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Abstract

The question of whether subcubic graphs have finite packing chromatic number or not is still open although positive responses are known for some subclasses, including subcubic trees, base-3 Sierpiski graphs and hexagonal lattices. In this paper, we answer positively to the question for some subcubic outerplanar graphs. We provide asymptotic bounds depending on structural properties of the weak dual of the outerplanar graphs and determine sharper bounds on some classes of subcubic outerplanar graphs.

Keywords: packing colouring, packing chromatic number, outerplanar graphs, subcubic graphs.

AMS Subject Classification:

1 Introduction

Let G be a *graph* and let $N(u) = \{u \in V(G)|\ uv \in E(G)\}$ be the *neighborhood* of a vertex u. Let $d_G(u, v)$ denote the usual shortest path distance between two vertices u and v in G. A *k-colouring* c of G is a map from $V(G)$ to $\{1, \dots k\}$ such that for every pair of adjacent vertices (u, v), we have $c(u) \neq c(v)$. A k-colouring is a *k-packing colouring* if $c(u) = c(v)$ implies $d_G(u, v) > c(u)$ for every two distinct vertices u and v. The smallest integer k such that there exists a k-colouring is called the *chromatic number* and is denoted by χ and the smallest integer

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k such that there exists a k-packing colouring is the *packing chromatic number* and is denoted by χ_ρ. An induced subgraph A of G is *packing colourable with colours* $\{k, \ldots, k+\ell\}$, for two positive integers k and ℓ, if there exists a $(\ell+1)$-colouring of A such that $d_G(u,v) > c(u)+k-1$, for every two distinct vertices u and v satisfying $c(u) = c(v)$. The packing chromatic number of a class of graphs is *finite* if there exist an integer ℓ such that $\chi_\rho(G) \leq \ell$, for every graph G of this class.

Let P_∞ denote the two-way infinite path and let P_∞^+ denote the one-way infinite path (with vertex set $\mathbb{N}$). Let $\mathbb{Z}^2$ denote the planar square lattice, $\mathscr{T}$ denote the planar triangular lattice and $\mathscr{H}$ denote the planar hexagonal lattice.

Packing colourings of graphs are inspired from frequency planning in wireless systems. The concept of packing colouring emphasized the fact that signals can have different powers, providing a model for the frequency assignment problem. The packing chromatic number of lattices has been studied by several authors: Soukal and Holub [12] proved that $\chi_\rho(\mathbb{Z}^2) \leq 17$, Ekstein *et al.* [3] that $12 \leq \chi_\rho(\mathbb{Z}^2)$; Fiala *et al.* [5] showed that $\chi_\rho(\mathscr{H}) \leq 7$, $\chi_\rho(\mathbb{Z}^2 \square P_2) = \infty$ and $\chi_\rho(\mathscr{H} \square P_6) = \infty$, Korže and Vesel [10] proved that $\chi_\rho(\mathscr{H}) \geq 7$ and Finbow and Rall [7] proved that $\chi_\rho(\mathscr{T}) = \infty$. Moreover, Togni [13] has proven that distance graphs with two parameters have finite packing chromatic number. More generally, determining if a graph has packing chromatic number at most 4 is a NP-complete problem [9] and determining if a tree has packing chromatic number at most k (with input the tree and k) is also a NP-complete problem [6].

Goddard et al. in [9] derived the following results for some basic classes of graphs. These facts will be used in the following sections. Note that in some cases, the number of colour can be decreased (depending on k and the number of vertices).

Proposition 1. *Let k be a positive integer. The three following properties holds.*

i) Every cycle has packing chromatic number at most 4;

ii) There is a packing colouring of P_∞ with colours $\{k, k+1, \ldots, 3k+2\}$ [9];

iii) If $k > 34$, then there is a packing colouring of P_∞ with colours $\{k, k+1, \ldots, 3k-1\}$ [9].

1.1 Definitions and notation about outerplanar graphs, block graphs and shortest paths

Let G be a graph and A a subgraph of G. We denote by $G-A$ the subgraph of G after deletion of all vertices of A from G and all edges incident to each vertex of A and we denote by $G[A]$

the graph $G - (V(G) \setminus A)$. Specifically, for $x \in V(G)$, $G - x$ denotes the subgraph of G after deletion of x and all edges incident to x from G.

An outerplanar graph G can be represented by a boundary cycle C containing all vertices of G, with non-crossing chords dividing the interior of C into faces. With respect to this fact we can define a *face* of G as the graph induced by a cycle without chord and an *internal face* of G as a face of an outerplanar graph G which contains more than two chords. A face F of G is called an *end face* if F contains only one chord and all the remaining edges of F belong to C.

The *weak dual* of G, denoted by $\mathcal{T}_G$, is the graph with vertex set the faces (without the outer face) of G and edge set $\{FF' |\ F \text{ and } F' \text{ have a common chord}\}$. We denote by u_F the vertex of $\mathcal{T}_G$ corresponding to the face F of G and sometime we identify a specified face and the vertex of $\mathcal{T}_G$ which corresponds to this specified face of G. It is well known that the weak dual of an outerplanar graph is a forest and that that the weak dual of a 2-connected outerplanar graph is a tree. Note that an end face of an outerplanar graph G corresponds to a leaf of $\mathcal{T}_G$ and that an internal face of G corresponds to a vertex of degree at least 3 in $\mathcal{T}_G$. Obviously, every end face of a 2-connected outerplanar graph contains at least one vertex of degree 2.

Let G be a graph. The *block graph* of G is the graph obtained from G by replacing each maximal 2-connected subgraph A of G by a vertex u_A which is adjacent to the vertices which were adjacent to a vertex of A, i.e. to vertices of the set $\{u \in V(G - A) | uv \in E(G),\ v \in V(A)\}$.

Let G be a graph and let $X, Y \subseteq V(G)$ be two subsets of vertices. A *shortest X,Y-path* is a shortest path in G between two vertices $u \in X$ and $v \in Y$ such that there exist no other vertices $u' \in X$ and $v' \in Y$ with $d_G(u', v') < d_G(u, v)$. If X only contains a vertex u, then we write u instead of $\{u\}$.

1.2 Motivation and organization

The question of whether subcubics graphs have finite packing chromatic number or not was raised by Goddard et al. [9]. The largest value of the packing chromatic number of a cubic graph known so far is 13 [8]. Various classes of subcubic graphs [1, 2] have been studied since then. It is known that subcubic trees have packing chromatic number at most 7 [11] and that there exist trees with maximum degree 4 and packing chromatic number as large as possible [11]. Outerplanar graphs form a class of structured graphs which is generally easy to colour and containing the class of trees. Our aim is to prove that for some classes of subcubic outerplanar graphs, every graph has a packing chromatic number bounded by a constant. We define these classes by giving restrictions on the weak dual of outerplanar graphs.

The second section presents our bound for 2-connected subcubic outerplanar graphs without internal face, i.e. when the weak dual is a path. Afterwards, in the third section, we use results from Section 2 in order to determine asymptotic bounds for some larger classes of subcubic outerplanar graphs. In the fourth section, we present sharper bounds for some specific classes of subcubic outerplanar graphs. Finally, in a last section, we present lower bounds for the packing chromatic number of subcubic outerplanar graphs and give concluding remarks. Table 1 summarizes the main results of this paper.

Condition on the subcubic outerplanar graph G	ℓ	Section
G is 2-connected and $\mathcal{T}_G$ is a path	15	2
G is 2-connected and $\mathcal{T}_G$ contains at most k vertices of degree at least 3	$17 \times 6^{3k} - 2$	3
G is connected and $\mathcal{T}_G$ contains at most k' vertices	$9 \times 6^{k'} - 2$	3
G is 2-connected and $\mathcal{T}_G$ contains one vertex of degree at least 3	51	4
The block graph of G is a path and $\Delta(\mathcal{T}_G) \leq 2$	611	4

Table 1: Classes of subcubic outerplanar graphs and values of ℓ for which every graph G satisfies $\chi_\rho(G) \leq \ell$.

2 2-connected subcubic outerplanar graphs when the weak dual is a path

The following observation will be used in order to construct some useful shortest path in 2-connected subcubic outerplanar graphs. Moreover, it gives a description of 2-connected outerplanar graphs with the weak dual of these graphs.

