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Convergence for PDEs with an arbitrary odd order spatial derivative term

Clémentine Courtès¹

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Abstract

We compute the rate of convergence of forward, backward and central finite difference θ -schemes for linear PDEs with an arbitrary odd order spatial derivative term. We prove convergence of the first or second order for smooth and less smooth initial data.

1 Introduction

We study in this paper linear partial differential equations with an arbitrary odd order spatial derivative term, which read

$$\partial_t u + \partial_x^{2p+1} u = 0, \quad (1)$$

with $p \in \mathbb{N}$. The particular case $p = 0$ corresponds to the advection equation with a unit constant speed $\partial_t u + \partial_x u = 0$ and describes the passive advection of scalar field carried at constant speed. The case $p = 1$ leads to Airy equation $\partial_t u + \partial_x^3 u = 0$ that models the propagation of long waves in shallow water [7] and derives from a linearization of the Korteweg-de Vries equation [3]. We especially focus on the initial value problem where (1) is considered with the initial condition $u|_{t=0} = u_0$. We deal with the numerical approach of this Cauchy problem and study the convergence of several finite difference schemes. Our concern here is to find a rate of convergence without assuming the smoothness of the initial data.

For this purpose, we use the finite difference method to discretize (1) in $\mathbb{R} \times [0, T]$. We choose to deal with a uniform time and space discretization. Let $\Delta t > 0$ and $\Delta x > 0$ be the time and space steps, we note $t^n = n\Delta t$ for all $n \in \{0, \dots, N\}$ where $N = \lfloor \frac{T}{\Delta t} \rfloor$ and $x_j = j\Delta x$ for all $j \in \mathbb{Z}$. We denote by $(v_j^n)_{(j,n)}$ the discrete unknowns defined by

$$\begin{cases} v_j^{n+1} + \theta \Delta t (D_{\bullet}^{2p+1} v)_j^{n+1} = v_j^n - (1 - \theta) \Delta t (D_{\bullet}^{2p+1} v)_j^n, \forall (j, n) \in \mathbb{Z} \times \{0, \dots, N\}, \\ v_j^0 = \frac{1}{\Delta x} \int_{x_j}^{x_{j+1}} u_0(y) dy, \forall j \in \mathbb{Z}, \end{cases} \quad (2)$$

with

$$(D_{\bullet}^{2p+1} v)_j^n = (D_+^{2p+1} v)_j^n = \sum_{k=0}^{2p+1} \frac{\binom{2p+1}{k} (-1)^k}{\Delta x^{2p+1}} v_{p-k+j+1}^n \quad (\text{forward scheme}), \quad (3a)$$

$$\text{or} \quad (D_{\bullet}^{2p+1} v)_j^n = (D_-^{2p+1} v)_j^n = \sum_{k=0}^{2p+1} \frac{\binom{2p+1}{k} (-1)^k}{\Delta x^{2p+1}} v_{p-k+j}^n \quad (\text{backward scheme}), \quad (3b)$$

$$\text{or} \quad (D_{\bullet}^{2p+1} v)_j^n = (D_c^{2p+1} v)_j^n = \frac{1}{2} (D_+^{2p+1} v + D_-^{2p+1} v)_j^n \quad (\text{central scheme}). \quad (3c)$$

The parameter θ belongs to $[0, 1]$ and we recover the explicit scheme for $\theta = 0$ and the implicit scheme for $\theta = 1$.

Notations 1. We denote by $\mathbb{H}^s(\mathbb{R})$ (with $s > 0$) the Sobolev space defined with the norm $\|u\|_{\mathbb{H}^s(\mathbb{R})} = \left(\int_{\mathbb{R}} (1 + |\xi|^2)^s |\widehat{u}(\xi)|^2 d\xi \right)^{\frac{1}{2}}$, where \widehat{u} is the Fourier transform of u . Moreover, we use the standard $\ell^\infty(0, N; \ell_\Delta^2(\mathbb{Z}))$ space whose norm is $\|v\|_{\ell^\infty(0, N; \ell_\Delta^2(\mathbb{Z}))} = \sup_{n \in \{0, \dots, N\}} \sqrt{\sum_{j \in \mathbb{Z}} \Delta x |v_j^n|^2}$. Lastly, we note $A \lesssim B$ when $A \leq CB$ where C is a constant independent of Δx and Δt .

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2 Order of accuracy for an initial datum in $\mathbb{H}^{4p+2}(\mathbb{R})$

We hereafter find some condition on θ , Δt and Δx for the schemes to be consistent and stable, to conclude the convergence study according to the Lax-Richtmyer theorem [6].

2.1 Consistency estimate

In Sect. 2, we suppose the initial datum regular enough to compute all the needed derivatives and the Taylor expansions up to the desired order. Indeed, supposing u_0 regular is sufficient to ensure the same regularity for $u(t, \cdot)$ for all $t \in [0, T]$ because of the following result.

Remark 1. Let u be a solution of (1), then by linearity of the equation all the derivatives of u verify (1) too and by Fourier transform, the \mathbb{L}^2 -norm of all its derivatives are conserved : $\|\partial_x^k u(t, \cdot)\|_{\mathbb{L}^2(\mathbb{R})} = \|\partial_x^k u_0\|_{\mathbb{L}^2(\mathbb{R})}$, for all $k \in \mathbb{N}$. Thus, $u_0 \in \mathbb{H}^{4p+2}(\mathbb{R})$ implies $u(t, \cdot) \in \mathbb{H}^{4p+2}(\mathbb{R})$, $\forall t \in [0, T]$.

Definition 1. For all $(j, n) \in \mathbb{Z} \times \{0, \dots, N\}$, we note $(u_\Delta)_j^n = \frac{1}{\Delta x} \int_{x_j}^{x_{j+1}} u(t^n, y) dy$ with u the exact solution of the Cauchy-problem (1) from u_0 . For all $(j, n) \in \mathbb{Z} \times \{0, \dots, N\}$, the consistency error is defined as

$$\epsilon_j^n = \frac{(u_\Delta)_j^{n+1} - (u_\Delta)_j^n}{\Delta t} + \theta (D_{\bullet}^{2p+1} u_\Delta)_j^{n+1} + (1 - \theta) (D_{\bullet}^{2p+1} u_\Delta)_j^n,$$

with D_{\bullet}^{2p+1} defined by (3a)–(3c).

