Convergence rates in the central limit theorem for weighted sums of Bernoulli random fields

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CONVERGENCE RATES IN THE CENTRAL LIMIT THEOREM FOR WEIGHTED SUMS OF BERNOULLI RANDOM FIELDS

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Abstract. We prove moment inequalities for a class of functionals of i.i.d. random fields. We then derive rates in the central limit theorem for weighted sums of such randoms fields via an approximation by m-dependent random fields.

1. Introduction and main results

1.1. **Rosenthal's inequality.** By Theorem 3 in [Ros70], we know that for any $p > 2$, there exists a constant C depending only on p such that for any finite sequence $(Y_i)_{i=1}^n$ of independent centered random variables with a finite moment of order p, then

$$\left\|\sum_{i=1}^n Y_i\right\|_p \leqslant C\left(\left(\sum_{i=1}^n \|Y_i\|_2^2\right)^{1/2} + \left(\sum_{i=1}^n \|Y_i\|_p^p\right)^{1/p}\right), \tag{1.1.1}$$

where $\|Y\|_q := \left(\mathbb{E}\left[|Y|^q\right]\right)^{1/q}$ for $q \geqslant 1$.

In order to give explicit constants, we will use the following version of Rosenthal's inequality due to Johnson, Schechtman and Zinn [JSZ85]: if $(Y_i)_{i=1}^n$ are independent centered random variables with a finite moment of order $p \geqslant 2$, then

$$\left\|\sum_{i=1}^n Y_i\right\|_p \leqslant \frac{14.5p}{\log p}\left(\left(\sum_{i=1}^n \|Y_i\|_2^2\right)^{1/2} + \left(\sum_{i=1}^n \|Y_i\|_p^p\right)^{1/p}\right). \tag{1.1.2}$$

Various extension of Rosenthal-type inequalities have been obtained under mixing conditions [Sha95, Rio00] or projective conditions [PUW07, Rio09, MP13]. We are interested by extensions of (1.1.2) to the setting of dependent random fields.

One of the main application of such an inequality is the estimate of the convergence rates in the central limit theorem for random fields which can be expressed as a functional of an i.i.d. one. The method consists in approximating the considered random field by an m-dependent one, and in controlling the approximation with the help of the established moment inequality.

1.2. **Notations and conventions.** In all the paper, we shall use the following notations.

(N.1) For a positive integer d, the set $\{1, \dots, d\}$ is denoted by $[d]$.

(N.2) The coordinatewise order is denoted by $\preccurlyeq$, that is, for $\mathbf{i} = (i_q)_{q=1}^d \in \mathbb{Z}^d$ and $\mathbf{j} = (j_q)_{q=1}^d \in \mathbb{Z}^d$, $\mathbf{i} \preccurlyeq \mathbf{j}$ means that $i_k \leqslant j_k$ for any $k \in [d]$.

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(N.3) For $k \in [d]$, $\mathbf{e_k}$ denotes the element of $\mathbb{Z}^d$ whose qth coordinate is 1 and all the others are zero. Moreover, we write $\mathbf{0} = (0, \dots, 0)$ and $\mathbf{1} = (1, \dots, 1)$.
(N.4) For $\mathbf{n} = (n_k)_{k=1}^d \in \mathbb{N}^d$, we write the product $\prod_{k=1}^d n_q$ as $|\mathbf{n}|$.
(N.5) The cardinality of a set I is denoted by $|I|$.
(N.6) For a real number x, we denote by $[x]$ the unique integer such that $[x] \leqslant x < [x] + 1$.
(N.7) We write Φ for the cumulative distribution function of a standard normal law.
(N.8) If Λ is a subset of $\mathbb{Z}^d$ and $\mathbf{k} \in \mathbb{Z}^d$, then $\Lambda - \mathbf{k}$ is defined as $\{\mathbf{l} - \mathbf{k}, \mathbf{l} \in \Lambda\}$.
(N.9) For $q \geqslant 1$, we denote by $\ell^q\left(\mathbb{Z}^d\right)$ the space of sequences $\mathbf{a} := (a_\mathbf{i})_{\mathbf{i} \in \mathbb{Z}^d}$ such that $\|\mathbf{a}\|_{\ell^q} := \left(\sum_{\mathbf{i} \in \mathbb{Z}^d} |a_\mathbf{i}|^q\right)^{1/q} < +\infty$.
(N.10) For $\mathbf{i} = (i_q)_{q=1}^d$, the quantity $\|\mathbf{i}\|_\infty$ is defined as $\max_{1 \leqslant q \leqslant d} |i_q|$.

Let $(Y_\mathbf{i})_{\mathbf{i} \in \mathbb{Z}^d}$ be a random field. The sum $\sum_{\mathbf{i} \in \mathbb{Z}^d} Y_\mathbf{i}$ is understood as the $\mathbb{L}^p$ limit of the sequence $(S_k)_{k \geqslant 1}$ where $S_k = \sum_{\mathbf{i} \in \mathbb{Z}^d, \|\mathbf{i}\|_\infty \leqslant k} Y_\mathbf{i}$.

1.3. **Physical dependence measure.** In the sequel, we will be interested in the so-called Bernoulli random fields.

Definition 1.1. *Let $d \geqslant 1$ be an integer. The random field $(X_\mathbf{n})_{\mathbf{n} \in \mathbb{Z}^d}$ is said to be Bernoulli if there exists an i.i.d. random field $(\varepsilon_\mathbf{i})_{\mathbf{i} \in \mathbb{Z}^d}$ and a measurable function $f \colon \mathbb{R}^{\mathbb{Z}^d} \to \mathbb{R}$ such that $X_\mathbf{n} = f\left((\varepsilon_{\mathbf{n}-\mathbf{i}})_{\mathbf{i} \in \mathbb{Z}^d}\right)$ for each $\mathbf{n} \in \mathbb{Z}^d$.*

Following [Wu05] we define the physical dependence measure.

Definition 1.2. *Let $(X_\mathbf{i})_{\mathbf{i} \in \mathbb{Z}^d} := (f(\varepsilon_{\mathbf{i}-\mathbf{j}}))_{\mathbf{j} \in \mathbb{Z}^d}$ be a Bernoulli random field, $p \geqslant 1$ and $(\varepsilon'_\mathbf{u})_{\mathbf{u} \in \mathbb{Z}^d}$ be an i.i.d. random field which is independent of the i.i.d. random field $(\varepsilon_\mathbf{u})_{\mathbf{u} \in \mathbb{Z}^d}$ and has the same distribution as $(\varepsilon_\mathbf{u})_{\mathbf{u} \in \mathbb{Z}^d}$. For $\mathbf{i} \in \mathbb{Z}^d$, we introduce the physical dependence measure*

$$\delta_{\mathbf{i},p} := \|X_\mathbf{i} - X_\mathbf{i}^*\|_p \tag{1.3.1}$$

where $X_\mathbf{i}^* = f\left(\varepsilon^*_{\mathbf{i}-\mathbf{j}}\right)$ and $\varepsilon^*_\mathbf{u} = \varepsilon_\mathbf{u}$ if $\mathbf{u} \neq \mathbf{0}$, $\varepsilon^*_\mathbf{0} = \varepsilon'_\mathbf{0}$.

