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# A DISCRETE FRAMEWORK TO FIND THE OPTIMAL MATCHING BETWEEN MANIFOLD-VALUED CURVES

ALICE LE BRIGANT

ABSTRACT. This paper introduces a detailed discrete framework to study curves on a manifold of constant sectional curvature regardless of parameterization. This model results from the discretization of the elastic metric  $G^{1,1/2}$  studied in the square root velocity framework extended to smooth manifold-valued curves, and is itself a Riemannian structure on the product manifold  $M^{n+1}$  of "discrete curves" given by  $n + 1$  points. We show that the discrete energy of a discretization of size  $n$  of a path of smooth curves converges to the continuous energy as  $n \rightarrow \infty$ . We also study the quotient structure of the space of unparameterized curves (or shapes) of the continuous model, and characterize the associated horizontal subspace of the tangent bundle. We introduce a simple algorithm that constructs the horizontal geodesic between two parameterized curves using a canonical decomposition of a path in a principal bundle. Illustrations are given for curves in the hyperbolic plane  $M = \mathbb{H}^2$ , the plane  $M = \mathbb{R}^2$  and the sphere  $M = \mathbb{S}^2$ .

## 1. INTRODUCTION

The study of curves and their shapes is a research area with numerous and varied applications, which is why it has known a great deal of activity over the past few years. These curves can be closed or open, and take their values in a Euclidean space or more generally in a Riemannian manifold. To name a few examples, closed plane curves are central in shape analysis of objects [18]; the study of trajectories on the Earth requires to deal with open curves on the sphere [20], and in signal processing, locally stationary Gaussian processes can be represented by open curves in the hyperbolic plane, seen as the statistical manifold of Gaussian densities [7], [8]. Here we are concerned with the study of open curves in a manifold  $M$  of constant sectional curvature.

There are naturally many ways to go about comparing curves in a manifold. One way is to see the space of manifold-valued curves as an (infinite-dimensional) manifold itself, and equip it with a Riemannian metric  $G$ . Then two curves can be compared using the induced geodesic distance, and the geodesics are the "optimal deformations" from one curve to another. The advantage of this strategy is that it provides all the convenient tools of the Riemannian framework. An interesting property for our metric, from the point of view of the applications, is reparameterization invariance: for closed curves, this amounts to considering only the shape of an object; for an open curve representing the evolution in time of a given process, this allows us to analyze it regardless of speed or pace. A common strategy is to consider the quotient space  $\mathcal{S}$  of curves modulo reparameterization, where two curves are considered identical if they pass through the same points of  $M$  but at different speeds, or equivalently when one can be obtained by reparameterizing the other. This quotient space is often called the *shape space*. If the scalar product  $G$  is the same at all points of  $M$  which project on the same "shape", then  $G$  induces a Riemannian structure on the quotient space, which allows us to compare curves regardless of their parameterization.

Since the simplest metric one can think of, the  $L^2$ -metric (slightly modified to stay constant along the fibers), induces a vanishing distance on the quotient space [9], different classes of metrics have been studied to perform shape analysis. The large class of Sobolev metrics involves higher order derivatives to overcome the vanishing problem of the  $L^2$ -metric [10]. To study plane curves, the authors of [12] introduced the family of *elastic metrics*  $G^{a,b}$ ,

parameterized by two constants  $a$  and  $b$  which control the degree of bending and stretching of the curve. In [13], the authors show that for a certain choice of parameters, the metric can be mapped to the  $L^2$ -metric using the so-called *square root velocity* (SRV) framework, where a curve is represented by its speed renormalized by the square root of its norm. A similar idea was introduced in [16] and used in [17], where a Sobolev metric is also mapped to an  $L^2$ -metric. The SRV framework was generalized in [2] for any elastic metric with weights  $a$  and  $b$  satisfying a certain relation. A quotient structure for the metric used in [13] is carefully developed in [6], where the authors prove that if at least one of two curves is piecewise-linear, then there exists a minimizing geodesic between the two, and give a precise algorithm to solve the matching problem. In [3], it is proven that in the same framework, there always exists a minimizing geodesic between two  $C^1$  plane curves. Another approach is proposed in [15], where the authors restrict to arc-length parameterized curves and characterize the horizontal space of the quotient structure for these curves in the elastic framework.

Concerning manifold-valued curves, the geodesic equations for Sobolev metrics in the space of curves and in the shape space were given in [1] in terms of the gradient of the metric with respect to itself. The SRV framework developed in [13] was generalized to curves in a manifold in [19] and [7], and to curves in a Lie group in [4]. Both metrics in [19] and [7] coincide with the metric of [13] in the flat case, however in [19] the computations are moved to the tangent spaces to the origins of the curves, whereas in [7] they are done directly in the base manifold. In [19] the geodesics are computed using geodesic shooting in the product space  $\mathbb{C}$  of tangent spaces, and retrieved in the space of curves using covariant integral. The pairwise registration of two curves is done using dynamic programming, and these methods are applied to curves in the space of symmetric positive definite matrices for visual-speech recognition and hand-gesture classification. The specific case of spherical trajectories is detailed in [20] : the geodesics of the product space  $\mathbb{C}$  are shown to have a circular arc for a baseline curve, which allows the authors to find the geodesic between two elements of  $\mathbb{C}$  by solving a minimization problem on a reduced space. On the other hand, in [7] the geodesics are computed using a geodesic shooting method that requires the equations of Jacobi fields for the induced geometry, and this is applied to curves in the hyperbolic plane for radar signal processing applications. However the quotient structure is not studied.

The aim of this paper is to give a detailed discretization of the Riemannian structure introduced in [7] that is itself a Riemannian structure on the finite-dimensional manifold  $M^{n+1}$  of "discrete curves". We show that the discrete energy of a discretization of size  $n$  of a path of smooth curves converges to the continuous energy as  $n \rightarrow \infty$ . Moreover, the problem of the quotient structure is addressed. The horizontal space of the quotient structure for an elastic metric  $G^{a,b}$ , and in particular for our metric, is given for any curve (not only arc-length parameterized). To find the optimal matching between two curves, we propose a simple algorithm that iteratively approaches the horizontal geodesic between one of the two curves and the fiber of the other. This yields the "closest parameterization" of the second curve with respect to the fixed parameterization of the first curve, and thereby an optimal matching between the two. We restrict ourselves to open curves in a manifold  $M$  of constant sectional curvature  $K$ , and all the formulas are derived for the two-dimensional examples of the hyperbolic plane  $M = \mathbb{H}^2$  ( $K = -1$ ), the Euclidean plane  $M = \mathbb{R}^2$  ( $K = 0$ ) and the sphere  $M = \mathbb{S}^2$  ( $K = +1$ ). After reminding the continuous model previously introduced in [7], Section 2 describes the horizontal space of the quotient structure and a way to compute horizontal geodesics. In Section 3, we introduce the discretization, and give the convergence result toward the continuous model, which is later proved in Section 5. Section 4 shows results of simulations in the three settings of positive, zero and negative curvature.

## 2. THE CONTINUOUS MODEL

**2.1. Some notations.** Let  $(M, \langle \cdot, \cdot \rangle)$  be a Riemannian manifold of constant sectional curvature  $K$ . We first introduce a few notations. The norm associated to the Riemannian metric  $\langle \cdot, \cdot \rangle$  is denoted by  $|\cdot|$  and the Levi-Civita connection by  $\nabla$ . If  $t \mapsto c(t)$  is a curve in  $M$  and  $t \mapsto w(t) \in T_{c(t)}M$  a vector field along  $c$ , we denote by  $c_t := dc/dt = c'$  the derivative of  $c$  with respect to  $t$  and by  $\nabla_t w := \nabla_{c_t} w$ ,  $\nabla_t^2 w := \nabla_{c_t} \nabla_{c_t} w$  the first and second order covariant derivatives of  $w$  along  $c$ . We use various notations depending on the context to denote parallel transport according to connection  $\nabla$ . If  $u \in T_{c(t_1)}M$  is a tangent vector to  $M$  in  $c(t_1)$ , the parallel transport of  $u$  from  $c(t_1)$  to  $c(t_2)$  along  $c$  is denoted by  $P_c^{t_1, t_2}(u)$ , or when there is no ambiguity on the choice of the curve  $c$ ,  $u^{t_1, t_2}$ , or even  $u^\parallel$  to lighten notations in some cases. We associate to each curve  $c$  its renormalized speed vector field  $v := c'/|c'|$ , and to each vector field  $t \mapsto w(t)$  along  $c$ , its tangential and normal components  $w^T := \langle w, v \rangle v$  and  $w^N := w - w^T$ . Finally, for all  $x \in M$  we denote by  $\exp_x^M : T_x M \rightarrow M$  the exponential map on  $M$  and by  $\log_x^M : M \rightarrow T_x M$  its inverse map.

## 2.2. The space of smooth parameterized curves.

**2.2.1. The Riemannian structure.** We represent open oriented curves in  $M$  by smooth immersions, i.e. smooth curves with velocity that doesn't vanish. The set  $\mathcal{M}$  of smooth immersions in  $M$  is an open submanifold of the Fréchet manifold  $C^\infty([0, 1], M)$  and its tangent space at a point  $c$  is the set of infinitesimal deformations of  $c$ , which can be seen as vector fields along the curve  $c$  in  $M$

$$\begin{aligned} \mathcal{M} &= \text{Imm}([0, 1], M) = \{c \in C^\infty([0, 1], M) : c'(t) \neq 0 \ \forall t \in [0, 1]\}, \\ T_c \mathcal{M} &= \{w \in C^\infty([0, 1], TM) : w(t) \in T_{c(t)}M \ \forall t \in [0, 1]\}. \end{aligned}$$

Reparametrizations are represented by increasing diffeomorphisms  $\phi : [0, 1] \rightarrow [0, 1]$  (so that they preserve the end points of the curves), and their set is denoted by  $\text{Diff}^+([0, 1])$ . We adopt the so-called square root velocity (SRV) representation, i.e. we represent each curve  $c \in \mathcal{M}$  by the pair formed by its starting point  $x$  and its speed vector field renormalized by the square root of its norm, via the bijection  $\mathcal{M} \rightarrow M \times T\mathcal{M}$

$$c \mapsto \left( x := c(0), q := \frac{c'}{\sqrt{|c'|}} \right).$$

The inverse of this function is simply given by  $M \times T\mathcal{M} \ni (x, q) \mapsto \pi_{\mathcal{M}}(q) \in \mathcal{M}$ , if  $\pi_{\mathcal{M}}$  is the canonical projection  $T\mathcal{M} \rightarrow \mathcal{M}$ . The renormalization of the speed vector field in  $q$  allows us to define a reparameterization invariant metric, as we will see shortly. For any tangent vector  $w \in T_c \mathcal{M}$ , consider a path of curves  $s \mapsto c^w(s) \in \mathcal{M}$  such that  $c^w(0) = c$  and  $c_s^w(0) := \partial c^w / \partial s(0) = w$ . We denote by  $q^w := c_t^w / |c_t^w|^{1/2}$  the square root velocity representation of  $c^w$ . With these notations, we equip  $\mathcal{M}$  with a Riemannian metric  $G$ , defined at point  $c \in \mathcal{M}$  for two tangent vectors  $w, z \in T_c \mathcal{M}$  by

$$(1) \quad G_c(w, z) = \langle w(0), z(0) \rangle + \int_0^1 \langle \nabla_s q^w(0, t), \nabla_s q^z(0, t) \rangle dt.$$

This definition does not depend on the choice of  $c^w$  and  $c^z$  and we can reformulate this scalar product in terms of (covariant) derivatives of  $w$  and  $z$ . Indeed, note that  $\nabla_s q^w(0, t) = \nabla_{c_s^w(0, t)}(c_t^w / |c_t^w|^{1/2}) = |c_t^w(0, t)|^{-1/2} (\nabla_{c_s^w(0, t)} c_t^w - 1/2 (\nabla_{c_s^w(0, t)} c_t^w)^T)$ , which gives after inverting the derivatives according to  $s$  and  $t$ ,

$$\nabla_s q^w(0, t) = |c'|^{-1} (\nabla_t w^N + \frac{1}{2} \nabla_t w^T).$$

The scalar product can then be rewritten

$$(2) \quad G_c(w, z) = \langle w(0), z(0) \rangle + \int_0^1 (\langle \nabla_t w^N, \nabla_t z^N \rangle + \frac{1}{4} \langle \nabla_t w^T, \nabla_t z^T \rangle) |c'|^{-1} dt,$$

$$(2) \quad G_c(w, z) = \langle w(0), z(0) \rangle + \int_0^1 (\langle \nabla_\ell w^N, \nabla_\ell z^N \rangle + \frac{1}{4} \langle \nabla_\ell w^T, \nabla_\ell z^T \rangle) d\ell,$$

where  $d\ell = |c'(t)|dt$  and  $\nabla_\ell = \frac{1}{|c'(t)|} \nabla_t$  respectively denote integration and covariant derivation according to arc length. This metric belongs to the class of so-called *elastic metrics* parameterized by any  $a, b \in \mathbb{R}$ , which can be defined for manifold-valued curves as

$$G_c^{a,b}(w, z) = \langle w(0), z(0) \rangle + \int_0^1 a^2 (\langle \nabla_\ell w^N, \nabla_\ell z^N \rangle + b^2 \langle \nabla_\ell w^T, \nabla_\ell z^T \rangle) d\ell.$$

With formulation (2) it is clear that  $G = G^{1, \frac{1}{2}}$  is invariant under the action of reparameterizing the curve and its tangent vectors by any increasing diffeomorphism  $\phi \in \text{Diff}^+([0, 1])$

$$(3) \quad G_{c \circ \phi}(w \circ \phi, z \circ \phi) = G_c(w, z).$$

This *reparameterization invariance* property will allow us to induce a Riemannian structure on the quotient space as we will see in Section 2.3.

**2.2.2. Geodesics between parameterized curves.** Two curves  $c_0, c_1 \in \mathcal{M}$  can be compared using the geodesic distance induced by  $G$ , i.e. by computing the length of the shortest path of curves  $[0, 1] \ni s \mapsto c(s) \in \mathcal{M}$  from  $c_0$  to  $c_1$

$$(4) \quad d_G(c_0, c_1) = \inf \{L(c) : c(0) = c_0, c(1) = c_1\},$$

where the length of a path of curves  $c$  can be written in terms of its SRV representation  $(x, q) : s \mapsto (x(s) := c(s, 0), q(s) := c_t(s)/|c_t(s)|^{1/2}) \in M \times T\mathcal{M}$  as

$$L(c) = \int_0^1 \sqrt{|x_s(s)|^2 + \int_0^1 |\nabla_s q(s, t)|^2 dt} ds.$$

Note that here - and in all that follows - we indifferently use the notations  $c(s, t) = c(s)(t)$ ,  $q(s, t) = q(s)(t)$  for all  $s, t \in [0, 1]$ . Now we recall a result shown in [7], which characterizes the geodesic paths of  $\mathcal{M}$ , i.e. those which achieve the infimum in (4), by searching for the critical points of the energy functional  $E : \mathcal{M} \rightarrow \mathbb{R}_+$ ,

$$(5) \quad E(c) = \int_0^1 \left( |x'(s)|^2 + \int_0^1 |\nabla_s q(s, t)|^2 dt \right) ds.$$

**Proposition 1** (Geodesic equations). *A geodesic path  $[0, 1] \ni s \mapsto c(s) \in \mathcal{M}$  for  $G$ , or more specifically its SRV representative  $s \mapsto (x(s), q(s))$ , verifies the equations*

$$(6) \quad \begin{aligned} \nabla_s x_s(s) + r(s, 0) &= 0 \quad \forall s \in [0, 1], \\ \nabla_s \nabla_s q(s, t) + |q(s, t)| (r(s, t) + r(s, t)^T) &= 0 \quad \forall t, s \in [0, 1], \end{aligned}$$

where  $x(s) = c(s, 0)$ ,  $q(s) = c_t(s)/\sqrt{|c_t(s)|}$  and the vector field  $r$  is given by

$$r(s, t) = \int_t^1 \mathcal{R}(q, \nabla_s q) c_s(s, \tau)^{\tau, t} d\tau, \quad t \in [0, 1].$$

**Remark 1.** In the flat case  $M = \mathbb{R}^d$ , the curvature term  $r$  vanishes and we obtain  $\nabla_s x_s(s) = 0$ ,  $\nabla_s \nabla_s q(s, t) = 0$  for all  $s$  and  $t$ . This means that the geodesic between two curves  $(x_0, q_0)$  and  $(x_1, q_1)$  in the SRV representation space  $M \times T\mathcal{M}$  is composed of a straight line  $s \mapsto x(s)$  and an  $L^2$ -geodesic  $s \mapsto q(s, \cdot)$ . This is illustrated in simulations of Section 4.

A possibility to construct the geodesics of  $\mathcal{M}$  is to use geodesic shooting. By solving the geodesic equations (6) we can construct the geodesic path starting from a given curve  $c_0$  at a given speed  $u \in T_{c_0}\mathcal{M}$  - this is the exponential map on  $\mathcal{M}$ . Given two curves  $c_0, c_1 \in \mathcal{M}$ , we can iteratively find the appropriate initial speed  $u$  which will make the geodesic land on  $c_1$ . Let us denote by  $\log_c^{L^2}$  the inverse of the exponential map for the  $L^2$ -metric on  $\mathcal{M}$  - it is simply given by  $\log_{c_0}^{L^2}(c_1)(t) = \log_{c_0(t)}^M(c_1(t))$  for  $t \in [0, 1]$ . The steps of the geodesic shooting algorithm are the following.

**Algorithm 1** (Geodesic shooting in  $\mathcal{M}$ ). Let  $c_0, c_1 \in \mathcal{M}$ . Set  $u = \log_{c_0}^{L^2}(c_1)$  and repeat until convergence :

- (1) compute the geodesic  $s \mapsto c(s)$  starting from  $c_0$  at speed  $u$  by solving the geodesic equations (6),
- (2) evaluate the difference  $j := \log_{c(1)}^{L^2}(c_1)$  between the target curve  $c_1$  and the extremity  $c(1)$  of the obtained geodesic,
- (3) compute the initial derivative  $\nabla_s J(0)$  of the Jacobi field  $s \mapsto J(s)$  along  $c$  verifying  $J(0) = 0$  and  $J(1) = j$ ,
- (4) correct the shooting direction  $u = u + \nabla_s J(0)$ .

This algorithm requires the characterization of the Jacobi fields for  $G$  on  $\mathcal{M}$ , and a way to deduce the initial derivative  $\nabla_s J(0)$  of a Jacobi field from its initial and final values  $J(0), J(1)$ . Concerning these two points, we refer the reader to [7] : the Jacobi fields of  $\mathcal{M}$  are shown to be solutions of a linear PDE, which can be solved to obtain the final value  $J(1)$  of a Jacobi field  $J$  along a path of curves  $c$  knowing its initial conditions  $J(0)$  and  $\nabla_s J(0)$ . If we consider only Jacobi fields with initial value  $J(0) = 0$ , then the function  $\phi : T_{c(0)}\mathcal{M} \rightarrow T_{c(1)}\mathcal{M}$ ,  $\nabla_s J(0) \mapsto J(1)$  is a linear bijection between two vector spaces and its inverse map can be computed by considering the image of a basis of  $T_{c(0)}\mathcal{M}$ . The equations characterizing the Jacobi fields in the discrete setting will be given in Section 3.

### 2.3. The space of unparameterized curves.

**2.3.1. The quotient structure.** In order to compare curves regardless of parameterization, we consider the quotient  $\mathcal{S} = \mathcal{M}/\text{Diff}^+([0, 1], M)$  of the space of curves by the diffeomorphisms group. This quotient is not a manifold, as it has singularities, i.e. points with non trivial isotropy group. If we get rid of these singularities and restrict ourselves to elements of  $\mathcal{M}$  on which the diffeomorphism group acts freely, then the space of free immersions  $\mathcal{M}_f$ , the quotient shape space  $\mathcal{S}_f = \mathcal{M}_f/\text{Diff}^+([0, 1])$  and the group of diffeomorphisms  $\text{Diff}^+([0, 1])$  form a principal bundle, the fibers of which are the sets of all the curves that are identical modulo reparameterization, i.e. that project on the same "shape". We denote by  $\pi : \mathcal{M}_f \rightarrow \mathcal{S}_f$  the projection of the fiber bundle and by  $\bar{c} := \pi(c) \in \mathcal{S}_f$  the shape of a curve  $c \in \mathcal{M}_f$ . The tangent bundle can then be decomposed

$$T\mathcal{M}_f = \text{Ver} \oplus \text{Hor}$$

into a vertical subspace consisting of all vectors tangent to the fibers of  $\mathcal{M}_f$  over  $\mathcal{S}_f$ , that is, those which have an action of reparameterizing the curve without changing its shape

$$\text{Ver}_c = \ker T_c\pi = \{mv = mc'/|c'| : m \in C^\infty([0, 1], \mathbb{R}), m(0) = m(1) = 0\},$$

and a horizontal subspace defined as the orthogonal of the vertical subspace according to  $G$

$$\text{Hor}_c = (\text{Ver}_c)^{\perp G} = \{h \in T_c\mathcal{M}_f : G_c(h, mv) = 0, \forall m \in C^\infty([0, 1], \mathbb{R}), m(0) = m(1) = 0\}.$$

If  $G$  is constant along the fibers, i.e. verifies property (3), then there exists a Riemannian metric  $\bar{G}$  on the shape space  $\mathcal{S}_f$  such that  $\pi$  is a Riemannian submersion from  $(\mathcal{M}_f, G)$  to  $(\mathcal{S}_f, \bar{G})$ ,

$$G_c(w^{hor}, z^{hor}) = \bar{G}_{\pi(c)}(T_c\pi(w), T_c\pi(z)),$$

where  $w^{hor}$  and  $z^{hor}$  are the horizontal parts of  $w$  and  $z$ , as well as the horizontal lifts of  $T_c\pi(w)$  and  $T_c\pi(z)$ , respectively. This expression defines  $\bar{G}$  in the sense that it does not depend on the choice of the representatives  $c$ ,  $w$  and  $z$  ([11], §29.21). If a geodesic for  $G$  has a horizontal initial speed, then its speed vector stays horizontal at all times - we say it is a horizontal geodesic - and projects on a geodesic of the shape space for  $\bar{G}$  ([11], §26.12). To compute the distance between two shapes  $\bar{c}_0$  and  $\bar{c}_1$  in the quotient space we choose a representative  $c_0$  of  $\bar{c}_0$  and compute the distance (in  $\mathcal{M}_f$ ) to the closest representative of  $\bar{c}_1$

$$\bar{d}(\bar{c}_0, \bar{c}_1) = \inf \{ d(c_0, c_1 \circ \phi) \mid \phi \in \text{Diff}^+([0, 1]) \}.$$

By definition, the distance in the quotient space allows us to compare curves regardless of parameterization

$$\bar{d}(\overline{c_0 \circ \phi}, \overline{c_1 \circ \psi}) = \bar{d}(\bar{c}_0, \bar{c}_1), \quad \forall \phi, \psi \in \text{Diff}^+([0, 1]).$$

We now characterize the horizontal subspace for any elastic metric  $G^{a,b}$  and in particular for our metric  $G^{1, \frac{1}{2}}$ , and give the decomposition of a tangent vector.

