Convergence rates in the central limit theorem for weighted sums of Bernoulli random fields

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CONVERGENCE RATES IN THE CENTRAL LIMIT THEOREM FOR WEIGHTED SUMS OF BERNOULLI RANDOM FIELDS

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Abstract. We prove moment inequalities for a class of functionals of i.i.d. random fields. We then derive rates in the central limit theorem for weighted sums of such randoms fields via an approximation by m-dependent random fields.

1. Introduction and main results

1.1. **Notations and conventions.** In all the paper, we shall use the following notations.

(N.1) For a positive integer d, the set $\{1,\dots,d\}$ is denoted by $[d]$.

(N.2) The coordinatewise order is denoted by $\preccurlyeq$, that is, for $\mathbf{i}=(i_q)_{q=1}^d\in\mathbb{Z}^d$ and $\mathbf{j}=(j_q)_{q=1}^d\in\mathbb{Z}^d$, $\mathbf{i}\preccurlyeq\mathbf{j}$ means that $i_q\leqslant j_q$ for any $q\in[d]$.

(N.3) For $q\in[d]$, $\mathbf{e_q}$ denotes the element of $\mathbb{Z}^d$ whose qth coordinate is 1 and all the others are zero. Moreover, we write $\mathbf{0}=(0,\dots,0)$ and $\mathbf{1}=(1,\dots,1)$.

(N.4) If $\mathbf{l}=\sum_{q=1}^d l_q\mathbf{e_q}$ is an element of $\mathbb{Z}^d$ and J a subset of $[d]$, then $\mathbf{l_J}$ is the element of $\mathbb{Z}^d$ defined by $\mathbf{l_J}=\sum_{q\in J}l_q\mathbf{e_q}$.

(N.5) For $\mathbf{n}=(n_q)_{q=1}^d\in\mathbb{N}^d$, we write the product $\prod_{q=1}^d n_q$ as $|\mathbf{n}|$.

(N.6) The cardinality of a set I is denoted by $|I|$.

(N.7) For a real number x, we denote by $[x]$ the unique integer such that $[x]\leqslant x<[x]+1$.

(N.8) We write Φ for the cumulative distribution function of a standard normal law.

(N.9) If Λ is a subset of $\mathbb{Z}^d$ and $\mathbf{k}\in\mathbb{Z}^d$, then $\Lambda-\mathbf{k}$ is defined as $\{\mathbf{l}-\mathbf{k},\mathbf{l}\in\Lambda\}$.

(N.10) For $q\geqslant 1$, we denote by $\ell^q\left(\mathbb{Z}^d\right)$ the space of sequences $\mathbf{a}:=(a_{\mathbf{i}})_{\mathbf{i}\in\mathbb{Z}^d}$ such that $\|\mathbf{a}\|_{\ell^q}:=\left(\sum_{\mathbf{i}\in\mathbb{Z}^d}|a_{\mathbf{i}}|^q\right)^{1/q}<+\infty$.

(N.11) For $\mathbf{i}=(i_q)_{q=1}^d$, the quantity $\|\mathbf{i}\|_\infty$ is defined as $\max_{1\leqslant q\leqslant d}|i_q|$.

Let $(Y_{\mathbf{i}})_{\mathbf{i}\in\mathbb{Z}^d}$ be a random field. The sum $\sum_{\mathbf{i}\in\mathbb{Z}^d}Y_{\mathbf{i}}$ is understood as the $\mathbb{L}^p$ limit of the sequence $(S_k)_{k\geqslant 1}$ where $S_k=\sum_{\mathbf{i}\in\mathbb{Z}^d,\|\mathbf{i}\|_\infty\leqslant k}Y_{\mathbf{i}}$.

1.2. **Rosenthal's inequality.** By Theorem 3 in [Ros70], we know that for any $p>2$, there exists a constant C depending only on p such that for any finite sequence $(Y_i)_{i=1}^N$ of independent centered random variables with a finite moment of order p, then

$$\left\|\sum_{i=1}^n Y_i\right\|_p\leqslant C\left(\left(\sum_{i=1}^n\mathbb{E}\left[Y_i^2\right]\right)^{1/2}+\left(\sum_{i=1}^n\mathbb{E}\left[|Y_i|^p\right]\right)^{1/p}\right). \tag{1.2.1}$$

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In order to give explicit constants, we will use the following version of Rosenthal's inequality due to Johnson, Schechtman and Zinn [JSZ85]: if $(Y_i)_{i=1}^N$ are independent centered random variables with a finite moment of order $p \geqslant 2$, then

$$\left\| \sum_{i=1}^n Y_i \right\|_p \leqslant \frac{14.5p}{\log p} \left(\left(\sum_{i=1}^n \mathbb{E}\left[Y_i^2\right] \right)^{1/2} + \left(\sum_{i=1}^n \mathbb{E}\left[|Y_i|^p\right] \right)^{1/p} \right). \tag{1.2.2}$$

Various extension of Rosenthal-type inequalities have been obtained under mixing conditions [Sha95, Rio00] or projective conditions [PUW07, Rio09, MP13].

1.3. **Physical dependence measure.** In the sequel, we will be interested in the so-called Bernoulli random fields.

Definition 1.1. *Let $d \geqslant 1$ be an integer. The random field $(X_{\mathbf{n}})_{\mathbf{n} \in \mathbb{Z}^d}$ is said to be Bernoulli if there exists an i.i.d. random field $(\varepsilon_{\mathbf{i}})_{\mathbf{i} \in \mathbb{Z}^d}$ and a measurable function $f \colon \mathbb{R}^{\mathbb{Z}^d} \to \mathbb{R}$ such that $X_{\mathbf{n}} = f\left((\varepsilon_{\mathbf{n}-\mathbf{i}})_{\mathbf{i} \in \mathbb{Z}^d}\right)$ for each $\mathbf{n} \in \mathbb{Z}^d$.*

Following [Wu05] we define the physical dependence measure.

Definition 1.2. *Let $(X_{\mathbf{i}})_{\mathbf{i} \in \mathbb{Z}^d} := \left(f\left(\varepsilon_{\mathbf{i}-\mathbf{j}}\right)\right)_{\mathbf{j} \in \mathbb{Z}^d}$ be a Bernoulli random field, $p \geqslant 1$ and $(\varepsilon'_{\mathbf{u}})_{\mathbf{u} \in \mathbb{Z}^d}$ be an i.i.d. random field which is independent of the i.i.d. random field $(\varepsilon_{\mathbf{u}})_{\mathbf{u} \in \mathbb{Z}^d}$ and has the same distribution as $(\varepsilon_{\mathbf{u}})_{\mathbf{u} \in \mathbb{Z}^d}$. For $\mathbf{i} \in \mathbb{Z}^d$, we introduce the physical dependence measure*

$$\delta_{\mathbf{i},p} := \left\| X_{\mathbf{i}} - X_{\mathbf{i}}^* \right\|_p \tag{1.3.1}$$

where $X_{\mathbf{i}}^* = f\left(\varepsilon^*_{\mathbf{i}-\mathbf{j}}\right)$ and $\varepsilon^*_{\mathbf{u}} = \varepsilon_{\mathbf{u}}$ if $\mathbf{u} \neq \mathbf{0}$, $\varepsilon^*_{\mathbf{0}} = \varepsilon'_{\mathbf{0}}$.

In [EVW13, BD14], various examples of Bernoulli random fields are given, for which the physical dependence measure is either computed or estimated. Proposition 1 of [EVW13] also gives the following moment inequality: if Γ is a finite subset of $\mathbb{Z}^d$, $(a_{\mathbf{i}})_{\mathbf{i} \in \Gamma}$ is a family of real numbers and $p \geqslant 2$, then for any Bernoulli random field $(X_{\mathbf{n}})_{\mathbf{n} \in \mathbb{Z}^d}$,

$$\left\| \sum_{\mathbf{i} \in \Gamma} a_{\mathbf{i}} X_{\mathbf{i}} \right\|_p \leqslant \left(2p \sum_{\mathbf{i} \in \Gamma} a_{\mathbf{i}}^2 \right)^{1/2} \cdot \sum_{\mathbf{j} \in \mathbb{Z}^d} \delta_{\mathbf{j},p}. \tag{1.3.2}$$

This was used in [EVW13, BD14] in order to establish functional central limit theorems. Truquet [Tru10] also obtained an inequality in this spirit. However, we cannot recover Rosenthal's inequality (1.2.1) from (1.3.2). If $(X_{\mathbf{i}})_{\mathbf{i} \in \mathbb{Z}^d}$ is i.i.d. and centered, (1.2.1) would give

$$\left\| \sum_{\mathbf{i} \in \Gamma} a_{\mathbf{i}} X_{\mathbf{i}} \right\|_p \leqslant C \left(\sum_{\mathbf{i} \in \Gamma} a_{\mathbf{i}}^2 \right)^{1/2} \sum_{\mathbf{j} \in \mathbb{Z}^d} \delta_{\mathbf{j},2} + C \left(\sum_{\mathbf{i} \in \Gamma} |a_{\mathbf{i}}|^p \right)^{1/p} \sum_{\mathbf{j} \in \mathbb{Z}^d} \delta_{\mathbf{j},p}. \tag{1.3.3}$$

In the case of linear processes, equality $\delta_{\mathbf{j},p} \leqslant K \delta_{\mathbf{j},2}$ holds for a constant K which does not depend on $\mathbf{j}$. However, there are processes for which such an inequality does not hold.

Example 1.3. Let $(\varepsilon_{\mathbf{i}})_{\mathbf{i} \in \mathbb{Z}^d}$ be an i.i.d. random field and for each $\mathbf{k} \in \mathbb{Z}^d$, let $f_{\mathbf{k}} \colon \mathbb{R} \to \mathbb{R}$ be a function such that the random variable $Z_{\mathbf{k}} := f_{\mathbf{k}}\left(\varepsilon_{\mathbf{0}}\right)$ is centered and has a finite moment of order p, and $\sum_{\mathbf{k} \in \mathbb{Z}^d} \left\| Z_{\mathbf{k}} \right\|_2^2 < +\infty$. . Define $X_{\mathbf{n}} := \lim_{N \to +\infty} \sum_{-N\mathbf{1} \preccurlyeq \mathbf{j} \preccurlyeq N\mathbf{1}} f_{\mathbf{k}}\left(\varepsilon_{\mathbf{n}-\mathbf{k}}\right)$, where the limit is taken in $\mathbb{L}^2$. Then $X_{\mathbf{i}} - X_{\mathbf{i}}^* = f_{\mathbf{i}}\left(\varepsilon_{\mathbf{0}}\right) - f_{\mathbf{i}}\left(\varepsilon'_{\mathbf{0}}\right)$ hence $\delta_{\mathbf{i},2}$ is of order $\left\| Z_{\mathbf{k}} \right\|_2$ while $\delta_{\mathbf{i},p}$ is of order $\left\| Z_{\mathbf{k}} \right\|_p$.

Consequently, having the ℓ^p-norm instead of the ℓ^2-norm of the $(a_{\mathbf{i}})_{\mathbf{i}\in\Gamma}$ is more suitable. The goal of this paper is to get Rosenthal-like inequalities for weighted sums of Bernoulli random fields. In the one dimensional case, probability and moment inequalities have been established in [LXW13] for maxima of partial sums of Bernoulli sequences. The techniques used therein permit to derive results for weighted sums of such sequences.

1.4. **Moment inequalities.** Our first inequality reads as follows.

