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► **To cite this version:**

Roland Bacher. The Stern Sequence and Moments of Minkowski's Question Mark Function. 2017.  
hal-01490725

**HAL Id: hal-01490725**

**<https://hal.science/hal-01490725v1>**

Preprint submitted on 15 Mar 2017

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# The Stern Sequence and Moments of Minkowski's Question Mark Function

Roland Bacher

March 15, 2017

*Abstract*<sup>1</sup>: We use properties of the Stern Sequence for numerical computations of moments  $\int_0^1 t^n d?(t)$  associated to Minkowski's Question Mark function.

## 1 Introduction

Minkowski's question mark function  $x \mapsto ?(x)$  and its inverse function, Conway's box function  $x \mapsto \square(x)$ , are related to continued fraction expansions, transcendence properties and probabilistic distributions of rationals in the Calkin-Wilf tree. Denjoy proved apparently that  $?(x)$  is monotonic continuous and singular (derivable on a set of full measure with zero derivative on this set), see [4]. Using a functional equation satisfied by  $?(x)$ , Alkauskas investigated the sequence  $m_0, m_1, \dots, m_n = \int_0^1 x^n d?(s)$  of moments of the probability density  $d?$  in a series of articles. Denoting by  $\square(y)$  the reciprocal function, known as *Conway's Box function*, of the increasing homeomorphism  $? : [0, 1] \rightarrow [0, 1]$ , the substitution  $t = ?(x)$  (with  $d?(x) = dt$  and  $x = \square(t)$ ) yields

$$m_n = \int_0^1 (\square(t))^n dt . \quad (1)$$

In the present paper we link these moments to the Stern sequence (which underlies the Calkin-Wilf tree)  $s(0) = 0, s(1) = 1, s(2n) = s(n), s(2n + 1) = s(n) + s(n + 1), n \geq 1$ . This gives new proofs for many results of Alkauskas, see for example [1], [2], [3]. It also leads to the discovery of some new properties.

The sequel of the paper is organized as follows:

Section 2 links the Stern sequence with Conway's Box function  $\square$  appearing in (1).

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<sup>1</sup>Keywords: Minkowski's Question Mark Function, Conway's Box Function, Stern sequence, Farey Sequence, Continued Fraction. Math. class: Primary: 11A55, Secondary: 11B57.

Section 3 recalls properties of Minkowski's question mark function.

Section 4 lists a few well-known identities among binomial coefficients and elements of the Stern sequence for later use.

Section 5 presents a set of linear relations obtained by considering Riemann sums for  $\int_0^1 \square(x)^n dx$ . These relations differ from the relations found by Alkauskas: they are perhaps slightly simpler but more interestingly, a crude spectral analysis of the underlying linear operator  $T$  is easy.  $T$  has a unique eigenvector  $(m_0, m_1, m_2, \dots)$  of eigenvalue 1. All other eigenvalues belong to the closed complex disc of radius  $1/2$ . The maximal error of the associated algorithm is thus roughly halved at each iteration.

Section 6 discusses a different set of Riemann sums which leads to linear relations used by Alkauskas.

We extend in Section 7 the moment-function  $n \mapsto m_n$  to an entire function  $z \mapsto m_z$  for  $z \in \mathbb{C}$ .

A computation of the derivative of this function at 0 to high accuracy suggests the conjectural identities

$$\log 2 = \sum_{n=1}^{\infty} \frac{m_n}{2n} \left(1 + \frac{1}{2^{n-1}}\right) = \sum_{n=1}^{\infty} \frac{m_n}{n} \left(\frac{1}{2^n} - (-1)^n\right)$$

given in Section 8.

Section 9 introduces a third type of Riemann sums, particularly well suited for asymptotic computations. The resulting asymptotic formula

$$m_n \sim \sum_{j=0}^{\infty} \frac{(\log 2)^j}{j!} m_j \sum_{h=2}^{\infty} \frac{1}{2^h} \left(1 - \frac{1}{h}\right)^n \quad (2)$$

is the object of Section 10. It is more complicated but experimentally more accurate than Alkauskas's asymptotic formula given in [3]. Alkauskas's formula can however be deduced from (2) by a simple application of Laplace's method.

Section 10.2 derives a second asymptotic formula related to (2) by a finer subdivision in the underlying Riemann sum. Since this should lead to slightly more accurate results, we consider (admittedly in a not completely rigorous way) in Section 10.3 the difference between the two formulae as a measure of accuracy for (2).

Section 11 is devoted to values  $m_{-n}$  of moments at negative integers. This leads to a sequence of identities among  $m_0, m_1, m_2, \dots$ . The two initial identities are

$$\sum_{j=0}^{\infty} m_j = \frac{5}{2} \quad \text{and} \quad \sum_{j=1}^{\infty} j m_j = m_2 + \frac{11}{2}.$$

Finally, Section 12 discusses the starting point of this work: asymptotics for  $\prod_{j=2^n}^{2^{n+1}} s_j$  allowing to compute some geometric means for values of the Stern sequence.

## 2 Conway's box function

We denote by  $\mathcal{D} = \mathbb{Z}[1/2] \cap [0, 1]$  the subset of all rational dyadic numbers in  $[0, 1]$ . The restriction to  $\mathcal{D}$  of *Conway's Box function*  $\square$  is recursively defined as follows:  $\square(0) = 0, \square(1) = 1$  and

$$\square\left(\frac{2m+1}{2^{n+1}}\right) = \frac{a+c}{b+d}$$

if  $\square\left(\frac{m}{2^n}\right) = \frac{a}{b}$  and  $\square\left(\frac{m+1}{2^n}\right) = \frac{c}{d}$  where  $a, b$ , respectively  $c, d$ , are coprime natural numbers. The values  $\square(m/16)$  for  $m = 0, \dots, 16$  are:

$m$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$\square\left(\frac{m}{16}\right)$	$\frac{0}{1}$	$\frac{1}{5}$	$\frac{1}{4}$	$\frac{2}{7}$	$\frac{1}{3}$	$\frac{3}{8}$	$\frac{2}{5}$	$\frac{3}{7}$	$\frac{1}{2}$	$\frac{4}{7}$	$\frac{3}{5}$	$\frac{5}{8}$	$\frac{2}{3}$	$\frac{5}{7}$	$\frac{3}{4}$	$\frac{4}{5}$	$\frac{1}{1}$

Values of  $\square$  for arguments in  $\mathcal{D}$  are easy to compute as follows: We define the *Stern-sequence*  $s(0), s(1), s(2), \dots$  recursively by  $s(0) = 0, s(1) = 1, s(2n) = s(n)$  and  $s(2n+1) = s(n) + s(n+1)$ . Its first coefficients are given by

$n$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$s(n)$	0	1	1	2	1	3	2	3	1	4	3	5	2	5	3	4
$n$	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31
$s(n)$	1	5	4	7	3	8	5	7	2	7	5	8	3	7	4	5

The main tool used in this paper is the following simple observation which defines  $\square$  on  $\mathcal{D}$  in terms of the Stern-sequence:

**Proposition 2.1.** *We have*

$$\square\left(\frac{m}{2^n}\right) = \frac{s(m)}{s(2^n + m)}$$

for all natural integers  $m, n$  such that  $0 \leq m \leq 2^n$ .

We leave the easy proof to the reader.  $\square$

Since  $\frac{a}{b} < \frac{c}{d}$  with  $bd > 0$  implies  $\frac{a}{b} < \frac{a+c}{b+d} < \frac{c}{d}$ , the function  $\square$  is strictly increasing. Induction on  $k$  shows

$$\square\left(\frac{m}{2^n} \pm \frac{1}{2^{n+k}}\right) = \frac{ka+c}{kb+d}, \quad k \geq 1 \quad (3)$$

if  $\square\left(\frac{m}{2^n}\right) = \frac{a}{b}$  and  $\square\left(\frac{m}{2^n} \pm \frac{1}{2^n}\right) = \frac{c}{d}$  (with  $a, b$  and  $c, d$  pairs of coprime natural numbers). In particular,  $\square$  extends to a strictly increasing continuous function (still denoted)  $\square : [0, 1] \rightarrow [0, 1]$ . Since

$$\lim_{k \rightarrow \infty} \frac{\square\left(\frac{m}{2^n} \pm \frac{1}{2^{n+k}}\right) - \square\left(\frac{m}{2^n}\right)}{\pm 2^{-n-k}} = \frac{2^{n+k}}{b(kb+d)} = \infty,$$

the function  $\square$  has a vertical tangent at dyadic arguments.

**Proposition 2.2.** *We have*

$$\square(x) = 1 - \square(1 - x) . \quad (4)$$

*Proof.* Continuity of Conway's Box function implies that it is enough to prove Proposition 2.2 for all dyadic rationals of the form  $\frac{m}{2^n}$ . This is done by induction using the trivial identity  $\frac{a+c}{b+d} = 1 - \frac{b-a+d-c}{b+d}$ .  $\square$

**Corollary 2.3.** *The function  $x \mapsto 2\square(x - \frac{1}{2}) - 1$  is symmetric.*

Thus we have

$$0 = \int_0^1 (\square(x) - 1/2)^{2n+1} dx = \sum_{k=0}^{2n+1} \binom{2n+1}{k} (-2)^{k-2n-1} m_k \quad (5)$$

for every odd natural number  $2n + 1$ . This can be restated as:

**Corollary 2.4.** *For all  $n \geq 0$  we have the identity*

$$m_{2n+1} = \frac{1}{2^{2n+1}} \sum_{k=0}^{2n} (-2)^k \binom{2n+1}{k} m_k . \quad (6)$$

*In particular,  $m_{2n+1}$  is a  $\mathbb{Z}[\frac{1}{2}]$ -linear combination of  $m_0, m_2, m_4, \dots, m_{2n}$ .*

### 3 Minkowski's question mark function

Given an irrational real number  $x$  in  $(0, 1)$  with continued fraction expansion given by

$$x = [0; a_1, a_2, a_3, \dots] = \frac{1}{a_1 + \frac{1}{a_2 + \dots}},$$

Minkowski's question mark function is defined by

$$?(x) = -2 \sum_{k=1}^{\infty} \frac{(-1)^k}{2^{a_1 + \dots + a_k}} . \quad (7)$$

**Proposition 3.1.** *Minkowski's question mark function is an increasing homeomorphism of  $[0, 1]$  such that  $\square \circ ?(x) = ? \circ \square(x) = x$ .*

*Proof (given for the sake of self-containedness).* Since  $\square$  is an increasing homeomorphism of  $[0, 1]$ , it is enough to prove that  $\square \circ ?(x) = x$  for every rational number  $x$  in  $[0, 1]$ . We show this by induction on the length  $n$  of the continued fraction expansion  $x = [0; a_1, a_2, \dots, a_n]$  of  $x$ . The result clearly holds for  $n = 0$  (corresponding to  $x = 0$ ) and for  $n = 1$  (corresponding to the inverse of a non-zero natural integer). Writing  $\frac{p_k}{q_k} = [0; a_1, \dots, a_k]$  we have

$$?\left(\frac{p_{n-1}}{q_{n-1}}\right) = -2 \sum_{j=1}^{n-1} \frac{(-1)^j}{2^{a_1 + \dots + a_j}} = \frac{m}{2^{a_1 + \dots + a_{n-1} - 1}}$$

for a suitable natural number  $m$ . We also have

$$? \left( \frac{p_{n-2}}{q_{n-2}} \right) = ? \left( \frac{p_{n-1}}{q_{n-1}} \right) - \frac{(-1)^n}{2^{a_1 + \dots + a_{n-1} - 1}}.$$

Using the induction hypothesis  $\frac{p_k}{q_k} = \square \circ ? \left( \frac{p_k}{q_k} \right)$  for  $k < n$  and applying (3) to

$$? \left( \frac{p_n}{q_n} \right) = \frac{m}{2^{a_1 + \dots + a_{n-1} - 1}} - \frac{(-1)^n}{2^{a_n} 2^{a_1 + \dots + a_{n-1} - 1}}$$

we get

$$\square \circ ? \left( \frac{p_n}{q_n} \right) = \frac{a_n p_{n-1} + p_{n-2}}{a_n q_{n-1} + q_{n-2}} = \frac{p_n}{q_n}.$$

□

The graph of  $?$  is well-known to behave in a self-similar way as shown by the following well-known result:

**Proposition 3.2.** *We have*

$$?(1-x) = 1-?(x) \tag{8}$$

and

$$? \left( \frac{x}{1+x} \right) = \frac{1}{2}?(x) \tag{9}$$

for all  $x \in [0, 1]$ .