**Proposition 2** (Horizontal subspace and decomposition of a vector). *Let  $c \in \mathcal{M}$  be a smooth immersion. A tangent vector  $h \in T_c\mathcal{M}$  is horizontal for the elastic metric  $G^{a,b}$  if and only if*

$$((a/b)^2 - 1) \langle \nabla_t h, \nabla_t v \rangle - \langle \nabla_t \nabla_t h, v \rangle + |c'|^{-1} \langle \nabla_t c', v \rangle \langle \nabla_t h, v \rangle = 0.$$

In particular, for  $a = 2b = 1$  we obtain

$$\text{Hor}_c = \{ h \in T_c\mathcal{M} : 3 \langle \nabla_t h, \nabla_t v \rangle - \langle \nabla_t \nabla_t h, v \rangle + |c'|^{-1} \langle \nabla_t c', v \rangle \langle \nabla_t h, v \rangle = 0, \forall t \in [0, 1] \}.$$

Any tangent vector  $w \in T_c\mathcal{M}$  can be decomposed in horizontal and vertical components  $w = w^{hor} + w^{ver}$  given by

$$w^{ver} = mv, \quad w^{hor} = w - mv,$$

where the real function  $m \in C^\infty([0, 1], \mathbb{R})$  verifies  $m(0) = m(1) = 0$  and

$$(7) \quad m'' - \langle \nabla_t c' / |c'|, v \rangle m' - 4|\nabla_t v|^2 m = \langle \nabla_t \nabla_t w, v \rangle - 3 \langle \nabla_t w, \nabla_t v \rangle - \langle \nabla_t c' / |c'|, v \rangle \langle \nabla_t w, v \rangle.$$

*Proof.* Let  $h \in T_c\mathcal{M}$  be a tangent vector. It is horizontal if and only if it is orthogonal to any vertical vector, that is any vector of the form  $mv$  with  $m \in C^\infty([0, 1], \mathbb{R})$  such that  $m(0) = m(1) = 0$ . We have  $\nabla_t(mv) = m'v + m\nabla_tv$  and since  $\langle \nabla_tv, v \rangle = 0$  we get  $\nabla_t(mv)^N = m\nabla_tv$  and  $\nabla_t(mv)^T = m'v$ . The scalar product can then be written

$$\begin{aligned} G_c^{a,b}(h, mv) &= \langle h(0), m(0)v(0) \rangle + \int_0^1 (a^2 \langle \nabla_t h^N, \nabla_t(mv)^N \rangle + b^2 \langle \nabla_t h^T, \nabla_t(mv)^T \rangle) |c'|^{-1} dt \\ &= \int_0^1 (a^2 m \langle \nabla_t h, \nabla_t v \rangle + b^2 m' \langle \nabla_t h, v \rangle) |c'|^{-1} dt \\ &= \int_0^1 a^2 m \langle \nabla_t h, \nabla_t v \rangle |c'|^{-1} dt - \int_0^1 b^2 m \frac{d}{dt} (\langle \nabla_t h, v \rangle |c'|^{-1}) dt \\ &= \int_0^1 m |c'| \left( (a^2 - b^2) \langle \nabla_t h, \nabla_t v \rangle - b^2 \langle \nabla_t \nabla_t h, v \rangle + b^2 \langle \nabla_t c', v \rangle \langle \nabla_t h, v \rangle \right) |c'|^{-1} dt, \end{aligned}$$

by integration by parts. The vector  $h$  is horizontal if and only if  $G_c(h, mv) = 0$  for all such  $m$ , and so dividing by  $b^2$  and multiplying by  $|c'|$  gives the desired equation. Now consider a tangent vector  $w$  and a real function  $m : [0, 1] \rightarrow \mathbb{R}$  such that  $m(0) = m(1) = 0$ . Then according to the above,  $w - mv$  is horizontal if and only if it verifies

$$3 \langle \nabla_t(w - mv), \nabla_t v \rangle - \langle \nabla_t \nabla_t(w - mv), v \rangle + |c'|^{-1} \langle \nabla_t c', v \rangle \langle \nabla_t(w - mv), v \rangle = 0,$$

i.e., since  $\langle \nabla_tv, v \rangle = 0$ ,  $\langle \nabla_t \nabla_tv, v \rangle = -|\nabla_tv|^2$  and  $\nabla_t \nabla_t(mv) = m''v + 2m'\nabla_tv + m\nabla_t \nabla_tv$ ,  $3 \langle \nabla_tw, \nabla_tv \rangle - 3|\nabla_tv|^2 m - \langle \nabla_t \nabla_tw, v \rangle + m'' - m|\nabla_tv|^2 + |c'|^{-1} \langle \nabla_t c', v \rangle (\langle \nabla_tw, v \rangle - m') = 0$ , which is what we wanted.  $\square$

2.3.2. *Computing geodesics in the shape space.* Recall that the geodesic path  $s \mapsto \bar{c}(s)$  between the shapes of two curves  $c_0$  and  $c_1$  is the projection of the horizontal geodesic  $s \mapsto c_h(s)$  linking  $c_0$  to the fiber of  $c_1$  in  $\mathcal{M}$  - i.e. such that  $c_h(0) = c_0$ ,  $c_h(1) \in \pi^{-1}(\bar{c}_1)$  and  $\partial_s c_h(s) \in \text{Hor}_{c_h(s)}$  for all  $s \in [0, 1]$ ,

$$\bar{c} = \pi(c_h).$$

In saying so we suppose that this horizontal geodesic  $c_h$  exists. The end point of  $c_h$  then gives the optimal reparameterization  $c_1 \circ \phi$  of the target curve  $c_1$  with respect to the initial curve  $c_0$ , i.e. such that

$$\bar{d}(\bar{c}_0, \bar{c}_1) = d(c_0, c_1 \circ \phi).$$

Here we propose a method to approach the horizontal geodesic  $c_h$ . To that end we decompose any path of curves  $s \mapsto c(s)$  in  $\mathcal{M}$  into a horizontal path composed with a path of reparameterizations,  $c(s) = c^{hor}(s) \circ \phi(s)$ , or equivalently

$$(8) \quad c(s, t) = c^{hor}(s, \phi(s, t)) \quad \forall s, t \in [0, 1],$$

where the path  $[0, 1] \ni s \mapsto c^{hor}(s) \in \mathcal{M}$  is such that  $c_s^{hor}(s) \in \text{Hor}_{c^{hor}(s)}$  for all  $s \in [0, 1]$ , and  $[0, 1] \ni s \mapsto \phi(s) \in \text{Diff}^+([0, 1])$  is a path of increasing diffeomorphisms. The horizontal and vertical parts of the speed vector of  $c$  can be expressed in terms of this decomposition. Indeed, by taking the derivative of (8) with respect to  $s$  and  $t$  we obtain

$$(9a) \quad c_s(s) = c_s^{hor}(s) \circ \phi(s) + \phi_s(s) \cdot c_t^{hor}(s) \circ \phi(s),$$

$$(9b) \quad c_t(s) = \phi_t(s) \cdot c_t^{hor}(s) \circ \phi(s),$$

and so with the notation  $v^{hor}(s, t) := c_t^{hor}(s, t)/|c_t^{hor}(s, t)|$ , since  $\phi_t > 0$ , (9b) gives

$$v(s) = v^{hor}(s) \circ \phi(s).$$

We can see that the first term on the right-hand side of Equation (9a) is horizontal. Indeed, for any  $m : [0, 1] \rightarrow C^\infty([0, 1], \mathbb{R})$  such that  $m(s, 0) = m(s, 1) = 0$  for all  $s$ , since  $G$  is reparameterization invariant we have

$$\begin{aligned} G(c_s^{hor}(s) \circ \phi(s), m(s) \cdot v(s)) &= G(c_s^{hor}(s) \circ \phi(s), m(s) \cdot v^{hor}(s) \circ \phi(s)) \\ &= G(c_s^{hor}(s), m(s) \circ \phi(s)^{-1} \cdot v^{hor}(s)) \\ &= G(c_s^{hor}(s), \tilde{m}(s) \cdot v^{hor}(s)) \end{aligned}$$

with  $\tilde{m}(s) = m(s) \circ \phi(s)^{-1}$ . Since  $\tilde{m}(s, 0) = \tilde{m}(s, 1) = 0$  for all  $s$ , the vector  $\tilde{m}(s) \cdot v^{hor}(s)$  is vertical and its scalar product with the horizontal vector  $c_s^{hor}(s)$  vanishes. On the other hand, the second term on the right hand-side of Equation (9a) is vertical, since it can be written

$$\phi_s(s) \cdot c_t^{hor} \circ \phi(s) = m(s) \cdot v(s),$$

with  $m(s) = |c_t(s)|\phi_s(s)/\phi_t(s)$  verifying  $m(s, 0) = m(s, 1) = 0$  for all  $s$ . Finally, the vertical and horizontal parts of the speed vector  $c_s(s)$  are given by

$$(10a) \quad c_s(s)^{ver} = m(s) \cdot v(s) = |c_t(s)|\phi_s(s)/\phi_t(s) \cdot v(s),$$

$$(10b) \quad c_s(s)^{hor} = c_s(s) - m(s) \cdot v(s) = c_s^{hor}(s) \circ \phi(s).$$

**Definition 1.** We call  $c^{hor}$  the *horizontal part* of the path  $c$  with respect to  $G$ .

**Proposition 3.** *The horizontal part of a path of curves  $c$  is at most the same length as  $c$*

$$L_G(c^{hor}) \leq L_G(c).$$

*Proof.* Since the metric  $G$  is reparameterization invariant, the squared norm of the speed vector of the path  $c$  at time  $s \in [0, 1]$  is given by

$$\begin{aligned} \|c_s(s, \cdot)\|_G^2 &= \|c_s^{hor}(s, \phi(s, \cdot))\|_G^2 + |\phi_s(s, \cdot)|^2 \|c_t^{hor}(s, \phi(s, \cdot))\|_G^2 \\ &= \|c_s^{hor}(s, \cdot)\|_G^2 + |\phi_s(s, \cdot)|^2 \|c_t^{hor}(s, \cdot)\|_G^2, \end{aligned}$$

where  $\|\cdot\|_G^2 := G(\cdot, \cdot)$ . This gives  $\|c_s^{hor}(s)\|_G \leq \|c_s(s)\|_G$  for all  $s$  and so  $L_G(c^{hor}) \leq L_G(c)$ .  $\square$



Now we will see how the horizontal part of a path of curves can be computed.

**Proposition 4** (Characterization of the horizontal part of a path). *Let  $s \mapsto c(s)$  be a path in  $\mathcal{M}$ . Then its horizontal part is given by  $c^{hor}(s, t) = c(s, \phi(s)^{-1}(t))$ , where the path of diffeomorphisms  $s \mapsto \phi(s)$  is solution of the PDE*

$$(11) \quad \phi_s(s, t) = m(s, t)/|c_t(s, t)| \cdot \phi_t(s, t),$$

and where  $m(s) : [0, 1] \rightarrow \mathbb{R}$ ,  $t \mapsto m(s, t)$  is solution for all  $s$  of the ODE

$$m_{tt} - \langle \nabla_t c_t / |c_t|, v \rangle m_t - 4|\nabla_t v|^2 m = \langle \nabla_t \nabla_t c_s, v \rangle - 3\langle \nabla_t c_s, \nabla_t v \rangle - \langle \nabla_t c_t / |c_t|, v \rangle \langle \nabla_t c_s, v \rangle.$$

*Proof.* We have seen in Equation (10a) that the vertical part of  $c_s(s)$  can be written as  $m(s) \cdot v(s)$  where  $m(s) = |c_t(s)|\phi_s(s)/\phi_t(s)$ , and as the norm of the vertical part of  $c_s(s)$ ,  $m(s)$  is solution of the ODE (7) for all  $s$ .  $\square$

If we take the horizontal part of the geodesic linking two curves  $c_0$  and  $c_1$ , we will obtain a horizontal path linking  $c_0$  to the fiber of  $c_1$  which will no longer be a geodesic path. However this path reduces the distance between  $c_0$  and the fiber of  $c_1$ , and gives a "better" representative  $\tilde{c}_1 = c_1 \circ \phi(1)$  of the target curve. By computing the geodesic between  $c_0$  and this new representative  $\tilde{c}_1$ , we are guaranteed to reduce once more the distance to the fiber. The algorithm that we propose simply iterates these two steps.

**Algorithm 2** (Constructing horizontal geodesics in a principal bundle). Let  $c_0, c_1 \in \mathcal{M}$ . Set  $\tilde{c}_1 = c_1$  and repeat until convergence :

- Construct the geodesic  $s \mapsto c(s)$  between  $c_0$  and  $\tilde{c}_1$  (e.g. using geodesic shooting).
- Compute the horizontal part  $s \mapsto c^{hor}(s)$  of  $c$  and set  $\tilde{c}_1 = c^{hor}(1)$ .

This algorithm yields an approximation of the horizontal geodesic  $c_h = c^{hor}$  between  $c_0$  and the fiber of  $c_1$  as well as the optimal reparameterization  $\tilde{c}_1 = c_h(1)$  of  $c_1$ . Before we test Algorithms 1 and 2, we first introduce a formal discretization of the continuous model presented so far.

### 3. THE DISCRETE MODEL

**3.1. The Riemannian structure.** Applications usually give access to a finite number of observations of a continuous process and provide series of points instead of continuous curves. It is therefore important to discretize the model presented above and to consider the product manifold  $M^{n+1}$  as the space of "discrete curves" given by  $n + 1$  points, for a fixed  $n \in \mathbb{N}^*$ . Its tangent space at a given point  $\alpha = (x_0, \dots, x_n)$  is given by

$$T_\alpha M^{n+1} = \{w = (w_0, \dots, w_n) : w_k \in T_{x_k} M, k = 0, \dots, n\}.$$

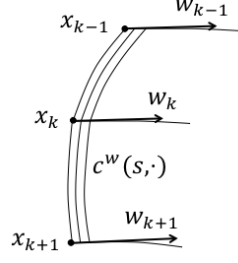
Assuming that there exists a connecting geodesic between  $x_k$  and  $x_{k+1}$  for all  $k$  – which seems reasonable considering that the points  $x_k$  should be "close" since they correspond to the discretization of a continuous curve – we use the following notations

$$(12) \quad \tau_k = \log_{x_k}^M x_{k+1}, \quad q_k = \sqrt{n} \tau_k / \sqrt{|\tau_k|}, \quad v_k = \tau_k / |\tau_k|,$$

as well as  $w_k^T = \langle w_k, v_k \rangle v_k$  and  $w_k^N = w_k - w_k^T$  to refer to the tangential and normal components of a tangent vector  $w_k \in T_{x_k} M$ . Given a tangent vector  $w \in T_\alpha M^{n+1}$ , we consider a path of piecewise geodesic curves  $[0, 1]^2 \ni (s, t) \mapsto c^w(s, t) \in M$  such that  $c^w(0, \frac{k}{n}) = x_k$  for  $k = 0, \dots, n$ ,  $c^w(s, \cdot)$  is a geodesic of  $M$  on the interval  $[\frac{k}{n}, \frac{k+1}{n}]$  for all  $s \in [0, 1]$  and  $k$  – and in particular  $c_t^w(0, \frac{k}{n}) = n\tau_k$  – and such that  $c_s^w(0, \frac{k}{n}) = w_k$ . Then we define the scalar product between  $w$  and  $z$  by

$$(13) \quad G_\alpha^n(w, z) = \langle w_0, z_0 \rangle + \frac{1}{n} \sum_{k=0}^{n-1} \langle \nabla_s q^w(0, \frac{k}{n}), \nabla_s q^z(0, \frac{k}{n}) \rangle.$$

This definition is a discrete analog of (1), and just as in the continuous case, it does not depend on the choices of  $c^w$  and  $c^z$ . Indeed, we can also obtain a discrete analog of (2).



**Proposition 5.** *The scalar product between two tangent vectors  $w, z \in T_\alpha M^{n+1}$  can also be written*

$$G_\alpha^n(w, z) = \langle w_0, z_0 \rangle + \sum_{k=0}^{n-1} \left( \langle (D_\tau w)_k^N, (D_\tau z)_k^N \rangle + \frac{1}{4} \langle (D_\tau w)_k^T, (D_\tau z)_k^T \rangle \right) |\tau_k|^{-1},$$

where  $D_\tau : T_\alpha M^{n+1} \rightarrow T_\alpha M^{n+1}$ ,  $w \mapsto D_\tau w = ((D_\tau w)_0, \dots, (D_\tau w)_n)$  is defined by

$$(D_\tau w)_k := \frac{1}{n} \nabla_t c_s^w(0, \frac{k}{n}) = (w_{k+1}^\parallel - w_k)^T + b_k^{-1} (w_{k+1}^\parallel - a_k w_k)^N,$$

and the coefficients  $a_k$  and  $b_k$  take the following values depending on the sectional curvature  $K$  of the base manifold  $M$

$$(14) \quad a_k = \begin{cases} \cosh |\tau_k| & \text{if } K = -1, \\ 1 & \text{if } K = 0, \\ \cos |\tau_k| & \text{if } K = +1, \end{cases} \quad b_k = \begin{cases} \sinh |\tau_k| / |\tau_k| & \text{if } K = -1, \\ 1 & \text{if } K = 0, \\ \sin |\tau_k| / |\tau_k| & \text{if } K = +1. \end{cases}$$

**Remark 2.** Notice that in the flat case our definition gives  $(D_\tau w)_k = w_{k+1} - w_k$ . In the non-flat case, when the discretization gets "thinner", i.e.  $n \rightarrow \infty$  and  $|\tau_k| \rightarrow 0$  while  $n|\tau_k|$  stays bounded for all  $0 \leq k \leq n$ , we get  $(D_\tau w)_k \underset{n \rightarrow \infty}{=} w_{k+1}^\parallel - w_k + o(1)$ .

Before we prove this proposition, let us recall a well-known result about Jacobi fields that will prove useful to derive the equations in the discrete case.

**Lemma 1.** *Let  $\gamma : [0, 1] \rightarrow M$  be a geodesic of a manifold  $M$  of constant sectional curvature  $K$ , and  $J$  a Jacobi field along  $\gamma$ . Then the parallel transport of  $J(t)$  along  $\gamma$  from  $\gamma(t)$  to  $\gamma(0)$  is given by*

$$J(t)^{t,0} = J^T(0) + \tilde{a}_k(t) J^N(0) + t \nabla_t J^T(0) + \tilde{b}_k(t) \nabla_t J^N(0), \quad \forall t \in [0, 1].$$

where

$$\tilde{a}_k(t) = \begin{cases} \cosh (|\gamma'(0)|t) & \text{if } K = -1, \\ 1 & \text{if } K = 0, \\ \cos (|\gamma'(0)|t) & \text{if } K = +1, \end{cases} \quad \tilde{b}_k(t) = \begin{cases} \sinh (|\gamma'(0)|t) / |\gamma'(0)| & \text{if } K = -1, \\ t & \text{if } K = 0, \\ \sin (|\gamma'(0)|t) / |\gamma'(0)| & \text{if } K = +1. \end{cases}$$

*Proof of Lemma 1.* For the sake of completeness, the proof is reminded in the appendix.  $\square$

*Proof of Proposition 5.* Let  $\alpha \in M^{n+1}$  be a "discrete curve" and  $w, z \in T_\alpha M^{n+1}$  tangent vectors at  $\alpha$ . Consider a path of piecewise geodesic curves  $s \mapsto c^w(s)$  that verifies all the conditions given above to define  $G^n(w, z)$ , and set  $(D_\tau w)_k := \frac{1}{n} \nabla_t c_s^w(0, \frac{k}{n})$ . Then by definition, the vector field  $J_k(u) = c_s^w(s, \frac{k+u}{n})$ ,  $u \in [0, 1]$  is a Jacobi field along the geodesic linking  $x_k$  to  $x_{k+1}$ , verifying  $J_k(0) = w_k$ ,  $J_k(1) = w_{k+1}$  and  $\nabla_u J_k(0) = (D_\tau w)_k$ . Applying Lemma 1 gives

$$w_{k+1}^\parallel = w_k^T + a_k w_k^N + (D_\tau w)_k^T + b_k (D_\tau w)_k^N.$$

Taking the tangential part and then the normal parts on both sides gives

$$\begin{aligned} (w_{k+1}^\parallel)^T &= w_k^T + (D_\tau w)_k^T, \\ (w_{k+1}^\parallel)^N &= a_k w_k^N + b_k (D_\tau w)_k^N. \end{aligned}$$

and so  $(D_\tau w)_k = (D_\tau w)_k^T + (D_\tau w)_k^N = (w_{k+1}^\parallel - w_k)^T + b_k^{-1}(w_{k+1}^\parallel - a_k w_k)^N$ . Finally, we observe that the covariant derivative involved in the definition of  $G^n$  can be written

$$\nabla_s q^w(0, \frac{k}{n}) = |c_t^w(0, \frac{k}{n})|^{-\frac{1}{2}} (\nabla_s c_t^w(0, \frac{k}{n}) - \frac{1}{2} \nabla_s c_t^w(0, \frac{k}{n})^T) = |n\tau_k|^{-\frac{1}{2}} (n(D_\tau w)_k - \frac{1}{2}n(D_\tau w)_k^T),$$

i.e.

$$\nabla_s q^w(0, \frac{k}{n}) = (n/|\tau_k|)^{1/2} ((D_\tau w)_k^N + \frac{1}{2}(D_\tau w)_k^T).$$

Injecting this into (13) gives the desired formula for the scalar product.  $\square$

Now we present our main result, that is, the convergence of the discrete model toward the continuous model.

