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# A new cumulative distribution function based on $m$ existing ones

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## Abstract

In this note, we present a new cumulative distribution function using sums and products of  $m$  existing cumulative distribution functions. Properties of such function are justified and using it we propose many distributions that exhibit various shapes for their probability density and hazard rate functions.

*Keywords:* Cumulative distribution function transformations, Probability density function, Statistical distributions, Hazard rate function.

2000 MSC: 60E05, 62E15.

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## 1. Introduction

In literature, several transformations exist to obtain a new cumulative distribution function (cdf) using other(s) well-known cdf(s). The most famous of them is the power transformation introduced by [3]. Using a cdf  $F(x)$ , the considered cdf is  $G(x) = (F(x))^\alpha$ ,  $\alpha \geq 1$ . For extensions and applications, see [4], [11] and [12], and the references therein. Another popular transformation is the quadratic rank transmutation map (QRTM) introduced by [13], where the considered cdf is  $G(x) = (1 + \lambda)F(x) - \lambda(F(x))^2$ ,  $\lambda \in [-1, 1]$ . Recent developments can be found in [1, 2], [5] and [6], and the references therein. Modern ideas include the DUS transformation proposed by [7]:  $G(x) = \frac{1}{e-1}(e^{F(x)} - 1)$ , the SS transformation introduced by [8]:  $G(x) = \sin\left(\frac{\pi}{2}F(x)\right)$  and the MG transformation studied by [9]:  $G(x) = e^{1 - \frac{1}{F(x)}}$ . An interesting approach is also given by the M transformation developed by [10], where using two cdfs  $F_1(x)$  and  $F_2(x)$ , the considered cdf is  $G(x) = \frac{F_1(x) + F_2(x)}{1 + F_1(x)}$ . In particular, [10] showed that the M transformation has great applications in data analysis. With specific cdfs  $F_1(x)$  and  $F_2(x)$ , it can better fit real data in comparison to some exploited distributions.

In this study, we propose a generalized version of the M transformation, called GM transformation. It is constructed from sums and products of  $m$  cdfs with  $m \geq 1$ . In comparison to the M transformation, it offers more possibility of cdf, mainly thanks to more flexibility on the denominator term. Then new distributions are derived, with the associated probability density function (pdf) and hazard rate function (hrf). In particular, some graphs of such functions related to new distributions based on Weibull and Cauchy with normal distributions are given and showing a wide variety of shapes, curves and asymmetries.

The note is organized as follows. In Section 2, we present our new transformation. Sections 3 and 4 apply it with specific well-known distributions, defining the associated pdfs and hrfs with some plots. Section 5 is devoted to the proof of our theorem.

## 2. GM transformation

Let  $m \geq 1$  be an integer,  $F_1(x), \dots, F_m(x)$  be  $m$  cdfs of continuous distribution(s) with common support and  $\delta_1, \dots, \delta_m$  be  $m$  binary numbers, i.e.  $\delta_k \in \{0, 1\}$  for any  $k \in \{1, \dots, m\}$ .

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We introduce the following transformation of  $F_1(x), \dots, F_m(x)$ :

$$G(x) = \frac{\sum_{k=1}^m F_k(x)}{m-1 + \prod_{k=1}^m (F_k(x))^{\delta_k}}, \quad (1)$$

with the imposed value  $\delta_m = 0$  in the special case where  $m = 1$ . The support of  $G(x)$  is the common one of  $F_1(x), \dots, F_m(x)$ .

The role of  $\delta_1, \dots, \delta_m$  is to activate or not the chosen cdfs in the product in the denominator. For examples, taking  $m = 2$ ,  $\delta_1 = 1$ , and  $\delta_2 = 1$ , the function (1) becomes  $G(x) = \frac{F_1(x)+F_2(x)}{1+F_1(x)F_2(x)}$ . Taking  $m = 3$ ,  $\delta_1 = 1$ ,  $\delta_2 = 1$  and  $\delta_3 = 0$ , the function (1) becomes  $G(x) = \frac{F_1(x)+F_2(x)+F_3(x)}{2+F_1(x)F_2(x)}$ ;  $F_3(x)$  is excluded of the denominator.

The following result motivates the interest of (1).

**Theorem 1.** *The function  $G(x)$  (1) possesses the properties of a cdf.*

The proof of Theorem 1 is given in Section 5.

Let us now present some immediate examples. Taking  $m = 1$  (so  $\delta_1 = 0$ ), we obtain the simple cdf  $G(x) = F_1(x)$ . The choice  $\delta_1 = \dots = \delta_m = 0$  gives an uniform mixture of cdfs:  $G(x) = \frac{1}{m} \sum_{k=1}^m F_k(x)$ . Finally, for  $m = 2$ ,  $\delta_1 = 1$  and  $\delta_2 = 0$ , we obtain the M transformation introduced by [10]:  $G(x) = \frac{F_1(x)+F_2(x)}{1+F_1(x)}$ .