Proposition 1. Assume $u_0 \in \mathbb{H}^{4p+2}(\mathbb{R})$ (and $u_0 \in \mathbb{H}^{6p+3}(\mathbb{R})$ if $\theta = \frac{1}{2}$) then, for the forward or backward finite difference schemes (3a) and (3b), the following consistency inequality holds

$$\|\epsilon\|_{\ell^\infty(0, N; \ell_\Delta^2(\mathbb{Z}))} \lesssim \Delta t \left| \frac{1}{2} - \theta \right| \left(\|\partial_x^{4p+2} u_0\|_{\mathbb{L}^2(\mathbb{R})} + \Delta x \|\partial_x^{2p+2} u_0\|_{\mathbb{L}^2(\mathbb{R})} + \Delta t^2 \|\partial_x^{6p+3} u_0\|_{\mathbb{L}^2(\mathbb{R})} \right).$$

For the central finite difference scheme (3c), the consistency inequality is as follows

$$\|\epsilon\|_{\ell^\infty(0, N; \ell_\Delta^2(\mathbb{Z}))} \lesssim \Delta t \left| \frac{1}{2} - \theta \right| \left(\|\partial_x^{4p+2} u_0\|_{\mathbb{L}^2(\mathbb{R})} + \Delta x^2 \|\partial_x^{2p+3} u_0\|_{\mathbb{L}^2(\mathbb{R})} + \Delta t^2 \|\partial_x^{6p+3} u_0\|_{\mathbb{L}^2(\mathbb{R})} \right).$$

Before proving the previous result, we state a useful lemma.

Lemma 1. For all ℓ and p in \mathbb{N} , there exists $\xi \in]-p, p+1[$ such that

$$\sum_{k=0}^{2p+1} \binom{2p+1}{k} (-1)^k (p-k+1)^\ell = \begin{cases} 0 & \text{if } \ell < 2p+1, \\ \ell! & \text{if } \ell = 2p+1, \\ \frac{\ell!}{(\ell-2p)!} \xi^{\ell-2p-1} & \text{if } \ell > 2p+1. \end{cases}$$

Proof of Lemma 1. Let $(x_{j-p}, \dots, x_{j+p+1})$ be $2p+2$ points regularly spaced of h , we recall the divided difference of order $2p+2$ of a smooth function f :

$$(2p+1)! f[x_{j-p}, \dots, x_{j+p+1}] = \frac{\sum_{k=0}^{2p+1} \binom{2p+1}{k} (-1)^k f(x_{p-k+j+1})}{h^{2p+1}}. \quad (4)$$

Moreover, we recall the existence of $\xi \in]\min(x_{j-p}, \dots, x_{j+p+1}), \max(x_{j-p}, \dots, x_{j+p+1})[$ such as

$$(2p+1)! f[x_{j-p}, \dots, x_{j+p+1}] = f^{(2p+1)}(\xi).$$

For more details, please refer to [4]. Lemma 1 is a consequence of the two previous equations with $f : y \mapsto y^\ell$, $h = 1$, $j = 0$ and $x_i = i$ for $i \in \mathbb{Z}$. □

Proof of Proposition 1. For $u_0 \in \mathbb{H}^{4p+2}(\mathbb{R})$ and for the forward finite difference scheme, one has

$$(D_{\bullet}^{2p+1} u_\Delta)_j^n = (D_+^{2p+1} u_\Delta)_j^n = \sum_{k=0}^{2p+1} \frac{\binom{2p+1}{k} (-1)^k}{\Delta x^{2p+1}} \left(\frac{1}{\Delta x} \int_{x_j}^{x_{j+1}} u(t^n, y + (p-k+1)\Delta x) dy \right).$$

Using a Taylor expansion (in space) up to order $2p + 2$ and exchanging the two sums inside leads to

$$(D_+^{2p+1}u_\Delta)_j^n = \frac{1}{\Delta x} \int_{x_j}^{x_{j+1}} \sum_{\ell=0}^{2p+1} \frac{\partial_x^\ell u(t^n, y)}{\ell!} \frac{(\sum_{k=0}^{2p+1} \binom{2p+1}{k} (-1)^k (p-k+1)^\ell \Delta x^\ell)}{\Delta x^{2p+1}} dy + (R_+)_j^n, \quad (5)$$

where

$$(R_+)_j^n = \frac{1}{\Delta x} \int_{x_j}^{x_{j+1}} \int_y^{y+(p-k+1)\Delta x} \frac{\partial_x^{2p+2} u(t^n, z)}{(2p+1)!} \left(\frac{\sum_{k=0}^{2p+1} \binom{2p+1}{k} (-1)^k (y + (p-k+1)\Delta x - z)^{2p+1}}{\Delta x^{2p+1}} \right) dz dy.$$

For simplicity, we will only use $\|R_+^n\|_{\ell_\Delta^2} \lesssim \Delta x \|\partial_x^{2p+2} u(t^n, \cdot)\|_{\mathbb{L}^2(\mathbb{R})}$. Equation (5) is simplified thanks to Lemma 1. Eventually, we obtain $(D_+^{2p+1}u_\Delta)_j^n = \frac{1}{\Delta x} \int_{x_j}^{x_{j+1}} \partial_x^{2p+1} u(t^n, y) dy + (R_+)_j^n$. Similarly, by adapting the previous computation, one has

$$(D_+^{2p+1}u_\Delta)_j^{n+1} = \frac{1}{\Delta x} \left(\int_{x_j}^{x_{j+1}} \partial_x^{2p+1} u(t^n, y) dy + \int_{x_j}^{x_{j+1}} \Delta t \partial_t \partial_x^{2p+1} u(t^n, y) dy + \int_{x_j}^{x_{j+1}} \int_{t^n}^{t^{n+1}} \partial_t^2 \partial_x^{2p+1} u(s, y) (t^{n+1} - s) ds dy \right) + (R_+)_j^{n+1}.$$

