On doubly universal functions

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ON DOUBLY UNIVERSAL FUNCTIONS

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Abstract. Let (λ_n) be a strictly increasing sequence of positive integers. Inspired by the notions of topological multiple recurrence and disjointness in dynamical systems, Costakis and Tsirivas have recently established that there exist power series $\sum_{k\geq 0} a_k z^k$ with radius of convergence 1 such that the pairs of partial sums $\{(\sum_{k=0}^{n} a_k z^k, \sum_{k=0}^{\lambda_n} a_k z^k) : n = 1, 2, \dots\}$ approximate all pairs of polynomials uniformly on compact subsets $K \subset \{z \in \mathbb{C} : |z| > 1\}$, with connected complement, if and only if $\limsup_n \frac{\lambda_n}{n} = +\infty$. In the present paper, we give a new proof of this statement avoiding the use of advanced tools of potential theory. It allows to obtain the algebraic genericity of the set of such power series and to study the case of doubly universal infinitely differentiable functions. Further we show that the Cesàro means of partial sums of power series with radius of convergence 1 cannot be frequently universal.

1. Introduction

For a simply connected domain $\Omega \subset \mathbb{C}$, we will denote by $H(\Omega)$ the space of all holomorphic functions on Ω. Let $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$. For $f \in H(\mathbb{D})$, we denote by $S_n(f)$ the n-th partial sum of its Taylor development with center 0. In 1996 Nestoridis proved that there exist functions $f \in H(\mathbb{D})$ such that for every compact set $K \subset \mathbb{C}$ with K^c connected and $K \cap \mathbb{D} = \emptyset$ and for every function $h \in A(K)$, where $A(K) := H(\mathring{K}) \cap C(K)$, there exists a sequence of positive integers (λ_n) such that $\sup_{z\in K} |S_{\lambda_n}(f)(z) - h(z)| \to 0$ as $n \to +\infty$ [19]. Such functions are called *universal Taylor series*. The partial sums of its Taylor development diverge in a maximal way. In the following, the set of universal Taylor series will be denoted by $\mathcal{U}(\mathbb{D}, 0)$. We refer the reader to [3] and the references therein for its properties. In particular we know that $\mathcal{U}(\mathbb{D}, 0)$ is a G_δ dense subset of $H(\mathbb{D})$, endowed with the topology of uniform convergence on all compact subsets of $\mathbb{D}$, and contains a dense vector subspace apart from 0. Notice that we know C^∞ versions of Nestoridis result (see for instance [3, 8, 12, 18, 20]). Inspired by the notion of topological multiple recurrence and disjointness in dynamical systems, Costakis and Tsirivas introduced the following new form of universality [11].

Definition 1.1. Let (λ_n) be a strictly increasing sequence of positive integers. A function $f \in H(\mathbb{D})$ belongs to the class $\mathcal{U}(\mathbb{D}, (\lambda_n), 0)$ if for every compact set $K \subset \mathbb{C} \setminus \mathbb{D}$ with connected complement and for every pair of functions $(g_1, g_2) \in A(K) \times A(K)$, there exists a subsequence of positive integers (μ_n) such that

$$\sup_{z\in K} |S_{\mu_n}(f)(z) - g_1(z)| \to 0 \text{ and } \sup_{z\in K} |S_{\lambda_{\mu_n}}(f)(z) - g_2(z)| \to 0, \text{ as } n \to +\infty.$$

Such a function will be called *doubly universal Taylor series with respect to the sequences* (n), (λ_n).

Using tools from potential theory they proved that the set $\mathcal{U}(\mathbb{D}, (\lambda_n), 0)$ is non-empty if and only if $\limsup_n \frac{\lambda_n}{n} = +\infty$. Moreover they obtained that the existence of a doubly universal series implies topological genericity of such series. In the present paper we show that the advanced knowledge of potential theory does not play a dominant role to obtain the proof of the implication $\mathcal{U}(\mathbb{D}, (\lambda_n), 0) \neq \emptyset \Rightarrow \limsup_n \frac{\lambda_n}{n} = +\infty$. Instead we employ some polynomial inequalities which were recently used to study the densities of approximation subsequences of universal Taylor series in the sense of Nestoridis (see [16, 17]). It seems quite natural that the arithmetic structure of subsequences along which the partial sums possess the universal approximation property is connected with the above notion of disjointness. As a consequence, we obtain that the set of doubly

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universal Taylor series is densely lineable, i.e. contains a dense vector subspace except 0. This concept gives some information about the algebraic structure of the set of such series. Several authors were recently interested in this phenomenon (see for instance [4]). Further, since we avoid the use of potential theory in a large way, we extend the aforementioned Costakis-Tsirivas result to the case of the sequence of partial sums of Taylor development at 0 of infinitely differentiable functions on $\mathbb{R}$. This generalization uses in an essential way classical Bernstein polynomials of given continuous functions on intervals of the type $[0, A]$. In particular, these specific polynomials possess a useful property in our context: we control both their degree and their valuation provided that the associated function vanishes on a neighborhood of zero. Finally we return to the connection between doubly universality and topological recurrence. In a recent note, Costakis and Parissis proved that every frequently Cesàro operators is topologically multiply recurrent [9]. In our context, we show that the Cesàro means of partial sums of a real or complex power series cannot be frequently universal series. So the doubly universality, which is related to the topological multiply recurrence, does not imply the frequent Cesàro universality.
The paper is organized as follows. In Section 2 we give a new shorter proof of the implication $\limsup_n \frac{\lambda_n}{n} < +\infty \Rightarrow \mathcal{U}(\mathbb{D}, (\lambda_n), 0) = \emptyset$ and we establish the algebraic genericity of the set $\mathcal{U}(\mathbb{D}, (\lambda_n), 0)$. In Section 3, we are interested in the case of doubly universal infinitely differentiable functions with respect to an increasing sequence (λ_n) of positive integers. We establish both the topological and algebraic genericity of the set of such functions provided that $\limsup_n \frac{\lambda_n}{n} = +\infty$ again. In Section 4, we study the frequent Cesàro universal series and finally we give an example, in a different context, of the existence of doubly universal series with respect to an increasing sequence $(\lambda_n) \neq \mathbb{N}$ of positive integers without additional assumption.

2. Doubly universal Taylor series in the complex plane

In this section, we begin by giving a proof with rather elementary arguments of the fact that $\limsup_n \frac{\lambda_n}{n} < +\infty$ implies that $\mathcal{U}(\mathbb{D}, (\lambda_n), 0) = \emptyset$. To do this, let us recall the nice Turán inequality [21], which estimates the global behavior of a polynomial on a circle $\{z \in \mathbb{C} : |z| = r\}$ by its supremum on subsets of $\{z \in \mathbb{C} : |z| = r\}$.

Lemma 2.1. *Let Q be a polynomial of arbitrary degree which possesses only n non zero coefficients. Then for any $r > 0$ and any δ $(0 < \delta < 2\pi)$*

$$\sup_{|z|=r} |Q(z)| \leq \left(\frac{4\pi e}{\delta}\right)^n \sup_{|t|\leq \delta/2} |Q(re^{it})|.$$

For $r > 0$ and $0 < \delta < 2\pi$, $\Gamma_{r,\delta}$ will be the set

$$\Gamma_{r,\delta} = \left\{ z \in \mathbb{C};\ |z| = r \text{ and } -\frac{\delta}{2} \leq \arg(z) \leq \frac{\delta}{2} \right\}$$

and $C_\delta = \frac{4\pi e}{\delta}$ the constant of the above Turán inequality.

