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Cyrille Chenavier

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# A Lattice Formulation of the $F_4$ Completion Procedure

Cyrille Chenavier \*

## Abstract

We write a procedure for constructing noncommutative Gröbner bases. Reductions are done by particular linear projectors, called reduction operators. The operators enable us to use a lattice construction to reduce simultaneously each  $S$ -polynomial into a unique normal form. We write an implementation as well as an example to illustrate our procedure. Moreover, the lattice construction is done by Gaussian elimination, which relates our procedure to the  $F_4$  algorithm for constructing commutative Gröbner bases.

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## 1 Introduction

Since they were introduced by Buchberger during his thesis [9, 10], Gröbner bases have made possible the study of several problems in computer science [6, Chapter 6] and mathematics [8], like algebraic geometry, computations with ideals or solving decision problems, for instance. In order to enlarge the number of application scopes of Gröbner bases, improvements of the Buchberger algorithm for constructing Gröbner bases were developed. For instance, several choices during the algorithm (choose a critical pair, choose a reduction of a  $S$ -polynomial) have

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an influence on its complexity and were investigated [?, ?, ?, ?]. Another issue is to avoid computations of unnecessary critical pairs [11, ?, ?, ?, ?], that is, critical pairs for which the  $S$ -polynomials reduce into zero.

In [14, 15], Faugère proposed new algorithms, namely the  $F_4$  and  $F_5$  algorithms, for constructing Gröbner bases. The  $F_5$  algorithm treats the case of unnecessary critical pairs. Its efficiency is studied in [?]. For the  $F_4$  algorithm, the method consists in reducing simultaneously several  $S$ -polynomials using linear algebra techniques. Indeed, at each step of the algorithm, this is not one but many  $S$ -polynomials which are reduced into normal forms. Hence, we do not choose one critical pair but we select some of them.

In this paper, we are interested in noncommutative Gröbner bases. The analogous of the  $F_4$  procedure<sup>1</sup> exists [?] and is written in the system MAGMA [?]. Our purpose is to provide a lattice formulation of the (noncommutative)  $F_4$  procedure. More precisely, we write a procedure, analogous to  $F_4$ , where we interpret the addition of relations as a lattice construction. For that, we use the lattice formulation of the completion introduced in [13].

## A Lattice Formulation of Completion

**Noncommutative Gröbner Bases and Completion.** First, we recall how are defined noncommutative Gröbner bases and how a classical completion algorithm, such as the noncommutative version of the Buchberger's one, works.

Given a set  $X$ , let  $X^*$  be the set of noncommutative monomials, that we identify with words, over  $X$ . We fix a monomial order on  $X^*$ . For every noncommutative polynomial  $f \in \mathbb{K}X^*$ , we write  $\text{lm}(f)$  the greatest word, with respect to the fixed monomial order, occurring in the decomposition of  $f$ . Given a two-sided ideal  $I$  of  $\mathbb{K}X^*$ , a subset  $R$  of  $I$  is called a noncommutative Gröbner basis of  $I$  if the following statement holds

$$\forall f \in I, \exists g \in R \text{ such that } \text{lm}(g) \text{ is a sub-word of } \text{lm}(f).$$

Equivalently, that means that the reduction system induced by the rules

$$\text{lm}(f) \longrightarrow r(f),$$

where  $f \in R$  and  $r(f)$  is the remainder of  $f$  for the fixed monomial order, is confluent. For that, it is necessary and sufficient to reduce the  $S$ -polynomials of critical pairs into zero.

Let  $I$  be a two-sided ideal of  $\mathbb{K}X^*$  and let  $R$  be a generating set of  $I$ . The set  $R$  is thought as a set of generating relations of the algebra  $\mathbb{K}X^*/I$ . In order to complete  $R$  into a noncommutative Gröbner basis, we add new relations to it. These new relations are used to reduce  $S$ -polynomials into zero. Consider for instance the set  $X = \{x, y, z\}$  and the deg-lex order on  $X^*$  induced by  $x < y < z$ . Consider the two-sided ideal  $I$  spanned by

$$R = \{yz - x, zx - xy\}.$$

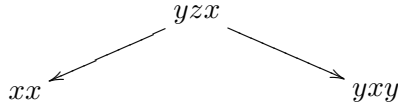
The associated reduction system is spanned by the following two rules

$$yz \longrightarrow x \text{ and } zx \longrightarrow xy.$$

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<sup>1</sup>We say "procedure" instead of "algorithm" when we deal with noncommutative algebras because the procedure has no reason to terminate in this case (see 3.1.8).

We have one critical pair



for which the  $S$ -polynomial  $yxy - xx$  cannot be reduced. In order to reduce this  $S$ -polynomial into zero, we need to add the rule

$$yxy \longrightarrow xx, \quad (1)$$

that is, we add the relation  $yxy - xx$  to  $R$ . The rule (1) creates new critical pairs for which we need to add new relations. In this case, it turns out that after a finite number of steps, all the  $S$ -polynomials reduce into zero [16], that is,  $I$  admits a finite noncommutative Gröbner basis.

**Reduction Operators.** Our lattice approach to completion requires *reduction operators*. The latter are linear projectors describing reductions on a vector space admitting a *well-ordered* basis, that is, a basis equipped with a well-founded total strict order. Typically, the vector space is a set of noncommutative polynomials and the well-order is a monomial order.

Let  $\mathbb{K}$  be a commutative field. Given a well-ordered set  $(G, <)$ , every non-zero vector  $v$  of the vector space  $\mathbb{K}G$  spanned by  $G$  admits a greatest element in its decomposition with respect to  $G$ . This greatest element is written  $\text{lg}(v)$ . A reduction operator relative to  $(G, <)$  is an idempotent linear endomorphism  $T$  of  $\mathbb{K}G$  such that for every  $g \in G$ , we have

$$T(g) = g \text{ or } \text{lg}(T(g)) < g.$$

In [5], Bergman uses reduction operators as a language to formalize reductions in a free algebra. This approach to reductions has applications in homological algebra [2, 3, 4, 12, 17]. These works are based on a lattice structure on the set of reduction operators. This structure provides a lattice formulation of confluence from which we deduce the one of completion.

**Lattice Formulations of Confluence and Completion.** In [13, Proposition 2.1.14], it is shown that the kernel map induces a bijection between the set  $\mathbf{RO}(G, <)$  of reduction operators and the set  $\mathcal{L}(\mathbb{K}G)$  of subspaces of  $\mathbb{K}G$ :

$$\ker : \mathbf{RO}(G, <) \xrightarrow{1:1} \mathcal{L}(\mathbb{K}G). \quad (2)$$

The one-to-one correspondence (2) induces a lattice structure on  $\mathbf{RO}(G, <)$ , where the order  $\preceq$ , the lower bound  $\wedge$  and the upper bound  $\vee$  are defined by

- $T_1 \preceq T_2$  if  $\ker(T_2) \subseteq \ker(T_1)$ ,
- $T_1 \wedge T_2 = \ker^{-1}(\ker(T_1) + \ker(T_2))$ ,
- $T_1 \vee T_2 = \ker^{-1}(\ker(T_1) \cap \ker(T_2))$ .

Given a subset  $F$  of  $\mathbf{RO}(G, <)$ , we let

$$\wedge F = \ker^{-1} \left( \sum_{T \in F} \ker(T) \right).$$

The set  $F$  is said to be *confluent* if we have

$$\text{im}(\wedge F) = \bigcap_{T \in F} \text{im}(T).$$

In [13, Corollary 2.3.9], it is shown that  $F$  is confluent if and only if the reduction relation on  $\mathbb{K}G$  defined by

$$v \xrightarrow{F} T(v),$$

for every  $T \in F$  and every  $v \notin \text{im}(T)$ , is confluent.

From this point of view on confluence, we deduce that some lattice constructions in  $\mathbf{RO}(G, <)$  are interpreted as a completion procedure. Indeed, one defines a particular operator

$$C^F = (\wedge F) \vee (\vee \overline{F}),$$

where  $\overline{F}$  is a subset of  $\mathbf{RO}(G, <)$  defined from  $F$  and  $\vee \overline{F}$  is the upper bound of this set. Recall from [13, Theorem 3.2.6] that the set

$$F \cup \{C^F\} \subset \mathbf{RO}(G, <),$$

is confluent.

**Example.** As an illustration of the constructions presented above, we consider as previously the rules

$$yz \longrightarrow \quad \text{and} \quad zx \longrightarrow xy,$$

oriented with respect to the deg-lex order induced by  $x < y < z$ . We have seen that we have the following unique critical pair

$$\begin{array}{ccc} & yzx & \\ & \swarrow \quad \searrow & \\ xx & & yxy \end{array} \quad (3)$$

In terms of reduction operators, the reduction  $yzx \longrightarrow xx$  in (3) is done by  $T_1 \in \mathbf{RO}(X^*, <)$  defined for every word  $w$  by

$$T_1(w) = \begin{cases} xx, & \text{if } w = yzx \\ w, & \text{otherwise.} \end{cases}$$

The reduction  $yzx \longrightarrow yxy$  in (3) is done by  $T_2 \in \mathbf{RO}(X^*, <)$  defined for every word  $w$  by

$$T_2(w) = \begin{cases} yxy, & \text{if } w = yzx \\ w, & \text{otherwise.} \end{cases}$$

Hence, the reduction of the  $S$ -polynomial of (3) into zero is made by completion of the following pair

$$P = (T_1, T_2).$$

Indeed, the operators  $\wedge P$  and  $C^P$  are defined for every word  $w$  by

$$(\wedge P)(w) = \begin{cases} xx, & \text{if } w = yzx \\ xx, & \text{if } w = yxy \\ w, & \text{otherwise} \end{cases} \quad \text{and} \quad C^P(w) = \begin{cases} xx, & \text{if } w = yxy \\ w, & \text{otherwise.} \end{cases}$$

**Principle of our Procedure.** The previous example illustrates the fact that the operator  $C^P$  is the operator which enables us to reduce all  $S$ -polynomials into zero. This fact is general, so that our completion procedure consists in computing successively an operator  $C^F$  where  $F$  is a subset of  $\mathbf{RO}(G, <)$  which is used to compute normal forms of  $S$ -polynomials. In order to describe formally our completion procedure, we need to relate reduction operators to noncommutative Gröbner bases. This formal link requires *presentations by operators*.

## Presentations by Operators

**Confluent Presentations by Operators.** Let  $\mathbf{A}$  be an algebra. A presentation by operator of  $\mathbf{A}$  is a triple  $\langle (X, <) \mid S \rangle$ , where

- $X$  is a set and  $<$  is a monomial order on  $X^*$ ,
- $S$  is a reduction operator relative to  $(X^*, <)$ ,
- we have an isomorphism of algebras

$$\mathbf{A} \simeq \frac{\mathbb{K}X^*}{I(\ker(S))},$$

where  $I(\ker(S))$  is the two-sided ideal of  $\mathbb{K}X^*$  spanned by  $\ker(S)$ .