In order to compute the difference $\frac{(u_\Delta)_j^{n+1} - (u_\Delta)_j^n}{\Delta t}$ that appears in Definition 1, we perform a Taylor expansion (in time) up to order 3. Gathering all those results together yields

$$\begin{aligned} \epsilon_j^n &= \frac{\Delta t}{2\Delta x} \int_{x_j}^{x_{j+1}} \partial_t^2 u(t^n, x) dx + \frac{1}{\Delta t \Delta x} \int_{x_j}^{x_{j+1}} \int_{t^n}^{t^{n+1}} \partial_t^3 u(s, y) \frac{(t^{n+1} - s)^2}{2} ds dy \\ &+ \frac{\theta \Delta t}{\Delta x} \int_{x_j}^{x_{j+1}} \partial_t \partial_x^{2p+1} u(t^n, y) dy + \frac{\theta}{\Delta x} \int_{x_j}^{x_{j+1}} \int_{t^n}^{t^{n+1}} \partial_t^2 \partial_x^{2p+1} u(s, y) (t^{n+1} - s) ds dy \\ &+ \theta (R_+)_j^{n+1} + (1 - \theta) (R_+)_j^n. \end{aligned}$$

The conclusion comes from the relation $\partial_t u(t, x) = -\partial_x^{2p+1} u(t, x)$, the Cauchy-Schwarz inequality and the conservation of the \mathbb{L}^2 -norm (cf. Remark 1). \square

Remark 2. The regularity $\mathbb{H}^{4p+2}(\mathbb{R})$ (or $\mathbb{H}^{6p+3}(\mathbb{R})$ if $\theta = \frac{1}{2}$) comes from the Taylor expansion in time and is essential in this proof.

Remark 3. We follow exactly the same guidelines for the backward finite difference scheme. For the central finite difference scheme, we need to perform a Taylor expansion in space up to order $2p + 3$ to obtain $(D_c^{2p+1}u_\Delta)_j^n = \frac{1}{\Delta x} \int_{x_j}^{x_{j+1}} \partial_x^{2p+1} u(t^n, y) dy + (R_c)_j^n$, with $\|R_c^n\|_{\ell_\Delta^2} \lesssim \Delta x^2 \|\partial_x^{2p+3} u_0\|_{\mathbb{L}^2(\mathbb{R})}$.

2.2 Stability

We note, for all $(v_j^n)_{j \in \mathbb{Z}}$ and $\xi \in [0, 1]$, $\widehat{V}^n(\xi) = \sum_{k \in \mathbb{Z}} v_k^n e^{2i\pi k \xi}$ in $\mathbb{L}^2([0, 1])$ with the equivalence of the norms $:\sum_{j \in \mathbb{Z}} \Delta x |v_j^n|^2 = \Delta x \int_0^1 |\widehat{V}^n(\xi)|^2 d\xi$. Eventually, we define the shift operator \mathcal{S}^ℓ by $\mathcal{S}^\ell v^n = (v_{j+\ell}^n)_{j \in \mathbb{Z}}$ thus, $\widehat{\mathcal{S}^\ell V^n} = e^{-2i\pi \ell \xi} \widehat{V}^n$.

Definition 2. A scheme is said to be stable in $\ell_\Delta^2(\mathbb{Z})$, if there exists a constant C independent of Δt and Δx such that, for $(v_j^n)_{(j,n)}$ verifying Equation (2),

$$\|v^{n+1}\|_{\ell_\Delta^2(\mathbb{Z})} \leq (1 + C\Delta t) \|v^n\|_{\ell_\Delta^2(\mathbb{Z})}, \quad \forall n \in \{0, \dots, N\}.$$

Proposition 2. For small Δt and Δx , the stability under the Courant-Friedrich-Lewy condition (in short CFL cond.) is explained in Table 1.

The following computation will simplify the proof of Proposition 2.

Lemma 2. One has, for all $\xi \in [0, 1]$,

$$\sum_{k=0}^{2p+1} \binom{2p+1}{k} (-1)^k e^{-2i\pi(p-k+1)\xi} = e^{-i\pi\xi} (-2i\sin(\pi\xi))^{2p+1}.$$

	p even		p odd
	$p = 0$ (Advection)	$p \neq 0$	
Forward scheme	stable under the CFL cond. $\Delta t(1 - 2\theta) \leq -\Delta x$	unconditionally unstable	stable under the CFL cond. $\Delta t(1 - 2\theta) \leq \frac{\Delta x^{2p+1}}{2^{2p}}$
Backward scheme	stable under the CFL cond. $\Delta t(1 - 2\theta) \leq \Delta x$	stable under the CFL cond. $\Delta t(1 - 2\theta) \leq \frac{\Delta x^{2p+1}}{2^{2p}}$	unconditionally unstable
Central scheme	stable under the CFL cond. $\Delta t(1 - 2\theta) \leq 2C\Delta x^2$	stable under the CFL cond. $\Delta t(1 - 2\theta) \leq 2C\frac{\Delta x^{4p+2}}{2^{4p}}$	stable under the CFL cond. $\Delta t(1 - 2\theta) \leq 2C\frac{\Delta x^{4p+2}}{2^{4p}}$

Table 1: Stability results for finite difference θ -schemes

Proof of Lemma 2. A proof may be found in Lemma 1.1 of [1]. □

Proof of Proposition 2. The forward finite difference scheme (3a) leads to

$$\begin{aligned} \widehat{U^{n+1}}(\xi) & \left(1 + \frac{\theta \Delta t}{(\Delta x)^{2p+1}} \sum_{k=0}^{2p+1} \binom{2p+1}{k} (-1)^k e^{-2i\pi(p-k+1)\xi} \right) \\ & = \widehat{U^n}(\xi) \left(1 - \frac{(1-\theta)\Delta t}{(\Delta x)^{2p+1}} \sum_{k=0}^{2p+1} \binom{2p+1}{k} (-1)^k e^{-2i\pi(p-k+1)\xi} \right), \text{ for any } \xi \text{ in } [0, 1]. \end{aligned}$$

