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# Time-varying Sampled-data Observer with Asynchronous Measurements (Extended Version)

Antonino Sferlazza, *Member IEEE*, Sophie Tarbouriech, *Member IEEE*, Luca Zaccarian, *Fellow Member IEEE*.

**Abstract**—In this paper a time-varying observer for a linear continuous-time plant with asynchronous sampled measurements is proposed. The observer is contextualized in the hybrid systems framework providing an elegant setting for the proposed solution. In particular some theoretical tools are provided, in terms of LMIs, certifying asymptotic stability of a certain compact set where the estimation error is zero. We consider sampled asynchronous measurements that occur at arbitrary times in a certain window with an upper and lower bound. The design procedure, that we propose for the selection of the time-varying gain, is based on a constructive algorithm that is guaranteed to find a solution to an infinite-dimensional LMI whenever a feasible solution exists. Finally a numerical example shows the effectiveness of the proposed approach.

**Index Terms**—Sample data observer, discrete asynchronous measurements, hybrid systems, linear systems, linear matrix inequalities.

## I. INTRODUCTION

In the last years, the design of observers for systems with sampled measurements has received great attention. This interest is motivated by many engineering applications, such as sampled-data systems, quantized systems, networked systems, localization of mobile vehicles, etc. [1], [2], [3]. In these cases, the output is available only at sampling instants, and, for this reason, classical observer structures cannot be used.

This problem is not new in control engineering and there are many works in the literature dealing with these issues, providing several solutions. In particular this problem has been considered in a stochastic framework, and particular Kalman filters have been developed for these purposes. For example in [4], a Kalman filter with intermittent observations is developed starting from the discrete Kalman filtering formulation, and modeling the input of the observation as a random process. A similar approach is followed in [5] where the observations are available according to a Bernoulli process. Further convergence analysis and boundedness analysis on the estimation error have been recently analyzed in [6] and [7]. Other examples are given in [8] where the intermittent observations are considered in the development of an unscented Kalman

filter. Another stochastic approach, is proposed in [9], where an intrinsic filtering on the special orthogonal group  $SO(3)$  is shown. Here the problem of the continuous-time dynamics with discrete measurements is cast into a rigorous stochastic and geometric framework.

A deterministic approach has been followed in other works. For example recently in [10], the  $\mathcal{H}_2$  state estimation problem in the context of sampled-data linear systems is presented with a fixed sampling-rate, and a time-invariant injection gain is computed in order to optimize the  $\mathcal{H}_2$  performance index for the estimation error dynamics. Another interesting approach is proposed in [11] where the finite time convergence of an observer is proven for linear systems with sampled measurements. Subsequently in [12] and [13], a similar method has been proposed, but for nonlinear Lipschitz systems with sampled measurements. This approach has been extended in [14] by means of new conditions in terms of linear matrix inequalities (LMIs). Also [15] deals with the same problem, developing an observer for the same class of systems, but the computation of the injection gain is based on different conditions. Moreover, also the structure of the observer is different from the previous cited works, since in [15] a high gain observers is proposed. In [16] and [17], nonlinear uniformly observable single output systems are addressed. Finally, by using Lyapunov tools adapted to impulsive systems, some classes of systems with both sampled and delayed outputs are addressed in [18] and [19].

Recently, different approaches have been proposed using the hybrid system formalism of [20]. The use of the hybrid formalism provides a natural setting for the modeling of this type of observers, where both continuous-time and discrete-time dynamics coexist. Indeed a sampled-data observer can be modeled by a "flow map", which describes the continuous-time dynamics when the measurement is not available, while the measurement can be considered as a discrete event and can be modeled by a suitable "jump map". This kind of formalism is applied in [21], [22], where the estimation of the state of a linear time-invariant system is proposed, with asynchronous measurements and a constant output error injection gain. In the same context [23], proposes a hybrid observer for linear systems producing an estimate that converges to the plant state in finite time. These concepts have also been applied to distributed systems such as in [24], where the problem of estimating the state of a linear time-invariant plant is addressed in a distributed fashion over networks allowing only intermittent transmission of information.

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In this paper we address the presence of asynchronous sampled measurements for continuous-time plants using a hybrid formalism. Differently from existing results, a design procedure based on a constructive solution to an infinite-dimensional LMI is given, leading to a time-varying observer gain. Since the proposed hybrid time-varying observer is based on a solution to an infinite-dimensional LMI, an algorithm is proposed, which is proven to find a solution to the infinite-dimensional problem in a finite number of iterations, whenever the problem admits one. The given conditions are shown to be nonconservative in the special case of a periodic sampling.