In [EVW13, BD14], various examples of Bernoulli random fields are given, for which the physical dependence measure is either computed or estimated. Proposition 1 of [EVW13] also gives the following moment inequality: if Γ is a finite subset of $\mathbb{Z}^d$, $(a_\mathbf{i})_{\mathbf{i} \in \Gamma}$ is a family of real numbers and $p \geqslant 2$, then for any Bernoulli random field $(X_\mathbf{n})_{\mathbf{n} \in \mathbb{Z}^d}$,

$$\left\|\sum_{\mathbf{i} \in \Gamma} a_\mathbf{i} X_\mathbf{i}\right\|_p \leqslant \left(2p \sum_{\mathbf{i} \in \Gamma} a_\mathbf{i}^2\right)^{1/2} \cdot \sum_{\mathbf{j} \in \mathbb{Z}^d} \delta_{\mathbf{j},p}. \tag{1.3.2}$$

This was used in [EVW13, BD14] in order to establish functional central limit theorems. Truquet [Tru10] also obtained an inequality in this spirit. However, we cannot recover Rosenthal's inequality (1.1.1) from (1.3.2). If $(X_\mathbf{i})_{\mathbf{i} \in \mathbb{Z}^d}$ is i.i.d. and centered, (1.1.1) would give

$$\left\|\sum_{\mathbf{i} \in \Gamma} a_\mathbf{i} X_\mathbf{i}\right\|_p \leqslant C \left(\sum_{\mathbf{i} \in \Gamma} a_\mathbf{i}^2\right)^{1/2} \sum_{\mathbf{j} \in \mathbb{Z}^d} \delta_{\mathbf{j},2} + C \left(\sum_{\mathbf{i} \in \Gamma} |a_\mathbf{i}|^p\right)^{1/p} \sum_{\mathbf{j} \in \mathbb{Z}^d} \delta_{\mathbf{j},p}. \tag{1.3.3}$$

In the case of linear processes, equality $\delta_{\mathbf{j},p} \leqslant K \delta_{\mathbf{j},2}$ holds for a constant K which does not depend on $\mathbf{j}$. However, there are processes for which such an inequality does not hold.

Example 1.3. Let $(\varepsilon_{\mathbf{i}})_{\mathbf{i}\in\mathbb{Z}^d}$ be an i.i.d. random field and for each $\mathbf{k}\in\mathbb{Z}^d$, let $f_{\mathbf{k}}\colon\mathbb{R}\to\mathbb{R}$ be a function such that the random variable $Z_{\mathbf{k}}:=f_{\mathbf{k}}(\varepsilon_{\mathbf{0}})$ is centered and has a finite moment of order p, and $\sum_{\mathbf{k}\in\mathbb{Z}^d}\|Z_{\mathbf{k}}\|_2^2<+\infty$. Define $X_{\mathbf{n}}:=\lim_{N\to+\infty}\sum_{-N\mathbf{1}\preccurlyeq\mathbf{j}\preccurlyeq N\mathbf{1}}f_{\mathbf{k}}(\varepsilon_{\mathbf{n}-\mathbf{k}})$, where the limit is taken in $\mathbb{L}^2$. Then $X_{\mathbf{i}}-X_{\mathbf{i}}^*=f_{\mathbf{i}}(\varepsilon_{\mathbf{0}})-f_{\mathbf{i}}(\varepsilon'_{\mathbf{0}})$ hence $\delta_{\mathbf{i},2}$ is of order $\|Z_{\mathbf{i}}\|_2$ while $\delta_{\mathbf{i},p}$ is of order $\|Z_{\mathbf{i}}\|_p$.

Consequently, having the ℓ^p-norm instead of the ℓ^2-norm of the $(a_{\mathbf{i}})_{\mathbf{i}\in\Gamma}$ is more suitable. The goal of this paper is to get Rosenthal-like inequalities for weighted sums of Bernoulli random fields. In the one dimensional case, probability and moment inequalities have been established in [LXW13] for maxima of partial sums of Bernoulli sequences. The techniques used therein permit to derive results for weighted sums of such sequences.

1.4. **Moment inequality.** We now give a Rosenthal-like inequality for weighted sums of Bernoulli random fields in terms of the physical dependence measure.

Theorem 1.4. *Let $\{\varepsilon_{\mathbf{i}},\mathbf{i}\in\mathbb{Z}^d\}$ be an i.i.d. set of random variables. Then for any measurable function $f\colon\mathbb{R}^{\mathbb{Z}^d}\to\mathbb{R}$ such that $X_{\mathbf{j}}:=f((X_{\mathbf{j}-\mathbf{i}}))_{i\in\mathbb{Z}^d}$ has a finite moment of order $p\geqslant 2$ and is centered, and any $(a_{\mathbf{i}})_{\mathbf{i}\in\mathbb{Z}^d}\in\ell^2(\mathbb{Z}^d)$,*

$$\left\|\sum_{\mathbf{i}\in\mathbb{Z}^d}a_{\mathbf{i}}X_{\mathbf{i}}\right\|_p\leqslant\frac{14.5p}{\log p}\left(\sum_{\mathbf{i}\in\mathbb{Z}^d}a_{\mathbf{i}}^2\right)^{1/2}\sum_{j=0}^{+\infty}(4j+4)^{d/2}\|X_{\mathbf{0},j}\|_2 + \frac{14.5p}{\log p}\left(\sum_{\mathbf{i}\in\mathbb{Z}^d}|a_{\mathbf{i}}|^p\right)^{1/p}\sum_{j=0}^{+\infty}(4j+4)^{d(1-1/p)}\|X_{\mathbf{0},j}\|_p, \quad (1.4.1)$$

where for $j\geqslant 1$,

$$X_{\mathbf{0},j}=\mathbb{E}[X_{\mathbf{0}}\mid\sigma\{\varepsilon_{\mathbf{u}},\|\mathbf{u}\|_\infty\leqslant j\}]-\mathbb{E}[X_{\mathbf{0}}\mid\sigma\{\varepsilon_{\mathbf{u}},\|\mathbf{u}\|_\infty\leqslant j-1\}] \quad (1.4.2)$$

and $X_{\mathbf{0},0}=\mathbb{E}[X_{\mathbf{0}}\mid\sigma\{\varepsilon_{\mathbf{0}}\}]$.

Corollary 1.5. *Let $\{\varepsilon_{\mathbf{i}},\mathbf{i}\in\mathbb{Z}^d\}$ be an i.i.d. set of random variables. Then for any measurable function $f\colon\mathbb{R}^{\mathbb{Z}^d}\to\mathbb{R}$ such that $X_{\mathbf{j}}:=f((X_{\mathbf{j}-\mathbf{i}}))_{i\in\mathbb{Z}^d}$ has a finite moment of order $p\geqslant 2$ and is centered, and any $(a_{\mathbf{i}})_{\mathbf{i}\in\mathbb{Z}^d}\in\ell^2(\mathbb{Z}^d)$,*

$$\left\|\sum_{\mathbf{i}\in\mathbb{Z}^d}a_{\mathbf{i}}X_{\mathbf{i}}\right\|_p\leqslant\sqrt{2}\frac{14.5p}{\log p}\left(\sum_{\mathbf{i}\in\mathbb{Z}^d}a_{\mathbf{i}}^2\right)^{1/2}\sum_{j=0}^{+\infty}(4j+4)^{d/2}\left(\sum_{\|\mathbf{i}\|_\infty=j}\delta_{\mathbf{i},2}^2\right)^{1/2} + \sqrt{2}\frac{14.5p}{\log p}\sqrt{p-1}\left(\sum_{\mathbf{i}\in\mathbb{Z}^d}|a_{\mathbf{i}}|^p\right)^{1/p}\sum_{j=0}^{+\infty}(4j+4)^{d(1-1/p)}\left(\sum_{\|\mathbf{i}\|_\infty=j}\delta_{\mathbf{i},p}^2\right)^{1/2}. \quad (1.4.3)$$

Remark 1.6. Like in [EM02], the fact that the constants are explicit allows to derive Kahane-Khintchine inequalities, that is, inequalities on the Orlicz norm associated to the Young function

$$\psi_\beta\colon x\mapsto\exp\left((x+h_\beta)^\beta\right)-\exp(h_\beta), \quad (1.4.4)$$

where $\beta>0$ and $h_\beta=((1-\beta)/\beta)^{1/\beta}$ if $0<\beta<1$ and 0 otherwise.

Remark 1.7. Using Remark 7 in [JS88] and Rosenthal's inequality in weak $\mathbb{L}^p$-spaces, it is possible to obtain a version of Theorem 1.4 where the $\mathbb{L}^p$ norm is replaced by $\|X\|_{p,\infty} := \sup_{A\in\mathcal{F}} \mathbb{P}(A)^{1/p-1}\mathbb{E}\left[|X|\,\mathbf{1}_A\right]$.

Remark 1.8. We can bound the $\mathbb{L}^q$-norm of $X_{\mathbf{0},j}$ by $\sum_{\mathbf{i}\in\mathbb{Z}^d:\|i\|_\infty=j} \|\|$ and formulate a version of inequality (1.4.1) where the right hand side is expressed in terms of the coefficients of physical dependence measure. The obtained result is not directly comparable to (1.3.2) because of the presence of the ℓ^p-norm of the coefficients.