The two sums are simplified thanks to Lemma 2. We finally obtain $\widehat{U^{n+1}}(\xi) = A_+(\xi)\widehat{U^n}(\xi)$, with A_+ the amplification coefficient defined by, $\forall \xi \in [0, 1]$

$$A_+(\xi) = \frac{\left(1 - \frac{(1-\theta)\Delta t}{(\Delta x)^{2p+1}} e^{-i\pi\xi} (-i)^{2p+1} (2\sin(\pi\xi))^{2p+1} \right)}{\left(1 + \frac{\theta\Delta t}{(\Delta x)^{2p+1}} e^{-i\pi\xi} (-i)^{2p+1} (2\sin(\pi\xi))^{2p+1} \right)}. \quad (6)$$

We are looking for a condition ensuring $|A_+(\xi)|^2 < (1 + C\Delta t)^2$ for any ξ in $[0, 1]$.

Case 1 : Assume that the parameter p of the spatial derivative is even. If $p \neq 0$, the stability condition leads to $\frac{2^{2p}\Delta t}{(\Delta x)^{2p+1}} (\sin(\pi\xi))^{2p}(1 - 2\theta) \leq -1$, (cf. [1]) which is impossible for all $\xi \in [0, 1]$: thus the forward finite difference scheme is unconditionally unstable for p even and non zero. On the contrary, assuming $p = 0$ means that the forward finite difference scheme is stable under CFL condition : $\Delta t(1 - 2\theta) \leq -\Delta x$ (which implies $\theta > \frac{1}{2}$).

Case 2 : In this case, the parameter p of the spatial derivative is odd, then the sufficient condition becomes $\frac{\Delta t}{(\Delta x)^{2p+1}} (2\sin(\pi\xi))^{2p}(1 - 2\theta) \leq 1$ (cf. [1]). Then the forward finite difference scheme is stable under the CFL condition $\Delta t(1 - 2\theta) \leq \frac{\Delta x^{2p+1}}{2^{2p}}$. Table 1 is a straightforward consequence. □

Remark 4. For the backward finite difference scheme, the only difference in the amplification coefficient is $e^{i\pi\xi}$ instead of $e^{-i\pi\xi}$ (in both the numerator and denominator). The parity needed for the stability changes because of that difference. For the central finite difference scheme, $e^{-i\pi\xi}$ is replaced with $\cos(\pi\xi)$ in the numerator and the denominator of the amplification coefficient.

2.3 Error estimates

We define the convergence error as follows.

Definition 3. For all $j \in \mathbb{Z}$ and $n \in \{0, \dots, N\}$, for u the analytical solution of (1) from u_0 and $(v_j^n)_{(j,n)}$ the numerical solution of (2), the convergence error is denoted by e_j^n and defined by $e_j^n = \frac{1}{\Delta x} \int_{x_j}^{x_{j+1}} u(t^n, y) dy - v_j^n$.

We are now able to state the main result of this section.

Theorem 1. For an initial datum $u_0 \in \mathbb{H}^{4p+2}(\mathbb{R})$ (and $u_0 \in \mathbb{H}^{6p+3}(\mathbb{R})$ if $\theta = \frac{1}{2}$), the error estimate of the forward finite difference scheme (3a) (if p is odd) or of the backward finite difference scheme (3b) (if p is even) satisfies

$$\|e\|_{\ell^\infty(0,N;\ell_\Delta^2(\mathbb{Z}))} \lesssim \Delta t \left| \frac{1}{2} - \theta \right| \left\| \partial_x^{4p+2} u_0 \right\|_{\mathbb{L}^2(\mathbb{R})} + \Delta x \left\| \partial_x^{2p+2} u_0 \right\|_{\mathbb{L}^2(\mathbb{R})} + \Delta t^2 \left\| \partial_x^{6p+3} u_0 \right\|_{\mathbb{L}^2(\mathbb{R})}.$$

For the central finite difference scheme (3c), the convergence rate becomes

$$\|e\|_{\ell^\infty(0,N;\ell_\Delta^2(\mathbb{Z}))} \lesssim \Delta t \left| \frac{1}{2} - \theta \right| \left\| \partial_x^{4p+2} u_0 \right\|_{\mathbb{L}^2(\mathbb{R})} + \Delta x^2 \left\| \partial_x^{2p+3} u_0 \right\|_{\mathbb{L}^2(\mathbb{R})} + \Delta t^2 \left\| \partial_x^{6p+3} u_0 \right\|_{\mathbb{L}^2(\mathbb{R})}.$$

All those results are gathered in Table 4.

Proof. We suppose p odd, so we work with the forward finite difference scheme. The case p even, with the backward scheme is similar. The definition of the convergence error implies

$$\widehat{e^{n+1}}(\xi) = A_+(\xi) \widehat{e^n}(\xi) + \frac{\Delta t}{1 + \frac{\theta \Delta t}{\Delta x^{2p+1}} e^{-i\pi\xi} (-i)^{2p+1} (2\sin(\pi\xi))^{2p+1}} \widehat{\epsilon^n}(\xi).$$

One has $\left\| \frac{1}{1 + \frac{\theta \Delta t}{\Delta x^{2p+1}} e^{-i\pi\xi} (-i)^{2p+1} (2\sin(\pi\xi))^{2p+1}} \right\|_{\mathbb{L}^\infty([0,1])} \leq 1$ and the stability condition gives $\|A_+\|_{\mathbb{L}^\infty([0,1])} \leq 1 + C\Delta t$. Thus, we obtain the following estimate, by discrete Grönwall lemma

$$\|e^{n+1}\|_{\ell_\Delta^2} \leq (1 + C\Delta t) \|e^n\|_{\ell_\Delta^2} + \Delta t \|\epsilon^n\|_{\ell_\Delta^2} \leq \dots \leq e^{CT} \|e^0\|_{\ell_\Delta^2} + \Delta t e^{CT} \sum_{k=0}^n \|\epsilon^k\|_{\ell_\Delta^2}.$$

The initial condition v_j^0 , Eq. (2), together with the consistency error conclude the proof. \square

Remark 5. As expected, for the particular case $\theta = \frac{1}{2}$ (the so-called Crank-Nicolson case), the rate of convergence in time is better as illustrated in Table 4, provided $u_0 \in \mathbb{H}^{6p+3}(\mathbb{R})$ (and not only in $\mathbb{H}^{4p+2}(\mathbb{R})$).