For this reason, we will call (1) as the *GM* transformation (as Generalization of the M transformation). To the best of our knowledge, it is new in literature.

New cdfs can also be derived by the *GM* transformation and existing transformations. Some of them using only one cdf are described below.

- For any cdf  $F$  of continuous distribution with support equal to  $\mathbb{R}$  or  $[0, +\infty)$  or  $(-\infty, 0)$  and any real numbers  $\beta_1, \dots, \beta_m$ , where  $\beta_k > 0$  for any  $k \in \{1, \dots, m\}$ , the *GM* transformation includes the following cdf:

$$G(x) = \frac{\sum_{k=1}^m F(\beta_k x)}{m-1 + \prod_{k=1}^m (F(\beta_k x))^{\delta_k}}.$$

- Combining the *GM* transformation and the power transformation introduced by [3], for any cdf  $F$  of continuous distribution and any real numbers  $\alpha_1, \dots, \alpha_m$ , where  $\alpha_k \geq 1$  for any  $k \in \{1, \dots, m\}$ , we obtain the cdf:

$$G(x) = \frac{\sum_{k=1}^m (F(x))^{\alpha_k}}{m-1 + \prod_{k=1}^m (F(x))^{\delta_k \alpha_k}}.$$

- Combining the *GM* transformation and the transformation using QRTM introduced by [13], for any cdf  $F$  of continuous distribution and any real numbers  $\lambda_1, \dots, \lambda_m$ , where  $\lambda_k \in [-1, 1]$  for any  $k \in \{1, \dots, m\}$ , we obtain the cdf:

$$G(x) = \frac{\sum_{k=1}^m ((1 + \lambda_k)F(x) - \lambda_k(F(x))^2)}{m-1 + \prod_{k=1}^m ((1 + \lambda_k)F(x) - \lambda_k(F(x))^2)^{\delta_k}}.$$

Others interesting combinations are possible according to the problem. Thanks to their adaptability, with a specific  $F(x)$ , these cdfs are of interest from the theoretical and applied aspects.

### 3. A particular case with some related new distributions

If we chose  $F_1(x) = \dots = F_m(x) = F(x)$  and  $\delta_1, \dots, \delta_m$  such that  $\sum_{k=1}^m \delta_k = q$  with  $q \in \{0, \dots, m\}$ , the GM transformation yields the following cdf:

$$G(x) = \frac{mF(x)}{m-1 + (F(x))^q}.$$

Let  $f$  be an associated pdf to  $F$ . Then an associated pdf to  $G$  is given by

$$g(x) = \frac{m(m-1 - (q-1)(F(x))^q)f(x)}{(m-1 + (F(x))^q)^2}.$$

The associated hrf is given by

$$h(x) = \frac{m(m-1 - (q-1)(F(x))^q)f(x)}{(m-1 + (F(x))^q)(m-1 + (F(x))^q - mF(x))}.$$

**Remark 1.** For this special case, note that  $G$  is still a cdf for any real numbers  $m > 1$  and  $q$  such that  $q \in [0, m)$ .

The case  $m = 2$  and  $q = 1$  corresponds to a particular case of the M transformation studied in [10]. New distributions arise from these functions. Some of them with potential interest are presented below.

- Considering the uniform distribution on  $[0, 1]$ , we have  $F(x) = x\mathbf{1}_{[0,1]}(x) + \mathbf{1}_{(1,+\infty)}(x)$ ,

$$G(x) = \frac{mx}{m-1+x^q}\mathbf{1}_{[0,1]}(x) + \mathbf{1}_{(1,+\infty)}(x), \quad g(x) = \frac{m(m-1-(q-1)x^q)}{(m-1+x^q)^2}\mathbf{1}_{[0,1]}(x)$$

and

$$h(x) = \frac{m(m-1-(q-1)x^q)}{(m-1+x^q)(m-1+x^q-mx)}\mathbf{1}_{[0,1]}(x).$$

- Considering the exponential distribution with parameter  $\lambda > 0$ , we have

$$F(x) = (1 - e^{-\lambda x})\mathbf{1}_{[0,+\infty)}(x),$$

$$G(x) = \frac{m(1 - e^{-\lambda x})}{m-1 + (1 - e^{-\lambda x})^q}\mathbf{1}_{[0,+\infty)}(x), \quad g(x) = \frac{m\lambda(m-1-(q-1)(1 - e^{-\lambda x})^q)e^{-\lambda x}}{(m-1 + (1 - e^{-\lambda x})^q)^2}\mathbf{1}_{[0,+\infty)}(x)$$

and

$$h(x) = \frac{m\lambda(m-1-(q-1)(1 - e^{-\lambda x})^q)e^{-\lambda x}}{(m-1 + (1 - e^{-\lambda x})^q)((1 - e^{-\lambda x})^q + me^{-\lambda x} - 1)}\mathbf{1}_{[0,+\infty)}(x).$$