The operator  $S$  of such a presentation does not describe all the reductions that can be applied to a given word. For that, we need to consider the "extensions" of  $S$ , that is, the operators defined for every pair of integers  $(n, m)$  by

$$S_{n,m} = \text{Id}_{\mathbb{K}X^{(\leq n+m-1)}} \oplus \left( \text{Id}_{\mathbb{K}X^{(n)}} \otimes S \otimes \text{Id}_{\mathbb{K}X^{(m)}} \right).$$

Explicitly, for every  $w \in X^*$ ,  $S_{n,m}(w)$  is defined by the following two conditions:

- Assume that the length of  $w$  is strictly smaller than  $n + m$ . Then,  $S_{n,m}(w) = w$ .
- Assume that the length of  $w$  is greater than or equal to  $n + m$ . We let  $w = w_1 w_2 w_3$ , where  $w_1$  and  $w_3$  have length  $n$  and  $m$ , respectively. Then,  $S_{n,m}(w) = w_1 S(w_2) w_3$ .

The strict order  $<$  being monomial, that guarantees that each operator  $S_{n,m}$  is a reduction operator relative to  $(X^*, <)$ . The presentation  $\langle (X, <) \mid S \rangle$  is said to be *confluent* if the set

$$\left\{ S_{n,m} \mid (n, m) \in \mathbb{N}^2 \right\} \subset \mathbf{RO}(X^*, <),$$

is confluent. In [13, Proposition 3.3.10] the formal link with noncommutative Gröbner bases is given:  $\langle (X, <) \mid S \rangle$  is confluent if and only if

$$\left\{ w - S(w) \mid w \notin \text{im}(S) \right\},$$

is a noncommutative Gröbner basis of  $I(\ker(S))$  with respect to  $<$ .

**Completion Procedure.** Let  $\mathbf{A}$  be an algebra and let  $\langle\langle X, < \mid S \rangle\rangle$  be a presentation by operator of  $\mathbf{A}$ . Our completion procedure consists in executing instructions of a loop in which we add new relations to the current presentation by operator. This procedure has to return a confluent presentation by operator of  $\mathbf{A}$ .

Let  $d$  be an integer and let  $\langle\langle X, < \mid S^d \rangle\rangle$  be the presentation by operator of  $\mathbf{A}$  at the beginning of the  $d$ -th iteration of the loop. The reduction of  $S$ -polynomials of this current presentation into normal forms is done by a set

$$F_d \subset \mathbf{RO}(X^*, <).$$

We want to complete the presentation  $\langle\langle X, < \mid S^d \rangle\rangle$  by

$$\left\{ w - C^{F_d}(w) \mid w \notin \text{im}(C^{F_d}) \right\}. \quad (4)$$

Hence, the new presentation of  $\mathbf{A}$  is  $\langle\langle X, < \mid S^{d+1} \rangle\rangle$ , where

$$S^{d+1} = S^d \wedge C^{F_d},$$

since a generating set of  $\ker(S^{d+1})$  is the union of a basis of  $\ker(S^d)$  and (4). Let  $S'$  be the lower bound of all the operators  $S^d$  so constructed:

$$S' = \bigwedge_{d \in \mathbb{N}} S^d.$$

The presentation  $\langle\langle X, < \mid S' \rangle\rangle$  is called the *completed presentation* of  $\langle\langle X, < \mid S \rangle\rangle$ . The main result of the paper states that such a presentation is confluent:

**Theorem 3.2.7.** *Let  $\mathbf{A}$  be an algebra and let  $\langle\langle X, < \mid S \rangle\rangle$  be a presentation by operator of  $\mathbf{A}$ . The completed presentation of  $\langle\langle X, < \mid S \rangle\rangle$  is a confluent presentation of  $\mathbf{A}$ .*

In 3.2.8, we illustrate with an example the behaviour of our procedure. For that, we use the implementation of various constructions of reduction operators given in Section 4.

**Relation with the  $F_4$  Procedure.** We end this introduction by explaining the link with  $F_4$ . We consider the previous example:

$$R = \{yzx - x, zx - xy\},$$

together with the only critical pair

$$\begin{array}{ccc} & yzx & \\ \swarrow & & \searrow \\ xx & & yxy \end{array}$$

The two reductions are done by the relations

$$f_1 = yzx - xx \quad \text{and} \quad f_2 = yzx - yxy.$$

Let  $M$  be the matrix of  $\{f_1, f_2\}$  with respect to  $yzx > yxy > xx$ :

$$M = \begin{pmatrix} 1 & 0 & -1 \\ 1 & -1 & 0 \end{pmatrix}.$$

By Gaussian elimination, we obtain

$$\text{Gauss}(M) = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \end{pmatrix}.$$

The rows of this matrix provide the relations

$$\tilde{f}_1 = yzx - xx \quad \text{and} \quad \tilde{f}_2 = yxy - xx.$$

Then, the  $F_4$  procedure add the relation  $\tilde{f}_2$  to  $R$  since its leading word  $yxy$  cannot be reduced by  $R$ . In the general case, the  $F_4$  procedure works as follows:

1. let  $M$  be the matrix associated with reductions of  $S$ -polynomials into normal forms,
2. we add to  $R$  the rows of  $\text{Gauss}(M)$  admitting a normal form as leading words.

The link between this method and the one we develop comes from the fact that we also create new relations by Gaussian elimination and we take into account the ones for which the leading words are normal forms. Indeed, consider the two reduction operators  $T_1$  and  $T_2$  as in the example of the previous section. These operators only act on the vector space spanned by the totally ordered finite set

$$xx < yxy < yzx.$$

Their restrictions to this vector space are defined by the following matrices

$$T_1 = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad T_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Moreover, letting  $P = (T_1, T_2)$ , the operators  $\wedge P$  and  $C^P$  can also be described by the matrices of their restrictions to the same vector space:

$$\wedge P = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad C^P = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

By definition of  $\wedge P$ , its kernel has the union of a basis of  $T_1$  and a basis of  $T_2$  as a generating set. The Gaussian elimination applied to this generating set provides

$$\{yzx - xx, yxy - xx\},$$

as a basis of  $\ker(\wedge P)$ . We write

$$\text{Obs}^P = \left\{ w \in X^* \mid w \notin \text{im}(\wedge P) \quad \text{and} \quad w \in \text{im}(T_1) \cap \text{im}(T_2) \right\}.$$

In fact,  $\text{Obs}^P$  is the set of leading words of an element of the kernel of  $\wedge P$  which are normal forms for the current presentation by operator. Moreover,  $C^P$  is defined for every word  $w$  by

$$C^P(w) = \begin{cases} \wedge P(w), & \text{if } w \in \text{Obs}^P \\ w, & \text{otherwise.} \end{cases}$$

In the general case, our procedure works as follows:



1. let  $F$  be a set of reduction operators associated with reductions of  $S$ -polynomials into normal forms,
2. compute  $\wedge F$  by Gaussian elimination and then  $C^F$  using the elements of the kernel of  $\wedge F$  admitting a normal form as leading words.

## Organisation

Section 2.1 is a recollection of results from [13]: we recall the definitions and properties of reduction operators, their confluence and completion used in the sequel. In Section 2.2, we define presentations by operators and the confluence property of such presentations. We give a criterion in terms of  $S$ -polynomials for a presentation by operator to be confluent. In Section 3.1, we write our completion procedure and define completed presentations. In Section 3.2, we show, using the criterion in terms of  $S$ -polynomials, that a completed presentation is confluent. We also write an example to show how our procedure works. This example was treated with an implementation of various constructions of reduction operators. This implementation is written in Section 4.

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## 2 Reduction Operators

### 2.1 Lattice Structure and Completion

**2.1.1. Conventions and Notations.** Throughout the paper, we fix a commutative field  $\mathbb{K}$ . We say vector space instead of  $\mathbb{K}$ -vector space. Let  $X$  be a set. We denote by  $\mathbb{K}X$  the vector space with basis  $X$ : its non-zero elements are the finite formal linear combinations of elements of  $X$  with coefficients in  $\mathbb{K}$ . An element of  $X$  is called a *generator* of  $\mathbb{K}X$ . By construction of  $\mathbb{K}X$ , for every  $v \in \mathbb{K}X \setminus \{0\}$ , there exist a unique finite set  $\text{supp}(v) \subseteq X$  and a unique family of non-zero scalars  $(\lambda_x)_{x \in \text{supp}(v)}$  such that

$$v = \sum_{x \in \text{supp}(v)} \lambda_x x.$$

The set  $\text{supp}(v)$  is called the *support* of  $v$ .

**2.1.2. Leading Generators and Leading Coefficients.** Let  $(G, <)$  be a *well-ordered set*, that is,  $G$  is a set and  $<$  is a well-founded total strict order on  $G$ . The strict order  $<$  being total, every non-empty finite subset of  $G$  admits a greatest element. In particular, for every  $v \in \mathbb{K}G \setminus \{0\}$ , the support of  $v$  admits a maximum. We write

$$\text{lg}(v) = \max(\text{supp}(v)) \quad \text{and} \quad \text{lc}(v) = \lambda_{\text{lg}(v)}.$$

The elements  $\text{lg}(v)$  and  $\text{lc}(v)$  are called the *leading generator* and the *leading coefficient* of  $v$ , respectively. We extend  $<$  into a partial strict order on  $\mathbb{K}G$  in the following way: we have  $u < v$  if one of the following two conditions is fulfilled

- $u = 0$  and  $v \neq 0$ ,
- $u \neq 0$ ,  $v \neq 0$  and  $\lg(u) < \lg(v)$ .

Throughout Section 2.1 we fix a well-ordered set  $(G, <)$ .

**2.1.3. Reduction Operators.** A *reduction operator relative to  $(G, <)$*  is an idempotent linear endomorphism  $T$  of  $\mathbb{K}G$  such that for every  $g \in G$ , we have  $T(g) \leq g$ . We denote by  $\mathbf{RO}(G, <)$  the set of reduction operators relative to  $(G, <)$ . Given  $T \in \mathbf{RO}(G, <)$ , a generator  $g$  is said to be  *$T$ -reduced* if  $T(g)$  is equal to  $g$ . We denote by  $\text{Red}(T)$  the set of  $T$ -reduced generators and by  $\text{Nred}(T)$  the complement of  $\text{Red}(T)$  in  $G$ .

**2.1.4. Kernels of Reduction Operators.** Recall from [13, Proposition 2.1.14] that the restriction of the kernel map to the set of reduction operators

$$\begin{aligned} \ker : \mathbf{RO}(G, <) &\longrightarrow \{\text{subspaces of } \mathbb{K}G\}, \\ T &\longmapsto \ker(T) \end{aligned} \tag{5}$$

is a bijection. The inverse of (5) is written  $\ker^{-1}$ .