3 Less smooth initial data

The previous order of accuracy is obtained for initial data u_0 at least in $\mathbb{H}^{4p+2}(\mathbb{R})$ (or $\mathbb{H}^{6p+3}(\mathbb{R})$ if $\theta = \frac{1}{2}$). In this section, our aim is to relax this hypothesis to obtain rates of convergence for non-smooth initial data, for example, $u_0 \in \mathbb{H}^m(\mathbb{R})$ with $m > 0$. We detail only the case $\theta \neq \frac{1}{2}$ but state the Crank-Nicolson results in Table 4.

3.1 Initial datum in $\mathbb{H}^m(\mathbb{R})$ with $m \geq 2p + 2$

As explained previously (Remark 2), the regularity of u_0 is determined by the Taylor expansion in time. A first step is then to deal with the time term in error estimates. The following proposition provides that the time error prevails until $u_0 \in \mathbb{H}^{2p+2}(\mathbb{R})$, for which the spatial error becomes predominant.

Proposition 3. Assume $u_0 \in \mathbb{H}^m(\mathbb{R})$ with $m \geq 2p + 2$, and let us fix $M = \min(m, 4p + 2)$, then the error estimate for the forward (respectively backward) finite difference scheme, if p is odd (respectively even), yields

$$\|e\|_{\ell^\infty(0,N;\ell_\Delta^2(\mathbb{Z}))} \lesssim \Delta t^{\frac{M}{4p+2}} \|\partial_x^M u_0\|_{\mathbb{L}^2(\mathbb{R})} + \Delta x \|\partial_x^{2p+2} u_0\|_{\mathbb{L}^2(\mathbb{R})}.$$

For the central difference scheme, we suppose $m \geq 2p + 3$, and one has (for the same M)

$$\|e\|_{\ell^\infty(0,N;\ell_\Delta^2(\mathbb{Z}))} \lesssim \Delta t^{\frac{M}{4p+2}} \|\partial_x^M u_0\|_{\mathbb{L}^2(\mathbb{R})} + \Delta x^2 \|\partial_x^{2p+3} u_0\|_{\mathbb{L}^2(\mathbb{R})}.$$

Before proving this result, we introduce a regularization of u_0 thanks to mollifiers $(\varphi^\delta)_{\delta>0}$. Let χ be a \mathcal{C}^∞ -function such that

- $0 \leq \chi \leq 1$,
- $\chi \equiv 1$ in $[-\frac{1}{2}, \frac{1}{2}]$ and $\text{supp}(\chi) \subset [-1, 1]$ (where supp is its support),
- $\chi(-\xi) = \chi(\xi)$, $\forall \xi \in [-1, 1]$.

Let φ be such as $\widehat{\varphi}(\xi) = \chi(\xi)$ and for all $\delta > 0$, we define φ^δ such that $\widehat{\varphi^\delta}(\xi) = \chi(\delta\xi)$, which implies $\varphi^\delta = \frac{1}{\delta} \varphi(\frac{\cdot}{\delta})$. Eventually,

- let u^δ be the solution of (1) with $u_0^\delta = u_0 \star \varphi^\delta$ as initial data, where \star stands for the convolution product.
- We denote then $((v^\delta)_j^n)_{(n,j) \in \{0, \dots, N\} \times \mathbb{Z}}$ the numerical solution obtained by applying the numerical scheme (2) from u_0^δ .
- The unknowns u and $(v_j^n)_{(n,j) \in \{0, \dots, N\} \times \mathbb{Z}}$ are always the exact and numerical solutions starting from the initial data u_0 .

Lemma 3. *Assume $u_0 \in \mathbb{H}^r(\mathbb{R})$ with $r > 0$ then the following upper bound holds, for $0 \leq \ell \leq r \leq s$,*

$$\left\| u_0 - u_0^\delta \right\|_{\mathbb{H}^\ell(\mathbb{R})} \lesssim \delta^{r-\ell} \|u_0\|_{\mathbb{H}^r(\mathbb{R})} \quad \text{and} \quad \left\| u_0^\delta \right\|_{\mathbb{H}^s(\mathbb{R})} \lesssim \frac{1}{\delta^{s-r}} \|u_0\|_{\mathbb{H}^r(\mathbb{R})}.$$

Proof. Lemma 3 is proved in [2] and follows from very classical arguments (see also [5]). \square

Proof of Proposition 3. We are now able to prove Proposition 3. The triangular inequality applied to the convergence error gives $\|e^n\|_{\ell_\Delta^2} \lesssim E_1 + E_2 + E_3$ with

$$E_1 = \left(\sum_{j \in \mathbb{Z}} \Delta x \left(\frac{1}{\Delta x} \int_{x_j}^{x_{j+1}} u(t^n, x) - u^\delta(t^n, x) dx \right)^2 \right)^{\frac{1}{2}}, \quad (7)$$

$$E_2 = \left(\sum_{j \in \mathbb{Z}} \Delta x \left(\frac{1}{\Delta x} \int_{x_j}^{x_{j+1}} u^\delta(t^n, x) dx - (v^\delta)_j^n \right)^2 \right)^{\frac{1}{2}}, \quad (8)$$

$$E_3 = \left(\sum_{j \in \mathbb{Z}} \Delta x \left((v^\delta)_j^n - v_j^n \right)^2 \right)^{\frac{1}{2}}. \quad (9)$$

Cauchy–Schwarz inequality together with the conservation of the \mathbb{L}^2 -norm (Remark 1) yield $E_1 \leq \|u_0 - u_0^\delta\|_{\mathbb{L}^2(\mathbb{R})} \lesssim \delta^M \|\partial_x^M u_0\|_{\mathbb{L}^2(\mathbb{R})}$. The latest inequality comes from Lemma 3 with $\ell = 0$ and $r = M$.