*Proof.* Identity (8) follows from Proposition 2.2 and Proposition 3.1. Identity 9 follows from the Definition (7) applied to  $\frac{x}{1+x} = \frac{1}{1+1/x} = [0; 1 + a_1, a_2, a_3, \dots]$ . □

The aim of this paper is to study the moments

$$m_z = \int_0^1 \square(t)^z dt = \int_0^1 x^z d?(x)$$

of the probability measure  $d?$  associated to the distribution function  $?(x) = \int_0^x d?(t)$ . The inequalities

$$\sum_{n=1}^{\infty} \frac{1}{2^n} \left(1 - \frac{1}{n}\right)^j \leq m_j \leq 2 \sum_{n=1}^{\infty} \frac{1}{2^n} \left(1 - \frac{1}{n}\right)^j$$

coming from the evaluation  $\square(1 - 2^{-m}) = 1 - \frac{1}{m+1}$ , and the trivial upper bound  $\left| \binom{-z-j+1}{j} \right| \leq \binom{|z|+j-1}{j} \leq (|z|+j)^{|z|}$  show that  $z \mapsto m_z$  is an entire function of  $\mathbb{C}$ .

The function  $m_z$  is also given by the expression

$$m_z = \sum_{k=0}^{\infty} \binom{z+k-1}{k} \gamma(k+z) m_k$$

(see (24)) where  $\gamma(z)$  is the entire function defined by

$$\gamma(z) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{1}{(1+n)^z}.$$

We give the series expansion of the entire function  $z \mapsto m_z$  at  $z = 0$  and study the asymptotics of  $m_z$  for real  $z \rightarrow \pm\infty$ .

**Proposition 3.3.** *We have the identities*

$$m_z = \sum_{j=0}^{\infty} \binom{z}{j} (-1)^j m_j = \sum_{j=0}^{\infty} \binom{-z+j-1}{j} m_j$$

where  $\binom{z}{j} = \frac{z(z-1)(z-2)\cdots(z-j+1)}{j!}$ .

The main contribution to  $m_{-k}$  given by Proposition 3.3 corresponds to indices  $j$  such that  $\frac{k+j}{j} e^{-\sqrt{(\log 2)/j}} \sim 1$  yielding  $j \sim \frac{k^2}{\log 2}$ .

Thus we have for example

$$\begin{aligned} m_{-1} &= \sum_{n=0}^{\infty} m_n \\ m_{-2} &= \sum_{n=0}^{\infty} (n+1) m_n \\ m_{-3} &= \frac{1}{2} \sum_{n=0}^{\infty} (n+1)(n+2) m_n \end{aligned}$$

and more generally

$$m_{-n} = \sum_{k=0}^{\infty} \binom{k+n-1}{n-1} m_k.$$

*Proof of Proposition 3.3.* Proposition 2.2 implies the equalities

$$m_z = \int_0^1 \square(t)^z dt = \int_0^1 (1-\square(t))^z dt = \sum_{j=0}^{\infty} \binom{z}{j} (-1)^j m_j = \sum_{j=0}^{\infty} \binom{-z+j-1}{j} m_j$$

which hold for all  $z \in \mathbb{C}$  since  $\square(t) \in (0, 1)$  for  $t \in (0, 1)$ .  $\square$

## 4 A few useful identities

Almost all results of this paper are based on a few trivial identities, recorded in this Section for later use.

### 4.1 Binomial coefficients

**Lemma 4.1.** *We have the series expansion*

$$\frac{1}{(1-x)^n} = \sum_{k=0}^{\infty} \binom{n+k-1}{k} x^k = \sum_{k=0}^{\infty} \binom{n+k-1}{n-1} x^k \quad (10)$$

for  $x$  in the open complex unit-disc.

*Proof.* Apply the equality  $\binom{-n}{k} = (-1)^k \binom{n+k-1}{k}$  (where  $\binom{x}{k} = \frac{x(x-1)\cdots(x-k+1)}{k!}$ ) to Newton's identity  $(1+(-x))^{-n} = \sum_{k=0}^{\infty} \binom{-n}{k} (-x)^k$  or use induction on  $n$ .  $\square$

**Remark 4.2.** *Lemma 4.1 has the following nice combinatorial proof:  $\frac{1}{(1-x)^n}$  is the generating series for colouring Easter eggs with  $n$  different colours (or, equivalently, for the number of monomials in  $n$  commuting variables). The  $k$ -th coefficient is thus given by  $\binom{k+n-1}{n-1}$ .*

**Lemma 4.3.** *We have*

$$\sum_{l=k}^j \binom{j}{l} \binom{l}{k} x^l = \binom{j}{k} x^k (x+1)^{j-k}$$

In particular, for  $x = \frac{-1}{2}$  we get

$$\sum_{l=k}^j \binom{j}{l} \binom{l}{k} \frac{1}{(-2)^l} = \binom{j}{k} \frac{(-1)^k}{2^j}$$

*Proof.* Compare the coefficients  $\binom{j}{l} \binom{l}{k}$  and  $\binom{j}{k} \binom{j-k}{l-k}$  of  $x^l$  of both sides.  $\square$

### 4.2 Identities for the Stern sequence

We recall that the Stern sequence  $s : \mathbb{N} \rightarrow \mathbb{N}$  is recursively defined by  $s(0) = 0, s(1) = 1, s(2n) = s(n)$  and  $s(2n+1) = s(n) + s(n+1)$  for  $n \geq 1$ .

**Proposition 4.4.** *For all  $n \geq 0$  and for all  $r$  such that  $0 \leq r \leq 2^n$ , the Stern sequence satisfies the identities*

$$s(2^n + r) = s(2^n - r) + s(r), \quad (11)$$

$$s(2^n + r) = s(2^{n+1} - r), \quad (12)$$

$$s(r) = 2s(2^n + r) - s(3 \cdot 2^n + r). \quad (13)$$



*Proof.* The identities hold for  $n = 0$  and  $r \in \{0, 1\}$ . Since  $s(2m) = s(m)$  they hold for  $r$  even by induction. For odd  $r = 2t + 1 < 2^{n+1}$ , we sum the identities corresponding to  $(n - 1, t)$  and  $(n - 1, t + 1)$  which hold by induction. The definition  $s(2m + 1) = s(m) + s(m + 1)$  and induction implies the identities for odd  $r$ .  $\square$

The main idea of this paper is to apply Lemma 4.1 to the trivial identities

$$\frac{\alpha s + \beta S}{\gamma s + \delta S} = \frac{\alpha}{\delta} \frac{s}{S} \frac{1}{\left(1 + \frac{\gamma s}{\delta S}\right)} + \frac{\beta}{\delta} \frac{1}{\left(1 + \frac{\gamma s}{\delta S}\right)}, \quad (14)$$

$$\frac{\alpha s + \beta S}{\gamma s + \delta S} = \frac{\alpha}{\gamma} + \left(\frac{\beta}{\delta} - \frac{\alpha}{\gamma}\right) \frac{1}{\left(1 + \frac{\gamma s}{\delta S}\right)}, \quad (15)$$

$$\frac{\alpha s + \beta S}{\gamma s + \delta S} = \frac{\beta}{\delta} + \left(\frac{\alpha}{\delta} - \frac{\beta\gamma}{\delta^2}\right) \frac{s}{S} \frac{1}{\left(1 + \frac{\gamma s}{\delta S}\right)}. \quad (16)$$

## 5 A simple set of linear equations for $m_{\mathbb{N}}$

**Theorem 5.1.** *The sequence  $m_0 = 1, m_1 = \frac{1}{2}, m_2, \dots$  of moments defined by  $m_n = \int_0^1 \square(t)^n dt$  (see (1)) satisfies the equalities*

$$m_n = \frac{1}{2^n} \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} \phi_{2k} \quad (17)$$

where

$$\phi_n = \sum_{k=0}^{\infty} \binom{n+k-1}{k} \frac{m_{n+k}}{2^{n+k}}. \quad (18)$$

**Remark 5.2.** *Since the increasing function*

$$k \mapsto \frac{\binom{n+k-1}{k} 2^{n+k+1}}{2^{n+k} \binom{n+k}{k+1}} = 2 \frac{k+1}{n+k}$$

(for  $k > 0$  and  $n$  a fixed natural integer) equals 1 for  $k = n - 2$  and since the moments  $m_n$  are slowly decreasing, the main contribution to  $\phi_n$  comes asymptotically from summands with indices  $k$  roughly equal to  $n$ .

The main contribution to  $\phi_n$  is thus given by moments of the form  $m_{2n+l}$  with  $l$  an element of  $\mathbb{Z}$  of small absolute value.

Similarly, the main contribution to  $m_n$  in Formula (17) corresponds asymptotically to indices  $k \sim n/4$ . and involves thus mainly moments of the form  $m_{n+l}$  for  $l$  a small integer.

Theorem 5.1 is an immediate consequence of the following result.

**Proposition 5.3.** *For all  $n \in \mathbb{N}$  we have the identities*

$$\int_0^{1/2} \square(t)^n dt = \frac{1}{2^{n+1}} \sum_{k=0}^n \binom{n}{k} (-1)^k \phi_k$$

and

$$\int_{1/2}^1 \square(t)^n dt = \frac{1}{2^{n+1}} \sum_{k=0}^n \binom{n}{k} \phi_k$$

with  $\phi_k$  defined by Formula (18).

**Lemma 5.4.** *We have*

$$\phi_n = 2 \int_{1/2}^1 (2\square(t) - 1)^n dt . \quad (19)$$

for  $\phi_n$  defined by Formula (18).

Corollary 2.3 shows that Lemma 5.4 can be restated as  $\phi_n = \int_0^1 |2\square(t) - 1|^n dt$ .

*Proof of Lemma 5.4.* Proposition 2.1 and the definition of Riemann sums show that we have

$$\begin{aligned} 2 \int_{1/2}^1 (2\square(t) - 1)^n dt &= 2 \lim_{l \rightarrow \infty} \frac{1}{2^{l+1}} \sum_{r=0}^{2^l} \left( 2 \frac{s(2^l + r)}{s(2^{l+1} + 2^l + r)} - 1 \right)^n \\ &= \lim_{l \rightarrow \infty} \frac{1}{2^l} \sum_{r=0}^{2^l} \left( \frac{2s(2^l + r) - s(3 \cdot 2^l + r)}{s(3 \cdot 2^l + r)} \right)^n . \end{aligned}$$

Using (13) we get

$$2 \int_{1/2}^1 (2\square(t) - 1)^n dt = \lim_{l \rightarrow \infty} \frac{1}{2^l} \sum_{r=0}^{2^l} \left( \frac{s(r)}{2s(2^l + r) - s(r)} \right)^n \quad (20)$$

or equivalently

$$2 \int_{1/2}^1 (2\square(t) - 1)^n dt = \lim_{l \rightarrow \infty} \frac{1}{2^l} \sum_{r=0}^{2^l} \left( \frac{s(r)}{2s(2^l + r)} \right)^n \left( \frac{1}{1 - \frac{s(r)}{2s(2^l + r)}} \right)^n .$$

Applying (10) we have

$$2 \int_{1/2}^1 (2\square(t) - 1)^n dt = \sum_{j=0}^{\infty} \binom{n+j-1}{j} \frac{m_{n+j}}{2^{n+j}}$$

which ends the proof.  $\square$

*Proof of Proposition 5.3.* Using Lemma 5.4 we have

$$\begin{aligned} \frac{1}{2^{n+1}} \sum_{k=0}^n \binom{n}{k} (-1)^k \phi_k &= \frac{1}{2^n} \int_{1/2}^1 \sum_{k=0}^n \binom{n}{k} (1 - 2\Box(t))^k dt \\ &= \int_{1/2}^1 (1 - \Box(t))^n dt \end{aligned}$$

which equals  $\int_0^{1/2} \Box(t)^n dt$  by (4). This proves the first equality.

The proof of the second equality is similar and left to the reader.  $\square$

## 5.1 Spectral properties

Theorem 5.1 expresses the moment-vector  $(m_0, m_1, m_2, \dots)$  as a fixed point of a continuous linear operator  $T$  acting on the vector space  $l^\infty(\mathbb{R})$  of real bounded sequences. We study here a few spectral properties of  $T$ . They imply in particular uniqueness of the fixed point  $(m_0, m_1, \dots)$  satisfying  $m_0 = 1$ .