Observation 2. *Let G be a 2-connected outerplanar graph that is not a cycle. Then G contains at least two end faces. Moreover, if G contains no internal face, then G has exactly two end faces.*

Proof. Considering the weak dual, $\mathcal{T}_G$ is a tree since G is connected. Therefore each 2-connected outerplanar graph that is not a cycle contains at least two end faces. In particular, if G has no internal face, then $\mathcal{T}_G$ is a path (the converse also holds). Hence, $\mathcal{T}_G$ has at least two leaves, implying that G has exactly two end faces. ■

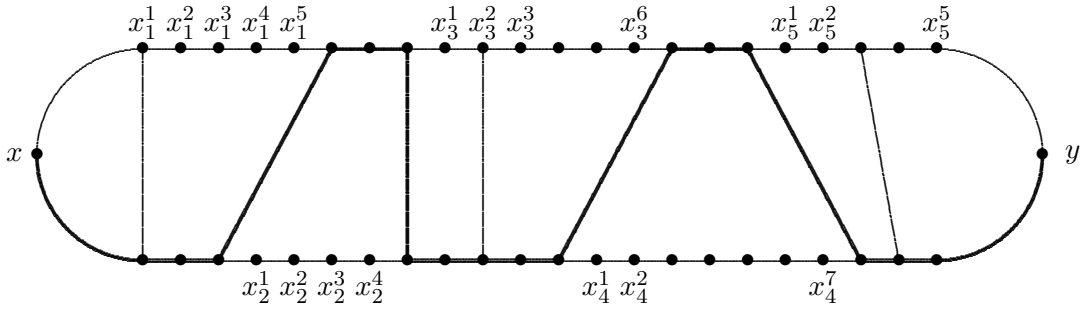

Figure 1: Structure of G in Lemma 3.

We begin this section with the following lemma. This lemma will be used in Sections 3 and 4.

Lemma 3. *Let G be a 2-connected outerplanar graph with $\Delta(\mathcal{T}_G) \leq 2$ and $\Delta(G) \leq 3$. Let x, y be a pair of vertices of degree 2 in G such that x belong to one of the end faces of G and y to the other one, and let P be a shortest x, y-path in G. Then $G - P$ can be packing coloured with colours $\{1, 2, 3, 4\}$.*

Proof. Let C denote the boundary cycle of G. If there is no chord in G, then G is a cycle and, by Proposition 1.i) , $\chi_\rho(G) \leq 4$.

Thus we may assume that C contains some chords in G. Note that $C - P$ is not necessarily connected, but each component of $C - P$ is a path. Let D_i, $i = 1, \ldots, k$ denote the components of $C - P$ in an ordering from x to y, and let ℓ_i denote the length of D_i. We further denote the vertices of each D_i by $x_i^1, x_i^2, \ldots, x_i^{\ell_i}$ in an ordering starting from a vertex of D_i which is closest to x in D_i. The described structure is shown in Fig. 1, where the thick x, y-path depicts P.

We colour each component D_i of $C - P$ with a pattern $1, 2, 1, 3$ starting from x_i^1 ($i = 1, 2, \ldots, k$), i.e., for each odd j, x_i^j is coloured with colour 1, for each j divisible by 4, x_i^j obtains colour 3, and, for each even j not divisible by 4, we colour vertex x_i^j with colour 2, $j = 1, 2, \ldots, \ell_i$. We denote by χ the defined colouring. Since P is a shortest path and $\Delta(G) \leq 3$, there is no collision between any pair of vertices coloured with colour 1 or 2, respectively. Suppose to the contrary that there is a pair of clashing vertices a and b coloured with colour 1. Clearly a and b belong to the same component D_i of $C - P$ and $ab \in E(G) \setminus E(C)$ by the definition of χ. But then we get a contradiction with the fact that D_i is a path. Now suppose that there is a pair of clashing vertices a and b coloured with colour 2, i.e., $d_G(a, b) \leq 2$. Again, a and b must belong to the same component D_i of $C - P$, otherwise $d_G(a, b) > 2$ and we obtain a contradiction. Analogously as for colour 1 we can show that $ab \notin E(G)$. Thus a and b must have a common

neighbour c in G. From the definition of χ, $c \notin V(D_i)$, and $c \notin P$ since $\Delta(G) \leq 3$. Therefore the only possible collision in the defined colouring χ could be between vertices coloured with colour 3. Analogously as for colours 1 and 2, any pair of clashing vertices a and b cannot be at distance one or two apart. Therefore any such collision happens for a and b with $d_G(a,b) = 3$. We will check and modify collisions in the defined colouring χ of the components $D_1, D_2, \ldots, D_k$ of $C - P$ one-by-one starting from D_1 and from the vertex x_i^1 in each D_i. Note that, in each step of the modification process, we check the modified colouring, not the original one. The following possible collisions can occur:

Case 1: a and b belong to different components D_i and D_j of $C - P$, $i, j \in \{1, 2, \ldots, k\}$. Since $d_G(a,b) = 3$, (up to a symmetry) $a = x_i^{\ell_i}$, b belongs to the chord of $D_j \cup P$ which is closest to a, and $j = i+1$, otherwise we get $d_G(a,b) > 3$. Then we can modify the colouring χ of D_j by recolouring vertices x_j^s, $s = 3, \ldots, l_j - 1$, with $\chi(x_j^s) := \chi(x_j^{s+1})$ and we set $\chi(x_j^{l_j}) \in \{1, 2, 3\}$ depending on the continuation of the pattern $1, 2, 1, 3$ in D_j.

Case 2: a and b belong to the same component D_i. Since $d_G(a,b) = 3$, a and b must belong to consecutive chords of $P \cup D_i$ and there is no vertex between these chords on P. We call such a pair of vertices coloured with colour 3 a *critical pair*. Consider a critical pair a and b such that $a = x_i^m$, $b = x_i^n$, $m < n$, and that there is no critical pair a' and a with $a' = x_i^o$, $o < m$. Then we modify the colouring of the vertices of D_i starting at vertex a by $\underline{3}, 1, 4, 1, \underline{2}, 1, 3, 1, 2, \ldots$ instead of $\underline{3}, 1, 2, 1, \underline{3}, 1, 2, 1, 3 \ldots$, i.e., we recolour the vertex x_i^{m+2} with colour 4 and we switch colours 2 and 3 of the vertices x_i^j for even $j > m+2$. Note that the underlined colours represent the critical pair a and b. It is easy to verify that vertices coloured with colour 4 are mutually at distance more than 4 apart, implying that there is no collision between any pair of vertices coloured with colour 4.

After these modifications we obtain a colouring of all the vertices of $G-P$ with colours $\{1, 2, 3, 4\}$ satisfying the distance constraints of the packing colouring.

■

Theorem 4. *If G is a 2-connected subcubic outerplanar graph with $\Delta(\mathcal{T}_G) \leq 2$, then $\chi_\rho(G) \leq 15$.*

Proof. Let x, y be any pair of vertices of degree 2 in G belonging to distinct end faces of G. Let P be a shortest x, y-path in G. By Lemma 3, the vertices of $V(G) \setminus V(P)$ can be coloured with colours from $\{1, 2, 3, 4\}$. Then the colouring can be completed in a packing 15-packing

colouring of G by colouring the vertices along the path P starting at x and using a packing colouring of the infinite path (since P is a shortest path in G, then the distance between any pair of vertices of P is the same on P and on G). For this, we repeat the pattern Q with colours from $\{5, \ldots, 15\}$ of length 36 along the vertices of P starting at x:

$$5, 6, 7, 9, 13, 12, 5, 8, 6, 10, 7, 11, 5, 9, 14, 6, 8, 15, 5, 7, 13, 10, 6, 11, 5, 8, 9, 7, 12, 6, 5, 14, 10, 15, 8, 11$$

It is easy to check that any two colours i in this repeating sequence are separated by at least i integers. ■

Note that the previous pattern was found by a computer search.

3 Asymptotic results for subcubic outerplanar graphs

In the paper, the main open problem is about the finiteness of the packing chromatic number of subcubic outerplanar graphs, i.e, we ask whether the packing chromatic number of an outerplanar graph depends on the order of the graph or not. In this section we prove that, for 2-connected graphs with a fixed number of internal faces and for graphs with a fixed number of faces, it does not depend on the order of graphs.

We begin this section by proving the following useful lemma that will be also used in Section 4. We recall that the weak dual of a 2-connected outerplanar graph is a tree and that u_F is the vertex of the weak dual corresponding to the face F.

Lemma 5. *There exists a packing colouring of P_∞^+ with colours $\{5, \ldots, 15\}$ such that the first vertex x along the path is at distance at least $\lceil (i-5)/2 \rceil$ of a vertex of colour i.*

Proof. By considering the pattern from Theorem 4 starting at x, we can easily check that the first six vertices of P_∞^+ satisfy the property. Since the colours used in the pattern from Theorem 4 are bounded by 15, the other vertices (other than the first six vertices) satisfy the property as well. ■

Let i, j and k be positive integers. Let $r_{i,j}^k$ be an integer such that $r_{i,j}^k \equiv i - j \pmod{k}$ minimizing the absolute value. The value $|r_{i,j}^k|$ corresponds to the distance between two vertices i and j in a cycle C_k with vertex set $\{0, \ldots, k-1\}$ (the vertices are enumerated along the

cycle). A subset of vertices A of a graph G preserve *cycle distance* if there exists an ordering $v_A^0, \ldots, v_A^{|A|-1}$ of the vertices of A satisfying $d_G(v_A^j, v_A^{j'}) \geq |r_{j,j'}^{|A|}|$, for integers $0 \leq j < j' \leq |A|-1$.

