2).$$

(Strictly speaking, this is not an equality, but a canonical identification.) For that, we will use the fusion procedure.

Recall the setup of Section 7, and in particular diagram (7.3).

Let $\mathcal{A}_1, \mathcal{A}_2$ in $P_{G\mathcal{O}}(\mathrm{Gr}_G, \mathbf{k})$, and set $\mathcal{B} := (\tau^\circ \mathcal{A}_1) \star_X (\tau^\circ \mathcal{A}_2)$. Then if $f : \mathrm{Gr}_{G, X^2} \rightarrow X^2$ is the natural map, Lemma 7.4 and (7.5) translate to the following properties: for each $k \in \mathbf{Z}$,

- the k -th cohomology sheaf of the complex $(f_* \mathcal{B})|_{\Delta_X}[-2]$ is locally constant on Δ_X , with stalk $H^k(\mathrm{Gr}_G, \mathcal{A}_1 \star \mathcal{A}_2)$;
- the k -th cohomology sheaf of the complex $(f_* \mathcal{B})|_U[-2]$ is locally constant on U , with stalk $H^k(\mathrm{Gr}_G \times \mathrm{Gr}_G, \mathcal{A}_1 \boxtimes \mathcal{A}_2)$.

From there, we will be able to deduce the desired identification

$$H^k(\mathrm{Gr}_G, \mathcal{A}_1 \star \mathcal{A}_2) \cong \bigoplus_{i+j=k} H^i(\mathrm{Gr}_G, \mathcal{A}_1) \otimes H^j(\mathrm{Gr}_G, \mathcal{A}_2)$$

as soon as we know that $\mathcal{H}^{k-2}(f_* \mathcal{B})$ is locally constant on the whole space X^2 .

We now prove this fact. Set $\tilde{\mathcal{B}} := (\tau^\circ \mathcal{A}_1) \tilde{\boxtimes} (\tau^\circ \mathcal{A}_2)$, so that $\mathcal{B} = m_* \tilde{\mathcal{B}}$. Hence if we set $\tilde{f} = f \circ m$, we have $f_* \mathcal{B} = \tilde{f}_* \tilde{\mathcal{B}}$. For $\lambda, \mu \in X_*(T)^+$, set

$$\mathrm{Gr}_{G, X}^\lambda = \tau^{-1}(\mathrm{Gr}_G^\lambda), \quad \mathrm{Gr}_{G, X}^\mu = \tau^{-1}(\mathrm{Gr}_G^\mu),$$

and define $\mathrm{Gr}_{G, X}^\lambda \tilde{\times} \mathrm{Gr}_{G, X}^\mu \subset \mathrm{Gr}_{G, X} \tilde{\times} \mathrm{Gr}_{G, X}$ by the requirement

$$q^{-1}(\mathrm{Gr}_{G, X}^\lambda \tilde{\times} \mathrm{Gr}_{G, X}^\mu) = p^{-1}(\mathrm{Gr}_{G, X}^\lambda \times \mathrm{Gr}_{G, X}^\mu).$$

(This definition makes sense, since $\mathrm{Gr}_{G, X}^\mu$ is stable by the left action of $G_{X, \mathcal{O}}$). Then

$$\tilde{\mathcal{F}} = \{\mathrm{Gr}_{G, X}^\lambda \tilde{\times} \mathrm{Gr}_{G, X}^\mu \mid \lambda, \mu \in X_*(T)^+\}$$

is a stratification of $\mathrm{Gr}_{G, X} \tilde{\times} \mathrm{Gr}_{G, X}$, and $\tilde{\mathcal{B}}$ is $\tilde{\mathcal{F}}$ -constructible. To show that the cohomology sheaves of $\tilde{f}_* \tilde{\mathcal{B}}$ are locally constant, it suffices by dévissage¹⁰ to check that for each $k \in \mathbf{Z}$ and each stratum $S \in \tilde{\mathcal{F}}$, the sheaf $\mathcal{H}^k \tilde{f}_* \mathbf{k}_S$ is locally constant.

¹⁰More precisely, one uses the following claim: the complexes \mathcal{M} such that the cohomology sheaves $\mathcal{H}^k \tilde{f}_* \mathcal{M}$ are local systems form a full *triangulated* subcategory of $D^b(\mathrm{Gr}_{G, X} \tilde{\times} \mathrm{Gr}_{G, X}, \mathbf{k})$. To prove this claim, consider a distinguished triangle $\mathcal{M}' \rightarrow \mathcal{M} \rightarrow \mathcal{M}'' \xrightarrow{[1]}$ with \mathcal{M}' and \mathcal{M}'' in the subcategory. The long exact sequence in cohomology expresses $\mathcal{H}^k \tilde{f}_* \mathcal{M}$ as an extension of $\ker(\mathcal{H}^k \tilde{f}_* \mathcal{M}'' \rightarrow \mathcal{H}^{k+1} \tilde{f}_* \mathcal{M}')$ by $\mathrm{coker}(\mathcal{H}^{k-1} \tilde{f}_* \mathcal{M}'' \rightarrow \mathcal{H}^k \tilde{f}_* \mathcal{M}')$, hence as an extension of two local systems. Therefore $\mathcal{H}^k \tilde{f}_* \mathcal{M}$ is a local system for each k , which means that \mathcal{M} belongs to our subcategory.

Let $\widehat{\text{Gr}}_{G,X^2}$ be the ind-scheme representing the functor

$$R \mapsto \left\{ (\mathcal{F}_1, \nu_1, \mu_1, x_1, x_2) \left| \begin{array}{l} (x_1, x_2) \in X^2(R) \\ \mathcal{F}_1 \text{ } G\text{-bundle on } X_R \\ \nu_1 \text{ trivialization of } \mathcal{F}_1 \text{ on } X_R \setminus x_1 \\ \mu_1 \text{ trivialization of } \mathcal{F}_1 \text{ on } \mathcal{D}_{x_2,R} \end{array} \right. \right\} / \text{isomorphism}$$

There is a natural map $q' : \widehat{\text{Gr}}_{G,X^2} \rightarrow \text{Gr}_{G,X} \times X$, that simply forgets μ_1 . The group scheme $G_{X,\mathcal{O}}$ acts on $\widehat{\text{Gr}}_{G,X^2}$ by twisting μ_1 , and q' is a bundle for this action. On the other hand, $\text{Gr}_{G,X}$ classifies data $(\mathcal{F}_2, \nu_2, x_2)$ with \mathcal{F}_2 a G -bundle on \mathcal{D}_{x_2} and ν_2 a trivialization on $\mathcal{D}_{x_2}^\times$ (see §7.1), so the group scheme $G_{X,\mathcal{O}}$ acts on $\text{Gr}_{G,X}$ by twisting ν_2 . A direct inspection shows then that

$$\begin{array}{c} \text{Gr}_{G,X} \widetilde{\times} \text{Gr}_{G,X} = \widehat{\text{Gr}}_{G,X^2} \times^{G_{X,\mathcal{O}}} \text{Gr}_{G,X} \\ \downarrow q' \\ \text{Gr}_{G,X} \times X \end{array}$$

is an associated space to our torsor q' , hence is a locally trivial fibration.