Now we state [11, Proposition 4.5] and we furnish a simple proof.

Proposition 2.2. *Let (λ_n) be a strictly increasing sequence of positive integers. Assume that $\limsup_n \left(\frac{\lambda_n}{n}\right) < +\infty$. Then the set $\mathcal{U}(\mathbb{D}, (\lambda_n), 0)$ is empty.*

Proof. The proof is based on the use of Turán's inequality. We argue by contradiction. Take $f = \sum_{n\geq 0} a_n z^n$ in $\mathcal{U}(\mathbb{D}, (\lambda_n), 0)$. Since we have $\limsup_n \left(\frac{\lambda_n}{n}\right) < +\infty$, there exists $d > 0$ such that

$$\forall n \in \mathbb{N},\ \lambda_n \leq dn. \tag{1}$$

Let $r > 0$ and $0 < \delta < 2\pi$. Fix a compact set $K \subset \mathbb{C} \setminus \mathbb{D}$ with connected complement. Let us choose $R > 0$ so that

$$\frac{R}{C_\delta^d} > \sup_{z\in K} |z|. \tag{2}$$

Clearly the set $K_{R,\delta} := \Gamma_{R,\delta} \cup K$ is a compact set with connected complement. Since $f \in \mathcal{U}(\mathbb{D}, (\lambda_n), 0)$ there exists an increasing (μ_n) of positive integers such that

$$\sup_{z\in K_{r,\delta}} |S_{\mu_n}(f)(z) - 1| \to 0 \text{ and } \sup_{z\in K_{r,\delta}} |S_{\lambda_{\mu_n}}(f)(z)| \to 0, \text{ as } n \to +\infty.$$

Therefore, for any $0 < \varepsilon < 1$, we can find $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$,

$$\sup_{z\in K_{r,\delta}} |S_{\mu_n}(f)(z) - 1| < \varepsilon/4 \text{ and } \sup_{z\in K_{r,\delta}} |S_{\lambda_{\mu_n}}(f)(z)| < \varepsilon/4. \tag{3}$$

In particular, we have, for every $n \geq n_0$,

$$\sup_{z\in\Gamma_{R,\delta}} \left| \sum_{j=0}^{\lambda_{\mu_n}} a_j z^j \right| < \varepsilon/4.$$

Using Lemma 2.1 and Cauchy estimates, we get for every $n \geq n_0$ and $j = 0, \ldots, \lambda_{\mu_n}$,

$$|a_j|^{1/j} \leq \frac{\varepsilon^{1/j}}{4^{1/j} R} C_\delta^{\lambda_{\mu_n}/j}. \tag{4}$$

Taking into account (1), we get for every $n \geq n_0$ and $j = 1 + \mu_n, \ldots, \lambda_{\mu_n}$,

$$|a_j|^{1/j} \leq \frac{\varepsilon^{1/j}}{4^{1/j}} \frac{C_\delta^d}{R}. \tag{5}$$

By (5) and (2), we deduce that there exists a positive integer $n_2 \geq n_0$ such that for $n \geq n_2$ the following estimate holds

$$\sup_{z\in K} |\sum_{j=1+\mu_n}^{\lambda_{\mu_n}} a_j z^j| < \varepsilon/4.$$

Finally using the inequality $\sup_{z\in K} |S_{\lambda_{\mu_n}}(f)(z)| < \varepsilon/4$, we have, for all $n \geq n_2$,

$$\sup_{z\in K} |S_{\mu_n}(f)(z)| = \sup_{z\in K} |S_{\lambda_{\mu_n}}(f)(z) - \sum_{j=1+\mu_n}^{\lambda_{\mu_n}} a_j z^j| \leq \sup_{z\in K} |S_{\lambda_{\mu_n}}(f)(z)| + \sup_{z\in K} |\sum_{j=1+\mu_n}^{\lambda_{\mu_n}} a_j z^j| < \varepsilon/2. \tag{6}$$

Combining (3) with (6) we obtain

$$1 \leq \sup_{z\in K} |S_{\mu_n}(f)(z) - 1| + \sup_{z\in K} |S_{\mu_n}(f)(z)| \leq 3\varepsilon/4,$$

which is a contradiction. This completes the proof of the proposition. □

Further we are interested in the algebraic structure of $\mathcal{U}(\mathbb{D}, (\lambda_n), 0)$. First let us define the set of doubly universal Taylor series along a given subsequence.

Definition 2.3. Let (λ_n) and $\mu = (\mu_n)$ be increasing sequences of positive integers. A function $f \in H(\mathbb{D})$ belongs to the class $\mathcal{U}^{(\mu)}(\mathbb{D}, (\lambda_n), 0)$ if for every compact set $K \subset \mathbb{C} \setminus \mathbb{D}$ with connected complement and for every pair of functions $(g_1, g_2) \in A(K) \times A(K)$, there exists a subsequence of positive integers $(\nu_n) \subset \mu$ such that

$$\sup_{z\in K} |S_{\nu_n}(f)(z) - g_1(z)| \to 0 \text{ and } \sup_{z\in K} |S_{\lambda_{\nu_n}}(f)(z) - g_2(z)| \to 0, \text{ as } n \to +\infty.$$

Remark 2.4. Arguing as in the proof of Proposition 2.2, we obtain that the existence of universal elements in $\mathcal{U}^{(\mu)}(\mathbb{D}, (\lambda_n), 0)$ implies $\limsup_n \frac{\lambda_{\mu_n}}{\mu_n} = +\infty$. On the other hand, the hypothesis $\limsup_n \frac{\lambda_{\mu_n}}{\mu_n} = +\infty$ implies that the set $\mathcal{U}^{(\mu)}(\mathbb{D}, (\lambda_n), 0)$ is G_δ and dense in $H(\mathbb{D})$. The proof works as in [11, Proposition 4.1] with obvious modifications.

Moreover a careful examination of the proof of Proposition 2.2 gives the following lemma.

Lemma 2.5. *Let (λ_n) and $\mu = (\mu_n)$ be increasing sequences of positive integers. Let f be in $\mathcal{U}^{(\mu)}(\mathbb{D}, (\lambda_n), 0)$. For every compact set $K \subset \mathbb{C}\backslash\mathbb{D}$, with K^c connected, and for every pair of functions $(g_1, g_2) \in A(K) \times A(K)$, with $g_1 \neq g_2$, there exists a subsequence (ν_n) of μ with $\limsup_n \frac{\lambda_{\nu_n}}{\nu_n} = +\infty$ such that*

$$\sup_{z\in K} |S_{\nu_n}(f)(z) - g_1(z)| \to 0 \text{ and } \sup_{z\in K} |S_{\lambda_{\nu_n}}(f)(z) - g_2(z)| \to 0, \text{ as } n \to +\infty.$$

Proof. As in the proof of Proposition 2.2, let us choose $R > 0$ so that $\frac{R}{C_\delta^d} > \sup_{z\in K} |z|$ and consider the compact set $K_{R,\delta} := \Gamma_{R,\delta} \cup K$. Since $f \in \mathcal{U}^{(\mu)}(\mathbb{D}, (\lambda_n), 0)$ there exists an increasing $(\nu_n) \subset \mu$ of positive integers such that

$$\sup_{z\in K_{r,\delta}} |S_{\nu_n}(f)(z) - g_1(z)| \to 0 \text{ and } \sup_{z\in K_{r,\delta}} |S_{\lambda_{\nu_n}}(f)(z) - g_2(z)| \to 0, \text{ as } n \to +\infty.$$

Arguing as in the end of the proof of Proposition 2.2, we deduce that we have necessary $\limsup_n \frac{\lambda_{\nu_n}}{\nu_n} = +\infty$. □

Combining Remark 2.4 with Lemma 2.5 we get that the set $\mathcal{U}(\mathbb{D}, (\lambda_n), 0) \cup \{0\}$ is algebraically generic.