**2.1.5. Lattice Structure.** We consider the binary relation on  $\mathbf{RO}(G, <)$  defined by

$$T_1 \preceq T_2 \text{ if and only if } \ker(T_2) \subseteq \ker(T_1).$$

This relation is reflexive and transitive. The map (5) being a bijection,  $\preceq$  is also anti-symmetric, so that it is an order relation on  $\mathbf{RO}(G, <)$ . Moreover, we have the equivalence:

$$T_1 \preceq T_2 \text{ if and only if } T_1 \circ T_2 = T_1. \tag{6}$$

Let us equip  $\mathbf{RO}(G, <)$  with a lattice structure. The lower bound  $T_1 \wedge T_2$  and the upper bound  $T_1 \vee T_2$  of two elements  $T_1$  and  $T_2$  of  $\mathbf{RO}(G, <)$  are defined in the following manner:

$$T_1 \wedge T_2 = \ker^{-1} \left( \ker(T_1) + \ker(T_2) \right) \text{ and } T_1 \vee T_2 = \ker^{-1} \left( \ker(T_1) \cap \ker(T_2) \right).$$

Recall from [13, Lemma 2.1.18] that we have the following implication

$$T_1 \preceq T_2 \implies \text{Red}(T_1) \subseteq \text{Red}(T_2). \tag{7}$$

**2.1.6. Notations.** From now on and until the end of Section 2.1, we fix a non-empty set

$$F \subseteq \mathbf{RO}(G, <).$$

Moreover, we let

$$\text{Red}(F) = \bigcap_{T \in F} \text{Red}(T) \text{ and } \wedge F = \ker^{-1} \left( \sum_{T \in F} \ker(T) \right).$$

**2.1.7. Obstructions.** For every  $T \in F$ , we have  $\wedge F \preceq T$ . Thus, from (7),  $\text{Red}(\wedge F)$  is included in  $\text{Red}(T)$  for every  $T \in F$ , so that  $\text{Red}(\wedge F)$  is included in  $\text{Red}(F)$ . We write

$$\text{Obs}^F = \text{Red}(F) \setminus \text{Red}(\wedge F). \quad (8)$$

**2.1.8. Confluence.** The set  $F$  is said to be *confluent* if  $\text{Obs}^F$  is the empty set. The link between this algebraic notion of confluence and the classical one coming from rewriting theory appears in [13, Corollary 2.3.9].

In Section 3.2, we use two characterisations of the confluence property in terms of reduction operators, namely the *Church-Rosser property* and the *Newman's Lemma*.

**2.1.9. Church-Rosser Property.** Recall from [13, Theorem 2.2.5] that  $F$  is confluent if and only if it has the *Church-Rosser property*: for every  $v \in \mathbb{K}G$ , there exist  $T_1, \dots, T_r \in F$  such that

$$(\wedge F)(v) = (T_r \circ \dots \circ T_1)(v).$$

The link between this algebraic notion of Church-Rosser property and the classical one coming from rewriting theory appears in [13, Proposition 2.3.8].

**2.1.10. Newman's Lemma.** The set  $F$  is said to be *locally confluent* if for every  $v \in \mathbb{K}G$  and for every pair  $(T, T')$  of elements of  $F$ , there exist  $v' \in \mathbb{K}G$  and  $T_1, \dots, T_r, T'_1, \dots, T'_k \in F$  such that

$$\begin{aligned} v' &= (T_r \circ \dots \circ T_1)(T(v)) \\ &= (T'_k \circ \dots \circ T'_1)(T'(v)). \end{aligned}$$

Recall from [13, Proposition 2.2.12] that  $F$  is confluent if and only if it is locally confluent.

**2.1.11. Complement.** A *complement* of  $F$  is an element  $C$  of  $\mathbf{RO}(G, <)$  such that

- $(\wedge F) \wedge C = \wedge F$ ,
- $\text{Obs}^F \subseteq \text{Nred}(C)$ .

Recall from [13, Proposition 3.2.2] that a reduction operator  $C$  satisfying  $(\wedge F) \wedge C = \wedge F$  is a complement of  $F$  if and only if  $F \cup \{C\}$  is a confluent subset of  $\mathbf{RO}(G, <)$ .

**2.1.12. The  $F$ -Complement.** The  *$F$ -complement* is the operator

$$C^F = (\wedge F) \vee (\vee \overline{F}),$$

where  $\vee \overline{F}$  is equal to  $\ker^{-1}(\mathbb{K}\text{Red}(F))$ . Recall from [13, Theorem 3.2.6] that the  $F$ -complement is a complement of  $F$ .

## 2.2 Presentations by Operators

**2.2.1. Algebras.** An unitary associative  $\mathbb{K}$ -algebra is a  $\mathbb{K}$ -vector space  $\mathbf{A}$  equipped with a  $\mathbb{K}$ -linear map, called multiplication,  $\mu : \mathbf{A} \otimes \mathbf{A} \rightarrow \mathbf{A}$  which is associative and for which there exists a unit  $1_{\mathbf{A}}$ . We say algebra instead of unitary associative  $\mathbb{K}$ -algebra. Given a set  $X$ , let  $X^*$  be the set of words over  $X$ . This set admits a monoid structure, where the multiplication is given by concatenation of words and the unit is the empty word, written 1. The free algebra over  $X$  is the vector space  $\mathbb{K}X^*$  spanned by  $X^*$  equipped with the multiplication induced by the one of the monoid  $X^*$ .

From now on, we fix an algebra  $\mathbf{A}$ .

**2.2.2. Monomial Orders.** Let  $X$  be a set. A *monomial order* on  $X^*$  is a well-founded total strict order  $<$  on  $X^*$  such that the following conditions are fulfilled:

- $1 < w$  for every word  $w$  different from 1,
- for every  $w_1, w_2, w, w' \in X^*$  such that  $w < w'$ , we have  $w_1 w w_2 < w_1 w' w_2$ .

In particular,  $(X^*, <)$  is a well-ordered set. In the sequel, given an element  $f \in \mathbb{K}X^*$ , we write  $\text{lm}(f)$  (for *leading monomial*) instead of  $\text{lg}(f)$ .

**2.2.3. The Deg-lex Order.** Let  $X$  be a set and let  $<$  be total well-founded strict order on  $X$ . The *deg-lex* order on  $X^*$  induced by  $<$ , still written  $<$ , is defined by  $x_1 \cdots x_n < y_1 \cdots y_m$  if one of the following two conditions is fulfilled

- $n < m$ ,
- $n = m$  and there exists  $k \in \{2, \dots, n\}$  such that  $x_i = y_i$  for every  $i \in \{1, \dots, k-1\}$  and  $x_k < y_k$ .

Recall from [1, Lemme 2.4.3] that  $<$  being total and well-founded,  $(X^*, <_{\text{deg-lex}})$  is a well-ordered set. Moreover, the deg-lex order  $<$  is monomial by definition.

**2.2.4. Exemple.** Let  $X = \{x, y, z\}$  such ordered:  $x < y < z$ . Then, we have  $x < yz$  and  $xy < zx$ .

**2.2.5. Definition.** A *presentation by operator* of  $\mathbf{A}$  is a triple  $\langle (X, <) \mid S \rangle$ , where

- $X$  is a set and  $<$  is a monomial order on  $X^*$ ,
- $S$  is a reduction operator relative to  $(X^*, <)$ ,
- we have an isomorphism of algebras

$$\mathbf{A} \simeq \frac{\mathbb{K}X^*}{I(\ker(S))},$$

where  $I(\ker(S))$  is the two-sided ideal of  $\mathbb{K}X^*$  spanned by  $\ker(S)$ .

**2.2.6. Confluent Presentations.** Let  $X$  be a set and let  $n$  be an integer. We denote by  $X^{(n)}$  and  $X^{(\leq n)}$  the set of words of length  $n$  and of length smaller or equal to  $n$ , respectively. Let  $\langle\langle X, < \mid S \rangle\rangle$  be a presentation by operator of  $\mathbf{A}$ . For every integers  $n$  and  $m$  such that  $(n, m)$  is different from  $(0, 0)$ , we let

$$S_{n,m} = \text{Id}_{\mathbb{K}X^{(\leq n+m-1)}} \oplus \left( \text{Id}_{\mathbb{K}X^{(n)}} \otimes S \otimes \text{Id}_{\mathbb{K}X^{(m)}} \right).$$

Explicitly, for every  $w \in X^*$ ,  $S_{n,m}(w)$  is defined by the following two conditions:

- Assume that the length of  $w$  is strictly smaller than  $n + m$ . Then,  $S_{n,m}(w) = w$ .
- Assume that the length of  $w$  is greater or equal to  $n + m$ . We let  $w = w_1 w_2 w_3$ , where  $w_1$  and  $w_3$  have length  $n$  and  $m$ , respectively. Then,  $S_{n,m}(w) = w_1 S(w_2) w_3$ .

We also let  $S_{0,0} = S$ . The set

$$\{S_{n,m} \mid (n, m) \in \mathbb{N}^2\},$$

is called the *reduction family* of  $\langle\langle X, < \mid S \rangle\rangle$ . Recall from [13, Lemma 3.3.6] that each  $S_{n,m}$  is a reduction operator relative to  $(X^*, <)$ , so that the reduction family of a presentation is a subset of  $\mathbf{RO}(X^*, <)$ .

A *confluent presentation by operator* of  $\mathbf{A}$  is a presentation by operator of  $\mathbf{A}$  such that its reduction family is confluent.

**2.2.7. Example.** Let  $X = \{x, y, z\}$  and let  $<$  be the deg-lex order induced by  $x < y < z$ . Consider the algebra presented by  $\langle\langle X, < \mid S \rangle\rangle$ , where  $S$  is defined on the basis  $X^*$  by

$$S(w) = \begin{cases} x, & \text{if } w = yz \\ xy, & \text{if } w = zx \\ w, & \text{otherwise.} \end{cases}$$

Let  $F$  be the reduction family of this presentation. Every sub-word of  $xyx$  is  $S$ -reduced, so that  $xyx$  belongs to  $\text{Red}(F)$ . Moreover, we have

$$\begin{aligned} yxy - xx &= (yxy - yzx) - (xx - yzx) \\ &= (yS(zx) - yzx) - (S(yz)x - yzx) \\ &= A + B \end{aligned}$$

where

$$A = (S_{1,0} - \text{Id}_{\mathbb{K}X^*})(yzx) \quad \text{and} \quad B = (\text{Id}_{\mathbb{K}X^*} - S_{0,1})(yzx).$$

The operators  $S_{1,0}$  and  $S_{0,1}$  being idempotent,  $A$  and  $B$  belong to  $\ker(S_{1,0})$  and  $\ker(S_{0,1})$ , respectively. Hence,  $yxy - xx$  is included in  $\ker(S_{1,0}) + \ker(S_{0,1})$ . The latter being included in  $\ker(\wedge F)$ , we have

$$(\wedge F)(yxy) = (\wedge F)(xx).$$

In particular,  $xx$  being smaller than  $xyx$  for  $<$ ,  $xyx$  is not  $\wedge F$ -reduced, so that  $xyx$  belongs to  $\text{Obs}^F$ . Thus, the latter is non empty, that is,  $\langle\langle X, < \mid S \rangle\rangle$  is not confluent.