Take $\lambda, \mu \in X_*(T)^+$, and set $S = \text{Gr}_{G,X}^\lambda \widetilde{\times} \text{Gr}_{G,X}^\mu$. The base change corresponding to the inclusion $\text{Gr}_{G,X}^\lambda \hookrightarrow \text{Gr}_{G,X}$ and the fiber change corresponding to the inclusion $\text{Gr}_{G,X}^\mu \hookrightarrow \text{Gr}_{G,X}$ show that the natural map

$$\begin{array}{c} S = \text{Gr}_{G,X}^\lambda \widetilde{\times} \text{Gr}_{G,X}^\mu \\ \downarrow q' \\ \text{Gr}_{G,X}^\lambda \times X \end{array}$$

is a locally trivial fibration with fiber $\text{Gr}_{G,X}^\mu$. It follows that the cohomology sheaves of $q'_*\underline{\mathbf{k}}_S$ are locally constant on $\text{Gr}_{G,X}^\lambda \times X$. Further, the projection $\text{Gr}_{G,X}^\lambda \rightarrow X$ is also a locally trivial fibration, and by a last dévissage argument, we conclude that $\tilde{f}_*\underline{\mathbf{k}}_S$ has locally constant cohomology sheaves, as desired.

8.2 Compatibility with the commutativity constraint

By construction, the fiber functor F factors through a functor from $P_{G_{\mathcal{O}}}(\text{Gr}_G, \mathbf{k})$ to the category $\text{Vect}_{\mathbf{k}}(\mathbf{Z})$ of \mathbf{Z} -graded \mathbf{k} -vector spaces. We can endow the latter with the usual structure of tensor category or with the supersymmetric structure; the difference between the two structures is the definition of the commutativity constraint, which involves a sign in the super case (see in particular Example 2.9(3)).

Recall the notion of even and odd components of Gr_G from §3.1. It follows in particular from Theorem 5.6(1) that if \mathcal{A} is supported on an even (resp. odd) component of Gr_G then $H^\bullet(\text{Gr}_G, \mathcal{A})$ is concentrated in even (resp. odd) degrees. Looking closely at the constructions in §8.1, one can check that the functor F maps the commutativity constraint on $P_{G_{\mathcal{O}}}(\text{Gr}_G, \mathbf{k})$ defined in §7.6 to the supersymmetric commutativity constraint on $\text{Vect}_{\mathbf{k}}(\mathbf{Z})$. (This is related

to the fact that the canonical isomorphism $(\text{swap}_U)^*((\tau^\circ \mathcal{F}_1 \boxtimes \tau^\circ \mathcal{F}_2)|_U) \cong (\tau^\circ \mathcal{F}_2 \boxtimes \tau^\circ \mathcal{F}_1)|_U$ involves some signs.)

However, to be in a position to apply the Tannakian reconstruction theorem from Section 2, we need to make sure that F maps the commutativity constraint on $P_{G_{\mathcal{O}}}(\text{Gr}_G, \mathbf{k})$ to the usual (unsigned) commutativity constraint on $\text{Vect}_{\mathbf{k}}(\mathbf{Z})$. One solution consists in altering the commutativity constraint on $P_{G_{\mathcal{O}}}(\text{Gr}_G, \mathbf{k})$ by an appropriate sign. In fact, due to the change of parity introduced in the functor τ° , one must multiply the isomorphism of §7.6 by -1 for the summands of the perverse sheaves \mathcal{F}_1 and \mathcal{F}_2 supported on even components of Gr_G . This is the commutativity constraint that we will consider below.

8.3 Compatibility with the weight functors

We have noticed in §8.2 that the fiber functor $F : P_{G_{\mathcal{O}}}(\text{Gr}_G, \mathbf{k}) \rightarrow \text{Vect}_{\mathbf{k}}$ in fact factors through the category $\text{Vect}_{\mathbf{k}}(\mathbf{Z})$ of \mathbf{Z} -graded \mathbf{k} -vector spaces. We can enhance this result using the weight functors of §5.2. In fact by Theorem 5.6(1) we have a commutative diagram

$$\begin{array}{ccc} & & \text{Vect}_{\mathbf{k}}(X_*(T)) \\ & \nearrow \bigoplus_{\mu} F_{\mu} & \downarrow \text{forget} \\ P_{G_{\mathcal{O}}}(\text{Gr}_G, \mathbf{k}) & & \text{Vect}_{\mathbf{k}} \\ & \searrow F & \end{array}$$

where $\text{Vect}_{\mathbf{k}}(X_*(T))$ is the category of $X_*(T)$ -graded \mathbf{k} -vector spaces. Recall from Example 2.9(1) that $\text{Vect}_{\mathbf{k}}(X_*(T))$ is a tensor category.

Proposition 8.1. *The functor $\bigoplus_{\mu} F_{\mu}$ is a tensor functor.*

Proof. We need to provide an identification

$$F_{\mu}(\mathcal{A}_1 \star \mathcal{A}_2) = \bigoplus_{\mu_1 + \mu_2 = \mu} F_{\mu_1}(\mathcal{A}_1) \otimes F_{\mu_2}(\mathcal{A}_2) \quad (8.1)$$

for each $\mu \in X_*(T)$ and all $\mathcal{A}_1, \mathcal{A}_2 \in P_{G_{\mathcal{O}}}(\text{Gr}_G, \mathbf{k})$.

Recall how the weight functors F_{μ} are defined (see Remark 5.7). We choose a maximal torus and a Borel subgroup $T \subset B \subset G$. Then $T \subset G_{\mathcal{K}}$ acts on $\text{Gr}_G = G_{\mathcal{K}}/G_{\mathcal{O}}$ with fixed points

$$(\text{Gr}_G)^T = \{L_{\mu} \mid \mu \in X_*(T)\}.$$

We pick a dominant regular cocharacter $\eta \in X_*(T)$, which provides a one-parameter subgroup $\mathbf{G}_m \subset T$ and a \mathbf{C}^{\times} -action on Gr_G with fixed points $(\text{Gr}_G)^T$. For $\mu \in X_*(T)$, the attractive variety relative to the fixed point L_{μ} is

$$S_{\mu} = \{x \in \text{Gr}_G \mid \eta(a) \cdot x \rightarrow L_{\mu} \text{ when } a \rightarrow 0\}$$