We denote by  $l^\infty = l^\infty(\mathbb{R})$  the real Banach space of bounded sequences with norm  $\|v\|_\infty = \sup_{n \in \mathbb{N}} |v_n|$  for  $v = (v_0, v_1, \dots)$  in  $l^\infty$ . We set

$$\|U\| = \sup_{v \in l^\infty, \|v\|_\infty = 1} \|U(v)\|$$

for the norm  $\|U\|$  of an endomorphism  $U \in \text{End}(l^\infty)$ . Similarly, we consider the norm

$$\|L\| = \sup_{v \in l^\infty, \|v\|_\infty = 1} |L(v)|$$

of a continuous linear form  $L : l^\infty \rightarrow \mathbb{R}$ .

Formulae (17) and (18) suggest to consider the sequence of operators

$$v = (v_0, v_1, \dots) \mapsto T_n(v) = \frac{1}{2^n} \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} \sum_{l=0}^{\infty} \binom{2k+l-1}{l} \frac{v_{2k+l}}{2^{2k+l}}. \quad (21)$$

**Proposition 5.5.** *Formula (21) defines continuous linear forms  $T_0, T_1, T_2, \dots$  of norm  $\|T_0\| = 1$  and  $\|T_n\| = \frac{1}{2}$  for  $n \geq 1$ .*

We define an endomorphism  $T : l^\infty \rightarrow l^\infty$  of the vector-space  $l^\infty$  by setting  $T = (T_0, T_1, T_2, \dots)$ . Proposition 5.5 and  $T_0(v_0, v_1, \dots) = v_0$  imply the following result:

**Corollary 5.6.** *The restriction of the linear operator  $T = (T_0, T_1, T_2, \dots)$  to the subspace  $l_0^\infty$  formed by all bounded sequences  $(v_0, v_1, v_2, \dots)$  starting with  $v_0 = 0$  yields an endomorphism of  $l_0^\infty$  whose spectrum is contained in  $\{z \in \mathbb{C} \mid |z| \leq \frac{1}{2}\}$ .*

In particular, the linear map

$$v \mapsto T(v) = (T_0(v), T_1(v), \dots)$$

defines a bounded linear operator of  $l^\infty$  which has a unique eigenvector of eigenvalue 1 of the form  $(1, \frac{1}{2}, \dots)$ .

The coordinates  $(m_0, m_1, \dots) = (1, \frac{1}{2}, \dots)$  of the unique eigenvector of eigenvalue 1 of  $T$  are of course the moments  $m_n = \int_0^1 x^n d?x$  of the density function associated to Minkowski's question-mark function ?.

*Proof of Proposition 5.5.* For  $v \in l^\infty$  such that  $\|v\|_\infty \leq 1$ , we have

$$|T_n(v)| \leq \frac{1}{2^n} \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} \frac{1}{2^{2k}} \sum_{j=0}^{\infty} \binom{2k-1+j}{j} \frac{1}{2^j}$$

with equality if and only if  $v$  is (up to a sign) the vector  $\mathbf{1} = (1, 1, 1, \dots)$  with all coefficients equal to 1.

Applying (10) we have thus

$$\begin{aligned} \|T_n\| &= |T_n(\mathbf{1})| \\ &= \frac{1}{2^n} \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} \frac{1}{2^{2k}} \left( \frac{1}{1 - \frac{1}{2}} \right)^{2k} \\ &= \frac{1}{2^n} \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} \\ &= \frac{1}{2^n} \left( \frac{(1+1)^n + (1-1)^n}{2} \right) \\ &= \begin{cases} 1 & \text{if } n = 0 \\ \frac{1}{2} & \text{if } n \geq 1 \end{cases} \end{aligned}$$

which completes the proof. □

**Remark 5.7.** Laplace's method shows that the coefficient

$$\frac{1}{2^n} \sum_{k=1}^{\lfloor n/2 \rfloor} \binom{n}{2k} \binom{m-1}{2k-1} \frac{1}{2^m}$$

of  $v_m$  in  $T_n$  given by Formula (21) is asymptotically equal to

$$\frac{1}{2} \frac{1}{\sqrt{2\pi n \mu (1+\mu)}} \left( \frac{((1+\mu)/2)^{1+\mu}}{\mu^\mu} \right)^n$$

for  $\mu = \frac{m}{n}$  having a bounded logarithm. This coefficient is asymptotically maximal for  $\mu = 1$  and decays exponentially fast otherwise. We have

$$\lim_{n \rightarrow \infty} n \int_0^\infty \frac{1}{2} \frac{1}{\sqrt{2\pi n \mu (1 + \mu)}} \left( \frac{((1 + \mu)/2)^{1+\mu}}{\mu^\mu} \right)^n d\mu = \frac{1}{2}$$

in agreement with Proposition 5.5.

**Remark 5.8.** The linear operator  $T$  has an unbounded eigenvector of eigenvalue  $\frac{1}{2}$  given by  $w = (0, 0, 2, 3, 4, 5, 6, 7, \dots)$  as can be seen as follows: We have  $T_0(w) = T_1(w) = 0$ . For  $n \geq 2$ , Formula (21) with  $w = (0, 0, 2, 3, 4, 5, 6, \dots)$  boils down to

$$T_n(w) = \frac{1}{2^n} \sum_{k=1}^{\lfloor n/2 \rfloor} \binom{n}{2k} \sum_{l=0}^{\infty} \binom{2k+l-1}{l} \frac{2k+l}{2^{2k+l}}.$$

Computing the derivative  $2k \frac{x^{2k-1}}{(1-x)^{2k+1}}$  of  $\left(\frac{x}{1-x}\right)^{2k}$  at  $x = \frac{1}{2}$  either directly or using the series expansion (10) given by Lemma 4.1 we get the identity

$$4k = \sum_{l=0}^{\infty} \binom{2k+l-1}{l} \frac{2k+l}{2^{2k+l}}.$$

For  $n \geq 2$  we have thus

$$\begin{aligned} T_n(w) &= \frac{1}{2^n} \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} 4k \\ &= \frac{1}{2^n} ((1+x)^n + (1-x)^n)' \Big|_{x=1} \\ &= \frac{n}{2}. \end{aligned}$$

## 5.2 Computational aspects

Theorem 5.1 is useful for computing numerical approximations of the first moments  $m_0, m_1, \dots, m_N$  of Minkowski's question mark function.

This can be done by computing an approximation  $(\tilde{m}_0, \tilde{m}_1, \dots, \tilde{m}_N)$  of the unique attracting fixed point  $(\tilde{m}_0, \tilde{m}_1, \dots)$  of the form  $(1, \dots)$  of the linear operator  $T \circ \pi_N$  where  $\pi_N : l^\infty \rightarrow l^\infty$  is the projection defined by

$$\pi_N(x_0, x_1, \dots, x_N, x_{N+1}, \dots) = (x_0, x_1, \dots, x_N, 0, 0, 0, \dots).$$

The error  $|\tilde{m}_i - m_i|$  is of order  $O(m_{N+1}) = O\left(N^{1/4} e^{-2\sqrt{N \log 2}}\right)$ , see Formula (42).

Since the distance to the fixed point is essentially divided by 2 under each iteration of  $T \circ \pi_N$ , the complexity of the resulting algorithm is roughly of order  $O\left(\sqrt{N/\log 2} N^2\right)$  if aiming at maximal accuracy.

More precisely, the algorithm can be implemented as follows:

```

010  $\tilde{m}_0 := 1,$ 
020 For  $n = 1, 2, 3, \dots, N$  do:
030    $\tilde{m}_n := 0,$ 
040 End of loop over  $n,$ 
050 Iterate the following loop:
060   For  $n = 0, 2, 4, \dots, 2\lfloor N/2 \rfloor$  do:
070      $b := \frac{1}{2^n},$ 
080      $\tilde{\phi}_n := 0,$ 
090     For  $k = 0, 1, 2, \dots, N - n$  do:
100        $\tilde{\phi}_n := \tilde{\phi}_n + b\tilde{m}_{n+k},$ 
110        $b := \frac{n+k}{2(k+1)}b,$ 
120     End of loop over  $k,$ 
130   End of loop over  $n,$ 
140   For  $n = 1, 2, 3, \dots, N$  do:
150      $b := \frac{1}{2^n},$ 
160      $\tilde{m}_n := 0,$ 
170     For  $k = 0, 1, 2, \dots, \lfloor N/2 \rfloor$  do:
180        $\tilde{m}_n := \tilde{m}_n + b\tilde{\phi}_{2k},$ 
190        $b := \frac{(n-2k-1)(n-2k)}{(2k+1)(2k+2)}b,$ 
200     End of loop over  $k,$ 
210   End of loop over  $n,$ 
220 End of outer loop (starting at 050).
```

**Comments:**

1. Computations should be done over the real numbers with sufficient accuracy (maximal achievable accuracy is of order  $O(m_{N+1})$ , see Section 10 for estimations).
2. The range and increment of the loop-variable  $n$  in line 060 is due to the fact that  $m_1, \dots, m_N$  depend only on  $\phi_0, \phi_2, \phi_4, \dots, \phi_{2\lfloor N/2 \rfloor}$  in Formula (17).
3. Instructions 070 and 150 need a loop in many programming languages.
4. The variable  $b$  in line 070, 100, 110 corresponds to the factor  $\binom{n+k-1}{k} \frac{1}{2^{n+k}}$  in Formula (18).
5. The variable  $b$  in line 150, 180, 190 corresponds to the factor  $\frac{1}{2^n} \binom{n}{2k}$  in Formula (17).
6. Maximal possible accuracy is achieved by iterating the outer loop (instructions 060-210) roughly  $2\sqrt{N/\log 2}$  times, see Corollary 10.2.

7. Using a known sequence of good approximations for  $m_1, \dots, m_N$  instead of 0 when initializing  $\tilde{m}_1, \dots, \tilde{m}_N$  (instruction 030) decreases the number of useful (i.e. leading to significantly better precision) iterations for the outer loop.
8. A progressive increase of  $N$  (starting from some small initial value) during the iteration of the outer loop yields a small speedup.

## 6 Formulae of Alkauskas

Theorem 5.1 is based on Riemann sums for the integral

$$A = 2 \int_{1/2}^1 (2\Box(t) - 1)^n dt$$

obtained by subdividing the interval  $[\frac{1}{2}, 1]$  into  $2^l$  sub-intervals of equal length  $\frac{1}{2^{l+1}}$ .

In this section we give a new proof of some formulae obtained by Alkauskas by considering the infinite subdivision

$$[0, 1] = \{0\} \cup \dots \cup \left[ \frac{1}{2^h}, \frac{1}{2^{h-1}} \right] \cup \left[ \frac{1}{2^{h-1}}, \frac{1}{2^{h-2}} \right] \cup \dots \cup \left[ \frac{1}{4}, \frac{1}{2} \right] \cup \left[ \frac{1}{2}, 1 \right]$$

suggested by the easy evaluations  $\Box\left(\frac{1}{2^h}\right) = \frac{1}{h+1}$ .

**Theorem 6.1.** *We have*

$$m_n = \sum_{h=1}^{\infty} \frac{1}{2^h} \frac{1}{(h+1)^n} \sum_{k=0}^{\infty} \binom{k+n-1}{n-1} \frac{m_k}{(h+1)^k} \quad (22)$$

and

$$m_n = \sum_{h=1}^{\infty} \frac{1}{2^h} \frac{1}{h^n} \sum_{k=0}^{\infty} \binom{k+n-1}{n-1} \frac{m_k}{(-h)^k}. \quad (23)$$

**Remark 6.2.** *From a computational point of view it is perhaps useful to rewrite the formulae of Theorem 6.1 as*

$$m_n = \sum_{k=0}^{\infty} \binom{k+n-1}{n-1} \gamma_{k+n} m_k \quad (24)$$

and

$$m_n = \sum_{k=0}^{\infty} \binom{k+n-1}{n-1} (-1)^k c_{k+n} m_k \quad (25)$$

where

$$\gamma_n = \sum_{k=1}^{\infty} \frac{1}{2^k (k+1)^n} = 2\text{Li}_n\left(\frac{1}{2}\right) - 1$$

and

$$c_n = \sum_{k=1}^{\infty} \frac{1}{2^k k^n} = \text{Li}_n\left(\frac{1}{2}\right)$$

where  $\text{Li}_n(x) = \sum_{k=1}^{\infty} \frac{x^k}{k^n}$  for  $x$  in the open complex unit-disc.

Formula (22) (or (24)) should be preferred over (23) (or (25)). It converges faster (under iteration) and positivity of all coefficients ensures numerical stability.

Precomputing (and storing) the constants  $\gamma_k$  and using (24) needs only twice as much memory but provides a significant speed-up.

Formula (25) has been used by Alkauskas for numerical computations of the first values of  $m_n$ , see Appendix A3 of [1] or Proposition 5 of [2].