Preliminary results in the directions of this paper have been presented in [25]. As compared to [25], we reformulate completely the numerical algorithm, which is here proven to always lead to a solution, whenever the (infinite-dimensional) observer design conditions are feasible. Moreover we include a new symbolic example where the approach is nonconservative and establish necessary and sufficient feasibility conditions for the periodic sampling case.

The paper is organized as follows. In Section II the problem statement is formalized by fitting our problem in the hybrid systems framework. Then in Section III we provide analysis and synthesis conditions and discuss feasibility issues also using a symbolic example. In Section IV a numerical algorithm is proposed in order to solve the infinite-dimensional LMIs given in Section III in a finite number of steps. Conclusion are in Section VI.

*Notation:*  $\mathbb{R}^n$  denotes the  $n$ -dimensional Euclidean space.  $\mathbb{R}_{\geq 0}$  denotes the set of nonnegative real numbers.  $\mathbb{Z}$  denotes the set of all integers, while  $\mathbb{Z}_{\geq 0}$  denotes the set of nonnegative integers.  $\mathbb{B}$  denotes the closed unit ball, of appropriate dimension, in the Euclidean norm.  $I_q$  denotes the identity matrix of order  $q \in \mathbb{Z}_{\geq 0}$ .  $\lambda_m(S)$  and  $\lambda_M(S)$  denote, respectively, the minimum and the maximum eigenvalues of a positive definite symmetric matrix  $S$ .  $x^+$  denotes the state of a hybrid system after a jump.  $\|x\|$  denotes the Euclidean norm of a vector  $x \in \mathbb{R}^n$ .  $\lceil \cdot \rceil$  and  $\lfloor \cdot \rfloor$  denote, respectively, the smallest integer upper bound and the greatest integer lower bound of their arguments.

## II. PROBLEM STATEMENT

In this work we consider a class of systems described by the following equation:

$$\dot{x} = Ax + Bu, \quad (1)$$

where  $x \in \mathbb{R}^n$  is the state of the system,  $u : [0, \infty) \rightarrow \mathbb{R}^q$  is a known input that belongs to the class of locally bounded measurable functions,  $A \in \mathbb{R}^{n \times n}$ , and  $B \in \mathbb{R}^{n \times q}$ . Let us assume that an output of system (1) is accessible at discrete instants of time, resulting in a sequence of  $m$  dimensional vectors  $y_k$ ,  $k \in \mathbb{Z}_{\geq 1}$  defined as:

$$y_k := Cx(t_k), \quad (2)$$

where  $C \in \mathbb{R}^{m \times n}$  is full row rank and  $t_k$ ,  $k \in \mathbb{Z}_{\geq 1}$ , is a sequence of increasing non-negative real numbers that satisfies the following assumption:

*Assumption 1:* There exist scalars  $T_m$  and  $T_M$ , with  $0 < T_m \leq T_M$ , such that:

$$T_m \leq |t_{k+1} - t_k| \leq T_M, \quad \forall k \in \mathbb{Z}_{\geq 1}. \quad (3)$$

Assumption 1 considers the case of asynchronous discrete-time measurements with a sampling interval lower and upper bounded by two known positive constants  $T_m$  and  $T_M$ . Note that  $T_m$  must be strictly greater than zero to avoid Zeno behaviors in the hybrid model developed below.

Taking inspiration from the hybrid systems formalism of [20], it is possible to represent the sampled-data system associated with this setting as follows:

$$\begin{cases} \dot{x} = Ax + Bu, \\ \dot{\tau} = 1, \end{cases} \quad (x, \tau) \in \mathcal{C}_x := \mathbb{R}^n \times [0, T_M], \quad (4a)$$

$$\begin{cases} x^+ = x, \\ \tau^+ = 0, \end{cases} \quad (x, \tau) \in \mathcal{D}_x := \mathbb{R}^n \times [T_m, T_M], \quad (4b)$$

$$y = Cx, \quad (4c)$$

where the variable  $\tau$  is a timer keeping track of the elapsed time since the last sample, and the impulsive nature of the available measurement is represented by the extra property that output  $y$  is only available at jump times. With model (4), it follows that for any sequence  $y_k$  in (2), satisfying (3), there exists a solution to (4) such that  $y_k = y(t_k, k)$ ,  $k \in \mathbb{Z}_{\geq 1}$ , and vice versa.

Constraining the jump set to be included in the set where  $\tau \in [T_m, T_M]$  ensures that Assumption 1 is verified as clarified in the next statement, which is a corollary of [26, Props 1.1 & 1.2, page 747].