(see the proof of Theorem 5.1), and for each $\mathcal{A} \in P_{G_{\mathcal{O}}}(\text{Gr}_G, \mathbf{k})$,

$$H_c^k(S_{\mu}, \mathcal{A}) \neq 0 \implies k = \langle 2\rho, \mu \rangle.$$

For $\mu \in X_*(T)$ and $\mathcal{A} \in \mathbb{P}_{G_O}(\mathrm{Gr}_G, \mathbf{k})$, we have $F_\mu(\mathcal{A}) := H_c^{(2\rho, \mu)}(\overline{S}_\mu, \mathcal{A})$. We get adjunction maps (see Remark 5.7)

$$\begin{array}{ccc} & H_c^{(2\rho, \mu)}(S_\mu, \mathcal{A}) & \\ & \downarrow \wr & \\ H^{(2\rho, \mu)}(\mathrm{Gr}_G, \mathcal{A}) & \longrightarrow & H_c^{(2\rho, \mu)}(\overline{S}_\mu, \mathcal{A}); \end{array}$$

moreover for each $k \in \mathbf{Z}$, there is a decomposition

$$H^k(\mathrm{Gr}_G, \mathcal{A}) = \bigoplus_{\substack{\mu \in X_*(T) \\ \langle 2\rho, \mu \rangle = k}} F_\mu(\mathcal{A}).$$

We need to insert this construction in the reasoning in §8.1. The various spaces considered in §7.4 carry an action of T . Specifically, this action twists ν in $\mathrm{Gr}_{G,X}$ and Gr_{G,X^2} , and twists ν_1 on $\mathrm{Gr}_{G,X} \times \mathrm{Gr}_{G,X}$ and $\mathrm{Gr}_{G,X} \tilde{\times} \mathrm{Gr}_{G,X}$. The maps q and m in the diagram (7.3) and the isomorphism π in the diagram (7.4) are T -equivariant.

To each pair $(\mu_1, \mu_2) \in X_*(T)^2$ corresponds a connected component $\tilde{C}_{(\mu_1, \mu_2)}$ of the set of T -fixed points in $\mathrm{Gr}_{G,X} \tilde{\times} \mathrm{Gr}_{G,X}$, namely

$$\tilde{C}_{(\mu_1, \mu_2)} = \{([t^{\mu_1}, L_{\mu_2}], x, x) \mid x \in X\} \cup \{(L_{\mu_1}, L_{\mu_2}, x_1, x_2) \mid (x_1, x_2) \in U\}$$

where $[t^{\mu_1}, L_{\mu_2}]$ is seen as a point in $G_{\mathcal{K}_x} \times^{G_{O_x}} \mathrm{Gr}_{G,x}$, identified with the fiber of the twisted product $\mathrm{Gr}_{G,X} \tilde{\times} \mathrm{Gr}_{G,X}$ over a point $(x, x) \in \Delta_X$, and (L_{μ_1}, L_{μ_2}) is likewise seen as a point in $\mathrm{Gr}_{G,x_1} \times \mathrm{Gr}_{G,x_2}$, identified with the fiber of $\mathrm{Gr}_{G,X} \tilde{\times} \mathrm{Gr}_{G,X}$ over $(x_1, x_2) \in U$ thanks to the map $\pi \circ m \circ q$ in (7.4). Moreover the projection $\tilde{C}_{\mu_1, \mu_2} \rightarrow X^2$ is an isomorphism. (Recall that we have chosen $X = \mathbb{A}^1$, so that we have a canonical identification $\mathrm{Gr}_{G,x} \cong \mathrm{Gr}_G$ for any x .)

The map $m : \mathrm{Gr}_{G,X} \tilde{\times} \mathrm{Gr}_{G,X} \rightarrow \mathrm{Gr}_{G,X^2}$ glues together along the diagonal Δ_X the various connected components $\tilde{C}_{(\mu_1, \mu_2)}$ for which $\mu_1 + \mu_2$ is the same. Therefore, to each $\mu \in X_*(T)$ corresponds a connected component

$$C_\mu := \bigsqcup_{\mu_1 + \mu_2 = \mu} m(\tilde{C}_{(\mu_1, \mu_2)})$$

of the set of T -fixed points in Gr_{G,X^2} .

Our dominant regular character $\eta \in X_*(T)$ defines a \mathbf{C}^\times -action on Gr_{G,X^2} . Denote the attractive variety around C_μ by $S_\mu(X^2)$. Over a point $(x, x) \in \Delta_X$, the fiber of the map $S_\mu(X^2) \rightarrow X^2$ is the semi-infinite orbit S_μ , viewed as a subvariety of $\mathrm{Gr}_{G,x}$ thanks to the isomorphism $\mathrm{Gr}_{G,X^2}|_{\Delta_X} = \mathrm{Gr}_{G,X}$. Over a point $(x_1, x_2) \in U$, the fiber of $S_\mu(X^2) \rightarrow X^2$ is the union

$$\bigsqcup_{\mu_1 + \mu_2 = \mu} S_{\mu_1} \times S_{\mu_2} \subset \mathrm{Gr}_{G,x_1} \times \mathrm{Gr}_{G,x_2},$$

where we use the isomorphism $\pi : \mathrm{Gr}_{G,X^2}|_U \xrightarrow{\sim} (\mathrm{Gr}_{G,X} \times \mathrm{Gr}_{G,X})|_U$ of §7.5.

Consider again $\mathcal{B} := (\tau^\circ \mathcal{A}_1) \star_X (\tau^\circ \mathcal{A}_2)$ and consider the maps depicted in the following diagram

$$\begin{array}{ccc}
S_\mu(X^2) & \xrightarrow{\tilde{s}'_\mu} \overline{S_\mu(X^2)} & \xrightarrow{\tilde{s}''_\mu} \mathrm{Gr}_{G, X^2} \\
& \searrow \tilde{s}_\mu & \downarrow f \\
& & X^2.
\end{array}$$

The stalks of the complex of sheaves $(f\tilde{s}_\mu)_!(\tilde{s}_\mu)^*\mathcal{B}$ can be computed by base change. Using Lemma 7.4 and (7.5), we obtain:

- The sheaf $\mathcal{H}^{k-2}(f\tilde{s}_\mu)_!(\tilde{s}_\mu)^*\mathcal{B}$ is locally constant on Δ_X , with stalk $H_c^k(S_\mu, \mathcal{A}_1 \star \mathcal{A}_2)$, so is $F_\mu(\mathcal{A}_1 \star \mathcal{A}_2)$ if $k = \langle 2\rho, \mu \rangle$ and is zero otherwise.
- The sheaf $\mathcal{H}^{k-2}(f\tilde{s}_\mu)_!(\tilde{s}_\mu)^*\mathcal{B}$ is locally constant on U , with stalk isomorphic to

$$\bigoplus_{\mu_1 + \mu_2 = \mu} H_c^k(S_{\mu_1} \times S_{\mu_2}, \mathcal{A}_1 \boxtimes \mathcal{A}_2),$$

so is

$$\bigoplus_{\mu_1 + \mu_2 = \mu} F_{\mu_1}(\mathcal{A}_1) \otimes F_{\mu_2}(\mathcal{A}_2)$$

if $k = \langle 2\rho, \mu \rangle$ and is zero otherwise. One may here observe the role of Remark 5.5(2) in our argument: it allows us to ignore the action a that appeared during the proof of the identification (7.5).