Theorem 1.4. *Let $\left\{\varepsilon_{\mathbf{i}}, \mathbf{i}\in\mathbb{Z}^d\right\}$ be an i.i.d. set of random variables. Then for any measurable function $f\colon \mathbb{R}^{\mathbb{Z}^d}\to\mathbb{R}$ such that $X_{\mathbf{j}} := f\left(\left(X_{\mathbf{j}-\mathbf{i}}\right)\right)_{i\in\mathbb{Z}^d}$ has a finite moment of order $p\geqslant 2$ and is centered, and any $(a_{\mathbf{i}})_{\mathbf{i}\in\mathbb{Z}^d}\in\ell^2\left(\mathbb{Z}^d\right)$,*

$$\left\|\sum_{\mathbf{i}\in\mathbb{Z}^d} a_{\mathbf{i}}X_{\mathbf{i}}\right\|_p \leqslant 2^{d/2}\frac{14.5p}{\log p}\left(\sum_{\mathbf{i}\in\mathbb{Z}^d} a_{\mathbf{i}}^2\right)^{1/2}\sum_{\mathbf{j}\in\mathbb{N}^d}\left\|\pi_{\mathbf{j}}\right\|_2 + 2^{d\frac{2p-1}{p}}\frac{14.5p}{\log p}(p-1)^{d/2}\left(\sum_{\mathbf{i}\in\mathbb{Z}^d}\left|a_{\mathbf{i}}\right|^p\right)^{1/p}\sum_{\mathbf{j}\in\mathbb{N}^d}\left|\mathbf{j}+\mathbf{1}\right|^{1/2-1/p}\left\|\pi_{\mathbf{j}}\right\|_p, \quad (1.4.1)$$

where for $\mathbf{j}\in\mathbb{N}^d$, denoting $S_{\mathbf{j}} := \{q\in[d]\mid j_q\neq 0\}$,

$$\pi_{\mathbf{j}} = \sum_{I\subset S_{\mathbf{j}}}(-1)^{|I|}\,\mathbb{E}\left[X_{\mathbf{0}}\mid\sigma\left(\varepsilon_{\mathbf{u}}, -\mathbf{j}-\sum_{i\in I}\mathbf{e}_{\mathbf{i}}\preccurlyeq\mathbf{u}\preccurlyeq\mathbf{j}+\sum_{i\in I}\mathbf{e}_{\mathbf{i}}\right)\right]. \quad (1.4.2)$$

The random variable $\pi_{\mathbf{j}}$ is linked to projection operators introduced in [VW14] in the following way. Define for an integrable random variable X the operators

$$P_l^{(q,0)}\colon\mathbb{L}^1\to\mathbb{L}^1,\quad P_l^{(q,0)}(X) = \mathbb{E}\left[X\mid\sigma\left(\varepsilon_{\mathbf{u}}, \mathbf{u}\in\mathbb{Z}^d, u_q\leqslant l\right)\right] \quad (1.4.3)$$

$$P_l^{(q,1)}\colon\mathbb{L}^1\to\mathbb{L}^1,\quad P_l^{(q,1)}(X) = \mathbb{E}\left[X\mid\sigma\left(\varepsilon_{\mathbf{u}}, \mathbf{u}\in\mathbb{Z}^d, u_q\geqslant l\right)\right]. \quad (1.4.4)$$

In this way,

$$\begin{aligned}\pi_{\mathbf{j}} &= \prod_{q\in S_{\mathbf{j}}}\left(P_j^{(q,0)}P_{-j_q}^{(q,1)} - P_{j_q-1}^{(q,0)}P_{-j_q+1}^{(q,1)}\right)(X_{\mathbf{0}})\\ &= \prod_{q\in S_{\mathbf{j}}}\left(P_{j_q}^{(q,0)}P_{-j_q}^{(q,1)} - P_{j_q}^{(q,0)}P_{-j_q+1}^{(q,1)} + P_{j_q}^{(q,0)}P_{-j_q+1}^{(q,1)} - P_{j_q-1}^{(q,0)}P_{-j_q+1}^{(q,1)}\right)(X_{\mathbf{0}})\\ &= \sum_{I\subset S_{\mathbf{j}}}(-1)^{|I|}\prod_{q\in I}\left(P_{j_q}^{(q,0)}P_{-j_q}^{(q,1)} - P_{j_q}^{(q,0)}P_{-j_q+1}^{(q,1)}\right)\prod_{q'\in S_{\mathbf{j}}\setminus I}\left(P_{j_q}^{(q',0)}P_{-j_q+1}^{(q',1)} - P_{j_q-1}^{(q',0)}P_{-j_q+1}^{(q',1)}\right)(X_{\mathbf{0}})\end{aligned}$$

hence for any $r\geqslant 1$,

$$\left\|\pi_{\mathbf{j}}\right\|_r \leqslant \sum_{I\subset S_{\mathbf{j}}}\left\|\prod_{q\in I}\left(P_{-j_q}^{(q,1)} - P_{-j_q+1}^{(q,1)}\right)\prod_{q'\in S_{\mathbf{j}}\setminus I}\left(P_{j_q}^{(q',0)} - P_{j_q-1}^{(q',0)}\right)(X_{\mathbf{0}})\right\|_r. \quad (1.4.5)$$

We denote for simplicity for $\mathbf{j}\in\mathbb{N}^d$ and $I\subset S_{\mathbf{j}}$,

$$P_{\mathbf{j}}^{(I)} = \prod_{q\in I}\left(P_{-j_q}^{(q,1)} - P_{-j_q+1}^{(q,1)}\right)\prod_{q'\in S_{\mathbf{j}}\setminus I}\left(P_{j_q}^{(q',0)} - P_{j_q-1}^{(q',0)}\right)(X_{\mathbf{0}}). \quad (1.4.6)$$

Notice that when I is empty, this corresponds to the operators defined in [VW14]. When I is not empty, $P^{(I)}$ also corresponds to these operators but when the family $(\varepsilon_{\mathbf{i}})_{\mathbf{i}\in\mathbb{Z}^d}$ is replaced by $(\varepsilon_{\mathbf{i}^I})_{\mathbf{i}\in\mathbb{Z}^d}$ where $\mathbf{i}^I_q=-i_q$ if $q\notin I$ and $\mathbf{i}^I_q=i_q$ if $q\in I$.

We shall denote

$$P_{\mathbf{i}}:=\prod_{q=1}^{d}\left(P^{(q,0)}_{i_q}-P^{(q,0)}_{i_q-1}\right),\quad \mathbf{i}\in\mathbb{Z}^d. \tag{1.4.7}$$

Remark 1.5. Using the arguments in the proof of Theorem 1 of [Wu05], we infer that $\|\pi_j\|_q\leqslant\delta_{j+1,q}+\delta_{-j-1,q}$ for any $j\in\mathbb{N}$, hence the right hand side of (1.4.1) can be bounded by the physical measure dependence when $d=1$. For d larger than 2, it is not clear to us how the coefficients $\pi_{\mathbf{j}},\mathbf{j}\in\mathbb{N}^d$ compare with the physical dependence measure.

We now give a Rosenthal-like inequality for weighted sums of Bernoulli random fields in terms of the physical dependence measure.

Theorem 1.6. *Let $\left\{\varepsilon_{\mathbf{i}},\mathbf{i}\in\mathbb{Z}^d\right\}$ be an i.i.d. set of random variables. Then for any measurable function $f\colon\mathbb{R}^{\mathbb{Z}^d}\to\mathbb{R}$ such that $X_{\mathbf{j}}:=f\left((X_{\mathbf{j}-\mathbf{i}})\right)_{i\in\mathbb{Z}^d}$ has a finite moment of order $p\geqslant 2$ and is centered, and any $(a_{\mathbf{i}})_{\mathbf{i}\in\mathbb{Z}^d}\in\ell^2\left(\mathbb{Z}^d\right)$,*

$$\left\|\sum_{\mathbf{i}\in\mathbb{Z}^d}a_{\mathbf{i}}X_{\mathbf{i}}\right\|_p\leqslant\frac{14.5p}{\log p}\left(\sum_{\mathbf{i}\in\mathbb{Z}^d}a_{\mathbf{i}}^2\right)^{1/2}\sum_{\mathbf{j}\in\mathbb{N}^d}\left(4\left\|\mathbf{j}\right\|_\infty+4\right)^{d/2}\delta_{\mathbf{j},2}$$
$$+\frac{14.5p}{\log p}\left(\sum_{\mathbf{i}\in\mathbb{Z}^d}\left|a_{\mathbf{i}}\right|^p\right)^{1/p}\sum_{\mathbf{j}\in\mathbb{N}^d}\left(4\left\|\mathbf{j}\right\|_\infty+4\right)^{d(1-1/p)}\delta_{\mathbf{j},p}. \tag{1.4.8}$$

Remark 1.7. Like in [EM02], we could derive Kahane-Khintchine inequalities. However, the result may not be optimal due to application of Lemma 2.4, since sharpness of the constant $(p-1)^{d/2}$ is not clear when the filtration is generated by i.i.d. random variables.

Remark 1.8. Using Remark 7 in [JS88] and Rosenthal's inequality in weak $\mathbb{L}^p$-spaces, it is possible to obtain a version of Theorems 1.4 and 1.6 where the $\mathbb{L}^p$ norm is replaced by $\|X\|_{p,\infty}:=\sup_{A\in\mathcal{F}}\mathbb{P}(A)^{1/p-1}\mathbb{E}\left[X\mathbf{1}_A\right]$.

1.5. **Convergence rates in the central limit theorem for weighted sums.** Let $(X_{\mathbf{j}})_{\mathbf{j}\in\mathbb{Z}^d}=f\left((\varepsilon_{\mathbf{j}-\mathbf{i}})_{\mathbf{i}\in\mathbb{Z}^d}\right)$ be a centered square integrable Bernoulli random field and for any positive integer n, let $b_n:=(b_{n,\mathbf{i}})_{\mathbf{i}\in\mathbb{Z}^d}$ be an element of $\ell^2\left(\mathbb{Z}^d\right)$. We are interested in the asymptotic behavior of the sequence $(S_n)_{n\geqslant 1}$ defined by

$$S_n:=\sum_{\mathbf{i}\in\mathbb{Z}^d}b_{n,\mathbf{i}}X_{\mathbf{i}}. \tag{1.5.1}$$

Let us denote for $\mathbf{k}\in\mathbb{Z}^d$ the map $\tau_{\mathbf{k}}\colon\ell^2\left(\mathbb{Z}^d\right)\to\ell^2\left(\mathbb{Z}^d\right)$ defined by $\tau_{\mathbf{k}}\left((x_{\mathbf{i}})_{\mathbf{i}\in\mathbb{Z}^d}\right):=(x_{\mathbf{i}+\mathbf{k}})_{\mathbf{i}\in\mathbb{Z}^d}$.

In [KVW16], Corollary 2.6 states the following. Assume that the series $\sum_{\mathbf{i}\in\mathbb{Z}^d}\|P_{\mathbf{i}}\left(X_{\mathbf{0}}\right)\|_2$ converges and that for any $q\in[d]$,

$$\frac{1}{\|b_n\|_{\ell^2}}\left\|\tau_{\mathbf{e_q}}\left(b_n\right)-b_n\right\|_{\ell^2}=0. \tag{1.5.2}$$

Then the series $\sum_{\mathbf{i}\in\mathbb{Z}^d} |\mathrm{Cov}\,(X_{\mathbf{0}}, X_{\mathbf{i}})|$ converges and denoting

$$\sigma := \left(\sum_{\mathbf{i}\in\mathbb{Z}^d} \mathrm{Cov}\,(X_{\mathbf{0}}, X_{\mathbf{i}})\right)^{1/2}, \tag{1.5.3}$$

the sequence $(S_n/\left\|b_n\right\|_{\ell^2})_{n\geqslant 1}$ converges in distribution to a centered normal distribution with variance σ^2. The argument rests on an approximation by an m-dependent random field. The same conclusion holds if we combine (1.5.2) with $\sum_{\mathbf{i}\in\mathbb{Z}^d} \delta_{\mathbf{i},2} < +\infty$. We are interested in rates in the previous central limit theorems. We shall state it in terms of the Wu's coefficients, which are more convenient than those given by (1.4.2). Nevertheless, an analogous statement can be derived by similar arguments for the latter coefficients. It seems difficult to link the assumption $\sum_{\mathbf{i}\in\mathbb{Z}^d} |\mathrm{Cov}\,(X_{\mathbf{0}}, X_{\mathbf{i}})| < \infty$ with the projectors defined in (1.4.6). However, by Lemma 2.4 in [KVW16], the inequality

$$\sum_{\mathbf{i}\in\mathbb{Z}^d} |\mathrm{Cov}\,(X_{\mathbf{0}}, X_{\mathbf{i}})| \leqslant \left(\sum_{\mathbf{j}\in\mathbb{Z}^d} \left\|P_{\mathbf{j}}\,(X_{\mathbf{0}})\right\|_2\right)^2 \tag{1.5.4}$$

holds, where $P_{\mathbf{j}}$ is defined by (1.4.7).