Theorem 2.6. *Let (λ_n) be a strictly increasing sequence of positive integers such that $\limsup_n \left(\frac{\lambda_n}{n}\right) = +\infty$. The set $\mathcal{U}(\mathbb{D}, (\lambda_n), 0) \cup \{0\}$ contains a dense vector subspace of $H(\mathbb{D})$.*

Proof. We proceed as in the proof of [3, Theorem 3] with essential modifications. Let us fix a dense sequence $(h_l)_l$ in $H(\mathbb{D})$. In the following, $d_{H(\mathbb{D})}$ denotes the standard metric of $H(\mathbb{D})$. Let (K_m) be a family of compact sets with connected complement and $K_m \cap \mathbb{D} = \emptyset$ for every $m \in \mathbb{N}$ such that every compact subset $K \subset \{z \in \mathbb{C} : |z| \geq 1\}$, with K^c connected, is contained in some K_n, $n \in \mathbb{N}$ [19, Lemma 2.1]. We construct a sequence $(f_l)_l$ in $H(\mathbb{D})$ and sequences $\mu^{k,l}$ of positive integers satisfying the following conditions, for any $k, l \geq 1$,

- $\mu^{k,l}$ is a subsequence of $\mu^{k,l-1}$, with $\mu^{k,0} = \mathbb{N}$,
- $d_{H(\mathbb{D})}(f_l, h_l) < 2^{-l}$,
- $\limsup_n \frac{\lambda_{\mu_n^{k,l}}}{\mu_n^{k,l}} = +\infty$,
- f_l belongs to $\bigcap_{k\geq 1} \mathcal{U}^{(\mu^{k,l-1})}(\mathbb{D}, (\lambda_n), 0)$,
- $\sup_{z\in K_k} |S_{\lambda_{\mu_n^{k,l}}}(f_l)| \to 0$, $\sup_{z\in K_k} |S_{\mu_n^{k,l}}(f_l) - 1| \to 0$, and $d_{H(\mathbb{D})}(S_{\mu_n^{k,l}}(f_l, \zeta), f_l) \to 0$, as $n \to +\infty$.

To do this, observe that first we can choose f_1 in the dense set $\mathcal{U}(\mathbb{D}, (\lambda_n), 0)$ so that $d_{H(\mathbb{D})}(f_1, h_1) < 2^{-1}$. Therefore, applying Lemma 2.5, for any $k \geq 1$, one may find a subsequence $\mu^{k,1}$ with $\limsup_n \frac{\lambda_{\mu_n^{k,1}}}{\mu_n^{k,1}} = +\infty$, such that $\sup_{K_k} |S_{\lambda_{\mu_n^{k,1}}}(f_1)| \to 0$ and $\sup_{K_k} |S_{\mu_n^{k,1}}(f_1) - 1| \to 0$ as $n \to +\infty$.

At step 2, we choose $f_2 \in \bigcap_{k\geq 1} \mathcal{U}^{(\mu^{k,1})}(\mathbb{D}, (\lambda_n), 0)$, which is a G_δ and dense subset of $H(\mathbb{D})$, with $d_{H(\mathbb{D})}(f_2, h_2) < 2^{-2}$. In particular, according to Lemma 2.5 for any $k \geq 1$, there exists a subsequence $\mu^{k,2}$ of $\mu^{k,1}$, with $\limsup_n \frac{\lambda_{\mu_n^{k,2}}}{\mu_n^{k,2}} = +\infty$ such that $\sup_{K_k} |S_{\lambda_{\mu_n^{k,2}}}(f_2)| \to 0$ and $\sup_{K_k} |S_{\mu_n^{k,2}}(f_2) - 1| \to 0$ as $n \to +\infty$. Then we repeat the same arguments to construct $(f_l)_l$ in $H(\mathbb{D})$ and sequences $\mu^{k,l}$ of positive integers satisfying the above properties. To finish the proof, it is sufficient to check that the linear span of the (f_l) is both dense in $H(\mathbb{D})$ and contained in $\mathcal{U}(\mathbb{D}, (\lambda_n), 0)$, except for the zero function. The density is clear by construction. Moreover let $f = \alpha_1 f_1 + \cdots + \alpha_m f_m$, with $\alpha_m \neq 0$. Let us consider two polynomials g_1, g_2 and a compact set K with connected complement and $K \cap \Omega = \emptyset$. There exists k such that $K \subset K_k$. Since $f_m \in \mathcal{U}^{(\mu^{k,m-1})}(\mathbb{D}, (\lambda_n), 0)$, there exists a sequence (γ_n) of positive integers with $(\gamma_n) \subset \mu^{k,m-1}$ such that

(7)

$$\sup_{z\in K_k} |S_{\lambda_{\gamma_n}}(\alpha_m f_m)(z) - g_1(z)| \to 0 \text{ and } \sup_{z\in K_k} |S_{\gamma_n}(\alpha_m f_m)(z) - (g_2(z) - \sum_{i=1}^{m-1} \alpha_i)| \to 0 \text{ as } n \to +\infty.$$

Observe that (γ_n) is a subsequence of any $\mu^{k,l}$ for $l \leq m-1$. Hence by construction we have, for any $l \leq m-1$,

$$\sup_{z\in K_k} |S_{\lambda_{\gamma_n}}(\alpha_l f_l)(z)| \to 0 \text{ and } \sup_{z\in K_k} |S_{\gamma_n}(\alpha_l f_l)(z) - \alpha_l| \to 0 \text{ as } n \to +\infty. \tag{8}$$

Finally from (7) and (8) we get

$$\sup_{z\in K_k} |S_{\lambda_{\gamma_n}}(f)(z) - g_1(z)| \to 0$$

and

$$\sup_{z\in K_k} |S_{\gamma_n}(f)(z) - g_2(z)| \leq \sup_{z\in K_k} |S_{\gamma_n}(\alpha_m f_m)(z) - (g_2(z) - \sum_{i=1}^{m-1} \alpha_i)| + \sum_{i=1}^{m-1} \sup_{z\in K_k} |S_{\gamma_n}(\alpha_i f_i)(z) - \alpha_i| \to 0$$

as $n \to +\infty$, which implies that f belongs to $\mathcal{U}(\mathbb{D}, (\lambda_n), 0)$. □