**2.2.8. Noncommutative Gröbner Bases.** In [13], confluent presentations by operators are related to *noncommutative Gröbner bases*. The latter offer a formalism to define terminating and confluent rewriting systems presenting an algebra. We recall how are defined noncommutative Gröbner bases.

Let  $X$  be a set and let  $<$  be a monomial order on  $X^*$ . Given a subset  $E$  of  $\mathbb{K}X^*$ , we let

$$\text{lm}(E) = \{\text{lm}(f) \mid f \in E\}.$$

Let  $I$  be a two-sided ideal of  $\mathbb{K}X^*$ . A subset  $R$  of  $I$  is called a noncommutative Gröbner basis of  $I$  if the semi-group ideal spanned by  $\text{lm}(R)$  is equal to  $\text{lm}(I)$ . In other words,  $R$  is a Gröbner basis of  $I$  if and only if for every  $w \in \text{lm}(I)$ , there exist  $w' \in \text{lm}(R)$  and  $w_1, w_2 \in X^*$  such that  $w$  is equal to  $w_1 w' w_2$ . The link between confluent presentations by operators and noncommutative Gröbner bases is as follows: let  $\langle (X, <) \mid S \rangle$  be a presentation by operator and let

$$R = \left\{ w - S(w) \mid w \in \text{Nred}(S) \right\}.$$

Then,  $\langle (X, <) \mid S \rangle$  is confluent if and only if  $R$  is a noncommutative Gröbner basis of  $I(\ker(S))$  [13, Proposition 3.3.10].

**2.2.9. Ambiguities.** In Section 3.1 we formulate a procedure to construct confluent presentations by operators. The proof of the soundness of this procedure (Section 3.2) requires *critical branchings*, introduced in 2.2.11. The latter are the analogous notion of *ambiguities* for Gröbner bases. We recall how ambiguities are defined and how they characterise noncommutative Gröbner bases.

Fix a set  $X$ , a subset  $R$  of  $\mathbb{K}X^*$  and a monomial order  $<$ . An ambiguity of  $R$  with respect to  $<$  is a tuple  $b = (w_1, w_2, w_3, f, g)$  where

- $w_1, w_2, w_3$  are words such that  $w_2 \neq 1$ ,
- $f, g$  belong to  $R$ ,

such that  $b$  satisfies one of the following two conditions:

1.  $w_1 w_2 = \text{lm}(f)$  and  $w_2 w_3 = \text{lm}(g)$ ,
2.  $w_1 w_2 w_3 = \text{lm}(f)$  and  $w_2 = \text{lm}(g)$ .

We write

1.  $\text{SP}(b) = f w_3 - w_1 g$  if  $b$  is of the form 1,
2.  $\text{SP}(b) = f - w_1 g w_3$  if  $b$  is of the form 2.

The ambiguity  $b$  is said to be *solvable relative to  $<$*  if there exists a decomposition

$$\text{SP}(b) = \sum_{i=1}^n \lambda_i w_i f_i w'_i, \tag{9}$$

where, for every  $i \in \{1, \dots, n\}$

- $\lambda_i$  is a non-zero scalar,
- $w_i, w'_i \in X^*$  and  $f_i \in R$  are such that  $w_i \text{lm}(f_i) w'_i$  is strictly smaller than  $w_1 w_2 w_3$  for  $<$ .

Recall from the Diamond Lemma [5, Theorem 1.2] that  $R$  is a noncommutative Gröbner basis of  $I(R)$  if and only if every critical branching of  $R$  with respect to  $<$  is solvable relative to  $<$ .

**2.2.10. Convention.** From now on and until the end of Section 2.2, we fix a presentation by operator  $\langle (X, <) \mid S \rangle$  of  $\mathbf{A}$ . We consider the notations of 2.2.6:  $S_{0,0} = S$  and for every pair of integers  $(n, m)$  such that  $n + m$  is different from 0, we let

$$S_{n,m} = \text{Id}_{\mathbb{K}X^{(\leq n+m-1)}} \oplus \left( \text{Id}_{\mathbb{K}X^{(n)}} \otimes S \otimes \text{Id}_{\mathbb{K}X^{(m)}} \right).$$

Finally, we let

$$R = \left\{ w - S(w) \mid w \in \text{Nred}(S) \right\},$$

as in 2.2.8.

**2.2.11. Critical Branchings.** A *critical branching* of  $\langle (X, <) \mid S \rangle$  is a triple

$$b = (w, (n, m), (n', m')) \in X^* \times \mathbb{N}^2 \times \mathbb{N}^2,$$

such that

- $w \in \text{Nred}(S_{n,m}) \cap \text{Nred}(S_{n',m'})$ ,
- $n = 0$  or  $n' = 0$ ,
- $m = 0$  or  $m' = 0$ ,
- $n + n' + m + m'$  is strictly smaller than the length of  $w$ .

The word  $w$  is called the *source* of  $b$ .

**2.2.12. Remark.** The roles of  $(n, m)$  and  $(n', m')$  being symmetric, we do not distinguish  $(w, (n, m), (n', m'))$  and  $(w, (n', m'), (n, m))$ .

**2.2.13.  $S$ -polynomials.** Let  $b = (w, (n, m), (n', m'))$  be a critical branching of  $\langle (X, <) \mid S \rangle$ . The  *$S$ -polynomial* of  $b$  is the following element of  $\mathbb{K}X^*$ :

$$\text{SP}(b) = S_{n,m}(w) - S_{n',m'}(w).$$

**2.2.14. Example.** Consider the algebra of Example 2.2.7. We have one critical branching:

$$b_1 = (yzx, (1, 0), (0, 1)).$$

Moreover, we have

$$\text{SP}(b_1) = yxy - xx.$$

**2.2.15. Definition.** Let  $w \in X^*$  and let  $f \in \mathbb{K}X^*$ . We say that  $f$  admits a  *$(S, w)$ -type decomposition* if it admits a decomposition

$$f = \sum_{i=1}^n \lambda_i w_i^1 (w_i - S(w_i)) w_i^2,$$

where, for every  $i \in \{1, \dots, n\}$

- $\lambda_i$  is a non-zero scalar,
- $w_i^1, w_i^2$  and  $w_i$  are words such that  $w_i \in \text{Nred}(S)$  and  $w_i^1 w_i w_i^2$  is strictly smaller than  $w$ .

**2.2.16. Lemma.** *There is a one-to-one correspondence  $b \mapsto \tilde{b}$  between critical branchings of  $\langle\langle X, < \rangle \mid S \rangle$  and ambiguities of  $R$  with respect to  $<$ . Moreover, a critical branching  $b$  of source  $w$  admits a  $(S, w)$ -type decomposition if and only if  $\tilde{b}$  is solvable relative to  $<$ .*

*Proof.* Let us show the first part of the lemma. Let  $b = (w, (n, m), (n', m'))$  be a critical branching of  $\langle\langle X, < \rangle \mid S \rangle$ . In order to define  $\tilde{b}$ , we distinguish four cases according to the values of  $n$  and  $m$ :

**Case 1:**  $(n, m)$  is equal to  $(0, 0)$ . We write

$$w = w_1 w_2 w_3,$$

where the lengths of  $w_1$  and  $w_3$  are equal to  $n'$  and  $m'$ , respectively. By definition of a critical branching,  $w$  and  $w_2$  belong to  $\text{Nred}(S)$  and we let

$$\tilde{b} = (w_1, w_2, w_3, w - S(w), w_1 (w_2 - S(w_2)) w_3).$$

By definition of a critical branching,  $n + n' + m + m' = n' + m'$  is strictly smaller than the length of  $w$ . In particular,  $w_2$  is not the empty word, so that the tuple  $\tilde{b}$  is an ambiguity of  $R$  with respect to  $<$  of the form 2.

**Case 2:**  $n$  is equal to 0 and  $m$  is different from 0. By definition of a critical branching,  $m'$  is equal to 0. If  $n'$  is also equal to 0, the pair  $(n', m')$  is equal to  $(0, 0)$ , so that we exchange the roles of  $(n, m)$  and  $(n', m')$  and we recover the first case. If  $n'$  is different from 0, we write

$$w = w_1 w_2 w_3,$$

where the lengths of  $w_1$  and  $w_3$  are equal to  $n'$  and  $m$ , respectively. In particular,  $b$  being a critical branching,  $w_1 w_2$  and  $w_2 w_3$  belong to  $\text{Nred}(S)$  and  $w_2$  is different from the empty word. Hence,

$$\tilde{b} = (w_1, w_2, w_3, w_1 w_2 - S(w_1 w_2), w_2 w_3 - S(w_2 w_3)),$$

is an ambiguity of  $R$  with respect to  $<$ .

**Case 3:**  $n$  is different from 0 and  $m$  is equal to 0. By definition of a critical branching,  $n'$  is equal to 0. Exchanging the roles of  $(n, m)$  and  $(n', m')$ , we recover the second case.

**Case 4:**  $n$  and  $m$  are different from 0. By definition of a critical branching, the pair  $(n', m')$  is equal to  $(0, 0)$ . Exchanging the roles of  $(n, m)$  and  $(n', m')$ , we recover the first case.

We have a well-defined map  $b \mapsto \tilde{b}$  between critical branchings of  $\langle\langle X, < \rangle \mid S \rangle$  and ambiguities of  $R$  with respect to  $<$ . Now, we define an inverse  $\tilde{b} \mapsto b$ . Let  $\tilde{b} = (w_1, w_2, w_3, f, g)$  be an ambiguity of  $R$  with respect to  $<$  and let  $w = w_1 w_2 w_3$ .

- If  $\tilde{b}$  is an ambiguity of the form 1, let  $n$  and  $m'$  be the lengths of  $w_1$  and  $w_3$ , respectively. The word  $w_2$  being non-empty,  $n + m'$  is strictly smaller than the length of  $w$ , so that

$$b = (w, (n, 0), (0, m')),$$

is a critical branching of  $\langle\langle X, < \rangle \mid S \rangle$ .



- If  $\tilde{b}$  is of the form 2, let  $n$  and  $m$  be the lengths of  $n$  and  $m$ , respectively. Then,

$$b = (w, (n, m), (0, 0)),$$

is a critical branching of  $\langle\langle X, < \mid S \rangle\rangle$ .