1.5. **Convergence rates in the central limit theorem for weighted sums.** Let $(X_{\mathbf{j}})_{\mathbf{j}\in\mathbb{Z}^d} = f\left((\varepsilon_{\mathbf{j}-\mathbf{i}})_{\mathbf{i}\in\mathbb{Z}^d}\right)$ be a centered square integrable Bernoulli random field and for any positive integer n, let $b_n := (b_{n,\mathbf{i}})_{\mathbf{i}\in\mathbb{Z}^d}$ be an element of $\ell^2\left(\mathbb{Z}^d\right)$. We are interested in the asymptotic behavior of the sequence $(S_n)_{n\geqslant 1}$ defined by

$$S_n := \sum_{\mathbf{i}\in\mathbb{Z}^d} b_{n,\mathbf{i}} X_{\mathbf{i}}. \tag{1.5.1}$$

Let us denote for $\mathbf{k}\in\mathbb{Z}^d$ the map $\tau_{\mathbf{k}}\colon \ell^2\left(\mathbb{Z}^d\right)\to\ell^2\left(\mathbb{Z}^d\right)$ defined by $\tau_{\mathbf{k}}\left((x_{\mathbf{i}})_{\mathbf{i}\in\mathbb{Z}^d}\right) := (x_{\mathbf{i}+\mathbf{k}})_{\mathbf{i}\in\mathbb{Z}^d}$.

In [KVW16], Corollary 2.6 gives the following result: under an appropriated dependence condition on the random field $(X_{\mathbf{i}})_{\mathbf{i}\in\mathbb{Z}^d}$ and under the following condition on the weights: for any $q\in[d]$,

$$\frac{1}{\|b_n\|_{\ell^2}}\left\|\tau_{\mathbf{e_q}}(b_n)-b_n\right\|_{\ell^2}=0, \tag{1.5.2}$$

the series $\sum_{\mathbf{i}\in\mathbb{Z}^d}\left|\mathrm{Cov}\left(X_{\mathbf{0}},X_{\mathbf{i}}\right)\right|$ converges and denoting

$$\sigma := \left(\sum_{\mathbf{i}\in\mathbb{Z}^d}\mathrm{Cov}\left(X_{\mathbf{0}},X_{\mathbf{i}}\right)\right)^{1/2}, \tag{1.5.3}$$

the sequence $\left(S_n/\left\|b_n\right\|_{\ell^2}\right)_{n\geqslant 1}$ converges in distribution to a centered normal distribution with variance σ^2. The argument rests on an approximation by an m-dependent random field.

Theorem 1.9. *Let $p>2$, $p' := \min\{p,3\}$ and let $(X_{\mathbf{j}})_{\mathbf{j}\in\mathbb{Z}^d} = f\left((\varepsilon_{\mathbf{j}-\mathbf{i}})_{\mathbf{i}\in\mathbb{Z}^d}\right)$ be a centered Bernoulli random field with a finite moment of order p and for any positive integer n, let $b_n := (b_{n,\mathbf{i}})_{\mathbf{i}\in\mathbb{Z}^d}$ be an element of $\ell^2\left(\mathbb{Z}^d\right)$ such that for any $n\geqslant 1$, the set $\left\{\mathbf{k}\in\mathbb{Z}^d, b_{n,\mathbf{k}}\neq 0\right\}$ is finite and nonempty, $\lim_{n\to+\infty}\|b_n\|_{\ell^2}=+\infty$ and (1.5.2) holds for any $q\in[d]$. Assume that for some positive α and β, the following series are convergent:*

$$C_2(\alpha) := \sum_{i=0}^{+\infty}(i+1)^{d/2+\alpha}\left\|X_{\mathbf{0},i}\right\|_2 \text{ and } C_p(\beta) := \sum_{i=0}^{+\infty}(i+1)^{d(1-1/p)+\beta}\left\|X_{\mathbf{0},i}\right\|_p. \tag{1.5.4}$$

Let S_n be defined by (1.5.1),

$$\Delta_n := \sup_{t\in\mathbb{R}}\left|\mathbb{P}\left\{\frac{S_n}{\|b_n\|_{\ell^2}}\leqslant t\right\}-\Phi(t/\sigma)\right| \text{ and} \tag{1.5.5}$$

$$\varepsilon_n := \sum_{\mathbf{j}\in\mathbb{Z}^d}\mathbb{E}\left[X_{\mathbf{0}}X_{\mathbf{j}}\right]\left(\sum_{\mathbf{i}\in\mathbb{Z}^d}\frac{b_{n,\mathbf{i}}b_{n,\mathbf{i}+\mathbf{j}}}{\|b_n\|_{\ell^2}}-1\right). \tag{1.5.6}$$

Assume that $\sum_{\mathbf{i}\in\mathbb{Z}^d}|\mathrm{Cov}\left(X_{\mathbf{0}},X_{\mathbf{i}}\right)|$ *is finite and that* σ *be given by* (1.5.3) *is positive. Let* $\gamma>0$ *and let*

$$n_0:=\inf\left\{N\geqslant 1\mid\forall n\geqslant N,\sqrt{\sigma^2+\varepsilon_n}-20\left(\log 2\right)^{-1}C_2\left(\alpha\right)\left(\left[\|b_n\|_{\ell^2}\right]^{\gamma}\right)^{-\alpha}\geqslant\sigma/2\right\}. \tag{1.5.7}$$

Then for each $n\geqslant n_0$,

$$\begin{aligned}\Delta_n\leqslant{}&150\left(20\left(\left[\|b_n\|_{\ell^2}\right]+21\right)^{\gamma}+21\right)^{\left(p'-1\right)d}\|X_{\mathbf{0}}\|_{p'}^{p'}\left(\frac{\|b_n\|_{\ell^{p'}}}{\|b_n\|_{\ell^2}}\right)^{p'}\left(\sigma/2\right)^{-p'}\\&+\left(2\frac{|\varepsilon_n|}{\sigma^2}+80\left(\log 2\right)^{-1}\frac{\|b_n\|_{\ell^2}^{-\gamma\alpha}}{\sigma^2}C_2\left(\alpha\right)^2\right)\left(2\pi e\right)^{-1/2}\\&+\left(\frac{14.5p}{\sigma\log p}4^{d/2}\|b_n\|_{\ell^2}^{-\gamma\alpha}C_2\left(\alpha\right)\right)^{\frac{p}{p+1}}+\left(\frac{\|b_n\|_{\ell^p}}{\sigma\|b_n\|_{\ell^2}}\frac{14.5p}{\log p}4^{d(1-1/p)}\|b_n\|_{\ell^2}^{-\gamma\beta}C_p\left(\beta\right)\right)^{\frac{p}{p+1}}.\end{aligned} \tag{1.5.8}$$

In particular, there exists a constant κ *such that for all* $n\geqslant n_0$,

$$\Delta_n\leqslant\kappa\left(\|b_n\|_{\ell^2}^{\gamma\left(p'-1\right)d-p'}\|b_n\|_{\ell^{p'}}^{p'}+|\varepsilon_n|+\|b_n\|_{\ell^2}^{-\gamma\alpha\frac{p}{p+1}}+\|b_n\|_{\ell^p}^{\frac{p}{p+1}}\|b_n\|_{\ell^2}^{-\frac{p}{p+1}(\gamma\beta+1)}\right). \tag{1.5.9}$$

Remark 1.10. If (1.5.2), $\lim_{n\to+\infty}\|b_n\|_{\ell^2}=+\infty$ and the family $(\delta_{\mathbf{i},2})_{\mathbf{i}\in\mathbb{Z}^d}$ is summable, then the sequence $(\varepsilon_n)_{n\geqslant 1}$ converges to 0 hence n_0 is well-defined. However, it is not clear to us whether the finiteness of of $C_2\left(\alpha\right)$ combined with (1.5.2) and $\lim_{n\to+\infty}\|b_n\|_{\ell^2}=+\infty$ imply that $\sum_{\mathbf{j}\in\mathbb{Z}^d}|\mathbb{E}\left[X_{\mathbf{0}}X_{\mathbf{j}}\right]|$ is finite. Nevertheless, we can use show an analogous result in terms of $\delta_{\mathbf{i},p}$ coefficients by changing the following in the statement of Theorem 1.9:

(1) the definition of $C_2\left(\alpha\right)$ should be replaced by

$$C_2\left(\alpha\right):=\sqrt{2}\sum_{j=0}^{+\infty}\left(j+1\right)^{d/2+\alpha}\left(\sum_{\|\mathbf{i}\|=j}\delta_{\mathbf{j},2}^2\right)^{1/2}; \tag{1.5.10}$$

(2) the definition of $C_p\left(\beta\right)$ should be replaced by

$$C_p\left(\beta\right):=\sqrt{2\left(p-1\right)}\sum_{j=0}^{+\infty}\left(j+1\right)^{d(1-1/p)+\beta}\left(\sum_{\|\mathbf{i}\|=j}\delta_{\mathbf{j},2}^2\right)^{1/2}. \tag{1.5.11}$$

In this case, the convergence of $\sum_{\mathbf{i}\in\mathbb{Z}^d}|\mathrm{Cov}\left(X_{\mathbf{0}},X_{\mathbf{i}}\right)|$ holds (cf. Proposition 2 in [EVW13]).