Theorem 1.9. *Let $p > 2$, $p' := \min\{p, 3\}$ and let $(X_{\mathbf{j}})_{\mathbf{j}\in\mathbb{Z}^d} = f\left((\varepsilon_{\mathbf{j}-\mathbf{i}})_{\mathbf{i}\in\mathbb{Z}^d}\right)$ be a centered Bernoulli random field with a finite moment of order p and for any positive integer n, let $b_n := (b_{n,\mathbf{i}})_{\mathbf{i}\in\mathbb{Z}^d}$ be an element of $\ell^2\left(\mathbb{Z}^d\right)$ such that for any $n \geqslant 1$, the set $\left\{\mathbf{k}\in\mathbb{Z}^d, b_{n,\mathbf{k}} \neq 0\right\}$ is finite, $\lim_{n\to+\infty} \left\|b_n\right\|_{\ell^2} = +\infty$ and (1.5.2) holds for any $q \in [d]$. Assume that for some positive α and β, the following series are convergent:*

$$C_2 := \sum_{\mathbf{i}\in\mathbb{Z}^d} \left(\|\mathbf{i}\|_\infty + 1\right)^{d/2+\alpha} \delta_{\mathbf{i},2} \text{ and } C_p := \sum_{\mathbf{i}\in\mathbb{Z}^d} \left(\|\mathbf{i}\|_\infty + 1\right)^{d(1-1/p)+\beta} \delta_{\mathbf{i},p}. \tag{1.5.5}$$

Let S_n be defined by (1.5.1),

$$\Delta_n := \sup_{t\in\mathbb{R}} \left|\mathbb{P}\left\{\frac{S_n}{\left\|b_n\right\|_{\ell^2}} \leqslant t\right\} - \Phi\,(t/\sigma)\right| \text{ and} \tag{1.5.6}$$

$$\varepsilon_n := \sum_{\mathbf{j}\in\mathbb{Z}^d} \mathbb{E}\left[X_{\mathbf{0}} X_{\mathbf{j}}\right] \left(\sum_{\mathbf{i}\in\mathbb{Z}^d} \frac{b_{n,\mathbf{i}} b_{n,\mathbf{i}+\mathbf{j}}}{\left\|b_n\right\|_{\ell^2}} - 1\right). \tag{1.5.7}$$

Assume that σ be given by (1.5.3) is positive. Let $\gamma > 0$ and let

$$n_0 := \inf\left\{N \geqslant 1 \mid \forall n \geqslant N, \sqrt{\sigma^2 + \varepsilon_n} - 20\,(\log 2)^{-1} C_2\left(\left[\left\|b_n\right\|_{\ell^2}\right]^\gamma\right)^{-\alpha} \geqslant \sigma/2\right\}. \tag{1.5.8}$$

Then for each $n \geqslant n_0$,

$$\begin{aligned}\Delta_n \leqslant{} & 150\left(20\left(\left[\left\|b_n\right\|_{\ell^2}\right] + 21\right)^\gamma + 21\right)^{(p'-1)d} \left\|X_{\mathbf{0}}\right\|_{p'}^{p'} \left(\frac{\left\|b_n\right\|_{\ell^{p'}}}{\left\|b_n\right\|_{\ell^2}}\right)^{p'} (\sigma/2)^{-p'} \\ & + \left(2\frac{|\varepsilon_n|}{\sigma^2} + 80\,(\log 2)^{-1} \frac{\left\|b_n\right\|_{\ell^2}^{-\gamma\alpha}}{\sigma^2} C_2^2\right)(2\pi e)^{-1/2} \\ & + \left(\frac{14.5p}{\sigma\log p} 4^{d/2} \left\|b_n\right\|_{\ell^2}^{-\gamma\alpha} C_2\right)^{\frac{p}{p+1}} + \left(\frac{\left\|b_n\right\|_{\ell^p}}{\sigma\left\|b_n\right\|_{\ell^2}} \frac{14.5p}{\log p} 4^{d(1-1/p)} \left\|b_n\right\|_{\ell^2}^{-\gamma\beta} C_p\right)^{\frac{p}{p+1}}.\end{aligned} \tag{1.5.9}$$

In particular, there exists a constant κ such that for all $n \geqslant n_0$,

$$\Delta_n \leqslant \kappa \left(\|b_n\|_{\ell^2}^{\gamma\left(p'-1\right)d-p'} \|b_n\|_{\ell^{p'}}^{p'} + |\varepsilon_n| + \|b_n\|_{\ell^2}^{-\gamma\alpha\frac{p}{p+1}} + \|b_n\|_{\ell^p}^{\frac{p}{p+1}} \|b_n\|_{\ell^2}^{-\frac{p}{p+1}(\gamma\beta+1)} \right). \quad (1.5.10)$$

Observe that (1.5.2), the fact that $\lim_{n\to+\infty} \|b_n\|_{\ell^2} = +\infty$ and finiteness of C_2 imply that the sequence $(\varepsilon_n)_{n\geqslant 1}$ converges to 0 hence n_0 is well-defined.

Recall notation (N.9). Let $(\Lambda_n)_{n\geqslant 1}$ be a sequence of subsets of $\mathbb{Z}^d$. The choice $b_{n,\mathbf{j}} = 1$ if $\mathbf{j} \in \Lambda_n$ and 0 otherwise yields the following corollary for set-indexed partial sums.

Corollary 1.10. *Let $(X_{\mathbf{i}})_{\mathbf{i}\in\mathbb{Z}^d}$ be a centered Bernoulli random field with a finite moment of order $p \geqslant 2$, $p' := \min\{p, 3\}$ and let $(\Lambda_n)_{n\geqslant 1}$ be a sequence of subset of $\mathbb{Z}^d$ such that $|\Lambda_n| \to +\infty$ and for any $\mathbf{k} \in \mathbb{Z}^d$, $\lim_{n\to+\infty} |\Lambda_n \cap (\Lambda_n - \mathbf{k})| / |\Lambda_n| = 0$. Assume that the series defined in (1.5.5) are convergent for some positive α and β and that σ defined by (1.5.3) is positive. Let $\gamma > 0$ and n_0 be defined by (1.5.8). There exists a constant κ such that for any $n \geqslant n_0$,*

$$\sup_{t\in\mathbb{R}} \left| \mathbb{P}\left\{ \frac{\sum_{\mathbf{i}\in\Lambda_n} X_{\mathbf{i}}}{|\Lambda_n|^{1/2}} \leqslant t \right\} - \Phi(t/\sigma) \right|$$
$$\leqslant \kappa \left(|\Lambda_n|^q + \sum_{\mathbf{j}\in\mathbb{Z}^d} |\mathbb{E}[X_{\mathbf{0}} X_{\mathbf{j}}]| \left| \frac{|\Lambda_n \cap (\Lambda_n - \mathbf{j})|}{|\Lambda_n|} - 1 \right| \right), \quad (1.5.11)$$

where

$$q := \max\left\{ \frac{\gamma(p'-1)d - p'}{2} + 1; -\gamma\alpha\frac{p}{2(p+1)}; \frac{1-p-p\gamma\beta}{p+1} \right\}. \quad (1.5.12)$$

1.6. **Rate of convergence of kernel estimates in a regression model.** We consider the following regression model:

$$Y_{\mathbf{i}} = g\left(\frac{\mathbf{i}}{n}\right) + X_{\mathbf{i}}, \quad \mathbf{i} \in \Lambda_n := \{1, \ldots, n\}^d, \quad (1.6.1)$$

where $g\colon [0,1]^d \to \mathbb{R}$ is an unknown smooth function and $(X_{\mathbf{i}})_{\mathbf{i}\in\mathbb{Z}^d}$ is a zero mean stationary Bernoulli random field. Let K be a probability kernel defined on $\mathbb{R}^d$ and let $(h_n)_{n\geqslant 1}$ be a sequence of positive numbers which converges to zero and which satisfies

$$\lim_{n\to+\infty} nh_n = +\infty \text{ and } \lim_{n\to+\infty} nh_n^{d+1} = 0. \quad (1.6.2)$$

We estimate the function g by the kernel estimator g_n defined by

$$g_n(\mathbf{x}) = \frac{\sum_{\mathbf{i}\in\Lambda_n} Y_{\mathbf{i}} K\left(\frac{\mathbf{x}-\mathbf{i}/n}{h_n}\right)}{\sum_{\mathbf{i}\in\Lambda_n} K\left(\frac{\mathbf{x}-\mathbf{i}/n}{h_n}\right)}, \quad x \in [0,1]^d, \quad (1.6.3)$$

where

$$\Lambda_n = \{1, \ldots, n\}^d. \quad (1.6.4)$$

We make the following assumptions on the regression function g and the probability kernel K:

(A1) The probability kernel K fulfils $\int_{\mathbb{R}^d} K(\mathbf{u})\,\mathrm{d}\mathbf{u} = 1$, is symmetric, non-negative, supported by $[-1,1]^d$. Moreover, there exist positive constants r, c and C such that for any $\mathbf{x}, \mathbf{y} \in [-1,1]^d$, $|K(\mathbf{x}) - K(\mathbf{y})| \leqslant r \|\mathbf{x} - \mathbf{y}\|_\infty$ and $c \leqslant K(\mathbf{x}) \leqslant C$.

(A2) There exists a positive constant B such that the absolute values of all the derivatives of first order of g are bounded by B on $[0,1]^d$.

Assumption (A2) will not be used in the following result. However, by Proposition 1 in [EM07], this guarantees that

$$\sup_{\mathbf{x}\in[0,1]^d}\sup_{g\in\operatorname{Lip}(B)}\left|\mathbb{E}\left[g_n\left(\mathbf{x}\right)\right]-g\left(\mathbf{x}\right)\right|=O\left(h_n\right), \tag{1.6.5}$$

where $\operatorname{Lip}(B)$ denotes the collection of all B-Lipschitz functions on $\mathbb{R}^d$.

Theorem 1.11. *Let $p>2$, $p':=\min\{p,3\}$ and let $\left(X_{\mathbf{j}}\right)_{\mathbf{j}\in\mathbb{Z}^d}=f\left(\left(\varepsilon_{\mathbf{j}-\mathbf{i}}\right)_{\mathbf{i}\in\mathbb{Z}^d}\right)$ be a centered Bernoulli random field with a finite moment of order p. Assume that for some positive α and β, the following series are convergent:*

$$C_2:=\sum_{\mathbf{i}\in\mathbb{Z}^d}\left(\|\mathbf{i}\|_\infty+1\right)^{d/2+\alpha}\delta_{\mathbf{i},2} \text{ and } C_p:=\sum_{\mathbf{i}\in\mathbb{Z}^d}\left(\|\mathbf{i}\|_\infty+1\right)^{d(1-1/p)+\beta}\delta_{\mathbf{i},p}. \tag{1.6.6}$$

Let $g_n(\mathbf{x})$ be defined by (1.6.3), $(h_n)_{n\geqslant1}$ be a sequence which converges to 0 and satisfies (1.6.2),

$$\Delta_n:=\sup_{t\in\mathbb{R}}\left|\mathbb{P}\left\{\left(nh_n\right)^{d/2}\left(g_n(\mathbf{x})-\mathbb{E}\left[g_n(\mathbf{x})\right]\right)\leqslant t\right\}-\Phi\left(\frac{t}{\sigma\left\|K\right\|_2}\right)\right|, \tag{1.6.7}$$

$$A_n:=\left(nh_n\right)^{d/2}\left(\sum_{\mathbf{i}\in\Lambda_n}K^2\left(\frac{\mathbf{x}-\mathbf{i}/n}{h_n}\right)\right)^{1/2}\|K\|_{\mathbb{L}^2(\mathbb{R}^d)}^{-1}\left(\sum_{\mathbf{i}\in\Lambda_n}K\left(\frac{\mathbf{x}-\mathbf{i}/n}{h_n}\right)\right)^{-1/2} \text{ and} \tag{1.6.8}$$

$$\varepsilon_n:=\sum_{\mathbf{j}\in\mathbb{Z}^d}\mathbb{E}\left[X_{\mathbf{0}}X_{\mathbf{j}}\right]\left(\sum_{\mathbf{i}\in\Lambda_n\cap(\Lambda_n-\mathbf{j})}\frac{K\left(\frac{\mathbf{x}-\mathbf{i}/n}{h_n}\right)K\left(\frac{\mathbf{x}-(\mathbf{i}-\mathbf{j})/n}{h_n}\right)}{\sum_{\mathbf{k}\in\Lambda_n}K^2\left(\frac{\mathbf{x}-\mathbf{k}/n}{h_n}\right)}-1\right). \tag{1.6.9}$$