3. Doubly universal infinitely differentiable functions

First let us introduce some notations and terminology. We consider the set $C_0^\infty(\mathbb{R})$ of functions $f \in C^\infty(\mathbb{R})$ with $f(0) = 0$. Its topology is defined by the seminorms $\sup_{x\in[-m,m]} |f^{(j)}(x)|$, $j, m \in \mathbb{N}$ and the associated standard translation-invariant metric will be denoted by $d_{C_0^\infty(\mathbb{R})}$. Moreover we will consider the classical space $\mathbb{R}^\mathbb{N}$ endowed with the metric $d_{\mathbb{R}^\mathbb{N}}$ defined by $d_{\mathbb{R}^\mathbb{N}}((u_n), (v_n)) = \sum_{n\geq 0} 2^{-n}(\max_{0\leq j\leq n} |u_j - v_j|/(1 + \max_{0\leq j\leq n} |u_j - v_j|))$. The metric space $(\mathbb{R}^\mathbb{N}, d_{\mathbb{R}^\mathbb{N}})$ is complete. As far as we know Fekete exhibited the first example of universal series by showing that there exists a formal power series $\sum_{n\geq 1} a_n x^n$ with the following property: for every continuous function g on $[-1, 1]$ with $g(0) = 0$ there exists an increasing sequence (λ_n) of positive integers such that $\sup_{x\in[-1,1]} |\sum_{k=1}^{\lambda_n} a_k x^k - g(x)| \to 0$, as $n \to +\infty$ [20]. A slight modification of Fekete's proof combined with Borel's theorem allows to obtain C^∞-function whose partial sums of its Taylor series around 0 approximate every continuous functions vanishing at 0 locally uniformly in $\mathbb{R}$ (see [12]). In the present section, we are going to obtain a natural extension of the results of Section 2 to the case of Fekete functions, exploiting the fact that we did not need to use advanced tools of potential theory to study the class of doubly (complex) universal Taylor series.

First of all, let us mention a useful inequality for polynomials in many variables between the complex and the real sup-norms [1, 13].

Theorem 3.1. *There exists a constant $C > 1$ such that, for any polynomial P of degree n in k variables with real coefficients, we have*

$$\sup_{|z_1|=\cdots=|z_k|=1} |P(z_1, \ldots, z_k)| \leq C^n \sup_{x_1,\ldots,x_k\in[-1,1]} |P(x_1, \ldots, x_k)|.$$

In Theorem 3.1, we can choose the constant C to be $1 + \sqrt{2}$ [1, 13].

Definition 3.2. Let (λ_n) be a strictly increasing sequence of positive integers. A function $f \in C_0^\infty(\mathbb{R})$ belongs to the class $\mathcal{U}(C_0^\infty(\mathbb{R}), (\lambda_n))$ if for every compact set $K \subset \mathbb{R}$ and for every pair (h_1, h_2) of continuous functions $h_1, h_2 : \mathbb{R} \to \mathbb{R}$ vanishing at zero, there exists a subsequence of positive integers (μ_n) such that

$$\sup_{x\in K} \left| \sum_{k=0}^{\lambda_{\mu_n}} \frac{f^{(k)}(0)}{k!} x^k - h_1(x) \right| \to 0 \text{ and } \sup_{x\in K} \left| \sum_{k=0}^{\mu_n} \frac{f^{(k)}(0)}{k!} x^k - h_2(x) \right| \to 0 \text{ as } n \to +\infty.$$

Now we can state a version of Proposition 2.2 in this context.

Proposition 3.3. *Let (λ_n) be a strictly increasing sequence of positive integers. Assume that $\limsup_n \left(\frac{\lambda_n}{n}\right) < +\infty$. Then the set $\mathcal{U}(C_0^\infty(\mathbb{R}), (\lambda_n))$ is empty.*

Proof. We argue by contradiction. Take f in $\mathcal{U}(C_0^\infty(\mathbb{R}), (\lambda_n))$ and set, for every $k \geq 0$, $a_k = \frac{f^{(k)}(0)}{k!}$. Since we have $\limsup_n \left(\frac{\lambda_n}{n}\right) < +\infty$, there exists $d > 0$ such that

$$\forall n \in \mathbb{N},\ \lambda_n \leq dn. \tag{9}$$

Let us fix

$$0 < \varepsilon < \frac{1}{2C^d}, \tag{10}$$

where C is the absolute constant given by Theorem 3.1. Since f belongs to $\mathcal{U}(C_0^\infty(\mathbb{R}), (\lambda_n))$, there exists a subsequence (μ_n) of positive integers such that, for any $n \geq n_0$,

$$\sup_{x\in[-1,1]} |\sum_{j=1}^{\mu_n} a_j x^j - x| < \varepsilon/4 \text{ and } \sup_{x\in[-1,1]} |\sum_{j=1}^{\lambda_{\mu_n}} a_j x^j| < \varepsilon/4. \tag{11}$$

Using Theorem 3.1 we get, for every $n \geq n_0$,

$$\sup_{|z|=1} \left| \sum_{j=1}^{\lambda_{\mu_n}} a_j z^j \right| \leq C^{\lambda_{\mu_n}} \frac{\varepsilon}{4}. \tag{12}$$

It follows from Cauchy's formula, for every $n \geq n_0$ and $j = 0, \ldots, \lambda_{\mu_n}$,

$$|a_j|^{1/j} \leq \frac{\varepsilon^{1/j}}{4^{1/j}} C^{\lambda_{\mu_n}/j}. \tag{13}$$

From (9) we deduce, for $j = 1 + \mu_n, \ldots, \lambda_{\mu_n}$,

$$|a_j|^{1/j} \leq \frac{\varepsilon^{1/j}}{4^{1/j}} C^d, \tag{14}$$

and therefore we get, for $n \geq n_0$,

$$\sup_{x\in[-1/(2C^d),1/(2C^d)]} |\sum_{j=1+\mu_n}^{\lambda_n} a_j x^j| \leq \frac{\varepsilon}{4} \sum_{j=1+\mu_n}^{\lambda_n} \frac{1}{2^j} \leq \frac{\varepsilon}{4}.$$

Finally using (11) we have, for all $n \geq n_0$,

$$\sup_{x\in[-1/(2C^d),1/(2C^d)]} |\sum_{j=1}^{\mu_n} a_j x^j| \leq \sup_{x\in[-1/(2C^d),1/(2C^d)]} |\sum_{j=1}^{\lambda_{\mu_n}} a_j x^j| + \sup_{x\in[-1/(2C^d),1/(2C^d)]} |\sum_{j=1+\mu_n}^{\lambda_{\mu_n}} a_j x^j| < \varepsilon/2. \tag{15}$$

Combining (11) with (15) we obtain, for $n \geq n_0$,

$$\begin{aligned} \frac{1}{2C^d} = \sup_{x\in[-1/(2C^d),1/(2C^d)]} |x| &\leq \sup_{x\in[-1/(2C^d),1/(2C^d)]} |\sum_{j=1}^{\mu_n} a_j x^j| + \sup_{x\in[-1/(2C^d),1/(2C^d)]} |\sum_{j=1}^{\mu_n} a_j x^j - x| \\ &\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{4} < \varepsilon. \end{aligned}$$

This last inequality gives a contradiction with (10). This completes the proof. □

To obtain the converse result, we will follow the main ideas of the proof of [11, Proposition 4.1]. First we need a quantitative approximation polynomial lemma which will play the role of [11, Theorem 2.1]. We have to approximate a given continuous function vanishing at 0 by polynomials whose both degrees and valuations are imposed. Exploiting the form of the classical Bernstein polynomials, we begin with the case where the approximation takes place on compact subsets of $[0, +\infty)$.