In particular,

$$\mathcal{H}^{k-2}(f\tilde{s}_\mu)_!(\tilde{s}_\mu)^*\mathcal{B} \neq 0 \implies k = \langle 2\rho, \mu \rangle.$$

Given $\mu \in X_*(T)$, denote the sheaf $\mathcal{H}^{\langle 2\rho, \mu \rangle - 2}(f\tilde{s}_\mu)_!(\tilde{s}_\mu)^*\mathcal{B}$ by $\mathcal{L}_\mu(\mathcal{B})$. Adjunction yields maps

$$\begin{array}{ccc}
& \mathcal{H}^{\langle 2\rho, \mu \rangle - 2}(f\tilde{s}_\mu)_!(\tilde{s}_\mu)^*\mathcal{B} & \\
& \downarrow & \\
\mathcal{H}^{\langle 2\rho, \mu \rangle - 2}f_*\mathcal{B} & \longrightarrow & \mathcal{L}_\mu(\mathcal{B}).
\end{array}$$

Since this is true over any point of X^2 , the vertical arrow is an isomorphism and the horizontal arrow is an epimorphism; moreover for each $k \in \mathbf{Z}$ the projections provide an isomorphism

$$\mathcal{H}^{k-2}f_*\mathcal{B} \cong \bigoplus_{\substack{\mu \in X_*(T) \\ \langle 2\rho, \mu \rangle = k}} \mathcal{L}_\mu(\mathcal{B}).$$

We see here that $\mathcal{L}_\mu(\mathcal{B})$ is a direct summand of the local system $\mathcal{H}^{k-2}f_*\mathcal{B}$ (see §8.1), so it is a local system itself. As we saw, its stalk over a point in Δ_X is $F_\mu(\mathcal{A}_1 \star \mathcal{A}_2)$ and its stalk over a point in U is $F_{\mu_1}(\mathcal{A}_1) \otimes F_{\mu_2}(\mathcal{A}_2)$. We thus obtain the desired identification (8.1). \square

9 Identification of the dual group

At this point, we have constructed the convolution product \star on $P_{G_{\mathcal{O}}}(\mathrm{Gr}_G, \mathbf{k})$, a faithful exact functor $F : P_{G_{\mathcal{O}}}(\mathrm{Gr}_G, \mathbf{k}) \rightarrow \mathrm{Vect}_{\mathbf{k}}$, an associativity constraint, a commutativity constraint, and a unit object $U = \mathbf{IC}_0$ such that:

- (a) $F \circ \star = \otimes \circ (F \otimes F)$ and $F(U) = \mathbf{k}$;
- (b) F maps the associativity constraint, the commutativity constraint and the unit constraint of $P_{G_{\mathcal{O}}}(\mathrm{Gr}_G, \mathbf{k})$ to the corresponding constraints of $\mathrm{Vect}_{\mathbf{k}}$;
- (c) If $F(L)$ has dimension 1, then there exists L^{-1} such that $L \otimes L^{-1} \cong U$.

(For the last property, one observes that for $L = \mathbf{IC}_{\lambda}$ to satisfy $\dim F(L) = 1$, by Proposition 5.8 λ must be orthogonal to each root $\alpha \in \Phi$, so $\mathrm{Gr}^{\lambda} = \{L_{\lambda}\}$, and we can take $L^{-1} = \mathbf{IC}_{-\lambda}$ since $-\lambda$ is dominant.)

Tannakian reconstruction (see Theorem 2.7) then gives us an affine group scheme \tilde{G} over \mathbf{k} and an equivalence S which fits in the following commutative diagram:

$$\begin{array}{ccc} P_{G_{\mathcal{O}}}(\mathrm{Gr}_G, \mathbf{k}) & \xrightarrow[\quad S \quad]{\quad \sim \quad} & \mathrm{Rep}_{\mathbf{k}}(\tilde{G}) \\ & \searrow F & \swarrow \omega \\ & \mathrm{Vect}_{\mathbf{k}} & \end{array}$$

where ω is the forgetful functor. We now need to identify \tilde{G} .

9.1 First step: \tilde{G} is a split connected reductive algebraic group over \mathbf{k}

Lemma 9.1. *The group scheme \tilde{G} is algebraic.*

Proof. Choose a finite set of generators $\lambda_1, \dots, \lambda_n$ of the monoid $X_*(T)^+$ of dominant cocharacters. Then for any nonnegative integral linear combination $\lambda = k_1\lambda_1 + \dots + k_n\lambda_n$, the sheaf \mathbf{IC}_{λ} appears as a direct summand of the convolution product $\mathbf{IC}_{\lambda_1}^{k_1} \star \dots \star \mathbf{IC}_{\lambda_n}^{k_n}$. Therefore $\mathcal{X} := \mathbf{IC}_{\lambda_1} \oplus \dots \oplus \mathbf{IC}_{\lambda_n}$ is a tensor generator of the category $P_{G_{\mathcal{O}}}(\mathrm{Gr}_G, \mathbf{k})$; namely, any object of $P_{G_{\mathcal{O}}}(\mathrm{Gr}_G, \mathbf{k})$ appears as a subquotient of a direct sum of tensor powers of \mathcal{X} . Thus $\mathrm{Rep}_{\mathbf{k}}(\tilde{G})$ admits a tensor generator, which implies that \tilde{G} is algebraic by Proposition 2.11(1). \square

Lemma 9.2. *The group scheme \tilde{G} is connected.*

Proof. If λ is a nonzero cocharacter of T , then the objects $\mathbf{IC}_{m\lambda}$ are all different for $m \in \mathbf{Z}$. It follows that for any nontrivial object \mathcal{X} in $P_{G_{\mathcal{O}}}(\mathrm{Gr}_G, \mathbf{k})$, the full subcategory formed by subquotients of direct sums $\mathcal{X}^{\oplus n}$ cannot be stable under \star . The same property then also holds for the tensor category $\mathrm{Rep}_{\mathbf{k}}(\tilde{G})$. This in turn implies that \tilde{G} is connected by Proposition 2.11(2). \square

Lemma 9.3. *The connected \mathbf{k} -algebraic group \tilde{G} is reductive.*

Proof. This directly follows from the semisimplicity of the category $\text{Rep}_{\mathbf{k}}(\tilde{G})$ (which itself follows from Theorem 4.2), see Proposition 2.11(3). \square

We now explain the construction of a split maximal torus in \tilde{G} .