The proofs that the two composites of  $b \mapsto \tilde{b}$  and  $\tilde{b} \mapsto b$  are identities are left to the reader.

Let us show the second part of the lemma. Given a critical branching  $b$ ,  $\text{SP}(b)$  and  $\text{SP}(\tilde{b})$  are equal. Letting  $w$  the source of  $w$ , a  $(S, w)$ -type decomposition of  $\text{SP}(b)$  is precisely a decomposition of the form (9) in 2.2.9. That shows the second part of the lemma.  $\square$

**2.2.17. Proposition.** *The presentation  $\langle\langle X, < \mid S \rangle\rangle$  is confluent if and only if for every critical branching  $b$  of source  $w$ ,  $\text{SP}(b)$  admits a  $(S, w)$ -type decomposition.*

*Proof.* The two-sided ideal  $I(R)$  spanned by  $R$  is equal to  $I(\ker(S))$ . Hence, from [13, Proposition 3.3.10] (see 2.2.8),  $\langle\langle X, < \mid S \rangle\rangle$  is confluent if and only if  $R$  is a noncommutative Gröbner basis of  $I(R)$ . From the Diamond Lemma [5, Theorem 1.2], (see 2.2.9),  $\langle\langle X, < \mid S \rangle\rangle$  is confluent if and only if every ambiguity of  $R$  with respect to  $<$  is solvable relative to  $<$ . Thus, from Lemma 2.2.16,  $\langle\langle X, < \mid S \rangle\rangle$  is confluent if and only if for every critical branching  $b$  of source  $w$ ,  $\text{SP}(b)$  admits a  $(S, w)$ -type decomposition.  $\square$

### 3 Completion Procedure

In Section 3.1, we formulate a procedure to construct confluent presentations by operators. In Section 3.2 we prove the soundness of this procedure.

Let us fix the conventions and notations used throughout Section 3:

- $\mathbf{A}$  is an algebra and  $\langle\langle X, < \mid S \rangle\rangle$  is a presentation by operator of  $\mathbf{A}$ ,
- we simply say reduction operator instead of reduction operator relative to  $(X^*, <)$ ,
- we consider the notations of 2.2.6: given a reduction operator  $T$ , we let  $T_{0,0} = T$  and for every pair of integers  $(n, m)$  different from  $(0, 0)$ , we let

$$T_{n,m} = \text{Id}_{\mathbb{K}X^{(\leq n+m-1)}} \oplus \left( \text{Id}_{\mathbb{K}X^{(n)}} \otimes T \otimes \text{Id}_{\mathbb{K}X^{(m)}} \right).$$

- Let  $f \in \mathbb{K}X^*$ . We write:

$$T(f) = \ker^{-1}(\mathbb{K}f).$$

Explicitly, letting

$$\bar{f} = \frac{1}{\text{lc}(f)} f,$$

$T(f)$  is defined on the basis  $X^*$  of  $\mathbb{K}X^*$  in the following way

$$(T(f))(w) = \begin{cases} \text{lm}(\bar{f}) - \bar{f}, & \text{if } w = \text{lm}(\bar{f}) \\ w, & \text{otherwise.} \end{cases}$$

### 3.1 Formulation

**3.1.1. Reduction Method.** Our procedure requires a method called **Reduction** with inputs a finite subset  $E$  of  $\mathbb{K}X^*$  and a reduction operator  $U$  and with output a finite set of reduction operators.  $\text{Reduction}(E, U)$  is defined as follows:

1. Let

$$M = \left( \bigcup_{f \in E} \text{supp}(f) \right) \setminus \text{lm}(E) \quad \text{and} \quad F = \{T(f) \mid f \in E\}.$$

2. **while**  $\exists w_1 w w_2 \in M$  such that  $w \in \text{Nred}(U)$ ,

(a) we add  $T(w_1(w - U(w))w_2)$  to  $F$ ,

(b) we remove  $w_1 w w_2$  from  $M$ ,

(c) we add  $\text{supp}(w_1 U(w)w_2)$  to  $M$ .

3.  $\text{Reduction}(E, U)$  returns the set  $F$  obtained when the loop **while** is over.

**3.1.2. Remark.** We consider the notations of 3.1.1. Let  $w_1 w w_2 \in M$  such that  $w \in \text{Nred}(U)$ . The strict order  $<$  being monomial, the elements of  $\text{supp}(w_1 U(w)w_2)$  are strictly smaller than  $w_1 w w_2$  for  $<$ . Hence,  $<$  being well-founded and  $E$  being finite, the loop **while** is executed a finite number of times, so that **Reduction** returns a result.

**3.1.3. Completion Procedure.** In the procedure, we assume that the presentation  $\langle\langle X, < \mid S \rangle\rangle$  is *finite*, that is,  $X$  is a finite set and the kernel of  $S$  is finite-dimensional. In particular, the set of critical branchings of  $\langle\langle X, < \mid S \rangle\rangle$  is finite.

---

**Algorithm 1** Completion procedure

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**Initialisation:**

- $d := 0$ ,
- $S^d := S$ ,
- $Q_d := \emptyset$  and  $P_d := \left\{ \text{critical branchings of } \langle (X, <) \mid S^d \rangle \right\}$ ,
- $E_d := \left\{ w - S^d_{n,m}(w) \mid (w, (n, m), (n', m')) \in P_d \right\}$ .

- 1: **while**  $Q_d \neq P_d$  **do**
  - 2:    $F_d := \text{Reduction}(E_d, S^d)$ ;
  - 3:    $S^{d+1} := S^d \wedge C^{F_d}$ ;
  - 4:    $Q_{d+1} := P_d$ ;
  - 5:    $d = d + 1$ ;
  - 6:    $P_d := \left\{ \text{critical branchings of } \langle (X, <) \mid S^d \rangle \right\}$ ;
  - 7:    $E_d := \left\{ w - S^d_{n,m}(w) \mid (w, (n, m), (n', m')) \in P_d \setminus Q_d \right\}$ ;
  - 8: **end while**
- 

**3.1.4. Remark.** The first instruction of the loop **while** makes sense if and only if the set  $E_d$  is finite. For that, it is sufficient that the kernel of  $S^d$  is finite-dimensional since, if it is so,  $P_d$  is a finite set. Moreover, the last instruction of the loop **while** makes sense if and only if  $Q_d$  is included in  $P_d$ . We show that  $\ker(S^d)$  is finite-dimensional and that  $Q_d$  is included in  $P_d$  in Lemma 3.1.5.

**3.1.5. Lemma.** *Let  $d$  be an integer.*

1. *The kernels of  $S^d$  and  $C^{F_d}$  are finite-dimensional.*
2. *The set  $Q_d$  is included in  $P_d$ .*

*Proof.* We show Point 1 by induction on  $d$ . The kernel of  $S^0 = S$  is finite-dimensional by hypotheses in 3.1.3. Let  $d \in \mathbb{N}$ , and assume that the kernel of  $S^d$  is finite-dimensional. We let

$$M_d = \bigcup_{f \in E_d} \text{supp}(f),$$

the union of words appearing in  $E_d$ . The elements of  $F_d$  are only acting on  $M_d$ , so that we have the inclusion

$$\ker(C^{F_d}) \subset \mathbb{K}M_d. \quad (10)$$

The kernel of  $S^d$  being finite-dimensional by induction hypothesis, the set of critical branchings of  $\langle\langle X, < \mid S^d \rangle\rangle$  is finite. Hence,  $E_d$  and  $M_d$  are finite sets, so that  $\ker(C^{F_d})$  is finite-dimensional from (10). Moreover, by definition of  $\wedge$ , we have

$$\ker(S^{d+1}) = \ker(S^d) + \ker(C^{F_d}),$$

so that  $\ker(S^{d+1})$  is finite-dimensional.

Let us show Point 2. By construction,  $Q_d$  is equal to  $P_{d-1}$ , that is,  $Q_d$  is the set of critical branchings of  $\langle\langle X, < \mid S^{d-1} \rangle\rangle$ . Let  $(w, (n, m), n', m')$  be such a critical branching. In particular, we have

$$w \in \text{Nred}\left(\left(S^{d-1}\right)_{n,m}\right) \cap \text{Nred}\left(\left(S^{d-1}\right)_{n',m'}\right). \quad (11)$$

Moreover, by construction,  $S^d$  is smaller than  $S^{d-1}$  for  $\preceq$ . Thus, from implication (7) of 2.1.5, we have

$$\text{Nred}\left(S^{d-1}\right) \subset \text{Nred}\left(S^d\right). \quad (12)$$

From (11) and (12),  $w$  belongs to  $\text{Nred}\left(S^d_{n,m}\right) \cap \text{Nred}\left(S^d_{n',m'}\right)$ . In particular,  $(w, (n, m), (n', m'))$  is a critical branching of  $\langle\langle X, < \mid S^{d+1} \rangle\rangle$ , that is, it belongs to  $P_d$ . Thus,  $Q_d$  is included in  $P_d$ .  $\square$

**3.1.6. Effectiveness.** In order to execute the procedure 3.1.3, we have to compute a lower bound of reduction operators relative to the infinite set  $X^*$ . Hence, the implementation of  $\ker^{-1}$  for totally ordered finite sets in Section 4 cannot be used in this context, *a priori*. However, from Lemma 3.1.5, the kernels of  $S^d$  and  $C^{F_d}$  are finite-dimensional, so that these two operators can be computed by restrictions over finite-dimensional subspaces of  $\mathbb{K}X^*$ . We illustrate how works such a computation in Example 3.2.8.

**3.1.7. Completed Presentations.** If the procedure 3.1.3 terminates after  $d$  iterations of the loop **while**, we let  $S^n = S^d$  for every integer  $n \geq d$ . Hence, the sequence  $(S^d)_{d \in \mathbb{N}}$  is well-defined if 3.1.3 terminates or not (in 3.1.8 we explain why it has no reason to terminate, *a priori*). We let

$$\bar{S} = \bigwedge_{d \in \mathbb{N}} S^d.$$

The triple  $\langle\langle X, < \mid \bar{S} \rangle\rangle$  is called the *completed presentation* of  $\langle\langle X, < \mid S \rangle\rangle$ . The main result of the paper is Theorem 3.2.7 which states that  $\langle\langle X, < \mid \bar{S} \rangle\rangle$  is a confluent presentation by operator of  $\mathbf{A}$ . The proof of this result is done in Section 3.2.

**3.1.8. Non-termination.** As said in 3.1.7,  $\langle\langle X, < \mid \bar{S} \rangle\rangle$  is a confluent presentation of  $\mathbf{A}$ , so that

$$R = \left\{ w - \bar{S}(w) \mid w \in \text{Nred}(\bar{S}) \right\},$$

is a noncommutative Gröbner basis of  $I(\ker(S))$ . Moreover, if 3.1.3 terminates, then  $\bar{S} = S^d$  for some integer  $d$ , so that the kernel of  $\bar{S}$  is finite-dimensional in this case. Hence, if 3.1.3 terminates,  $I(\ker(S))$  admits a finite noncommutative Gröbner basis. However, it is well-known [18, Section 1.3] that there exist finitely generated ideal of  $\mathbb{K}X^*$  which do not admit any finite noncommutative Gröbner basis. For this reason, 3.1.3 does not terminate in general.