- Considering the logistic distribution with parameters  $\mu \in \mathbb{R}$  and  $s > 0$ , we have

$$F(x) = \left(1 + e^{-\left(\frac{x-\mu}{s}\right)}\right)^{-1}, \quad x \in \mathbb{R},$$

$$G(x) = \frac{m\left(1 + e^{-\left(\frac{x-\mu}{s}\right)}\right)^{-1}}{m-1 + \left(1 + e^{-\left(\frac{x-\mu}{s}\right)}\right)^{-q}}, \quad g(x) = \frac{m\left(m-1-(q-1)\left(1 + e^{-\left(\frac{x-\mu}{s}\right)}\right)^{-q}\right)e^{-\left(\frac{x-\mu}{s}\right)}}{s\left(1 + e^{-\left(\frac{x-\mu}{s}\right)}\right)^2\left(m-1 + \left(1 + e^{-\left(\frac{x-\mu}{s}\right)}\right)^{-q}\right)^2}$$

and

$$h(x) = \frac{m\left(m-1-(q-1)\left(1 + e^{-\left(\frac{x-\mu}{s}\right)}\right)^{-q}\right)e^{-\left(\frac{x-\mu}{s}\right)}}{s\left(1 + e^{-\left(\frac{x-\mu}{s}\right)}\right)^2\left(m-1 + \left(1 + e^{-\left(\frac{x-\mu}{s}\right)}\right)^{-q}\right)\left(m-1 + \left(1 + e^{-\left(\frac{x-\mu}{s}\right)}\right)^{-q} - m\left(1 + e^{-\left(\frac{x-\mu}{s}\right)}\right)^{-1}\right)}.$$

- Considering the Cauchy distribution with parameters  $x_0 \in \mathbb{R}$  and  $a > 0$ , we have

$$F(x) = \frac{1}{\pi} \arctan\left(\frac{x-x_0}{a}\right) + \frac{1}{2}, \quad x \in \mathbb{R},$$

$$G(x) = \frac{m\left(\frac{1}{\pi} \arctan\left(\frac{x-x_0}{a}\right) + \frac{1}{2}\right)}{m-1 + \left(\frac{1}{\pi} \arctan\left(\frac{x-x_0}{a}\right) + \frac{1}{2}\right)^q}$$

and

$$g(x) = \frac{ma(m-1-(q-1)\left(\frac{1}{\pi} \arctan\left(\frac{x-x_0}{a}\right) + \frac{1}{2}\right)^q)}{\pi((x-x_0)^2 + a^2)\left(m-1 + \left(\frac{1}{\pi} \arctan\left(\frac{x-x_0}{a}\right) + \frac{1}{2}\right)^q\right)^2}.$$

- Considering the normal distribution with parameters  $\mu \in \mathbb{R}$  and  $\sigma > 0$ , we have

$$F(x) = \int_{-\infty}^x \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(t-\mu)^2}{2\sigma^2}} dt = \Phi(x), \quad x \in \mathbb{R},$$

$$G(x) = \frac{m\Phi(x)}{m-1 + (\Phi(x))^q},$$

$$g(x) = \frac{(m-1-(q-1)(\Phi(x))^q) e^{-\frac{(x-\mu)^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}(m-1 + (\Phi(x))^q)^2}$$

and

$$h(x) = \frac{m(m-1-(q-1)(\Phi(x))^q) e^{-\frac{(x-\mu)^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}(m-1 + (\Phi(x))^q)(m-1 + (\Phi(x))^q - m\Phi(x))}.$$

- Considering the Weibull distribution with parameters  $k > 0$  and  $\lambda > 0$ , we have

$$F(x) = \left(1 - e^{-\left(\frac{x}{\lambda}\right)^k}\right) \mathbf{1}_{[0,+\infty)}(x),$$

$$G(x) = \frac{m\left(1 - e^{-\left(\frac{x}{\lambda}\right)^k}\right)}{m-1 + \left(1 - e^{-\left(\frac{x}{\lambda}\right)^k}\right)^q} \mathbf{1}_{[0,+\infty)}(x), \quad (2)$$

$$g(x) = \frac{m\frac{k}{\lambda}\left(\frac{x}{\lambda}\right)^{k-1}\left(m-1-(q-1)\left(1 - e^{-\left(\frac{x}{\lambda}\right)^k}\right)^q\right) e^{-\left(\frac{x}{\lambda}\right)^k}}{\left(m-1 + \left(1 - e^{-\left(\frac{x}{\lambda}\right)^k}\right)^q\right)^2} \mathbf{1}_{[0,+\infty)}(x) \quad (3)$$

and

$$h(x) = \frac{m\frac{k}{\lambda}\left(\frac{x}{\lambda}\right)^{k-1}\left(m-1-(q-1)\left(1 - e^{-\left(\frac{x}{\lambda}\right)^k}\right)^q\right) e^{-\left(\frac{x}{\lambda}\right)^k}}{\left(m-1 + \left(1 - e^{-\left(\frac{x}{\lambda}\right)^k}\right)^q\right)\left(\left(1 - e^{-\left(\frac{x}{\lambda}\right)^k}\right)^q + me^{-\left(\frac{x}{\lambda}\right)^k} - 1\right)} \mathbf{1}_{[0,+\infty)}(x). \quad (4)$$