Since  $\gamma_n \sim \frac{1}{2^{n+1}}$  for large  $n$ , the arguments of Remark 5.2 show that the main contribution to  $m_n$  in Formula (24) corresponds asymptotically to summands  $k \sim n$  involving  $m_{n-a}, \dots, m_{n+a}$ .

**Proposition 6.3.** *Setting*

$$I_h(n) = \int_{2^{-h}}^{2^{-h+1}} \square(t)^n dt . \quad (26)$$

we have

$$I_h(n) = \frac{1}{2^h} \sum_{k=0}^{\infty} \binom{k+n-1}{n-1} \frac{m_k}{(h+1)^{k+n}} \quad (27)$$

and

$$I_h(n) = \frac{1}{2^h} \sum_{k=0}^{\infty} \binom{k+n-1}{n-1} (-1)^k \frac{m_k}{h^{k+n}} . \quad (28)$$

**Lemma 6.4.** *We have*

$$\square\left(\frac{1}{2^h} + \frac{r}{2^{h+l}}\right) = \frac{s(2^l + r)}{(h+1)s(2^l + r) - s(r)} \quad (29)$$

and

$$\square\left(\frac{1}{2^h} + \frac{r}{2^{h+l}}\right) = \frac{s(2^l + r)}{hs(2^l + r) + s(2^l - r)} \quad (30)$$

for  $0 \leq r \leq 2^l$ .



**Remark 6.5.** More generally, if

$$\square\left(\frac{q}{2^h}\right) = \frac{a}{b} \text{ and } \square\left(\frac{q+1}{2^h}\right) = \frac{c}{d}$$

with  $(a, b) \in \mathbb{N}^2$  and  $(c, d) \in \mathbb{N}^2$  pairs of relatively prime natural numbers, then

$$\begin{aligned} \square\left(\frac{q}{2^h} + \frac{r}{2^{h+l}}\right) &= \frac{as(2^l+r) + (c-a)s(r)}{bs(2^l+r) + (d-b)s(r)} \\ &= \frac{cs(2^l+r) + (a-c)s(2^l-r)}{ds(2^l+r) + (b-d)s(2^l-r)} \end{aligned}$$

for  $l \in \mathbb{N}$  and for  $r$  such that  $0 \leq r \leq 2^l$ . One can then apply (14), (15), (16) (or a similar identity) with  $S = (2^l + r)$ ,  $s = s(r)$  in order to get Riemann sums for  $\int_{a/b}^{c/d} \square(t)^n dt$ .

*Proof of Lemma 6.4.* An induction on  $h$  establishes the formula for  $l = 0$  (and  $r \in \{0, 1\}$ ).

An induction on  $l$  (for constant  $h$ ) ends the proof.  $\square$

*Proof of Proposition 6.3.* We have

$$I_h(n) = \frac{1}{2^h} \lim_{l \rightarrow \infty} \frac{1}{2^l} \sum_{r=0}^{2^l} \square\left(\frac{1}{2^h} + \frac{r}{2^{h+l}}\right)^n.$$

By (29) we have

$$\begin{aligned} I_h(n) &= \frac{1}{2^h} \lim_{l \rightarrow \infty} \frac{1}{2^l} \sum_{r=0}^{2^l} \left( \frac{s(2^l+r)}{(h+1)s(2^l+r) - s(r)} \right)^n \\ &= \lim_{l \rightarrow \infty} \frac{1}{2^{l+h}} \sum_{r=0}^{2^l} \frac{1}{(h+1)^n} \left( \frac{1}{1 - \frac{s(r)}{(h+1)s(2^l+r)}} \right)^n \end{aligned}$$

and (10) implies now

$$\begin{aligned} I_h(n) &= \lim_{l \rightarrow \infty} \frac{1}{2^{l+h}} \sum_{r=0}^{2^l} \frac{1}{(h+1)^n} \sum_{k=0}^{\infty} \binom{k+n-1}{n-1} \left( \frac{s(r)}{(h+1)s(2^l+r)} \right)^k \\ &= \frac{1}{2^h} \sum_{k=0}^{\infty} \binom{k+n-1}{n-1} \frac{m_k}{(h+1)^{k+n}}. \end{aligned}$$

This proves the first equality.

The second equality follows from (12) applied to (30) yielding the identities

$$\begin{aligned}
I_h(n) &= \frac{1}{2^h} \lim_{l \rightarrow \infty} \frac{1}{2^l} \sum_{r=0}^{2^l} \left( \frac{s(2^l + r)}{hs(2^l + r) + s(r)} \right)^n \\
&= \lim_{l \rightarrow \infty} \frac{1}{2^{l+h}} \sum_{r=0}^{2^l} \frac{1}{h^n} \left( \frac{1}{1 + \frac{s(r)}{hs(2^l+r)}} \right)^n \\
&= \frac{1}{2^h} \sum_{k=0}^{\infty} \binom{k+n-1}{n-1} (-1)^k \frac{m_k}{h^{k+n}} .
\end{aligned}$$

□

*Proof of Theorem 6.1.* Follows from  $m_n = \sum_{h=1}^{\infty} I_h(n)$  where  $I_h(n)$  is evaluated using Proposition 6.3. □

## 7 Holomorphicity of $m_x$

**Theorem 7.1.** (i) The map  $n \mapsto m_n$  extends to an entire function  $x \mapsto m_x$ .

(ii) The series expansion of  $x \mapsto m_x$  at  $x = 0$  is given by

$$\sum_{n=0}^{\infty} \frac{x^n}{n!} \sum_{k=n}^{\infty} c_{n,k} m_k \tag{31}$$

where

$$\sum_{k=n}^{\infty} c_{n,k} x^k = (\log(1-x))^n . \tag{32}$$

Equivalently, the numbers  $c_{n,k}$  are given by the equality

$$c_{n,k} = (-1)^k \frac{n!}{k!} s(k, n) \tag{33}$$

where the numbers  $s(k, m)$  defined by  $\sum_{m=0}^k s(k, m) x^m = x(x-1)(x-2) \cdots (x-k+1)$  are Stirling numbers of the first kind.

**Remark 7.2.** The rational numbers  $c_{n,k}$  defined by (32) are given by the recursive formulae  $c_{0,0} = 1$ ,  $c_{0,k} = 0$  if  $k > 0$  and

$$c_{n+1,k} = - \sum_{j=1}^{k-1} \frac{c_{n,k-j}}{j} , n > 0 .$$

They are also defined by the equality

$$c_{n,k} = (-1)^n \sum_{a_1, \dots, a_n \geq 1, a_1 + \dots + a_n = k} \frac{1}{a_1 \cdot a_2 \cdots a_n} .$$

*Proof of Theorem 7.1.* Extending formula (26) by considering

$$I_h(x) = \int_{2^{-h}}^{2^{-h+1}} e^{x \log(\square(t))} dt$$

for arbitrary  $x \in \mathbb{C}$  (where  $\log(\square(t)) \in \mathbb{R}$  denotes the usual logarithm of the strictly positive real number  $\square(t)$ ), the inequalities

$$\frac{1}{h+1} = \square(2^{-h}) \leq \square(t) \leq \square(2^{-h+1}) = \frac{1}{h}, \quad t \in [2^{-h}, 2^{-h+1}]$$

show

$$|I_h(x)| \leq \frac{1}{2^h} \max_{t \in [\frac{1}{1+h}, \frac{1}{h}]} |t^x| \leq \frac{(1+h)^{|x|}}{2^h}.$$

This implies

$$\left| \sum_{h=1}^{\infty} I_h(x) \right| \leq \sum_{h=1}^{\infty} \frac{(1+h)^{|x|}}{2^h} < \infty.$$

The map  $x \mapsto m_x = \sum_{h=1}^{\infty} I_h(x)$  defines thus an entire function which coincides with  $m_x$  for  $x \in \mathbb{N}$ .

Using the symmetry  $\square(x) = 1 - \square(x)$  we have

$$m_x = \lim_{l \rightarrow \infty} \frac{1}{2^l} \sum_{k=1}^{2^l} \left( \frac{s(k)}{s(2^l + k)} \right)^x = \lim_{l \rightarrow \infty} \frac{1}{2^l} \sum_{k=0}^{2^l-1} \left( 1 - \frac{s(k)}{s(2^l + k)} \right)^x.$$

The  $n$ -th derivative of  $m_x$  at  $x = 0$  evaluates thus to

$$\begin{aligned} & \lim_{l \rightarrow \infty} \frac{1}{2^l} \sum_{k=0}^{2^l-1} \left( \log \left( 1 - \frac{s(k)}{s(2^l + k)} \right) \right)^n \\ &= \lim_{l \rightarrow \infty} \frac{1}{2^l} \sum_{k=0}^{2^l-1} \left( - \sum_{j=1}^{\infty} \frac{1}{j} \left( \frac{s(k)}{s(2^l + k)} \right)^j \right)^n \end{aligned}$$

which proves formula (31). □

**Remark 7.3.** *Holomorphicity of  $x \mapsto m_x$  can also be proved using Proposition 3.3.*

## 8 Two conjectural relations

The derivative of the holomorphic function  $x \mapsto m_x$  (see Theorem 7.1) is given by

$$- \sum_{n=1}^{\infty} \frac{m_n}{n} \sim -0.7924251285954891181912115152998913988894127820438$$

at the origin  $x = 0$ . It coincides experimentally with the number

$$-2 \left( \log 2 - \sum_{n=1}^{\infty} \frac{m_n}{n2^n} \right)$$

leading to the following conjectural identity.

**Conjecture 8.1.** *We have*

$$\log 2 = \sum_{n=1}^{\infty} \frac{m_n}{2n} \left( 1 + \frac{1}{2^{n-1}} \right). \quad (34)$$

A variation is given by

**Conjecture 8.2.**

$$\log 2 = \sum_{n=1}^{\infty} \frac{m_n}{n} \left( \frac{1}{2^n} - (-1)^n \right). \quad (35)$$

## 9 A third set of formulae

In this section we consider the partition

$$[0, 1] \setminus \{1\} = \left[0, \frac{1}{2}\right] \cup \left[\frac{1}{2}, \frac{3}{4}\right] \cup \left[\frac{3}{4}, \frac{7}{8}\right] \cup \left[\frac{7}{8}, \frac{15}{16}\right] \cup \dots$$

The resulting identities, well suited for computing asymptotics, are given by the following result:

**Theorem 9.1.**

$$\begin{aligned} m_n &= \frac{1}{2} \sum_{j=0}^{\infty} \binom{n+j-1}{j} (-1)^j m_{n+j} \\ &+ \sum_{h=2}^{\infty} \frac{1}{2^h} \left( \frac{h-1}{h} \right)^n \sum_{k=0}^n \binom{n}{k} \left( \frac{1}{h(h-1)} \right)^k \sum_{j=0}^{\infty} \binom{k-1+j}{j} (-1)^j \frac{m_{k+j}}{h^j} \end{aligned}$$

and

$$m_n = \sum_{h=1}^{\infty} \frac{1}{2^h} \left( \frac{h}{h+1} \right)^n \sum_{k=0}^n \binom{n}{k} \left( \frac{-1}{h(h+1)} \right)^k \sum_{j=0}^{\infty} \binom{k-1+j}{j} \frac{m_{k+j}}{(h+1)^j}.$$

**Remark 9.2.** *Only terms of order  $h \sim \sqrt{n/\log 2} + O(n^{1/4})$  yield large contributions to the first sum of the formulae in Theorem 9.1. Corresponding terms of the second sum (over  $k$ ) for such contributions decay exponentially fast. Terms of the third sum (over  $j$ ) decay also exponentially fast for fixed  $h > 1$  and for  $k$  small.*

We set

$$J_h(n) = \int_{1-2^{-h+1}}^{1-2^{-h}} \square(t)^n dt . \quad (36)$$

**Proposition 9.3.** *We have for all  $h \in \mathbb{N}$ ,  $h \geq 1$  the identities*

$$J_h(n) = \frac{1}{2^h} \left( \frac{h-1}{h} \right)^n \sum_{k=0}^n \binom{n}{k} \left( \frac{1}{h(h-1)} \right)^k \sum_{j=0}^{\infty} \binom{k-1+j}{j} (-1)^j \frac{m_{k+j}}{h^j} \quad (37)$$

and

$$J_h(n) = \frac{1}{2^h} \left( \frac{h}{h+1} \right)^n \sum_{k=0}^n \binom{n}{k} \left( \frac{-1}{h(h+1)} \right)^k \sum_{j=0}^{\infty} \binom{k-1+j}{j} \frac{m_{k+j}}{(h+1)^j} . \quad (38)$$

Observe that (37) boils down to

$$J_1(n) = \frac{1}{2} \sum_{j=0}^{\infty} \binom{n+j-1}{j} (-1)^j m_{n+j} \quad (39)$$

for  $h = 1$ .