As in §8.3, we denote by $\text{Vect}_{\mathbf{k}}(X_*(T))$ the category of finite dimensional $X_*(T)$ -graded \mathbf{k} -vector spaces. This is a tensor category, and the weight functors provide us with a factorization of F as

$$\text{P}_{G_{\mathcal{O}}}(\text{Gr}_G, \mathbf{k}) \xrightarrow{F'} \text{Vect}_{\mathbf{k}}(X_*(T)) \xrightarrow{\text{forget}} \text{Vect}_{\mathbf{k}},$$

see §8.3. Let \tilde{T} be the unique split \mathbf{k} -torus such that $X^*(\tilde{T}) = X_*(T)$; then $\text{Vect}_{\mathbf{k}}(X_*(T)) \cong \text{Rep}_{\mathbf{k}}(\tilde{T})$ canonically, and F' induces a tensor functor $F_{\tilde{T}} : \text{Rep}_{\mathbf{k}}(\tilde{G}) \rightarrow \text{Rep}_{\mathbf{k}}(\tilde{T})$. There is then a commutative diagram

$$\begin{array}{ccc} \text{P}_{G_{\mathcal{O}}}(\text{Gr}_G, \mathbf{k}) & \xrightarrow{F'} & \text{Vect}_{\mathbf{k}}(X_*(T)) \\ \downarrow \wr & & \parallel \\ \text{Rep}_{\mathbf{k}}(\tilde{G}) & \xrightarrow{F_{\tilde{T}}} & \text{Rep}_{\mathbf{k}}(\tilde{T}) \end{array}$$

commuting with the forgetful functors to $\text{Vect}_{\mathbf{k}}$. Thus $F_{\tilde{T}}$ is induced by a unique morphism $\varphi : \tilde{T} \rightarrow \tilde{G}$ of algebraic groups, see Proposition 2.10.

Each character $\lambda \in X^*(\tilde{T})$ appears in at least one $F_{\tilde{T}}(\mathbf{IC}_{\mu})$ (one can here choose μ as the dominant W -conjugate of λ and use Theorem 5.1 and Proposition 5.8). It follows that φ is an embedding of a closed subgroup, so that \tilde{T} can be considered as a split torus in \tilde{G} . Furthermore, the rank of \tilde{G} is the transcendence degree of the field of fractions of the Grothendieck ring of $\text{Rep}_{\mathbf{k}'}(\tilde{G}')$, where \mathbf{k}' is an algebraic closure of \mathbf{k} and \tilde{G}' is the group scheme obtained from \tilde{G} by extension of scalars from \mathbf{k} to \mathbf{k}' . In view of the description of the simple objects in $\text{P}_{G_{\mathcal{O}}}(\text{Gr}_G, \mathbf{k})$, this rank is equal to the rank of the \mathbf{Z} -module $X_*(T)$, so is equal to the dimension of \tilde{T} . It follows that \tilde{T} is a maximal torus of \tilde{G} .

9.2 Second step: Identification of the root datum of (\tilde{G}, \tilde{T})

We first determine a “canonical” Borel subgroup in \tilde{G} .

Consider the sum $2\rho \in X^*(T)$ of the positive roots of G . Then there exists a Borel subgroup $\tilde{B} \subset \tilde{G}$ that contains \tilde{T} and such that 2ρ is a dominant coweight for the choice of positive roots of \tilde{G} given by the \tilde{T} -weights in the Lie algebra of \tilde{B} .

Lemma 9.4. *For such a choice of Borel subgroup \tilde{B} , hence of positive roots, the dominant weights for \tilde{T} are exactly the dominant coweights of T (for the choice of the positive roots as the T -weights in the Lie algebra of B).*

Proof. Given $\lambda \in X_*(T)^+$ (that is, dominant for $T \subset B \subset G$), let $V = \mathbf{S}(\mathbf{IC}_{\lambda})$ be the simple \tilde{G} -module corresponding to \mathbf{IC}_{λ} . By Proposition 5.8 the maximal value of $\langle 2\rho, \mu \rangle$ for μ a weight of V is obtained for $\mu = \lambda$, and only for this weight. Therefore λ is dominant for $\tilde{T} \subset \tilde{B} \subset \tilde{G}$, and is then the highest weight of V .

Conversely, let $\mu \in X^*(\check{T})$ be dominant for $\check{T} \subset \check{B} \subset \check{G}$. Let V be the simple \check{G} -module of highest weight μ . Then $V = \mathbf{S}(\mathbf{IC}_\lambda)$ for a unique λ dominant for $T \subset B \subset G$, and by the first step $\lambda = \mu$. Thus μ is dominant for $T \subset B \subset G$. \square

This claim implies in particular that \check{B} is uniquely determined; that is, no root of (\check{G}, \check{T}) is orthogonal to 2ρ . From now on we fix this choice of Borel subgroup in \check{G} , and hence of positive roots of \check{G} with respect to \check{T} .

Lemma 9.4 implies that the simple root directions of $T \subset B \subset G$ are the simple coroot directions of $\check{T} \subset \check{B} \subset \check{G}$:

$$\{\mathbf{Q}_+ \alpha \mid \alpha \text{ simple coroot of } \check{T} \subset \check{B} \subset \check{G}\} = \{\mathbf{Q}_+ \alpha \mid \alpha \text{ simple root of } T \subset B \subset G\}. \quad (9.1)$$

Lemma 9.5. *The set of simple roots of \check{G} coincides with the set of simple coroots of G .*

Proof. Let \check{G} be the connected reductive \mathbf{k} -group which is Langlands dual to G , i.e. whose root datum is dual to that of (G, T) . Then \check{T} is also a maximal torus in \check{G} . Choose the positive roots of (\check{G}, \check{T}) as the positive coroots of $T \subset B \subset G$, so that the dominant weights of (\check{G}, \check{T}) are $X_*(T)^+$.

Given $\lambda \in X_*(T)^+$, we can consider the simple \check{G} -module $V_\lambda(\check{G})$ with highest weight λ , and the simple \check{G} -module $V_\lambda(\check{G}) = \mathbf{S}(\mathbf{IC}_\lambda)$ with highest weight λ . The point is that these two \check{T} -modules have the same weights; specifically, the set of weights of both of these modules is

$$\left\{ \mu \in X_*(T) \mid \begin{array}{l} \mu - \lambda \text{ is in the coroot lattice of } (G, T) \\ \text{and } \mu \text{ is in the convex hull of } W\lambda \end{array} \right\},$$

see again Proposition 5.8. (Note however that we do not know yet that these two \check{T} -modules have the same character.)

We now observe that

$$\{\lambda - \mu \mid \lambda \in X_*(T)^+, \mu \text{ a weight of } V_\lambda(\check{G})\}$$

is the \mathbf{N} -span of the positive roots of (\check{G}, \check{T}) . The argument just above shows that this is also the \mathbf{N} -span of the positive roots of (\check{G}, \check{T}) . Looking then at the indecomposable elements of this monoid, we deduce that the simple roots of \check{G} are the simple roots of \check{G} , i.e. the simple coroots of G . \square

We can finally conclude.

Theorem 9.6. *The group \check{G} is Langlands dual to G ; more precisely the root datum of \check{G} with respect to \check{T} is dual to the root datum of G with respect to T .*

Proof. By construction $X^*(\check{T})$ is dual to $X^*(T)$. What remains to be proved is that the roots and coroots of \check{G} , together with the canonical bijection between these two sets, coincide with the coroots and roots of G , together with their canonical bijection.