Recall notation (N.8). Let $(\Lambda_n)_{n\geqslant 1}$ be a sequence of subsets of $\mathbb{Z}^d$. The choice $b_{n,\mathbf{j}}=1$ if $\mathbf{j}\in\Lambda_n$ and 0 otherwise yields the following corollary for set-indexed partial sums.

Corollary 1.11. *Let* $(X_{\mathbf{i}})_{\mathbf{i}\in\mathbb{Z}^d}$ *be a centered Bernoulli random field with a finite moment of order* $p\geqslant 2$, $p':=\min\left\{p,3\right\}$ *and let* $(\Lambda_n)_{n\geqslant 1}$ *be a sequence of subset of* $\mathbb{Z}^d$ *such that* $|\Lambda_n|\to+\infty$ *and for any* $\mathbf{k}\in\mathbb{Z}^d$, $\lim_{n\to+\infty}|\Lambda_n\cap(\Lambda_n-\mathbf{k})|/|\Lambda_n|=0$. *Assume that the series defined in* (1.5.4) *are convergent for some positive* α *and* β, *that* $\sum_{\mathbf{i}\in\mathbb{Z}^d}|\mathrm{Cov}\left(X_{\mathbf{0}},X_{\mathbf{i}}\right)|$ *is finite and that* σ *defined by* (1.5.3) *is positive. Let* $\gamma>0$ *and* n_0 *be defined by* (1.5.7). *There exists a constant* κ *such that for any* $n\geqslant n_0$,

$$\sup_{t\in\mathbb{R}}\left|\mathbb{P}\left\{\frac{\sum_{\mathbf{i}\in\Lambda_n}X_{\mathbf{i}}}{|\Lambda_n|^{1/2}}\leqslant t\right\}-\Phi\left(t/\sigma\right)\right|$$
$$\leqslant\kappa\left(|\Lambda_n|^q+\sum_{\mathbf{j}\in\mathbb{Z}^d}\left|\mathbb{E}\left[X_{\mathbf{0}}X_{\mathbf{j}}\right]\right|\left|\frac{|\Lambda_n\cap(\Lambda_n-\mathbf{j})|}{|\Lambda_n|}-1\right|\right),\quad(1.5.12)$$

where

$$q:=\max\left\{\frac{\gamma\left(p'-1\right)d-p'}{2}+1;-\gamma\alpha\frac{p}{2\left(p+1\right)};\frac{1-p-p\gamma\beta}{p+1}\right\}.\quad(1.5.13)$$

1.6. **Rate of convergence of kernel estimates in a regression model.** We consider the following regression model:

$$Y_{\mathbf{i}}=g\left(\frac{\mathbf{i}}{n}\right)+X_{\mathbf{i}},\quad\mathbf{i}\in\Lambda_n:=\{1,\dots,n\}^d,\quad(1.6.1)$$

where $g\colon[0,1]^d\to\mathbb{R}$ is an unknown smooth function and $(X_{\mathbf{i}})_{\mathbf{i}\in\mathbb{Z}^d}$ is a zero mean stationary Bernoulli random field. Let K be a probability kernel defined on $\mathbb{R}^d$ and let $(h_n)_{n\geqslant1}$ be a sequence of positive numbers which converges to zero and which satisfies

$$\lim_{n\to+\infty}nh_n=+\infty\text{ and }\lim_{n\to+\infty}nh_n^{d+1}=0.\quad(1.6.2)$$

We estimate the function g by the kernel estimator g_n defined by

$$g_n\left(\mathbf{x}\right)=\frac{\sum_{\mathbf{i}\in\Lambda_n}Y_{\mathbf{i}}K\left(\frac{\mathbf{x}-\mathbf{i}/n}{h_n}\right)}{\sum_{\mathbf{i}\in\Lambda_n}K\left(\frac{\mathbf{x}-\mathbf{i}/n}{h_n}\right)},\quad x\in[0,1]^d,\quad(1.6.3)$$

where

$$\Lambda_n=\{1,\dots,n\}^d.\quad(1.6.4)$$

We make the following assumptions on the regression function g and the probability kernel K:

- (A1) The probability kernel K fulfills $\int_{\mathbb{R}^d}K\left(\mathbf{u}\right)\mathrm{d}\mathbf{u}=1$, is symmetric, non-negative, supported by $[-1,1]^d$. Moreover, there exist positive constants r, c and C such that for any $\mathbf{x},\mathbf{y}\in[-1,1]^d$, $|K\left(\mathbf{x}\right)-K\left(\mathbf{y}\right)|\leqslant r\left\|\mathbf{x}-\mathbf{y}\right\|_\infty$ and $c\leqslant K\left(\mathbf{x}\right)\leqslant C$.
- (A2) There exists a positive constant B such that the absolute values of all the derivatives of first order of g are bounded by B on $[0,1]^d$.

Assumption (A2) will not be used in the following result. However, by Proposition 1 in [EM07], this guarantees that

$$\sup_{\mathbf{x}\in[0,1]^d}\sup_{g\in\mathrm{Lip}(B)}\left|\mathbb{E}\left[g_n\left(\mathbf{x}\right)\right]-g\left(\mathbf{x}\right)\right|=O\left(h_n\right),\quad(1.6.5)$$

where $\mathrm{Lip}\left(B\right)$ denotes the collection of all B-Lipschitz functions on $\mathbb{R}^d$.

Theorem 1.12. *Let $p>2$, $p':=\min\{p,3\}$ and let $(X_{\mathbf{j}})_{\mathbf{j}\in\mathbb{Z}^d}=f\left((\varepsilon_{\mathbf{j}-\mathbf{i}})_{\mathbf{i}\in\mathbb{Z}^d}\right)$ be a centered Bernoulli random field with a finite moment of order p. Assume that for some positive α and β, the following series are convergent:*

$$C_2\left(\alpha\right):=\sum_{i=0}^{+\infty}\left(i+1\right)^{d/2+\alpha}\left\|X_{\mathbf{0},i}\right\|_2\text{ and }C_p\left(\beta\right):=\sum_{i=0}^{+\infty}\left(i+1\right)^{d(1-1/p)+\beta}\left\|X_{\mathbf{0},i}\right\|_p.\quad(1.6.6)$$

Let $g_n(\mathbf{x})$ *be defined by* (1.6.3), $(h_n)_{n\geqslant 1}$ *be a sequence which converges to* 0 *and satisfies* (1.6.2),

$$\Delta_n := \sup_{t\in\mathbb{R}} \left| \mathbb{P}\left\{ (nh_n)^{d/2} \left(g_n(\mathbf{x}) - \mathbb{E}\left[g_n(\mathbf{x}) \right] \right) \leqslant t \right\} - \Phi\left(\frac{t}{\sigma \left\| K \right\|_2} \right) \right|, \tag{1.6.7}$$

$$A_n := (nh_n)^{d/2} \left(\sum_{\mathbf{i}\in\Lambda_n} K^2\left(\frac{\mathbf{x}-\mathbf{i}/n}{h_n} \right) \right)^{1/2} \left\| K \right\|_{\mathbb{L}^2(\mathbb{R}^d)}^{-1} \left(\sum_{\mathbf{i}\in\Lambda_n} K\left(\frac{\mathbf{x}-\mathbf{i}/n}{h_n} \right) \right)^{-1/2} \quad \textit{and} \tag{1.6.8}$$

$$\varepsilon_n := \sum_{\mathbf{j}\in\mathbb{Z}^d} \mathbb{E}\left[X_{\mathbf{0}} X_{\mathbf{j}} \right] \left(\sum_{\mathbf{i}\in\Lambda_n\cap(\Lambda_n-\mathbf{j})} \frac{K\left(\frac{\mathbf{x}-\mathbf{i}/n}{h_n} \right) K\left(\frac{\mathbf{x}-(\mathbf{i}-\mathbf{j})/n}{h_n} \right)}{\sum_{\mathbf{k}\in\Lambda_n} K^2\left(\frac{\mathbf{x}-\mathbf{k}/n}{h_n} \right)} - 1 \right). \tag{1.6.9}$$