For this case, particularly rich, we denote the associated distribution by  $GM_W(m, q, k, \lambda)$ . The case  $m = 2$  and  $q = 1$  corresponds to the distribution  $M_W(k, \lambda)$  introduced by [10]. Our distribution has the advantage to offer more flexibility thanks to the additional parameters  $m$  and  $q$ , opening the door to many applications in data analysis. In order to illustrate the potential of applicability of  $GM_W(m, q, k, \lambda)$ , some graphs of the associated cdf, pdf and hrf are presented in Figures 1, 2 and 3 showing various shapes, curves and asymmetries.

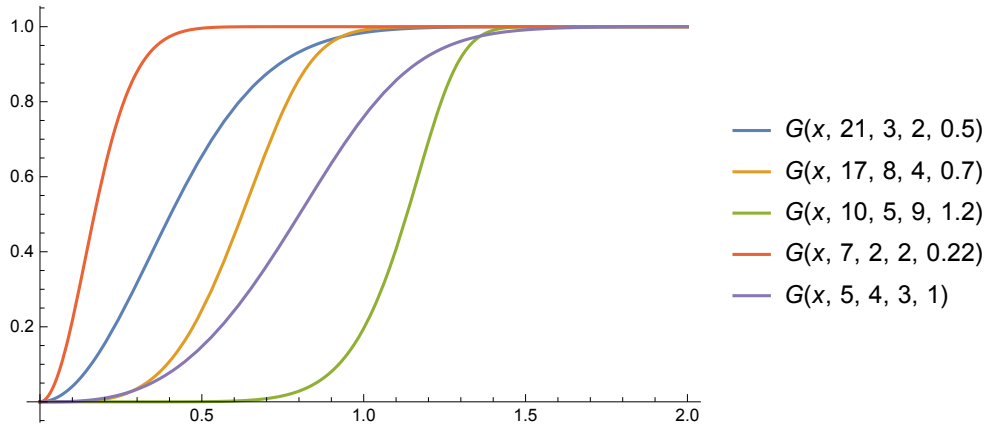


Figure 1: Some cdfs  $G(x) = G(x, m, q, k, \lambda)$  (2) associated to the distribution  $GM_W(m, q, k, \lambda)$ .

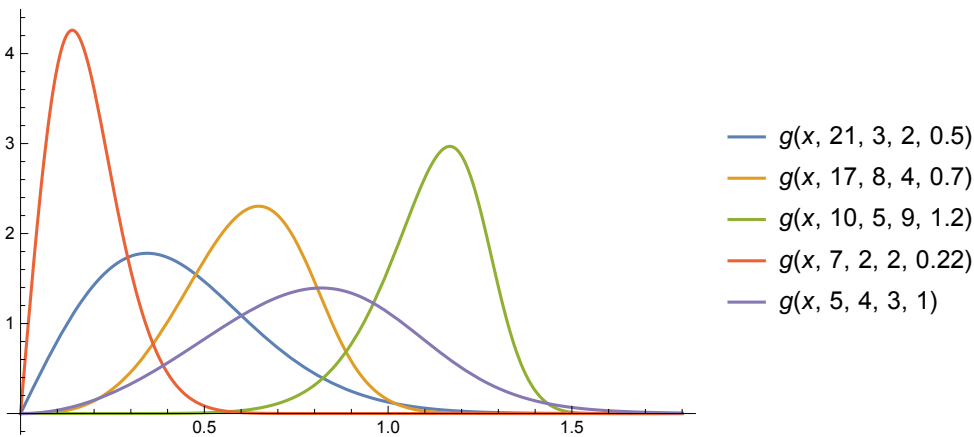


Figure 2: Some pdfs  $g(x) = g(x, m, q, k, \lambda)$  (3) associated to the distribution  $GM_W(m, q, k, \lambda)$ .