*Proof of Proposition 9.3.* Identity (11) of Proposition 4.4 implies

$$J_h(n) = \int_{2^{-h}}^{2^{-h+1}} (1 - \square(t))^n dt .$$

Using Formula (30) of Lemma 6.4 we get

$$\begin{aligned} J_h(n) &= \frac{1}{2^h} \lim_{l \rightarrow \infty} \frac{1}{2^l} \sum_{r=0}^{2^l} \left( 1 - \frac{s(2^l+r)}{hs(2^l+r) + s(2^l-r)} \right)^n \\ &= \frac{1}{2^h} \lim_{l \rightarrow \infty} \frac{1}{2^l} \sum_{r=0}^{2^l} \left( 1 - \frac{s(2^l+r)}{hs(2^l+r) + s(r)} \right)^n \\ &= \lim_{l \rightarrow \infty} \frac{1}{2^{h+l}} \sum_{r=0}^{2^l} \left( \frac{h-1}{h} + \frac{1}{h^2} \left( \frac{s(r)}{s(2^l+r) + \frac{s(r)}{h}} \right) \right)^n \\ &= \lim_{l \rightarrow \infty} \frac{1}{2^{h+l}} \left( \frac{h-1}{h} \right)^n \sum_{r=0}^{2^l} \sum_{k=0}^n \binom{n}{k} \frac{1}{h^k (h-1)^k} \left( \frac{s(r)}{s(2^l+r) + \frac{1}{h}s(r)} \right)^k . \end{aligned}$$

Using the identity

$$\begin{aligned} \left( \frac{s(r)}{s(2^l + r) + \frac{s(r)}{h}} \right)^k &= \left( \frac{s(r)}{s(2^l + r)} \right)^k \left( \frac{1}{1 + \frac{1}{h} \frac{s(r)}{s(2^l + r)}} \right)^k \\ &= \sum_{j=0}^{\infty} \binom{k-1+j}{j} \frac{(-1)^j}{h^j} \left( \frac{s(r)}{s(2^l + r)} \right)^{k+j} \end{aligned}$$

obtained by applying formula (10), we get the first equation.

Starting with

$$\begin{aligned} J_h(n) &= \frac{1}{2^h} \lim_{l \rightarrow \infty} \frac{1}{2^l} \sum_{r=0}^{2^l} \left( 1 - \frac{s(2^l + r)}{(h+1)s(2^l + r) - s(r)} \right)^n \\ &= \lim_{l \rightarrow \infty} \frac{1}{2^{h+l}} \sum_{r=0}^{2^l} \left( \frac{h}{h+1} - \frac{1}{(h+1)^2} \left( \frac{s(r)}{s(2^l + r) - \frac{s(r)}{h+1}} \right) \right)^n \\ &= \lim_{l \rightarrow \infty} \frac{1}{2^{h+l}} \left( \frac{h}{h+1} \right)^n \sum_{r=0}^{2^l} \sum_{k=0}^n \binom{n}{k} \frac{(-1)^k}{h^k (h+1)^k} \left( \frac{s(r)}{s(2^l + r) - \frac{s(r)}{h+1}} \right)^k . \end{aligned}$$

and finishing as above yields the second identity.  $\square$

*Proof of Theorem 9.1.* Follows from Proposition 9.3 applied to the obvious identity  $m_n = \sum_{h=1}^{\infty} J_h(n)$ .  $\square$

## 10 Asymptotics

We set

$$\lambda = \sum_{n=0}^{\infty} \frac{(\log 2)^n}{n!} m_n . \quad (40)$$

Numerically,  $\lambda$  is approximately equal to

1.42815984554560290424313465212729430726822547802532544939052972 .

**Theorem 10.1.** *For every strictly positive  $\epsilon$  there exists a natural integer  $N$  such that*

$$\left| m_n - \lambda \sum_{h=2}^{\infty} \frac{1}{2^h} \left( \frac{h-1}{h} \right)^n \right| \leq \epsilon m_n$$

if  $n \geq N$ .

The error given by the asymptotic approximation

$$m_n \sim \lambda \sum_{h=2}^{\infty} \frac{1}{2^h} \left(1 - \frac{1}{h}\right)^n \quad (41)$$

in Theorem 10.1 is surprisingly small, see Section 10.3.

**Corollary 10.2.** *We have*

$$m_n \sim \lambda \frac{n^{1/4}}{(\log 2)^{3/4}} \sqrt{\frac{\pi}{2}} e^{-2\sqrt{n \log 2}} \quad (42)$$

for  $n \rightarrow \infty$ .

Corollary 10.2 is of course equivalent to Theorem 1 in [3]. The constant  $\lambda$  defined by (40) is related to the constant

$$c_0 = \int_0^1 2^t \left(1 - \frac{1}{2} t\right) dt = \frac{1}{\log 2} - \frac{1}{2} \int_0^1 2^{t?}(t) dt$$

in Theorem 1 of [3] by

$$\lambda = c_0 2 \log 2$$

and satisfies the following additional identities:

**Proposition 10.3.** *We have*

$$\lambda = 2 \sum_{n=0}^{\infty} \frac{(-\log 2)^n}{n!} m_n = \frac{4}{3} \sum_{n=0}^{\infty} \frac{(\log 2)^{2n}}{(2n)!} m_{2n} = 4 \sum_{n=0}^{\infty} \frac{(\log 2)^{2n+1}}{(2n+1)!} m_{2n+1} .$$

Observe that the constant  $\lambda$  appears also in the asymptotic expression  $\lambda \frac{n!}{(\log 2)^{n+1}}$  for  $m_{-n}$ , see [2] or Proposition 11.8.

**Remark 10.4.** *A computation of  $\lambda$  with high precision needs only relatively few initial values of  $m_0, m_1, m_2, \dots$ . I ignore however a direct approach for accurately computing only the first few values of  $m_2, m_3, m_4, \dots$ .*

**Proposition 10.5.** *We have*

$$\lim_{n \rightarrow \infty} \left( \frac{n^{1/4}}{(\log 2)^{3/4}} \sqrt{\frac{\pi}{2}} e^{-2\sqrt{n \log 2}} \right)^{-1} \left( \sum_{h=2}^{\infty} \frac{1}{2^h} \left(1 - \frac{1}{h}\right)^n \right) = 1 .$$

*Proof of Proposition 10.5.* We apply Laplace's method to  $\sum_{h=2}^{\infty} \frac{1}{2^h} \left(1 - \frac{1}{h}\right)^n$ .

The derivative

$$f'_n(x) = \frac{1}{2^x} \left(1 - \frac{1}{x}\right)^n \frac{(n + x(1-x) \log 2)}{x(x-1)}$$

of the function  $f_n(x) = \frac{1}{2^x} \left(1 - \frac{1}{x}\right)^n$  has roots given by the solutions of  $x^2 - x = \frac{n}{\log 2}$ .

Assuming  $x$  real and positive, the positive root of  $f'_n$  is given by

$$\rho = \frac{1 + \sqrt{1 + 4n/\log 2}}{2} = \sqrt{\frac{n}{\log 2}} + \frac{1}{2} + \frac{1}{8}\sqrt{\frac{\log 2}{n}} + O\left(\frac{1}{\sqrt{n^3}}\right)$$

and we have

$$f_n(\rho) = \frac{1}{\sqrt{2}}e^{-2\sqrt{n\log 2}} \left(1 + O\left(\frac{1}{\sqrt{n}}\right)\right).$$

A straightforward computation shows

$$f''_n(\rho) = \frac{1}{\sqrt{2}}e^{-2\sqrt{n\log 2}} \left(-2\frac{\sqrt{\log 2^3}}{\sqrt{n}} + O\left(\frac{1}{n}\right)\right). \quad (43)$$

Applying Laplace's method

$$\begin{aligned} \int_2^\infty f_n(h)dh &\sim f_n(\rho) \int_{-\infty}^\infty e^{-\frac{f''_n(\rho)}{2}t^2} dt \\ &= \sqrt{\frac{2\pi f_n(\rho)^3}{-f''_n(\rho)}} \end{aligned}$$

to the integral approximation  $\int_2^\infty f_n(h)dh$  of  $\sum_{h=2}^\infty f_n(h)$  we get the result.  $\square$

**Proposition 10.6.** *For every  $\epsilon > 0$  there exists a natural integer  $A$  such that*

$$0 \leq m_n - \sum_{h \in [\lfloor \sqrt{n/\log 2} - An^{1/4} \rfloor, \lfloor \sqrt{n/\log 2} + An^{1/4} \rfloor]} J_h(n) < \epsilon m_n$$

for all  $n$  large enough with  $J_h(n) = \int_{1-2^{-h+1}}^{1-2^{-h}} \square(t)^n dt$  given by (36).

*Proof.* The easy evaluation  $\square\left(1 - \frac{1}{2^h}\right) = 1 - \frac{1}{h+1}$  for  $h \in \mathbb{N}$  shows

$$1 - \frac{1}{h} \leq \square(t) \leq 1 - \frac{1}{h+1}$$

for  $t \in \left[1 - \frac{1}{2^{h-1}}, 1 - \frac{1}{2^h}\right]$  and we have

$$\frac{1}{2^h} \left(1 - \frac{1}{h}\right)^x \leq J_h(x) \leq \frac{1}{2^h} \left(1 - \frac{1}{h+1}\right)^x$$

for real positive  $x$ . Since the unique positive root of the logarithmic derivative

$$\frac{df/dh}{f} = -\log 2 + \frac{x}{h(h-1)}$$



with respect to  $h$  of  $f = \frac{1}{2^n} \left(1 - \frac{1}{h}\right)^x$  is given by  $h \sim \sqrt{x/\log 2}$  for large  $x$ , the decay of the function

$$s \longrightarrow \frac{1}{2\sqrt{x/\log 2 + sx^{1/4}}} \left(1 - \frac{1}{\sqrt{x/\log 2 + sx^{1/4}}}\right)^x$$

is exponentially fast in  $|s|$  for large  $x$ . This implies the result.  $\square$

*Proof of Theorem 10.1.* Setting

$$\tilde{J}_h(n) = 2^h \left(\frac{h}{h-1}\right)^n J_h(n)$$

formula (37) of Proposition 9.3 shows the identities

$$\begin{aligned} \tilde{J}_h(n) &= \sum_{k=0}^n \binom{n}{k} \frac{1}{h^k (h-1)^k} \sum_{j=0}^{\infty} \binom{k-1+j}{j} (-1)^j \frac{m_{k+j}}{h^j} \\ &= \sum_{k=0}^n \frac{(\log 2)^k}{k!} \frac{\prod_{j=1}^{k-1} \left(1 - \frac{j}{n}\right)}{\left(\frac{h(h-1)}{n} \log 2\right)^k} \sum_{j=0}^{\infty} \binom{k-1+j}{j} (-1)^j \frac{m_{k+j}}{h^j}. \end{aligned}$$

For  $k$  fixed and for  $h = \sqrt{n/2 \log 2} + O(n^{1/4})$  we have

$$\lim_{n \rightarrow \infty} \frac{\prod_{j=1}^{k-1} \left(1 - \frac{j}{n}\right)}{\left(\frac{h(h-1)}{n} \log 2\right)^k} = 1$$

and we get the asymptotics

$$\tilde{J}_h(n) \sim \sum_{k=0}^{\infty} \frac{(\log 2)^k}{k!} m_k = \lambda$$

for  $h = \sqrt{n/\log 2} + O(n^{1/4})$ .