Assume that $\sum_{\mathbf{i}\in\mathbb{Z}^d} \left| \mathrm{Cov}\left(X_{\mathbf{0}}, X_{\mathbf{i}} \right) \right|$ *is finite and that* $\sigma := \sum_{\mathbf{j}\in\mathbb{Z}^d} \mathrm{Cov}\left(X_{\mathbf{0}}, X_{\mathbf{j}} \right) > 0$. *Let* $n_1 \in \mathbb{N}$ *be such that for each* $n \geqslant n_1$,

$$\frac{1}{2} \leqslant (nh_n)^{-d} K\left(\frac{\mathbf{x}-\mathbf{i}/n}{h_n} \right) \leqslant \frac{3}{2} \quad \textit{and} \tag{1.6.10}$$

$$\frac{1}{2} \left\| K \right\|_{\mathbb{L}^2(\mathbb{R}^d)} \leqslant (nh_n)^{-d} K^2\left(\frac{\mathbf{x}-\mathbf{i}/n}{h_n} \right) \leqslant \frac{3}{2} \left\| K \right\|_{\mathbb{L}^2(\mathbb{R}^d)}. \tag{1.6.11}$$

Let n_0 *be the smallest integer for which for all* $n \geqslant n_0$,

$$\sqrt{\sigma^2+\varepsilon_n} - 20\left(\log 2 \right)^{-1} C_2(\alpha) \left(\left[\left(\sum_{\mathbf{i}\in\Lambda_n} K\left(\frac{1}{h_n}\left(\mathbf{x} - \frac{\mathbf{i}}{n} \right) \right)^2 \right)^{1/2} \right]^{\gamma} \right)^{-\alpha} \geqslant \sigma/2. \tag{1.6.12}$$

Then there exists a constant κ *such that for each* $n \geqslant \max\left\{ n_0, n_1 \right\}$,

$$\Delta_n \leqslant \kappa \left| A_n - 1 \right|^{\frac{p}{p+1}} + \left| \varepsilon_n \right| + \kappa \left(nh_n \right)^{\frac{d}{2}\left(\gamma\left(p'-1 \right)d - p' + 2 \right)} + \left(nh_n \right)^{-\frac{d}{2}\gamma\alpha\frac{p}{p+1}} + \left(nh_n \right)^{\frac{2d-p(\gamma\beta+1)}{2(p+1)}}. \tag{1.6.13}$$

Lemma 1 in [EMS10] shows that under (1.6.2), the sequence $(A_n)_{n\geqslant 1}$ goes to 1 as n goes to infinity and that the integer n_1 is well-defined. Moreover, a rate can be given in terms of K, by following the computations in the proof of that lemma.

2. Proofs

2.1. **Proof of Theorem 1.4.** We define for $j \geqslant 1$ and $\mathbf{i} \in \mathbb{Z}^d$,

$$X_{\mathbf{i},j} = \mathbb{E}\left[X_i \mid \sigma\left(\varepsilon_{\mathbf{u}}, \left\| \mathbf{u}-\mathbf{i} \right\|_\infty \leqslant j \right) \right] - \mathbb{E}\left[X_i \mid \sigma\left(\varepsilon_{\mathbf{u}}, \left\| \mathbf{u}-\mathbf{i} \right\|_\infty \leqslant j-1 \right) \right]. \tag{2.1.1}$$

In this way, by the martingale convergence theorem,

$$X_{\mathbf{i}} - \mathbb{E}\left[X_{\mathbf{i}} \mid \varepsilon_{\mathbf{i}} \right] = \lim_{N\to+\infty} \sum_{j=1}^{N} X_{\mathbf{i},j} \tag{2.1.2}$$

hence

$$\left\| \sum_{\mathbf{i}\in\mathbb{Z}^d} a_{\mathbf{i}} X_{\mathbf{i}} \right\|_p \leqslant \sum_{j=1}^{+\infty} \left\| \sum_{\mathbf{i}\in\mathbb{Z}^d} a_{\mathbf{i}} X_{\mathbf{i},j} \right\|_p + \left\| \sum_{\mathbf{i}\in\mathbb{Z}^d} a_{\mathbf{i}} \mathbb{E}\left[X_{\mathbf{i}} \mid \varepsilon_{\mathbf{i}} \right] \right\|_p. \tag{2.1.3}$$

Let us fix $j \geqslant 1$. We divide $\mathbb{Z}^d$ into blocks. For $\mathbf{v} \in \mathbb{Z}^d$, we define

$$A_{\mathbf{v}} := \prod_{q=1}^{d} \left([(2j+2)\, v_q, (2j+2)\,(v_q+1) - 1] \cap \mathbb{Z} \right), \tag{2.1.4}$$

and if K is a subset of $[d]$, we define

$$E_K := \left\{ \mathbf{v} \in \mathbb{Z}^d, v_q \text{ is even if and only if } q \in K \right\}. \tag{2.1.5}$$

Therefore, the following inequality takes place

$$\left\| \sum_{\mathbf{i} \in \mathbb{Z}^d} a_{\mathbf{i}} X_{\mathbf{i},j} \right\|_p \leqslant \sum_{K \subset [d]} \left\| \sum_{\mathbf{v} \in E_K} \sum_{\mathbf{i} \in A_{\mathbf{v}}} a_{\mathbf{i}} X_{\mathbf{i},j} \right\|_p . \tag{2.1.6}$$

Observe that the random variable $\sum_{\mathbf{i} \in A_{\mathbf{v}}} a_{\mathbf{i}} X_{\mathbf{i},j}$ is measurable for the σ-algebra generated by $\varepsilon_{\mathbf{u}}$, where $\mathbf{u}$ satisfies $(2j+2)\, v_q - (j+1) \leqslant u_q \leqslant j+1+(2j+2)\,(v_q+1)-1$ for all $q \in [d]$. Since the family $\left\{ \varepsilon_{\mathbf{u}}, \mathbf{u} \in \mathbb{Z}^d \right\}$ is independent, the family $\left\{ \sum_{\mathbf{i} \in A_{\mathbf{v}}} a_{\mathbf{i}} X_{\mathbf{i},j}, \mathbf{v} \in E_K \right\}$ is independent for each fixed $K \subset [d]$. Using inequality (1.1.2), it thus follows that

$$\begin{aligned} \left\| \sum_{\mathbf{v} \in E_K} \sum_{\mathbf{i} \in A_{\mathbf{v}}} a_{\mathbf{i}} X_{\mathbf{i},j} \right\|_p \leqslant{}& \frac{14.5p}{\log p} \left(\sum_{\mathbf{v} \in E_K} \left\| \sum_{\mathbf{i} \in A_{\mathbf{v}}} a_{\mathbf{i}} X_{\mathbf{i},j} \right\|_2^2 \right)^{1/2} \\ &+ \frac{14.5p}{\log p} \left(\sum_{\mathbf{v} \in E_K} \left\| \sum_{\mathbf{i} \in A_{\mathbf{v}}} a_{\mathbf{i}} X_{\mathbf{i},j} \right\|_p^p \right)^{1/p} . \end{aligned} \tag{2.1.7}$$