Lemma 3.4. *Let (l_n) and (m_n) be two strictly increasing sequences of positive integers such that $l_n \leq m_n$ and $\frac{m_n}{l_n} \to +\infty$ as $n \to +\infty$. Let $A > 0$. For every continuous function $h : \mathbb{R}_+ \to \mathbb{R}$, with $h(0) = 0$, there exists a sequence (P_n) of real polynomials of the form $P_n(x) = \sum_{k=l_n}^{m_n} c_{n,k} x^k$, such that*

$$\sup_{x\in[0,A]} |P_n(x) - h(x)| \to 0, \text{ as } n \to +\infty.$$

Proof. Let $\varepsilon > 0$. By continuity of the function h at 0, one can find $\eta > 0$ such that for all $x \in [0, \eta)$, $|h(x)| < \varepsilon/4$. Let us consider the continuous function $\tilde{h}$ defined on $\mathbb{R}_+$ by

$$\tilde{h}(x) = \begin{cases} 0 & \text{for } 0 \leq x \leq \eta/2 \\ \frac{2h(\eta)}{\eta}(x - \eta/2) & \text{for } \eta/2 \leq x \leq \eta \\ h(x) & \text{for } x \geq \eta \end{cases}$$

Then, for every $n \geq 1$, let us consider $B_{m_n}(\tilde{h})$ its Bernstein polynomial of degree m_n, given by

$$B_{m_n}(\tilde{h})(x) = \frac{1}{A^{m_n}} \sum_{k=0}^{m_n} \binom{m_n}{k} \tilde{h}\left(A\frac{k}{m_n}\right) x^k (A-x)^{m_n-k}.$$

Moreover since the sequence (l_n/m_n) converges to 0, there exists a positive integer N_1 such that, for every $n \geq N_1$, $Al_n/m_n \leq \eta/2$. Therefore by construction, for every $n \geq N_1$ and for $k = 0, 1, \ldots, l_n$,we have $\tilde{h}\left(A\frac{k}{m_n}\right) = 0$. The Bernstein polynomials of $\tilde{h}$ have the following form

$$B_{m_n}(\tilde{h})(x) = \frac{1}{A^{m_n}} \sum_{k=l_n}^{m_n} \binom{m_n}{k} \tilde{h}\left(A\frac{k}{m_n}\right) x^k (A-x)^{m_n-k} := \sum_{k=l_n}^{m_n} c_{n,k} x^k.$$

Obviously the function $\tilde{h}$ is continuous on $[0, A]$. So it is known that the sequence $(B_n(\tilde{h}))$ converges uniformly to $\tilde{h}$ on $[0, A]$ [5]. Thus we deduce the existence of a positive integer $N_2 > N_1$ such that, for every $n \geq N_2$,

$$\sup_{x \in [0,A]} |B_{m_n}(\tilde{h})(x) - \tilde{h}(x)| < \varepsilon/2.$$

Now, for $x \leq \eta/2$, since $\tilde{h}(x) = 0$, the triangle inequality gives

$$|B_{m_n}(\tilde{h})(x) - h(x)| \leq |B_{m_n}(\tilde{h})(x)| + |h(x)| < \varepsilon/2 + \varepsilon/4 < \varepsilon.$$

On the other hand, for $\eta/2 \leq x \leq \eta$, we have

$$|B_{m_n}(\tilde{h})(x) - h(x)| \leq |B_{m_n}(\tilde{h})(x) - \tilde{h}(x)| + |\tilde{h}(x) - h(x)| < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

Finally for $x \geq \eta$, we have $\tilde{h}(x) = h(x)$ and we get $|B_{m_n}(\tilde{h})(x) - h(x)| < \varepsilon$. This completes the proof. □

Then we extend Lemma 3.4 to the case of symmetric intervals $[-A, A]$.

Lemma 3.5. *Let (l_n) and (m_n) be two strictly increasing sequences of positive integers such that $l_n \leq m_n$ and $\frac{m_n}{l_n} \to +\infty$ as $n \to +\infty$. Let $A > 0$. For every continuous function $h : \mathbb{R} \to \mathbb{R}$, with $h(0) = 0$, there exists a sequence (P_n) of real polynomials of the form $P_n(x) = \sum_{k=l_n}^{m_n} c_{n,k} x^k$, such that*

$$\sup_{x \in [-A,A]} |P_n(x) - h(x)| \to 0, \text{ as } n \to +\infty.$$

Proof. Let $\varepsilon > 0$. By continuity of the function h at 0, one can find $\eta > 0$ such that for all $x \in [0, \eta)$, $|h(x)| < \varepsilon/4$. Let us consider the continuous function $\tilde{h}$ defined on $\mathbb{R}$ by

$$\tilde{h}(x) = \begin{cases} 0 & \text{for } |x| \leq \eta/2 \\ \frac{2h(\eta)}{\eta}(x - \eta/2) & \text{for } \eta/2 \leq x \leq \eta \\ \frac{-2h(-\eta)}{\eta}(x + \eta/2) & \text{for } -\eta \leq x \leq -\eta/2 \\ h(x) & \text{for } |x| \geq \eta \end{cases}$$

Observe that we have

$$\sup_{x \in [-A,A]} |h(x) - \tilde{h}(x)| \leq \varepsilon/2. \tag{16}$$

Define also the continuous function $g : \mathbb{R} \to \mathbb{R}$, $x \mapsto \tilde{h}(x)/x^2$. By classical Weierstrass approximation theorem one can find a polynomial P such that

$$\sup_{x \in [-A,A]} |P(x) - g(x)| < \varepsilon/4A^2.$$

We deduce the following inequality

$$\sup_{x\in[-A,A]} |x^2P(x) - \tilde{h}(x)| < \varepsilon/4. \tag{17}$$

Set $W(x) := x^2P(x)$. Observe that $W(0) = W'(0) = 0$. Let us write

$$W(x) = Q_1(x^2) + xQ_2(x^2)$$

where Q_1 and Q_2 are polynomials vanishing at 0. Then we apply Lemma 3.4 to find two sequences of polynomials $P_{n,1}$ and $P_{n,2}$ of the form

$$P_{n,1}(x) = \sum_{k=\lfloor l_n/2\rfloor+1}^{\lfloor m_n/2\rfloor} c_{n,k}^{(1)}x^k \text{ and } P_{n,2}(x) = \sum_{k=\lfloor l_n/2\rfloor}^{\lfloor (m_n-1)/2\rfloor} c_{n,k}^{(2)}x^k$$

such that

$$\sup_{x\in[0,A^2]} |P_{n,1}(x) - Q_1(x)| \to 0 \text{ and } \sup_{x\in[0,A^2]} |P_{n,2}(x) - Q_2(x)| \to 0, \text{ as } n \to +\infty.$$

For n large enough we get

$$\sup_{x\in[-A,A]} |P_{n,1}(x^2) - Q_1(x^2)| < \varepsilon/8 \text{ and } \sup_{x\in[-A,A]} |P_{n,2}(x^2) - Q_2(x^2)| < \varepsilon/8A.$$