Proposition 10.6 shows now

$$\begin{aligned} m_n &\sim_{\epsilon} \sum_{h=\lfloor \sqrt{n/\log 2} - An^{1/4} \rfloor}^{\lfloor \sqrt{n/\log 2} + An^{1/4} \rfloor} J_h(n) \\ &\sim_{\epsilon} \sum_{h=\lfloor \sqrt{n/\log 2} - An^{1/4} \rfloor}^{\lfloor \sqrt{n/\log 2} + An^{1/4} \rfloor} \frac{1}{2^h} \left(\frac{h-1}{h}\right)^n \tilde{J}_h(n) \\ &\sim_{\epsilon} \lambda \sum_{h=2}^{\infty} \frac{1}{2^h} \left(\frac{h-1}{h}\right)^n \end{aligned}$$

for  $n \rightarrow \infty$  and fixed  $A$  (depending on  $\epsilon$ ) with  $a \sim_{\epsilon} b$  denoting  $|a - b| < \epsilon a$  for arbitrary small  $\epsilon$  if  $n$  is large enough.  $\square$

*Proof of Proposition 10.3.* Working with formula (38) we get the asymptotics

$$\begin{aligned} m_n &\sim \sum_{h=1}^{\infty} \frac{1}{2^h} \left( \frac{h}{h+1} \right)^n \sum_{k=0}^{\infty} \frac{(-\log 2)^k}{k!} m_k \\ &= 2 \sum_{k=2}^{\infty} \frac{1}{2^k} \left( \frac{k-1}{k} \right)^n \sum_{k=0}^{\infty} \frac{(-\log 2)^k}{k!} m_k \end{aligned}$$

which imply the first equality by comparing with Theorem 10.1. The two other identities are easy consequences.  $\square$

*Proof of Corollary 10.2.* Follows from Theorem 10.1 and Proposition 10.5.  $\square$

### 10.1 Asymptotic formula for $\phi_n$

Using similar techniques, we get the asymptotic approximation

$$\phi_n \sim 2\lambda \sum_{h=3}^{\infty} \frac{1}{2^h} \left( 1 - \frac{2}{h} \right)^n \quad (44)$$

(where  $\lambda$  is given by (40)) for  $\phi_n = 2 \int_{1/2}^1 (2\Box(t) - 1)^n dt$ , see Formula (19) in Lemma 5.4. The relative error seems again to be of order  $O(\phi_n^{5/4})$  and has again (suitably normalized) a more or less periodic behaviour as a function of  $\sqrt{n}$ .

Using Laplace's method for the right side of (44) we get the simpler and less accurate expression

$$\phi_n \sim \lambda \frac{(2n)^{1/4} \sqrt{\pi}}{\log(2)^{3/4}} e^{-2\sqrt{2n \log 2}}. \quad (45)$$

### 10.2 A second asymptotic formula

The motivation for this section is the estimation of the order of the error in the asymptotic approximation (41).

A refinement of the Riemann sum underlying Formula (41) should yield a slightly more accurate approximation for  $m_n$ . The order of the difference between the two formulae should be a measure for the accuracy of (41).

We subdivide the interval underlying the integral  $J_h(n)$  defined by (36) into two intervals of equal lengths. We have  $J_h(n) = A_h(n) + B_h(n)$  where

$$A_h(n) = \int_{1-2^{-h+1}}^{1-3 \cdot 2^{-h-1}} \Box(t)^n dt \text{ and } B_h(n) = \int_{1-3 \cdot 2^{-h-1}}^{1-2^{-h}} \Box(t)^n dt .$$

We have

$$\begin{aligned}
A_h(n) &= \frac{1}{2^{h+1}} \lim_{l \rightarrow \infty} \frac{1}{2^l} \sum_{r=0}^{2^l} \left( 1 - \frac{s(2^l + r) + s(r)}{hs(2^l + r) + (h+1)s(r)} \right)^n \\
&= \frac{1}{2^{h+1}} \lim_{l \rightarrow \infty} \frac{1}{2^l} \sum_{r=0}^{2^l} \left( \frac{2h-1}{2h+1} - \frac{s(r)}{(2h+1)^2(s(2^l + r) - \frac{h+1}{2h+1}s(r))} \right)^n \\
&= \frac{1}{2^{h+1}} \left( \frac{2h-1}{2h+1} \right)^n \sum_{k=0}^n \binom{n}{k} \frac{(-1)^k}{(4h^2-1)^k} \lim_{l \rightarrow \infty} \frac{2^k}{2^l} \sum_{r=0}^{2^l} \left( \frac{s(r)}{2s(2^l + r) - \frac{2h+2}{2h+1}s(r)} \right)^k
\end{aligned}$$

which yields

$$\lim_{h \rightarrow \infty} A_h(n) = \frac{1}{2^{h+1}} \left( \frac{2h-1}{2h+1} \right)^n \sum_{k=0}^n \binom{n}{k} \frac{(-1)^k}{(4h^2-1)^k} 2^k \phi_k$$

by Identity (20).

For  $h = \sqrt{n/\log 2} + O(n^{1/4})$  we have thus

$$A_h(n) \sim \frac{1}{2^{h+1}} \left( \frac{2h-1}{2h+1} \right)^n \sum_{k=0}^{\infty} \frac{(-\log 2)^k}{2^k k!} \phi_k .$$

A similar calculation shows

$$B_h(n) \sim \frac{1}{2^{h+1}} \left( \frac{2h-1}{2h+1} \right)^n \sum_{k=0}^{\infty} \frac{(\log 2)^k}{2^k k!} \phi_k$$

for  $h = \sqrt{n/\log 2} + O(n^{1/4})$ .

We get thus for large  $n$  and  $h = \sqrt{n/\log 2} + O(n^{1/4})$  the approximation

$$J_h(n) \sim \frac{1}{2^h} \sum_{k=0}^{\infty} \frac{(\log 2)^{2k}}{2^{2k} (2k)!} \phi_{2k} .$$

Setting

$$\rho = \sum_{k=0}^{\infty} \frac{(\log 2)^{2k}}{2^{2k} (2k)!} \phi_{2k} \tag{46}$$

we have asymptotically

$$m_n \sim \rho \sum_{h=1}^{\infty} \frac{1}{2^h} \left( 1 - \frac{2}{2h+1} \right)^n .$$

Using Laplace's method we get the asymptotic approximation

$$\sum_{h=1}^{\infty} \frac{1}{2^h} \left( 1 - \frac{2}{2h+1} \right)^n \sim \frac{n^{1/4} \sqrt{\pi}}{(\log 2)^{3/4}} e^{-2\sqrt{n \log 2}} .$$

This shows

$$m_n \sim \rho \frac{n^{1/4} \sqrt{\pi}}{(\log 2)^{3/4}} e^{-2\sqrt{n \log 2}} \quad (47)$$

and implies the identity

$$\rho = \frac{\lambda}{\sqrt{2}} \quad (48)$$

as can be seen by comparing the two asymptotic approximations (42) and (47) of  $m_n$ .

The asymptotic formula

$$m_n \sim \lambda \sum_{h=1}^{\infty} \frac{1}{2^{h+1/2}} \left(1 - \frac{1}{h+1/2}\right)^n \quad (49)$$

should thus be slightly better than (41), see Figure 1 in Section 10.3.

### 10.3 An estimation for the error of the asymptotic formulae

Setting

$$x \mapsto S_x(n) = \sum_{h=1}^{\infty} \frac{1}{2^{h+x}} \left(1 - \frac{1}{h+x}\right)^n, \quad (50)$$

the asymptotic formulae (41) and (49) can be rewritten as  $m_n \sim \lambda S_0(n)$  and  $m_n \sim \lambda S_{1/2}(n)$ . Since  $x \mapsto S_x(n)$  is almost 1-periodic (for small positive  $x$  and huge fixed  $n$ ) and oscillates experimentally around the exact value of the integral

$$S_f(n) = \int_1^{\infty} \frac{1}{2^t} \left(1 - \frac{1}{t}\right)^n dt, \quad (51)$$

it is tempting to rescale the errors  $m_n - S_x(n)$  by the inverse of the factor

$$\kappa(n) = \sqrt{\left(S_0(n) - S_f(n)\right)^2 + \left(S_{1/4}(n) - S_f(n)\right)^2} \quad (52)$$

given by the ‘‘amplitude’’ of the almost 1-periodic function  $x \mapsto S_x(n) - S_f(n)$ .

The sequence  $S_f(n)$  of integrals is easy to compute recursively: We have

the initial values

$$\begin{aligned}
S_f(0) &= \int_1^\infty \frac{dt}{2^t} = \frac{1}{2 \log 2} \\
S_f(1) &= \int_1^\infty \frac{1}{2^t} \left(1 - \frac{1}{t}\right) dt \\
&= \frac{1}{2 \log 2} + \int_{-\infty}^{-\log 2} \frac{e^t}{t} dt? \\
&= \frac{1}{2 \log 2} - \int_{\log 2}^\infty \frac{e^{-t}}{t} dt? \\
&= \frac{1}{2 \log 2} + \text{Ei}(-\log 2)
\end{aligned}$$

(where  $\text{Ei}(x) = \int_{-\infty}^x \frac{e^t}{t} dt = -\int_{-x}^\infty \frac{e^{-t}}{t} dt$  is the exponential integral) and integration by parts yields the recursion relation

$$S_f(n) = \left(2 + \frac{\log 2}{n-1}\right) S_f(n-1) - S_f(n-2). \quad (53)$$

The normalized errors

$$E_0(n) = \frac{1}{\kappa(n)} \left( m_n - \lambda \sum_{h=2}^\infty \frac{1}{2^h} \left(1 - \frac{1}{h}\right)^n \right), \quad (54)$$

$$E_{1/2}(n) = \frac{1}{\kappa(n)} \left( m_n - \lambda \sum_{h=1}^\infty \frac{1}{2^{h+1/2}} \left(1 - \frac{1}{h+1/2}\right)^n \right), \quad (55)$$

$$E_f(n) = \frac{1}{\kappa(n)} \left( m_n - \lambda \int_1^\infty \frac{1}{2^t} \left(1 - \frac{1}{t}\right)^n dt \right) \quad (56)$$

are depicted in Figure 1 representing the points  $(\sqrt{n}, E_0(n))$ ,  $(\sqrt{n}, E_{1/2}(n))$  and  $(\sqrt{n}, E_f(n))$  for  $n$  in  $\{100, \dots, 400\}$ . Points on the smallest sinusoidal curve are associated to  $E_f$ , points on the sinusoidal curve of intermediate size to  $E_{1/2}$  and points on the largest curve to  $E_0$ . In all three cases the error seems to be close to a damped periodic function of  $\sqrt{n}$  of local amplitude  $O(\kappa(n))$ .

**Remark 10.7.** *The existence of the linear recurrence relation (53) implies the existence of asymptotic recurrence relations (given by the same formula) for the sequences  $m_n$  and  $S_x(n)$ .*

*The asymptotic linear recurrence formula for  $m_n$  can be improved into an affine asymptotic formula using ideas of the next Section.*

**Remark 10.8.** *It would be interesting to understand the asymptotic behaviour of the amplitude  $\kappa(n)$  given by Formula (52). (The number  $\kappa(n)$  is*

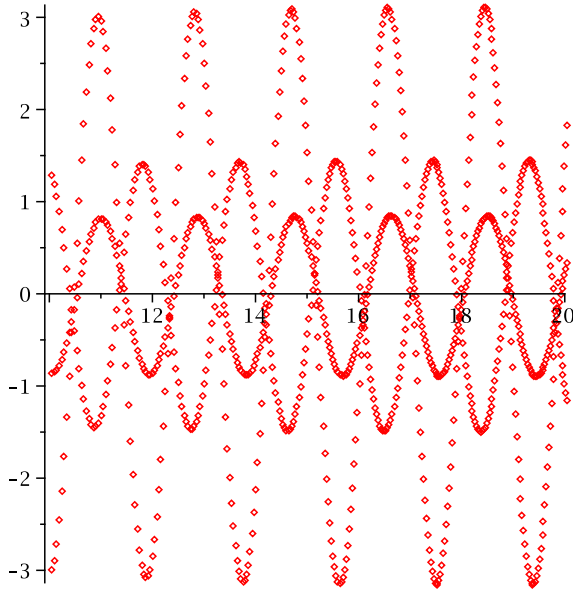


Figure 1: Error of asymptotical formulae

essentially the error term in Euler-MacLaurin's summation formula.) For moderate values of  $n$  it seems to be comparable to

$$\frac{\sqrt{n}}{\log n \log \log n} e^{-9/2\sqrt{n \log 2}}$$

which implies  $\kappa(n) < m_n^{9/4}$ . The accuracy of the asymptotic formulae  $m_n \sim \lambda S_*(n)$  (for  $*$  = 0, 1/2 and  $f$ ) is thus surprisingly high.