**Definition 2.** Let  $\alpha = (x_0, \dots, x_n) \in M^{n+1}$  be a discrete curve, and  $t \mapsto c(t) \in M$  a smooth curve. We say that  $\alpha$  is *the discretization of size  $n$  of  $c$*  when  $c(\frac{k}{n}) = x_k$  for all  $k = 0, \dots, n$ . If  $s \mapsto \alpha(s) = (x_0(s), \dots, x_n(s)) \in M^{n+1}$  is a path of discrete curves and  $s \mapsto c(s) \in \mathcal{M}$  a path of smooth curves, then  $\alpha$  is *the discretization of size  $n$  of  $c$*  when  $\alpha(s)$  is the discretization of  $c(s)$  for all  $s \in [0, 1]$ , i.e. when  $x_k(s) = c(s, \frac{k}{n})$  for all  $s$  and  $k$ . We will still use this term if  $c$  is not smooth, and speak of the only path of piecewise-geodesic curves of which  $\alpha$  is the discretization.

Consider a path  $[0, 1] \ni s \mapsto \alpha(s) = (x_0(s), \dots, x_n(s)) \in M^{n+1}$  of discrete curves. Defining  $\tau_k(s)$  and  $q_k(s)$  as in (12) for all  $s \in [0, 1]$ , the path  $\alpha$  can be represented by its SRV representation  $[0, 1] \rightarrow M \times T_\alpha M^{n+1}$ ,

$$(15) \quad s \mapsto \left( x_0(s), (q_k(s))_{0 \leq k \leq n-1} \right).$$

To compute the squared norm of its speed vector  $\alpha'(s)$ , consider the path of piecewise geodesic curves  $[0, 1]^2 \ni (s, t) \mapsto c(s, t) \in M$  such that  $c(s, \frac{k}{n}) = x_k(s)$  and  $c_t(s, \frac{k}{n}) = n\tau_k(s)$  for all  $s$  and  $k$ . Then, notice that we have

$$(16) \quad \nabla_s q(s, \frac{k}{n}) = \nabla_s q_k(s), \quad (D_\tau \alpha'(s))_k = \frac{1}{n} \nabla_t c_s(s, \frac{k}{n}) = \nabla_s \tau_k(s),$$

and so the squared norm of the speed vector of  $\alpha$  can be expressed in terms of the SRV representation

$$G^n(\alpha'(s), \alpha'(s)) = |x_0'(s)|^2 + \frac{1}{n} \sum_{k=0}^{n-1} |\nabla_s q_k(s)|^2.$$

In the following result, we show that if  $s \mapsto \alpha(s)$  is the discretization of a path  $s \mapsto c(s) \in \mathcal{M}$  of continuous curves, then its energy with respect to  $G^n$ ,

$$(17) \quad E^n(\alpha) = \frac{1}{2} \int_0^1 \left( |x_0'(s)|^2 + \frac{1}{n} \sum_{k=0}^{n-1} |\nabla_s q_k(s)|^2 \right) ds,$$

gets closer to the energy (5) of  $c$  with respect to  $G$  as the size of the discretization grows.

**Theorem 1** (Convergence of the discrete model to the continuous model). *Let  $s \mapsto c(s)$  be a  $C^1$ -path of  $C^2$ -curves with non vanishing derivative with respect to  $t$ . This path can be identified with an element  $(s, t) \mapsto c(s, t)$  of  $C^{1,2}([0, 1] \times [0, 1], M)$  such that  $c_t \neq 0$ . Consider the  $C^1$ -path in  $M^{n+1}$ ,  $s \mapsto \alpha(s) = (x_0(s), \dots, x_n(s))$ , that is the discretization of size  $n$  of  $c$ . Then there exists a constant  $\lambda > 0$  such that for  $n$  large enough, the difference between the energies of  $c$  and  $\alpha$  is bounded by*

$$|E(c) - E^n(\alpha)| \leq \frac{\lambda}{n} (\inf |c_t|)^{-1} |c_s|_{2,\infty}^2 (1 + |c_t|_{1,\infty})^3,$$

where  $E$  and  $E^n$  are the energies with respect to metrics  $G$  and  $G^n$  respectively and where

$$|c_t|_{1,\infty} := |c_t|_\infty + |\nabla_t c_t|_\infty,$$

$$|c_s|_{2,\infty} := |c_s|_\infty + |\nabla_t c_s|_\infty + |\nabla_t^2 c_s|_\infty,$$

if  $|w|_\infty := \sup_{s,t \in [0,1]} |w(s, t)|$  denotes the supremum over both  $s$  and  $t$  of a vector field  $w$  along  $c$ .

**Remark 3.** Note that since we assume that  $c$  is a  $C^1$ -path of  $C^2$ -curves, the following norms are bounded for  $i = 1, 2$ ,

$$|c_t|_\infty, |c_s|_\infty, |\nabla_t^i c_t|_\infty, |\nabla_t^i c_s|_\infty < \infty.$$

*Proof of Theorem 1.* For the sake of clarity, the proof is put off to Section 5.  $\square$

Now that we have established a formal Riemannian setting to study discrete curves defined by a series of points, and that we have studied its link to the continuous model, we need to derive the equations of the corresponding geodesics and Jacobi fields to apply the methods described in Section 2. For the sake of readability, we first introduce some notations.

### 3.2. Computing geodesics in the discrete setting.

**3.2.1. Notations.** The purpose of the notations that we introduce here is to lighten the equations derived in the rest of the paper. For any discrete curve  $\alpha = (x_0, \dots, x_n) \in M^{n+1}$  we define for all  $0 \leq k \leq n$ , using the coefficients  $a_k$  and  $b_k$  defined by (14) and (12), the functions  $f_k, g_k : T_{x_k}M \rightarrow T_{x_k}M$ ,

$$\begin{aligned} f_k &: w \mapsto w^T + a_k w^N, \\ g_k &: w \mapsto |q_k|(2w^T + b_k w^N). \end{aligned}$$

and for  $k = 0, \dots, n-1$ , the functions  $f_k^{(-)}, g_k^{(-)} : T_{x_{k+1}}M \rightarrow T_{x_k}M$  by

$$f_k^{(-)} = f_k \circ P_{\gamma_k}^{x_{k+1}, x_k}, \quad g_k^{(-)} = g_k \circ P_{\gamma_k}^{x_{k+1}, x_k},$$

where  $\gamma_k$  denotes the geodesic between  $x_k$  and  $x_{k+1}$ , which we previously assumed existed. Notice that when the discretization gets "thinner", that is  $n \rightarrow \infty$ ,  $|\tau_k| \rightarrow 0$  while  $n|\tau_k|$  stays bounded for all  $0 \leq k \leq n$ , we get in the non flat setting, for any fixed  $w \in T_{x_{k+1}}M$ ,  $f_k(w) = w + o(1/n)$  and  $g_k(w) = |q_k|(w + w^T) + o(1/n)$  - in the flat setting, these are always equalities. Now if we consider a path  $s \mapsto \alpha(s) = (x_0(s), \dots, x_n(s))$  of discrete curves, we can define for each  $s$  the functions

$$\begin{aligned} f_k(s), g_k(s) &: T_{x_k(s)}M \rightarrow T_{x_k(s)}M, \quad 0 \leq k \leq n, \\ f_k(s)^{(-)}, g_k(s)^{(-)} &: T_{x_{k+1}(s)}M \rightarrow T_{x_k(s)}M, \quad 0 \leq k \leq n-1, \end{aligned}$$

corresponding to the discrete curve  $\alpha(s)$ . It is of interest for the rest of this paper to compute the covariant derivatives of these maps with respect to  $s$ .

**Lemma 2.** *The first and second order covariant derivatives of  $f_k$  and  $g_k$  with respect to  $s$  are functions  $T_{x_k(s)}M \rightarrow T_{x_k(s)}M$  defined by*

$$\begin{aligned} \nabla_s f_k &: w \mapsto \partial_s a_k w^N + (1 - a_k)(\langle w, \nabla_s v_k \rangle v_k + \langle w, v_k \rangle \nabla_s v_k), \\ \nabla_s g_k &: w \mapsto \partial_s |q_k|/|q_k| g_k(w) + |q_k| \partial_s b_k w^N + |q_k|(2 - b_k)(\langle w, \nabla_s v_k \rangle v_k + \langle w, v_k \rangle \nabla_s v_k), \\ \nabla_s \nabla_s f_k &: w \mapsto \partial_s^2 a_k w^N - 2\partial_s a_k (\langle w, \nabla_s v_k \rangle v_k + \langle w, v_k \rangle \nabla_s v_k) \\ &\quad + (1 - a_k)(\langle w, \nabla_s \nabla_s v_k \rangle v_k + 2\langle w, \nabla_s v_k \rangle \nabla_s v_k + \langle w, v_k \rangle \nabla_s \nabla_s v_k), \\ \nabla_s \nabla_s g_k &: w \mapsto \partial_s (\partial_s |q_k|/|q_k|) g_k(w) + \partial_s |q_k|/|q_k| \nabla_s g_k(w) + (\partial_s |q_k| \partial_s b_k + |q_k| \partial_s^2 b_k) w^N \\ &\quad + |q_k|(2 - b_k)(\langle w, \nabla_s \nabla_s v_k \rangle v_k + 2\langle w, \nabla_s v_k \rangle \nabla_s v_k + \langle w, v_k \rangle \nabla_s \nabla_s v_k) \\ &\quad + (\partial_s |q_k|(2 - b_k) - 2|q_k| \partial_s b_k)(\langle w, \nabla_s v_k \rangle v_k + \langle w, v_k \rangle \nabla_s v_k). \end{aligned}$$

*Proof.* For any vector field  $s \mapsto w(s) \in T_{x_k(s)}M$  along  $s \mapsto x_k(s)$  we have by definition

$$\begin{aligned} \nabla_s (f_k(w)) &= \nabla_s f_k(w) + f_k(\nabla_s w), \\ \nabla_s (g_k(w)) &= \nabla_s g_k(w) + g_k(\nabla_s w), \\ \nabla_s \nabla_s (f_k(w)) &= \nabla_s \nabla_s f_k(w) + 2 \nabla_s f_k(\nabla_s w) + f_k(\nabla_s \nabla_s w), \\ \nabla_s \nabla_s (g_k(w)) &= \nabla_s \nabla_s g_k(w) + 2 \nabla_s g_k(\nabla_s w) + g_k(\nabla_s \nabla_s w). \end{aligned}$$

Noticing that  $\nabla_s(w^T) = (\nabla_s w)^T + \langle w, \nabla_s v_k \rangle v_k + \langle w, v_k \rangle \nabla_s v_k$  and  $\nabla_s(w^N) = \nabla_s w - \nabla_s(w^T)$ , the formulas given in Lemma 2 result from simple calculation.  $\square$

Using these functions, we can deduce the covariant derivatives of  $f_k^{(-)}$  and  $g_k^{(-)}$ . Denoting by  $\gamma_k(s)$  the geodesic of  $M$  linking  $x_k(s)$  to  $x_{k+1}(s)$  for all  $s \in [0, 1]$  and  $0 \leq k \leq n-1$ , we have the following result.

**Lemma 3.** *The covariant derivatives of the functions  $f_k^{(-)}$  and  $g_k^{(-)}$  with respect to  $s$  are functions  $T_{x_{k+1}(s)}M \rightarrow T_{x_k(s)}M$  given by*

$$\begin{aligned} \nabla_s(f_k^{(-)}) &: w \mapsto (\nabla_s f_k)^{(-)}(w) + f_k(\mathcal{R}(Y_k, \tau_k)(w_{k+1}^{\parallel})), \\ \nabla_s(g_k^{(-)}) &: w \mapsto (\nabla_s g_k)^{(-)}(w) + g_k(\mathcal{R}(Y_k, \tau_k)(w_{k+1}^{\parallel})), \end{aligned}$$

where  $(\nabla_s f_k)(s)^{(-)} = \nabla_s f_k(s) \circ P_{\gamma_k(s)}^{x_{k+1}(s), x_k(s)}$ ,  $(\nabla_s g_k)(s)^{(-)} = \nabla_s g_k(s) \circ P_{\gamma_k(s)}^{x_{k+1}(s), x_k(s)}$ , and

$$(18) \quad Y_k = (x_k')^T + b_k(x_k')^N + \frac{1}{2} \nabla_s \tau_k^T + K \frac{1 - a_k}{|\tau_k|^2} \nabla_s \tau_k^N,$$

if  $K$  is the sectional curvature of the base manifold.

*Proof.* The proof is given in Appendix A.  $\square$

**3.2.2. Geodesic equations and exponential map.** With these notations, we can characterize the geodesics for metric  $G^n$ . The geodesic equations can be derived in a similar way as in the continuous case, that is by searching for the critical points of the energy (17). We obtain the following characterization in terms of the SRV representation (15).

**Proposition 6** (Discrete geodesic equations). *A path  $s \mapsto \alpha(s) = (x_0(s), \dots, x_n(s))$  in  $M^{n+1}$  is a geodesic for metric  $G^n$  if and only if its SRV representation  $s \mapsto (x_0(s), (q_k(s))_k)$  verifies the following differential equations*

$$(19) \quad \begin{aligned} \nabla_s x_0' + \frac{1}{n} \left( R_0 + f_0^{(-)}(R_1) + \dots + f_0^{(-)} \circ \dots \circ f_{n-2}^{(-)}(R_{n-1}) \right) &= 0, \\ \nabla_s \nabla_s q_k + \frac{1}{n} g_k^{(-)} \left( R_{k+1} + f_{k+1}^{(-)}(R_{k+2}) + \dots + f_{k+1}^{(-)} \circ \dots \circ f_{n-2}^{(-)}(R_{n-1}) \right) &= 0, \end{aligned}$$

for all  $k = 0, \dots, n-1$ , with the notations (12) and  $R_k := \mathcal{R}(q_k, \nabla_s q_k) x_k'$ .

*Proof.* The proof is given in Appendix B.  $\square$

**Remark 4.** Let  $[0, 1] \ni s \mapsto c(s, \cdot) \in \mathcal{M}$  be a  $C^1$  path of smooth curves and  $[0, 1] \ni s \mapsto \alpha(s) \in M^{n+1}$  the discretization of size  $n$  of  $c$ . We denote as usual by  $q := c_t/|c_t|^{1/2}$  and  $(q_k)_k$  their respective SRV representations. When  $n \rightarrow \infty$  and  $|\tau_k| \rightarrow 0$  while  $n|\tau_k|$  stays bounded for all  $0 \leq k \leq n$ , the coefficients of the discrete geodesic equation (19) for  $\alpha$  converge to the coefficients of the continuous geodesic equation (6) for  $c$ , i.e.

$$\begin{aligned} \nabla_s x_0'(s) &= -r_0(s) + o(1), \\ \nabla_s \nabla_s q_k(s) &= -|q_k(s)|(r_k(s) + r_k(s)^T) + o(1), \end{aligned}$$

for all  $s \in [0, 1]$  and  $k = 0, \dots, n-1$ , where  $r_{n-1} = 0$  and for  $k = 1, \dots, n-2$ ,

$$r_k(s) := \frac{1}{n} \sum_{\ell=k+1}^{n-1} P_c^{\frac{1}{n}, \frac{k}{n}} \left( \mathcal{R}(q, \nabla_s q) c_s(s, \frac{\ell}{n}) \right) \xrightarrow{n \rightarrow \infty} r(s, \frac{k}{n}),$$

with the exception that the sum starts at  $\ell = 0$  for  $r_0$ . More details on this can be found in Appendix B.

**Remark 5.** Just as in the continuous case, when the base manifold is a Euclidean space, the curvature terms  $R_k$ 's vanish and we obtain

$$x_0''(s) = 0, \quad q_k''(s) = 0, \quad k = 0, \dots, n-1, \quad \forall s \in [0, 1]$$

i.e. the geodesics are those for which the SRV representations are  $L^2$ -geodesics.

Using equations (19) we can now build the exponential map, that is, an algorithm allowing us to approximate the geodesic of  $M^{n+1}$  starting from a point  $(x_0^0, \dots, x_n^0) \in M^{n+1}$  at speed  $(u_0, \dots, u_n)$  with  $u_k \in T_{x_k}M$  for all  $k = 0, \dots, n$ . In other words, we are looking for a path  $[0, 1] \ni s \mapsto \alpha(s) = (x_0(s), \dots, x_n(s))$  such that  $x_k(0) = x_k^0$  and  $x_k'(0) = u_k$  for all  $k$ , and that verifies the geodesic equations (19). Assume that we know at time  $s \in [0, 1]$  the values of  $x_k(s)$  and  $x_k'(s)$  for all  $k = 0, \dots, n$ . Then we propagate using

$$\begin{aligned} x_k(s + \epsilon) &= \log_{x_k(s)}^M \epsilon x_k'(s), \\ x_k'(s + \epsilon) &= (x_k'(s) + \epsilon \nabla_s x_k'(s))^{s, s+\epsilon}. \end{aligned}$$

In the following proposition, we see how we can compute the acceleration  $\nabla_s x_k'$  for each  $k$ .

**Proposition 7** (Discrete exponential map). *Let  $[0, 1] \ni s \mapsto \alpha(s) = (x_0(s), \dots, x_n(s))$  be a geodesic path in  $M^{n+1}$ . For all  $s \in [0, 1]$ , the coordinates of its acceleration  $\nabla_s \alpha'(s)$  can be iteratively computed in the following way*

$$\begin{aligned} \nabla_s x_0' &= -\frac{1}{n} \left( R_0 + f_0^{(-)}(R_1) + \dots + f_0^{(-)} \circ \dots \circ f_{n-2}^{(-)}(R_{n-1}) \right), \\ \nabla_s x_{k+1}' \parallel &= \nabla_s f_k(x_k') + f_k(\nabla_s x_k') + \frac{1}{n} \nabla_s g_k(\nabla_s q_k) + \frac{1}{n} g_k(\nabla_s \nabla_s q_k) + \mathcal{R}(\tau_k, Y_k)(x_{k+1}' \parallel), \end{aligned}$$

for  $k = 0, \dots, n-1$ , where the  $R_k$ 's are defined as in Proposition 6, the symbol  $\parallel$  denotes the parallel transport from  $x_{k+1}(s)$  back to  $x_k(s)$  along the geodesic linking them, the maps  $\nabla_s f_k$  and  $\nabla_s g_k$  are given by Lemma 2,  $Y_k$  is given by Equation (18) and

$$\begin{aligned} \nabla_s \tau_k &= (D_\tau \alpha')_k, \quad \nabla_s q_k = \sqrt{\frac{n}{|\tau_k|}} \left( \nabla_s \tau_k - \frac{1}{2} \nabla_s \tau_k^T \right), \quad \nabla_s v_k = \frac{1}{|\tau_k|} (\nabla_s \tau_k - \nabla_s \tau_k^T), \\ \nabla_s \nabla_s q_k &= -\frac{1}{n} g_k^{(-)} \left( R_{k+1} + f_{k+1}^{(-)}(R_{k+2}) + \dots + f_{k+1}^{(-)} \circ \dots \circ f_{n-2}^{(-)}(R_{n-1}) \right). \end{aligned}$$

*Proof.* The proof is given in Appendix B.  $\square$

The equations of Proposition 7 allow us to iteratively construct a geodesic  $s \mapsto \alpha(s)$  in  $M^{n+1}$  for metric  $G^n$  from the knowledge of its initial conditions  $\alpha(0)$  and  $\alpha'(0)$ . The next step is to construct geodesics under boundary constraints, i.e. to find the shortest path between two elements  $\alpha_0$  and  $\alpha_1$  of  $M^{n+1}$ .