Lemma 6. *Let G be graph and let $A \subseteq V(G)$ be a subset preserving cycle distance. The vertices of A can be packing-coloured with colours $\{\ell, \ldots, 6\ell+4\}$, for any integer ℓ.*

Proof. First, if $|A| \leq 5\ell+5$, then we can colour each vertex of A with a different colour from $\{\ell, \ldots, 6\ell+4\}$. Second, if $5\ell+5 < |A| \leq 6\ell+4$, then we begin by colour two vertices of A by each colour of $\{\ell, \ldots, 2\ell-1\}$. Since $d_G(v_A^i, v_A^{i+2\ell}) > 2\ell-1$, for $0 \leq i \leq \ell$, we can easily colour the vertices $v_A^0, \ldots, v_A^{\ell-1}, v_A^{2\ell}, \ldots, v^{3\ell-1}$ with colours from $\{\ell, \ldots, 2\ell-1\}$. We can colour the remaining uncoloured vertices with distinct colours of $\{2\ell, \ldots, 6\ell+4\}$. Third, suppose $|A| > 6\ell+4$. By Proposition 1.ii), we can colour P_∞ with colours from $\{\ell, \ldots, 3\ell+2\}$. Let C be a cycle with more than $6\ell+4$ vertices and let P' be an induced $P_{3\ell+2}$ in C. Since the distance between the two extremities of $C - P'$ is at least $3\ell+3$, we can colour the vertices of $C - P'$ with the colours $\{\ell, \ldots, 3\ell+2\}$ (using Proposition 1.ii)) and colour the vertices of P' with distinct colours of $\{3\ell+3, \ldots, 6\ell+4\}$. Hence, we can use the colouring of this cycle C in order to colour the vertices of A with colours from $\{\ell, \ldots, 6\ell+4\}$. ■

A subset of vertices A of a graph G is *decomposable* into k sets preserving cycle distance if there exist k sets of vertices A_1, A_2, ..., A_k, such that $A_1 \cup \ldots \cup A_k = A$ and for each integer i, A_i preserves cycle distance. The following lemma will be useful in order to prove Theorems 8 and 9.

Lemma 7. *Let G be graph and let $A \subseteq V(G)$ be a subset decomposable into k sets preserving cycle distance. The vertices of A can be packing-coloured with colours $\{\ell, \ldots, 6^k(\ell+1)-2\}$, for any integer ℓ.*

Proof. We proceed by induction on k. For $k = 1$, by Lemma 6, we can colour the vertices of A with colours $\{\ell, \ldots, 6\ell+4\}$. Now suppose that a subset $A \subset V(G)$ is decomposable into $k+1$ sets preserving cycle distance. Using induction hypothesis we can colour the vertices of A_1, ..., A_k with colours $\{\ell, \ldots, 6^k(\ell+1)-2\}$. For the vertices of A_{k+1}, by Lemma 6, we can use colours $\{6^k(\ell+1)-1, \ldots, \ell'\}$, where $\ell' = 6(6^k(\ell+1)-1)+4 = 6^{k+1}(\ell+1)-2$. We do not recolour the vertices from $\cup_{i=1}^{k}(A_i \cap A_{k+1})$ (in the case it is not empty). ■

The following theorem is one of our main result. It can be used in order to prove that some subcubic outerplanar graphs have finite packing chromatic number.

Theorem 8. *If G is a 2-connected outerplanar graph with $\Delta(G) \leq 3$ and with k internal faces, then $\chi_\rho(G) \leq 17 \times 6^{3k} - 2$.*

Proof. Let $F_1, \ldots, F_k$ denote the k distinct internal faces of G. By removing the vertices of degree at least 3 from $\mathcal{T}_G$, we obtain a disjoint union of paths. We distinguish two kinds of paths in this union. For two vertices u_{F_i} and u_{F_j}, $1 \leq i < j \leq k$, the path with one extremity adjacent to u_{F_i} and the other one to u_{F_j} is denoted by $U_{i,j}$. The paths with only one extremity adjacent to a vertex of degree 3, let say u_{F_i}, $1 \leq i \leq k$, are denoted by $U_i^1, \ldots, U_i^{\ell_i}$, where ℓ_i is the number of paths with one extremity adjacent to u_{F_i}. In this proof, we will consider differently the non internal faces depending on the kind of path they are in $\mathcal{T}_G$.

Let B be the set of vertices of G at distance at most one from an internal face. For an internal face F_i, $1 \leq i \leq k$, let B_i be the set of vertices of G not belonging to F_i and at distance (exactly) one from vertices of F_i. Note that $B = B_1 \cup \ldots \cup B_k$. Let i and k' be integers such that $1 \leq i \leq k$, $1 \leq k' \leq \ell_i$. We denote by $y_i^{k'}$ a vertex of G of degree 2 belonging to the face from $V(U_i^{k'})$ which is a leaf in $\mathcal{T}_G$. Let $P_i^{k'}$ be a shortest $B_i, y_i^{k'}$-path in G. We denote by $(p_i^{k'})_1, (p_i^{k'})_2, \ldots$ its vertices in an ordering starting from the vertex of B_i. Finally, let i and j be two integers, $1 \leq i < j \leq k$, such that $U_{i,j}$ is defined. We denote by $P_{i,j}$ a shortest B_i, B_j-path and denote by $p_{i,j}^1, \ldots, p_{i,j}^{k_{i,j}}$ its vertices in an ordering starting from the vertex of B_i.

Let P denote the set of vertices from the previous defined paths of G, i.e $P = \cup_{1 \leq i \leq k, 1 \leq k' \leq \ell_i} V(P_i^{k'})$ $\cup_{1 \leq i < j \leq k} V(P_{i,j})$. The proof is organized as follows. First, we will colour the vertices of $V(G) \setminus (B \cup P)$ with colours $\{1, 2, 3, 4\}$. Second, we will colour some vertices of P. Third, we will colour the remaining uncoloured vertices of G.

Step 1: Colouring the vertices of $V(G) \setminus (B \cup P)$ with colours $\{1, 2, 3, 4\}$.

We colour the vertices of $V(G) \setminus (B \cup P)$ by colouring one by one each connected component of $G - (B \cup P)$ in the same way than in the proof of Lemma 3, i.e. we use the pattern 1,2,1,3. Note that the distance between any two vertices from $V(G) \setminus (B \cup P)$ in two different connected component of $G - B$ is at least 5. Moreover, we proceed as in the proof of Lemma 3 to avoid clashing vertices of colour 3, i.e., we use colour 4.

Step 2: Colouring vertices of P.

Let i, j, i' and k' be integers such that $U_{i,j}$ and $U_i^{k'}$ are defined. Let r be any vertex of $\mathcal{T}_G$. Let $\overrightarrow{\mathcal{T}_G}$ be the digraph obtained from $\mathcal{T}_G$ by replacing each edge $uv \in E(\mathcal{T}_G)$, with $d(u, r) < d(v, r)$, by an arc from u to v. For the vertices of $P_{i'}^{k'}$, we use the pattern of Theorem 4 and Lemma 5 beginning with the third vertex of $P_{i'}^{k'}$, i.e the vertex $(p_{i'}^{k'})_3$. We

distinguish two cases in order to colour $P_{i,j}$:

i): there is a directed path from u_{F_i} to u_{F_j} in $\overrightarrow{\mathcal{T}_G}$. For the vertices of $P_{i,j}$, we use the pattern of Theorem 4 and Lemma 5, beginning with the third vertex of $P_{i,j}$, i.e the vertex $p^3_{i,j}$ and finishing by the vertex $p^{k_{i,j}-3}_{i,j}$

ii): there is a directed path from u_{F_j} to u_{F_i} in $\overrightarrow{\mathcal{T}_G}$. For the vertices of $P_{i,j}$, we use the same pattern as for case i), beginning with the vertex $p^{k_{i,j}-3}_{i,j}$ and finishing by $p^3_{i,j}$.

Note that every vertex of $V(\overrightarrow{\mathcal{T}_G})$ has in-degree at most one. This property, along with Lemma 5, ensure us that a vertex coloured with colour a in $P_{i,j}$, $a \in \{5, \ldots, 15\}$, is at distance at least $a+1$ from any other vertex coloured by a in $P_{i',j'}$, for $1 \leq i' < j' \leq k$.

Step 3: Colouring the remaining vertices of G.

Let $w_{i,j}$ be a vertex among $\{p^2_{i,j}, p^{k_{i,j}-2}_{i,j}\}$ at distance 2 from a vertex of $V(F_i)$ (when $U_{i,j}$ is defined). Let W_i be the set $\{w_{i,j}|\ U_{i,j}$ is defined, $1 \leq j \leq k\}$ and let D_i be the following set $\{(p^{k'}_i)_2|\ 1 \leq k' \leq \ell_i\} \cup W_i$. Since the sets $V(F_i)$, B_i and D_i , $1 \leq i \leq k$, are sets preserving cycle distance, then the subgraph induced by $V(F_1) \cup B_1 \cup D_1 \cup \ldots \cup V(F_k) \cup B_k \cup D_k$ is decomposable into $3k$ sets preserving cycle distance. Hence, using Lemma 7, the remaining uncoloured vertices can be coloured with colours $\{16, \ldots, 17 \times 6^{3k} - 2\}$.