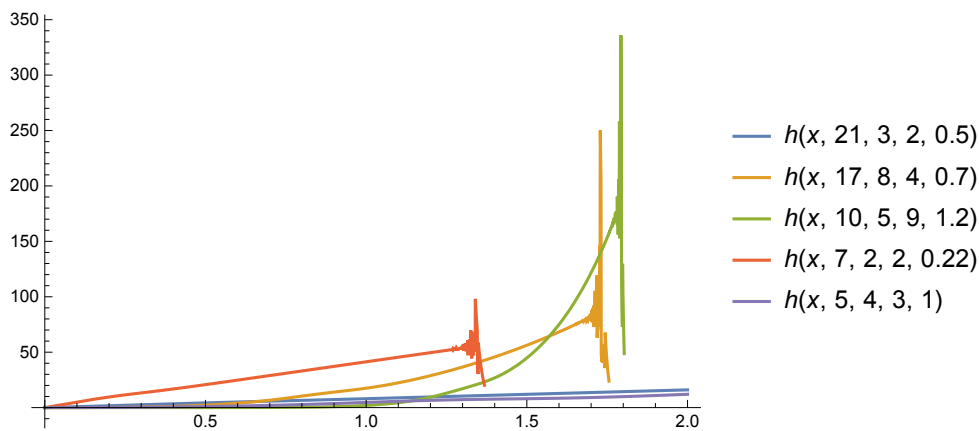


Figure 3: Some hrfs  $h(x) = h(x, m, q, k, \lambda)$  (4) associated to the distribution  $GM_W(m, q, k, \lambda)$ .

#### 4. Another case with some related new distributions

If we chose  $m = 2$  and  $\delta_1 = \delta_2 = 1$ , then the  $GM$  transformation is reduced to the following form

$$G(x) = \frac{F_1(x) + F_2(x)}{1 + F_1(x)F_2(x)}.$$

The main difference with  $G$  and the cdf proposed by [10] is the function  $F_2$  in the denominator, leading new cdf. The associated pdf is given by

$$g(x) = \frac{f_1(x)(1 - (F_2(x))^2) + f_2(x)(1 - (F_1(x))^2)}{(1 + F_1(x)F_2(x))^2}.$$

The associated hrf is given by

$$h(x) = \frac{f_1(x)(1 - (F_2(x))^2) + f_2(x)(1 - (F_1(x))^2)}{(1 + F_1(x)F_2(x))(1 - F_1(x))(1 - F_2(x))}.$$

New distributions can arise from the expressions above and some of them are presented below.

- Considering the cdf  $F_1$  of the power distribution with parameters  $\alpha > 0$  and the cdf  $F_2$  of the power distribution with parameters  $\beta > 0$ . Then we have  $F_1(x) = x^\alpha \mathbf{1}_{[0,1]}(x) + \mathbf{1}_{(1,+\infty)}(x)$ ,  $F_2(x) = x^\beta \mathbf{1}_{[0,1]}(x) + \mathbf{1}_{(1,+\infty)}(x)$ ,

$$G(x) = \frac{x^\alpha + x^\beta}{1 + x^{\alpha+\beta}} \mathbf{1}_{[0,1]}(x) + \mathbf{1}_{(1,+\infty)}(x),$$

$$g(x) = \frac{\alpha x^{\alpha-1}(1 - x^{2\beta}) + \beta x^{\beta-1}(1 - x^{2\alpha})}{(1 + x^{\alpha+\beta})^2} \mathbf{1}_{[0,1]}(x)$$

and

$$h(x) = \frac{\alpha x^{\alpha-1}(1 - x^{2\beta}) + \beta x^{\beta-1}(1 - x^{2\alpha})}{(1 + x^{\alpha+\beta})(1 - x^\alpha)(1 - x^\beta)} \mathbf{1}_{[0,1]}(x).$$

- Considering the cdf  $F_1$  of the Weibull distribution with parameters  $k_1 > 0$  and  $\lambda_1 > 0$  and the cdf  $F_2$  of the Weibull distribution with parameters  $k_2 > 0$  and  $\lambda_2 > 0$ . Then we have  $F_1(x) = \left(1 - e^{-\left(\frac{x}{\lambda_1}\right)^{k_1}}\right) \mathbf{1}_{[0,+\infty)}(x)$ ,  $F_2(x) = \left(1 - e^{-\left(\frac{x}{\lambda_2}\right)^{k_2}}\right) \mathbf{1}_{[0,+\infty)}(x)$ ,

$$G(x) = \frac{2 - e^{-\left(\frac{x}{\lambda_1}\right)^{k_1}} - e^{-\left(\frac{x}{\lambda_2}\right)^{k_2}}}{2 - e^{-\left(\frac{x}{\lambda_1}\right)^{k_1}} - e^{-\left(\frac{x}{\lambda_2}\right)^{k_2}} + e^{-\left(\frac{x}{\lambda_1}\right)^{k_1} - \left(\frac{x}{\lambda_2}\right)^{k_2}}} \mathbf{1}_{[0,+\infty)}(x),$$

$$g(x) = \frac{\frac{k_1}{\lambda_1} \left(\frac{x}{\lambda_1}\right)^{k_1-1} e^{-\left(\frac{x}{\lambda_1}\right)^{k_1}} \left(1 - \left(1 - e^{-\left(\frac{x}{\lambda_2}\right)^{k_2}}\right)^2\right)}{\left(2 - e^{-\left(\frac{x}{\lambda_1}\right)^{k_1}} - e^{-\left(\frac{x}{\lambda_2}\right)^{k_2}} + e^{-\left(\frac{x}{\lambda_1}\right)^{k_1} - \left(\frac{x}{\lambda_2}\right)^{k_2}}\right)^2} \mathbf{1}_{[0,+\infty)}(x)$$