Assume that $\sigma:=\sum_{\mathbf{j}\in\mathbb{Z}^d}\operatorname{Cov}\left(X_{\mathbf{0}},X_{\mathbf{j}}\right)>0$. Let $n_1\in\mathbb{N}$ be such that for each $n\geqslant n_1$,

$$\frac{1}{2}\leqslant\left(nh_n\right)^{-d}K\left(\frac{\mathbf{x}-\mathbf{i}/n}{h_n}\right)\leqslant\frac{3}{2} \text{ and} \tag{1.6.10}$$

$$\frac{1}{2}\|K\|_{\mathbb{L}^2(\mathbb{R}^d)}\leqslant\left(nh_n\right)^{-d}K^2\left(\frac{\mathbf{x}-\mathbf{i}/n}{h_n}\right)\leqslant\frac{3}{2}\|K\|_{\mathbb{L}^2(\mathbb{R}^d)}. \tag{1.6.11}$$

Then there exists a constant κ such that for each $n\geqslant\max\{n_0,n_1\}$,

$$\Delta_n\leqslant\kappa\left|A_n-1\right|^{\frac{p}{p+1}}+\left|\varepsilon_n\right|+\kappa\left(nh_n\right)^{\frac{d}{2}\left(\gamma\left(p'-1\right)d-p'+2\right)} + \left(nh_n\right)^{-\frac{d}{2}\gamma\alpha\frac{p}{p+1}}+\left(nh_n\right)^{\frac{2d-p(\gamma\beta+1)}{2(p+1)}}. \tag{1.6.12}$$

Lemma 1 in [EMS10] shows that under (1.6.2), the sequence $(A_n)_{n\geqslant1}$ goes to 1 as n goes to infinity and that the integer n_1 is well-defined. Moreover, a rate can be given in terms of K, by following the computations in the proof of that lemma.

2. Proofs

2.1. **Tools.**
We briefly review orthomartingales, following [Kho02].

Definition 2.1. *We say that $(\mathcal{F}_{\mathbf{i}})_{\mathbf{i}\in\mathbb{Z}^d}$ is a commuting filtration if for any integrable random variable Y and any $\mathbf{i},\mathbf{j}\in\mathbb{Z}^d$,*

$$\mathbb{E}\left[\mathbb{E}\left[Y \mid \mathcal{F}_{\mathbf{i}}\right] \mid \mathcal{F}_{\mathbf{j}}\right] = \mathbb{E}\left[Y \mid \mathcal{F}_{\min\{\mathbf{i},\mathbf{j}\}}\right]. \tag{2.1.1}$$

Definition 2.2. *A collection of random variables $\left\{M_{\mathbf{n}},\mathbf{n}\in\mathbb{Z}^d\right\}$ is said to be an orthomartingale random field with respect to the commuting filtration $(\mathcal{F}_{\mathbf{i}})_{\mathbf{i}\in\mathbb{Z}^d}$ if for all $\mathbf{n}\in\mathbb{Z}^d$, the random variable $M_{\mathbf{n}}$ is $\mathcal{F}_{\mathbf{n}}$-measurable, integrable and*

$$\text{for all } \mathbf{i},\mathbf{j}\in\mathbb{Z}^d \text{ such that } \mathbf{i}\preccurlyeq\mathbf{j}, \quad \mathbb{E}\left[M_{\mathbf{j}} \mid \mathcal{F}_{\mathbf{i}}\right] = M_{\mathbf{i}}. \tag{2.1.2}$$

Definition 2.3. *A collection of random variables $\left\{D_{\mathbf{n}},\mathbf{n}\in\mathbb{Z}^d\right\}$ (where only finite many of them are not zero) is said to be an orthomartingale differences random field with respect to the commuting filtration $(\mathcal{F}_{\mathbf{i}})_{\mathbf{i}\in\mathbb{Z}^d}$ if for all $\mathbf{n}\in\mathbb{Z}^d$, the random variable $D_{\mathbf{n}}$ is $\mathcal{F}_{\mathbf{n}}$-measurable, integrable and the random field $(M_{\mathbf{n}})_{\mathbf{n}\in\mathbb{Z}^d}$ defined by $M_{\mathbf{n}} := \sum_{\mathbf{i}\preccurlyeq\mathbf{n}} D_{\mathbf{i}}$ is an orthomartingale differences random field.*

Notice that $\left\{D_{\mathbf{n}},\mathbf{n}\in\mathbb{Z}^d\right\}$ is said to be an orthomartingale differences random field with respect to the commuting filtration $(\mathcal{F}_{\mathbf{i}})_{\mathbf{i}\in\mathbb{Z}^d}$ if and only if for each $\mathbf{n}\in\mathbb{Z}^d$ and each $q\in[d]$, $\mathbb{E}\left[M_{\mathbf{n}} \mid \mathcal{F}_{\mathbf{n}-\mathbf{e_q}}\right]=0$.

Observe that if $\left\{D_{\mathbf{n}},\mathbf{n}\in\mathbb{Z}^d\right\}$ is an orthomartingale differences random field, then for any fixed $(n_1,\dots,n_{d-1})\in\mathbb{Z}^{d-1}$, the random field $\left(D_{(n_1,\dots,n_{d-1},n)}\right)_{n\in\mathbb{Z}}$ is a martingale differences sequence for the filtration $(\mathcal{G}_n)_{n\in\mathbb{Z}}$, where

$$\mathcal{G}_n = \sigma\left(\bigcup_{n_1,\dots,n_{d-1}\in\mathbb{Z}^{d-1}} \mathcal{F}_{(n_1,\dots,n_{d-1},n)}\right). \tag{2.1.3}$$

Rio's version of Burkholder's inequality (Theorem 2.1 in [Rio09]) states the following: if $(X_n)_{n\geqslant 1}$ is a martingale differences sequence and $p\geqslant 2$, then

$$\left\|\sum_{i=1}^n X_i\right\|_p^2 \leqslant (p-1)\sum_{i=1}^n \left\|X_i\right\|_p^2. \tag{2.1.4}$$

With the previous observation, this can be extended to orthomartingale differences by induction on the dimension.

Lemma 2.4. *Let $(Y_{\mathbf{i}})_{\mathbf{m}\preccurlyeq\mathbf{i}\preccurlyeq\mathbf{n},\mathbf{i}\in\mathbb{Z}^d}$ be an orthomartingale differences random field with respect to the commuting filtration $(\mathcal{F}_{\mathbf{i}})_{\mathbf{m}\preccurlyeq\mathbf{i}\preccurlyeq\mathbf{n},\mathbf{i}\in\mathbb{Z}^d}$. Then*

$$\left\|\sum_{\mathbf{m}\preccurlyeq\mathbf{i}\preccurlyeq\mathbf{n}} Y_{\mathbf{i}}\right\|_p^2 \leqslant (p-1)^d \sum_{\mathbf{m}\preccurlyeq\mathbf{i}\preccurlyeq\mathbf{n}} \left\|Y_{\mathbf{i}}\right\|_p^2. \tag{2.1.5}$$

2.2. **Proof of Theorem 1.4.** We assume that there are only finitely many indexes $\mathbf{i}$ such that $a_{\mathbf{i}} \neq 0$. The general case can be deduced from the latter by considering for a fixed u the weights $a_{\mathbf{i}}^{(u)}$ instead of $a_{\mathbf{i}}$, where

$$a_{\mathbf{i}}^{(u)} = \begin{cases} a_{\mathbf{i}} & \text{if } \|\mathbf{i}\|_\infty \leqslant m, \\ 0 & \text{otherwise,} \end{cases} \tag{2.2.1}$$

and letting u going to infinity.

For $\mathbf{i} \in \mathbb{Z}^d$ and $\mathbf{v} \in \mathbb{N}^d$, we define

$$X_{\mathbf{i},\mathbf{v}} = \mathbb{E}\left[X_{\mathbf{i}} \mid \sigma\left(\varepsilon_{\mathbf{u}}, \mathbf{i}-\mathbf{v} \preccurlyeq \mathbf{u} \preccurlyeq \mathbf{i}+\mathbf{v}\right)\right] \tag{2.2.2}$$

and for a subset J of $[d]$ and a positive integer n,

$$X_{\mathbf{i},J}^{(n)} = \sum_{I \subset J} (-1)^{|I|} \sum_{\mathbf{0_J} \preccurlyeq \mathbf{k_J} \preccurlyeq n\mathbf{1_J}} X_{\mathbf{i},\mathbf{k_I}+\mathbf{1_I}}. \tag{2.2.3}$$

In this way,

$$X_{\mathbf{i},J}^{(n)} = \sum_{I \subset J} (-1)^{|I|} X_{\mathbf{i},(n+1)\mathbf{1_I}}, \tag{2.2.4}$$

hence

$$\begin{aligned} \sum_{J \subset [d]} (-1)^{|J|} X_{i,J}^{(n)} &= \sum_{J \subset [d]} (-1)^{|J|} \sum_{I \subset J} (-1)^{|I|} X_{\mathbf{i},(n+1)\mathbf{1_I}} \\ &= \sum_{I \subset [d]} (-1)^{|I|} X_{\mathbf{i},(n+1)\mathbf{1_I}} \sum_{I \subset J \subset [d]} (-1)^{|J|}. \end{aligned}$$

Taking into account that $\sum_{I \subset J \subset [d]} (-1)^{|J|}$ equals zero except when $I = [d]$, we derive that

$$\sum_{J \subset [d]} (-1)^{|J|} X_{\mathbf{i},J}^{(n)} = (-1)^d X_{\mathbf{i},(n+1)\mathbf{1}}. \tag{2.2.5}$$

Therefore, by martingale convergence theorem,

$$\left\| \sum_{\mathbf{i} \in \mathbb{Z}^d} a_{\mathbf{i}} X_{\mathbf{i}} \right\|_p = \lim_{n \to +\infty} \left\| \sum_{\mathbf{i} \in \mathbb{Z}^d} a_{\mathbf{i}} \sum_{J \subset [d]} (-1)^{|J|} X_{\mathbf{i},J}^{(n)} \right\|_p \tag{2.2.6}$$

$$\leqslant \lim_{n \to +\infty} \sum_{J \subset [d]} \left\| \sum_{\mathbf{i} \in \mathbb{Z}^d} a_{\mathbf{i}} X_{\mathbf{i},J}^{(n)} \right\|_p, \tag{2.2.7}$$

and by (2.2.3), we have.

$$\sum_{\mathbf{i} \in \mathbb{Z}^d} a_{\mathbf{i}} X_{\mathbf{i},J}^{(n)} = \sum_{\mathbf{0_J} \preccurlyeq \mathbf{k_J} \preccurlyeq n\mathbf{1_J}} \sum_{\mathbf{i} \in \mathbb{Z}^d} a_{\mathbf{i}} \sum_{I \subset J} (-1)^{|I|} X_{\mathbf{i},\mathbf{k_I}+\mathbf{1_I}}. \tag{2.2.8}$$

Using the triangular inequality, it follows that

$$\left\| \sum_{\mathbf{i} \in \mathbb{Z}^d} a_{\mathbf{i}} X_{\mathbf{i}} \right\|_p \leqslant \sum_{J \subset [d]} \sum_{\mathbf{0_J} \preccurlyeq \mathbf{k_J}} \left\| \sum_{\mathbf{i} \in \mathbb{Z}^d} a_{\mathbf{i}} \sum_{I \subset J} (-1)^{|I|} X_{\mathbf{i},\mathbf{k_I}+\mathbf{1_I}} \right\|_p. \tag{2.2.9}$$