**3.2.3. Jacobi fields and geodesic shooting.** As explained in Section 2.2.2 for the continuous model, we solve the boundary value problem using geodesic shooting. To do so, recall that we need to characterize the Jacobi fields for the metric  $G^n$ , since these play a role in the correction of the shooting direction at each iteration of the algorithm. Jacobi fields are vector fields that describe the way that geodesics spread out in the Riemannian manifold: for any geodesic  $s \mapsto \alpha(s)$  in  $M^{n+1}$  and Jacobi field  $s \mapsto J(s)$  along  $\alpha$ , there exists a family of geodesics  $(-\delta, \delta) \ni a \mapsto \alpha(a, \cdot)$  such that  $\alpha(0, s) = \alpha(s)$  for all  $s$  and

$$J(s) = \frac{\partial}{\partial a} \Big|_{a=0} \alpha(a, s).$$

**Proposition 8** (Discrete Jacobi fields). *Let  $[0, 1] \ni s \mapsto \alpha(s) = (x_0(s), \dots, x_n(s))$  be a geodesic path in  $M^{n+1}$ ,  $[0, 1] \ni s \mapsto J(s) = (J_0(s), \dots, J_n(s))$  a Jacobi field along  $\alpha$ , and  $(-\delta, \delta) \ni a \mapsto \alpha(a, \cdot)$  a corresponding family of geodesics, in the sense just described. Then  $J$  verifies the second order linear ODE*

$$\begin{aligned} \nabla_s \nabla_s J_0 &= \mathcal{R}(x_0', J_0) x_0' - \frac{1}{n} \left( \nabla_a R_0 + f_0^{(-)}(\nabla_a R_1) + \dots + f_0^{(-)} \circ \dots \circ f_{n-2}^{(-)}(\nabla_a R_{n-1}) \right) \\ &\quad - \frac{1}{n} \sum_{k=0}^{n-2} \sum_{\ell=0}^k f_0^{(-)} \circ \dots \circ \nabla_a (f_\ell^{(-)}) \circ \dots \circ f_k^{(-)}(R_{k+1}), \end{aligned}$$

$$\begin{aligned} \nabla_s \nabla_s J_{k+1}^\parallel &= f_k(\nabla_s \nabla_s J_k) + 2\nabla_s f_k(\nabla_s J_k) + \nabla_s \nabla_s f_k(J_k) + \frac{1}{n} g_k(\nabla_s \nabla_s \nabla_a q_k) \\ &\quad + \frac{2}{n} \nabla_s g_k(\nabla_s \nabla_a q_k) + \frac{1}{n} \nabla_s \nabla_s g_k(\nabla_a q_k) + 2\mathcal{R}(\tau_k, Y_k)(\nabla_s J_{k+1}^\parallel) \\ &\quad + \mathcal{R}(\nabla_s \tau_k, Y_k)(J_{k+1}^\parallel) + \mathcal{R}(\tau_k, \nabla_s Y_k)(J_{k+1}^\parallel) + \mathcal{R}(\tau_k, Y_k)\left(\mathcal{R}(Y_k, \tau_k)(J_{k+1}^\parallel)\right), \end{aligned}$$

for all  $0 \leq k \leq n-1$ , where  $R_k := \mathcal{R}(q_k, \nabla_s q_k)x_k'$  and the various covariant derivatives according to  $a$  can be expressed as functions of  $J$  and  $\nabla_s J$ ,

$$\begin{aligned} \nabla_a R_k &= \mathcal{R}(\nabla_a q_k, \nabla_s q_k)x_k' + \mathcal{R}(q_k, \nabla_s \nabla_a q_k + \mathcal{R}(J, x_k')q_k)x_k' + \mathcal{R}(q_k, \nabla_s q_k)\nabla_s J_k, \\ \nabla_a q_k &= \sqrt{\frac{n}{|\tau_k|}} \left( \nabla_a \tau_k - \frac{1}{2} \nabla_a \tau_k^T \right), \quad \nabla_a \tau_k = (D_\tau J)_k, \quad \nabla_a v_k = \frac{1}{|\tau_k|} (\nabla_a \tau_k - \nabla_a \tau_k^T), \\ \nabla_s \nabla_a q_k &= n g_k^{-1} ((\nabla_s J_{k+1})^\parallel + \mathcal{R}(Y_k, \tau_k)(J_{k+1}^\parallel) - \nabla_s f_k(J_k) - f_k(\nabla_s J_k)) \\ &\quad + n \nabla_s (g_k^{-1})(J_{k+1}^\parallel - f_k(J_k)), \\ \nabla_s \nabla_s \nabla_a q_k &= \mathcal{R}(\nabla_s x_k', J_k)q_k + \mathcal{R}(x_k', \nabla_s J_k)q_k + 2\mathcal{R}(x_k', J_k)\nabla_s q_k \\ &\quad - \frac{1}{n} \sum_{\ell=k+1}^{n-1} g_k^{(-)} \circ f_{k+1}^{(-)} \circ \dots \circ f_{\ell-1}^{(-)}(\nabla_a R_\ell) - \frac{1}{n} \sum_{\ell=k+1}^{n-1} \sum_{j=k}^{\ell-1} g_k^{(-)} \circ \dots \circ \nabla_a (f_j^{(-)}) \circ \dots \circ f_{\ell-1}^{(-)}(R_\ell), \\ \nabla_s Y_k &= (\nabla_s x_k')^T + b_k(\nabla_s x_k')^N + (1-b_k)(\langle x_k', \nabla_s v_k \rangle v_k + \langle x_k', v_k \rangle \nabla_s v_k) + \partial_s b_k(x_k')^N \\ &\quad + \frac{1}{2}(\nabla_s \nabla_s \tau_k)^T + K \frac{1-a_k}{|\tau_k|^2} (\nabla_s \nabla_s \tau_k)^N + \partial_s \left( K \frac{1-a_k}{|\tau_k|^2} \right) (\nabla_s \tau_k)^N \\ &\quad + \left( \frac{1}{2} - K \frac{1-a_k}{|\tau_k|^2} \right) (\langle \nabla_s \tau_k, \nabla_s v_k \rangle v_k + \langle \nabla_s \tau_k, v_k \rangle \nabla_s v_k), \end{aligned}$$

with the notation conventions  $f_{k+1}^{(-)} \circ \dots \circ f_{k-1}^{(-)} := Id$ ,  $\sum_{\ell=n}^{n-1} := 0$  and with the maps

$$\begin{aligned} \nabla_a (f_k^{(-)})(w) &= (\nabla_a f_k)^{(-)}(w) + f_k \left( \mathcal{R}(Z_k, \tau_k)(w_{k+1}^\parallel) \right), \\ \nabla_a (g_k^{(-)})(w) &= (\nabla_a g_k)^{(-)}(w) + g_k \left( \mathcal{R}(Z_k, \tau_k)(w_{k+1}^\parallel) \right), \\ \nabla_s (g_k^{-1})(w) &= \partial_s (|q_k|^{-1}) |q_k| g_k^{-1}(w) + |q_k|^{-1} \partial_s (b_k^{-1}) w^N \\ &\quad + |q_k|^{-1} (1/2 - b_k^{-1}) (\langle w, \nabla_s v_k \rangle v_k + \langle w, v_k \rangle \nabla_s v_k), \end{aligned}$$

and

$$Z_k = J_k^T + b_k J_k^N + \frac{1}{2} \nabla_a \tau_k^T + K \frac{1-a_k}{|\tau_k|^2} \nabla_a \tau_k^N.$$

*Proof.* The proof is given in Appendix B.  $\square$

The equations of Proposition 8 allow us to iteratively compute the Jacobi field  $J$  along a geodesic  $\alpha$  - and in particular, its end value  $J(1)$  - from the knowledge of the initial conditions  $\{J_k(0), 0 \leq k \leq n\}$  and  $\{\nabla_s J_k(0), 0 \leq k \leq n\}$ . Indeed, if at time  $s \in [0, 1]$  we have  $J_k(s)$  and  $\nabla_s J_k(s)$  for all  $k = 0, \dots, n$ , then we can propagate using

$$\begin{aligned} J_k(s + \epsilon) &= \left( J_k(s) + \epsilon \nabla_s J_k(s) \right)^{x_k, x_{k+1}}, \\ \nabla_s J_k(s + \epsilon) &= \left( \nabla_s J_k(s) + \epsilon \nabla_s \nabla_s J_k(s) \right)^{x_k, x_{k+1}}, \end{aligned}$$

where  $\nabla_s \nabla_s J_k(s)$  is deduced from  $\nabla_s \nabla_s J_{k-1}(s)$  using Proposition 8. We can now apply Algorithm 1, where we replace the smooth geodesic equations (6) by the discrete geodesic equations (19) and we solve them using the exponential map described in Proposition 7. Notice that in  $M^{n+1}$ , the  $k^{\text{th}}$  component of the  $L^2$ -logarithm map between two elements  $\alpha_0 = (x_0^0, \dots, x_n^0)$  and  $\alpha_1 = (x_0^1, \dots, x_n^1)$  is given by  $\log_{x_k^0}^M(x_k^1)$ .

**Algorithm 3** (Geodesic shooting in  $M^{n+1}$ ). Let  $(x_0^0, \dots, x_n^0), (x_0^1, \dots, x_n^1) \in M^{n+1}$ . Set  $u = \log_{\alpha_0}^{L^2}(\alpha_1)$  and repeat until convergence :

- (1) compute the geodesic  $s \mapsto \alpha(s)$  starting from  $\alpha_0$  at speed  $u$  using Proposition 7,
- (2) evaluate the difference  $j := \log_{\mathfrak{S}_{\alpha(1)}}^{L^2}(\alpha_1)$  between the target curve  $\alpha_1$  and the extremity  $\alpha(1)$  of the obtained geodesic,
- (3) compute the initial derivative  $\nabla_s J(0)$  of the Jacobi field  $s \mapsto J(s)$  along  $\alpha$  verifying  $J(0) = 0$  and  $J(1) = j$ ,
- (4) correct the shooting direction  $u = u + \nabla_s J(0)$ .

Recall that the map  $\phi : T_{\alpha(0)}\mathcal{M} \rightarrow T_{\alpha(1)}\mathcal{M}$ ,  $\nabla_s J(0) \mapsto J(1)$  associating to the initial derivative  $\nabla_s J(0)$  of a Jacobi field with initial value  $J(0) = 0$  its end value  $J(1)$ , is a linear bijection between two vector spaces which can be obtained using Proposition 8. Its inverse map can be computed by considering the image of a basis of  $T_{c(0)}\mathcal{M}$ .

**3.3. A discrete analog of unparameterized curves.** The final step in building our discrete model is to introduce a discretization of the quotient shape space. There seems to be no natural, intrinsic definition of the shape of a discrete curve, as by definition we are lacking information : we only have access to a finite number  $n + 1$  of points. Therefore to introduce our model, we will make the assumption that we know the equations of the underlying curves, that is, that for each discrete curve  $\alpha$ , we have access to the shape  $\bar{c}$  of the smooth curve  $c$  of which  $\alpha$  is the discretization. In applications, if we don't have access to this information, we can set  $\bar{c}$  to be the shape of an optimal interpolation. Recall that  $\alpha = (x_0, \dots, x_n)$  is the discretization of size  $n$  of  $t \mapsto c(t)$  if  $c(k/n) = x_k$  for all  $0 \leq k \leq n$ . For an element  $\bar{c}$  of the shape space  $\mathcal{S} = \mathcal{M}/\text{Diff}^+([0, 1])$ , we denote by

$$\text{Disc}_n(\bar{c}) := \{\alpha \in M^{n+1} : \exists c \in \pi^{-1}(\bar{c}), \alpha \text{ is the discretization of size } n \text{ of } c\}$$

the set of its discretizations, i.e. the set of elements of  $M^{n+1}$  that are discretizations of smooth curves with shape  $\bar{c}$ . Recall that  $\pi$  is the natural projection  $\mathcal{M} \rightarrow \mathcal{S}$ . We denote by  $\mathcal{M}_n$  the set of discrete curves of size  $n$  paired up with their underlying shapes

$$\mathcal{M}_n := \{(\alpha, \bar{c}) \in M^{n+1} \times \mathcal{S} : \alpha \in \text{Disc}_n(\bar{c})\}.$$

The goal, for two elements  $(\alpha_0, \bar{c}_0)$  and  $(\alpha_1, \bar{c}_1)$  of  $\mathcal{M}_n$ , is to redistribute the  $n + 1$  points on  $\bar{c}_1$  to minimize the discrete distance to the  $n + 1$  points  $\alpha_0$  on  $\bar{c}_0$ , i.e. to find the optimal discretization  $\alpha_1^{\text{opt}}$  of  $\bar{c}_1$

$$(20) \quad \alpha_1^{\text{opt}} = \text{argmin}\{d_n(\alpha_0, \alpha), \alpha \in \text{Disc}_n(\bar{c}_1)\},$$

where  $d_n$  is the geodesic distance associated to the discrete metric  $G^n$ . We want to approach this reparameterization  $\alpha_1^{\text{opt}}$  using Algorithm 2, i.e. by iteratively computing the "horizontal part" of the geodesic linking  $\alpha_0$  to an iteratively improved discretization  $\tilde{\alpha}_1$  of  $\bar{c}_1$ . To define the horizontal part of a path of discrete curves, we need two things : a notion of horizontality, and a notion of reparameterization. Let us start with the former. We define the discrete vertical and horizontal spaces in  $\alpha$  as the following subsets of  $T_\alpha M^{n+1}$

$$\text{Ver}_\alpha^n := \{mv : m = (m_k)_k \in \mathbb{R}^{n+1}\},$$

$$\text{Hor}_\alpha^n := \{h \in T_\alpha M^{n+1} : G^n(h, mv) = 0 \quad \forall m = (m_k)_k \in \mathbb{R}^{n+1}, m_0 = m_n = 0\},$$

where  $v = (v_k)_k$  is still defined by (12). Similarly to the continuous case, we can show the following result.

**Proposition 9** (Discrete horizontal space and decomposition of a vector). *Let  $\alpha \in M^{n+1}$  and  $h \in T_\alpha M^{n+1}$ . Then  $h \in \text{Hor}_\alpha^n$  if and only if*

$$\langle (D_\tau h)_k, v_k \rangle - 4 \frac{|\tau_k|}{|\tau_{k-1}|} \left\langle (D_\tau h)_{k-1}, b_{k-1}^{-1} v_k \right\rangle + \left( \frac{1}{4} - b_{k-1}^{-1} \right) \lambda_{k-1} v_{k-1} = 0.$$

Any tangent vector  $w \in T_\alpha M^{n+1}$  can be uniquely decomposed into a sum  $w = w^{\text{ver}} + w^{\text{hor}}$  where  $w^{\text{ver}} = mv \in \text{Ver}_\alpha^n$ ,  $w^{\text{hor}} = w - mv \in \text{Hor}_\alpha^n$  and the components  $(m_k)_k$  verify



$m_0 = m_1 = 0$  and the following recurrence relation

$$\begin{aligned} & \lambda_k m_{k+1} - \left(1 + 4 \frac{|\tau_k|}{|\tau_{k-1}|} (b_{k-1}^{-2} + \lambda_{k-1}^2 (\frac{1}{4} - b_{k-1}^{-2}))\right) m_k + \frac{|\tau_k|}{|\tau_{k-1}|} \lambda_{k-1} m_{k-1} \\ &= \langle (D_\tau w)_k, v_k \rangle - 4 \frac{|\tau_k|}{|\tau_{k-1}|} \left( b_{k-1}^{-1} \langle (D_\tau w)_{k-1}, v_k^\parallel \rangle + (\frac{1}{4} - b_{k-1}^{-1}) \lambda_{k-1} \langle (D_\tau w)_{k-1}, v_{k-1} \rangle \right), \end{aligned}$$

with the notation  $\lambda_k := \langle v_{k+1}^\parallel, v_k \rangle$ .

*Proof.* Let  $h \in T_\alpha \mathcal{M}$  be a tangent vector. It is horizontal if and only if it is orthogonal to any vertical vector, that is any vector of the form  $mv$  with  $m = (m_k)_k \in \mathbb{R}^{n+1}$  such that  $m_0 = m_n = 0$ . Recall that by definition

$$(D_\tau w)_k := (w_{k+1}^\parallel - w_k)^T + b_k^{-1} (w_{k+1}^\parallel - a_k w_k)^N,$$

and so with the notation  $\lambda_k := \langle v_{k+1}^\parallel, v_k \rangle$ , we get

$$\begin{aligned} (D_\tau(mv))_k^T &= m_{k+1} (v_{k+1}^\parallel)^T - m_k v_k = (m_{k+1} \lambda_k - m_k) v_k, \\ (D_\tau(mv))_k^N &= b_k^{-1} m_{k+1} (v_{k+1}^\parallel)^N = b_k^{-1} m_{k+1} (v_{k+1}^\parallel - \lambda_k v_k). \end{aligned}$$

The scalar product between  $h$  and  $mv$  can then be written

$$\begin{aligned} G_\alpha^n(h, mv) &= \sum_{k=0}^{n-1} \left( b_k^{-1} m_{k+1} \langle (D_\tau h)_k, v_{k+1}^\parallel - \lambda_k v_k \rangle + \frac{1}{4} (m_{k+1} \lambda_k - m_k) \langle (D_\tau h)_k, v_k \rangle \right) |\tau_k|^{-1} \\ &= \sum_{k=0}^{n-1} m_{k+1} |\tau_k|^{-1} \left( b_k^{-1} \langle (D_\tau h)_k, v_{k+1}^\parallel - \lambda_k v_k \rangle + \frac{1}{4} \lambda_k \langle (D_\tau h)_k, v_k \rangle \right) \\ &\quad - \frac{1}{4} \sum_{k=0}^{n-1} m_k |\tau_k|^{-1} \langle (D_\tau h)_k, v_k \rangle \end{aligned}$$

Changing the indices in the first sum and taking into account that  $m_0 = m_n = 0$ , we obtain

$$\sum_{k=1}^{n-1} m_k \left( |\tau_{k-1}|^{-1} \langle (D_\tau h)_{k-1}, b_{k-1}^{-1} v_k^\parallel + (\frac{1}{4} - b_{k-1}^{-1}) \lambda_{k-1} v_{k-1} \rangle - \frac{1}{4} |\tau_k|^{-1} \langle (D_\tau h)_k, v_k \rangle \right) = 0.$$

Since this is true for all such  $m$  the summand is equal to zero for all  $k$  and we get the desired equation. The decomposition of a tangent vector  $w$  into a vertical part  $mv$  and a horizontal part  $w - mv$  with  $m = (m_k)_k \in \mathbb{R}^{n+1}$  such that  $m_0 = m_n = 0$ , is then simply characterized by the fact that  $w - mv$  verifies this equation.  $\square$

Now let us define a discrete analog of the reparameterization action. Let us fix an integer  $p \in \mathbb{N}^*$  and set  $N := np$ . To each element  $(\alpha = (x_0, \dots, x_n), \bar{c})$  of  $\mathcal{M}_n$ , we associate the unique discretization  $\beta = (y_0, \dots, y_n) \in \text{Disc}_N(\bar{c})$  of size  $N$ , such that  $y_{kp} = x_k$  for  $k = 0, \dots, n$  and the  $p-1$  points  $\{y_\ell, kp < \ell < (k+1)p\}$  are regularly distributed on  $\bar{c}$  between  $y_{kp}$  and  $y_{(k+1)p}$  for all  $k$ . In other words,  $\beta$  is the discretization of size  $N$  of the only parameterized curve  $c \in \pi^{-1}(\bar{c})$  of which  $\alpha$  is a discretization and which is parametrized by arc length on the segments  $c|_{[k/n, (k+1)/n]}$ ,  $0 \leq k \leq n$ .

**Definition 3.** We call the discrete curve  $\beta \in \text{Disc}_N(\bar{c})$  the *refinement of size  $N$*  of  $(\alpha, \bar{c})$ .

The discrete analogs of the increasing diffeomorphisms of the continuous case are defined as increasing injections  $\phi : \{0, \dots, n\} \rightarrow \{0, \dots, N\}$  such that  $\phi(0) = 0$  and  $\phi(n) = N$ . Their set is denoted by  $\text{Inj}^+(n, N)$ . We then define the discrete analog of reparameterizing as the action  $\star : \text{Inj}^+(n, N) \times \mathcal{M}_n \rightarrow \mathcal{M}_n$ ,

$$\phi \star ((x_k)_k, \bar{c}) := ((\tilde{x}_k)_k, \bar{c}), \quad \text{with } \tilde{x}_k := y_{\phi(k)}, k = 0, \dots, n,$$

where  $(y_k)_k$  is the refinement of size  $N$  of  $((x_k)_k, \bar{c})$ . Note that the action of  $\phi$  is non transitive. This definition of reparameterization in the discrete case simply boils down to

redistributing the  $n + 1$  points on  $\bar{c}$  by choosing among the  $N + 1$  points of the refinement of  $\alpha$ , while preserving the order and keeping the extremities fixed. We can now define the horizontal part of a path of discrete curves.