#### 10.4 Increasing accuracy

The behaviour of the error-terms  $E_*(n)$  occurring in the previous Section suggests to try an asymptotic formula of the form

$$m_n \sim \lambda S_f(n) + a \left( S_0(n) - S_f(n) \right) + b \left( S_{1/4}(n) - S_f(n) \right) \quad (57)$$

with  $\lambda$  defined by (40) and  $S_*(n)$  as in the previous Section. Experimentally, such a formula seems to exist with

$$\begin{aligned} a &\sim -.521901056340432536774725873446, \\ b &\sim -.148755851763595338634628933193. \end{aligned}$$

The term  $\lambda S_f(n)$  is of course the principal contribution and plays the role of Formula (41) or (49). The two remaining terms  $a \left( S_0(n) - S_f(n) \right)$  and

$b(S_{1/4}(n) - S_f(n))$  sum up to a fairly regular (damped) oscillatory contribution of much lesser size. More precisely, its local amplitude should be asymptotically equal to  $\frac{\lambda\sqrt{a^2+b^2}}{\kappa(n)}$  with  $\kappa(n)$  given by Formula (52).

## 10.5 An improved algorithm

Accurate asymptotic approximations can be used for improving the algorithm given in Section 5.2. Indeed, the cutoff at  $N$  induces large relative errors for the last values of  $\tilde{\phi}_n$ . It is thus natural to compute  $\tilde{\phi}_0, \tilde{\phi}_2, \dots, \tilde{\phi}_{2\lfloor N/2 \rfloor}$  using the first  $N + 1$  values  $\tilde{m}_0, \dots, \tilde{m}_N$  and  $M - N$  additional values  $\tilde{m}_{N+1}, \dots, \tilde{m}_M$  given by asymptotic approximations of  $m_{N+1}, \dots, m_M$  (for  $M > N$  a suitable integer depending on  $N$  and on the accuracy of the chosen approximation).

We illustrate this by modifying the algorithm of Section 5.2 using high-level instructions in order to involve the asymptotic approximation (41) (the approximation (42) is of much lesser interest):

Add the lines

005 Precompute (and store) sufficiently accurate values  $\tilde{S}(n)$  of  $S_0(n)$  (or, slightly better, of  $S_f(n)$ ) for  $n = N + 1, \dots, M$ .

051 Compute  $\tilde{\lambda} := \sum_{n=0}^M \frac{(\log 2)^n}{n!} \tilde{m}_n$ ,

052 Set  $\tilde{m}_n = \lambda \tilde{S}(n)$  for  $n = N + 1, \dots, M$ .

at the obvious locations.

Replace 090 by

090 For  $k = 0, 1, 2, \dots, M - n$  do:

The resulting algorithm can easily be modified in order to work with other asymptotic approximations. The author used mainly (57) (this needs precomputations of approximations for  $S_f(n), S_0(n), S_{1/4}(n)$  with  $n$  in  $\{N + 1, \dots, M\}$ ).

Concerns using an algorithm based on a conjectural formula can be avoided by checking the final data using a single iteration of (the main loop in) the original algorithm (described in Section 5.2) with a sufficiently high value  $N' > N$  (with missing values replaced by their (conjecturally very accurate) approximations). The obtained data are exact up to an absolute error bounded by  $\max(|\epsilon|, m_{N'+1})$  with  $\epsilon$  denoting the maximal modification of  $\tilde{m}_1, \dots, \tilde{m}_{N'}$  during the final checking-run.

The improved version has smaller memory requirement and a much better running time : The (conjectural) accuracy of the used approximation should more than double the number of achievable correct digits for a given value of  $N$ . In order to achieve the same accuracy, the original algorithm has to be run with  $N$  multiplied by more than 4 which multiplies the running time of the main loop by more than  $16 = 4^2$ .

## 11 Values of $m$ at negative integers

**Proposition 11.1.** *The equality*

$$m_{-n} = m_n + \sum_{k=0}^{n-1} \binom{n}{k} (m_{-k} + m_k) \quad (58)$$

*holds for  $n \in \mathbb{N}$  a natural integer.*

**Remark 11.2.** *The generalization*

$$m_z = \frac{1}{2} \sum_{k=0}^{\infty} \binom{-z}{k} (m_{-k} + m_k)$$

*of Proposition 11.1 fails for arbitrary complex values of  $z$ . Indeed, Proposition 11.1 is based on the identity  $(1+x)^z = \sum_{k=0}^{\infty} \binom{z}{k} x^k$  for arbitrary  $x \in \mathbb{R}$  which breaks down if  $-z$  is not in  $\mathbb{N}$ .*

*Proof of Proposition 11.1.* We have for  $n \in \mathbb{N}$

$$m_{-n} = \lim_{l \rightarrow \infty} \frac{1}{2^l} \sum_{r=1}^{2^l} \left( \frac{s(2^l + r)}{s(r)} \right)^n$$

Using (11) we have

$$\begin{aligned} m_{-n} &= \lim_{l \rightarrow \infty} \frac{1}{2^l} \sum_{r=1}^{2^l} \left( \frac{s(r) + s(2^l - r)}{s(r)} \right)^n \\ &= \lim_{l \rightarrow \infty} \frac{1}{2^l} \sum_{r=1}^{2^l} \left( 1 + \frac{s(2^l - r)}{s(r)} \right)^n \\ &= \lim_{l \rightarrow \infty} \frac{1}{2^l} \left( \sum_{r=1}^{2^{l-1}} \left( 1 + \frac{s(2^l - r)}{s(r)} \right)^n + \sum_{r=2^{l-1}+1}^{2^l} \left( 1 + \frac{s(2^l - r)}{s(r)} \right)^n \right) \\ &= \lim_{l \rightarrow \infty} \frac{1}{2^l} \left( \sum_{r=1}^{2^{l-1}} \left( 1 + \frac{s(2^l - r)}{s(r)} \right)^n + \sum_{r=0}^{2^{l-1}-1} \left( 1 + \frac{s(r)}{s(2^l - r)} \right)^n \right) \end{aligned}$$

Using (12) we have thus

$$\begin{aligned} m_{-n} &= \lim_{l \rightarrow \infty} \frac{1}{2^l} \left( \sum_{r=1}^{2^{l-1}} \left( 1 + \frac{s(2^{l-1} + r)}{s(r)} \right)^n + \sum_{r=1}^{2^{l-1}} \left( 1 + \frac{s(r)}{s(2^{l-1} + r)} \right)^n \right) \\ &= \lim_{l \rightarrow \infty} \frac{1}{2^l} \sum_{k=0}^n \binom{n}{k} \sum_{r=1}^{2^{l-1}} \left( \left( \frac{s(2^{l-1} + r)}{s(r)} \right)^k + \left( \frac{s(r)}{s(2^{l-1} + r)} \right)^k \right) \\ &= \frac{1}{2} \sum_{k=0}^n \binom{n}{k} (m_{-k} + m_k) \end{aligned}$$



which implies the result.  $\square$

### 11.1 Matrices relating $m_{-\mathbb{N}}$ and $m_{\mathbb{N}}$

Identity (58) of Proposition 11.1 implies the existence of infinite lower diagonal triangular unipotent matrices  $A, B = A^{-1}$  with integral coefficients such that

$$\begin{pmatrix} m_0 \\ m_{-1} \\ m_{-2} \\ \vdots \end{pmatrix} = A \begin{pmatrix} m_0 \\ m_1 \\ m_2 \\ \vdots \end{pmatrix} \text{ and } \begin{pmatrix} m_0 \\ m_1 \\ m_2 \\ \vdots \end{pmatrix} = B \begin{pmatrix} m_0 \\ m_{-1} \\ m_{-2} \\ \vdots \end{pmatrix}.$$

The first few rows and columns of the matrices  $A, B = A^{-1}$  are

$$\begin{pmatrix} 1 & & & & & \\ 2 & 1 & & & & \\ 6 & 4 & 1 & & & \\ 26 & 18 & 6 & 1 & & \\ 150 & 104 & 36 & 8 & 1 & \end{pmatrix}, \begin{pmatrix} 1 & & & & & \\ -2 & 1 & & & & \\ 2 & -4 & 1 & & & \\ -2 & 6 & -6 & 1 & & \\ 2 & -8 & 12 & -8 & 1 & \end{pmatrix}$$

and their coefficients are described by the following result.

**Proposition 11.3.** *Let  $\sigma_n, n \in \mathbb{Z}$  be a sequence (with values in a commutative ring containing 1) indexed by the set  $\mathbb{Z}$  of all integers such that*

$$\sigma_{-n} - \sigma_n = \sum_{k=0}^{n-1} \binom{n}{k} (\sigma_{-k} + \sigma_k).$$

Then

$$\begin{aligned} \sigma_{-i} &= \sum_{j=0}^i \alpha_{i,j} \sigma_j \\ \sigma_i &= \sum_{j=0}^i \beta_{i,j} \sigma_{-j} \end{aligned}$$

for all  $i \in \mathbb{N}$  where  $\alpha_{i,j}, \beta_{i,j}, 0 \leq i, j$  are integers given by the formulae

$$\alpha_{i,j} = \binom{i}{j} \sum_{h=1}^{\infty} \frac{h^{i-j}}{2^h} \tag{59}$$

and

$$\beta_{i,j} = \begin{cases} 1 & \text{if } i = j \\ 2(-1)^{i+j} \binom{i}{j} & \text{otherwise.} \end{cases}$$

In particular, the matrices  $A$  and  $B$  with coefficients  $\alpha_{i,j}, \beta_{i,j}, 0 \leq i, j$  are mutually inverse lower triangular unipotent integral matrices.

*Proof.* We have  $\alpha_{i,i} = 1$  as required and the matrix  $A$  is clearly lower triangular. The proof is now by induction on the row-index  $i$  of the coefficients  $\alpha_{i,j}$  for  $A$ . Equation (58) of Proposition 11.1 shows that we have

$$\alpha_{i+1,j} = \sum_{l=0}^i \binom{i+1}{l} (\alpha_{l,j} + \delta_{l,j})$$

for  $i+1 \geq j$ , where  $\delta_{l,j} = 1$  if  $l = j$  and  $\delta_{l,j} = 0$  otherwise.

We get

$$\begin{aligned} \alpha_{i+1,j} &= \binom{i+1}{j} - \binom{i+1}{i+1} \binom{i+1}{j} \sum_{h=1}^{\infty} \frac{h^{i+1-j}}{2^h} \\ &\quad + \sum_{h=1}^{\infty} \sum_{k=0}^{i+1} \binom{i+1}{k} \binom{k}{j} \frac{h^{k-j}}{2^h} \\ &= \binom{i+1}{j} - \binom{i+1}{j} \sum_{h=1}^{\infty} \frac{h^{i+1-j}}{2^h} + \sum_{h=1}^{\infty} \frac{\binom{i+1}{j}^{i+1}}{2^h} \sum_{k=0}^{i+1} \binom{i+1-j}{k-j} h^{k-j} \\ &= \binom{i+1}{j} \left( 1 - \sum_{h=1}^{\infty} \frac{h^{i+1-j}}{2^h} + \sum_{h=1}^{\infty} \frac{(h+1)^{i+1-j}}{2^h} \right) \\ &= \binom{i+1}{j} \sum_{h=1}^{\infty} \frac{h^{i+1-j}}{2^h}. \end{aligned}$$

This implies the formula for the coefficients of  $A$ .

We prove the formula for the coefficients of the inverse matrix  $B = A^{-1}$  by computing the product  $AB$ . We have

$$\begin{aligned} \sum_{k=j}^i \alpha_{i,k} \beta_{k,j} &= 2 \sum_{k=j}^i \alpha_{i,k} (-1)^{k+j} \binom{k}{j} - \alpha_{i,j} \\ &= 2 \sum_{h=1}^{\infty} \sum_{k=j}^i \frac{h^{i-k}}{2^h} \binom{i}{k} \binom{k}{j} (-1)^{k+j} - \alpha_{i,j} \\ &= \sum_{h=1}^{\infty} \frac{(-1)^j h^i}{2^{h-1}} \sum_{k=j}^i \binom{i}{k} \binom{k}{j} \frac{1}{(-h)^k} - \alpha_{i,j}. \end{aligned}$$

Identity 4.3 of Lemma 4.3 shows that this simplifies to

$$\sum_{h=1}^{\infty} \frac{(h-1)^{i-j}}{2^{h-1}} \binom{i}{j} - \alpha_{i,j}.$$

This equals 1 if  $i = j$  and 0 for  $i > j$  by definition of  $\alpha_{i,j}$ .  $\square$

The sum appearing in (59) defines natural integers having a recursive definition:

**Proposition 11.4.** *The natural integers*

$$\gamma_n = \sum_{h=1}^{\infty} \frac{h^n}{2^h}$$

(appearing in (59)) have the recursive definition  $\gamma_0 = 1$  and

$$\gamma_n = 1 + \sum_{j=0}^{n-1} \binom{n}{j} \gamma_j$$

for  $n \geq 1$ .

The sequence of integers  $\gamma_0, \gamma_1, \dots$  starts as

$$1, 2, 6, 26, 150, 1082, 9366, 94586, 1091670, \dots ,$$

see sequence A629 of [5].