We are thus reduced to estimate the quantities

$$c_{J,\mathbf{k}} := \left\| \sum_{\mathbf{i}\in\mathbb{Z}^d} a_{\mathbf{i}} \sum_{I\subset J} (-1)^{|I|} X_{\mathbf{i},\mathbf{k}_{\mathbf{I}}+\mathbf{1}_{\mathbf{I}}} \right\|_p, \quad J\subset [d], \mathbf{k}\in\mathbb{Z}^d. \tag{2.2.10}$$

Let us fix $J\subset[d]$ and $\mathbf{k}\in\mathbb{Z}^d$. Define

$$A_{\mathbf{v}} := \prod_{q=1}^d A_{\mathbf{v},q}, \quad \mathbf{v}\in\mathbb{Z}^d, \tag{2.2.11}$$

where

$$A_{\mathbf{v},q} := \begin{cases} [(2k_q+2)\,v_j, (2k_q+2)\,(v_j+1)) & \text{if } q\in J,\\ \{v_q\} & \text{if } q\notin J, \end{cases} \tag{2.2.12}$$

and for a subset K of J,

$$E_K := \left\{\mathbf{v}\in\mathbb{Z}^d, v_q \text{ is even if and only if } q\in K\right\}. \tag{2.2.13}$$

By (2.2.10) and the fact that $\mathbb{Z}^d$ is the disjoint union $\bigcup_{K\subset J}\bigcup_{\mathbf{v}\in E_K} A_{\mathbf{v}}$, we derive that

$$c_{J,\mathbf{k}} \leqslant \sum_{K\subset J} \left\| \sum_{\mathbf{v}\in E_K}\sum_{\mathbf{i}\in A_{\mathbf{v}}} a_{\mathbf{i}} \sum_{I\subset J} (-1)^{|I|} X_{\mathbf{i},\mathbf{k}_{\mathbf{I}}+\mathbf{1}_{\mathbf{I}}} \right\|_p. \tag{2.2.14}$$

Lemma 2.5. *For any $K\subset J$, the collection of random variables*

$$\left\{ \sum_{\mathbf{i}\in A_{\mathbf{v}}} a_{\mathbf{i}} \sum_{I\subset J} (-1)^{|I|} X_{\mathbf{i},\mathbf{k}_{\mathbf{I}}+\mathbf{1}_{\mathbf{I}}}, \mathbf{v}\in E_K \right\} \tag{2.2.15}$$

is zero-mean and independent.

Proof. That $\sum_{\mathbf{i}\in A_{\mathbf{v}}} a_{\mathbf{i}} \sum_{I\subset J} (-1)^{|I|} X_{\mathbf{i},\mathbf{k}_{\mathbf{I}}+\mathbf{1}_{\mathbf{I}}}$ follows from the fact that each random variable $X_{\mathbf{i}}$, $\mathbf{i}\in\mathbb{Z}^d$ is centered. For a subset S of $\mathbb{Z}^d$, denote by $\mathcal{F}_S$ the σ-algebra generated by the random variables $\varepsilon_{\mathbf{s}}$, $s\in S$. Then for any $\mathbf{v}\in E_K$, the random variable $\sum_{\mathbf{i}\in A_{\mathbf{v}}} a_{\mathbf{i}} \sum_{I\subset J} (-1)^{|I|} X_{\mathbf{i},\mathbf{k}_{\mathbf{I}}+\mathbf{1}_{\mathbf{I}}}$ is $\mathcal{F}_{S_{\mathbf{v}}}$-measurable, where $S_{\mathbf{v}} = \prod_{q=1}^d S_{\mathbf{v},q}$ and

$$S_{\mathbf{v},q} := \begin{cases} [(2k_q+2)\,v_j - k_q - 1, (2k_q+2)\,(v_j+1)+k_q+1) & \text{if } q\in J,\\ \{v_q\} & \text{if } q\notin J. \end{cases} \tag{2.2.16}$$

If $\mathbf{v}$ and $\mathbf{v}'$ are two distinct elements of E_K, then $S_{\mathbf{v}}\cap S_{\mathbf{v}'}$ is empty. Since the set $\left\{\varepsilon_{\mathbf{u}}, \mathbf{u}\in\mathbb{Z}^d\right\}$ is independent, this proves Lemma 2.5. ☐

Combining Lemma 2.5 with inequality (1.2.2), we derive form (2.2.14)

$$c_{J,\mathbf{k}} \leqslant \sum_{K\subset J} \frac{14.5p}{\log p} \left(\sum_{\mathbf{v}\in E_K} \mathbb{E}\left[\left(\sum_{\mathbf{i}\in A_{\mathbf{v}}} a_{\mathbf{i}} \sum_{I\subset J} (-1)^{|I|} X_{\mathbf{i},\mathbf{k}_{\mathbf{I}}+\mathbf{1}_{\mathbf{I}}} \right)^2 \right] \right)^{1/2}$$
$$+ \sum_{K\subset J} \frac{14.5p}{\log p} \left(\sum_{\mathbf{v}\in E_K} \mathbb{E}\left[\left| \sum_{\mathbf{i}\in A_{\mathbf{v}}} a_{\mathbf{i}} \sum_{I\subset J} (-1)^{|I|} X_{\mathbf{i},\mathbf{k}_{\mathbf{I}}+\mathbf{1}_{\mathbf{I}}} \right|^p \right] \right)^{1/p}. \tag{2.2.17}$$

Lemma 2.6. *The random field* $(Y_{\mathbf{i}})_{\mathbf{i}\in\mathbb{Z}^d}$ *defined by*

$$Y_{\mathbf{i}} = \begin{cases} a_{\mathbf{i}} \sum_{I\subset J} (-1)^{|I|} X_{\mathbf{i},\mathbf{k_I}+\mathbf{1_I}} & \text{if } \mathbf{i}\in A_{\mathbf{v}} \\ 0 & \text{otherwise,} \end{cases} \tag{2.2.18}$$

is an orthomartingale differences random field with respect to the commuting filtration $(\mathcal{F}_{\mathbf{i}})_{\mathbf{i}\in\mathbb{Z}^d}$ *defined by*

$$\mathcal{F}_{\mathbf{i}} := \sigma\left(\varepsilon_{\mathbf{u}}, \mathbf{u} \preccurlyeq \mathbf{i} + \mathbf{k_J} + \mathbf{1_J}\right). \tag{2.2.19}$$

Proof. Measurability of $Y_{\mathbf{i}}$ with respect to $\mathcal{F}_{\mathbf{i}}$ follows by the definition of $X_{\mathbf{i},\mathbf{v}}$ given in (2.2.2). Let $\mathbf{i}\in A_{\mathbf{v}}$ and let us check that for any $q\in[d]$,

$$\mathbb{E}\left[Y_{\mathbf{i}} \mid \mathcal{F}_{\mathbf{i}-\mathbf{e_q}}\right] = 0. \tag{2.2.20}$$

If q does not belong to J, then (2.2.20) holds because $Y_{\mathbf{i}}$ is centered and measurable with respect to the σ-algebra generated by $\varepsilon_{\mathbf{u}}, u_q = i_q$, which is independent of $\mathcal{F}_{\mathbf{i}-\mathbf{e_q}}$. Assume now that q belongs to J. Then the equality

$$\begin{aligned}\mathbb{E}\left[Y_{\mathbf{i}} \mid \mathcal{F}_{\mathbf{i}-\mathbf{e_q}}\right] = a_{\mathbf{i}} &\sum_{I\subset J, q\notin I} (-1)^{|I|}\,\mathbb{E}\left[X_{\mathbf{i},\mathbf{k_I}+\mathbf{1_I}} \mid \mathcal{F}_{\mathbf{i}-\mathbf{e_q}}\right] \\ &- a_{\mathbf{i}} \sum_{I\subset J, q\notin I} (-1)^{|I|}\,\mathbb{E}\left[X_{\mathbf{i},\mathbf{k}_{\mathbf{I}\cup\{\mathbf{q}\}}+\mathbf{1}_{\mathbf{I}\cup\{\mathbf{q}\}}} \mid \mathcal{F}_{\mathbf{i}-\mathbf{e_q}}\right]\end{aligned} \tag{2.2.21}$$

holds. For a subset S of $\mathbb{Z}^d$, denote by $\mathcal{F}_S$ the σ-algebra generated by the random variables $\varepsilon_{\mathbf{s}}$, $s\in S$. By Proposition in [WW13], equality $\mathbb{E}\left[\mathbb{E}\left[Y \mid \mathcal{F}_S\right] \mid \mathcal{F}_{S'}\right] = \mathbb{E}\left[Y \mid \mathcal{F}_{S\cap S'}\right]$ takes place for any integrable random variable Y. Applying this to

$$S = \left\{\mathbf{u}\in\mathbb{Z}^d, \mathbf{i} - \mathbf{k}_{\mathbf{I}\cup\{\mathbf{q}\}} - \mathbf{1}_{\mathbf{I}\cup\{\mathbf{q}\}} \preccurlyeq \mathbf{u} \preccurlyeq \mathbf{i} + \mathbf{k}_{\mathbf{I}\cup\{\mathbf{q}\}} + \mathbf{1}_{\mathbf{I}\cup\{\mathbf{q}\}}\right\} \tag{2.2.22}$$

and $S' = \{\mathbf{u}, \mathbf{u}\preccurlyeq \mathbf{i}+\mathbf{k_J}+\mathbf{1_J}\}$, we derive (2.2.20). This ends the proof of Lemma 2.6. $\square$

Using Lemmas 2.4 and 2.6, we infer from (2.2.17) that

$$\begin{aligned} c_{J,\mathbf{k}} \leqslant &\sum_{K\subset J} \frac{14.5p}{\log p} \left(\sum_{\mathbf{v}\in E_K}\sum_{\mathbf{i}\in A_{\mathbf{v}}} a_{\mathbf{i}}^2\,\mathbb{E}\left[\left(\sum_{I\subset J}(-1)^{|I|} X_{\mathbf{i},\mathbf{k_I}+\mathbf{1_I}}\right)^2\right]\right)^{1/2} \\ &+ \sum_{K\subset J}\frac{14.5p}{\log p}(p-1)^{d/2}\left(\sum_{\mathbf{v}\in E_K}\left(\sum_{\mathbf{i}\in A_{\mathbf{v}}} |a_{\mathbf{i}}|^2 \left\|\sum_{I\subset J}(-1)^{|I|} X_{\mathbf{i},\mathbf{k_I}+\mathbf{1_I}}\right\|_p^2\right)^{p/2}\right)^{1/p}. \end{aligned} \tag{2.2.23}$$

Lemma 2.7. *The random field* $\left(\sum_{I\subset J}(-1)^{|I|} X_{\mathbf{i},\mathbf{k_I}+\mathbf{1_I}}\right)_{\mathbf{i}\in\mathbb{Z}^d}$ *is identically distributed and*

$$\left\|\sum_{I\subset J}(-1)^{|I|} X_{\mathbf{i},\mathbf{k_I}+\mathbf{1_I}}\right\|_q = \left\|\pi_{\mathbf{k}}\right\|_q, \quad q\in\{2,p\}. \tag{2.2.24}$$

Proof. Denote by $T^{\mathbf{i}}$ the map defined from $\mathbb{R}^{\mathbb{Z}^d}$ to itself by $T^{\mathbf{i}}\left((x_{\mathbf{j}})_{\mathbf{j}\in\mathbb{Z}^d}\right) = (x_{\mathbf{i}+\mathbf{j}})_{\mathbf{j}\in\mathbb{Z}^d}$. Then the equality

$$\sum_{I\subset J}(-1)^{|I|} X_{\mathbf{i},\mathbf{k_I}+\mathbf{1_I}} = \left(\sum_{I\subset J}(-1)^{|I|} X_{\mathbf{0},\mathbf{k_I}+\mathbf{1_I}}\right)\circ T^{\mathbf{i}} \tag{2.2.25}$$

holds. Observe that for any function $f\colon \mathbb{R}^{\mathbb{Z}^d} \to \mathbb{R}$, the random fields $\left(f\left(\varepsilon_{\mathbf{n}-\mathbf{i}}\right)\right)_{\mathbf{i}\in\mathbb{Z}^d}$ and $\left(f\circ T^l\left(\varepsilon_{\mathbf{n}-\mathbf{i}}\right)\right)_{\mathbf{i}\in\mathbb{Z}^d}$ have the same distribution. Equality (2.2.24) follows from the fact that $\pi_{\mathbf{k}} = \sum_{I\subset J}\left(-1\right)^{|I|} X_{\mathbf{0},\mathbf{k}_{\mathbf{I}}+\mathbf{1}_{\mathbf{I}}}$. This ends the proof of Lemma 2.7. $\square$