Thus by construction the polynomial $\tilde{W}(x) := P_{n,1}(x^2) + xP_{n,2}(x^2)$ has the following form

$$\tilde{W}(x) = \sum_{k=l_n}^{m_n} c_{n,k}x^k \tag{18}$$

and we have

$$\begin{aligned} \sup_{x\in[-A,A]} |\tilde{W}(x) - W(x)| &\leq \sup_{x\in[-A,A]} |P_{n,1}(x^2) - Q_1(x^2)| + \sup_{x\in[-A,A]} |x(P_{n,2}(x^2) - Q_2(x^2))| \\ &< \frac{\varepsilon}{8} + A\frac{\varepsilon}{8A} = \frac{\varepsilon}{4}. \end{aligned} \tag{19}$$

Finally combining the triangle inequality with (16), (17) and (19), we get

$$\begin{aligned} \sup_{[-A,A]} |h - \tilde{W}| &\leq \sup_{[-A,A]} |h - \tilde{h}| + \sup_{[-A,A]} |\tilde{h} - W| + \sup_{[-A,A]} |W - \tilde{W}| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} = \varepsilon. \end{aligned} \tag{20}$$

Thus the polynomial $\tilde{W}$ has the desired properties (given by (18) and (20)). This completes the proof. □

Next we introduce an intermediate result.

Definition 3.6. Let (a_n) and (b_n) be strictly increasing sequence of positive integers. A function $f \in C_0^\infty(\mathbb{R})$ belongs to the class $\mathcal{U}(C_0^\infty(\mathbb{R}), (a_n), (b_n))$ if for every compact set $K \subset \mathbb{R}$ and for every continuous function $h : \mathbb{R} \to \mathbb{R}$ vanishing at zero, there exists a subsequence of positive integers (μ_n) such that

$$\sup_{x\in K}\left|\sum_{k=0}^{a_{\mu_n}} \frac{f^{(k)}(0)}{k!}x^k - h(x)\right| \to 0 \text{ and } \sup_{x\in K}\left|\sum_{k=0}^{b_{\mu_n}} \frac{f^{(k)}(0)}{k!}x^k - h(x)\right| \to 0 \text{ as } n \to +\infty.$$

Proposition 3.7. *Let (a_n) and (b_n) be strictly increasing sequence of positive integers. Then the set $\mathcal{U}(C_0^\infty(\mathbb{R}), (a_n), (b_n))$ is G_δ and dense in $C_0^\infty(\mathbb{R})$.*

Proof. It suffices to combine the ideas of the proof of [11, Proposition 3.2] with the arguments of [18]. Let (f_j) be an enumeration of all the polynomials with coefficients in $\mathbb{Q}$ and $f_j(0) = 0$. Let us define the set

$$\begin{aligned} E(m,j,s,n) = \quad \{f \in C_0^\infty(\mathbb{R}) : \sup_{x\in[-m,m]} \left|\textstyle\sum_{k=1}^{a_n} \frac{f^{(k)}(0)}{k!}x^k - f_j(x)\right| < \tfrac{1}{s} \\ \text{and } \sup_{x\in[-m,m]} \left|\textstyle\sum_{k=1}^{b_n} \frac{f^{(k)}(0)}{k!}x^k - f_j(x)\right| < \tfrac{1}{s}\} \end{aligned}$$

for every $m, j, s, n \in \mathbb{N}^*$. Observe that $E(m, j, s, n)$ is an open set and the following description holds

$$\mathcal{U}(C_0^\infty(\mathbb{R}), (a_n), (b_n)) = \bigcap_{m,j,s\in\mathbb{N}^*} \bigcup_{n\in\mathbb{N}} E(m, j, s, n).$$

By Baire's category theorem it suffices to show that $\cup_{n\in\mathbb{N}} E(m, j, s, n)$ is dense in $C_0^\infty(\mathbb{R})$. To do this, let $m, j, s \in \mathbb{N}^*$, $\varepsilon > 0$ and g be a polynomial. We seek $f \in C_0^\infty(\mathbb{R})$ and $n \in \mathbb{N}$ such that $f \in E(m, j, s, n)$. Applying the proof of [7, Lemma 2.3], for any $\eta > 0$, we find $c_1, c_2, \dots, c_l$ in $\mathbb{R}$ such that

$$d_{\mathbb{R}^{\mathbb{N}}}((c_1, \dots, c_l, 0, \dots), 0) < \eta \text{ and } \sup_{x\in[-m,m]} |p(x) + g(x) - f_j(x)| < \eta,$$

with $p(x) = \sum_{k=1}^{l} c_k x^k$. Since the sequences (a_n) and (b_n) are strictly increasing, we fix $n \in \mathbb{N}$ such that $\min\{a_n, b_n\} > \max\{l, \deg(g)\}$. Moreover the linear Borel map $T_0 : C_0^\infty(\mathbb{R}) \to \mathbb{R}^{\mathbb{N}}, f \mapsto (f^{(k)}(0)/k!)$ is open. Hence with a previous good choice of $\eta < \varepsilon$ we find a function $w \in C_0^\infty(\mathbb{R})$ such that $T_0 w = p$ and $d_{C_0^\infty(\mathbb{R})}(w, 0) = d_{C_0^\infty(\mathbb{R})}(w + g, g) < \varepsilon$. So the function $f = w + g$ does the job. □

Proposition 3.8. *Let (λ_n) be a strictly increasing sequence of positive integers. Assume that $\limsup_n \frac{\lambda_n}{n} = +\infty$. Then the set $\mathcal{U}(C_0^\infty(\mathbb{R}), (\lambda_n))$ is G_δ and dense in $C_0^\infty(\mathbb{R})$ and contains a dense vector subspace apart from 0.*

Proof. Let (f_j) be an enumeration of all the polynomials with coefficients in $\mathbb{Q}$ vanishing at zero. Let us consider the sets

$$E(m, j_1, j_2, s, n) = \{f \in C_0^\infty(\mathbb{R}) : \begin{array}{l} \sup_{x\in[-m,m]} |\sum_{k=0}^{\lambda_n} \frac{f^{(k)}(0)}{k!} x^k - f_{j_1}(x)| < \frac{1}{s} \\ \text{and } \sup_{x\in[-m,m]} |\sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} x^k - f_{j_2}(x)| < \frac{1}{s}\}\end{array}$$

for every m, j_1, j_2, s, $n \in \mathbb{N}$. Weirstrass approximation theorem ensures that

$$\mathcal{U}(C_0^\infty(\mathbb{R}), (\lambda_n)) = \bigcap_{m,j_1,j_2,s\in\mathbb{N}} \bigcup_{n\in\mathbb{N}} E(m, j_1, j_2, s, n).$$