■

This following theorem is more general than Theorem 8 since it gives bounds for outerplanar graphs which are not 2-connected. However, since the parameter is the number of faces, the bound is weaker than that of Theorem 8.

Theorem 9. *If G is a connected outerplanar graph with k faces and $\Delta(G) \leq 3$, then $\chi_\rho(G) \leq 9 \times 6^k - 2$.*

Proof. Let $F_1, \ldots, F_k$ denote the k distinct faces (without the outer face) of G. Let O be the set of vertices which only belong to the outer face. We denote by $O_{i,j}$ the subset $\{u \in O|\ \exists u_i \in F_i$ and $\exists u_j \in F_j$ such that there exist an u_i,u-path and an u_j,u-path in $G[O \cup \{u_i, u_j\}]\}$, for $1 \leq i < j \leq k$.

First, we will colour the vertices of O with colours $\{1, \ldots, 7\}$. Second, we will colour the vertices of $\cup_{1 \leq i \leq k} V(F_i)$.

Step 1: Colouring the vertices of O with colours $\{1, \ldots, 7\}$.

Since G is connected, $\mathcal{T}_G$ is a forest and for each connected component of $\mathcal{T}_G$ (in the case $\mathcal{T}_G$ has several connected components), there exists a vertex u_{F_i} and another vertex outside the

connected component of u_{F_i}, let say u_{F_j}, such that $O_{i,j} \neq \emptyset$. Let $\mathcal{T}'_G$ be the graph obtained from $\mathcal{T}_G$ by adding an edge between two vertices u_{F_i} and u_{F_j} if $O_{i,j} \neq \emptyset$. Let r be a vertex of $\mathcal{T}_G$. Let $\overrightarrow{\mathcal{T}_G}$ be the digraph obtained from $\mathcal{T}'_G$ by replacing each edge $uv \in E(\mathcal{T}'_G)$, with $d(u,r) < d(v,r)$, by an arc from u to v.

Sloper [11] has determined a way to colour any subcubic tree with colours $\{1,\ldots,7\}$. In his paper, he considers a vertex of degree at most 2 as the root and he subdivides the tree into levels according to the distance to the root (a vertex at distance i from the root is at level i). Depending on the colour of the vertices in the ℓ first levels, Sloper gives a way to extend the colouring to the level $\ell+1$. In this colouring of subcubic trees, we can note that we can always begin by colouring the root with 1, the neighborhood of the root with colours 2 and 3, the vertices at even levels with colour 1 and the vertices at level 3 with colours $\{2,3,4,5\}$. When we say that we consider a vertex u of O as the root and use Sloper's pattern [11], we mean that we subdivide the connected component of u in O into levels following the distance to u, consider that u and the vertices at even levels are coloured with 1, that the vertices at level 1 are coloured with 2 and 3 and that the vertices at level 3 are coloured with colours $\{2,3,4,5\}$ and that we use the way to extend the colouring described by Sloper [11].

Let i and j be integers such that $O_{i,j} \neq \emptyset$. If there is an arc from u_{F_i} to u_{F_j} in $\overrightarrow{\mathcal{T}_G}$, then we consider the vertex of $O_{i,j}$ at shortest distance from F_i as the root and use Sloper's pattern to colour the vertices of $O_{i,j}$. If the arc is from u_{F_j} to u_{F_i} in $\overrightarrow{\mathcal{T}_G}$, then we consider the vertex of $O_{i,j}$ at shortest distance of F_j as the root and use Sloper's pattern to colour the vertices of $O_{i,j}$.

For the remaining uncoulored vertices of O, we consider the vertex at shortest distance of an inner face as the root. Note that every vertex of $V(\overrightarrow{\mathcal{T}_G})$ has in-degree at most one. This property ensures us that a vertex coloured with colour a in $O_{i,j}$, $a \in \{1,\ldots,7\}$, is at distance at least $a+1$ from any other vertex coloured by a in O.

Step 2: Colouring the remaining vertices of G.

The sets $V(F_1),\ldots,V(F_k)$ are sets preserving cycle distance. Hence, by Lemma 7, the remaining uncoloured vertices can be coloured with colours $\{8,\ldots,9\times 6^k-2\}$.

■

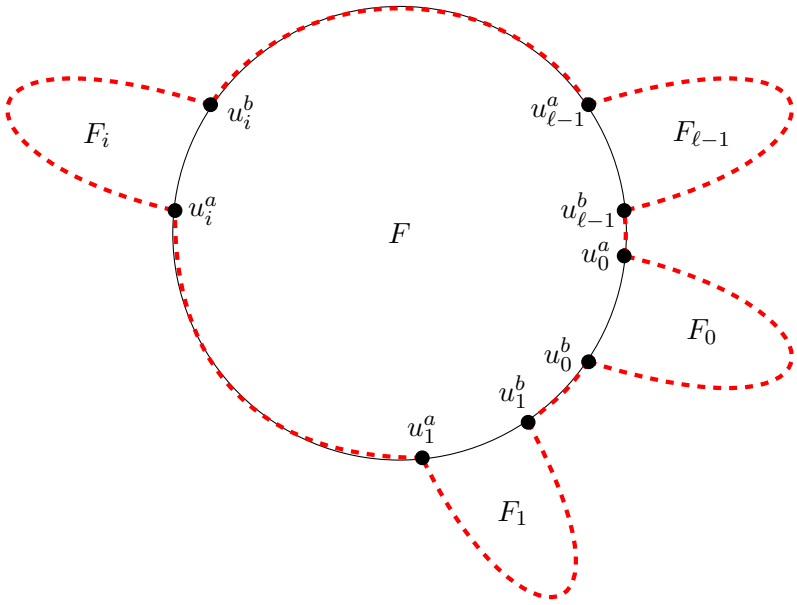

Figure 2: A 2-connected outerplanar graph with one internal face and its different subgraphs (C is represented by a dashed line).

4 Some 2-connected outerplanar graphs with finite packing chromatic number

In this section we consider some special classes of subcubic outerplanar graphs for which we show that their packing chromatic number is finite.

Theorem 10. *If G is a 2-connected subscubic outerplanar graph with exactly one internal face, then $\chi_\rho(G) \leq 51$.*

Proof. Suppose G is a 2-connected outerplanar graph with exactly one internal face and with $\Delta(G) \leq 3$. Let C denote the boundary cycle of G. Let F denote the internal face of G. Let $C' = \{v_0, \ldots, v_{N-1}\}$ denote the set of vertices which belong to F, with v_i adjacent to v_{i+1}, for $0 \leq i < N$. When N is odd, we suppose that v_{N-1} is a vertex such that $d(v_{N-1}) = 2$. Such a vertex exists since the number of vertices of degree 3 in C' is even. By removing the edges of $C \cap F$ from G, and by removing the isolated vertices from the resulting graph, we obtain a graph G' which is a disjoint union of 2-connected outerplanar graphs having no internal face.

Let F_0, …, $F_{\ell-1}$ denote the 2-connected components of G', enumerated in the clockwise order along the cycle C in G (for details, see Fig. 2). Note that, since any F_i contains no internal face,

each F_i has exactly two end faces. Let i be an integer with $0 \leq i < \ell$, and let u_i^a and u_i^b denote the two adjacent vertices of degree 3 in G which belong to $V(F_i) \cap C'$, as it is depicted in Fig. 2. Let y_i be a vertex of degree 2 in the end face of F_i which does not contain u_i^a. Let $x_i \in \{u_i^a, u_i^b\}$ denote a vertex at minimal distance from y_i. Finally, let P_i be a shortest x_i, y_i-path in G. We further denote the vertices of each P_i by $x_i, p_i^1, \ldots, y_i$ in an ordering starting from x_i. Let D_i^1, $\ldots$, $D_i^{k_i}$ denote the connected component of $F_i - P_i$ with D_i^1 containing a vertex among u_i^a and u_i^b and with D_i^k being at larger distance than D_i^{k-1} from x_i, $2 \leq k \leq k_i$.

The proof will be organized as follows. First, we will colour the vertices of C'. Second, we will colour the vertices of $\cup_{0 \leq i < \ell} F_i - P_i$ with colour 1, 2 and 3. The obtained colouring is possibly not a packing colouring of G. Third, we will recolour some vertices. Fourth, we recolour some vertices with colour 4 in order to have a packing colouring. Finally, we colour vertices of the paths $\cup_{0 \leq i < \ell} P_i \setminus \{x_i\}$.