By Lemma 2.7 and (2.2.23), we derive that

$$c_{J,\mathbf{k}} \leqslant \sum_{K\subset J}\frac{14.5p}{\log p}\left\|\pi_{\mathbf{k}}\right\|_2\left(\sum_{\mathbf{v}\in E_K}\sum_{\mathbf{i}\in A_{\mathbf{v}}} a_{\mathbf{i}}^2\right)^{1/2} + \sum_{K\subset J}\frac{14.5p}{\log p}\left(p-1\right)^{d/2}\left\|\pi_{\mathbf{k}}\right\|_2\left(\sum_{\mathbf{v}\in E_K}\left(\sum_{\mathbf{i}\in A_{\mathbf{v}}}\left|a_{\mathbf{i}}\right|^2\right)^{p/2}\right)^{1/p}. \tag{2.2.26}$$

Defining $B_K := \bigcup_{\mathbf{v}\in E_K} A_{\mathbf{v}}$, the equality

$$\sum_{\mathbf{v}\in E_K}\sum_{\mathbf{i}\in A_{\mathbf{v}}} a_{\mathbf{i}}^2 = \sum_{\mathbf{i}\in B_K} a_{\mathbf{i}}^2 \tag{2.2.27}$$

takes place, by disjointness of the family $\left(A_{\mathbf{v}}\right)_{\mathbf{v}\in E_K}$. Moreover, since $p\geqslant 2$, the function $t\mapsto t^{p/2}$ is convex on $\mathbb{R}_+$ hence by Jensen's inequality,

$$\sum_{\mathbf{v}\in E_K}\left(\sum_{\mathbf{i}\in A_{\mathbf{v}}}\left|a_{\mathbf{i}}\right|^2\right)^{p/2} = \sum_{\mathbf{v}\in E_K}\left|A_{\mathbf{v}}\right|^{p/2}\left(\frac{1}{\left|A_{\mathbf{v}}\right|}\sum_{\mathbf{i}\in A_{\mathbf{v}}}\left|a_{\mathbf{i}}\right|^2\right)^{p/2} \tag{2.2.28}$$

$$\leqslant \sum_{\mathbf{v}\in E_K}\left|A_{\mathbf{v}}\right|^{p/2-1}\sum_{\mathbf{i}\in A_{\mathbf{v}}}\left|a_{\mathbf{i}}\right|^p. \tag{2.2.29}$$

By (2.2.11) and (2.2.12), $\left|A_{\mathbf{v}}\right| = \prod_{q\in J}\left(2k_q+2\right)$ hence

$$\sum_{\mathbf{v}\in E_K}\left(\sum_{\mathbf{i}\in A_{\mathbf{v}}}\left|a_{\mathbf{i}}\right|^2\right)^{p/2} \leqslant \prod_{q\in J}\left(2k_q+2\right)^{p/2-1}\sum_{\mathbf{i}\in B_K}\left|a_{\mathbf{i}}\right|^p. \tag{2.2.30}$$

Plugging the bounds (2.2.27) and (2.2.30) into (2.2.26), we deduce

$$c_{J,\mathbf{k}} \leqslant \sum_{K\subset J}\frac{14.5p}{\log p}\left\|\pi_{\mathbf{k}}\right\|_2\left(\sum_{\mathbf{i}\in B_K} a_{\mathbf{i}}^2\right)^{1/2} + \sum_{K\subset J}\frac{14.5p}{\log p}\left(p-1\right)^{d/2}\prod_{q\in J}\left(2k_q+2\right)^{1/2-1/p}\left\|\pi_{\mathbf{k}}\right\|_2\left(\sum_{\mathbf{i}\in B_K}\left|a_{\mathbf{i}}\right|^p\right)^{1/p}. \tag{2.2.31}$$

Using concavity of the function $t \mapsto t^{1/q}$ on $\mathbb{R}_+$ for $q \in \{2, p\}$, we derive equality $\sum_{i=1}^N x_i^{1/q} \leqslant N^{\frac{q-1}{q}} \left(\sum_{i=1}^N x_i\right)^{1/q}$. Since $\mathbb{Z}^d$ is the disjoint union $\bigcup_{K \subset J} B_K$, we eventually get

$$c_{J,\mathbf{k}} \leqslant 2^{d/2} \frac{14.5p}{\log p} \|\pi_{\mathbf{k}}\|_2 \left(\sum_{\mathbf{i} \in \mathbb{Z}^d} a_{\mathbf{i}}^2\right)^{1/2} + 2^{d\frac{2p-1}{p}} \frac{14.5p}{\log p} (p-1)^{d/2} \prod_{q \in J} (k_q + 1)^{1/2 - 1/p} \|\pi_{\mathbf{k}}\|_p \left(\sum_{\mathbf{i} \in \mathbb{Z}^d} |a_{\mathbf{i}}|^p\right)^{1/p}. \tag{2.2.32}$$

Inequality (1.4.1) follows from (2.2.9), (2.2.10) and (2.2.32).

2.3. **Proof of Theorem 1.6.** We define for $j \in \mathbb{N}$ and $\mathbf{i} \in \mathbb{Z}^d$,

$$X_{\mathbf{i},j} = \mathbb{E}\left[X_i \mid \sigma\left(\varepsilon_{\mathbf{u}}, \|\mathbf{u} - \mathbf{i}\|_\infty \leqslant j + 1\right)\right] - \mathbb{E}\left[X_i \mid \sigma\left(\varepsilon_{\mathbf{u}}, \|\mathbf{u} - \mathbf{i}\|_\infty \leqslant j\right)\right]. \tag{2.3.1}$$

In this way, by martingale convergence,

$$X_{\mathbf{i}} - \mathbb{E}\left[X_{\mathbf{i}} \mid \varepsilon_{\mathbf{i}}\right] = \lim_{N \to +\infty} \sum_{j=0}^{N} X_{\mathbf{i},j} \tag{2.3.2}$$

hence

$$\left\|\sum_{\mathbf{i} \in \mathbb{Z}^d} a_{\mathbf{i}} X_{\mathbf{i}}\right\|_p \leqslant \sum_{j=0}^{+\infty} \left\|\sum_{\mathbf{i} \in \mathbb{Z}^d} a_{\mathbf{i}} X_{\mathbf{i},j}\right\|_p + \left\|\sum_{\mathbf{i} \in \mathbb{Z}^d} a_{\mathbf{i}} \mathbb{E}\left[X_{\mathbf{i}} \mid \varepsilon_{\mathbf{i}}\right]\right\|_p \tag{2.3.3}$$

Let us fix $j \in \mathbb{N}$. Like in the proof of Theorem 1.4, we shall divide $\mathbb{Z}^d$ into blocks. For $\mathbf{v} \in \mathbb{Z}^d$, we define

$$A_{\mathbf{v}} := \prod_{q=1}^{d} \left[(2j+2) v_q, (2j+2)(v_q+1)\right), \tag{2.3.4}$$

and if K is a subset of $[d]$, we define

$$E_K := \left\{\mathbf{v} \in \mathbb{Z}^d, v_q \text{ is even if and only if } q \in K\right\}. \tag{2.3.5}$$

Therefore, the following inequality takes place

$$\left\|\sum_{\mathbf{i} \in \mathbb{Z}^d} a_{\mathbf{i}} X_{\mathbf{i},j}\right\|_p \leqslant \sum_{K \subset [d]} \left\|\sum_{\mathbf{v} \in E_K} \sum_{\mathbf{i} \in A_{\mathbf{v}}} a_{\mathbf{i}} X_{\mathbf{i},j}\right\|_p \tag{2.3.6}$$

Observe that the random variable $\sum_{\mathbf{i} \in A_{\mathbf{v}}} a_{\mathbf{i}} X_{\mathbf{i},j}$ is measurable for the σ-algebra generated by $\varepsilon_{\mathbf{u}}$, where $\mathbf{u}$ satisfies $(2j+2) v_q - (j+1) \leqslant u_q \leqslant j + 1 + (2j+2)(v_q+1) - 1$ for all $q \in [d]$. Since the family $\left\{\varepsilon_{\mathbf{u}}, \mathbf{u} \in \mathbb{Z}^d\right\}$ is independent, the family $\left\{\sum_{\mathbf{i} \in A_{\mathbf{v}}} a_{\mathbf{i}} X_{\mathbf{i},j}, \mathbf{v} \in E_K\right\}$ is independent for all $K \subset [d]$. Using inequality (1.2.2), it thus follows that

$$\left\|\sum_{\mathbf{v} \in E_K} \sum_{\mathbf{i} \in A_{\mathbf{v}}} a_{\mathbf{i}} X_{\mathbf{i},j}\right\|_p \leqslant \frac{14.5p}{\log p} \left(\sum_{\mathbf{v} \in E_K} \left\|\sum_{\mathbf{i} \in A_{\mathbf{v}}} a_{\mathbf{i}} X_{\mathbf{i},j}\right\|_2^2\right)^{1/2} + \frac{14.5p}{\log p} \left(\sum_{\mathbf{v} \in E_K} \left\|\sum_{\mathbf{i} \in A_{\mathbf{v}}} a_{\mathbf{i}} X_{\mathbf{i},j}\right\|_p^p\right)^{1/p}. \tag{2.3.7}$$

Adapting the proof of Lemma 2.7, one can see that $\|X_{\mathbf{i},j}\|_q = \|X_{\mathbf{0},j}\|_q$ for $q \in \{2, p\}$, hence the triangle inequality yields

$$\left\| \sum_{\mathbf{v} \in E_K} \sum_{\mathbf{i} \in A_{\mathbf{v}}} a_{\mathbf{i}} X_{\mathbf{i},j} \right\|_p \leqslant \frac{14.5p}{\log p} \|X_{\mathbf{0},j}\|_2 \left(\sum_{\mathbf{v} \in E_K} \left(\sum_{\mathbf{i} \in A_{\mathbf{v}}} |a_{\mathbf{i}}| \right)^2 \right)^{1/2} + \frac{14.5p}{\log p} \|X_{\mathbf{0},j}\|_p \left(\sum_{\mathbf{v} \in E_K} \left(\sum_{\mathbf{i} \in A_{\mathbf{v}}} |a_{\mathbf{i}}| \right)^p \right)^{1/p}. \quad (2.3.8)$$

By Jensen's inequality, for $q \in \{2, p\}$,

$$\left(\sum_{\mathbf{i} \in A_{\mathbf{v}}} |a_{\mathbf{i}}| \right)^q \leqslant |A_{\mathbf{v}}|^{q-1} \sum_{\mathbf{i} \in A_{\mathbf{v}}} |a_{\mathbf{i}}|^q \leqslant (2j+2)^{d(q-1)} \sum_{\mathbf{i} \in A_{\mathbf{v}}} |a_{\mathbf{i}}|^q \quad (2.3.9)$$

and using $\sum_{i=1}^N x_i^{1/q} \leqslant N^{\frac{q-1}{q}} \left(\sum_{i=1}^N x_i \right)^{1/q}$, it follows that

$$\sum_{K \subset [d]} \left\| \sum_{\mathbf{v} \in E_K} \sum_{\mathbf{i} \in A_{\mathbf{v}}} a_{\mathbf{i}} X_{\mathbf{i},j} \right\|_p \leqslant \frac{14.5p}{\log p} \|X_{\mathbf{0},j}\|_2 \left(\sum_{\mathbf{i} \in \mathbb{Z}^d} a_i^2 \right)^{1/2} (4j+4)^{d/2} + \frac{14.5p}{\log p} \|X_{\mathbf{0},j}\|_p \left(\sum_{\mathbf{i} \in \mathbb{Z}^d} |a_i|^p \right)^{1/p} (4j+4)^{d(1-1/p)}. \quad (2.3.10)$$