Since the sets $E(m, j_1, j_2, s, n)$ are open, according to Baire's category theorem it suffices to prove that $\cup_{n\in\mathbb{N}} E(m, j_1, j_2, s, n)$ is dense in $C_0^\infty(\mathbb{R})$ for every m, j_1, j_2, $s \in \mathbb{N}$ to obtain that the set $\mathcal{U}(C_0^\infty(\mathbb{R}), (\lambda_n))$ is G_δ and dense in $C_0^\infty(\mathbb{R})$. To do this, we fix m, j_1, j_2, $s \in \mathbb{N}$, $\varepsilon > 0$ and $g \in C_0^\infty(\mathbb{R})$. Then it suffices to find $n \geq 0$ and $f \in E(m, j_1, j_2, s, n)$ such that

$$d_{C_0^\infty(\mathbb{R})}(f, g) < \varepsilon, \tag{21}$$

where $d_{C_0^\infty(\mathbb{R})}$ denotes the Fréchet distance in $C_0^\infty(\mathbb{R})$. By Weierstrass approximation theorem we can assume that g is a polynomial with $g(0) = 0$. Since $\limsup_n \frac{\lambda_n}{n} = +\infty$, there exists a strictly increasing sequence $(\mu_n) \subset \mathbb{N}$ such that $\frac{\lambda_{\mu_n}}{\mu_n} \to +\infty$ as $n \to +\infty$. We apply Lemma 3.5 for $h := f_{j_1} - f_{j_2}$, $l_n = 1 + \mu_n$ and $m_n = \lambda_{\mu_n}$. We obtain a sequence of polynomial (P_n) of the form $P_n(x) = \sum_{k=1+\mu_n}^{\lambda_{\mu_n}} c_{n,k} x^k$ which converges to $f_{j_1} - f_{j_2}$ uniformly on $[-m, m]$. There exists $N_1 \in \mathbb{N}$ such that for every $n \geq N_1$ the following inequality holds

$$\sup_{x\in[-m,m]} |P_n(x) - (f_{j_1}(x) - f_{j_2}(x))| < 1/2s.$$

Observe that the linear Borel map $T_0 : C_0^\infty(\mathbb{R}) \to \mathbb{R}^{\mathbb{N}}, f \mapsto (f^{(k)}(0)/k!)$ is open. Hence the image of every $\varepsilon/2$-neighborhood of 0 in $C_0^\infty(\mathbb{R})$ contains some η-neighborhood of 0 in $\mathbb{R}^{\mathbb{N}}$. Moreover, by construction we have

$$\text{val}(P_n) > \mu_n \text{ and } \mu_n \to +\infty, \text{ as } n \to +\infty, \tag{22}$$

where $\text{val}(P_n)$ denotes the valuation of the polynomial P_n. So the property (22) implies that the inequality $d_{\mathbb{R}^{\mathbb{N}}}(P_n, 0) < \eta$ holds for n large enough. Therefore one can find a positive integer $N_2 > N_1$ such that for $n \geq N_2$ there exists $u_n \in C_0^\infty(\mathbb{R})$ with $T_0 u_n = P_n$ and $d_{C_0^\infty(\mathbb{R})}(u_n, 0) < \varepsilon/2$.

On the other hand, applying Proposition 3.7 for $a_n = \mu_n$, $b_n = \lambda_{\mu_n}$, we find a function $w \in C_0^\infty(\mathbb{R})$ and a sufficiently large positive integer ν with $\mu_\nu > \deg(f_{j_2})$ such that

$$d_{C_0^\infty(\mathbb{R})}(w, g - f_{j_2}) < \varepsilon/2, \quad \sup_{x\in[-m,m]} |\sum_{k=1}^{\mu_\nu} \frac{w^{(k)}(0)}{k!}x^k| < 1/2s \text{ and } \sup_{x\in[-m,m]} |\sum_{k=1}^{\lambda_{\mu_\nu}} \frac{w^{(k)}(0)}{k!}x^k| < 1/2s.$$

Thus the function $f := w + u_\nu + f_{j_2}$ belongs to $E(m, j_1, j_2, s, \mu_\nu)$ and satisfies inequality (21). Indeed we have

$$\sup_{x\in[-m,m]} |\sum_{k=1}^{\mu_\nu} \frac{f^{(k)}(0)}{k}x^k - f_{j_2}(x)| = \sup_{x\in[-m,m]} |\sum_{k=1}^{\mu_\nu} \frac{w^{(k)}(0)}{k}x^k| < 1/2s,$$

$$\begin{aligned}\sup_{x\in[-m,m]} |\sum_{k=1}^{\lambda_{\mu_\nu}} \frac{f^{(k)}(0)}{k}x^k - f_{j_1}(x)| &\leq \sup_{x\in[-m,m]} |\sum_{k=1}^{\mu_\nu} \frac{w^{(k)}(0)}{k}x^k| + \sup_{x\in[-m,m]} |P_\nu(x) - (f_{j_1}(x) - f_{j_2}(x))| \\ &< 1/2s + 1/2s = 1/s,\end{aligned}$$

and

$$\begin{aligned}d_{C_0^\infty(\mathbb{R})}(f, g) = d_{C_0^\infty(\mathbb{R})}(w + u_\nu + f_{j_2}, g) &\leq d_{C_0^\infty(\mathbb{R})}(w, g - f_{j_2}) + d_{C_0^\infty(\mathbb{R})}(u_\nu, 0) \\ &< \varepsilon/2 + \varepsilon/2 = \varepsilon.\end{aligned}$$

Hence the set $\mathcal{U}(C_0^\infty(\mathbb{R}), (\lambda_n))$ is G_δ and dense in $C_0^\infty(\mathbb{R})$. Finally to prove that the set $\mathcal{U}(C_0^\infty(\mathbb{R}), (\lambda_n))$ contains a dense vector subspace, except 0, it suffices to write the analogue of Lemma 2.5 (which will be a corollary of Proposition 3.3) and to follow the proof of Theorem 2.6. □

Propositions 3.3 and 3.8 can be summarized as follows.

Theorem 3.9. *Let (λ_n) be a strictly increasing sequence of positive integers. The following assertions are equivalent*

(1) $\mathcal{U}(C_0^\infty(\mathbb{R}), (\lambda_n))$ is non-empty,
(2) $\limsup_n \frac{\lambda_n}{n} = +\infty$.

In addition, in the case $\limsup_n \frac{\lambda_n}{n} = +\infty$, the set $\mathcal{U}(C_0^\infty(\mathbb{R}), (\lambda_n))$ is a G_δ and dense subset of $C_0^\infty(\mathbb{R})$ and contains a dense vector subspace apart from 0.

4. Further development and remark

The notion of doubly universal series has connection with that of topological multiple recurrence in dynamical systems. We refer the reader to [11, page 22]. Recently Costakis and Parissis proved that a frequently Cesàro hypercyclic bounded linear operator T acting on an infinite dimensional separable Banach space over $\mathbb{C}$ is topologically multiply recurrent [9]. The notion of Cesàro hypercyclicity for an operator T was introduced in [14] and that of frequent Cesàro hypercyclicity in [10]. Let us introduce the set of frequent Cesàro universal series. For a power series $f = \sum_{j\geq 0} a_j z^j$, $\sigma_n(f) := \frac{1}{n+1}\sum_{j=0}^n S_j(f)$ denotes the sequence of Cesàro means of the partial sums of the Taylor expansion of f at 0. We know that the set $\mathcal{U}_{Ces}(\mathbb{D})$ of functions $f \in H(\mathbb{D})$ such that, for every compact set $K \subset \mathbb{C}$ with K^c connected and $K \cap \mathbb{D} = \emptyset$ and for every function $h \in A(K)$, there exists an increasing sequence (λ_n) of positive integers such that $\sup_{z\in K} |\sigma_{\lambda_n}(f)(z) - h(z)| \to 0$, as $n \to +\infty$, is a G_δ-subset of $H(\mathbb{D})$ (see [15] or for instance [3]).