$$+ \frac{\frac{k_2}{\lambda_2} \left(\frac{x}{\lambda_2}\right)^{k_2-1} e^{-\left(\frac{x}{\lambda_2}\right)^{k_2}} \left(1 - \left(1 - e^{-\left(\frac{x}{\lambda_1}\right)^{k_1}}\right)^2\right)}{\left(2 - e^{-\left(\frac{x}{\lambda_1}\right)^{k_1}} - e^{-\left(\frac{x}{\lambda_2}\right)^{k_2}} + e^{-\left(\frac{x}{\lambda_1}\right)^{k_1} - \left(\frac{x}{\lambda_2}\right)^{k_2}}\right)^2} \mathbf{1}_{[0,+\infty)}(x)$$

and

$$\begin{aligned}
h(x) &= \frac{\frac{k_1}{\lambda_1} \left(\frac{x}{\lambda_1}\right)^{k_1-1} e^{-\left(\frac{x}{\lambda_1}\right)^{k_1}} \left(1 - \left(1 - e^{-\left(\frac{x}{\lambda_2}\right)^{k_2}}\right)^2\right)}{\left(2 - e^{-\left(\frac{x}{\lambda_1}\right)^{k_1}} - e^{-\left(\frac{x}{\lambda_2}\right)^{k_2}} + e^{-\left(\frac{x}{\lambda_1}\right)^{k_1} - \left(\frac{x}{\lambda_2}\right)^{k_2}}\right) e^{-\left(\frac{x}{\lambda_1}\right)^{k_1}} e^{-\left(\frac{x}{\lambda_2}\right)^{k_2}}} \mathbf{1}_{[0,+\infty)}(x) \\
&+ \frac{\frac{k_2}{\lambda_2} \left(\frac{x}{\lambda_2}\right)^{k_2-1} e^{-\left(\frac{x}{\lambda_2}\right)^{k_2}} \left(1 - \left(1 - e^{-\left(\frac{x}{\lambda_1}\right)^{k_1}}\right)^2\right)}{\left(2 - e^{-\left(\frac{x}{\lambda_1}\right)^{k_1}} - e^{-\left(\frac{x}{\lambda_2}\right)^{k_2}} + e^{-\left(\frac{x}{\lambda_1}\right)^{k_1} - \left(\frac{x}{\lambda_2}\right)^{k_2}}\right) e^{-\left(\frac{x}{\lambda_1}\right)^{k_1}} e^{-\left(\frac{x}{\lambda_2}\right)^{k_2}}} \mathbf{1}_{[0,+\infty)}(x).
\end{aligned}$$

- Considering the cdf  $F_1$  of the Cauchy distribution with parameters 0 and 1 and the cdf  $F_2$  of the normal distribution with parameters  $\mu \in \mathbb{R}$  and  $\sigma > 0$ . Then we have  $F_1(x) = \frac{1}{\pi} \arctan(x) + \frac{1}{2}$ ,  $F_2(x) = \int_{-\infty}^x \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(t-\mu)^2}{2\sigma^2}} dt = \Phi(x)$ ,  $x \in \mathbb{R}$ ,

$$G(x) = \frac{\frac{1}{\pi} \arctan(x) + \frac{1}{2} + \Phi(x)}{1 + \left(\frac{1}{\pi} \arctan(x) + \frac{1}{2}\right) \Phi(x)}, \quad (5)$$

$$g(x) = \frac{\frac{1}{\pi(x^2+1)} (1 - (\Phi(x))^2) + \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \left(1 - \left(\frac{1}{\pi} \arctan(x) + \frac{1}{2}\right)^2\right)}{\left(1 + \left(\frac{1}{\pi} \arctan(x) + \frac{1}{2}\right) \Phi(x)\right)^2} \quad (6)$$

and

$$h(x) = \frac{\frac{1}{\pi(x^2+1)} (1 - (\Phi(x))^2) + \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \left(1 - \left(\frac{1}{\pi} \arctan(x) + \frac{1}{2}\right)^2\right)}{\left(1 + \left(\frac{1}{\pi} \arctan(x) + \frac{1}{2}\right) \Phi(x)\right) \left(\frac{1}{2} - \frac{1}{\pi} \arctan(x)\right) \Phi(-x)}. \quad (7)$$

Some graphs of these three functions for arbitrary values of  $(\mu, \sigma)$  are given in Figures 4, 5 and 6. Again, we see different kinds of shapes, curves and asymmetries, which can be of interest for the statistician in a analysis data context.