*Proof of Proposition 11.4.* We have

$$\begin{aligned} \gamma_n &= \frac{1}{2} + \frac{1}{2} \sum_{h=1}^{\infty} \frac{(h+1)^n}{2^h} \\ &= \frac{1}{2} + \frac{1}{2} \sum_{j=0}^n \binom{n}{j} \sum_{h=1}^{\infty} \frac{h^j}{2^h} \\ &= \frac{1}{2} + \frac{1}{2} \sum_{j=0}^n \binom{n}{j} \gamma_j \end{aligned}$$

which implies the result.  $\square$

**Remark 11.5.** *Lower triangular matrices with lower triangular coefficients  $\gamma_{i,j}$  of the form  $\binom{i}{j} c_{i-j}$  for some sequence  $c_0, c_1, \dots$  form a commutative algebra. Indeed, the map associating to such a matrix with coefficients  $\binom{i}{j} c_{i-j}$  the formal exponential power series  $\sum_{n=0}^{\infty} c_n \frac{x^n}{n!}$  defines an isomorphism of algebras onto the algebra of formal exponential power series (with product given by the obvious “bilinear” extension of  $\frac{x^i}{i!} \frac{x^j}{j!} = \binom{i+j}{i} \frac{x^{i+j}}{(i+j)!}$ ). The easy equality  $\sum_{n=0}^{\infty} \beta_{n,0} \frac{x^n}{n!} = 1 + 2 \sum_{n=1}^{\infty} (-1)^n \frac{x^n}{n!} = 2e^{-x} - 1$  shows thus the identity*

$$\sum_{n=0}^{\infty} \gamma_n \frac{x^n}{n!} = \sum_{n=0}^{\infty} a_{n,0} \frac{x^n}{n!} = \frac{1}{2e^{-x} - 1} .$$

Proposition 11.1 and Proposition 11.3 imply the following result:

**Corollary 11.6.** *We have*

$$m_{-n} = \sum_{k=0}^n \binom{n}{k} \gamma_{n-k} m_k \quad (60)$$

(with  $\gamma_n$  defined by Proposition 11.4) and

$$m_n = m_{-n} + 2 \sum_{k=0}^{n-1} (-1)^{n+k} \binom{n}{k} m_{-k} \quad (61)$$

for all  $n$  in  $\mathbb{N}$ .

Corollary 11.6 is better suited than Proposition 3.3 for computing values  $m_{-\mathbb{N}}$  using  $m_{\mathbb{N}}$ . It involves only finitely many terms of  $m_{\mathbb{N}}$  with coefficients which are decreasing. (The main contribution to  $m_{-n}$  given by the formula of Proposition 3.3 corresponds to summands indexed by integers close to  $\frac{n^2}{\log 2}$ .)

Combining Formula (60) of Corollary 11.6 with Proposition 3.3 we get:

**Corollary 11.7.** *We have for all  $n$  in  $\mathbb{N}$  the identity*

$$\sum_{k=0}^n \binom{n}{k} \gamma_{n-k} m_k = \sum_{j=0}^{\infty} \binom{n+j-1}{j} m_j. \quad (62)$$

Corollary 11.7 yields

$$\begin{aligned} 2m_0 + m_1 &= \sum_{j=0}^{\infty} m_j, \\ 6m_0 + 4m_1 + m_2 &= \sum_{j=0}^{\infty} (j+1)m_j, \\ 26m_0 + 18m_1 + 6m_2 + m_3 &= \sum_{j=0}^{\infty} \binom{j+2}{2} m_j, \\ 150m_0 + 104m_1 + 36m_2 + 8m_3 + m_4 &= \sum_{j=0}^{\infty} \binom{j+3}{3} m_j. \end{aligned}$$

Using the easy evaluation  $m_0 = 1, m_1 = \frac{1}{2}$ , the case  $n = 1$  yields the nice evaluation

$$\sum_{j=0}^{\infty} m_j = \frac{5}{2} \quad (63)$$

which can be used as an accuracy-check for numerical computations.

Similarly, using  $n = 2$ , we get the identity  $m_2 + 8 = \sum_{j=0}^{\infty} (j+1)m_j$ . Subtraction of (63) yields

$$\sum_{j=1}^{\infty} j m_j = m_2 + \frac{11}{2}. \quad (64)$$

## 11.2 Asymptotics for $m_{-n}$

**Proposition 11.8.** *We have*

$$\lim_{n \rightarrow \infty} m_{-n} \frac{(\log 2)^{n-1}}{n!} = \lambda$$

for  $\lambda = \sum_{k=0}^{\infty} \frac{(\log 2)^k}{k!} m_k$  given by (40).

The following easy result is probably well-known:

**Lemma 11.9.** *We have*

$$\lim_{n \rightarrow \infty} \frac{(\log 2)^{n+1}}{n!} \sum_{k=1}^{\infty} \frac{k^n}{2^k} = 1 .$$

*Proof of Lemma 11.9.* We apply Laplace's method to  $\int_1^{\infty} x^n 2^{-x} dx \sim \sum_{k=1}^{\infty} k^n 2^{-k}$ .

The derivative

$$\frac{d}{dx} \left( x^n e^{-x \log 2} \right) = \left( \frac{n}{x} - \log 2 \right) x^n e^{-x \log 2}$$

of  $x^n 2^{-x}$  has a unique strictly positive root at  $\frac{n}{\log 2}$  and second derivative  $-\frac{(\log 2)^2}{n} \left( \frac{n}{\log 2} \right)^n e^{-n}$  at the critical point  $x = \frac{n}{\log 2}$  corresponding to the maximum  $\left( \frac{n}{\log 2} \right)^n e^{-n}$  of the function  $x \mapsto x^n 2^{-x}$ .

Laplace's method yields thus the asymptotics

$$\begin{aligned} \sum_{k=1}^{\infty} \frac{k^n}{2^k} &\sim \left( \frac{n}{\log 2} \right)^n e^{-n} \int_{-\infty}^{\infty} e^{-\frac{(\log 2)^2}{2n} x^2} dx \\ &= \sqrt{\frac{2\pi n}{(\log 2)^2}} \left( \frac{n}{\log 2} \right)^n e^{-n} \\ &= \frac{1}{(\log 2)^{n+1}} \sqrt{2\pi n} \frac{n^n}{e^n} \\ &\sim \frac{n!}{(\log 2)^{n+1}} \end{aligned}$$

where the last asymptotic equivalence follows from Stirling's formula  $n! \sim \sqrt{2\pi n} \frac{n^n}{e^n}$ .  $\square$

*Proof of Proposition 11.8.* Using Corollary 11.6 and the asymptotics  $\sum_{k=1}^{\infty} \frac{k^n}{2^k} \sim \frac{n!}{(\log 2)^{n+1}}$  given by Lemma 11.9, we get the asymptotics

$$\begin{aligned} m_{-n} &= \sum_{k=0}^n \alpha_{n,k} m_k \sim \sum_{k=0}^n \binom{n}{k} \frac{(n-k)!}{(\log 2)^{n-k+1}} m_k \\ &= \frac{n!}{(\log 2)^{n+1}} \sum_{k=0}^n \frac{(\log 2)^k}{k!} m_k \sim \lambda \frac{n!}{(\log 2)^{n+1}} \end{aligned}$$

with  $\lambda$  given by (40).  $\square$

## 12 Geometric means for the Stern sequence

It is an easy exercise to compute the arithmetic mean  $\frac{1}{2^n} \sum_{j=2^n}^{2^{n+1}} s(j)$ .

The following result gives asymptotics for the geometric mean:

**Theorem 12.1.** *There exists a real constant  $\beta$  such that*

$$\lim_{n \rightarrow \infty} e^{-(n\alpha + \beta)2^n} \prod_{j=2^n}^{2^{n+1}} s(j) = 1$$

where

$$\alpha = \log 2 - \sum_{j=1}^{\infty} \frac{m_j}{j2^j}$$

$$\sim .39621256429774455909560575764994569944470639102190 .$$

**Remark 12.2.** *The constant  $\alpha$  is involved in the Hausdorff dimension of growth points for  $\varphi(x)$ , see Kinney or Alkauskas. See also Conjecture 8.1 for a conjectural manifestation of  $\alpha$ .*

*I am not aware of the existence of an efficient method for computing the value of  $\beta \sim -.0851895$  with high precision.*

**Lemma 12.3.** *Given an increasing function  $\varphi : [0, 1] \rightarrow [0, 1]$  and a strictly positive natural integer  $N$  we have*

$$\left| \int_0^1 \varphi(t) dt - \frac{1}{N} \sum_{k=0}^{N-1} \frac{\varphi\left(\frac{k}{N}\right) + \varphi\left(\frac{k+1}{N}\right)}{2} \right| \leq \frac{1}{2N} .$$

*Proof.* The error of the trapezoidal rule

$$\int_a^b \varphi(t) dt \sim (b-a) \frac{\varphi(a) + \varphi(b)}{2}$$

is bounded by  $\left| \frac{(b-a)(\varphi(b) - \varphi(a))}{2} \right|$  if  $\varphi$  is monotonous. □

*Proof of Theorem 12.1.* We consider

$$\begin{aligned} S(n) &= \frac{1}{2^n} \sum_{k=2^n}^{2^{n+1}} \log(s(k)) \\ &= \frac{1}{2^n} \sum_{k=1}^{2^n} \frac{\log(s(2^n + k - 1)) + \log(s(2^n + k))}{2} \end{aligned}$$

where the second identity follows from the evaluations  $s(2^n) = s(2^{n+1}) = 1$ . Using (12) we get

$$\begin{aligned}
S(n+1) &= \frac{1}{2^{n+1}} \sum_{k=1}^{2^{n+1}} \frac{\log(s(2^{n+1} + k - 1)) + \log(s(2^{n+1} + k))}{2} \\
&= \frac{1}{2^{n+1}} \sum_{k=1}^{2^n} \frac{\log(s(2^{n+1} + k - 1)) + \log(s(2^{n+1} + k))}{2} \\
&\quad + \frac{1}{2^{n+1}} \sum_{k=1+2^n}^{2^{n+1}} \frac{\log(s(2^{n+1} + k - 1)) + \log(s(2^{n+1} + k))}{2} \\
&= \frac{2}{2^{n+1}} \sum_{k=1}^{2^n} \frac{\log(s(3 \cdot 2^n + k - 1)) + \log(s(3 \cdot 2^n + k))}{2}
\end{aligned}$$

and (13) yields

$$\begin{aligned}
S(n+1) &= \frac{1}{2^n} \sum_{k=1}^{2^n} \frac{\log(2s(2^n + k - 1) - s(k - 1)) + \log(2s(2^n + k) - s(k))}{2} \\
&= \log(2) + \frac{1}{2^n} \sum_{k=1}^{2^n} \frac{\log(s(2^n + k - 1) - \frac{1}{2}s(k - 1)) + \log(s(2^n + k) - \frac{1}{2}s(k))}{2}
\end{aligned}$$

Using  $\frac{d^k}{dx^k} \log(u - vx) = -(k-1)! \frac{v^k}{(u-vx)^k}$  for  $k \geq 1$ , we get

$$S(n+1) = \log(2) + S(n) - \sum_{j=1}^{\infty} \frac{1}{j2^j} \frac{1}{2^n} \sum_{k=1}^{2^n} \frac{1}{2} \left( \left( \frac{s(k-1)}{s(2^n + k - 1)} \right)^j + \left( \frac{s(k)}{s(2^n + k)} \right)^j \right)$$

which implies

$$\lim_{n \rightarrow \infty} (S(n+1) - S(n)) = \log 2 - \sum_{j=1}^{\infty} \frac{m_j}{j2^j} = \alpha$$

by Proposition 2.1.

Lemma 12.3 shows

$$\begin{aligned}
&\left| \sum_{n=0}^{\infty} \sum_{j=1}^{\infty} \frac{1}{j2^j} \left( m_j - \frac{1}{2^n} \sum_{k=1}^{2^n} \frac{1}{2} \left( \left( \frac{s(k-1)}{s(2^n + k - 1)} \right)^j + \left( \frac{s(k)}{s(2^n + k)} \right)^j \right) \right) \right| \\
&\leq \sum_{n=0}^{\infty} \sum_{j=1}^{\infty} \frac{1}{2^j} \frac{1}{2^n} \leq \sum_{n=0}^{\infty} \sum_{j=1}^{\infty} \frac{1}{2^{n+j}} = 2.
\end{aligned}$$

This proves the existence of  $\beta$ . □

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