### 3.2 Soundness

The aim of this section is to show that a completed presentation is a confluent presentation by operator.

**3.2.1. Lemma.** *Let  $w \in X^*$  and let  $T$  and  $T'$  be two reduction operators such that  $T' \preceq T$ .*

1. *Let  $(n, m)$  be a pair of integers such that  $w$  is not  $T_{n,m}$ -reduced. Then,  $(T_{n,m} - T'_{n,m})(w)$  admits a  $(T', w)$ -type decomposition.*
2. *Let  $f \in \mathbb{K}X^*$  admitting a  $(T, w)$ -type decomposition. Then,  $f$  admits a  $(T', w)$ -type decomposition.*

*Proof.* Let us show Point 1. We let  $w = w^{(n)}w'w^{(m)}$ , where  $w^{(n)}$  and  $w^{(m)}$  have length  $n$  and  $m$ , respectively. Let

$$T(w') = \sum_{i=1}^k \lambda_i w_i, \quad (13)$$

be the decomposition of  $T(w')$  with respect to the basis  $X^*$ . By hypotheses,  $T'$  is smaller than  $T$ , so that  $T' \circ T$  is equal to  $T'$  (see Relation (6) of 2.1.5). Hence, we have

$$\begin{aligned} (T_{n,m} - T'_{n,m})(w) &= w^{(n)}(T(w') - T'(w'))w^{(m)} \\ &= w^{(n)}(T(w') - T'(T(w')))w^{(m)}. \end{aligned}$$

From (13), we obtain

$$(T_{n,m} - T'_{n,m})(w) = \sum_{i=1}^k \lambda_i w^{(n)}(w_i - T'(w_i))w^{(m)}. \quad (14)$$

By hypotheses,  $w$  is not  $T_{n,m}$ -reduced. Thus,  $w'$  is not  $T$ -reduced, so that each  $w_i$  is strictly smaller than  $w'$  for  $<$ . The strict order  $<$  being monomial, each  $w^{(n)}w_iw^{(m)}$  is strictly smaller than  $w^{(n)}w'w^{(m)} = w$ . Hence, (14) is a  $(T', w)$ -type decomposition of  $(T_{n,m} - T'_{n,m})(w)$ .

Let us show Point 2. Let

$$f = \sum_{i=1}^n \lambda_i w_i^1 (w_i - T(w_i)) w_i^2, \quad (15)$$

be a  $(T, w)$ -type decomposition of  $f$ . Letting

$$A = \sum_{i=1}^n \lambda_i w_i^1 (w_i - T'(w_i)) w_i^2 \quad \text{and} \quad B = \sum_{i=1}^n \lambda_i w_i^1 (T(w_i) - T'(w_i)) w_i^2,$$

we have

$$f = A - B.$$

The decomposition (15) being  $(T, w)$ -type, each  $w_i' = w_i^1 w_i w_i^2$  is strictly smaller than  $w$ . In particular,  $A$  is  $(T', w)$ -type. For every  $i \in \{1, \dots, n\}$ , let  $n_i$  and  $m_i$  be the lengths of  $w_i^1$  and  $w_i^2$ , respectively. We have

$$B = \sum_{i=1}^n \lambda_i (T_{n_i, m_i} - T'_{n_i, m_i})(w_i').$$

Each  $w_i$  being not  $T$ -reduced, each  $w'_i$  is not  $T_{n_i, m_i}$ -reduced. Hence, from Point 1 of the lemma, each  $(T_{n_i, m_i} - T'_{n_i, m_i})(w'_i)$  admits a  $(T', w'_i)$ -type decomposition, so that it admits a  $(T', w)$ -type decomposition since  $w'_i$  is strictly smaller than  $w$ . Hence,  $B$  admits a  $(T', w)$ -type decomposition, so that  $f$  also admits such a decomposition.  $\square$

**3.2.2. Notation.** For every integer  $d$ , let  $F_d$  be the reduction family of  $\langle (X, <) \mid S^d \rangle$ :

$$F_d = \left\{ \left( S^d \right)_{n, m} \mid (n, m) \in \mathbb{N}^2 \right\}.$$

**3.2.3. Lemma.** Let  $d$  be an integer, let  $(w, (n, m), (n', m')) \in P_d \setminus Q_d$  and let  $f$  be the  $S$ -polynomial of  $(w, (n, m), (n', m'))$ :

$$f = \left( S^d \right)_{n, m}(w) - \left( S^d \right)_{n', m'}(w).$$

1.  $(\wedge F_d)(f)$  is equal to 0.
2.  $f$  admits a  $(S^{d+1}, w)$ -type decomposition.

*Proof.* Let us show Point 1. By construction of  $E_d$ , we have

$$w - \left( S^d \right)_{n, m}(w) \in E_d \quad \text{and} \quad w - \left( S^d \right)_{n', m'}(w) \in E_d.$$

Hence, by definition of the method **Reduction**, the operators

$$T_1 = T \left( w - \left( S^d \right)_{n, m}(w) \right) \quad \text{and} \quad T_2 = T \left( w - \left( S^d \right)_{n', m'}(w) \right),$$

belong to  $F_d$ . In particular,

$$f = (w - S^d_{n, m}(w)) - (w - S^d_{n', m'}(w)),$$

belongs to the kernel of  $T_1 \wedge T_2$ . The latter is included in the kernel of  $\wedge F_d$ , so that Point 1 holds.

Let us show Point 2. The operator  $C^{F_d}$  being a complement of  $F_d$ , we have

$$\wedge (F_d \cup \{C^{F_d}\}) = \wedge F_d, \tag{16}$$

and  $F_d \cup \{C^{F_d}\}$  is confluent (see 2.1.11), that is, it has the Church-Rosser property (see 2.1.9). Hence, from Point 1 of the lemma and Relation (16), there exist  $T_1, \dots, T_r \in F_d \cup \{C^{F_d}\}$ , such that

$$(T_r \circ \dots \circ T_1)(f) = 0. \tag{17}$$

We let

$$f_1 = (\text{Id}_{\mathbb{K}X^*} - T_1)(f),$$

and for every  $k \in \{2, \dots, r\}$ ,

$$f_k = (\text{Id}_{\mathbb{K}X^*} - T_k)(T_{k-1} \circ \dots \circ T_1(f)).$$

From (17), we have

$$f = \sum_{k=1}^r f_k. \quad (18)$$

The tuple  $(w, (n, m), (n', m'))$  being a critical branching of  $\langle (X, <) \mid S^d \rangle$ , we have

$$w \in \text{Nred} \left( \left( S^d \right)_{n,m} \right) \cap \text{Nred} \left( \left( S^d \right)_{n',m'} \right),$$

so that the leading monomial of  $f$  is strictly smaller than  $w$ . Moreover, each  $T_i$  is either of the form  $T(w_1(w_2 - S^d(w_2))w_3)$ , or is equal to  $C^{F_d}$ . Hence, each  $f_i$  admits a  $(S^d, w)$ -type decomposition or a  $(C^{F_d}, w)$ -type decomposition. The reduction operators  $S^d$  and  $C^{F_d}$  being smaller than  $S^{d+1}$ , each  $f_i$  admits a  $(S^{d+1}, w)$ -type decomposition from Point 2 of Lemma 3.2.1, so that  $f$  admits a  $(S^{d+1}, w)$ -type decomposition from (18).  $\square$

**3.2.4. Proposition.** *Let  $d$  be an integer. For every  $(w, (n, m), (n', m')) \in Q_d$ , the  $S$ -polynomial*

$$\left( S^d \right)_{n,m}(w) - \left( S^d \right)_{n',m'}(w),$$

*admits a  $(S^d, w)$ -type decomposition.*

*Proof.* We show the proposition by induction on  $d$ . The set  $Q_0$  being empty, Proposition 3.2.4 holds for  $d = 0$ .

Assume that for every  $(w, (n, m), (n', m')) \in Q_d$ ,  $S^d_{n,m}(w) - S^d_{n',m'}(w)$  admits a  $(S_d, w)$ -type decomposition. We let

$$\begin{aligned} A &= \left( S^d \right)_{n',m'}(w) - \left( S^{d+1} \right)_{n',m'}(w) \quad , \\ B &= \left( S^d \right)_{n,m}(w) - \left( S^{d+1} \right)_{n,m}(w), \\ C &= \left( S^d \right)_{n,m}(w) - \left( S^d \right)_{n',m'}(w). \end{aligned}$$

We have

$$\left( S^{d+1} \right)_{n,m}(w) - \left( S^{d+1} \right)_{n',m'}(w) = A - B + C.$$

By construction,  $S^{d+1}$  is smaller than  $S^d$ . Moreover,  $(w, (n, m), (n', m'))$  being a critical branching, we have

$$w \in \text{Nred} \left( \left( S^d \right)_{n,m} \right) \cap \text{Nred} \left( \left( S^d \right)_{n',m'} \right).$$

Hence, from Point 1 of Lemma 3.2.1,  $A$  and  $B$  admit a  $(S^{d+1}, w)$ -type decomposition. It remains to show that  $C$  admits a  $(S^{d+1}, w)$ -type decomposition. By construction,  $Q_{d+1}$  is equal to  $P_d$ , so that it contains  $Q_d$  from Point 2 of Lemma 3.1.5. If  $(w, (n, m), (n', m'))$  does not belong to  $Q_d$ ,  $C$  admits a  $(S^{d+1}, w)$ -type decomposition from Lemma 3.2.3. If  $(w, (n, m), (n', m'))$  belongs to  $Q_d$ ,  $C$  admits a  $(S^d, w)$ -type decomposition by induction hypothesis. Hence, from Point 2 of Lemma 3.2.1,  $C$  admits a  $(S^{d+1}, w)$ -type decomposition.  $\square$

**3.2.5. Notation.** Recall that the completed presentation of  $\langle (X, <) \mid S \rangle$  is written  $\langle (X, <) \mid \bar{S} \rangle$ , where  $\bar{S}$  is the operator defined in 3.1.7. The last lemma we need to prove Theorem 3.2.7 is:

**3.2.6. Lemma.**

1. For every integer  $d$ , let  $I_d$  be the ideal spanned by  $\ker(S^d)$ . The sequence  $(I_d)_{d \in \mathbb{N}}$  is constant.
2. We have

$$\text{Nred}(\bar{S}) = \bigcup_{d \in \mathbb{N}} \text{Nred}(S^d). \quad (19)$$

*Proof.* Let us show Point 1. By definition of the method **Reduction**, the kernel of each element of  $F_d$  is included in  $I_d$ . In particular,

$$\ker(\wedge F_d) = \sum_{T \in F_d} \ker(T),$$

is also included in  $I_d$ . Moreover,  $C^{F_d}$  being a complement of  $F_d$ , it is smaller than  $\wedge F_d$ , that is its kernel is included in the one of  $\wedge F_d$ . In particular,  $\ker(C^{F_d})$  is included in  $I_d$ , so that

$$\ker(S^{d+1}) = \ker(S^d) + \ker(C^{F_d}),$$

is also included in  $I_d$ . Hence, the sequence  $(I_d)_{d \in \mathbb{N}}$  is not increasing. Moreover, the sequence  $(S^d)_{d \in \mathbb{N}}$  is not increasing by construction, which means that  $(\ker(S^d))_{d \in \mathbb{N}}$  is not decreasing. Hence,  $(I_d)_{d \in \mathbb{N}}$  constant.