Definition 4.1. A power series $f = \sum_{j\geq 0} a_j z^j$ of radius of convergence 1 is said to be a *frequently Cesàro universal series* if for every $\varepsilon > 0$, for every compact set $K \subset \mathbb{C} \setminus \mathbb{D}$ with connected complement, and any function $h \in A(K)$, we have

$$\underline{\text{dens}}\left\{n \in \mathbb{N};\ \sup_{z\in K} |\sigma_n(f)(z) - h(z)| < \varepsilon\right\} > 0.$$

The lower and upper densities of a subset A of $\mathbb{N}$ are respectively defined as follows

$$\underline{\text{dens}}(A) = \liminf_{N\to+\infty} \frac{\#\{n \in A : n \leq N\}}{N} \text{ and } \overline{\text{dens}}(A) = \limsup_{N\to+\infty} \frac{\#\{n \in A : n \leq N\}}{N}$$

where as usual # denotes the cardinality of the corresponding set.

According to a recent result we know that universal Taylor series cannot be frequently universal in the sense of Definition 4.1 where we replace the Cesàro operators σ_n by the partial sums S_n [16]. We state a similar result for the Cesàro universal series .

Theorem 4.2. *The set of frequently Cesàro universal series is empty.*

Proof. Let f be in $\mathcal{U}_{Ces}(\mathbb{D})$. According to [2, Corollary 4.4] f is a universal Taylor series. Let $K \subset \mathbb{C} \setminus \mathbb{D}$ be a compact set with connected complement and h be a non-zero polynomial. Then Theorem 3.3 of [16] ensures that there exists a subsequence (λ_n) of positive integers with $\overline{\text{dens}}(\lambda_n) = 1$ such that $\sup_{z\in K} |S_{\lambda_n}(f)(z) - h(z)| \to 0$, as $n \to +\infty$. According to Section 4 of [6] we have $\sup_{z\in K} |\sigma_{\lambda_n}(f)(z) - h(z)| \to 0$. We end as in the proof of [16, Theorem 3.3]. Indeed define the subset A of $\mathbb{N}$ by

$$A = \{n \in \mathbb{N};\ \sup_{z\in K} |\sigma_n(f)(z)| < d/2\},$$

where $d = \sup_{z\in K} |h(z)|$. Thus there exists an integer N large enough, such that, for every $n \geq N$, $\lambda_n \notin A$. Let us consider the sequence $\tilde{\lambda} = (\lambda_N, \lambda_{N+1}, \dots)$. Clearly $\overline{\text{dens}}(\tilde{\lambda}) = 1$. So the inclusion $A \subset \mathbb{N} \setminus \tilde{\lambda}$ implies

$$\underline{\text{dens}}(A) \leq \underline{\text{dens}}(\mathbb{N} \setminus \tilde{\lambda}).$$

But we have $\underline{\text{dens}}(\mathbb{N} \setminus \tilde{\lambda}) = 1 - \overline{\text{dens}}(\tilde{\lambda}) = 0$. Thus f cannot be a frequently Cesàro universal series. □

Therefore the sequence of operators given by Cesàro means of sequence of operators given by the partial sums (S_n) of the Taylor development at 0 of functions of $H(\mathbb{D})$ is not frequently universal even if the sequence of operators (S_n) is doubly universal. Since the notion of doubly universality has connection with that of topological recurrence, we can compare this result with the main result of [9].

Remark 4.3.
(1) Theorem 4.2 remains true in the case of Fekete universal functions. To see this, it suffices to argue as in the proof of Theorem 4.2 taking into account the results of [17].
(2) The proof of Theorem 4.2 shows that all the elements $f = \sum_{k\geq 0} a_k z^k$ of $\mathcal{U}_{Ces}(\mathbb{D})$ are 1-upper frequently Cesàro universal, i.e. for every compact set $K \subset \mathbb{C} \setminus \mathbb{D}$ with connected complement, and any function $h \in A(K)$, there exists an increasing sequence $\lambda = (\lambda_n)$ of positive integers with $\overline{\text{dens}}(\lambda) = 1$ such that $\sup_{z\in K} |\sigma_{\lambda_n}(f)(z) - h(z)| \to 0$ as $n \to +\infty$.

Let (λ_n) be a strictly increasing sequence of positive integers with $(\lambda_n) \neq \mathbb{N}$. We end the paper with the following remark, which shows that one can find examples of doubly universal series with respect to the given sequence (λ_n) without additional hypothesis. For instance, let us define the set $\mathcal{U}(\mathbb{R}^{\mathbb{N}}, (\lambda_n))$ of sequences $(a_n) \subset \mathbb{R}^{\mathbb{N}}$ satisfying the following universal property: for every pair of real numbers (l_1, l_2) there exists a subsequence (μ_n) of positive integers such that $|\sum_{k=0}^{\lambda_{\mu_n}} a_k - l_1| \to 0$ and $|\sum_{k=0}^{\mu_n} a_k - l_2| \to 0$, as $n \to +\infty$. Then $\mathcal{U}(\mathbb{R}^{\mathbb{N}}, (\lambda_n))$ is a G_δ and dense subset of $\mathbb{R}^{\mathbb{N}}$, endowed with its natural topology defined in Section 3. In particular we have $\mathcal{U}(\mathbb{R}^{\mathbb{N}}, (\lambda_n)) \neq \emptyset$. Indeed, let us consider

$$E(j_1, j_2, s, n) = \left\{ (a_n) \in \mathbb{R}^{\mathbb{N}} : |\sum_{k=0}^{\lambda_n} a_k - r_{j_1}| < \frac{1}{s} \text{ and } |\sum_{k=0}^{n} a_k - r_{j_2}| < \frac{1}{s} \right\}$$

for every j_1, j_2, s, $n \in \mathbb{N}$, where (r_j) is an enumeration of $\mathbb{Q}$. Obviously we have the following description

$$\mathcal{U}(\mathbb{R}^{\mathbb{N}}, (\lambda_n)) = \bigcap_{j_1, j_2, s\in\mathbb{N}} \bigcup_{n\in\mathbb{N}} E(j_1, j_2, s, n).$$

Since the sets $E(j_1, j_2, s, n)$ are open, according to Baire's category theorem it suffices to prove that $\cup_{n\in\mathbb{N}} E(j_1, j_2, s, n)$ is dense in $\mathbb{R}^{\mathbb{N}}$ for every j_1, j_2, $s \in \mathbb{N}$. Fix j_1, j_2, $s \in \mathbb{N}$, $\varepsilon > 0$ and $(b_n) \in \mathbb{R}^{\mathbb{N}}$. We seek $n \geq 0$ and $(a_k) \in E(j_1, j_2, s, n)$ such that $d_{\mathbb{R}^{\mathbb{N}}}((a_n), (b_n)) < \varepsilon$. Let us choose $n \in \mathbb{N}$ so that $\sum_{k\geq n} 2^{-k} < \varepsilon$ and $\lambda_n > n$. It is easy to check that the sequence (a_k) defined by $a_k = b_k$, for $k = 0, \dots, n-1$, $a_n = r_{j_2} - \sum_{k=0}^{n} b_k$, $a_k = 0$ for $k = n+1, \dots, \lambda_n - 1$ and $a_{\lambda_n} = r_{j_1} - r_{j_2}$ does the job.

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