By stationarity, one can see that $\left\| X_{\mathbf{i},j} \right\|_q = \left\| X_{\mathbf{0},j} \right\|_q$ for $q \in \{2, p\}$, hence the triangle inequality yields

$$\begin{aligned} \left\| \sum_{\mathbf{v} \in E_K} \sum_{\mathbf{i} \in A_{\mathbf{v}}} a_{\mathbf{i}} X_{\mathbf{i},j} \right\|_p \leqslant{}& \frac{14.5p}{\log p} \left\| X_{\mathbf{0},j} \right\|_2 \left(\sum_{\mathbf{v} \in E_K} \left(\sum_{\mathbf{i} \in A_{\mathbf{v}}} |a_{\mathbf{i}}| \right)^2 \right)^{1/2} \\ &+ \frac{14.5p}{\log p} \left\| X_{\mathbf{0},j} \right\|_p \left(\sum_{\mathbf{v} \in E_K} \left(\sum_{\mathbf{i} \in A_{\mathbf{v}}} |a_{\mathbf{i}}| \right)^p \right)^{1/p} . \end{aligned} \tag{2.1.8}$$

By Jensen's inequality, for $q \in \{2, p\}$,

$$\left(\sum_{\mathbf{i} \in A_{\mathbf{v}}} |a_{\mathbf{i}}| \right)^q \leqslant |A_{\mathbf{v}}|^{q-1} \sum_{\mathbf{i} \in A_{\mathbf{v}}} |a_{\mathbf{i}}|^q \leqslant (2j+2)^{d(q-1)} \sum_{\mathbf{i} \in A_{\mathbf{v}}} |a_{\mathbf{i}}|^q \tag{2.1.9}$$

and using $\sum_{i=1}^N x_i^{1/q} \leqslant N^{\frac{q-1}{q}} \left(\sum_{i=1}^N x_i \right)^{1/q}$, it follows that

$$\begin{aligned} \sum_{K \subset [d]} \left\| \sum_{\mathbf{v} \in E_K} \sum_{\mathbf{i} \in A_{\mathbf{v}}} a_{\mathbf{i}} X_{\mathbf{i},j} \right\|_p \leqslant{}& \frac{14.5p}{\log p} \left\| X_{\mathbf{0},j} \right\|_2 \left(\sum_{\mathbf{i} \in \mathbb{Z}^d} a_i^2 \right)^{1/2} (4j+4)^{d/2} \\ &+ \frac{14.5p}{\log p} \left\| X_{\mathbf{0},j} \right\|_p \left(\sum_{\mathbf{i} \in \mathbb{Z}^d} |a_i|^p \right)^{1/p} (4j+4)^{d(1-1/p)} . \end{aligned} \tag{2.1.10}$$

Combining (2.1.3), (2.1.6) and (2.1.10), we derive that

$$\left\|\sum_{\mathbf{i}\in\mathbb{Z}^d} a_{\mathbf{i}}X_{\mathbf{i}}\right\|_p \leqslant \frac{14.5p}{\log p}\sum_{j=1}^{+\infty}\|X_{\mathbf{0},j}\|_2\left(\sum_{\mathbf{i}\in\mathbb{Z}^d} a_i^2\right)^{1/2}(4j+4)^{d/2}$$
$$+\frac{14.5p}{\log p}\sum_{j=1}^{+\infty}\|X_{\mathbf{0},j}\|_p\left(\sum_{\mathbf{i}\in\mathbb{Z}^d}|a_i|^p\right)^{1/p}(4j+4)^{d(1-1/p)}+\left\|\sum_{\mathbf{i}\in\mathbb{Z}^d} a_{\mathbf{i}}\mathbb{E}\left[X_{\mathbf{i}}\mid\varepsilon_{\mathbf{i}}\right]\right\|_p. \quad (2.1.11)$$

In order to control the last term, we use inequality (1.1.2) and bound $\|\mathbb{E}\left[X_{\mathbf{i}}\mid\varepsilon_{\mathbf{i}}\right]\|_q$ by $\|X_{\mathbf{0},0}\|_q$ for $q\in\{1,2\}$. This ends the proof of Theorem 1.4.

Proof of Corollary 1.5.

Lemma 2.1. *For $q\in\{2,p\}$ and $j\in\mathbb{N}$, the following inequality holds*

$$\|X_{\mathbf{0},j}\|_q \leqslant \left(\sum_{\mathbf{i}\in\mathbb{Z}^d,\|\mathbf{i}\|_\infty=j}\delta_{\mathbf{i},q}^2\right)^{1/2}. \quad (2.1.12)$$

Proof. Let us write the set of elements of $\mathbb{Z}^d$ whose infinite norm is equal to j as $\{\mathbf{v_s},1\leqslant s\leqslant N_j\}$ where $N_j\in\mathbb{N}$. We also assume that $v_s-v_{s-1}\in\{\mathbf{e_k},1\leqslant k\leqslant d\}$ for all $s\in\{2,\dots,N_j\}$.

Denote

$$\mathcal{F}_s:=\sigma\left(\varepsilon_{\mathbf{u}},\|\mathbf{u}\|_\infty\leqslant j,\varepsilon_{\mathbf{v_t}},1\leqslant t\leqslant s\right), \quad (2.1.13)$$

and $\mathcal{F}_0:=\sigma\left(\varepsilon_{\mathbf{u}},\|\mathbf{u}\|_\infty\leqslant j\right)$. Then $X_{\mathbf{0},j}=\sum_{s=1}^{N_j}\mathbb{E}\left[X_{\mathbf{0}}\mid\mathcal{F}_s\right]-\mathbb{E}\left[X_{\mathbf{0}}\mid\mathcal{F}_{s-1}\right]$, from which it follows, by Theorem 2.1 in [Rio09], that

$$\|X_{\mathbf{0},j}\|_q^2\leqslant(q-1)\sum_{s=1}^{N_j}\|\mathbb{E}\left[X_{\mathbf{0}}\mid\mathcal{F}_s\right]-\mathbb{E}\left[X_{\mathbf{0}}\mid\mathcal{F}_{s-1}\right]\|_q^2. \quad (2.1.14)$$

Then using similar arguments as in the proof of Theorem 1 (i) in [Wu05] give the bound $\|\mathbb{E}\left[X_{\mathbf{0}}\mid\mathcal{F}_s\right]-\mathbb{E}\left[X_{\mathbf{0}}\mid\mathcal{F}_{s-1}\right]\|_q\leqslant\delta_{\mathbf{v_s},q}+\delta_{\mathbf{v_{s-1}},q}$. □

□

2.2. **Proof of Theorem 1.9.** Denote for a random variable Z the quantity

$$\delta(Z):=\sup_{t\in\mathbb{R}}\left|\mathbb{P}\left\{Z\leqslant t\right\}-\Phi(t)\right|. \quad (2.2.1)$$

We say that a random field $(Y_{\mathbf{i}})_{\mathbf{i}\in\mathbb{Z}^d}$ is m-dependent if the collections of random variables $(Y_{\mathbf{i}},\mathbf{i}\in A)$ and $(Y_{\mathbf{i}},\mathbf{i}\in B)$ are independent whenever $\inf\left\{\|\mathbf{a}-\mathbf{b}\|_\infty,\mathbf{a}\in A,\mathbf{b}\in B\right\}>m$. The proof of Theorem 1.9 will use the following tools.

(T.1) By Theorem 2.6 in [CS04], if I is a finite subset of $\mathbb{Z}^d$, $(Y_{\mathbf{i}})_{\mathbf{i}\in I}$ an m-dependent centered random field such that $\mathbb{E}\left[|Y_{\mathbf{i}}|^p\right]<+\infty$ for each $\mathbf{i}\in I$ and some $p\in(2,3]$ and $\mathrm{Var}\left(\sum_{\mathbf{i}\in I}Y_{\mathbf{i}}\right)=1$, then

$$\delta\left(\sum_{\mathbf{i}\in I}Y_{\mathbf{i}}\right)\leqslant 75\left(10m+1\right)^{(p-1)d}\sum_{\mathbf{i}\in I}\mathbb{E}\left[|Y_{\mathbf{i}}|^p\right]. \quad (2.2.2)$$

(T.2) By Lemma 1 in [EMO07], for any two random variables Z and Z' and $p \geqslant 1$,

$$\delta(Z+Z') \leqslant 2\delta(Z) + \|Z'\|_p^{\frac{p}{p+1}}. \tag{2.2.3}$$

Let $(\varepsilon_{\mathbf{u}})_{\mathbf{u}\in\mathbb{Z}^d}$ be an i.i.d. random field and let $f\colon \mathbb{R}^{\mathbb{Z}^d} \to \mathbb{R}$ be a measurable function such that for each $\mathbf{i} \in \mathbb{Z}^d$, $X_{\mathbf{i}} = f(\varepsilon_{\mathbf{i}-\mathbf{u}})$. Let $\gamma > 0$ and n_0 defined by (1.5.7).