Lemma 2.8. *For $q \in \{2, p\}$ and $j \in \mathbb{N}$, the following inequality holds*

$$\|X_{\mathbf{0},j}\|_q \leqslant \sum_{\mathbf{i} \in \mathbb{Z}^d, \|\mathbf{i}\|_\infty = j+1} \delta_{\mathbf{i},p}. \quad (2.3.11)$$

Proof. Let us write the set of elements of $\mathbb{Z}^d$ whose infinite norm is equal to $j+1$ as $\{\mathbf{v_s}, 1 \leqslant s \leqslant N_j\}$ where $N_j \in \mathbb{N}$ and denote

$$\mathcal{F}_s := \sigma\left(\varepsilon_{\mathbf{u}}, \|\mathbf{u}\|_\infty \leqslant j, \varepsilon_{\mathbf{v_t}}, 1 \leqslant t \leqslant s\right), \quad (2.3.12)$$

and $\mathcal{F}_0 := \sigma\left(\varepsilon_{\mathbf{u}}, \|\mathbf{u}\|_\infty \leqslant j\right)$. Then $X_{\mathbf{0},j} = \sum_{s=1}^{N_j} \mathbb{E}\left[X_{\mathbf{0}} \mid \mathcal{F}_s\right] - \mathbb{E}\left[X_{\mathbf{0}} \mid \mathcal{F}_{s-1}\right]$, from which it follows that

$$\|X_{\mathbf{0},j}\|_q \leqslant \sum_{s=1}^{N_j} \left\| \mathbb{E}\left[X_{\mathbf{0}} \mid \mathcal{F}_s\right] - \mathbb{E}\left[X_{\mathbf{0}} \mid \mathcal{F}_{s-1}\right] \right\|_q. \quad (2.3.13)$$

Let $Y_{\mathbf{0}}^{(s)} := f\left(\left(\varepsilon^*_{-\mathbf{u}}\right)_{\mathbf{u} \in \mathbb{Z}^d}\right)$, where $\varepsilon^*_{\mathbf{v}} = \varepsilon_{\mathbf{v}}$ if $\mathbf{v} \neq \mathbf{v_s}$ and $\varepsilon^*_{\mathbf{v_s}} = \varepsilon'_{\mathbf{v_s}}$, where $\left(\varepsilon'_{\mathbf{u}}\right)_{\mathbf{u} \in \mathbb{Z}^d}$ is an independent copy of $\left(\varepsilon_{\mathbf{u}}\right)_{\mathbf{u} \in \mathbb{Z}^d}$. Since $\varepsilon_{\mathbf{v_s}}$ is independent of the σ-algebra generated by $Y_{\mathbf{0}}^{(s)}$ and $\varepsilon^*_{\mathbf{v_s}}$, we have

$$\mathbb{E}\left[X_{\mathbf{0}} \mid \mathcal{F}_s\right] = \mathbb{E}\left[X_{\mathbf{0}} - Y_{\mathbf{0}}^{(s)} \mid \mathcal{F}_s\right] + \mathbb{E}\left[Y_{\mathbf{0}}^{(s)} \mid \mathcal{F}_{s-1}\right]. \quad (2.3.14)$$

If A belongs to $\mathcal{F}_{s-1}$, then $Y_{\mathbf{0}}^{(s)} \mathbf{1}_A$ and $X_{\mathbf{0}} \mathbf{1}_A$ have the same distribution hence $\mathbb{E}\left[Y_{\mathbf{0}}^{(s)} \mid \mathcal{F}_{s-1}\right] = \mathbb{E}\left[X_{\mathbf{0}} \mid \mathcal{F}_{s-1}\right]$. Consequently,

$$\|X_{\mathbf{0},j}\|_q \leqslant \sum_{s=1}^{N_j} \left\| X_{\mathbf{0}} - Y_{\mathbf{0}}^{(s)} \right\|_q \quad (2.3.15)$$

and using the shift introduced in the proof of Lemma 2.7, we derive that $\left\|X_{\mathbf{0}}-Y_{\mathbf{0}}^{(s)}\right\|_q \leqslant \delta_{-\mathbf{v_s},q}$ hence

$$\left\|X_{\mathbf{0},j}\right\|_q \leqslant \sum_{s=1}^{N_j} \delta_{-\mathbf{v_s},q}. \tag{2.3.16}$$

Since $\{-\mathbf{v_s}, 1 \leqslant s \leqslant N_j\} = \{\mathbf{v_s}, 1 \leqslant s \leqslant N_j\}$, we get the wanted conclusion. □

Combining (2.3.3), (2.3.6), (2.3.10) and (2.3.11), we derive that

$$\begin{aligned}\left\|\sum_{\mathbf{i}\in\mathbb{Z}^d} a_{\mathbf{i}} X_{\mathbf{i}}\right\|_p &\leqslant \frac{14.5p}{\log p}\sum_{j=1}^{+\infty}\sum_{\mathbf{i}\in\mathbb{Z}^d,\|\mathbf{i}\|_\infty=j}\delta_{\mathbf{i},2}\left(\sum_{\mathbf{i}\in\mathbb{Z}^d} a_i^2\right)^{1/2}(4j+4)^{d/2}\\ &+\frac{14.5p}{\log p}\sum_{j=1}^{+\infty}\sum_{\mathbf{i}\in\mathbb{Z}^d,\|\mathbf{i}\|_\infty=j}\delta_{\mathbf{i},p}\left(\sum_{\mathbf{i}\in\mathbb{Z}^d}|a_i|^p\right)^{1/p}(4j+4)^{d(1-1/p)}+\left\|\sum_{\mathbf{i}\in\mathbb{Z}^d} a_{\mathbf{i}}\mathbb{E}\left[X_{\mathbf{i}}\mid\varepsilon_{\mathbf{i}}\right]\right\|_p.\end{aligned} \tag{2.3.17}$$

In order to control the last term, we use inequality (1.2.2) and bound $\left\|\mathbb{E}\left[X_{\mathbf{i}}\mid\varepsilon_{\mathbf{i}}\right]\right\|_q$ by $\delta_{\mathbf{0},q}$. This ends the proof of Theorem 1.6.

2.4. **Proof of Theorem 1.9.** Denote for a random variable Z the quantity

$$\delta(Z) := \sup_{t\in\mathbb{R}}\left|\mathbb{P}\{Z\leqslant t\}-\Phi(t)\right|. \tag{2.4.1}$$

We say that a random field $(Y_{\mathbf{i}})_{\mathbf{i}\in\mathbb{Z}^d}$ is m-dependent if the collections of random variables $(Y_{\mathbf{i}}, \mathbf{i}\in A)$ and $(Y_{\mathbf{i}}, \mathbf{i}\in B)$ are independent whenever $\inf\{\|\mathbf{a}-\mathbf{b}\|_\infty, \mathbf{a}\in A, \mathbf{b}\in B\} > m$. The proof of Theorem 1.9 will use the following tools.

(T.1) By Theorem 2.6 in [CS04], if I is a finite subset of $\mathbb{Z}^d$, $(Y_{\mathbf{i}})_{\mathbf{i}\in I}$ an m-dependent centered random field such that $\mathbb{E}\left[|Y_{\mathbf{i}}|^p\right] < +\infty$ for each $\mathbf{i}\in I$ and some $p\in(2,3]$ and $\operatorname{Var}\left(\sum_{\mathbf{i}\in I} Y_{\mathbf{i}}\right) = 1$, then

$$\delta\left(\sum_{\mathbf{i}\in I} Y_{\mathbf{i}}\right) \leqslant 75\,(10m+1)^{(p-1)d}\sum_{\mathbf{i}\in I}\mathbb{E}\left[|Y_{\mathbf{i}}|^p\right]. \tag{2.4.2}$$

(T.2) By Lemma 1 in [EMO07], for any two random variables Z and Z' and $p\geqslant 1$,

$$\delta(Z+Z') \leqslant 2\delta(Z) + \|Z'\|_p^{\frac{p}{p+1}}. \tag{2.4.3}$$

Let $(\varepsilon_{\mathbf{u}})_{\mathbf{u}\in\mathbb{Z}^d}$ be an i.i.d. random field and let $f\colon \mathbb{R}^{\mathbb{Z}^d}\to\mathbb{R}$ be a measurable function such that for each $\mathbf{i}\in\mathbb{Z}^d$, $X_{\mathbf{i}} = f(\varepsilon_{\mathbf{i}-\mathbf{u}})$. Let $\gamma > 0$ and n_0 defined by (1.5.8).

Let $m := \left(\left[\|b_n\|_{\ell^2}\right]+1\right)^\gamma$ and let us define

$$X_{\mathbf{i}}^{(m)} := \mathbb{E}\left[X_{\mathbf{i}}\mid\sigma\left(\varepsilon_{\mathbf{u}}, \mathbf{i}-m\mathbf{1}\preccurlyeq\mathbf{u}\preccurlyeq\mathbf{i}+m\mathbf{1}\right)\right]. \tag{2.4.4}$$

Since the random field $(\varepsilon_{\mathbf{u}})_{\mathbf{u}\in\mathbb{Z}^d}$ is independent, the following properties hold.

(P.1) The random field $\left(X_{\mathbf{i}}^{(m)}\right)_{\mathbf{i}\in\mathbb{Z}^d}$ is $(2m+1)$-dependent.

(P.2) The random field $\left(X_{\mathbf{i}}^{(m)}\right)_{\mathbf{i}\in\mathbb{Z}^d}$ is identically distributed and $\left\|X_{\mathbf{i}}^{(m)}\right\|_{p'} \leqslant \|X_{\mathbf{0}}\|_{p'}$.

(P.3) For any $(a_{\mathbf{i}})_{\mathbf{i}\in\mathbb{Z}^d}\in\ell^2\left(\mathbb{Z}^d\right)$ and $q\geqslant 2$, the following inequality holds:

$$\left\|\sum_{\mathbf{i}\in\mathbb{Z}^d}a_{\mathbf{i}}\left(X_{\mathbf{i}}-X_{\mathbf{i}}^{(m)}\right)\right\|_q\leqslant\frac{14.5q}{\log q}\left(\sum_{\mathbf{i}\in\mathbb{Z}^d}a_{\mathbf{i}}^2\right)^{1/2}\sum_{\substack{\mathbf{j}\in\mathbb{N}^d\\ \|\mathbf{j}\|_\infty\geqslant m}}\left(4\|\mathbf{j}\|_\infty+4\right)^{d/2}\delta_{\mathbf{j},2}$$
$$+\frac{14.5q}{\log q}\left(\sum_{\mathbf{i}\in\mathbb{Z}^d}|a_{\mathbf{i}}|^q\right)^{1/q}\sum_{\substack{\mathbf{j}\in\mathbb{N}^d\\ \|\mathbf{j}\|_\infty\geqslant m}}\left(4\|\mathbf{j}\|_\infty+4\right)^{d(1-1/q)}\delta_{\mathbf{j},q}. \quad (2.4.5)$$

In order to prove (2.4.5), we follow the proof of Theorem 1.6 and start from the decomposition $X_{\mathbf{i}}-X_{\mathbf{i}}^{(m)}=\lim_{N\to+\infty}\sum_{j=m}^N X_{\mathbf{i},j}$ instead of (2.3.2).