**Definition 4.** The *horizontal part*  $(\alpha^{hor}, \bar{c})$  of a path  $s \mapsto (\alpha(s), \bar{c}(s)) \in \mathcal{M}_n$  is defined by

$$(\alpha(s), \bar{c}(s)) := \phi(s) \star (\alpha^{hor}(s), \bar{c}(s)), \quad \forall s \in [0, 1],$$

where  $\phi(s) \in \text{Inj}^+(n, N)$  verifies for all  $s \in [0, 1]$

$$(21) \quad \phi_s(s)(k) = \frac{m_k(s)}{|n\tau_k(s)|} \Delta\phi(s)(k), \quad k = 0, \dots, n,$$

with  $\Delta\phi(s)(k) = N/2(\phi(s)(k+1) - \phi(s)(k-1))$ ,  $1 \leq k \leq N-1$  and where  $m = (m_k)_k$  is the norm of the vertical component of  $\alpha'(s)$  and verifies

$$(22) \quad \begin{aligned} & \lambda_k m_{k+1} - \left( 1 + 4 \frac{|\tau_k|}{|\tau_{k-1}|} (b_{k-1}^2 + \lambda_{k-1}^2 (\frac{1}{4} - b_{k-1}^2)) \right) m_k + \frac{|\tau_k|}{|\tau_{k-1}|} \lambda_{k-1} m_{k-1} \\ & = \langle \nabla_s \tau_k, v_k \rangle - 4 \frac{|\tau_k|}{|\tau_{k-1}|} \left( b_{k-1} \langle \nabla_s \tau_{k-1}, v_k^\parallel \rangle + (\frac{1}{4} - b_{k-1}) \lambda_{k-1} \langle \nabla_s \tau_{k-1}, v_{k-1} \rangle \right). \end{aligned}$$

**Remark 6.** Equation (21) defining the path of "reparameterizations"  $\phi$  is merely a discretization of Equation (11). The recurrence relation (22) verified by the  $m_k$ 's translates the fact that  $m(s)v(s)$  is the vertical component of  $\alpha'(s)$ , and as such it verifies the recurrence relation of Proposition 9, with  $(D_\tau \alpha'(s))_k = \nabla_s \tau_k(s)$  (16).

To find the horizontal part of a path of curves  $\alpha$ , we proceed as follows. For all  $s \in [0, 1]$ , we compute the values of  $\phi(s)(k)$ ,  $0 \leq k \leq n$  (which correspond to the  $n + 1$  points  $x_k(s) = y_{kp}(s)$ ) and interpolate between these values in order to have a value corresponding to each point  $y_\ell(s)$ ,  $0 \leq \ell \leq N$ , of the refinement of  $\alpha(s)$ . The  $k^{th}$  coordinate  $x_k^{hor}(s)$  of the horizontal part of  $\alpha(s)$  is chosen to be the point  $y_\ell(s)$  whose value is closest to  $kp$ . Now we can go back to our initial problem, which was, given two pairs of discrete curves and their underlying shapes  $(\alpha_0, \bar{c}_0)$  and  $(\alpha_1, \bar{c}_1)$ , to find the optimal reparameterization (20) of  $\bar{c}_1$  while fixing  $\alpha_0$ . We propose the following algorithm.

**Algorithm 4** (Computing the optimal matching between the shapes of discrete curves). Let  $(\alpha_0, \bar{c}_0), (\alpha_1, \bar{c}_1) \in \mathcal{M}_n$ . Set  $\tilde{\alpha}_1 = \alpha_1$  and repeat until convergence :

- Construct the geodesic  $s \mapsto \alpha(s)$  between  $\alpha_0$  and  $\tilde{\alpha}_1$  using Algorithm 3.
- Compute the horizontal part  $s \mapsto \alpha^{hor}(s)$  of  $\alpha$  and set  $\tilde{\alpha}_1 = \alpha^{hor}(1)$ .

Output :  $\alpha_1^{opt} := \tilde{\alpha}_1$ .

#### 4. SIMULATIONS

We test Algorithms 3 and 4 in three settings : the negative-curvature case, when the base manifold is the hyperbolic half-plane  $\mathbb{H}^2$ , the flat case  $M = \mathbb{R}^2$ , and the positive-curvature case  $M = \mathbb{S}^2$ . For the geometry of  $\mathbb{H}^2$  and the useful algorithms such as the exponential map and the logarithm map, we refer the reader to [7]. Concerning the geometry of  $\mathbb{S}^2$ , we have used the same formulas as those given in appendix in [20]. Results of geodesic shooting (Algorithm 3) in these three settings can be found in Figure 1. We show the geodesic paths in  $M^{n+1}$  with respect to the discrete metric  $G^n$  (in blue) and the  $L^2$ -metric (in green) between pairs of curves (in black), for  $M = \mathbb{H}^2$  (top row),  $M = \mathbb{R}^2$  (middle row) and  $M = \mathbb{S}^2$  (bottom row). The pairs of curves considered in the hyperbolic half-plane and the plane are the same, and so the differences observed are due to the different geometries. The flat case ( $M = \mathbb{R}^2$ ) allows us to validate our geodesic shooting algorithm. Indeed, we have an explicit equation for the geodesics in that case as they are simply the projections in  $M^{n+1}$  of the  $L^2$ -geodesics between the square root velocity representations of the curves, as stated in Remark 5. We can see that these exact geodesics (shown in red) are close to the geodesics obtain by geodesic shooting (in blue). In the three settings (negative, flat and

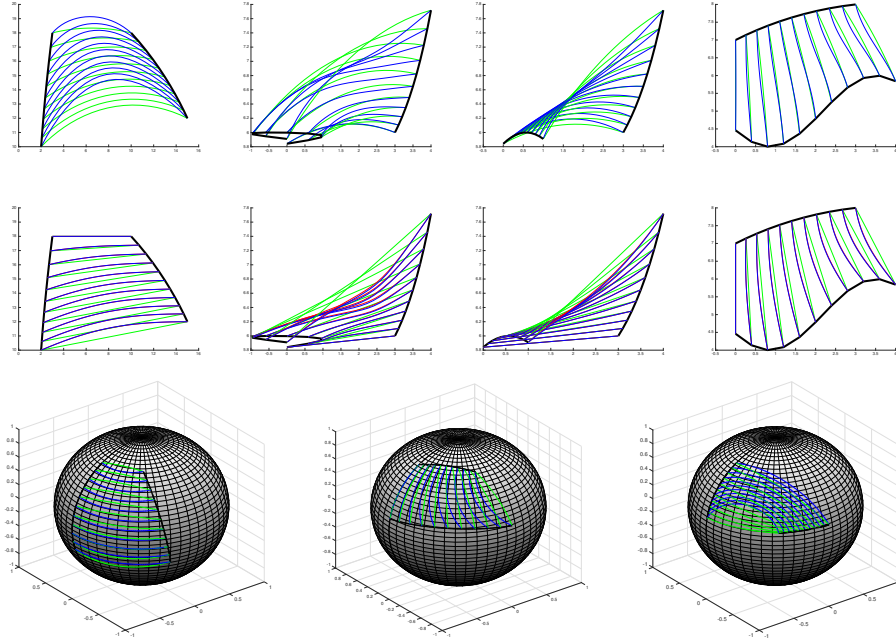


FIGURE 1. Geodesics (in blue) between parameterized curves in  $\mathbb{H}^2$  (top row),  $\mathbb{R}^2$  (middle row) and  $\mathbb{S}^2$  (bottom row) obtained by geodesic shooting. The  $L^2$ -geodesic is given in green for comparison. For plane curves, the geodesics computed using geodesic shooting are compared to the exact geodesics in red.

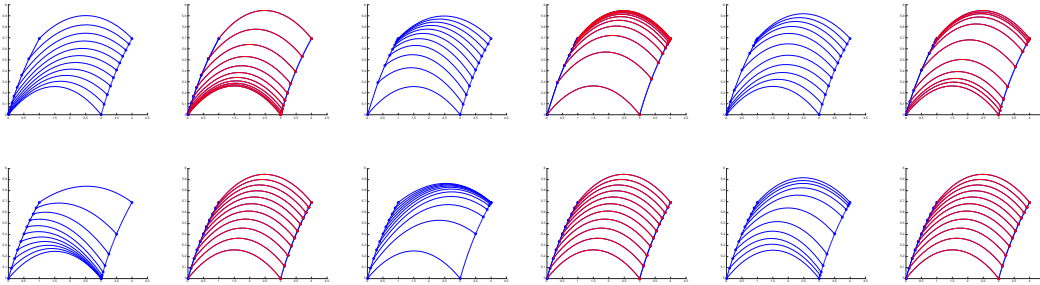


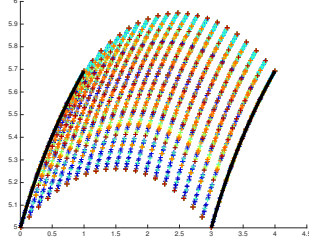
FIGURE 2. Geodesics between parameterized curves in  $\mathbb{H}^2$  (blue) and corresponding horizontal geodesics (red).

0.654	0.645	0.692	0.645	0.660	0.645
0.665	0.644	0.666	0.644	0.648	0.644

FIGURE 3. Length of the geodesics shown in Figure 2

positive curvature), we can observe that our metric has a tendency to "shrink" the curve as it optimally deforms from one state to another, compared to the  $L^2$ -metric.

We then tested Algorithm 4 to obtain geodesics between shapes of discrete curves. Results are shown in Figure 2, where the geodesics between parameterized curves of the hyperbolic half-plane are shown in blue and the horizontal geodesics are shown in red. We always consider the same pair of curves, which are identical modulo translation and parameterization,

FIGURE 4. Geodesics between the shapes of two elements of  $\mathbb{H}^{n+1}$  (in black)

and always fix the parameterization of the curve on the left-hand side, while searching for the optimal reparameterization of the curve on the right-hand side. On the top row, the points are "evenly distributed" along the latter, and on the top row, along the former. We can see that the red horizontal geodesics obtained as outputs of Algorithm 2 redistribute the points along the right-hand side curve in the way that seems natural : similarly to the distribution of the points on the left curve. The length of these geodesics are given in the table of Figure 3, in the same order as the corresponding images of Figure 2. The horizontal geodesics are always shorter than the initial geodesics, as expected, and have always approximatively the same length. This common length is the distance between the shapes of the two underlying curves. We also find that the underlying shapes of the horizontal geodesics are very similar. Testing other combinations of parameterizations for the same pair of shapes and overlapping the obtained horizontal geodesics gives an idea of the geodesic in the shape space, as shown in Figure 4.

## 5. PROOF OF THEOREM 1

We conclude this paper with the proof of Theorem 1. Let us first remind the result.

**Theorem 1.** *Let  $s \mapsto c(s)$  be a  $C^1$ -path of  $C^2$ -curves with non vanishing derivative with respect to  $t$ . This path can be identified with an element  $(s, t) \mapsto c(s, t)$  of  $C^{1,2}([0, 1] \times [0, 1], M)$  such that  $c_t \neq 0$ . Consider the  $C^1$ -path in  $M^{n+1}$ ,  $s \mapsto \alpha(s) = (x_0(s), \dots, x_n(s))$ , that is the discretization of size  $n$  of  $c$ . Then there exists a constant  $\lambda > 0$  such that for  $n$  large enough, the difference between the energies of  $c$  and  $\alpha$  is bounded by*

$$|E(c) - E^n(\alpha)| \leq \frac{\lambda}{n} (\inf |c_t|)^{-1} |c_s|_{2,\infty}^2 (1 + |c_t|_{1,\infty})^3,$$

where  $E$  and  $E^n$  are the energies with respect to metrics  $G$  and  $G^n$  respectively and where

$$\begin{aligned} |c_t|_{1,\infty} &:= |c_t|_\infty + |\nabla_t c_t|_\infty, \\ |c_s|_{2,\infty} &:= |c_s|_\infty + |\nabla_t c_s|_\infty + |\nabla_t^2 c_s|_\infty, \end{aligned}$$

and  $|w|_\infty := \sup_{s,t \in [0,1]} |w(s, t)|$  denotes the supremum over both  $s$  and  $t$  of a vector field  $w$  along  $c$ .

*Proof of Theorem 1.* To prove this result, we introduce the unique path  $\hat{c}$  of piecewise geodesic curves of which  $\alpha$  is the  $n$ -discretization. It is obtained by linking the points  $x_0(s), p_1(s), \dots, x_n(s)$  of  $\alpha$  by pieces of geodesics for all times  $s \in [0, 1]$

$$\begin{aligned} \hat{c}(s, \frac{k}{n}) &= c(s, \frac{k}{n}) = x_k(s), \\ \hat{c}(s, \cdot) \Big|_{[\frac{k}{n}, \frac{k+1}{n}]} &\text{ is a geodesic,} \end{aligned}$$

for  $k = 0, \dots, n$ . Then the difference between the energy of the path of curves  $E(c)$  and the discrete energy of the path of discrete curves  $E^n(\alpha)$  can be controlled in two steps :

$$|E(c) - E^n(\alpha)| \leq |E(c) - E(\hat{c})| + |E(\hat{c}) - E^n(\alpha)|.$$

**Step 1.** We first consider the difference between the continuous energies of the smooth and piecewise geodesic curves

$$\begin{aligned} |E(c) - E(\hat{c})| &= \left| \int_0^1 \int_0^1 (|\nabla_s q(s, t)|^2 - |\nabla_s \hat{q}(s, t)|^2) dt ds \right| \\ &\leq \int_0^1 \int_0^1 ||\nabla_s q(s, t)|^2 - |\nabla_s \hat{q}(s, t)|^2| dt ds \\ &\leq \int_0^1 \int_0^1 (|\nabla_s q(s, t)| + |\nabla_s \hat{q}(s, t)|) \cdot |\nabla_s q(s, t)^{t, \frac{k}{n}} - \nabla_s \hat{q}(s, t)^{t, \frac{k}{n}}| dt ds. \end{aligned}$$

Note that the parallel transports  $\nabla_s q(s, t)^{t, \frac{k}{n}}$  and  $\nabla_s \hat{q}(s, t)^{t, \frac{k}{n}}$  are performed along different curves  $c(s, \cdot)$  and  $\hat{c}(s, \cdot)$  respectively. Let us fix  $s \in [0, 1]$ ,  $0 \leq k \leq n$  and  $t \in [\frac{k}{n}, \frac{k+1}{n}]$ . Using the notation  $w^\parallel(t) := w(t)^{t, \frac{k}{n}}$  to denote the parallel transport of a vector field  $w$  from  $t$  to  $\frac{k}{n}$  along its baseline curve, the difference we need to control is

$$\begin{aligned} \|\nabla_s q^\parallel - \nabla_s \hat{q}^\parallel\| &= \left\| |c_t|^{-1/2} (\nabla_s c_t - \frac{1}{2} \nabla_s c_t^T)^\parallel - |\hat{c}_t|^{-1/2} (\nabla_s \hat{c}_t - \frac{1}{2} \nabla_s \hat{c}_t^T)^\parallel \right\| \\ &= \left\| (\nabla_s c_t - \frac{1}{2} \nabla_s c_t^T)^\parallel (|c_t|^{-1/2} - |\hat{c}_t|^{-1/2}) \right. \\ &\quad \left. + |\hat{c}_t|^{-1/2} ((\nabla_s c_t^\parallel - \nabla_s \hat{c}_t^\parallel) - \frac{1}{2} (\nabla_s c_t^\parallel - \nabla_s \hat{c}_t^\parallel)^T) \right\|. \end{aligned}$$

Since  $|w - \frac{1}{2} w^T| \leq |w|$  for any vector  $w$ , we can write

$$(23) \quad \|\nabla_s q^\parallel - \nabla_s \hat{q}^\parallel\| \leq |\nabla_s c_t| \cdot \left| |c_t|^{-1/2} - |\hat{c}_t|^{-1/2} \right| + |\hat{c}_t|^{-1/2} \|\nabla_s c_t^\parallel - \nabla_s \hat{c}_t^\parallel\|.$$

Let us first consider the difference  $|c_t^\parallel - \hat{c}_t^\parallel|$ . Since  $\hat{c}_t(s, t)^{t, \frac{k}{n}} = \hat{c}_t(s, \frac{k}{n})$ , we can write

$$|c_t(s, t)^{t, \frac{k}{n}} - \hat{c}_t(s, t)^{t, \frac{k}{n}}| \leq |c_t(s, t)^{t, \frac{k}{n}} - c_t(s, \frac{k}{n})| + |c_t(s, \frac{k}{n}) - \hat{c}_t(s, \frac{k}{n})|.$$

The first term is smaller than  $1/n \cdot |\nabla_t c_t|_\infty$ . To bound the second term, we place ourselves in a local chart  $(\phi, U)$  centered in  $c(s, \frac{k}{n})$ , such that  $c([0, 1] \times [0, 1]) \subset U$ . After identification with an open set of  $\mathbb{R}^d$  – where  $d$  is the dimension of the manifold  $M$  – using this chart, we get

$$\begin{aligned} |c_t(s, \frac{k}{n}) - \hat{c}_t(s, \frac{k}{n})| &\leq |c_t(s, \frac{k}{n}) - n(c(s, \frac{k+1}{n}) - c(s, \frac{k}{n}))| \\ &\quad + |\hat{c}_t(s, \frac{k}{n}) - n(c(s, \frac{k+1}{n}) - c(s, \frac{k}{n}))|. \end{aligned}$$

Since a geodesic locally looks like a straight line (see e.g. [5]) there exists a constant  $\lambda_1$  such that  $|\hat{c}_t(s, \frac{k}{n}) - n(c(s, \frac{k+1}{n}) - c(s, \frac{k}{n}))| \leq \lambda_1 |c(s, \frac{k+1}{n}) - c(s, \frac{k}{n})|^2$ , and so

$$|c_t(s, \frac{k}{n}) - \hat{c}_t(s, \frac{k}{n})| \leq \frac{1}{2n} |c_{tt}|_\infty + \frac{\lambda_1}{n} |c_t|_\infty^2.$$

The second derivative in  $t$  of the coordinates of  $c$  in the chart  $(U, \phi)$  can be written  $c_{tt}^\ell = \nabla_t c_t^\ell - \Gamma_{ij}^\ell c_t^i c_t^j$  for  $\ell = 1, \dots, d$ , and so there exists a constant  $\lambda_2$  such that  $|c_{tt}| \leq \lambda_2 (|\nabla_t c_t|_\infty + |c_t|_\infty^2)$ , and

$$(24) \quad |c_t^\parallel - \hat{c}_t^\parallel| \leq \frac{\lambda_3}{n} (|c_t|_{1, \infty} + |c_t|_{1, \infty}^2).$$

This means that for  $n$  large enough, we can write e.g.

$$(25) \quad \frac{1}{2} \inf |c_t| \leq |\hat{c}_t| \leq \frac{3}{2} |c_t|_\infty.$$

From (24) we can also deduce that

$$(26) \quad \left| |c_t|^{-\frac{1}{2}} - |\hat{c}_t|^{-\frac{1}{2}} \right| = \frac{||c_t| - |\hat{c}_t||}{|c_t|^{\frac{1}{2}} + |\hat{c}_t|^{\frac{1}{2}}} \leq \frac{|c_t^\parallel - \hat{c}_t^\parallel|}{|c_t|^{\frac{1}{2}} + |\hat{c}_t|^{\frac{1}{2}}} \leq \frac{\lambda_3}{n} (\inf |c_t|)^{-\frac{1}{2}} (|c_t|_{1, \infty} + |c_t|_{1, \infty}^2).$$

Let us now consider the difference  $|\nabla_s c_t^\parallel - \nabla_s \hat{c}_t^\parallel|$ . Since  $c_s(s, \frac{k}{n}) = \hat{c}_s(s, \frac{k}{n})$ , we get

$$\begin{aligned} & |\nabla_s c_t(s, t)^{t, \frac{k}{n}} - \nabla_s \hat{c}_t(s, t)^{t, \frac{k}{n}}| \\ & \leq \left| \nabla_t c_s(s, t)^{t, \frac{k}{n}} - \nabla_t c_s(s, \frac{k}{n}) \right| + \left| \nabla_t c_s(s, \frac{k}{n}) - n \left( c_s(s, \frac{k+1}{n})^{\frac{k+1}{n}, \frac{k}{n}} - c_s(s, \frac{k}{n}) \right) \right| \\ & + \left| \nabla_t \hat{c}_s(s, t)^{t, \frac{k}{n}} - \nabla_t \hat{c}_s(s, \frac{k}{n}) \right| + \left| \nabla_t \hat{c}_s(s, \frac{k}{n}) - n \left( \hat{c}_s(s, \frac{k+1}{n})^{\frac{k+1}{n}, \frac{k}{n}} - \hat{c}_s(s, \frac{k}{n}) \right) \right| \end{aligned}$$

and so

$$(27) \quad |\nabla_s c_t(s, t)^{t, \frac{k}{n}} - \nabla_s \hat{c}_t(s, t)^{t, \frac{k}{n}}| \leq \frac{3}{2n} |\nabla_t \nabla_t c_s(s, \tau)|_\infty + \frac{3}{2n} |\nabla_t \nabla_t \hat{c}_s(s, \tau)|_\infty.$$

We can decompose  $\nabla_t \nabla_t \hat{c}_s(s, t) = \nabla_t \nabla_s \hat{c}_t(s, t) = \nabla_s \nabla_t \hat{c}_t(s, t) + \mathcal{R}(\hat{c}_t, \hat{c}_s) \hat{c}_t(s, t)$ , and since  $\nabla_t \hat{c}_t(s, t) = 0$  and  $|\mathcal{R}(X, Y)Z| \leq |K| \cdot (|\langle Y, Z \rangle| |X| + |\langle X, Z, \rangle| |Y|) \leq 2|K| \cdot |X| \cdot |Y| \cdot |Z|$  by Cauchy Schwarz, we get using (25)

$$(28) \quad |\nabla_t \nabla_t \hat{c}_s| \leq 2 |\hat{c}_t|^2 |\hat{c}_s| \leq \frac{9}{2} |c_t|_\infty^2 |\hat{c}_s|.$$