Let α be a simple root of $T \subset B \subset G$. Let $\check{\alpha}$ be the corresponding coroot. By Lemma 9.5, $\check{\alpha}$ is a simple root of $\check{T} \subset \check{B} \subset \check{G}$. The corresponding coroot $\check{\check{\alpha}}$ of \check{G} is \mathbf{Q}_+ -proportional to a simple root of $T \subset B \subset G$ by (9.1). The conditions

$$\begin{cases} \langle \check{\check{\alpha}}, \check{\alpha} \rangle = 2, \\ \langle \check{\check{\alpha}}, \check{\beta} \rangle \leq 0 \quad \text{for } \check{\beta} \text{ simple root of } \check{T} \subset \check{B} \subset \check{G}, \check{\beta} \neq \check{\alpha} \end{cases}$$

then give $\check{\check{\alpha}} = \alpha$.

We thus have an identification

$$\left\{ \begin{array}{l} \text{simple roots of} \\ T \subset B \subset G \end{array} \right\} = \left\{ \begin{array}{l} \text{simple coroots of} \\ \check{T} \subset \check{B} \subset \check{G} \end{array} \right\}.$$

By Lemma 9.5 we also have

$$\left\{ \begin{array}{l} \text{simple coroots of} \\ T \subset B \subset G \end{array} \right\} = \left\{ \begin{array}{l} \text{simple roots of} \\ \check{T} \subset \check{B} \subset \check{G} \end{array} \right\},$$

and the bijections between simple roots and simple coroots are the same. We may thus identify the Weyl groups of G and \check{G} and extend the above bijections between simple roots and coroots to bijections between roots and coroots, compatible with the bijections between roots and coroots, which finishes the proof. \square

A Equivariant perverse sheaves and convolution

A.1 Equivariant perverse sheaves

Let X be a complex algebraic variety, let H be a connected¹¹ algebraic group acting on X , and consider a field \mathbf{k} .¹² Let

$$a, p : H \times X \rightarrow X, \quad e : X \rightarrow H \times X$$

be the maps defined by

$$p(g, x) = x, \quad a(g, x) = g \cdot x, \quad e(x) = (1, x).$$

Let also $p_{23} : H \times H \times X \rightarrow H \times X$ be the projection on the last two components, and $m : H \times H \rightarrow H$ be the multiplication map.

Let \mathcal{T} be a Whitney stratification of X whose strata are stable under the H -action. Then there are at least 3 “reasonable” definitions of the category of \mathcal{T} -constructible H -equivariant perverse sheaves on X :

¹¹This assumption is crucial; in case H is disconnected, only the first definition of equivariant perverse sheaves has favorable properties.

¹²More generally, the same results apply when \mathbf{k} is an arbitrary commutative Noetherian ring of finite global dimension.

1. the heart $\mathbf{P}_{\mathcal{T},H}^\#(X, \mathbf{k})$ of the perverse t -structure on the \mathcal{T} -constructible equivariant derived category $D_{\mathcal{T},H}^b(X, \mathbf{k})$ in the sense of Bernstein–Lunts, see [BL, §5];
2. the category $\mathbf{P}_{\mathcal{T},H}^b(X, \mathbf{k})$ whose objects are pairs (\mathcal{F}, ϑ) where $\mathcal{F} \in \mathbf{P}_{\mathcal{T}}(X, \mathbf{k})$ and $\vartheta : a^* \mathcal{F} \rightarrow p^* \mathcal{F}$ is an isomorphism such that

$$e^*(\vartheta) = \text{id}_{\mathcal{F}} \quad \text{and} \quad (m \times \text{id}_X)^*(\vartheta) = (p_{23})^*(\vartheta) \circ (\text{id}_H \times a)^*(\vartheta), \quad (\text{A.1})$$

and whose morphisms from (\mathcal{F}, ϑ) to $(\mathcal{F}', \vartheta')$ are morphisms $f : \mathcal{F} \rightarrow \mathcal{F}'$ in $\mathbf{P}_{\mathcal{T}}(X, \mathbf{k})$ such that the following diagram commutes:

$$\begin{array}{ccc} a^* \mathcal{F} & \xrightarrow{\vartheta} & p^* \mathcal{F} \\ a^*(f) \downarrow & & \downarrow p^*(f) \\ a^* \mathcal{F}' & \xrightarrow{\vartheta'} & p^* \mathcal{F}' \end{array};$$

3. the full subcategory $\mathbf{P}_{\mathcal{T},H}(X, \mathbf{k})$ of $\mathbf{P}_{\mathcal{T}}(X, \mathbf{k})$ consisting of objects \mathcal{F} such that there exists an isomorphism $p^* \mathcal{F} \cong a^* \mathcal{F}$.

There exists an obvious forgetful functor $\mathbf{P}_{\mathcal{T},H}^b(X, \mathbf{k}) \rightarrow \mathbf{P}_{\mathcal{T},H}(X, \mathbf{k})$. There exists also a canonical forgetful functor

$$\mathbf{P}_{\mathcal{T},H}^\#(X, \mathbf{k}) \rightarrow \mathbf{P}_{\mathcal{T},H}^b(X, \mathbf{k}). \quad (\text{A.2})$$

Indeed, fix a free H -space P and a smooth $\dim(X)$ -acyclic map $\pi : P \rightarrow X$ of relative dimension d (which exist thanks to the results of [BL, §3.1]), and let $q : P \rightarrow P/H$ be the quotient morphism. Then $\mathbf{P}_{\mathcal{T},H}^\#(X, \mathbf{k})$ is (by definition) equivalent to the category whose objects are the triples $(\mathcal{F}_P, \mathcal{F}_X, \beta)$ where $\mathcal{F}_P \in D^b(P/H, \mathbf{k})$, $\mathcal{F}_X \in \mathbf{P}_{\mathcal{T}}(X, \mathbf{k})$ and $\beta : q^* \mathcal{F}_P \rightarrow \pi^* \mathcal{F}_X$ is an isomorphism, and whose morphisms from $(\mathcal{F}_P, \mathcal{F}_X, \beta)$ to $(\mathcal{F}'_P, \mathcal{F}'_X, \beta')$ are the pairs (f_P, f_X) with $f_P : \mathcal{F}_P \rightarrow \mathcal{F}'_P$ and $f_X : \mathcal{F}_X \rightarrow \mathcal{F}'_X$ compatible (in the natural sense) with β and β' . Consider the following diagram, where all squares are commutative (and in fact Cartesian):

$$\begin{array}{ccccc} & & H \times X & \xleftarrow{\text{id}_H \times \pi} & H \times P & \xrightarrow{\text{id}_H \times \pi} & H \times X & & \\ & & \swarrow p & & \swarrow p & & \searrow a & & \searrow a \\ X & \xleftarrow{\pi} & P & & P & & P & \xrightarrow{\pi} & X \\ & & \searrow q & & \searrow q & & & & \\ & & & & P/H & & & & \end{array}$$