Let $m := ([\|b_n\|_{\ell^2}] + 1)^{\gamma}$ and let us define

$$X_{\mathbf{i}}^{(m)} := \mathbb{E}\left[X_{\mathbf{i}} \mid \sigma\left(\varepsilon_{\mathbf{u}}, \mathbf{i} - m\mathbf{1} \preccurlyeq \mathbf{u} \preccurlyeq \mathbf{i} + m\mathbf{1}\right)\right]. \tag{2.2.4}$$

Since the random field $(\varepsilon_{\mathbf{u}})_{\mathbf{u}\in\mathbb{Z}^d}$ is independent, the following properties hold.

(P.1) The random field $\left(X_{\mathbf{i}}^{(m)}\right)_{\mathbf{i}\in\mathbb{Z}^d}$ is $(2m+1)$-dependent.

(P.2) The random field $\left(X_{\mathbf{i}}^{(m)}\right)_{\mathbf{i}\in\mathbb{Z}^d}$ is identically distributed and $\left\|X_{\mathbf{i}}^{(m)}\right\|_{p'} \leqslant \|X_{\mathbf{0}}\|_{p'}$.

(P.3) For any $(a_{\mathbf{i}})_{\mathbf{i}\in\mathbb{Z}^d} \in \ell^2\left(\mathbb{Z}^d\right)$ and $q \geqslant 2$, the following inequality holds:

$$\begin{aligned}\left\|\sum_{\mathbf{i}\in\mathbb{Z}^d} a_{\mathbf{i}}\left(X_{\mathbf{i}} - X_{\mathbf{i}}^{(m)}\right)\right\|_q &\leqslant \frac{14.5q}{\log q}\left(\sum_{\mathbf{i}\in\mathbb{Z}^d} a_{\mathbf{i}}^2\right)^{1/2} \sum_{j\geqslant m} (4j+4)^{d/2} \|X_{\mathbf{0},j}\|_2 \\ &\quad + \frac{14.5q}{\log q}\left(\sum_{\mathbf{i}\in\mathbb{Z}^d} |a_{\mathbf{i}}|^q\right)^{1/q} \sum_{j\geqslant m} (4j+4)^{d(1-1/q)} \|X_{\mathbf{0},j}\|_q. \end{aligned} \tag{2.2.5}$$

In order to prove (2.2.5), we follow the proof of Theorem 1.4 and start from the decomposition $X_{\mathbf{i}} - X_{\mathbf{i}}^{(m)} = \lim_{N\to+\infty} \sum_{j=m}^{N} X_{\mathbf{i},j}$ instead of (2.1.2).

Define $S_n^{(m)} := \sum_{\mathbf{i}\in\mathbb{Z}^d} b_{n,\mathbf{i}} X_{\mathbf{i}}^{(m)}$. An application of (T.2) to $Z := S_n^{(m)} \|b_n\|_{\ell^2}^{-1} \sigma^{-1}$ and $Z' := \left(S_n - S_n^{(m)}\right)\|b_n\|_{\ell^2}^{-1}\sigma^{-1}$ yields

$$\Delta_n \leqslant 2\delta\left(\frac{S_n^{(m)}}{\sigma\|b_n\|_{\ell^2}}\right) + \sigma^{-\frac{p}{p+1}} \frac{1}{\|b_n\|_{\ell^2}^{\frac{p}{p+1}}} \left\|S_n - S_n^{(m)}\right\|_p^{\frac{p}{p+1}}. \tag{2.2.6}$$

Moreover,

$$\delta\left(\frac{S_n^{(m)}}{\delta\|b_n\|_{\ell^2}}\right) = \sup_{t\in\mathbb{R}}\left|\mathbb{P}\left\{\frac{S_n^{(m)}}{\delta\|b_n\|_{\ell^2}} \leqslant t\right\} - \Phi(t)\right| \tag{2.2.7}$$

$$= \sup_{u\in\mathbb{R}}\left|\mathbb{P}\left\{\frac{S_n^{(m)}}{\left\|S_n^{(m)}\right\|_2} \leqslant u\right\} - \Phi\left(u\frac{\left\|S_n^{(m)}\right\|_2}{\delta\|b_n\|_{\ell^2}}\right)\right| \tag{2.2.8}$$

$$\leqslant \delta\left(\frac{S_n^{(m)}}{\left\|S_n^{(m)}\right\|_2}\right) + \sup_{u\in\mathbb{R}}\left|\Phi\left(u\frac{\left\|S_n^{(m)}\right\|_2}{\sigma\|b_n\|_{\ell^2}}\right) - \Phi(u)\right|, \tag{2.2.9}$$

hence, by (P.1) and (T.1) applied with $Y_{\mathbf{i}} := X_{\mathbf{i}}^{(m)} / \left\|S_n^{(m)}\right\|_2$, p' instead of p and $2m+1$ instead of m, we derive that

$$\Delta_n \leqslant (I) + (II) + (III) \tag{2.2.10}$$

where

$$(I) := 150\,(20m+21)^{(p'-1)d} \sum_{\mathbf{i}\in\mathbb{Z}^d} |b_{n,\mathbf{i}}|^{p'} \left\|X_i^{(m)}\right\|_{p'}^{p'} \left\|S_n^{(m)}\right\|_2^{-p'}, \tag{2.2.11}$$

$$(II) := 2\sup_{u\in\mathbb{R}} \left|\Phi\left(u\frac{\left\|S_n^{(m)}\right\|_2}{\sigma\,\|b_n\|_{\ell^2}}\right) - \Phi(u)\right| \text{ and} \tag{2.2.12}$$

$$(III) := \sigma^{-\frac{p}{p+1}} \frac{1}{\|b_n\|_{\ell^2}^{\frac{p}{p+1}}} \left\|S_n - S_n^{(m)}\right\|_p^{\frac{p}{p+1}}. \tag{2.2.13}$$

By (P.2) and the reversed triangular inequality, the term (I) can be bounded in the following way

$$(I) \leqslant 150\,(20m+21)^{(p'-1)d} \|X_{\mathbf{0}}\|_{p'}^{p'} \|b_n\|_{\ell^{p'}}^{p'} \left(\|S_n\|_2 - \left\|S_n - S_n^{(m)}\right\|\right)^{-p'} \tag{2.2.14}$$

and by (P.2) with $q=2$, we obtain that

$$\left(\|S_n\|_2 - \left\|S_n - S_n^{(m)}\right\|\right)^{-p'} \leqslant \left(\|S_n\|_2 - 20\,(\log 2)^{-1}\, m^{-\alpha}\, \|b_n\|_{\ell^2}\, C_2(\alpha)\right)^{-p'}. \tag{2.2.15}$$

By (1.5.6), we have

$$\frac{\|S_n\|_2^2}{\|b_n\|_{\ell^2}^2} = \sigma^2 + \varepsilon_n, \tag{2.2.16}$$

and we eventually get

$$(I) \leqslant 150\,(20m+21)^{(p'-1)d} \|X_{\mathbf{0}}\|_{p'}^{p'} \left(\frac{\|b_n\|_{\ell^{p'}}}{\|b_n\|_{\ell^2}}\right)^{p'} \left(\sqrt{\sigma^2+\varepsilon_n} - 20\,(\log 2)^{-1}\, m^{-\alpha} C_2(\alpha)\right)^{-p'}.$$

Since $n \geqslant n_0$, we derive, in view of (1.5.7),

$$(I) \leqslant 150\,(20m+21)^{(p'-1)d} \|X_{\mathbf{0}}\|_{p'}^{p'} \left(\frac{\|b_n\|_{\ell^{p'}}}{\|b_n\|_{\ell^2}}\right)^{p'} (\sigma/2)^{-p'} \tag{2.2.17}$$