To bound  $|\hat{c}_s|$  we apply Lemma 1 to the Jacobi field  $J : [0, 1] \ni u \mapsto \hat{c}_s(s, \frac{k+u}{n})$  along the geodesic  $\gamma(u) = \hat{c}(s, \frac{k+u}{n})$ , that is

$$(29) \quad J(u)^{u, 0} = J(0)^T + a_k(u) J(0)^N + u \nabla_t J(0)^T + b_k(u) \nabla_t J(0)^N$$

where, since  $\gamma'(0) = \frac{1}{n} \hat{c}_t(s, \frac{k}{n}) = \tau_k(s)$ , the coefficients are defined by

$$a_k(u) = \begin{cases} \cosh(|\tau_k(s)|u), & \text{if } K = -1, \\ 1 & \text{if } K = 0, \\ \cos(|\tau_k(s)|u), & \text{if } K = +1, \end{cases} \quad b_k(u) = \begin{cases} \sinh(|\tau_k(s)|u) / |\tau_k(s)| & \text{if } K = -1, \\ u & \text{if } K = 0, \\ \sin(|\tau_k(s)|u) / |\tau_k(s)| & \text{if } K = +1. \end{cases}$$

This gives  $J(1)^{1,0} = J(0)^T + a_k(1) J(0)^N + \nabla_t J(0)^N + b_k(1) \nabla_t J(0)^N$  and so

$$\begin{aligned} \nabla_t J(0)^T &= (J(1)^{1,0} - J(0))^T \\ \nabla_t J(0)^N &= b_k(1)^{-1} (J(1)^{1,0} - a_k(1) J(0))^N. \end{aligned}$$

Injecting this into (29), we obtain since  $u = nt - k$  and  $\hat{c}_s(s, \frac{k}{n}) = c_s(s, \frac{k}{n})$ ,

$$(30) \quad \hat{c}_s(s, t)^{t, \frac{k}{n}} = c_s(s, \frac{k}{n})^T + a_k(nt - k) c_s(s, \frac{k}{n})^N + (nt - k) \left( c_s(s, \frac{k+1}{n})^{\frac{k+1}{n}, \frac{1}{n}} - c_s(s, \frac{k}{n}) \right)^T \\ + \frac{b_k(nt - k)}{b_k(1)} \left( c_s(s, \frac{k+1}{n})^{\frac{k+1}{n}, \frac{1}{n}} - a_k(1) c_s(s, \frac{k}{n}) \right)^N.$$

When  $n \rightarrow \infty$ ,  $a_k(1) \rightarrow 1$ ,  $b_k(1) \rightarrow 1$ , and since  $0 \leq nt - k \leq 1$ ,  $a_k(nt - k) \rightarrow 1$ ,  $b_k(nt - k) \rightarrow 1$  also. Therefore, for  $n$  large enough we can see that  $|\hat{c}_s| \leq \lambda_4 |c_s|_\infty$  for some constant  $\lambda_4$ . Injecting this into (28) gives

$$|\nabla_t \nabla_t \hat{c}_s|_\infty \leq \frac{9\lambda_4}{2} |c_t|_\infty^2 |c_s|_\infty,$$

and so we obtain the following bound for the difference (27)

$$(31) \quad |\nabla_s c_t^\parallel - \nabla_s \hat{c}_t^\parallel| \leq \frac{3}{2n} (|\nabla_t \nabla_t c_s|_\infty + \frac{9\lambda_4}{2} |c_t|_\infty^2 |c_s|_\infty) \leq \frac{\lambda_5}{n} |c_s|_{2,\infty} (1 + |c_t|_{1,\infty}^2),$$

for some constant  $\lambda_5$ . Injecting (25), (26) and (31) in Equation (23) we obtain

$$\begin{aligned} |\nabla_s q^\parallel - \nabla_s \hat{q}^\parallel| &\leq \frac{\lambda_3}{n} (\inf |c_t|)^{-\frac{1}{2}} |c_s|_{2,\infty} (|c_t|_{1,\infty} + |c_t|_{1,\infty}^2) \\ &\quad + \frac{\lambda_5 \sqrt{2}}{n} (\inf |c_t|)^{-\frac{1}{2}} |c_s|_{2,\infty} (1 + |c_t|_{1,\infty}^2), \\ (32) \quad |\nabla_s q^\parallel - \nabla_s \hat{q}^\parallel| &\leq \frac{\lambda_6}{n} (\inf |c_t|)^{-\frac{1}{2}} |c_s|_{2,\infty} (1 + |c_t|_{1,\infty})^2, \end{aligned}$$

for some constant  $\lambda_6$ . To conclude this first step, let us bound the sum

$$(33) \quad |\nabla_s q| + |\nabla_s \hat{q}| = |c_t|^{-\frac{1}{2}} |\nabla_s c_t - \frac{1}{2} \nabla_s c_t^T| + |\hat{c}_t|^{-\frac{1}{2}} |\nabla_s \hat{c}_t - \frac{1}{2} \nabla_s \hat{c}_t^T| \\ \leq (\inf |c_t|)^{-\frac{1}{2}} |\nabla_t c_s|_\infty + \sqrt{2} (\inf |c_t|)^{-\frac{1}{2}} |\nabla_t \hat{c}_s|_\infty.$$

Taking the derivative according to  $t$  on both sides of (30), we get since  $n|\tau_k(s)| = |\hat{c}_t(s, \frac{k}{n})|$ ,

$$\begin{aligned} \nabla_t \hat{c}_s(s, t)^{t, \frac{k}{n}} &= |\hat{c}_t(s, \frac{k}{n})| e_k(nt - k) c_s(s, \frac{k}{n})^N + n \left( c_s(s, \frac{k+1}{n})^\parallel - c_s(s, \frac{k}{n}) \right)^T \\ &\quad + n \frac{a_k(nt - k)}{b_k(1)} \left( c_s(s, \frac{k+1}{n})^\parallel - a_k(1) c_s(s, \frac{k}{n}) \right)^N, \end{aligned}$$

since  $b'_k(u) = a_k(u)$  and  $a'_k(u) = |\tau_k(s)| e_k(u) = \frac{1}{n} |\hat{c}_t(s, \frac{k}{n})| e_k(u)$ , where

$$e_k(u) = \begin{cases} \sinh(|\tau_k(s)|u), & \text{if } K = -1, \\ 0 & \text{if } K = 0, \\ -\sin(|\tau_k(s)|u), & \text{if } K = +1. \end{cases}$$

Since the coefficients  $e_k(nt - k)$ ,  $a_k(nt - k)/b_k(1)$  and  $a_k(1)$  are bounded for  $n$  large enough, and since  $|\hat{c}_t| \leq \frac{3}{2} |c_t|_\infty$ , we can write for some constant  $\lambda_7$ ,

$$(34) \quad |\nabla_t \hat{c}_s|_\infty \leq \lambda_7 (|\hat{c}_t|_\infty |c_s|_\infty + |\nabla_t c_s|_\infty) \leq \frac{3\lambda_7}{2} |c_s|_{2,\infty} (1 + |c_t|_{1,\infty}).$$

Inserting this into (33) gives

$$(35) \quad \begin{aligned} |\nabla_s q| + |\nabla_s \hat{q}| &\leq (\inf |c_t|)^{-1/2} (|\nabla_t c_s|_\infty + \frac{3\lambda_7}{\sqrt{2}} |c_s|_{2,\infty} (1 + |c_t|_{1,\infty})) \\ &\leq \lambda_8 (\inf |c_t|)^{-1/2} |c_s|_{2,\infty} (1 + |c_t|_{1,\infty}). \end{aligned}$$

Finally, we are able to bound the difference between the energies of the smooth and piecewise-geodesic paths by combining Equations (32) and (35)

$$|E(c) - E(\hat{c})| \leq \frac{\lambda_6 \lambda_8}{n} (\inf |c_t|)^{-1} |c_s|_{2,\infty}^2 (1 + |c_t|_{1,\infty})^3.$$

**Step 2.** Let us now consider the difference of energy between the path of piecewise geodesic curves and the path of discrete curves. Since  $\nabla_s q_k(s) = \nabla_s \hat{q}(s, \frac{k}{n})$  for all  $s \in [0, 1]$  and  $0 \leq k \leq n$ , we can write

$$\begin{aligned} |E(\hat{c}) - E^n(\alpha)| &= \left| \int_0^1 \left( \int_0^1 |\nabla_s \hat{q}(s, t)|^2 dt - \frac{1}{n} \sum_{k=0}^{n-1} |\nabla_s q_k(s)|^2 \right) ds \right| \\ &\leq \sum_{k=0}^{n-1} \int_0^1 \int_{\frac{k}{n}}^{\frac{k+1}{n}} \left| |\nabla_s \hat{q}(s, t)|^2 - |\nabla_s \hat{q}(s, \frac{k}{n})|^2 \right| dt ds \\ &\leq \sum_{k=0}^{n-1} \int_0^1 \int_{\frac{k}{n}}^{\frac{k+1}{n}} (|\nabla_s \hat{q}(s, t)| + |\nabla_s \hat{q}(s, \frac{k}{n})|) |\nabla_s \hat{q}(s, t)^{t, \frac{k}{n}} - \nabla_s \hat{q}(s, \frac{k}{n})| dt ds. \end{aligned}$$

We fix once again  $s \in [0, 1]$ ,  $0 \leq k \leq n$  and  $t \in [\frac{k}{n}, \frac{k+1}{n}]$ . Since  $|\hat{c}_t(s, t)| = |\hat{c}_t(s, \frac{k}{n})|$ , we get

$$\begin{aligned} |\nabla_s \hat{q}(s, t)^{t, \frac{k}{n}} - \nabla_s \hat{q}(s, \frac{k}{n})| &\leq \left| |\hat{c}_t(s, \frac{k}{n})|^{-\frac{1}{2}} \left( \nabla_s \hat{c}_t(s, t)^{t, \frac{k}{n}} - \nabla_s \hat{c}_t(s, \frac{k}{n}) \right) \right. \\ &\quad \left. - \frac{1}{2} (\nabla_s \hat{c}_t(s, t)^{t, \frac{k}{n}} - \nabla_s \hat{c}_t(s, \frac{k}{n}))^T \right| \\ &\leq |\hat{c}_t(s, \frac{k}{n})|^{-\frac{1}{2}} |\nabla_s \hat{c}_t(s, t)^{t, \frac{k}{n}} - \nabla_s \hat{c}_t(s, \frac{k}{n})|. \end{aligned}$$

Considering once again the Jacobi field  $J(u) := \hat{c}_s(s, \frac{k+u}{n})$ ,  $u \in [0, 1]$ , along the geodesic  $\gamma(u) = \hat{c}(s, \frac{k+u}{n})$ , Equation (29) gives

$$\hat{c}_s(s, t)^{t, \frac{k}{n}} = c_s(s, \frac{k}{n})^T + a_k(nt - k) c_s(s, \frac{k}{n})^N + (t - \frac{k}{n}) \nabla_t \hat{c}_s(s, \frac{k}{n})^T + b_k(nt - k) \frac{1}{n} \nabla_t \hat{c}_s(s, \frac{k}{n})^N.$$

Recall that  $b'_k(u) = a_k(u)$  and  $a'_k(u) = |\tau_k| e_k(u)$ , and so taking the derivative with respect to  $t$  and decomposing  $\nabla_t \hat{c}_s(s, \frac{k}{n})^T = \nabla_t \hat{c}_s(s, \frac{k}{n}) - \nabla_t \hat{c}_s(s, \frac{k}{n})^N$ , we obtain

$$\nabla_t \hat{c}_s(s, t)^{t, \frac{k}{n}} - \nabla_t \hat{c}_s(s, \frac{k}{n}) = |\hat{c}_t(s, \frac{k}{n})| e_k(nt - k) c_s(s, \frac{k}{n})^N + (a_k(nt - k) - 1) \nabla_t \hat{c}_s(s, \frac{k}{n})^N.$$

Noticing that  $\frac{e_k(nt-k)}{(nt-k)|\tau_k|} \rightarrow 1$  and  $\frac{a_k(nt-k)-1}{(nt-k)|\tau_k|} \rightarrow 0$  when  $n \rightarrow \infty$ , we can deduce that for  $n$  large enough,

$$\begin{aligned} |e_k(nt-k)| &\leq 2(nt-k)|\tau_k| \leq 2|\tau_k| = \frac{2}{n}|c_t| \leq \frac{2}{n}|c_t|_\infty, \\ |a_k(nt-k)-1| &\leq (nt-k)|\tau_k| \leq |\tau_k| = \frac{1}{n}|c_t| \leq \frac{1}{n}|c_t|_\infty. \end{aligned}$$

This gives

$$\int_{\frac{k}{n}}^{\frac{k+1}{n}} |\nabla_t \hat{c}_s(s, t)^{t, \frac{k}{n}} - \nabla_t \hat{c}_s(s, \frac{k}{n^2})| dt \leq \frac{2}{n^2} (|c_t|_\infty^2 |c_s|_\infty + |c_t|_\infty |\nabla_t \hat{c}_s|_\infty).$$

Recall from (34) and (35) that

$$\begin{aligned} |\nabla_t \hat{c}_s|_\infty &\leq \frac{3\lambda_7}{2} |c_s|_{2,\infty} (1 + |c_t|_{1,\infty}), \\ |\nabla_s \hat{q}|_\infty &\leq \frac{3\lambda_7}{\sqrt{2}} (\inf |c_t|)^{-\frac{1}{2}} |c_s|_{2,\infty} (1 + |c_t|_{1,\infty}), \end{aligned}$$

and so

$$\begin{aligned} &\int_{\frac{k}{n}}^{\frac{k+1}{n}} (|\nabla_s \hat{q}(s, t)| + |\nabla_s \hat{q}(s, \frac{k}{n})|) \cdot |\nabla_s \hat{q}(s, t)^{t, \frac{k}{n}} - \nabla_s \hat{q}(s, \frac{k}{n})| dt \\ &\leq 2|\nabla_s \hat{q}|_\infty \sqrt{2} (\inf |c_t|)^{-\frac{1}{2}} \int_{\frac{k}{n}}^{\frac{k+1}{n}} |\nabla_t \hat{c}_s(s, t)^{t, \frac{k}{n}} - \nabla_t \hat{c}_s(s, \frac{k}{n})| dt \\ &\leq 6\lambda_7 (\inf |c_t|)^{-1} |c_s|_{2,\infty} (1 + |c_t|_{1,\infty}) \frac{2}{n^2} (|c_t|_\infty^2 |c_s|_\infty + |c_t|_\infty \frac{3\lambda_7}{2} |c_s|_{2,\infty} (1 + |c_t|_{1,\infty})) \\ &\leq \frac{\lambda_9}{n^2} (\inf |c_t|)^{-1} |c_s|_{2,\infty}^2 (1 + |c_t|_{1,\infty})^3. \end{aligned}$$

Finally, we obtain

$$|E(\hat{c}) - E^n(\alpha)| \leq \frac{\lambda_9}{n} (\inf |c_t|)^{-1} |c_s|_{2,\infty}^2 (1 + |c_t|_{1,\infty})^3,$$

which completes the proof.  $\square$

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#### APPENDIX A

**Lemma 1.** *Let  $\gamma : [0, 1] \rightarrow M$  be a geodesic of a manifold  $M$  of constant sectional curvature  $K$ , and  $J$  a Jacobi field along  $\gamma$ . Then the parallel transport of  $J(t)$  along  $\gamma$  from  $\gamma(t)$  to  $\gamma(0)$  is given by*

$$J(t)^{t,0} = J^T(0) + \tilde{a}_k(t) J^N(0) + t \nabla_t J^T(0) + \tilde{b}_k(t) \nabla_t J^N(0), \quad \forall t \in [0, 1].$$

where

$$\tilde{a}_k(t) = \begin{cases} \cosh(|\gamma'(0)|t), & \text{if } K = -1, \\ 1 & \text{if } K = 0, \\ \cos(|\gamma'(0)|t), & \text{if } K = +1, \end{cases} \quad \tilde{b}_k(t) = \begin{cases} \sinh(|\gamma'(0)|t) / |\gamma'(0)| & \text{if } K = -1, \\ t & \text{if } K = 0, \\ \sin(|\gamma'(0)|t) / |\gamma'(0)| & \text{if } K = +1. \end{cases}$$

*Proof of Lemma 1.* As a Jacobi field along  $\gamma$ ,  $J$  satisfies the well-known equation

$$\nabla_t \nabla_t J(t) = -\mathcal{R}(J(t), \gamma'(t)) \gamma'(t).$$

If  $M$  is flat, we get  $\nabla_t \nabla_t J(t) = 0$  and so  $J(t) = J(0) + t \nabla_t J(0)$ . If not, we can decompose  $J$  in the sum  $J = J^T + J^N$  of two vector fields that parallel translate along  $\gamma$ , by projecting it in the basis  $(v = \gamma' / |\gamma'|, n)$ . Since  $\langle \nabla_t \nabla_t J(t), \gamma'(t) \rangle = 0$  and  $\gamma'$  is parallel along  $\gamma$ , we get by integrating twice that

$$\begin{aligned} \langle J(t), \gamma'(t) \rangle &= \langle \nabla_t J(0), \gamma'(0) \rangle t + \langle J(0), \gamma'(0) \rangle, \\ \langle J(t), v(t) \rangle &= \langle \nabla_t J(0), v(0) \rangle t + \langle J(0), v(0) \rangle. \end{aligned}$$



Since  $\nabla_t \nabla_t J^T(t) = \nabla_t \nabla_t \langle J(t), v(t) \rangle = \langle \nabla_t \nabla_t J(t), v(t) \rangle = 0$ , the normal component  $J^N$  is also a Jacobi field, that is it verifies

$$\nabla_t \nabla_t J^N(t) = -\mathcal{R}(J^N(t), \gamma'(t))\gamma'(t).$$

And since  $M$  has constant sectional curvature  $K$ , for any vector field  $w$  along  $\gamma$  we have

$$\langle \mathcal{R}(J^N, \gamma')\gamma', w \rangle = K (\langle \gamma', \gamma' \rangle \langle J^N, w \rangle - \langle J^N, \gamma' \rangle \langle \gamma', w \rangle) = \langle K |\gamma'|^2 J^N, w \rangle,$$

and the differential equation verified by  $J^N$  can be rewritten  $\nabla_t \nabla_t J^N(t) = -K |\gamma'(t)|^2 J^N(t)$ . Since the speed of the geodesic  $\gamma$  has constant norm, the solution to that differential equation is of the form

$$J^N(t) = \tilde{a}_k(t)J^N(0) + \tilde{b}_k(t)\nabla_t J(0)^N,$$

where the functions  $\tilde{a}_k(t)$ 's and  $\tilde{b}_k(t)$ 's depend on the value of  $K$  as defined in the lemma.  $\square$

**Lemma 3.** *The covariant derivatives of the functions  $f_k^{(-)}$  and  $g_k^{(-)}$  with respect to  $s$  are functions  $T_{x_{k+1}(s)}M \rightarrow T_{x_k(s)}M$  given by*

$$\nabla_s (f_k^{(-)}) : w \mapsto (\nabla_s f_k)^{(-)}(w) + f_k(\mathcal{R}(Y_k, \tau_k)(w_{k+1}^{\parallel})),$$

$$\nabla_s (g_k^{(-)}) : w \mapsto (\nabla_s g_k)^{(-)}(w) + g_k(\mathcal{R}(Y_k, \tau_k)(w_{k+1}^{\parallel})),$$

where  $(\nabla_s f_k)^{(-)}(s) = \nabla_s f_k(s) \circ P_{\gamma_k(s)}^{x_{k+1}(s), x_k(s)}$ ,  $(\nabla_s g_k)^{(-)}(s) = \nabla_s g_k(s) \circ P_{\gamma_k(s)}^{x_{k+1}(s), x_k(s)}$ , and

$$Y_k = (x_k')^T + b_k(x_k')^N + \frac{1}{2}\nabla_s \tau_k^T + K \frac{1 - a_k}{|\tau_k|^2} \nabla_s \tau_k^N,$$

if  $K$  is the sectional curvature of the base manifold.