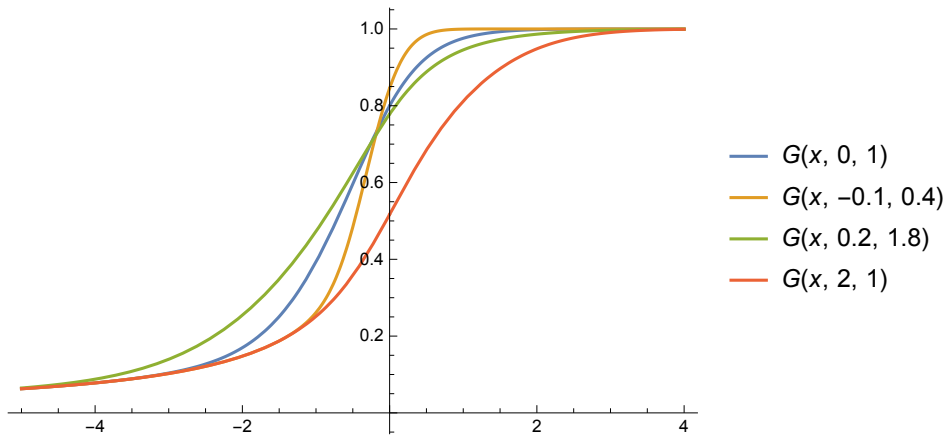


Figure 4: Some cdfs  $G(x) = G(x, \mu, \sigma)$  (5) with various values for  $\mu$  and  $\lambda$ .

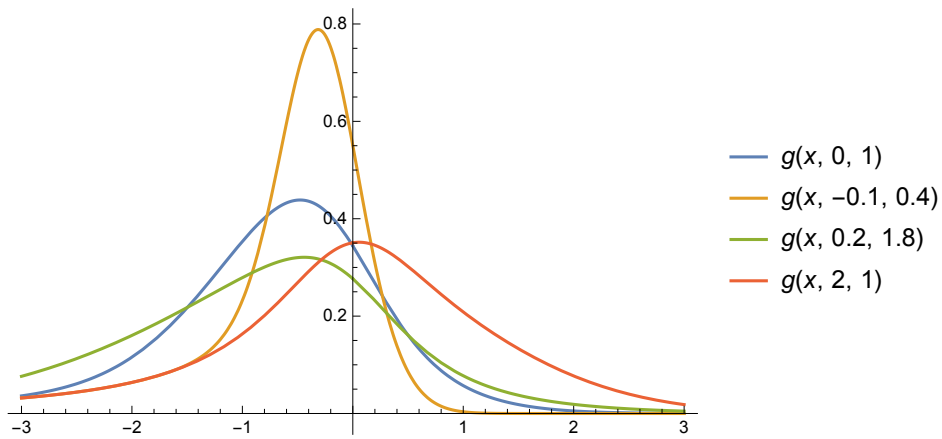


Figure 5: Some pdfs  $g(x) = g(x, \mu, \sigma)$  (6) with various values for  $\mu$  and  $\lambda$ .

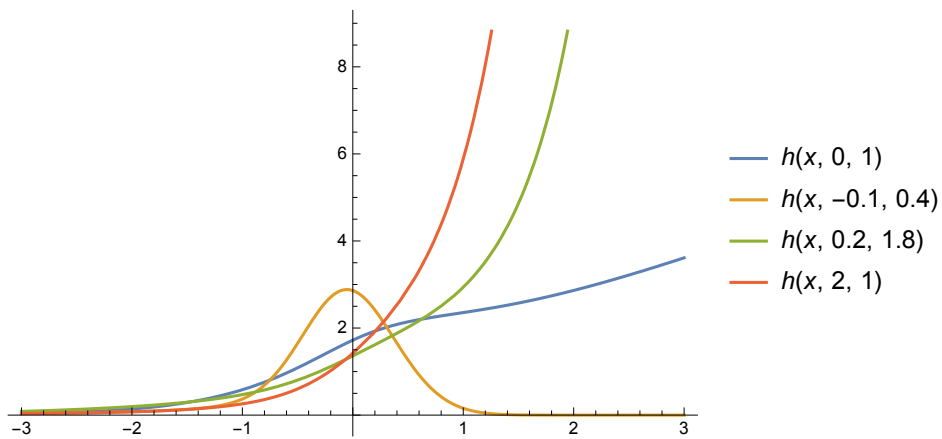


Figure 6: Some hrfs  $h(x) = h(x, \mu, \sigma)$  (6) with various values for  $\mu$  and  $\lambda$ .

## 5. Proofs

*Proof of Theorem 1.* For any  $k \in \{1, \dots, m\}$ , let  $f_k(x)$  be a pdf associated to the cdf  $F_k(x)$ . Recall that  $F_k(x)$  is continuous with  $F_k(x) \in [0, 1]$ ,  $\lim_{x \rightarrow +\infty} F_k(x) = 1$ ,  $\lim_{x \rightarrow -\infty} F_k(x) = 0$  and  $f_k(x) = F'_k(x)$  almost everywhere with  $f_k(x) \geq 0$ . Let us now investigate the sufficient conditions for  $G(x)$  to be a cdf.