Then we obtain a canonical isomorphism

$$(\text{id}_H \times \pi)^* p^* \mathcal{F}_X \cong p^* \pi^* \mathcal{F}_X \xrightarrow[\sim]{p^*(\beta^{-1})} p^* q^* \mathcal{F}_P \cong a^* q^* \mathcal{F}_P \xrightarrow[\sim]{a^*(\beta)} a^* \pi^* \mathcal{F}_X \cong (\text{id}_H \times \pi)^* a^* \mathcal{F}_X.$$

Since π is $\dim(X)$ -acyclic, the functor $(\text{id}_H \times \pi)^* : D^{[-\dim(X), 0]}(H \times X, \mathbf{k}) \rightarrow D^{[-\dim(X), 0]}(H \times P, \mathbf{k})$ is fully faithful, see [BL, Proposition 1.9.2]. Since $p^* \mathcal{F}_X$ and $a^* \mathcal{F}_X$ belong to the subcategory $D^{[-\dim(X), 0]}(H \times X, \mathbf{k})$, it follows that our isomorphism above is induced by an

isomorphism $\vartheta : a^* \mathcal{F}_X \rightarrow p^* \mathcal{F}_X$. We leave it to the reader to check that this isomorphism satisfies the conditions in (A.1); then we can define (A.2) as sending a triple $(\mathcal{F}_P, \mathcal{F}_X, \beta)$ to the pair $(\mathcal{F}_X, \vartheta)$, and a pair (f_P, f_X) to f_X .

The following result is well known, but not explicitly proved in the literature (to the best of our knowledge).

Proposition A.1. *The forgetful functors*

$$\mathbf{P}_{\mathcal{T}, H}^{\#}(X, \mathbf{k}) \rightarrow \mathbf{P}_{\mathcal{T}, H}^b(X, \mathbf{k}) \rightarrow \mathbf{P}_{\mathcal{T}, H}(X, \mathbf{k})$$

are equivalences of categories.

In view of this proposition, in the body of these notes we identify the three categories above, and denote them by $\mathbf{P}_{\mathcal{T}, H}(X, \mathbf{k})$.

In the proof of this proposition we will use the fact (see [BBD, Théorème 3.2.4]) that perverse sheaves form a stack for the smooth topology. In our particular case, if \mathcal{U} denotes the stratification on P whose strata are the subsets $\pi^{-1}(T)$ for $T \in \mathcal{T}$ and \mathcal{V} denotes the stratification on P/H whose strata are the subsets $q(U)$ with $U \in \mathcal{U}$, and if

$$r_1, r_2 : P \times_{P/H} P \rightarrow P, \quad r_{12}, r_{23}, r_{13} : P \times_{P/H} P \times_{P/H} P \rightarrow P \times_{P/H} P$$

are the natural projections, this means that the category $\mathbf{P}_{\mathcal{V}}(P/H, \mathbf{k})$ is equivalent, via the functor q^* , to the category whose objects are pairs (\mathcal{F}, σ) where $\mathcal{F} \in \mathbf{P}_{\mathcal{U}}(P, \mathbf{k})[-\dim(H)]$ and $\sigma : (r_1)^* \mathcal{F} \rightarrow (r_2)^* \mathcal{F}$ is an isomorphism such that $(r_{23})^*(\sigma) \circ (r_{12})^*(\sigma) = (r_{13})^*(\sigma)$, and whose morphisms $(\mathcal{F}, \sigma) \rightarrow (\mathcal{F}', \sigma')$ are morphisms $f : \mathcal{F} \rightarrow \mathcal{F}'$ such that $(r_2)^*(f) \circ \sigma = \sigma' \circ (r_1)^*(f)$.

With this result at hand we can give the proof of Proposition A.1.

Proof. The second functor is an equivalence by [Le, §4.2.10]. Hence what remains to prove is that the composition $\mathbf{P}_{\mathcal{T}, H}^{\#}(X, \mathbf{k}) \rightarrow \mathbf{P}_{\mathcal{T}, H}(X, \mathbf{k})$ is an equivalence.

First we show that this functor is faithful. Let $(f_P, f_X) : (\mathcal{F}_P, \mathcal{F}_X, \beta) \rightarrow (\mathcal{F}'_P, \mathcal{F}'_X, \beta')$ be a morphism in $\mathbf{P}_{\mathcal{T}, H}^{\#}(X, \mathbf{k})$ such that $f_X = 0$. Then by the compatibility of (f_P, f_X) with β and β' we deduce that $q^*(f_P) = 0$. Now it is easily seen that \mathcal{F}_P belongs to $\mathbf{P}_{\mathcal{V}}(P/H, \mathbf{k})[\dim(H) - d]$. Since q is smooth with connected fibers, the functor q^* is fully faithful on perverse sheaves (see [BBD, Proposition 4.2.5]); we deduce that $f_P = 0$, finishing the proof of faithfulness.

Next we prove that our functor is full. Let $(\mathcal{F}_P, \mathcal{F}_X, \beta)$ and $(\mathcal{F}'_P, \mathcal{F}'_X, \beta')$ be in $\mathbf{P}_{\mathcal{T}, H}^{\#}(X, \mathbf{k})$, and let $f : \mathcal{F}_X \rightarrow \mathcal{F}'_X$ be a morphism. To construct a morphism $f_P : \mathcal{F}_P \rightarrow \mathcal{F}'_P$ such that $\beta' \circ q^*(f_P) = \pi^*(f) \circ \beta$, we use the stack property recalled above: we remark that the morphism $(\beta')^{-1} \circ \pi^*(f) \circ \beta$ satisfies the descent condition, hence is of the form $q^*(f_P)$ for a unique isomorphism $f_P : \mathcal{F}_P \rightarrow \mathcal{F}'_P$.

Finally, we prove that our functor is essentially surjective. Let \mathcal{F} be in $\mathbf{P}_{\mathcal{T}, H}(X, \mathbf{k})$. Then there exists a (unique) isomorphism $\vartheta : a^*(\mathcal{F}) \rightarrow p^*(\mathcal{F})$ which satisfies the conditions (A.1). Identifying $H \times P$ with $P \times_{P/H} P$ via the morphism (a, p) , $(\text{id}_H \times \pi)^*(\vartheta)$ defines an isomorphism $\sigma : (r_1)^*(\pi^* \mathcal{F}) \rightarrow (r_2)^*(\pi^* \mathcal{F})$. Identifying $H \times H \times P$ with $P \times_{P/H} P \times_{P/H} P$ via $(g, h, x) \mapsto (ghx, hx, x)$, we see that the second condition in (A.1) guarantees that σ satisfies the descent condition, so that the pair $(\pi^* \mathcal{F}, \sigma)$ defines an object $\mathcal{F}_P \in D^b(P/H, \mathbf{k})$ such that $\pi^* \mathcal{F} \cong q^* \mathcal{F}_P$. \square