For the E_2 -term, we use the previous section (Sect. 2). Indeed, E_2 corresponds to the convergence error for a smooth initial data u_0^δ . Hence, one has

$$E_2 \lesssim \Delta t \|\partial_x^{4p+2} u_0^\delta\|_{\mathbb{L}^2(\mathbb{R})} + \Delta x \|\partial_x^{2p+2} u_0^\delta\|_{\mathbb{L}^2(\mathbb{R})} \lesssim \frac{\Delta t}{\delta^{4p+2-M}} \|\partial_x^M u_0\|_{\mathbb{L}^2(\mathbb{R})} + \Delta x \|\partial_x^{2p+2} u_0\|_{\mathbb{L}^2(\mathbb{R})},$$

where the latest inequality comes from Lemma 3 with $(s, r) = (4p+2, M)$ and $(s, r) = (2p+2, 2p+2)$.

Finally, the stability of the scheme gives the following estimate for E_3 :

$$E_3 = \|(v^\delta)^n - v^n\|_{\ell_\Delta^2} \leq \|u_0^\delta - u_0\|_{\mathbb{L}^2(\mathbb{R})}.$$

Thus, the convergence error is upper bounded by

$$\|e^n\|_{\ell_\Delta^2} \lesssim \delta^M \|\partial_x^M u_0\|_{\mathbb{L}^2(\mathbb{R})} + \frac{\Delta t}{\delta^{4p+2-M}} \|\partial_x^M u_0\|_{\mathbb{L}^2(\mathbb{R})} + \Delta x \|\partial_x^{2p+2} u_0\|_{\mathbb{L}^2(\mathbb{R})}.$$

Proposition 3 comes from the optimal choice for δ : $\delta = \Delta t^{\frac{1}{4p+2}}$. \square

Remark 6. *The result for the central finite difference scheme is proved exactly in the same way, with the same s , r , ℓ and δ .*

3.2 Initial datum in $\mathbb{H}^m(\mathbb{R})$ with $m \geq 0$

The main result of this paper is summarized in the following theorem where the initial data u_0 has any Sobolev regularity.

Theorem 2. *Assume that $u_0 \in \mathbb{H}^m(\mathbb{R})$ with $0 \leq m$, then, the forward (respectively backward) finite difference scheme if p is odd (respectively even) has the following error-estimate*

$$\|e\|_{\ell^\infty(0,N;\ell_\Delta^2(\mathbb{Z}))} \lesssim \Delta t^{\frac{\min(m,4p+2)}{4p+2}} \|\partial_x^{\min(4p+2,m)} u_0\|_{\mathbb{L}^2(\mathbb{R})} + \Delta x^{\frac{\min(m,2p+2)}{2p+2}} \|\partial_x^{\min(2p+2,m)} u_0\|_{\mathbb{L}^2(\mathbb{R})}.$$

The previous inequality becomes for the central finite difference scheme

$$\|e\|_{\ell^\infty(0,N;\ell_\Delta^2(\mathbb{Z}))} \lesssim \Delta t^{\frac{\min(m,4p+2)}{4p+2}} \|\partial_x^{\min(4p+2,m)} u_0\|_{\mathbb{L}^2(\mathbb{R})} + \Delta x^{\frac{\min(m,2p+3)}{2p+3}} \|\partial_x^{\min(2p+3,m)} u_0\|_{\mathbb{L}^2(\mathbb{R})}.$$

The previous results are summarized in Table 4.

Proof. Here again, we suppose that p is odd, we thus detail the proof for the forward finite difference scheme. We have already proved the case $m \geq 4p + 2$ in Sect. 2 and the case $2p + 2 \leq m \leq 4p + 2$ in Subsect. 3.1. Let us now focus on the case $0 \leq m \leq 2p + 2$.

The proof of Theorem 2 follows the same guidelines as the proof of Proposition 3. Let $u_0 \in \mathbb{H}^m(\mathbb{R})$, we regularize this initial data thanks to mollifiers $(\varphi^\delta)_{\delta>0}$ whose properties are listed in Subsect. 3.1. This involves introducing the same new unknowns u^δ , u_0^δ and $((v^\delta)_j^n)_{(j,n)}$.

The convergence error $(e_j^n)_{(j,n)}$ is upper bounded by the same E_1 , E_2 and E_3 , defined in (7)-(9). Lemma 3 with $\ell = 0$ and $r = m$ leads to $E_1 + E_3 \lesssim \delta^m \|\partial_x^m u_0\|_{\mathbb{L}^2(\mathbb{R})}$. By definition $u_0^\delta \in \mathbb{H}^k(\mathbb{R})$, $\forall k > 0$, therefore Proposition 3 applies with $k = 2p + 2$ for example and $M = \min(k, 2p + 2) = 2p + 2$. It gives the following estimate for E_2 : $E_2 \lesssim \Delta t^{\frac{p+1}{2p+1}} \|\partial_x^{2p+2} u_0^\delta\|_{\mathbb{L}^2(\mathbb{R})} + \Delta x \|\partial_x^{2p+2} u_0^\delta\|_{\mathbb{L}^2(\mathbb{R})} \lesssim \left(\Delta t^{\frac{p+1}{2p+1}} + \Delta x \right) \|\partial_x^{2p+2} u_0^\delta\|_{\mathbb{L}^2(\mathbb{R})}$. We then apply Lemma 3 with $s = 2p + 2$ and $r = m$. Finally, it yields

$$\|e^n\|_{\ell_\Delta^2} \lesssim \delta^m \|\partial_x^m u_0\|_{\mathbb{L}^2(\mathbb{R})} + \frac{\left(\Delta t^{\frac{p+1}{2p+1}} + \Delta x \right)}{\delta^{2p+2-m}} \|\partial_x^m u\|_{\mathbb{L}^2(\mathbb{R})}.$$

The conclusion comes from the optimal choice $\delta = \left(\Delta t^{\frac{p+1}{2p+1}} + \Delta x \right)^{\frac{1}{2p+2}}$.