Step 1: Colouring the vertices of C' with colours 1, 2, 29 to 45. For integers k, k', let $r_{k,k'}$ be an integer such that $r_{k,k'} \equiv k - k' \pmod{N}$ and $-\lfloor N/2 \rfloor \leq r_{k,k'} \leq \lfloor N/2 \rfloor$. We can note that v_k and $v_{k'}$ are at distance $|r_{k,k'}|$. We begin by a partitioning of C' in five subsets: $C'_1 = \{v_k |\ k \equiv 0 \pmod 2,\ 0 \leq k < N\}$, $C'_2 = \{v_k |\ k \equiv 1 \pmod 4,\ 0 \leq k < N\}$, $C'_3 = \{v_k |\ k \equiv 3 \pmod{12},\ 0 \leq k < N\}$, $C'_4 = \{v_k |\ k \equiv 7 \pmod{12},\ 0 \leq k < N\}$ and $C'_5 = \{v_k |\ k \equiv 11 \pmod{12},\ 0 \leq k < N\}$. Let m_j denotes the vertex with largest index in C'_j, for $j \in \{1, 2, 3, 4, 5\}$. We use the following patterns to colour the vertices of C'.

1. if $|C'_1| \equiv 0 \pmod 2$ ($|C'_1| \equiv 1 \pmod 2$, respectively), then we colour all vertices of C'_1 ($C'_1 \setminus \{m_1\}$, respectively) with colour 1;
2. we colour all vertices of $C'_2 \setminus \{m_2\}$ with colour 2;
3. if $|C'_3| \equiv 0 \pmod 4$ ($|C'_3| \equiv 1 \pmod 4$, respectively), then we use the pattern $29, 30, 35, 36$, $29, 30, 35, 36, \ldots, 29, 30, 35, 36$ to colour the vertices of C'_3 ($C'_3 \setminus \{m_3\}$, respectively);
4. if $|C'_3| \equiv 2 \pmod 4$ ($|C'_3| \equiv 3 \pmod 4$, respectively), then we use the pattern $29, 30, 35, 36$, $29, 30, 35, 36, \ldots, 29, 30, 35, 29, 30, 36$ to colour the vertices of C'_3 ($C'_3 \setminus \{m_3\}$, respectively);
5. if $|C'_4| \equiv 0 \pmod 4$ ($|C'_4| \equiv 1 \pmod 4$, respectively), then we use the pattern $31, 32, 37, 38$, $31, 32, 37, 38, \ldots, 31, 32, 37, 38$ to colour the vertices of C'_4 ($C'_4 \setminus \{m_4\}$, respectively);
6. if $|C'_4| \equiv 2 \pmod 4$ ($|C'_4| \equiv 3 \pmod 4$, respectively), then we use the pattern $31, 32, 37, 38, 31$, $32, 37, 38, \ldots, 31, 32, 37, 31, 32, 38$ to colour the vertices of C'_4 ($C'_4 \setminus \{m_4\}$, respectively);

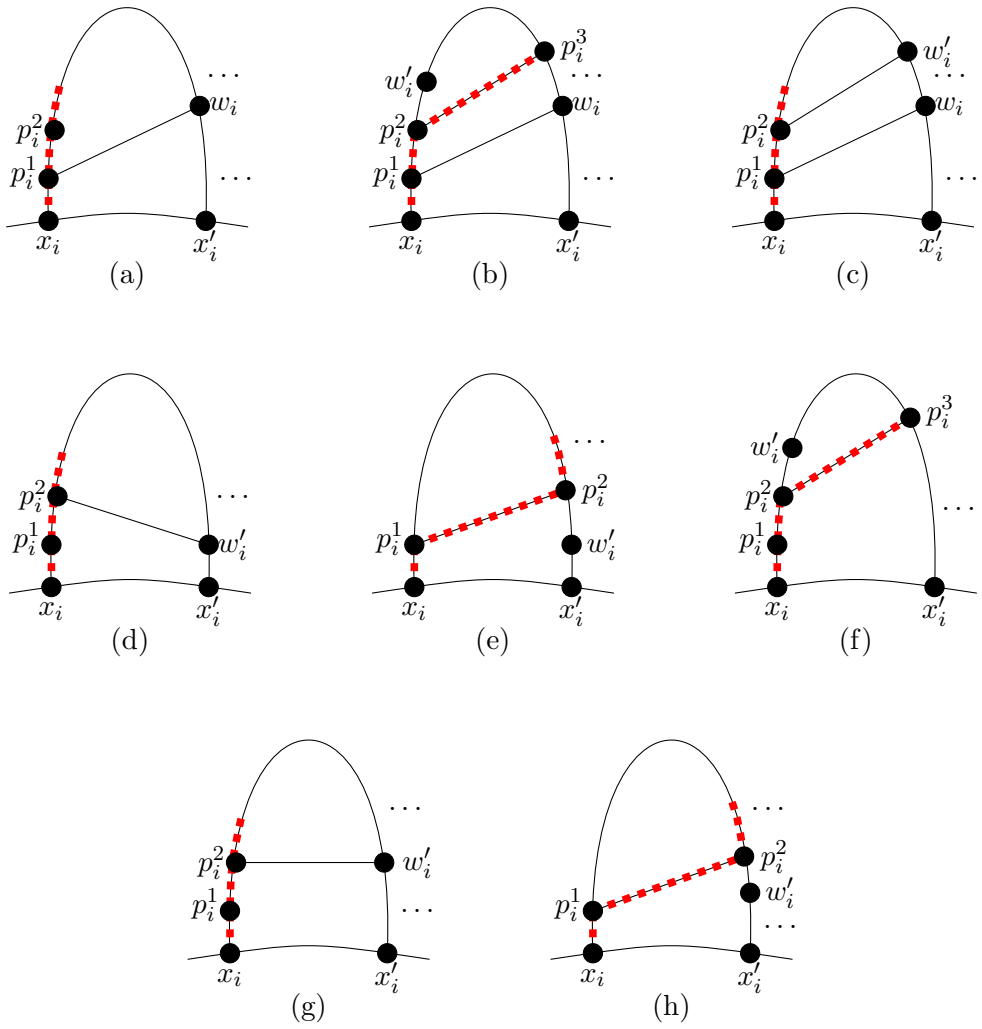

Figure 3: Eight configurations in step 3 (P_i is represented by a dashed line).

7. if $|C'_5| \equiv 0 \pmod 4$ ($|C'_5| \equiv 1 \pmod 4$, respectively), then we use the pattern 33, 34, 39,40, 33, 34,39, 40, . . . , 33,34, 39, 40 to colour the vertices of C'_5 ($C'_5 \setminus \{m_5\}$, respectively);
8. if $|C'_5| \equiv 2 \pmod 4$ ($|C'_5| \equiv 3 \pmod 4$, respectively), then we use the pattern 33, 34, 39,40, 33, 34,39, 40, . . . , 33,34, 39,33, 34, 40 to colour the vertices of C'_5 ($C'_5 \setminus \{m_5\}$, respectively);
9. when it is necessary, we use the colours 41, 42, 43, 44, 45 to colour the vertices of $\{m_k | \ 1 \leq k \leq 5\}$.

It can be verified that the vertices with the same colour in C' have enough mutual distance by calculating the distance for each pair of vertices in each pattern. For example, in the

pattern $29, 30, 35, 36$ of length four, two vertices with the same colour are at distance at least 48, since the pattern has length four and we colour vertices with the same remainder modulo 12. The same goes for the pattern $29, 30, 35, 29, 30, 36$ of length six. Note that for every pair of vertices (u_i^a, u_i^b), there is at least one vertex of colour 1.

Step 2: Colouring the vertices of $F_i - P_i$ with colours 1, 2 and 3, for every $i = 0, \ldots, \ell - 1$.

Let x_i' be the vertex among u_i^a and u_i^b different from x_i. Let ℓ_i be the order of D_i^1 and let $x_i', x_i^1, \ldots, x_i^{\ell_i - 1}$ be the vertices of D_i^1 in an ordering starting from x_i'. If x_i' is coloured with colour 1, then we use the pattern $3, 1, 2, 1$ to colour the vertices $x_i^1, \ldots, x_i^{\ell_i - 1}$. If x_i' is not coloured with colour 1, then we use the pattern $1, 3, 1, 2$ to colour the vertices $x_i^1, \ldots, x_i^{\ell_i - 1}$. We colour the remaining uncoloured vertices of $F_i - P_i$ in the same way than in the proof of Lemma 3. However, we do not change the colouring in order to avoid clashing vertices of colour 3.

Step 3: Recolouring some vertices in F_i, for every $i = 0, \ldots, \ell - 1$.

Note that no vertex of colour 2 can be at distance 2 from x_i', since we have used the patterns $1, 3, 1, 2$ and $3, 1, 2, 1$ to colour the vertices of $V(D_i^1) \setminus \{x_i\}$. We consider four cases depending on whether there exists a vertex w_i of colour 2 which is at distance 2 from x_i or not and depending on whether there exits a neighbor w_i' of colour 1 of p_i^2 or not. We can also note that w_i belongs to the chord delimiting the end face of F_i containing x_i and that p_i^2 can have at most one neighbor of colour 1, even if p_i^2 is y_i (since y_i has degree 2).