A.2 Convolution

Let H be a complex algebraic group, and let $K \subset H$ be a closed subgroup. Recall that the K -bundle given by the quotient morphism $H \rightarrow H/K$ is locally trivial for the analytic topology, see [Se]. (In all the cases we will consider, this morphism is in fact locally trivial for the Zariski topology.) We assume (for simplicity) that the decomposition of H/K into K -orbits forms a Whitney stratification, and consider the equivariant derived category $D_K^b(H/K, \mathbf{k})$ (constructible with respect to the stratification by K -orbits, which is automatic). This category admits a natural convolution bifunctor, constructed as follows. Consider the diagram

$$H/K \times H/K \xleftarrow{p} H \times H/K \xrightarrow{q} H \times^K H/K \xrightarrow{m} H/K, \quad (\text{A.3})$$

where $H \times^K H/K$ is the quotient of $H \times H/K$ by the action defined by $k \cdot (g, hK) = (gk^{-1}, khK)$ for $k \in K$ and $g, h \in H$, q is the quotient morphism, and the maps p and m are defined by

$$p(g, hK) = (gK, hK), \quad m([g, hK]) = ghK.$$

Since K acts freely on $H \times H/K$, by [BL, Theorem 2.6.2] the functor q^* induces an equivalence

$$D_K^b(H \times^K H/K, \mathbf{k}) \xrightarrow{\sim} D_{K \times K}^b(H \times H/K, \mathbf{k})$$

(where K acts on $H \times^K H/K$ via left multiplication on H , and $K \times K$ acts on $H \times H/K$ via $(k_1, k_2) \cdot (g, hK) = (k_1 g k_2^{-1}, k_2 h K)$). Now, consider some objects $\mathcal{F}_1, \mathcal{F}_2$ in $D_K^b(H/K, \mathbf{k})$. Then $p^*(\mathcal{F}_1 \boxtimes \mathcal{F}_2)$ defines in a natural way an object in $D_{K \times K}^b(H \times H/K, \mathbf{k})$. Hence there exists a unique object $\tilde{\mathcal{F}}_1 \boxtimes \tilde{\mathcal{F}}_2$ in $D_K^b(H \times^K H/K, \mathbf{k})$ such that

$$q^*(\tilde{\mathcal{F}}_1 \boxtimes \tilde{\mathcal{F}}_2) \cong p^*(\mathcal{F}_1 \boxtimes \mathcal{F}_2). \quad (\text{A.4})$$

We then set

$$\mathcal{F}_1 \star \mathcal{F}_2 := m_*(\tilde{\mathcal{F}}_1 \boxtimes \tilde{\mathcal{F}}_2).$$

It is a classical fact that this construction defines a monoidal structure on the category $D_K^b(H/K, \mathbf{k})$ (which does not, in general, restrict to a monoidal structure on $P_K(H/K, \mathbf{k})$).

Remark A.2. Since the maps p and q are smooth of relative dimension $\dim(K)$, we have canonical isomorphisms $p^! \cong p^*[\dim(K)]$ and $q^! \cong q^*[\dim(K)]$, so that the condition (A.4) can be replaced by $q^!(\tilde{\mathcal{F}}_1 \boxtimes \tilde{\mathcal{F}}_2) \cong p^!(\mathcal{F}_1 \boxtimes \mathcal{F}_2)$.

A.3 The case of Gr_G

The main object of study in these notes is the category $P_{G_{\mathcal{O}}}(\text{Gr}_G, \mathbf{k})$. This setting does not fit exactly in the framework of §§A.1–A.2 because $G_{\mathcal{K}}$ and $G_{\mathcal{O}}$ are not algebraic groups in the usual sense. But the category $D_{G_{\mathcal{O}}}^b(\text{Gr}_G, \mathbf{k})$ still makes sense, as follows.

For any $n \in \mathbf{Z}_{\geq 1}$, we denote by $H_n \subset G_{\mathcal{O}}$ the kernel of the morphism

$$G(\mathcal{O}) \rightarrow G(\mathcal{O}/t^n \mathcal{O})$$

induced by the quotient morphism $\mathcal{O} \rightarrow \mathcal{O}/t^n \mathcal{O}$. Note that if $m \geq n$, then H_m is a distinguished subgroup in H_n , and the quotient H_n/H_m is a unipotent group. If $X \subset \text{Gr}_G$ is a

closed finite union of $G_{\mathcal{O}}$ -orbits, there exists $n \in \mathbf{Z}_{\geq 1}$ such that H_n acts trivially on X . Then it makes sense to consider the equivariant derived category $D_{G_{\mathcal{O}}/H_n}^b(X, \mathbf{k})$. Since H_n/H_m is unipotent for any $m \geq n$, one can check using [BL, Theorem 3.7.3] that the functor

$$D_{G_{\mathcal{O}}/H_n}^b(X, \mathbf{k}) \rightarrow D_{G_{\mathcal{O}}/H_m}^b(X, \mathbf{k})$$

given by inverse image under the projection $G/H_m \rightarrow G/H_n$ is an equivalence of categories. Hence one can define the category $D_{G_{\mathcal{O}}}^b(X, \mathbf{k})$ to be $D_{G_{\mathcal{O}}/H_n}^b(X, \mathbf{k})$ for any n such that H_n acts trivially on X .

If $X \subset Y \subset \text{Gr}_G$ are closed finite unions of $G_{\mathcal{O}}$ -orbits, the direct image under the embedding $X \hookrightarrow Y$ induces a fully-faithful functor $D_{G_{\mathcal{O}}}^b(X, \mathbf{k}) \rightarrow D_{G_{\mathcal{O}}}^b(Y, \mathbf{k})$. Hence we can finally define $D_{G_{\mathcal{O}}}^b(\text{Gr}_G, \mathbf{k})$ as the union of the categories $D_{G_{\mathcal{O}}}^b(X, \mathbf{k})$ for all closed finite unions of $G_{\mathcal{O}}$ -orbits $X \subset \text{Gr}_G$.

A construction similar to that of §A.2 produces a convolution bifunctor \star on the category $D_{G_{\mathcal{O}}}^b(\text{Gr}_G, \mathbf{k})$. More precisely, if \mathcal{F}_1 and \mathcal{F}_2 are in $D_{G_{\mathcal{O}}}^b(\text{Gr}_G, \mathbf{k})$, one should choose a closed finite union of $G_{\mathcal{O}}$ -orbits $X \subset \text{Gr}_G$ such that \mathcal{F}_2 belongs to $D_{G_{\mathcal{O}}}^b(X, \mathbf{k})$, and $n \in \mathbf{Z}_{\geq 1}$ such that H_n acts trivially on X , and replace the diagram (A.3) by the similar diagram

$$\text{Gr}_G \times X \leftarrow G_{\mathcal{K}}/H_n \times X \rightarrow (G_{\mathcal{K}}/H_n) \times^{(G_{\mathcal{O}}/H_n)} X \rightarrow \text{Gr}_G,$$

and proceed as before. In the body of the paper, as in [MV], to avoid heavy notation we neglect these technical subtleties.

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