Let us show Point 2. The equality (19) means that the set

$$F = \{S^d \mid d \in \mathbb{N}\},$$

is confluent. From Newman's Lemma (see 2.1.10) in terms of reduction operators, it is sufficient to show that  $F$  is locally confluent. Let  $f \in \mathbb{K}X^*$  and let  $d$  and  $d'$  be two integers. Without lost of generalities, we may assume that  $d$  is greater or equal to  $d'$ . In particular,  $(S^d)_{d \in \mathbb{N}}$  being not increasing, from Relation (6) of 2.1.5, we have

$$S^d \circ S^{d'} = S^d.$$

Hence,  $S^d \circ S^{d'}(f)$  and  $S^d(f)$  are equal, so that  $F$  is locally confluent. □

**3.2.7. Theorem.** Let  $\mathbf{A}$  be an algebra and let  $\langle (X, <) \mid S \rangle$  be a presentation by operator of  $\mathbf{A}$ . The completed presentation of  $\langle (X, <) \mid S \rangle$  is a confluent presentation of  $\mathbf{A}$ .

*Proof.* Let  $\bar{S}$  be the operator define in 3.1.7.

First, we show that  $\langle (X, <) \mid \bar{S} \rangle$  is a presentation of  $\mathbf{A}$ . From Point 1 of Lemma 3.2.6, the ideal spanned by the kernels of the operators  $S^d$  is equal to the ideal  $I$  spanned by the kernel of  $S^0 = S$ . In particular, the ideal spanned by

$$\ker(\bar{S}) = \sum_{d \in \mathbb{N}} \ker(S^d),$$



is equal to  $I$ . Hence,  $\langle (X, <) \mid S \rangle$  being a presentation of  $\mathbf{A}$ ,  $\langle (X, <) \mid \bar{S} \rangle$  is also a presentation of  $\mathbf{A}$ .

Let us show that this presentation is confluent. From Proposition 2.2.17, it is sufficient to show that for each critical branching

$$b = (w, (n, m), (n', m')),$$

of  $\langle (X, <) \mid \bar{S} \rangle$ , the  $S$ -polynomial

$$\text{SP}(b) = (\bar{S}_{n,m} - \bar{S}_{n',m'})(w),$$

admits a  $(\bar{S}, w)$ -type decomposition. From Point 2 of Lemma 3.2.6, there exist integers  $d$  and  $d'$  such that

$$w \in \text{Nred} \left( \left( S^d \right)_{n,m} \right) \cap \text{Nred} \left( \left( S^{d'} \right)_{n',m'} \right).$$

Without loss of generalities, we may assume that  $d$  is greater or equal to  $d'$ . Hence,  $\text{Nred}(S^d)$  is included in  $\text{Nred}(S^{d'})$  from Relation (7) of 2.1.5, so that  $b$  is a critical branching of  $\langle (X, <) \mid S^d \rangle$ , that is it belongs to  $P_d = Q_{d+1}$ . We let

$$\begin{aligned} A_d &= \left( S^{d+1} \right)_{n',m'}(w) - \bar{S}_{n',m'}(w), \\ B_d &= \left( S^{d+1} \right)_{n,m}(w) - \bar{S}_{n,m}(w), \\ C_d &= \left( S^{d+1} \right)_{n,m}(w) - \left( S^{d+1} \right)_{n',m'}(w). \end{aligned}$$

We have

$$\text{SP}(b) = A_d - B_d + C_d. \tag{20}$$

From Proposition 3.2.4,  $b$  being an element of  $Q_{d+1}$ ,  $C_d$  admits a  $(S^{d+1}, w)$ -type decomposition, so that it admits a  $(\bar{S}, w)$ -type decomposition from Point 2 of Lemma 3.2.1. Moreover,  $S^{d+1}$  being smaller than  $S^d$ , from Relation (7) of 2.1.5, we have

$$w \in \text{Nred} \left( \left( S^{d+1} \right)_{n,m} \right) \cap \text{Nred} \left( \left( S^{d+1} \right)_{n',m'} \right).$$

The operator  $\bar{S}$  being smaller than  $S^{d+1}$ ,  $A_d$  and  $B_d$  also admit a  $(\bar{S}, w)$ -type decomposition from Point 1 of Lemma 3.2.1. Hence, from (20),  $\text{SP}(b)$  admits a  $(\bar{S}, w)$ -type decomposition.  $\square$

**3.2.8. Example.** We consider the algebra of Example 2.2.7: let  $X = \{x, y, z\}$ , let  $<$  be the deg-lex order induced by  $x < y < z$  and let  $\mathbf{A}$  be the algebra presented by  $\langle (X, <) \mid S \rangle$ , where  $S$  is defined for every word  $w$  by

$$S(w) = \begin{cases} x, & \text{if } w = yz \\ xy, & \text{if } w = zx \\ w, & \text{otherwise.} \end{cases}$$

We do not give details of computations of successive  $F$ -complements. These computations appear in Section 4.3.

We have one critical branching of  $\langle (X, <) \mid S \rangle$ :

$$P_0 = \left\{ b_1 = (yzx, (1, 0), (0, 1)) \right\},$$

and we have

$$E_0 = \left\{ yzx - xx, yzx - yxy \right\}.$$

The words  $xx$  and  $yxy$  having each sub-word  $S$ -reduced,  $\text{Reduction}(E_0, S)$  is equal to

$$F_0 = \left\{ T_1 = T(yzx - xx), T_2 = T(yzx - yxy) \right\}.$$

Given a word  $w$  different from  $xx$ ,  $yxy$  and  $yzx$ ,  $(C^{F_0})(w)$  is equal to  $w$ . Hence,  $C^{F_0}$  can be computed by its restriction to the subspace of  $\mathbb{K}X^*$  spanned by

$$G_1 = \{xx < yxy < yzx\}.$$

We identify  $T_1$  and  $T_2$  to their canonical matrices relative to  $G_1$ :

$$T_1 = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad T_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

We obtain

$$C^{F_0} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The operator

$$S^1 = S \wedge C^{F_0},$$

can be computed by restriction to the vector space spanned by

$$G_2 = \{x < xx < xy < yz < zx < yxy\}.$$

We have

$$S = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad C^{F_0} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

and we obtain that  $S^1$  is the operator defined for every word  $w$  by

$$S^1(w) = \begin{cases} x, & \text{if } w = yz \\ xy, & \text{if } w = zx \\ xx, & \text{if } w = yxy \\ w, & \text{otherwise.} \end{cases}$$

We have

$$P_1 = \left\{ b_1, b_2 = (yxyz, (2, 0), (0, 1)), b_3 = (yxyxy, (2, 0), (0, 2)) \right\}.$$

Hence,  $P_1 \setminus Q_1$  contains  $b_2$  and  $b_3$ , and we have

$$E_1 = \{yxyz - xxz, yxyz - yxx, yxyxy - xxxy, yxyxy - yxxx\}.$$

Each sub-word of the words  $xxz$ ,  $yxx$ ,  $xxxxy$  et  $yxxx$  are  $S$ -reduced, so that  $\text{Reduction}(E_1, S^1)$  is equal to

$$F_1 = \left\{ \begin{array}{l} T_3 = T(yxyz - xxz), T_4 = T(yxyz - yxx) \\ T_5 = T(yxyxy - xxxy), T_6 = T(yxyxy - yxxx) \end{array} \right\}.$$

The restriction of  $C^{F_1}$  to the subspace spanned by

$$\{xxz < yxx < xxxy < yxxx < yxyz < yxyxy\},$$

is

$$C^{F_1} = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

so that  $S^2$  is the operator defined for every word  $w$  by

$$S^2(w) = \begin{cases} x, & \text{if } w = yz \\ xy, & \text{if } w = zx \\ xx, & \text{if } w = yxy \\ xxz, & \text{if } w = yxx \\ xxxy, & \text{if } w = yxxx \\ w, & \text{otherwise,} \end{cases}$$

and

$$P_2 \setminus Q_2 = \{b_4 = (yxyxx, (2, 0), (0, 2)), b_5 = (yxyxxx, (2, 0), (0, 3)), b_6 = (yxxx, (0, 0), (0, 1))\}.$$

We have

$$E_2 = \left\{ \begin{array}{l} yxxx - xxzx, yxxx - xxxy, yxyxxx - yxxxxy \\ yxyxx - yxxxz, yxyxxx - xxxxx, yxyxx - xxxx \end{array} \right\},$$

and we check that

$$F_2 = \left\{ \begin{array}{l} T_7 = T(yxxx - xxzx), T_8 = T(yxxx - xxxy) \\ T_9 = T(yxyxxx - yxxxxy), T_{10} = T(yxyxx - yxxxz) \\ T_{11} = T(yxyxxx - xxxxx), T_{12} = T(yxyxx - xxxx) \\ T_{13} = T(xxzx - xxxy), T_{14} = T(yxxxz - xxxyz) \\ T_{15} = T(xxyyz - xxxx), T_{16} = T(yxxxxy - xxxxy) \\ T_{17} = T(xxyxy - xxxxx) \end{array} \right\}.$$

We obtain

$$C^{F_2} = \text{Id}_{\mathbb{K}X^*}.$$

Hence,  $S^3$  is equal to  $S_2$ , so that  $Q_3$  is equal to  $P_3$ . Thus,

$$\langle (X, <) \mid S^2 \rangle,$$

is a confluent presentation of  $\mathbf{A}$ .

## 4 Source Code and Example

### 4.1 Preliminaries

**4.1.1. Purpose.** Let  $(G, <)$  be a totally ordered finite set and let  $\mathbb{K}$  be a commutative field. In Section 4.2, we write the source code of the bijection (5) of 2.1.4, that is, of the map

$$\ker^{-1} : \mathcal{L}(\mathbb{K}G) \longrightarrow \mathbf{RO}(G, <),$$

mapping every subspace of  $\mathbb{K}G$  to the unique reduction operator relative to  $(G, <)$  with kernel this subspace. We deduce from this implementation the ones of the lower bound, the upper bound and the  $F$ -complement.