Case i) There exists a vertex w_i of colour 2 at distance 2 from x_i and p_i^2 has no neighbor of colour 1 (see Fig. 3(a)). We modify the colouring used for D_1^i by replacing the colouring starting from the vertex w_i by $\underline{4}, 2, 1, 3, 1, \ldots$ instead of $\underline{2}, 1, 3, 1, 2, \ldots$. Note that the underlined colour represents the colour of the vertex w_i.

Case ii) There exists a vertex w_i of colour 2 at distance 2 from x_i and p_i^2 has a neighbor w_i' of colour 1. Suppose that $p_i^2 p_i^3$ is a chord (see Fig. 3(b)). We modify the colouring used for D_1^i by replacing the colouring starting from the vertex w_i by $\underline{4}, 1, 2, 1, 3, \ldots$ instead of $\underline{2}, 1, 3, 1, 2, \ldots$ and the colouring used for D_2^i by replacing the colouring starting from the vertex w_i' by $\underline{2}, 1, 3, 1, \ldots$ instead of $\underline{1}, 2, 1, 3, \ldots$. Suppose that $p_i^2 p_i^3$ is not a chord (see Fig. 3(c)). We modify the colouring used for D_1^i by replacing the colouring starting from the vertex w_i by $\underline{4}, 2, 1, 3, 1, \ldots$ instead of $\underline{2}, 1, 3, 1, 2, \ldots$. Note that in the two last subcases the colour of w_i' is not 1 anymore.

Case iii) There does not exist a vertex of colour 2 at distance 2 from x_i and p_i^2 has a neighbor

w'_i of colour 1. Suppose that w'_i is a neighbor of x'_i (see Fig. 3(d,e)). We can note that x'_i is not coloured with colour 1 and that x_i is coloured with colour 1. Let P'_i be the path with vertex set x'_i, w'_i, p_i^2,p_i^3,.... We replace P_i by the path P'_i. Moreover we colour w'_i with colour 3. Now suppose that w'_i is at distance at least 2 from x'_i and that $p_i^2w'_i$ and $p_i^1p_i^2$ are not chords (see Fig. 3(f)). We modify the colouring used for D_2^i by replacing the colouring starting from the vertex w'_i by $\underline{2}, 1, 3, 1, \ldots$ instead of $\underline{1}, 2, 1, 3, \ldots$. If w'_i is at distance at least 2 from x'_i and $p_i^2w'_i$ is a chord (see Fig. 3(g)), we modify the colouring used for D_1^i by replacing the colouring starting from the vertex w'_i by $\underline{4}, 1, 2, 1, 3, \ldots$ instead of $\underline{1}, 2, 1, 3, \ldots$ or $\underline{1}, 3, 1, 2, \ldots$. Finally, if w'_i is at distance at least 2 from x'_i and $p_i^1p_i^2$ is a chord (see Fig. 3(h)), then we change the colour of w'_i by 4.

Case iv) There does not exist a vertex of colour 2 at distance 2 from x_i and p_i^2 has no neighbor w'_i of colour 1. In this last case, we do not recolour vertices.

Step 4: Recolouring some vertices with colour 4 in F_i, for every $i = 0, \ldots, \ell - 1$. We can note that, after the recolouring of step 3, as in the proof of Lemma 3, there is no possible pair of clashing vertices of colour 1 or of colour 2. The only possible pair a and b of clashing vertices have colour 3. If a and b belong to different component D_i^k and $D_i^{k'}$ of $F_i - P_i$, $k, k' \in \{1, 2, \ldots, k_i\}$, then we proceed as in Case 1 of the proof of Lemma 3. Note that we do not change the colour of the vertices of D_i^1 in this case. If the colouring used for D_i^2 has been changed in Step 3 and $a \in V(D_i^1)$, $b \in V(D_i^2)$, then we modify the colouring used for D_2^i by replacing the colouring starting from the vertex w'_i of Step 3 by $\underline{2}, 3, 1, 2, 1, 3 \ldots$ instead of $\underline{2}, 1, 3, 1, \ldots$. If a and b belong to the same component D_i^k of $F_i - P_i$, $k, \in \{1, 2, \ldots, k_i\}$, then we proceed as in Case 2 of the proof of Lemma 3. However, we have to check that the vertices coloured with colour 4 in Step 3 are at mutual distance at least 5 and that the vertices coloured with colour 4 in Step 3 are at distance at least 4 from the added vertices of colour 4 of Step 4.

Since we have used the patterns $1, 3, 1, 2$ and $3, 1, 2, 1$ to colour the vertices of $V(D_i^1) \setminus \{x_i\}$, the vertex w_i is at distance at least 2 from both x_i and x'_i in case i) and ii) of step 3 and that a vertex w'_i of colour 4 in case iii) of step 3 is at distance at least 2 from both x_i and x'_i. Thus, the vertices coloured with colour 4 in Step 3 are at mutual distance at least 5. Suppose that a is a vertex of colour 4 from Step 3 and that b is a vertex of colour 4 not in D_i^1. We can note that a is the vertex w_i or w'_i of Step 3. The minimal distance between

a vertex of D_i^1 and a vertex of D_i^2 is at least 3. Moreover, because we have proceeded as in Lemma 3, b is at distance at least 2 from a vertex at minimal distance of a. Hence, $d(a,b) \geq 5$. Now suppose that a is a vertex of colour 4 from Step 3 and that b is a vertex of colour 4 in D_i^1. Since, in every case, a is at distance at least 3 from another vertex of colour 3 in D_i^1, we obtain that $d(a,b) \geq 5$.

Step 5: Colouring the vertices of $P_i \setminus \{x_i\}$ with colours 16 to 28 and 46 to 51, for every $i = 0, \ldots, \ell-1$. We begin by colouring the vertex p_i^2 by 1, for each i between 1 and $\ell-1$. Since we have changed the colour of the eventual neighbor of p_i^2 with colour 1 in Step 3, there are no possible collisions.

For the vertices of P_i, we use the pattern of Theorem 4 and Lemma 5, beginning with the fourth vertex of P_i, i.e the vertex p_i^3.

Let $B = \{p_k^1 |\ 0 \leq k < \ell\}$. We finish by colouring the vertices from B with colours 16 to 28 and colours 46 to 51. For integers k, k', let $r'_{k,k'}$ be an integer such that $r'_{k,k'} \equiv k - k' \pmod{\ell}$ and $-\lfloor \ell/2 \rfloor \leq r'_{k,k'} \leq \lfloor \ell/2 \rfloor$. We can note that the vertices p_k^1 and $p_{k'}^1$ are at distance $2|r'_{k,k'}| + 1$. We begin by a partitioning of B in three subsets B_1, B_2 and B_3, with $B_j = \{p_k^1 |\ k \equiv j \pmod 3,\ 0 \leq k < \ell\}$, $j = 0, 1, 2$. Let m'_j, m''_j denote the two vertices with the two largest indices in B_j, for $j \in \{1, 2, 3\}$. We use the following patterns to colour the vertices of B.

1. if $|B_1| \equiv 0 \pmod 3$ ($|B_1| \equiv 1 \pmod 3$, $|B_1| \equiv 2 \pmod 3$, respectively), then we use the pattern $16, 17, 18, 16, 17, 18, \ldots, 16, 17, 18$ to colour the vertices B_1 ($B_1 \setminus \{m'_1\}$, $B_1 \setminus \{m'_1, m''_1\}$, respectively);
2. if $|B_2| \equiv 0 \pmod 5$ ($|B_2| \equiv 1 \pmod 5$, $|B_2| \equiv 2 \pmod 5$, respectively), then we use the pattern $19, 20, 21, 25, 26, 19, 20, 21, 25, 26, \ldots, 19, 20, 21, 25, 26$ to colour the vertices of B_2 ($B_2 \setminus \{m'_2\}$, $B_2 \setminus \{m'_2, m''_2\}$, respectively);
3. if $|B_2| \equiv 3 \pmod 5$ ($|B_2| \equiv 4 \pmod 5$, respectively), then we use the pattern $19, 20, 21, 25, 26, 19, 20, 21, 25, 26, \ldots, 19, 20, 21, 25, 26, 19, 20, 21, 25, 19, 20, 21, 26$ to colour the vertices of B_2 ($B_2 \setminus \{m'_2\}$, respectively);
4. if $|B_3| \equiv 0 \pmod 5$ ($|B_3| \equiv 1 \pmod 5$, $|B_3| \equiv 2 \pmod 5$, respectively), then we use the pattern $22, 23, 24, 27, 28, 22, 23, 24, 27, 28, \ldots, 22, 23, 24, 27, 28$ to colour the vertices of B_3 ($B_3 \setminus \{m'_3\}$, $B_3 \setminus \{m'_3, m''_3\}$, respectively);
5. if $|B_3| \equiv 3 \pmod 5$ ($|B_3| \equiv 4 \pmod 5$, respectively), then we use the pattern $22, 23, 24, 27, 28, 22, 23, 24, 27, 28, \ldots, 22, 23, 24, 27, 28, 22, 23, 24, 27, 22, 23, 24, 28$ to colour the ver-

tices of B_3 ($B_3 \setminus \{m'_3\}$, respectively);

6. when it is necessary, we use the colours 46, 47, 48, 49, 50, 51 to colour the vertices of $\{m'_k, m''_k |\ 1 \leq k \leq 3\}$.