*Proof of Lemma 3.* Fix  $0 \leq k \leq n$  and let  $w_{k+1} : s \mapsto w_{k+1}(s)$  be a vector field along the curve  $x_{k+1} : s \mapsto x_{k+1}(s)$ . By definition,

$$\nabla_s (f_k^{(-)}(w_{k+1})) = \nabla_s (f_k^{(-)})(w_{k+1}) + f_k^{(-)}(\nabla_s w_{k+1}),$$

$$\nabla_s (g_k^{(-)}(w_{k+1})) = \nabla_s (g_k^{(-)})(w_{k+1}) + g_k^{(-)}(\nabla_s w_{k+1}).$$

Consider the path of geodesics  $s \mapsto \gamma_k(s, \cdot)$  such that for all  $s \in [0, 1]$ ,  $\gamma_k(s, 0) = x_k(s)$ ,  $\gamma_k(s, 1) = x_{k+1}(s)$  and  $t \mapsto \gamma_k(s, t)$  is a geodesic. We denote by  $w_{k+1}^{\parallel}$  the vector field along the curve  $x_k$  obtained by parallel transporting back the vector  $w_{k+1}(s)$  along the geodesic  $\gamma_k(s, \cdot)$  for all  $s \in [0, 1]$ , i.e.  $w_{k+1}^{\parallel}(s) = P_{\gamma_k(s, \cdot)}^{1,0}(w_{k+1}(s))$ . We have

$$(36) \quad \nabla_s (f_k^{(-)}(w_{k+1})) = \nabla_s (f_k(w_{k+1}^{\parallel})) = \nabla_s f_k(w_{k+1}^{\parallel}) + f_k(\nabla_s (w_{k+1}^{\parallel})),$$

and so we need to compute  $\nabla_s (w_{k+1}^{\parallel})$ . Let  $V(s, t) := P_{\gamma_k(s, \cdot)}^{1,t}(w_{k+1})$  so that  $\nabla_s V(s, 1) = \nabla_s w_{k+1}$  and  $\nabla_s V(s, 0) = \nabla_s (w_{k+1}^{\parallel})$ , then

$$\begin{aligned} \nabla_s V(s, 1)^{1,0} &= \nabla_s V(s, 0) + \int_0^1 \nabla_t \nabla_s V(s, t)^{t,0} dt, \\ &= \nabla_s V(s, 0) + \int_0^1 \mathcal{R}(\partial_t \gamma_k^{t,0}, \partial_s \gamma_k^{t,0}) V(s, t)^{t,0} dt, \end{aligned}$$

since  $\nabla_t V = 0$ , and where  $\partial_t \gamma_k(s, t)^{t,0} = \tau_k(s)$ . We get, since  $\nabla R = 0$ ,

$$(37) \quad (\nabla_s w_{k+1})^{\parallel} = \nabla_s (w_{k+1}^{\parallel}) + \mathcal{R} \left( \tau_k, \int_0^1 \partial_s \gamma_k^{t,0} dt \right) (w_{k+1}^{\parallel}),$$

To find an expression for  $\partial_s \gamma_k^{t,0}$ , we consider the Jacobi field  $J(t) := \partial_s \gamma_k(s, t)$  along the geodesic  $t \mapsto \gamma_k(s, t)$ . The vector field  $J$  verifies

$$J(0) = x_k'(s), J(1) = x_{k+1}'(s), \nabla_t J(0) = \nabla_s \tau_k(s),$$

where the last equality results from the inversion  $\nabla_t \partial_s \gamma_k(s, 0) = \nabla_s \partial_t \gamma_k(s, 0)$  and  $\partial_t \gamma_k(s, 0) = \tau_k(s)$ . Applying Lemma 1 gives, for all  $k = 0, \dots, n-1$ ,

$$\partial_s \gamma_k(s, t)^{t,0} = x_k'(s)^T + a_k(s, t) x_k'(s)^N + t \nabla_s \tau_k(s)^T + b_k(s, t) \nabla_s \tau_k(s)^N.$$

with the coefficients

$$a_k(s, t) = \begin{cases} \cosh(|\tau_k(s)|t), & \text{if } K = -1, \\ 1 & \text{if } K = 0, \\ \cos(|\tau_k(s)|t), & \text{if } K = +1, \end{cases} \quad b_k(s, t) = \begin{cases} \sinh(|\tau_k(s)|t)/|\tau_k(s)| & \text{if } K = -1, \\ 1 & \text{if } K = 0, \\ \sin(|\tau_k(s)|t)/|\tau_k(s)| & \text{if } K = +1. \end{cases}$$

Integrating this and injecting it in (37) gives

$$(38) \quad \nabla_s(w_{k+1}^{\parallel}) = (\nabla_s w_{k+1})^{\parallel} + \mathcal{R}(Y_k, \tau_k)(w_{k+1}^{\parallel}),$$

where  $Y_k$  is defined by

$$Y_k = (x_k')^T + b_k(x_k')^N + \frac{1}{2} \nabla_s \tau_k^T + K \frac{1 - a_k}{|\tau_k|^2} \nabla_s \tau_k^N,$$

and injecting this in (36) finally gives,

$$\begin{aligned} \nabla_s(f_k^{(-)}(w_{k+1})) &= \nabla_s f_k(w_{k+1}^{\parallel}) + f_k((\nabla_s w_{k+1})^{\parallel}) + f_k(\mathcal{R}(Y_k, \tau_k)(w_{k+1}^{\parallel})) \\ &= (\nabla_s f_k)^{(-)}(w_{k+1}) + f_k^{(-)}(\nabla_s w_{k+1}) + f_k(\mathcal{R}(Y_k, \tau_k)(w_{k+1}^{\parallel})), \end{aligned}$$

which is what we wanted. The covariant derivative  $\nabla_s(g_k^{(-)}(w_{k+1}))$  can be computed in a similar way.  $\square$

## APPENDIX B

**Proposition 6** (Discrete geodesic equations). *A path  $s \mapsto \alpha(s) = (x_0(s), \dots, x_n(s))$  in  $M^{n+1}$  is a geodesic for metric  $G^n$  if and only if its SRV representation  $s \mapsto (x_0(s), (q_k(s))_k)$  verifies the following differential equations*

$$\begin{aligned} \nabla_s x_0' + \frac{1}{n} \left( R_0 + f_0^{(-)}(R_1) + \dots + f_0^{(-)} \circ \dots \circ f_{n-2}^{(-)}(R_{n-1}) \right) &= 0, \\ \nabla_s \nabla_s q_k + \frac{1}{n} g_k^{(-)} \left( R_{k+1} + f_{k+1}^{(-)}(R_{k+2}) + \dots + f_{k+1}^{(-)} \circ \dots \circ f_{n-2}^{(-)}(R_{n-1}) \right) &= 0, \end{aligned}$$

for all  $k = 0, \dots, n-1$ , with the notations (12) and  $R_k := \mathcal{R}(q_k, \nabla_s q_k) x_k'$ .

*Proof of Proposition 6.* We consider a variation  $(-\delta, \delta) \ni a \mapsto \alpha(a, \cdot) = (x_0(a, \cdot), \dots, x_n(a, \cdot))$  of this curve which coincides with  $\alpha$  for  $a = 0$ , i.e.  $\alpha(0, s) = \alpha(s)$  for all  $s \in [0, 1]$ , and which preserves the end points of  $\alpha$ , i.e.  $\alpha(a, 0) = \alpha(0)$  and  $\alpha(a, 1) = \alpha(1)$  for all  $a \in (-\delta, \delta)$ . The energy of this variation with respect to metric  $G^n$  can be seen as a real function of the variable  $a$  and is given by

$$E^n(a) = \frac{1}{2} \int_0^1 \left( |\partial_s x_0(a, s)|^2 + \frac{1}{n} \sum_{k=0}^{n-1} |\nabla_s q_k(a, s)|^2 \right) ds,$$

and its derivative with respect to  $a$  is

$$\begin{aligned} (E^n)'(a) &= \int_0^1 \left( \left\langle \partial_a \partial_s x_0, \partial_s x_0 \right\rangle + \frac{1}{n} \sum_{k=0}^{n-1} \left\langle \nabla_a \nabla_s q_k, \nabla_s q_k \right\rangle \right) ds, \\ &= \int_0^1 \left( \left\langle \partial_s \partial_a x_0, \partial_s x_0 \right\rangle + \frac{1}{n} \sum_{k=0}^{n-1} \left\langle \nabla_s \nabla_a q_k + \mathcal{R}(\partial_a x_k, \partial_s x_k) q_k, \nabla_s q_k \right\rangle \right) ds, \\ &= - \int_0^1 \left( \left\langle \nabla_s(\partial_s x_0), \partial_a x_0 \right\rangle + \frac{1}{n} \sum_{k=0}^{n-1} \left\langle \nabla_s \nabla_s q_k, \nabla_a q_k \right\rangle + \left\langle \mathcal{R}(q_k, \nabla_s q_k) \partial_s x_k, \partial_a x_k \right\rangle \right) ds, \end{aligned}$$

where we integrate by parts to obtain the third line from the second. The goal is to express  $\partial_a x_k$  in terms of  $\partial_a x_0$  and  $\nabla_a q_\ell$ ,  $\ell = 0, \dots, k$ . That way, the only elements that depend on

$a$  once we take  $a = 0$  are  $(\partial_a x_0, \nabla_a q_0, \dots, \nabla_a q_{n-1})$  which can be chosen *independently* to be whatever we want. Let us fix  $0 \leq k \leq n-1$  and  $s \in [0, 1]$  and consider the path of geodesics  $a \mapsto \gamma_k(a, \cdot)$  such that  $\gamma_k(a, 0) = x_k(a, s)$ ,  $\gamma_k(a, 1) = x_{k+1}(a, s)$  and  $\partial_t \gamma_k(a, 0) = \tau_k(a, s) = \log_{x_k(a, s)}(x_{k+1}(a, s))$ . Then by definition, for each  $a \in [0, 1]$ ,  $t \mapsto J(a, t) := \partial_a \gamma_k(a, t)$  is a Jacobi field along the geodesic  $t \mapsto \gamma_k(a, t)$  of  $M$ , and so Lemma 1 gives

$$(39) \quad \partial_a x_{k+1}^{\parallel} = \partial_a x_k^T + a_k \partial_a x_k^N + \nabla_a \tau_k^T + b_k \nabla_s \tau_k^N,$$

where  $\partial_a x_{k+1}^{\parallel}$  denotes the parallel transport of  $\partial_a x_{k+1}$  from  $x_{k+1}(s)$  to  $x_k(s)$  along the geodesic. Differentiation of  $q_k = \sqrt{n} \tau_k / |\tau_k|$  gives  $\nabla_s q_k = \sqrt{n} |\tau_k|^{-1/2} (\nabla_s \tau_k - \frac{1}{2} \nabla_s \tau_k^T)$  and taking the tangential part on both sides yields  $\nabla_s q_k^T = \sqrt{n} |\tau_k|^{-1/2} \frac{1}{2} \nabla_s \tau_k^T$ , and so finally  $\nabla_s \tau_k = |\tau_k|^{1/2} / \sqrt{n} (\nabla_s q_k + \nabla_s q_k^T) = |q_k|/n (\nabla_s q_k + \nabla_s q_k^T)$ . Injecting this in (39) and noticing that  $\langle f_k(w), z \rangle = \langle w, f_k(z) \rangle$  and  $\langle g_k(w), z \rangle = \langle w, g_k(z) \rangle$  for any pair of vectors  $w, z$  gives

$$(40) \quad \partial_a x_{k+1}^{\parallel} = f_k(\partial_a x_k) + \frac{1}{n} g_k(\nabla_a q_k),$$

$$(41) \quad \langle w_{k+1}, \partial_a x_{k+1} \rangle = \langle f_k^{(-)}(w_{k+1}), \partial_a x_k \rangle + \frac{1}{n} \langle g_k^{(-)}(w_{k+1}), \nabla_a q_k \rangle,$$

for any tangent vector  $w_{k+1} \in T_{x_{k+1}} M$ . From equation (41) we can deduce, for  $k = 1, \dots, n$ ,

$$\langle w_k, \partial_a x_k \rangle = \left\langle f_0^{(-)} \circ \dots \circ f_{k-1}^{(-)}(w_k), \partial_a x_0 \right\rangle + \frac{1}{n} \sum_{\ell=0}^{k-1} \left\langle g_\ell^{(-)} \circ f_{\ell+1}^{(-)} \circ \dots \circ f_{k-1}^{(-)}(w_k), \nabla_a q_\ell \right\rangle.$$

With the notation  $R_k := \mathcal{R}(q_k, \nabla_s q_k) x_k'$  we get

$$\langle R_k, \partial_a x_k \rangle = \left\langle f_0^{(-)} \circ \dots \circ f_{k-1}^{(-)}(R_k), \partial_a x_0 \right\rangle + \frac{1}{n} \sum_{\ell=0}^{k-1} \left\langle g_\ell^{(-)} \circ f_{\ell+1}^{(-)} \circ \dots \circ f_{k-1}^{(-)}(R_k), \nabla_a q_\ell \right\rangle,$$

and we can then write the derivative of the energy for  $a = 0$  in the following way

$$(E^n)'(0) = - \int_0^1 \left( \left\langle \nabla_s x_0' + \frac{1}{n} \sum_{k=0}^{n-1} f_0^{(-)} \circ \dots \circ f_{k-1}^{(-)}(R_k), \partial_a x_0 \right\rangle + \frac{1}{n^2} \sum_{k=1}^{n-1} \sum_{\ell=0}^{k-1} \left\langle g_\ell^{(-)} \circ f_{\ell+1}^{(-)} \circ \dots \circ f_{k-1}^{(-)}(R_k), \nabla_a q_\ell \right\rangle + \frac{1}{n} \sum_{k=0}^{n-1} \left\langle \nabla_s \nabla_s q_k, \nabla_a q_k \right\rangle \right) ds,$$

where in the first sum we use the notation convention  $f_0 \circ \dots \circ f_{-1} := \text{Id}$ . Noticing that the double sum can be rewritten

$$\sum_{\ell=0}^{n-2} \sum_{k=\ell+1}^{n-1} \left\langle g_\ell^{(-)} \circ f_{\ell+1}^{(-)} \circ \dots \circ f_{k-1}^{(-)}(R_k), \nabla_a q_\ell \right\rangle,$$

we obtain

$$(42) \quad (E^n)'(0) = - \int_0^1 \left( \left\langle \nabla_s x_0' + \frac{1}{n} \sum_{k=0}^{n-1} f_0^{(-)} \circ \dots \circ f_{k-1}^{(-)}(R_k), \partial_a x_0 \right\rangle + \frac{1}{n} \sum_{k=0}^{n-1} \left\langle \nabla_s \nabla_s q_k + \frac{1}{n} \sum_{\ell=k+1}^{n-1} g_\ell^{(-)} \circ f_{\ell+1}^{(-)} \circ \dots \circ f_{\ell-1}^{(-)}(R_\ell), \nabla_a q_k \right\rangle \right) ds,$$

where in the last sum we use the convention  $\sum_{\ell=n}^{n-1} = 0$ . Since this quantity has to vanish for any choice of  $(\partial_a x_0(0, \cdot), \nabla_a q_0(0, \cdot), \dots, \nabla_a q_{n-1}(0, \cdot))$ , the geodesic equations for the discrete

metric are

$$\begin{aligned}\nabla_s x_0' + \frac{1}{n} \sum_{k=0}^{n-1} f_0^{(-)} \circ \cdots \circ f_{k-1}^{(-)}(R_k) &= 0, \\ \nabla_s \nabla_s q_k + \frac{1}{n} \sum_{\ell=k+1}^{n-1} g_k^{(-)} \circ f_{k+1}^{(-)} \circ \cdots \circ f_{\ell-1}^{(-)}(R_\ell) &= 0,\end{aligned}$$

for all  $k = 0, \dots, n-1$ , with the conventions  $\sum_{\ell=n}^{n-1} = 0$  and  $f_0 \circ \cdots \circ f_{-1} := \text{Id}$ .  $\square$

**Remark 4.** Let  $[0, 1] \ni s \mapsto c(s, \cdot) \in \mathcal{M}$  be a  $C^1$  path of smooth curves and  $[0, 1] \ni s \mapsto \alpha(s) \in M^{n+1}$  a discretization of size  $n$  of  $c$ . We denote as usual by  $q := c_t/|c_t|^{1/2}$  and  $(q_k)_k$  their respective SRV representations. When  $n \rightarrow \infty$  and  $|\tau_k| \rightarrow 0$  while  $n|\tau_k|$  stays bounded for all  $0 \leq k \leq n$ , the coefficients of the discrete geodesic equation (19) for  $\alpha$  converge to the coefficients of the continuous geodesic equation (6) for  $c$ , i.e.

$$\begin{aligned}\nabla_s x_0'(s) &= -r_0(s) + o(1), \\ \nabla_s \nabla_s q_k(s) &= -|q_k(s)|(r_k(s) + r_k(s)^T) + o(1),\end{aligned}$$

for all  $s \in [0, 1]$  and  $k = 0, \dots, n-1$ , where  $r_{n-1} = 0$  and for  $k = 1, \dots, n-2$ ,

$$r_k(s) := \frac{1}{n} \sum_{\ell=k+1}^{n-1} P_c^{\frac{\ell}{n}, \frac{k}{n}}(\mathcal{R}(q, \nabla_s q)c_s(s, \frac{\ell}{n})) \xrightarrow{n \rightarrow \infty} r(s, \frac{k}{n}),$$

with the exception that the sum starts at  $\ell = 0$  for  $r_0$ .

*Proof of Remark 4.* This is due to three arguments : (1) at the limit,  $f_k(w) = w + o(1/n)$  and  $g_k(w) = |q_k|(w + w^T) + o(1/n)$ , (2) parallel transport along a piecewise geodesic curve uniformly converges to the parallel transport along the limit curve, and (3) the discrete curvature term  $R_k(s)$  converges to the continuous curvature term  $\mathcal{R}(q, \nabla_s q)c_s(s, \frac{k}{n})$  for all  $k$ . Indeed, let  $\hat{c}$  be the unique piecewise geodesic curve of which  $\alpha$  is the discretization, i.e.  $c(\frac{k}{n}) = \hat{c}(\frac{k}{n}) = x_k$  for all  $k = 0, \dots, n$  and  $\hat{c}$  is a geodesic on each segment  $[\frac{k}{n}, \frac{k+1}{n}]$ . Defining

$$\begin{aligned}\hat{r}_k &:= \frac{1}{n}(R_{k+1} + f_{k+1}^{(-)}(R_{k+2}) + \cdots + f_{k+1}^{(-)} \circ \cdots \circ f_{n-2}^{(-)}(R_{n-1})) \quad \text{for } k = 1, \dots, n-2, \\ \hat{r}_0 &:= \frac{1}{n}(R_0 + f_0^{(-)}(R_1) + \cdots + f_0^{(-)} \circ \cdots \circ f_{n-2}^{(-)}(R_{n-1})) \quad \text{and } \hat{r}_{n-1} := 0,\end{aligned}$$

the geodesic equations can be written in terms of the vectors  $\hat{r}_k$

$$\begin{aligned}\nabla_s x_0'(s) + \hat{r}_0(s) &= 0, \\ \nabla_s \nabla_s q_k(s) + g_k^{(-)}(\hat{r}_k(s)) &= 0.\end{aligned}$$

We can show that for any  $0 \leq k \leq \ell \leq n-2$  and any vector  $w \in T_{x_{\ell+1}}M$ ,

$$\begin{aligned}\left| f_k^{(-)} \circ \cdots \circ f_\ell^{(-)}(w) - P_c^{\frac{\ell+1}{n}, \frac{k}{n}}(w) \right| &\leq \sum_{j=k}^{\ell} |a_j - 1| \cdot \left| f_{j+1}^{(-)} \circ \cdots \circ f_\ell^{(-)}(w) \right| \\ &\quad + \left| P_{\hat{c}}^{\frac{\ell+1}{n}, \frac{k}{n}}(w) - P_c^{\frac{\ell+1}{n}, \frac{k}{n}}(w) \right|.\end{aligned}$$

Since  $|a_j - 1|/|\tau_k|^2 \rightarrow 0$  when  $n \rightarrow \infty$  and  $n|\tau_k|$  stays bounded, we have for all  $0 \leq j \leq n$  and  $n$  large enough  $|a_j - 1| \leq \frac{1}{n^2}$ , and using the fact that parallel transport along a piecewise geodesic curve uniformly converges to the parallel transport along the limit curve, we get

$$\left| f_k^{(-)} \circ \cdots \circ f_\ell^{(-)}(w) - P_c^{\frac{\ell+1}{n}, \frac{k}{n}}(w) \right| \rightarrow 0$$

when  $n \rightarrow \infty$ . Now, denoting by  $R(s, t) := \mathcal{R}(q, \nabla_s q)c_s(s, t)$  the curvature term involved in the continuous geodesic equations, we have since  $x_k'(s) = c_s(s, \frac{k}{n})$  and  $|\mathcal{R}(X, Y)Z| \leq |K| \cdot (|\langle Y, Z \rangle||X| + |\langle X, Z \rangle||Y|) \leq 2|K| \cdot |X| \cdot |Y| \cdot |Z|$  by Cauchy Schwarz,

$$\begin{aligned} |R_k - R(\frac{k}{n})| &\leq |\mathcal{R}(q_k - q(\frac{k}{n}), \nabla_s q_k)x_k'| + |\mathcal{R}(q(\frac{k}{n}), \nabla_s q_k - \nabla_s q(\frac{k}{n}))x_k'| \\ &\leq |q_k - q(\frac{k}{n})| \cdot |\nabla_s q_k| \cdot |x_k'| + |q(\frac{k}{n})| \cdot |\nabla_s q_k - \nabla_s q(\frac{k}{n})| \cdot |x_k'| \end{aligned}$$

Let us show that both summands of this upper bound tend to 0 when  $n \rightarrow \infty$ .