In order to bound (II), we argue as in [YWLH12] (p. 456). Doing similar computations as in [EM14] (p. 272), we obtain that

$$(II) \leqslant (2\pi e)^{-1/2} \left(\inf_{k\geqslant 1} a_k\right)^{-1} \left|a_n^2 - 1\right|, \tag{2.2.18}$$

where $a_n := \left\|S_n^{(m)}\right\|_2 \sigma^{-1} \|b_n\|_{\ell^2}^{-1}$. Observe that for any n, by (P.2),

$$a_n \geqslant \frac{\|S_n\|_2 - \left\|S_n - S_n^{(m)}\right\|_2}{\sigma\,\|b_n\|_{\ell^2}} \geqslant \frac{\sqrt{\sigma^2+\varepsilon_n} - 20\,(\log 2)^{-1}\, C_2(\alpha)\, m^{-\alpha}}{\sigma} \tag{2.2.19}$$

and using again (P.2) combined with Theorem 1.4 for $p=2$,

$$\left|a_n^2-1\right| = \left|\frac{\left\|S_n^{(m)}\right\|_2^2}{\sigma^2\left\|b_n\right\|_{\ell^2}^2}-1\right| \tag{2.2.20}$$

$$\leqslant \left|\frac{\|S_n\|_2^2}{\sigma^2\left\|b_n\right\|_{\ell^2}^2}-1\right| + \frac{\left|\left\|S_n^{(m)}\right\|_2^2-\|S_n\|_2^2\right|}{\sigma^2\left\|b_n\right\|_{\ell^2}^2} \tag{2.2.21}$$

$$\leqslant \frac{|\varepsilon_n|}{\sigma^2} + \frac{\left|\left\|S_n^{(m)}\right\|_2-\|S_n\|_2\right|\left(\left\|S_n^{(m)}\right\|_2+\|S_n\|_2\right)}{\sigma^2\left\|b_n\right\|_{\ell^2}^2} \tag{2.2.22}$$

$$\leqslant \frac{|\varepsilon_n|}{\sigma^2} + \frac{\left\|S_n^{(m)}-S_n\right\|_2\left(\left\|S_n^{(m)}\right\|_2+\|S_n\|_2\right)}{\sigma^2\left\|b_n\right\|_{\ell^2}^2} \tag{2.2.23}$$

$$\leqslant \frac{|\varepsilon_n|}{\sigma^2} + 40\left(\log 2\right)^{-1}\frac{m^{-\alpha}}{\sigma^2}C_2\left(\alpha\right)^2. \tag{2.2.24}$$

This leads to the estimate

$$(II) \leqslant \frac{(2\pi e)^{-1/2}}{\sqrt{\sigma^2+\varepsilon_n}-20\left(\log 2\right)^{-1}C_2\left(\alpha\right)m^{-\alpha}}\left(\frac{|\varepsilon_n|}{\sigma}+40\left(\log 2\right)^{-1}\frac{m^{-\alpha}}{\sigma}C_2\left(\alpha\right)^2\right), \tag{2.2.25}$$

and since $n\geqslant n_0$, we derive, in view of (1.5.7),

$$(II) \leqslant \left(2\frac{|\varepsilon_n|}{\sigma^2}+80\left(\log 2\right)^{-1}\frac{\left\|b_n\right\|_{\ell^2}^{-\gamma\alpha}}{\sigma^2}C_2\left(\alpha\right)^2\right)(2\pi e)^{-1/2}. \tag{2.2.26}$$

The estimate of (III) rests on (P.3):

$$(III) \leqslant \sigma^{-\frac{p}{p+1}}\left(\frac{14.5p}{\log p}\sum_{j\geqslant m}\left(4j+4\right)^{d/2}\left\|X_{\mathbf{0},j}\right\|_2\right)^{\frac{p}{p+1}}$$
$$+\sigma^{-\frac{p}{p+1}}\left\|b_n\right\|_{\ell^2}^{-\frac{p}{p+1}}\left\|b_n\right\|_{\ell^p}^{\frac{p}{p+1}}\left(\frac{14.5p}{\log p}\sum_{j\geqslant m}\left(4j+4\right)^{d(1-1/p)}\left\|X_{\mathbf{0},j}\right\|_p\right)^{\frac{p}{p+1}} \tag{2.2.27}$$

hence

$$(III) \leqslant \left(\frac{14.5p}{\sigma\log p}4^{d/2}\left\|b_n\right\|_{\ell^2}^{-\gamma\alpha}C_2\left(\alpha\right)\right)^{\frac{p}{p+1}} + \left(\frac{\left\|b_n\right\|_{\ell^p}}{\sigma\left\|b_n\right\|_{\ell^2}}\frac{14.5p}{\log p}4^{d(1-1/p)}\left\|b_n\right\|_{\ell^2}^{-\gamma\beta}C_p\left(\beta\right)\right)^{\frac{p}{p+1}}. \tag{2.2.28}$$

The combination of (2.2.10), (2.2.17), (2.2.26) and (2.2.28) gives (1.5.8).

2.3. **Proof of Theorem 1.12.** Since the random variables $X_{\mathbf{i}}$ are centered, we derive by definition of $g_n\left(\mathbf{x}\right)$ that

$$\left(nh_n\right)^{d/2}\left(g_n(\mathbf{x})-\mathbb{E}\left[g_n(\mathbf{x})\right]\right) = \left(nh_n\right)^{d/2}\frac{\sum_{\mathbf{i}\in\Lambda_n}X_{\mathbf{i}}K\left(\frac{\mathbf{x}-\mathbf{i}/n}{h_n}\right)}{\sum_{\mathbf{i}\in\Lambda_n}K\left(\frac{\mathbf{x}-\mathbf{i}/n}{h_n}\right)}. \tag{2.3.1}$$

Keeping in mind (1.6.4), we define

$$b_{n,\mathbf{i}} = K\left(\frac{1}{h_n}\left(\mathbf{x} - \frac{\mathbf{i}}{n}\right)\right), \quad \mathbf{i} \in \Lambda_n \tag{2.3.2}$$

and $b_{n,\mathbf{i}} = 0$ otherwise. Denote $b_n = (b_{n,\mathbf{i}})_{\mathbf{i}\in\mathbb{Z}^d}$ and $\|b_n\|_{\ell^2} := \left(\sum_{\mathbf{i}\in\mathbb{Z}^d} b_{n,\mathbf{i}}^2\right)^{1/2}$. In this way, by (2.3.1) and (1.6.8),

$$\frac{1}{\|K\|_{\mathbb{L}^2(\mathbb{R}^d)}\,\sigma}\left(nh_n\right)^{d/2}\left(g_n(\mathbf{x}) - \mathbb{E}\left[g_n(\mathbf{x})\right]\right) = \frac{1}{\sigma}\sum_{\mathbf{i}\in\mathbb{Z}^d} b_{n\mathbf{i}} X_{\mathbf{i}} \left\|b_n\right\|_{\ell^2}^{-1} A_n. \tag{2.3.3}$$

Applying (T.2) to $Z = \sum_{\mathbf{i}\in\mathbb{Z}^d} b_{n\mathbf{i}} X_{\mathbf{i}} \left\|b_n\right\|_{\ell^2}^{-1}$ and $Z' = \sum_{\mathbf{i}\in\mathbb{Z}^d} b_{n\mathbf{i}} X_{\mathbf{i}} \left\|b_n\right\|_{\ell^2}^{-1} \sigma^{-1}\left(A_n - 1\right)$ and using Proposition 1 in [EVW13], we derive that

$$\Delta_n \leqslant 2\Delta_n' + 3\left\|(2p)^{1/2}\,\sigma^{-1}\sum_{\mathbf{j}\in\mathbb{Z}^d}\delta_{\mathbf{j},p}\right\|_{\frac{p}{p+1}}^{\frac{p}{p+1}} \left|A_n - 1\right|^{\frac{p}{p+1}}, \tag{2.3.4}$$

where

$$\Delta_n' = \sup_{t\in\mathbb{R}}\left|\mathbb{P}\left\{Z \leqslant t\right\} - \Phi\left(\frac{t}{\sigma}\right)\right|. \tag{2.3.5}$$

We then use Theorem 1.9 to handle Δ_n' (which is allowed, by (A1)). Using boundedness of K, we control the ℓ^p and $\ell^{p'}$ norms , by a constant times the ℓ^2-norm. This ends the proof of Theorem 1.12.

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