Note that in some cases these graphs (depending on the size of B and C') can be coloured with less than 51 colours. ■

Proposition 11. *There is a packing colouring of even vertices of P_∞ with colours $\{k, k+1, \ldots, 2k-1\}$.*

Proof. For the colouring of the vertices of the path P_∞ we use pattern $1, k, 1, k+1, 1, k+2, \ldots, 1, 2k-1$ and after deleting colour 1 we get required colouring. Note that the distance between any pair of vertices coloured with the same colour in two consecutive copies of this pattern is $2k$. ■

Theorem 12. *Let G be a connected outerplanar graph with no internal face and with $\Delta(G) \leq 3$, such that the block graph of G is a path. Then $\chi_\rho(G) \leq$*

Proof. Let G be a graph, B_G the block graph of G and let $B_1, \ldots, B_k$ denote the blocks of G such that B_i, B_{i+1} are consecutive in B_G, $i = 1, \ldots, k-1$. Let C_i denote the boundary cycle of B_i, $i = 1, \ldots, k$. Since G contains no internal face, each B_i contains no internal face as well, implying that every B_i contains exactly two end faces. Let x_i denote any vertex of degree 2 in one of the end faces of B_i and y_i any vertex of degree 2 in the other end face of B_i. Now choose a shortest x_1, x_k-path P in G. Let $P_i = P \cap B_i$, $i = 1, \ldots, k$ and let $\bar{x}_i$ and $\bar{y}_i$ be the end vertices of P_i. Obviously P goes through all the blocks of G, hence P_i is nonempty for each $i = 1, \ldots, k$, every P_i is a path since the block graph of G is a path, and each P_i is a shortest path in G since P is shortest in G. Among all possible choices of the vertices x_i, y_i we choose x'_i and y'_i in such a way that the x'_i, P-path and the y'_i, P-path are shortest possible, respectively, for each $i = 1, \ldots, k$. Let Q_1^y be a shortest y'_1, P-path in G, Q_k^y a shortest y'_k, P-path in G and, analogously for each $i = 2, \ldots, k-1$, let Q_i^x, Q_i^y be a shortest x'_i, P-path, or y'_i, P-path, respectively. Note that some of the paths Q_i^x, Q_i^y may be trivial or empty. The structure of the graph G is depicted in Fig. 4 (the thick path represents the path P).

Consider each block B_i separately. We show that, in each B_i, we can colour vertices of $C_i - (P_i \cup Q_i^x \cup Q_i^y)$ with colours $\{1, 2, 3, 4\}$. Let C_i^1 be a shortest cycle in B_i containing all

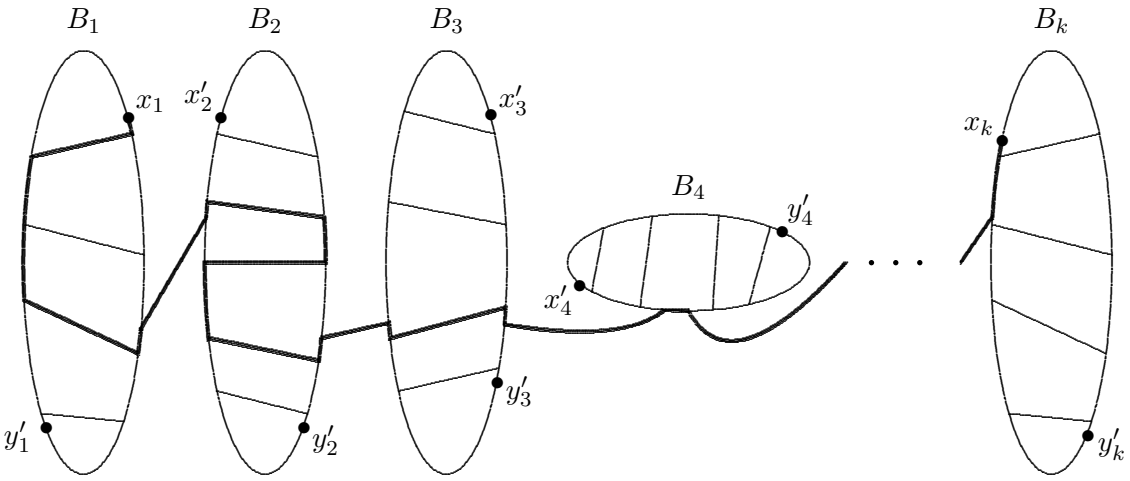

Figure 4: Structure of the graph G in Theorem 12.

vertices of P_i. If $C_i^1 = C_i$, then we can colour all vertices of $B_i - V(P_i)$ with colours $\{1, 2, 3, 4\}$ by Lemma 3 since $\bar{x}_i$ and $\bar{y}_i$ belong to different end faces of B_i and P_i is a shortest $\bar{x}_i, \bar{y}_i$-path in B_i.

Thus we may assume that there is a chord h in B_i such that $h \in E(C_i^1)$ (see Fig. 5). Up to a symmetry suppose that $x'_i \notin C_i^1$. Let x_i^1 and x_i^2 denote the end vertices of h and $B_i^x = G\left[x_i^1, C_i, x'_i, C_i, x_i^2, x_i^1\right]$. By the proof of Lemma 3, we can colour all vertices of $B_i^x - V(Q_i^x)$ with the pattern $1, 2, 1, 3$ since Q_i^x goes through one end vertex of h, which is of degree 2 in B_i^x and Q_i^x is shortest possible. Note that this colouring is not necessarily a packing colouring, but after we colour all the vertices of $B_i - (V(P_i) \cup V(Q_i^x) \cup V(Q_i^y))$ we will modify the colouring we obtain analogously as it was described in the proof of Lemma 3.

Now we consider the subgraph $G[C_i^1]$. In both end faces of $G[C_i^1]$, there is a vertex of degree 2 in $G[C_i^1]$ belonging to P_i. Let y_i^1 and y_i^2 denote such vertices. Since P is shortest in G, the subpath $y_i^1 P_i y_i^2$ is shortest in G too. Now we can colour the vertices of $C_i^1 - V(P_i) - V(Q_i^x)$ with colours $1, 2, 3$ as it is mentioned in the proof of Lemma 3, distinguishing the following cases:

- if neither $h \in E(Q_i^x)$ nor $h \in E(P_i)$ (see Fig. 5(a)), then we follow up the colouring of the vertices of $B_i^x - V(Q_i^x)$, i.e., if, e.g., we finished the colouring of the vertices of $B_i^x - V(Q_i^x)$ with $\ldots, 1, 2, 1,$ we continue the colouring of the vertices of $C_i^1 - V(P_i) - V(Q_i^x)$ with $3, 1, 2, \ldots$;

- if $h \in E(Q_i^x)$ or $h \in E(P_i)$ (see Fig. 5 (b) or (c), respectively), then we colour the vertices of $C_i^1 - V(P_i) - V(Q_i^x)$ starting with the pattern $1, 2, 1, 3$.

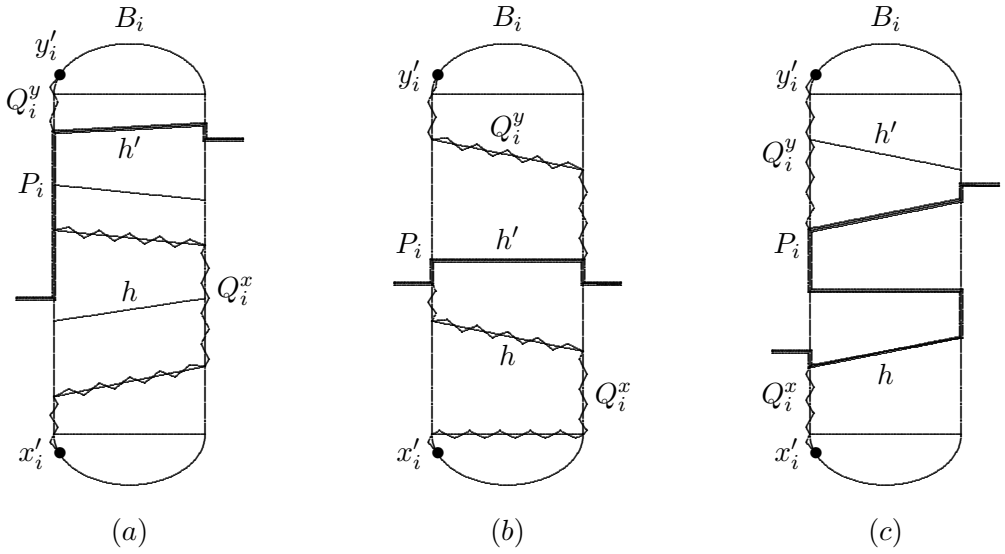

Figure 5: Structure of a block of the graph G in Theorem 12.