- Since  $\sum_{k=1}^m F_k(x)$  and  $m-1 + \prod_{k=1}^m (F_k(x))^{\delta_k}$  are continuous functions with  $m-1 + \prod_{k=1}^m (F_k(x))^{\delta_k} \neq 0$ ,  $G(x)$  is a continuous function of  $x$ .
- Let us prove that  $G(x) \in [0, 1]$ . Owing to  $\sum_{k=1}^m F_k(x) \geq 0$  and  $m-1 + \prod_{k=1}^m (F_k(x))^{\delta_k} > 0$ , we have  $G(x) \geq 0$ . On the other hand, using the inequality:  $\prod_{k=1}^m (1-x_k) \geq 1 - \sum_{k=1}^m x_k$ ,  $x_k \in [0, 1]$ , with  $x_k = 1 - (F_k(x))^{\delta_k} \in [0, 1]$  and observing that  $(F_k(x))^{\delta_k} \geq F_k(x)$ , we obtain

$$\prod_{k=1}^m (F_k(x))^{\delta_k} \geq 1 - \sum_{k=1}^m (1 - (F_k(x))^{\delta_k}) = 1 - m + \sum_{k=1}^m (F_k(x))^{\delta_k} \geq 1 - m + \sum_{k=1}^m F_k(x).$$

Hence  $G(x) \leq 1$ .

- Let us prove that  $G'(x) \geq 0$ . For any derivable function  $u(x)$ , note that  $((u(x))^{\delta_k})' = \delta_k u'(x)$  since  $\delta_k \in \{0, 1\}$ . Therefore we have  $G'(x) = \frac{A(x)}{B(x)}$  almost everywhere, where

$$A(x) = \left( \sum_{k=1}^m f_k(x) \right) \left( m-1 + \prod_{k=1}^m (F_k(x))^{\delta_k} \right) - \left( \sum_{k=1}^m F_k(x) \right) \left( \sum_{k=1}^m \delta_k f_k(x) \prod_{\substack{u=1 \\ u \neq k}}^m (F_u(x))^{\delta_u} \right)$$

and

$$B(x) = \left( m-1 + \prod_{k=1}^m (F_k(x))^{\delta_k} \right)^2.$$

We have  $B(x) > 0$ . Let us now investigate the sign of  $A(x)$ . The following decomposition holds:  $A(x) = A_1(x) + A_2(x)$ , where

$$A_1(x) = \sum_{k=1}^m \delta_k f_k(x) \left( m-1 + \prod_{u=1}^m (F_u(x))^{\delta_u} - \left( \sum_{v=1}^m F_v(x) \right) \prod_{\substack{u=1 \\ u \neq k}}^m (F_u(x))^{\delta_u} \right)$$

and

$$A_2(x) = \sum_{k=1}^m (1 - \delta_k) f_k(x) \left( m-1 + \prod_{k=1}^m (F_k(x))^{\delta_k} \right).$$

Since  $A_2(x) \geq 0$  as a sum of positive terms, let us focus on the sign of  $A_1(x)$ . Observe that, if  $\delta_k = 1$ , we have  $F_k(x) \prod_{\substack{u=1 \\ u \neq k}}^m (F_u(x))^{\delta_u} = \prod_{u=1}^m (F_u(x))^{\delta_u}$ . If  $\delta_k = 0$ , the  $k$ -th term in the sum of

$A_1(x)$  is zero. Therefore we can write

$$A_1(x) = \sum_{k=1}^m \delta_k f_k(x) \left( m-1 - \left( \sum_{\substack{v \neq k \\ v=1}}^m F_v(x) \right) \prod_{\substack{u=1 \\ u \neq k}}^m (F_u(x))^{\delta_u} \right).$$

Since  $F_v(x) \prod_{\substack{u=1 \\ u \neq k}}^m (F_u(x))^{\delta_u} \leq 1$ , we have  $m - 1 - \left( \sum_{\substack{u=1 \\ u \neq k}}^m F_u(x) \right) \prod_{\substack{u=1 \\ u \neq k}}^m (F_u(x))^{\delta_u} \geq 0$ , implying that  $A_1(x) \geq 0$ . Therefore  $A(x) \geq 0$ , so  $G'(x) \geq 0$ .

- Let us now investigate  $\lim_{x \rightarrow -\infty} G(x)$  and  $\lim_{x \rightarrow +\infty} G(x)$ . If  $m \geq 2$ , we have  $m - 1 + \prod_{k=1}^m (F_k(x))^{\delta_k} \geq m - 1 > 0$ . Since  $\lim_{x \rightarrow -\infty} \sum_{k=1}^m F_k(x) = 0$ , we have  $\lim_{x \rightarrow -\infty} G(x) = 0$ . If  $m = 1$ , recall that we have imposed  $\delta_m = 0$ , so  $\lim_{x \rightarrow -\infty} G(x) = \lim_{x \rightarrow -\infty} F_1(x) = 0$ . On the other hand, for any  $m \geq 1$ , we have  $\lim_{x \rightarrow +\infty} G(x) = \frac{m}{m - 1 + 1} = 1$ .

□

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