$$\begin{aligned} |q_k - q(\frac{k}{n})| &= \left| |n\tau_k|^{-1/2} n\tau_k - |c_t(\frac{k}{n})|^{-1/2} c_t(\frac{k}{n}) \right| \\ &\leq \left| |n\tau_k|^{-1/2} - |c_t(\frac{k}{n})|^{-1/2} \right| \cdot |n\tau_k| + |c_t(\frac{k}{n})|^{-1/2} |n\tau_k - c_t(\frac{k}{n})| \\ &= \frac{|n\tau_k| - |c_t(\frac{k}{n})|}{|n\tau_k|^{1/2} + |c_t(\frac{k}{n})|^{1/2}} \cdot |n\tau_k| + |c_t(\frac{k}{n})|^{-1/2} |n\tau_k - c_t(\frac{k}{n})| \\ &\leq \left( \frac{|n\tau_k|}{|n\tau_k|^{1/2} + |c_t(\frac{k}{n})|^{1/2}} + |c_t(\frac{k}{n})|^{-1/2} \right) |n\tau_k - c_t(\frac{k}{n})| \end{aligned}$$

and since the portion of  $c(s, \cdot)$  on the segment  $[\frac{k}{n}, \frac{k+1}{n}]$  is close to a geodesic at the limit,  $|n\tau_k - c_t(\frac{k}{n})| \rightarrow 0$  when  $n \rightarrow \infty$ , and so does  $|q_k(s) - q(\frac{k}{n})|$ . Similarly,

$$\begin{aligned} |\nabla_s q_k - \nabla_s q(\frac{k}{n})| &= \left| |n\tau_k|^{-1/2} (n\nabla_s \tau_k - \frac{1}{2} n\nabla_s \tau_k^T) - |c_t|^{-1/2} (\nabla_s c_t(\frac{k}{n}) - \frac{1}{2} \nabla_s c_t(\frac{k}{n})^T) \right| \\ &\leq \left| |n\tau_k|^{-1/2} - |c_t(\frac{k}{n})|^{-1/2} \right| \cdot |n\nabla_s \tau_k| + |c_t|^{-1/2} |n\nabla_s \tau_k - \nabla_s c_t(\frac{k}{n})|, \end{aligned}$$

where once again  $\left| |n\tau_k|^{-1/2} - |c_t(\frac{k}{n})|^{-1/2} \right| \rightarrow 0$  and  $|n\nabla_s \tau_k|$  is bounded. The last term can be bounded, for  $n$  large enough, by

$$\begin{aligned} |n\nabla_s \tau_k - \nabla_s c_t(\frac{k}{n})| &\leq |n\nabla_s \tau_k - n(c_s(\frac{k+1}{n})^\parallel - c_s(\frac{k}{n}))| + |\nabla_t c_s(\frac{k}{n}) - n(c_s(\frac{k+1}{n})^\parallel - c_s(\frac{k}{n}))| \\ &\leq n|1 - b_k^{-1}| \cdot |c_s(\frac{k+1}{n})^\parallel - c_s(\frac{k}{n})| + \frac{1}{n} |\nabla_t \nabla_t c_s|_\infty \\ &\leq \frac{1}{n} (|\nabla_t c_s|_\infty + |\nabla_t \nabla_t c_s|_\infty), \end{aligned}$$

since  $\nabla_s \tau_k = (D_\tau \alpha')_k = (x_{k+1}^\parallel - x_k)^T + b_k^{-1}(x_{k+1}' - x_k')$  and  $b_k^{-1} \rightarrow 1$ . Finally, we can see that

$$\begin{aligned} |\hat{r}_0(s) - r_0(s)| &\leq \frac{1}{n} |R_0 - R(0)| + \frac{1}{n} \sum_{\ell=0}^{n-2} \left| f_0^{(-)} \circ \dots \circ f_\ell^{(-)}(R_{\ell+1}) - P_c^{\frac{\ell+1}{n}, 0}(R_{\ell+1}) \right| \\ &\quad + \frac{1}{n} \sum_{\ell=0}^{n-2} |R_{\ell+1} - R(\frac{\ell+1}{n})| \end{aligned}$$

goes to 0 when  $n \rightarrow \infty$ . We can show in a similar way that  $|g_k^{(-)}(\hat{r}_k) - |q_k|(r_k + r_k^T)| \rightarrow 0$  when  $n \rightarrow \infty$ .  $\square$

**Proposition 7** (Discrete exponential map). *Let  $[0, 1] \ni s \mapsto \alpha(s) = (x_0(s), \dots, x_n(s))$  be a geodesic path in  $M^{n+1}$ . For all  $s \in [0, 1]$ , the coordinates of its acceleration  $\nabla_s \alpha'(s)$  can be iteratively computed in the following way*

$$\begin{aligned} \nabla_s x_0' &= -\frac{1}{n} \left( R_0 + f_0^{(-)}(R_1) + \dots + f_0^{(-)} \circ \dots \circ f_{n-2}^{(-)}(R_{n-1}) \right), \\ \nabla_s x_{k+1}^\parallel &= \nabla_s f_k(x_k') + f_k(\nabla_s x_k') + \frac{1}{n} \nabla_s g_k(\nabla_s q_k) + \frac{1}{n} g_k(\nabla_s \nabla_s q_k) + \mathcal{R}(\tau_k, Y_k)(x_{k+1}^\parallel), \end{aligned}$$

for  $k = 0, \dots, n-1$ , where the  $R_k$ 's are defined as in Proposition 6, the symbol  $\cdot^\parallel$  denotes the parallel transport from  $x_{k+1}(s)$  back to  $x_k(s)$  along the geodesic linking them, the maps

$\nabla_s f_k$  and  $\nabla_s g_k$  are given by Lemma 2,  $Y_k$  is given by Equation (18) and

$$\begin{aligned}\nabla_s \tau_k &= (D_\tau \alpha')_k, \quad \nabla_s q_k = \sqrt{\frac{n}{|\tau_k|}} \left( \nabla_s \tau_k - \frac{1}{2} \nabla_s \tau_k^T \right), \quad \nabla_s v_k = \frac{1}{|\tau_k|} (\nabla_s \tau_k - \nabla_s \tau_k^T), \\ \nabla_s \nabla_s q_k &= -\frac{1}{n} g_k^{(-)} \left( R_{k+1} + f_{k+1}^{(-)}(R_{k+2}) + \dots + f_{k+1}^{(-)} \circ \dots \circ f_{n-2}^{(-)}(R_{n-1}) \right).\end{aligned}$$

*Proof of Proposition 7.* For all  $s \in [0, 1]$ , we initialize  $\nabla_s x_k'(s)$  for  $k = 0$  using the first geodesic equation in (19); the difficulty lies in deducing  $\nabla_s x_{k+1}'(s)$  from  $\nabla_s x_k'(s)$ . Just as we have previously obtained (40), we can obtain by replacing the derivatives with respect to  $a$  by derivatives with respect to  $s$

$$(43) \quad \begin{aligned}x_{k+1}'^{\parallel} &= x_k'^T + a_k x_k'^N + \nabla_s q_k^T + b_k \nabla_s \tau_k^N, \\ x_{k+1}'^{\parallel} &= f_k(x_k') + \frac{1}{n} g_k(\nabla_s q_k),\end{aligned}$$

and by differentiating with respect to  $s$

$$(44) \quad \nabla_s (x_{k+1}'^{\parallel}) = \nabla_s f_k(x_k') + f_k(\nabla_s x_k') + \frac{1}{n} \nabla_s g_k(\nabla_s q_k) + \frac{1}{n} g_k(\nabla_s \nabla_s q_k).$$

We have already computed (38) the covariant derivative of a vector field  $s \mapsto w_{k+1}(s)^{\parallel} \in T_{x_k(s)}M$  and so we can write

$$\nabla_s (x_{k+1}'^{\parallel}) = (\nabla_s x_{k+1}')^{\parallel} + \mathcal{R}(Y_k, \tau_k)(x_{k+1}'^{\parallel}),$$

where  $Y_k$  is defined by Equation (18). Together with Equation (44), this gives the desired equation for  $\nabla_s x_{k+1}'^{\parallel}$ . Finally,  $\nabla_s \tau_k = (D_\tau \alpha')_k$  results directly from (43),  $\nabla_s \nabla_s q_k$  is deduced from the second geodesic equation and the remaining equations follow from simple computation.  $\square$

**Proposition 8** (Discrete Jacobi fields). *Let  $[0, 1] \ni s \mapsto \alpha(s) = (x_0(s), \dots, x_n(s))$  be a geodesic path in  $M^{n+1}$ ,  $[0, 1] \ni s \mapsto J(s) = (J_0(s), \dots, J_n(s))$  a Jacobi field along  $\alpha$ , and  $(-\delta, \delta) \ni a \mapsto \alpha(a, \cdot)$  a corresponding family of geodesics, in the sense just described. Then  $J$  verifies the second order differential equations*

$$\begin{aligned}\nabla_s \nabla_s J_0 &= \mathcal{R}(x_0', J_0)x_0' - \frac{1}{n} \left( \nabla_a R_0 + f_0^{(-)}(\nabla_a R_1) + \dots + f_0^{(-)} \circ \dots \circ f_{n-2}^{(-)}(\nabla_a R_{n-1}) \right) \\ &\quad - \frac{1}{n} \sum_{k=0}^{n-2} \sum_{\ell=0}^k f_0^{(-)} \circ \dots \circ \nabla_a (f_\ell^{(-)}) \circ \dots \circ f_k^{(-)}(R_{k+1}), \\ \nabla_s \nabla_s J_{k+1}^{\parallel} &= f_k(\nabla_s \nabla_s J_k) + 2\nabla_s f_k(\nabla_s J_k) + \nabla_s \nabla_s f_k(J_k) + \frac{1}{n} g_k(\nabla_s \nabla_s \nabla_a q_k) \\ &\quad + \frac{2}{n} \nabla_s g_k(\nabla_s \nabla_a q_k) + \frac{1}{n} \nabla_s \nabla_s g_k(\nabla_a q_k) + 2\mathcal{R}(\tau_k, Y_k)(\nabla_s J_{k+1}^{\parallel}) \\ &\quad + \mathcal{R}(\nabla_s \tau_k, Y_k)(J_{k+1}^{\parallel}) + \mathcal{R}(\tau_k, \nabla_s Y_k)(J_{k+1}^{\parallel}) + \mathcal{R}(\tau_k, Y_k)(\mathcal{R}(Y_k, \tau_k)(J_{k+1}^{\parallel})),\end{aligned}$$

for all  $0 \leq k \leq n-1$ , where  $R_k := \mathcal{R}(q_k, \nabla_s q_k) x_k'$  and the various covariant derivatives according to  $a$  can be expressed as functions of  $J$  and  $\nabla_s J$ ,

$$\begin{aligned} \nabla_a R_k &= \mathcal{R}(\nabla_a q_k, \nabla_s q_k) x_k' + \mathcal{R}(q_k, \nabla_s \nabla_a q_k + \mathcal{R}(J, x_k') q_k) x_k' + \mathcal{R}(q_k, \nabla_s q_k) \nabla_s J_k, \\ \nabla_a q_k &= \sqrt{\frac{n}{|\tau_k|}} \left( \nabla_a \tau_k - \frac{1}{2} \nabla_a \tau_k^T \right), \quad \nabla_a \tau_k = (D_\tau J)_k, \quad \nabla_a v_k = \frac{1}{|\tau_k|} (\nabla_a \tau_k - \nabla_a \tau_k^T), \\ \nabla_s \nabla_a q_k &= n g_k^{-1} ((\nabla_s J_{k+1})^\parallel + \mathcal{R}(Y_k, \tau_k)(J_{k+1}^\parallel) - \nabla_s f_k(J_k) - f_k(\nabla_s J_k)) \\ &\quad + n \nabla_s (g_k^{-1})(J_{k+1}^\parallel - f_k(J_k)), \\ \nabla_s \nabla_s \nabla_a q_k &= \mathcal{R}(\nabla_s x_k', J_k) q_k + \mathcal{R}(x_k', \nabla_s J_k) q_k + 2\mathcal{R}(x_k', J_k) \nabla_s q_k \\ &\quad - \frac{1}{n} \sum_{\ell=k+1}^{n-1} g_k^{(-)} \circ f_{k+1}^{(-)} \circ \dots \circ f_{\ell-1}^{(-)} (\nabla_a R_\ell) - \frac{1}{n} \sum_{\ell=k+1}^{n-1} \sum_{j=k}^{\ell-1} g_k^{(-)} \circ \dots \circ \nabla_a (f_j^{(-)}) \circ \dots \circ f_{\ell-1}^{(-)} (R_\ell), \\ \nabla_s Y_k &= (\nabla_s x_k')^T + b_k (\nabla_s x_k')^N + (1 - b_k) (\langle x_k', \nabla_s v_k \rangle v_k + \langle x_k', v_k \rangle \nabla_s v_k) + \partial_s b_k (x_k')^N \\ &\quad + \frac{1}{2} (\nabla_s \nabla_s \tau_k)^T + K \frac{1 - a_k}{|\tau_k|^2} (\nabla_s \nabla_s \tau_k)^N + \partial_s \left( K \frac{1 - a_k}{|\tau_k|^2} \right) (\nabla_s \tau_k)^N \\ &\quad + \left( \frac{1}{2} - K \frac{1 - a_k}{|\tau_k|^2} \right) (\langle \nabla_s \tau_k, \nabla_s v_k \rangle v_k + \langle \nabla_s \tau_k, v_k \rangle \nabla_s v_k), \end{aligned}$$

with the notation conventions  $f_{k+1}^{(-)} \circ \dots \circ f_{k-1}^{(-)} := Id$ ,  $\sum_{\ell=n}^{n-1} := 0$  and with the maps

$$\begin{aligned} \nabla_a (f_k^{(-)})(w) &= (\nabla_a f_k)^{(-)}(w) + f_k \left( \mathcal{R}(Z_k, \tau_k)(w_{k+1}^\parallel) \right), \\ \nabla_a (g_k^{(-)})(w) &= (\nabla_a g_k)^{(-)}(w) + g_k \left( \mathcal{R}(Z_k, \tau_k)(w_{k+1}^\parallel) \right), \\ \nabla_s (g_k^{-1})(w) &= \partial_s (|q_k|^{-1}) |q_k| g_k^{-1}(w) + |q_k|^{-1} \partial_s (b_k^{-1}) w^N \\ &\quad + |q_k|^{-1} (1/2 - b_k^{-1}) (\langle w, \nabla_s v_k \rangle v_k + \langle w, v_k \rangle \nabla_s v_k), \end{aligned}$$

and

$$Z_k = J_k^T + b_k J_k^N + \frac{1}{2} \nabla_a \tau_k^T + K \frac{1 - a_k}{|\tau_k|^2} \nabla_a \tau_k^N.$$

*Proof of Proposition 8.* For all  $a \in (-\delta, \delta)$ ,  $\alpha(a, \cdot)$  verifies the geodesic equations (19). Taking the covariant derivative of these equations according to  $a$  we obtain

$$\begin{aligned} \nabla_a \nabla_s \partial_s x_0 + \frac{1}{n} \sum_{k=0}^{n-1} \nabla_a \left( f_0^{(-)} \circ \dots \circ f_{k-1}^{(-)} (\mathcal{R}(q_k, \nabla_s q_k) \partial_s x_k) \right) &= 0, \\ (45) \quad \nabla_a \nabla_s \nabla_s q_k + \frac{1}{n} \sum_{\ell=k+1}^{n-1} \nabla_a \left( g_k^{(-)} \circ f_{k+1}^{(-)} \circ \dots \circ f_{\ell-1}^{(-)} (\mathcal{R}(q_\ell, \nabla_s q_\ell) \partial_s x_\ell) \right) &= 0. \end{aligned}$$

Since for  $a = 0$ ,  $\nabla_a \nabla_s \partial_s x_0 = \nabla_s \nabla_s J_0 + \mathcal{R}(J_0, \partial_s x_0) \partial_s x_0$ , we get

$$\nabla_s \nabla_s J_0 = \mathcal{R}(\partial_s x_0, J_0) \partial_s x_0 - \frac{1}{n} \sum_{k=0}^{n-1} \nabla_a \left( f_0^{(-)} \circ \dots \circ f_{k-1}^{(-)} (\mathcal{R}(q_k, \nabla_s q_k) \partial_s x_k) \right),$$

and the differentiation

$$\nabla_a (f_0^{(-)} \circ \dots \circ f_{k-1}^{(-)} (R_k)) = f_0^{(-)} \circ \dots \circ f_{k-1}^{(-)} (\nabla_a R_k) + \sum_{\ell=0}^{k-1} f_0^{(-)} \circ \dots \circ \nabla_a (f_\ell^{(-)}) \circ \dots \circ f_{k-1}^{(-)} (R_k)$$

gives the desired equation for  $\nabla_s \nabla_s J_0$ . Now we will try to deduce  $\nabla_s \nabla_s J_{k+1}$  from (45). If  $J_{k+1}^\parallel(s)$  denotes the parallel transport of the vector  $J_{k+1}(s)$  from  $x_{k+1}(s)$  back to  $x_k(s)$  along the geodesic that links them, we know from (40) that

$$(46) \quad J_{k+1}^\parallel = f_k(J_k) + \frac{1}{n} g_k(\nabla_a q_k).$$

We also know from (38) that

$$(47) \quad (\nabla_s J_{k+1})^\parallel = \nabla_s(J_{k+1}^\parallel) + \mathcal{R}(\tau_k, Y_k)(J_{k+1}^\parallel),$$

and by iterating

$$\begin{aligned} (\nabla_s \nabla_s J_{k+1})^\parallel &= \nabla_s((\nabla_s J_{k+1})^\parallel) + \mathcal{R}(\tau_k, Y_k)((\nabla_s J_{k+1})^\parallel) \\ &= \nabla_s \nabla_s(J_{k+1}^\parallel) + \nabla_s(\mathcal{R}(\tau_k, Y_k)(J_{k+1}^\parallel)) + \mathcal{R}(\tau_k, Y_k)((\nabla_s J_{k+1})^\parallel) \end{aligned}$$

Developping and injecting Equation (46) in the latter gives

$$\begin{aligned} (\nabla_s \nabla_s J_{k+1})^\parallel &= \nabla_s \nabla_s(f_k(J_k)) + \frac{1}{n} \nabla_s \nabla_s(g_k(\nabla_a q_k)) + \mathcal{R}(\nabla_s \tau_k, Y_k)(J_{k+1}^\parallel) \\ &\quad + \mathcal{R}(\tau_k, \nabla_s Y_k)(J_{k+1}^\parallel) + \mathcal{R}(\tau_k, Y_k)(\mathcal{R}(Y_k, \tau_k)(J_{k+1}^\parallel)) + 2\mathcal{R}(\tau_k, Y_k)((\nabla_s J_{k+1})^\parallel). \end{aligned}$$

Developping the covariant derivatives  $\nabla_s \nabla_s(f_k(J_k))$  and  $\nabla_s \nabla_s(g_k(\nabla_a q_k))$  gives the desired formula. Now let us explicit the different terms involved in these differential equations. Since  $\nabla \mathcal{R} = 0$  and  $\nabla_a \partial_s x_k = \nabla_s \partial_a x_k$ , we have

$$\begin{aligned} \nabla_a R_k &= \mathcal{R}(\nabla_a q_k, \nabla_s q_k) \partial_s x_k + \mathcal{R}(q_k, \nabla_a \nabla_s q_k) \partial_s x_k + \mathcal{R}(q_k, \nabla_s q_k) \nabla_s J_k \\ &= \mathcal{R}(\nabla_a q_k, \nabla_s q_k) x_k' + \mathcal{R}(q_k, \nabla_s \nabla_a q_k + \mathcal{R}(J, x_k') q_k) x_k' + \mathcal{R}(q_k, \nabla_s q_k) \nabla_s J_k. \end{aligned}$$

By taking the inverse of (46) we get

$$\nabla_a q_k = n g_k^{-1}(J_{k+1}^\parallel - f_k(J_k)),$$

and taking the derivative according to  $s$  on both sides and injecting Equation (47) gives

$$\begin{aligned} \nabla_s \nabla_a q_k &= n g_k^{-1}((\nabla_s J_{k+1})^\parallel + \mathcal{R}(Y_k, \tau_k)(J_{k+1}^\parallel)) - \nabla_s f_k(J_k) - f_k(\nabla_s J_k) \\ &\quad + n \nabla_s(g_k^{-1})(J_{k+1}^\parallel - f_k(J_k)). \end{aligned}$$

To obtain  $\nabla_s \nabla_s \nabla_a q_k$ , notice that

$$\begin{aligned} \nabla_s \nabla_s \nabla_a q_k &= \nabla_s \nabla_a \nabla_s q_k + \nabla_s(\mathcal{R}(\partial_s x_k, J_k) q_k), \\ &= \nabla_a \nabla_s \nabla_s q_k + \mathcal{R}(\partial_s x_k, J_k) \nabla_s q_k + \nabla_s(\mathcal{R}(\partial_s x_k, J_k) q_k), \end{aligned}$$

and injecting Equation (45) with

$$\begin{aligned} \nabla_a(g_k^{(-)} \circ f_{k+1}^{(-)} \circ \dots \circ f_{\ell-1}^{(-)}(R_\ell)) &= g_k^{(-)} \circ f_{k+1}^{(-)} \circ \dots \circ f_{\ell-1}^{(-)}(\nabla_a R_\ell) \\ &\quad + \sum_{j=k}^{\ell-1} g_k^{(-)} \circ \dots \circ \nabla_a(f_j^{(-)}) \circ \dots \circ f_{\ell-1}^{(-)}(R_\ell), \end{aligned}$$

gives us the desired formula.  $\nabla_s Y_k$  results from simple differentiation, and differentiating the maps  $f_k^{(-)}$  and  $g_k^{(-)}$  with respect to  $a$  is completely analogous to the the computations of Lemma 3. Finally, the inverse of  $g_k$  is given by  $g_k^{-1} : T_{x_k} M \rightarrow T_{x_k} M$ ,

$$g_k^{-1} : w \mapsto |q_k|^{-1} (b_k^{-1} w + (\frac{1}{2} - b_k^{-1}) w^T),$$

and since  $\nabla_s(w^T) = (\nabla_s w)^T + \langle w, \nabla_s v_k \rangle v_k + \langle w, v_k \rangle \nabla_s v_k$ , it is straightforward to verify that  $\nabla_s(g_k^{-1})(w) = \nabla_s(g_k^{-1}(w)) - g_k^{-1}(\nabla_s w)$  gives

$$\begin{aligned} \nabla_s(g_k^{-1})(w) &= \partial_s(|q_k|^{-1}) |q_k| g_k^{-1}(w) + |q_k|^{-1} \partial_s(b_k^{-1}) w^N \\ &\quad + |q_k|^{-1} (1/2 - b_k^{-1}) (\langle w, \nabla_s v_k \rangle v_k + \langle w, v_k \rangle \nabla_s v_k). \end{aligned}$$

□



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