Define $S_n^{(m)}:=\sum_{\mathbf{i}\in\mathbb{Z}^d}b_{n,\mathbf{i}}X_{\mathbf{i}}^{(m)}$. An application of (T.2) to $Z:=S_n^{(m)}\|b_n\|_{\ell^2}^{-1}\sigma^{-1}$ and $Z':=\left(S_n-S_n^{(m)}\right)\|b_n\|_{\ell^2}^{-1}\sigma^{-1}$ yields

$$\Delta_n\leqslant 2\delta\left(\frac{S_n^{(m)}}{\sigma\|b_n\|_{\ell^2}}\right)+\sigma^{-\frac{p}{p+1}}\frac{1}{\|b_n\|_{\ell^2}^{\frac{p}{p+1}}}\left\|S_n-S_n^{(m)}\right\|_p^{\frac{p}{p+1}}. \quad (2.4.6)$$

Moreover,

$$\delta\left(\frac{S_n^{(m)}}{\delta\|b_n\|_{\ell^2}}\right)=\sup_{t\in\mathbb{R}}\left|\mathbb{P}\left\{\frac{S_n^{(m)}}{\delta\|b_n\|_{\ell^2}}\leqslant t\right\}-\Phi(t)\right| \quad (2.4.7)$$
$$=\sup_{u\in\mathbb{R}}\left|\mathbb{P}\left\{\frac{S_n^{(m)}}{\left\|S_n^{(m)}\right\|_2}\leqslant u\right\}-\Phi\left(u\frac{\left\|S_n^{(m)}\right\|_2}{\delta\|b_n\|_{\ell^2}}\right)\right| \quad (2.4.8)$$
$$\leqslant\delta\left(\frac{S_n^{(m)}}{\left\|S_n^{(m)}\right\|_2}\right)+\sup_{u\in\mathbb{R}}\left|\Phi\left(u\frac{\left\|S_n^{(m)}\right\|_2}{\sigma\|b_n\|_{\ell^2}}\right)-\Phi(u)\right|, \quad (2.4.9)$$

hence, by (P.1) and (T.1) applied with $Y_{\mathbf{i}}:=X_{\mathbf{i}}^{(m)}/\left\|S_n^{(m)}\right\|_2$, p' instead of p and $2m+1$ instead of m, we derive that

$$\Delta_n\leqslant(I)+(II)+(III) \quad (2.4.10)$$

where

$$(I):=150\left(20m+21\right)^{(p'-1)d}\sum_{\mathbf{i}\in\mathbb{Z}^d}|b_{n,\mathbf{i}}|^{p'}\left\|X_i^{(m)}\right\|_{p'}^{p'}\left\|S_n^{(m)}\right\|_2^{-p'}, \quad (2.4.11)$$
$$(II):=2\sup_{u\in\mathbb{R}}\left|\Phi\left(u\frac{\left\|S_n^{(m)}\right\|_2}{\sigma\|b_n\|_{\ell^2}}\right)-\Phi(u)\right| \text{ and} \quad (2.4.12)$$
$$(III):=\sigma^{-\frac{p}{p+1}}\frac{1}{\|b_n\|_{\ell^2}^{\frac{p}{p+1}}}\left\|S_n-S_n^{(m)}\right\|_p^{\frac{p}{p+1}}. \quad (2.4.13)$$

By (P.2) and the reversed triangular inequality, the term (I) can be bounded in the following way

$$(I)\leqslant 150\left(20m+21\right)^{(p'-1)d}\|X_{\mathbf{0}}\|_{p'}^{p'}\|b_n\|_{\ell^{p'}}^{p'}\left(\|S_n\|_2-\left\|S_n-S_n^{(m)}\right\|\right)^{-p'} \quad (2.4.14)$$

and by (P.2) with $q = 2$, we obtain that

$$\left(\|S_n\|_2 - \left\|S_n - S_n^{(m)}\right\|\right)^{-p'} \leqslant \left(\|S_n\|_2 - 20\,(\log 2)^{-1}\, m^{-\alpha}\, \|b_n\|_{\ell^2}\, C_2\right)^{-p'}. \tag{2.4.15}$$

By (1.5.7), we have

$$\frac{\|S_n\|_2^2}{\|b_n\|_{\ell^2}^2} = \sigma^2 + \varepsilon_n, \tag{2.4.16}$$

and we eventually get

$$(I) \leqslant 150\,(20m+21)^{(p'-1)d}\, \|X_{\mathbf{0}}\|_{p'}^{p'} \left(\frac{\|b_n\|_{\ell^{p'}}}{\|b_n\|_{\ell^2}}\right)^{p'} \left(\sqrt{\sigma^2+\varepsilon_n} - 20\,(\log 2)^{-1}\, m^{-\alpha} C_2\right)^{-p'}.$$

Since $n \geqslant n_0$, we derive, in view of (1.5.8),

$$(I) \leqslant 150\,(20m+21)^{(p'-1)d}\, \|X_{\mathbf{0}}\|_{p'}^{p'} \left(\frac{\|b_n\|_{\ell^{p'}}}{\|b_n\|_{\ell^2}}\right)^{p'} (\sigma/2)^{-p'} \tag{2.4.17}$$

In order to bound (II), we argue as in [YWLH12] (p. 456). Doing similar computations as in [EM14] (p. 272), we obtain that

$$(II) \leqslant (2\pi e)^{-1/2} \left(\inf_{k\geqslant 1} a_k\right)^{-1} \left|a_n^2 - 1\right|, \tag{2.4.18}$$

where $a_n := \left\|S_n^{(m)}\right\|_2 \sigma^{-1} \|b_n\|_{\ell^2}^{-1}$. Observe that for any n, by (P.2),

$$a_n \geqslant \frac{\|S_n\|_2 - \left\|S_n - S_n^{(m)}\right\|_2}{\sigma\, \|b_n\|_{\ell^2}} \geqslant \frac{\sqrt{\sigma^2+\varepsilon_n} - 20\,(\log 2)^{-1}\, C_2 m^{-\alpha}}{\sigma} \tag{2.4.19}$$

and using again (P.2) combined with Theorem 1.6 for $p = 2$,

$$\left|a_n^2 - 1\right| = \left|\frac{\left\|S_n^{(m)}\right\|_2^2}{\sigma^2\, \|b_n\|_{\ell^2}^2} - 1\right| \tag{2.4.20}$$

$$\leqslant \left|\frac{\|S_n\|_2^2}{\sigma^2\, \|b_n\|_{\ell^2}^2} - 1\right| + \frac{\left|\left\|S_n^{(m)}\right\|_2^2 - \|S_n\|_2^2\right|}{\sigma^2\, \|b_n\|_{\ell^2}^2} \tag{2.4.21}$$

$$\leqslant \frac{|\varepsilon_n|}{\sigma^2} + \frac{\left|\left\|S_n^{(m)}\right\|_2 - \|S_n\|_2\right| \left(\left\|S_n^{(m)}\right\|_2 + \|S_n\|_2\right)}{\sigma^2\, \|b_n\|_{\ell^2}^2} \tag{2.4.22}$$

$$\leqslant \frac{|\varepsilon_n|}{\sigma^2} + \frac{\left\|S_n^{(m)} - S_n\right\|_2 \left(\left\|S_n^{(m)}\right\|_2 + \|S_n\|_2\right)}{\sigma^2\, \|b_n\|_{\ell^2}^2} \tag{2.4.23}$$

$$\leqslant \frac{|\varepsilon_n|}{\sigma^2} + 40\,(\log 2)^{-1}\, \frac{m^{-\alpha}}{\sigma^2} C_2^2. \tag{2.4.24}$$

This leads to the estimate

$$(II) \leqslant \frac{(2\pi e)^{-1/2}}{\sqrt{\sigma^2+\varepsilon_n} - 20\,(\log 2)^{-1}\, C_2 m^{-\alpha}} \left(\frac{|\varepsilon_n|}{\sigma} + 40\,(\log 2)^{-1}\, \frac{m^{-\alpha}}{\sigma} C_2^2\right), \tag{2.4.25}$$

and since $n \geqslant n_0$, we derive, in view of (1.5.8),

$$(II) \leqslant \left(2\frac{|\varepsilon_n|}{\sigma^2} + 80\left(\log 2\right)^{-1}\frac{\|b_n\|_{\ell^2}^{-\gamma\alpha}}{\sigma^2}C_2^2\right)\left(2\pi e\right)^{-1/2}. \tag{2.4.26}$$

The estimate of (III) rests on (P.2):

$$(III) \leqslant \sigma^{-\frac{p}{p+1}}\left(\frac{14.5p}{\log p}\sum_{\substack{\mathbf{j}\in\mathbb{N}^d\\ \|\mathbf{j}\|_\infty\geqslant m}}\left(4\|\mathbf{j}\|_\infty+4\right)^{d/2}\delta_{\mathbf{j},2}\right)^{\frac{p}{p+1}}$$

$$+\sigma^{-\frac{p}{p+1}}\|b_n\|_{\ell^2}^{-\frac{p}{p+1}}\|b_n\|_{\ell^p}^{\frac{p}{p+1}}\left(\frac{14.5p}{\log p}\sum_{\substack{\mathbf{j}\in\mathbb{N}^d\\ \|\mathbf{j}\|_\infty\geqslant m}}\left(4\|\mathbf{j}\|_\infty+4\right)^{d(1-1/p)}\delta_{\mathbf{j},p}\right)^{\frac{p}{p+1}} \tag{2.4.27}$$

hence

$$(III) \leqslant \left(\frac{14.5p}{\sigma\log p}4^{d/2}\|b_n\|_{\ell^2}^{-\gamma\alpha}C_2\right)^{\frac{p}{p+1}} + \left(\frac{\|b_n\|_{\ell^p}}{\sigma\|b_n\|_{\ell^2}}\frac{14.5p}{\log p}4^{d(1-1/p)}\|b_n\|_{\ell^2}^{-\gamma\beta}C_p\right)^{\frac{p}{p+1}}. \tag{2.4.28}$$

The combination of (2.4.10), (2.4.17), (2.4.26) and (2.4.28) gives (1.5.9).

2.5. **Proof of Theorem 1.11.** Since the random variables $X_{\mathbf{i}}$ are centered, we derive by definition of $g_n(\mathbf{x})$ that

$$(nh_n)^{d/2}\left(g_n(\mathbf{x}) - \mathbb{E}\left[g_n(\mathbf{x})\right]\right) = (nh_n)^{d/2}\frac{\sum_{\mathbf{i}\in\Lambda_n}X_{\mathbf{i}}K\left(\frac{\mathbf{x}-\mathbf{i}/n}{h_n}\right)}{\sum_{\mathbf{i}\in\Lambda_n}K\left(\frac{\mathbf{x}-\mathbf{i}/n}{h_n}\right)}. \tag{2.5.1}$$

Keeping in mind (1.6.4), we define

$$b_{n,\mathbf{i}} = K\left(\frac{1}{h_n}\left(\mathbf{x}-\frac{\mathbf{i}}{n}\right)\right), \quad \mathbf{i}\in\Lambda_n \tag{2.5.2}$$

and $b_{n,\mathbf{i}} = 0$ otherwise. Denote $b_n = (b_{n,\mathbf{i}})_{\mathbf{i}\in\mathbb{Z}^d}$ and $\|b_n\|_{\ell^2} := \left(\sum_{\mathbf{i}\in\mathbb{Z}^d}b_{n,\mathbf{i}}^2\right)^{1/2}$. In this way, by (2.5.1) and (1.6.8),

$$\frac{1}{\|K\|_{\mathbb{L}^2(\mathbb{R}^d)}\sigma}(nh_n)^{d/2}\left(g_n(\mathbf{x}) - \mathbb{E}\left[g_n(\mathbf{x})\right]\right) = \frac{1}{\sigma}\sum_{\mathbf{i}\in\mathbb{Z}^d}b_{n\mathbf{i}}X_{\mathbf{i}}\|b_n\|_{\ell^2}^{-1}A_n. \tag{2.5.3}$$

Applying (T.2) to $Z = \sum_{\mathbf{i}\in\mathbb{Z}^d}b_{n\mathbf{i}}X_{\mathbf{i}}\|b_n\|_{\ell^2}^{-1}$ and $Z' = \sum_{\mathbf{i}\in\mathbb{Z}^d}b_{n\mathbf{i}}X_{\mathbf{i}}\|b_n\|_{\ell^2}^{-1}\sigma^{-1}(A_n-1)$ and using Proposition 1 in [EVW13], we derive that

$$\Delta_n \leqslant 2\Delta_n' + 3\left\|(2p)^{1/2}\sigma^{-1}\sum_{\mathbf{j}\in\mathbb{Z}^d}\delta_{\mathbf{j},p}\right\|^{\frac{p}{p+1}}|A_n-1|^{\frac{p}{p+1}}, \tag{2.5.4}$$

where

$$\Delta_n' = \sup_{t\in\mathbb{R}}\left|\mathbb{P}\left\{Z\leqslant t\right\} - \Phi\left(\frac{t}{\sigma}\right)\right|. \tag{2.5.5}$$

We then use Theorem 1.9 to handle Δ_n' (which is allowed, by (A1)). Using boundedness of K, we control the ℓ^p and $\ell^{p'}$ norms , by a constant times the ℓ^2-norm. This ends the proof of Theorem 1.11.

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