**4.1.2. Reduction Operators and Gaussian Elimination.** Let  $V$  be a subspace of  $\mathbb{K}G$ . The set  $G$  being finite, the Gaussian elimination provides a unique basis  $\mathcal{B}$  of  $V$  satisfying the following conditions:

- for every  $e \in \mathcal{B}$ ,  $\text{lc}(e)$  is equal to 1,
- given two different elements  $e$  and  $e'$  of  $\mathcal{B}$ ,  $\text{lg}(e')$  does not belong to the support of  $e$ .

Let  $T$  be the endomorphism of  $\mathbb{K}G$  defined on the basis  $G$  in the following way:

$$T(g) = \begin{cases} \text{lg}(e) - e, & \text{if } g = \text{lg}(e) \text{ for } e \in \mathcal{B} \\ g, & \text{otherwise.} \end{cases} \quad (21)$$

We check that  $T$  is a reduction operator relative to  $(G, <)$ . Moreover, the kernel of  $T$  is equal to  $V$ , so that  $\ker^{-1}(V)$  is equal to  $T$ .

**4.1.3. Organisation of the Source Code.** In our implementation, the subspaces of  $\mathbb{K}G$  are represented by lists of vectors: given such a list, the associated subspace is the one spanned by this list. Our implementation of  $\ker^{-1}$  works as follows: consider a list of vector  $L$ , we compute the basis  $\mathcal{B}$  such as 4.1.2 of  $\mathbb{K}L$  using Gaussian elimination, then we compute  $\ker^{-1}(\mathbb{K}L)$  using (21). For that, we define several intermediate methods. We use SageMath software <sup>2</sup>, written in Python.

### 4.2 Source Code

**4.2.1. Basic Methods.** We first introduce several basic methods used in the sequel. We assume that the ground field  $\mathbb{K}$  is the field of rational numbers  $\mathbb{Q}$ .

---

<sup>2</sup><http://www.sagemath.org>

```

1  def f(u,v,a) :
      return v-a*u

      def notZero(v) :
5         V=VectorSpace(QQ,len(v))
          return v!=V.zero()

def lg(v): # returns lg(v)
9     k=0
    for i in [0..len(v)-1]:
11        if v[i]!=0: k=i+1
    return k

def order(u,v):
15    if lg(u)>lg(v): return int(-1)
    elif lg(u)==lg(v): return int(0)
17    else: return int(1)

19 def leadingVector(L): # returns v of L such that lg(v) is maximal
    v=L[0]
21    for i in [1..len(L)-1]:
        if lg(L[i])>lg(L[i-1]): v=L[i]
23    return v

25 def dimension(L): # returns the dimension of the subspace spanned by L
    A=matrix(L)
27    V=A.image()
    return dim(V)

```

**4.2.2. Pivot.** The following method takes as inputs a list of vector  $L$  and a vector  $v$ . It replaces each element  $v'$  of  $L$  by a vector  $v' - \lambda v$ , where the scalar  $\lambda$  is chosen in such a way that the coefficient of  $\text{lg}(v)$  in  $v' - \lambda v$  vanishes. After we have made this procedure on every vector of  $L$ , we remove from the list the zero vectors.

```

def pivot(L,v):
2     k,G=lg(v),[]
    for i in [0..len(L)-1]:
4         G=G+[f(v,L[i],L[i][k-1]/v[k-1])]
    return filter(notZero,G)+[v]

```

**4.2.3. Ordered Basis.** Let  $L$  be a list of vectors. We wish to construct a list of vectors  $L'$  satisfying the following conditions:

- $\mathbb{K}L'$  is equal to  $\mathbb{K}L$ ,
- the leading generators of the elements of  $L'$  are pairwise distinct,
- given two distinct elements  $v$  and  $v'$  of  $L'$ , the leading generator of  $v$  does not belong to  $\text{supp}(v')$ .

For that, we define an intermediate method with input a list of vectors using as pivot a leading vector of this list, that is, a vector of this list with maximal leading generator. This intermediate method returns the list such obtained, where the elements are ordered with respect to their leading generators. This list satisfies the following conditions

1.  $\mathbb{K}L'$  is equal to  $\mathbb{K}L$ ,
2.  $L'$  does not contain any zero vector,
3. for every integer  $i$ , the  $i$ -th element of  $L'$  as a leading generator not greater than the  $i + 1$ -th,
4. the list  $L'$  contains exactly one vector with maximal leading generator.

```

1 def orderedBasisStep1(L):
    G=pivot(L,leadingVector(L))
3     G.sort(cmp=ordre)
    return G

def orderedBasis(L):
7     n,G=dimension(L),orderedBasisStep1(L)
    for i in [1..n-1]:
9         G=pivot(G,G[i])
        G.sort(cmp=order)
11    return G

```

**4.2.4. Gaussian Elimination.** The list returned by the method of 4.2.3 is not the one obtained by Gaussian elimination: it remains to divide each vector of this list by its leading generator. This is the purpose of the following method:

```

1 def reducedBasis(L):
    G=orderedBasis(L)
3     n=len(G)
    H=[]
5     for i in [0..n-1]:
        v,k=G[i],lg(G[i])
7         H=H+[1/v[k-1]*v]
    return H

```

**4.2.5. Reverse Order.** The method of 4.2.4 returns a reduced basis, where the elements are written in the non-decreasing order with respect to their leading generators. In the following method, we return the reduced basis written in the reverse order:

```

def reverseReducedBasis(L):
2     if L==[]: return L
        else: G=reducedBasis(L); G.reverse(); return G

```

**4.2.6. Reduction Operator.** The following method takes as input a list of vectors  $L$  and returns  $\ker^{-1}(\mathbb{K}L)$ :

```

1 def operator(G):
    L=reverseReducedBasis(G)
3     n=len(L[0])
    V=VectorSpace(QQ,n)
5     v=V.zero()
    G=(lg(L[0])-1)*[v]+[L[0]]
7     k=len(L)
    for i in [1..k-1]:
9         G=G+(lg(L[i])-lg(L[i-1])-1)*[v]+[L[i]]
    G=G+(n-lg(L[k-1]))*[v]
11    return identity_matrix(QQ,n)-matrix(G).transpose()

```

**4.2.7. Lower Bound and Upper Bound.** The following two method compute the lower bound and the upper bound of two reduction operators:

```

1 def lowerBound(T_1,T_2):
    V_1,V_2=kernel(T_1.transpose()),kernel(T_2.transpose())
3     G_1,G_2=basis(V_1),basis(V_2)
    L_1,L_2=reverseReducedBasis(G_1),reverseReducedBasis(G_2)
5     G=L_1+L_2
    L=reverseReducedBasis(G)
7     return operator(L)

9 def upperBound(T_1,T_2):
    V_1,V_2=kernel(T_1.transpose()),kernel(T_2.transpose())
11    V=V_1.intersection(V_2)
    G=basis(V)
13    L=reverseReducedBasis(G)
    return operator(L)

```

**4.2.8.  $F$ -complement.** By definition of the  $F$ -complement, we need an intermediate method with input a reduction operator  $T$  and output  $\ker^{-1}(\mathbb{K}\text{Red}(T))$ . We define this method before defining the one of the  $F$ -complement:

```

def tilde(T):
2     n,L=T.nrows(),[]
    for i in [0..n-1]:
4         j,k=i,n-i-1
            if T[i,i]==1: L=L+[vector(j*[0]+[1]+k*[0])]
6     return operator(L)

8 def complement(L):
    n,C,T=len(L),L[0],tilde(L[0])
10    for i in [1..n-1]: C=lowerBound(C,L[i])
    for j in [1..n-1]: T=upperBound(T,tilde(L[j]))
12    return lowerBound(C,T)

```

### 4.3 Example

In this section, we compute the successive  $F$ -complements of Example 3.2.8. Recall that given  $f \in \mathbb{K}X^*$ , we let

$$T(f) = \ker^{-1}(\mathbb{K}f).$$

**4.3.1. First Step.** We have

$$F_0 = \{T_1, T_2\},$$

where

$$T_1 = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad T_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

We obtain

$$C^{F_0} = \text{complement}([T_1, T_2])$$

$$= \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

**4.3.2. Second Step.** We have

$$F_1 = \left\{ \begin{array}{l} T_3 = T(yxyz - xxz), T_4 = T(yxyz - yxx) \\ T_5 = T(yxyxy - xxxy), T_6 = T(yxyxy - yxxx) \end{array} \right\}.$$

For the deg-lex order induced by  $x < y < z$ , we have

$$xxz < yxx < xxxy < yxxx < yxyz < yxyxy. \quad (22)$$

The matrices of  $T_3, T_4, T_5$  and  $T_6$  relative to (22) are

$$T_3 = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad T_4 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$T_5 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad T_6 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$



We obtain

$$C^{F_1} = \text{complement} ([T_3, T_4, T_5, T_6])$$

$$= \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

**4.3.3. Third Step.** We have

$$F_2 = \left\{ \begin{array}{l} T_7 = T(yxxx - xxzx), T_8 = T(yxxx - xxy) \\ T_9 = T(yxyxxx - yxxxxy), T_{10} = T(yxyxx - yxxxz) \\ T_{11} = T(yxyxxx - xxxxx), T_{12} = T(yxyxx - xxx) \\ T_{13} = T(xxzx - xxy), T_{14} = T(yxxxz - xxyyz) \\ T_{15} = T(xxyyz - xxx), T_{16} = T(yxxxxy - xxyxy) \\ T_{17} = T(xxyxy - xxxxx) \end{array} \right\}.$$

For every integer  $n$ , we write

$$x^n = \underbrace{x \cdots x}_{n \text{ letters}}.$$

We have

$$x^4 < x^3y < x^2zx < yx^3 < x^5 < x^3yz < yx^3z < yxyx^2 < x^3yxyx < yx^4y < yxyx^3. \quad (23)$$

We write matrices of  $T_7, \dots, T_{17}$  in the basis (23). We obtain that the  $F_2$ -complement is the identity matrix of size 11.

**Conclusion.** We wrote a lattice formulation of the  $F_4$  completion procedure. However, our procedure does not take into account that there exist unnecessary critical branchings. For instance, in the second step of the example developed in 3.2.8, we reduced the  $S$ -polynomials of critical branchings with source  $xyz$  and  $yxyxy$ . However, it turns out that it is sufficient to reduce the first one to obtain a confluent presentation by operator. Hence, a natural further work is to avoid reductions of unnecessary critical branchings, that is, we should relate reduction operators to  $F_5$  completion procedure. Another further work is to exploit the lattice formulation of completion to obtain applications in homological algebra using *Brown reduction* [7].

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