□

Remark 7. *The backward scheme, with p even, is very similar. The central finite difference scheme is proved with the same method except for the variable k , which is taken $k = 2p + 3$ for that scheme.*

4 Numerical results

In order to illustrate numerically the previous results, we perform two sets of examples : on the one hand, we compute the numerical rate of convergence of various equations for a fix initial data and one the other hand, the equation is fixed and we test different initial data.

In all examples, the computational domain is set to $[0, 50]$ subdivided into J cells with

$$J \in \{800, 1600, 3200, 6400, 12800, 25600, 51200, 102400\}$$

and the numerical simulation is performed up to time $T = 0.1$. Not to have a too restricted Courant-Friedrich-Lewy condition, we implement the implicit scheme ($\theta = 1$) and impose $\Delta t = \Delta x$. The convergence error is computed between the solution with J cells and a 'reference' solution with $2J$ cells in space.

Since the indicator function belongs to $\mathbb{H}^s(\mathbb{R})$ for all $s < \frac{1}{2}$, we build test functions in $\mathbb{H}^s(\mathbb{R})$ with $s < \frac{1}{2} + k$ by integrating k -times the indicator function. Such functions will be denoted in $\mathbb{H}^{\frac{1}{2}+k-}(\mathbb{R})$.

The first test consists of fixing u_0 in $\mathbb{H}^{\frac{3}{2}-}$ or $\mathbb{H}^{\frac{5}{2}-}$ and compute the convergence rate for $p = 0$ (advection equation), $p = 1$ (Airy equation) and $p = 2$. The numerical results are gathered in Tables 2 and 3 and correctly match with the expected theoretical rates. For the second sample of examples, the equation is fixed ($p = 0$ for Fig. 1-left and $p = 1$ for Fig. 1-right) whereas the Sobolev regularity of the initial data is fluctuating. As shown in Fig. 1, the theoretical rates are represented by the line and the numerical rates correspond to the dot. The exponent of the Sobolev regularity of u_0 is shown in the x-axis. Again, the different rates match very well, which tends to indicate that the convergence orders we have proven are optimal.

Δx	$p = 0$ (Advection)		$p = 1$ (Airy)		$p = 2$ (Fifth-order derivative)	
	\mathbb{L}^2 -error	order	\mathbb{L}^2 -error	order	\mathbb{L}^2 -error	order
$6.250 \cdot 10^{-2}$					$2.985 \cdot 10^{-3}$	
$3.125 \cdot 10^{-2}$					$2.757 \cdot 10^{-3}$	0.115
$1.563 \cdot 10^{-2}$					$2.441 \cdot 10^{-3}$	0.175
$7.813 \cdot 10^{-3}$	$1.194 \cdot 10^{-4}$		$1.348 \cdot 10^{-3}$		$2.176 \cdot 10^{-3}$	0.166
$3.906 \cdot 10^{-3}$	$7.381 \cdot 10^{-5}$	0.694	$1.125 \cdot 10^{-3}$	0.261	$1.961 \cdot 10^{-3}$	0.149
$1.953 \cdot 10^{-3}$	$4.471 \cdot 10^{-5}$	0.723	$9.670 \cdot 10^{-4}$	0.219		
$9.766 \cdot 10^{-4}$	$2.664 \cdot 10^{-5}$	0.747	$7.968 \cdot 10^{-4}$	0.279		
$4.883 \cdot 10^{-4}$	$1.585 \cdot 10^{-5}$	0.749	$6.572 \cdot 10^{-4}$	0.278		
theoretical		0.750		0.250		0.150

Table 2: For a Sobolev regularity $\mathbb{H}^{\frac{3}{2}-}$

Δx	$p = 0$ (Advection)		$p = 1$ (Airy)		$p = 2$ (Fifth-order derivative)	
	\mathbb{L}^2 -error	order	\mathbb{L}^2 -error	order	\mathbb{L}^2 -error	order
$6.250 \cdot 10^{-2}$					$3.388 \cdot 10^{-2}$	
$3.125 \cdot 10^{-2}$					$3.639 \cdot 10^{-2}$	0.093
$1.563 \cdot 10^{-2}$					$3.032 \cdot 10^{-2}$	0.263
$7.813 \cdot 10^{-3}$	$2.586 \cdot 10^{-3}$		$1.011 \cdot 10^{-2}$		$2.528 \cdot 10^{-2}$	0.262
$3.906 \cdot 10^{-3}$	$1.347 \cdot 10^{-3}$	0.940	$7.507 \cdot 10^{-3}$	0.430	$2.138 \cdot 10^{-2}$	0.242
$1.953 \cdot 10^{-3}$	$6.873 \cdot 10^{-4}$	0.971	$6.267 \cdot 10^{-3}$	0.261		
$9.766 \cdot 10^{-4}$	$3.437 \cdot 10^{-4}$	0.999	$4.401 \cdot 10^{-3}$	0.510		
$4.883 \cdot 10^{-4}$	$1.719 \cdot 10^{-4}$	1.000	$3.074 \cdot 10^{-3}$	0.520		
theoretical		1.000		0.417		0.250