In both possibilities we coloured all the vertices of $(B_i^x \cup C_i^1) - (V(Q_i^x) \cup V(P_i))$ with colours $\{1,2,3\}$. If $y'_i \in V(C_i^1)$, then we coloured all the vertices of $B_i \setminus (V(P_i) \cup V(Q_i^x))$. Hence we may assume that $y'_i \notin C_i^1$, i.e., there is a chord $h' \neq h$ of B_i in C_i^1. Analogously as for the extension of the colouring of the vertices of $B_i^x - V(Q_i^x)$ to $G[V(C_i^1) \setminus (V(P_i) \cup V(Q_i^x))]$, we colour the vertices of $B_i - (V(B_i^x) \cup V(C_i^1))$ either by an extension of the pattern $1,2,1,3$ from the colouring of the vertices of if neither $h' \in E(P_i)$ nor $h' \in E(Q_i^y)$, or we start the colouring with the pattern $1,2,1,3$ if $h' \in E(P_i)$ or $h' \in E(Q_i^y)$. Thus we coloured all the vertices of $B_i - (V(P_i) \cup V(Q_i^x) \cup V(Q_i^y))$ with colours $\{1,2,3\}$. Now, using the same method as it is given in the proof of Lemma 3, we can modify the above defined colouring using colour 4 if necessary, thus we obtain a packing colouring of all the vertices of $B_i - (V(P_i) \cup V(Q_i^x) \cup V(Q_i^y))$ with colours $1,2,3,4$.

Now we consider two consecutive blocks B_i, B_{i+1}. Neither the neighbour z_i^2 of the last vertex of P_i in B_i which does not belong to P_i, nor of the first vertex z_{i+1}^1 of P_{i+1} in B_{i+1} which does not belong to P_{i+1}, is coloured with colour 4 (which follows from the moditications given in the proof of Lemma 3). But $\mathrm{dist}_G(z_i^2, z_{i+1}^1)$ can be equal to three and, possibly, both z_i^2, z_{i+1}^1 can be coloured with colour 3, therefore we delete colouring of z_{i+1}^1 in each B_i, $i = 1, \ldots, k-1$.

Now we colour the paths Q_i^x, Q_i^y, $i = 1,2,\ldots,k$. Rename these paths Q_i^x, Q_i^y with Q_j, $j = 1, \ldots, 2k-2$ as they leave the path P starting from B_1. Note that for Q_i^x or Q_i^y empty we set the corresponding Q_j also empty. The distance from P to any vertex of Q_j in G is the same as on

Q_j, hence each Q_j is a shortest path in G between P and the relevant vertex x_i or y_i, respectively. Thus, by Lemma 5, we can colour the vertices of each path Q_{2i-1} $(i = 1, 2, \ldots, k-1)$ with the same pattern $5, 6, \ldots$ using colours $5, 6, \ldots, 15$, starting at the vertex of Q_{2i-1} at distance two from P. Analogously we can colour the vertices of the paths Q_{2i} $(i = 1, 2, \ldots, k-1)$ using the same pattern $5, 6, \ldots$ using colours $5, 6, \ldots, 15$, starting at the vertex of Q_{2i} at distance three from P. Then the distance between vertices on distinct paths Q_m, Q_n coloured with colour 5 is at least $3 + 1 + 2$, the distance between vertices on distinct paths Q_m, Q_n coloured with colour 6 is at least $4 + 1 + 3$, etc. Therefore the defined colouring of the paths Q_j, $j = 1, \ldots, 2k-2$ is a packing colouring.

Now we colour the remaining vertices of G. We start with a colouring of the path P with a pattern using colours $16, 18, \ldots, 50$ by Proposition 1.ii). Then we colour all uncoloured vertices of Q_j $(j = 1, \ldots, 2k+2)$ at distance one from P with colours $51, 55, \ldots, 152$ by Proposition 1.iii). For the remaining vertices of Q_{2i} at distance two from P, the distance between any such vertices on Q_{2m} and Q_{2n} $(m, n \in \{1, 2, \ldots, k-1\}, m \neq n)$ is at least $2|m-n|+4$. Hence we can colour these vertices with colours $153, \ldots, 305$ by Proposition 11. Finally we colour the vertices z^1_{i+1} $(i = 1, \ldots, k-1)$ with colours $306, \ldots, 611$ by Proposition 11 since $\mathrm{dist}_G(z^1_m, z^1_n) \geq 2|m-n|+1$. ■

Remark that it is possible to decrease the upper bound of 611 of the last theorem by using colouring patterns found by computer instead of using Proposition 1.

5 Concluding remarks

In the previous sections, we have determined some classes of outerplanar graphs with finite packing chromatic number. As for lower bounds, we are (only) able to state the following:

Proposition 13. *There exists 2-connected subcubic outerplanar graphs G without internal faces and with chromatic number 5.*

Proof. It has been proven in [9] that $\chi_\rho(G) = 5$ for $G = P_n \Box P_2$ and $n \geq 6$. ■

Proposition 14. *There exists a 2-connected subcubic outerplanar graph with packing chromatic number 7.*

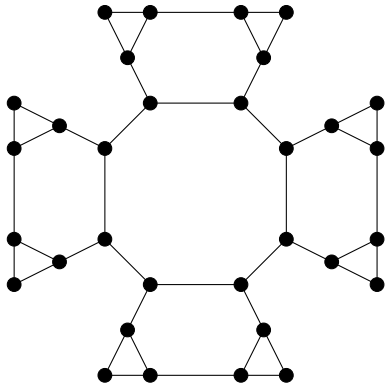

Figure 6: An outerplanar subcubic graph with packing chromatic number 7.

Proof. We have checked, by computer, that the graph G illustrated in Figure 6 has packing chromatic number 7 by verifying that every proper colouring of G with 6 colours is not a packing colouring and by determining a packing colouring of G with 7 colours.

■

Brešar et al. [1] have proven that for any finite graph G, the graph $G \boxtimes P_\infty$ has finite packing chromatic number. The degree of $G \boxtimes P_\infty$ can be arbitrary large. This property illustrates the fact that the degree of a graph is not the only parameter to consider in order to have finite packing chromatic number. Maybe the fact that the weak dual is a path (and is not any tree) helps to bound the packing chromatic number. It remains an open question to determine if the packing chromatic number of subcubic graphs is finite or not.

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References

[1] B. Brešar, S. Klavžar, and D. F. Rall. On the packing chromatic number of Cartesian products, hexagonal lattice, and trees, *Discrete Applied Mathematics* **155**(17):2303–2311, 2007.

[2] B. Brešar, S. Klavžar, and D. F. Rall. Packing Chromatic Number of Base-3 Sierpiski Graphs, *Graphs and Combinatorics* **32**(4):1315–1327, 2016.

[3] J. Ekstein, J. Fiala, P. Holub, B. Lidický, The packing chromatic number of the square lattice is at least 12, *arXiv*:1003.2291v1, 2010.

[4] J. Fiala, P. A. Golovach, Complexity of the packing coloring problem for trees, *Discrete Applied Mathematics* **158**:771–778, 2010.

[5] J. Fiala, S. Klavžar, B. Lidický, The packing chromatic number of infinite product graphs, *European J. Combin.* **30**(5):1101–1113, 2009.

[6] J. Fiala, P. A. Golovach. Complexity of the packing coloring problem for trees, *Discrete Appl. Math.*, **158**(7):771–778, 2010.

[7] A.S. Finbow, D.F. Rall, On the packing chromatic number of some lattices, *Discrete Appl. Math.* **158**:1224-1228, 2010.

[8] N. Gastineau, O. Togni, *S*-Packing Colorings of Cubic Graphs, *Discrete Mathematics* **339**:2461–2470, 2016.

[9] W. Goddard, S. M. Hedetniemi, S. T. Hedetniemi, J. M. Harris, D. F. Rall, Broadcast chromatic numbers of graphs, *Ars Combinatoria* **86**:33-49, 2008.

[10] D. Korže, A. Vesel, On the packing chromatic number of square and hexagonal lattice, *Ars Mathematica Contemporanea* **7**:13–22, 2014.

[11] C. Sloper, Broadcast-coloring of trees, *Reports in Informatics* **233**:1-11, 2002.

[12] R. Soukal, P. Holub, A Note on Packing Chromatic Number of the Square Lattice, *The Electronic Journal of Combinatorics* **17**:447–468, 2010.

[13] O. Togni, On packing Colorings of Distance Graphs, *Discrete Applied Mathematics* **167**:280–289, 2014.