Table 3: For a Sobolev regularity $\mathbb{H}^{\frac{5}{2}-}$

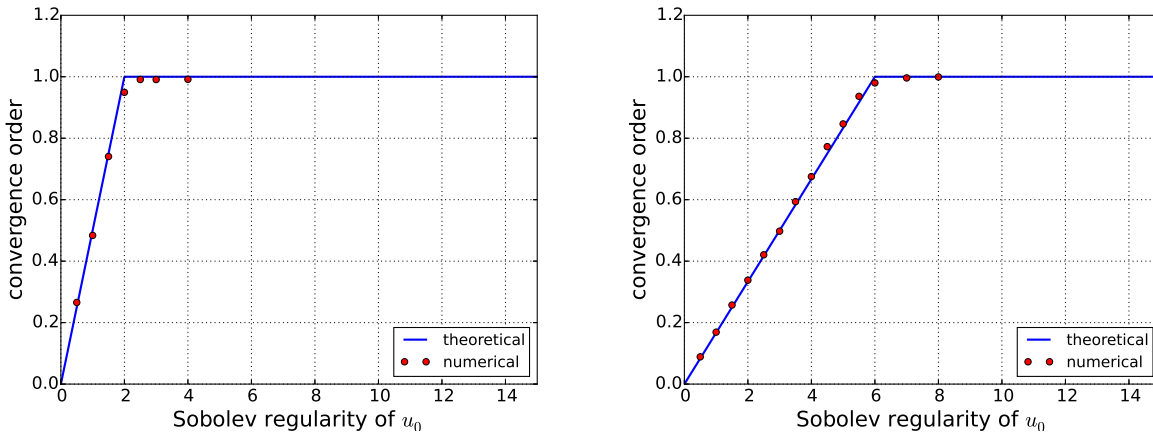


Figure 1: Numerical versus theoretical orders— left : Advection equation ($p = 0$), right : Airy equation ($p = 1$)

For $\theta \neq \frac{1}{2}$	p even		p odd
	$p = 0$ (Advection)		
Forward scheme	$\Delta t^{\frac{\min(m,2)}{2}} \ \partial_x^{\min(m,2)} u_0\ _{\mathbb{L}^2}$	$\Delta t^{\frac{\min(m,2)}{2}} \ \partial_x^{\min(m,2)} u_0\ _{\mathbb{L}^2}$	$\Delta t^{\frac{\min(m,4p+2)}{4p+2}} \ \partial_x^{\min(m,4p+2)} u_0\ _{\mathbb{L}^2}$ $+ \Delta x^{\frac{\min(m,2p+2)}{2p+2}} \ \partial_x^{\min(m,2p+2)} u_0\ _{\mathbb{L}^2}$
Backward scheme	$\Delta t^{\frac{\min(m,2)}{2}} \ \partial_x^{\min(m,2)} u_0\ _{\mathbb{L}^2}$	$\Delta t^{\frac{\min(m,2)}{2}} \ \partial_x^{\min(m,2)} u_0\ _{\mathbb{L}^2}$	$\Delta t^{\frac{\min(m,4p+2)}{4p+2}} \ \partial_x^{\min(m,4p+2)} u_0\ _{\mathbb{L}^2}$ $+ \Delta x^{\frac{\min(m,2p+2)}{2p+2}} \ \partial_x^{\min(m,2p+2)} u_0\ _{\mathbb{L}^2}$
Central scheme	$\Delta t^{\frac{\min(m,2)}{2}} \ \partial_x^{\min(m,2)} u_0\ _{\mathbb{L}^2}$	$\Delta t^{\frac{\min(m,2)}{2}} \ \partial_x^{\min(m,2)} u_0\ _{\mathbb{L}^2}$	$\Delta t^{\frac{\min(m,4p+2)}{4p+2}} \ \partial_x^{\min(m,4p+2)} u_0\ _{\mathbb{L}^2}$ $+ \Delta x^2 \frac{\min(m,2p+3)}{2p+3} \ \partial_x^{\min(m,2p+3)} u_0\ _{\mathbb{L}^2}$ $+ \Delta x^2 \frac{\min(m,2p+3)}{2p+3} \ \partial_x^{\min(m,2p+3)} u_0\ _{\mathbb{L}^2}$
For $\theta = \frac{1}{2}$	(Crank-Nicolson case)		
Forward scheme	$\Delta t^2 \frac{\min(m,3)}{3} \ \partial_x^{\min(m,3)} u_0\ _{\mathbb{L}^2(\mathbb{R})}$	$\Delta t^2 \frac{\min(m,3)}{3} \ \partial_x^{\min(m,3)} u_0\ _{\mathbb{L}^2(\mathbb{R})}$	$\Delta t^2 \frac{\min(m,6p+3)}{6p+3} \ \partial_x^{\min(m,6p+3)} u_0\ _{\mathbb{L}^2(\mathbb{R})}$ $+ \Delta x^{\frac{\min(m,2p+2)}{2p+2}} \ \partial_x^{\min(m,2p+2)} u_0\ _{\mathbb{L}^2(\mathbb{R})}$
Backward scheme	$\Delta t^2 \frac{\min(m,3)}{3} \ \partial_x^{\min(m,3)} u_0\ _{\mathbb{L}^2(\mathbb{R})}$	$\Delta t^2 \frac{\min(m,3)}{3} \ \partial_x^{\min(m,3)} u_0\ _{\mathbb{L}^2(\mathbb{R})}$	$\Delta t^2 \frac{\min(m,6p+3)}{6p+3} \ \partial_x^{\min(m,6p+3)} u_0\ _{\mathbb{L}^2(\mathbb{R})}$ $+ \Delta x^{\frac{\min(m,2p+2)}{2p+2}} \ \partial_x^{\min(m,2p+2)} u_0\ _{\mathbb{L}^2(\mathbb{R})}$
Central scheme	$\Delta t^2 \frac{\min(m,3)}{3} \ \partial_x^{\min(m,3)} u_0\ _{\mathbb{L}^2(\mathbb{R})}$	$\Delta t^2 \frac{\min(m,3)}{3} \ \partial_x^{\min(m,3)} u_0\ _{\mathbb{L}^2(\mathbb{R})}$	$\Delta t^2 \frac{\min(m,6p+3)}{6p+3} \ \partial_x^{\min(m,6p+3)} u_0\ _{\mathbb{L}^2(\mathbb{R})}$ $+ \Delta x^2 \frac{\min(m,2p+3)}{2p+3} \ \partial_x^{\min(m,2p+3)} u_0\ _{\mathbb{L}^2(\mathbb{R})}$ $+ \Delta x^2 \frac{\min(m,2p+3)}{2p+3} \ \partial_x^{\min(m,2p+3)} u_0\ _{\mathbb{L}^2(\mathbb{R})}$

Table 4: Error estimates for $u_0 \in \mathbb{H}^m(\mathbb{R})$

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