*Proposition 1:* Consider any solution to (4) and denote by  $t_k$ ,  $k \in \mathbb{Z}_{\geq 1}$ , its jump times. Then the sequence  $t_k$  satisfies Assumption 1. Moreover, consider any sequence  $t_k$ ,  $k \in \mathbb{Z}_{\geq 1}$ , satisfying Assumption 1. For each  $x_0 \in \mathbb{R}^n$  there exists  $\tau_0 \in [0, T_M]$  such that a solution  $\phi$  to (4) with  $\phi(0, 0) = (x_0, \tau_0)$  has jump times coinciding with  $t_k$ ,  $k \in \mathbb{Z}_{\geq 1}$ .

In this paper, we propose an observer whose structure implicitly complies with the restriction specified in Assumption 1 on the available output. Our observer is capable of providing an asymptotic estimate of the plant state, regardless of the sequence of times  $t_k$  at which the sampled output is available. The hybrid structure of the proposed observer is the following:

$$\begin{cases} \dot{\hat{x}} = A\hat{x} + Bu, \\ \hat{x}^+ = \hat{x} + K(\tau)(y - C\hat{x}), \end{cases} \quad \begin{aligned} &(\hat{x}, x, \tau) \in \mathbb{R}^n \times \mathcal{C}_x, \\ &(\hat{x}, x, \tau) \in \mathbb{R}^n \times \mathcal{D}_x, \end{aligned} \quad (5)$$

where the matrix function  $K : [T_m, T_M] \rightarrow \mathbb{R}^{n \times m}$  corresponds to the time-varying gain of the observer responsible for the *discrete* output injection term. It is clear that with dynamics (5), and due to Proposition 1, output  $y$  is only used at the sampling instants  $t_k$  compliant with Assumption 1.

The design of the time-varying gain  $K(\cdot)$  will be performed in the next section. Note that as compared to a standard LTI Luenberger architecture (such as the one used in [21]), observer (5) is based on an injection term that depends on the elapsed time since the last measurement. Such an elapsed time is known to the observer by way of state  $\tau$  in (4).

### III. STABILITY CONDITIONS GAIN SELECTION

One of the main goals of this work is to give design rules to select the gain function  $K(\cdot)$  in (5) such that the estimation error  $e := x - \hat{x}$  converges asymptotically to zero. Such a property is well characterized in terms of the stability of the following error dynamics, issued from (4)-(5):

$$\begin{cases} \dot{e} = Ae, \\ \dot{\tau} = 1, \end{cases} \quad (e, \tau) \in \mathcal{C} := \mathbb{R}^n \times [0, T_M], \quad (6a)$$

$$\begin{cases} e^+ = (I - K(\tau)C)e, \\ \tau^+ = 0, \end{cases} \quad (e, \tau) \in \mathcal{D} := \mathbb{R}^n \times [T_m, T_M]. \quad (6b)$$

We first present an analysis result certifying asymptotic stability of the compact set:

$$\mathcal{A} := \{(e, \tau) : e = 0, \tau \in [0, T_M]\}, \quad (7)$$

corresponding to the set where the estimation error is zero. Then we will design  $K(\cdot)$  inducing Global Asymptotic Stability (GAS) of  $\mathcal{A}$ , corresponding to Lyapunov stability (for each  $\epsilon > 0$ ,  $\exists \delta > 0$  such that  $|e(0, 0)| \leq \delta \Rightarrow |e(t, j)| \leq \epsilon$  for all  $(t, j) \in \text{dom } e$ ) and convergence ( $\lim_{t+j \rightarrow \infty} |e(t, j)| = 0$ ). Due to the developments in [20, Chapter 7], and compactness of  $\mathcal{A}$ , GAS is actually equivalent to Uniform Global Asymptotic Stability (UGAS) defined in [20, Chapter 3] involving Lyapunov stability, uniform global boundedness and uniform global attractivity.

Lemma 1 below is an extension of [21, Theorem 1] to the case of a time-varying injection gain  $K(\cdot)$ .

*Lemma 1:* Assume that there exists a matrix  $P = P^\top > 0$ , and a continuous matrix function  $\tau \rightarrow K(\tau)$  such that:

$$\begin{bmatrix} e^{(-A^\top \tau)} P e^{(-A\tau)} & P \\ P & P \end{bmatrix} > \begin{bmatrix} 0 & \star \\ PK(\tau)C & 0 \end{bmatrix}, \quad \forall \tau \in [T_m, T_M]. \quad (8)$$

Then set  $\mathcal{A}$  in (7) is uniformly globally asymptotically stable (UGAS) for the error dynamics in (6).

*Proof:* Consider the Lyapunov function:

$$V(e, \tau) = e^\top e^{(-A^\top \tau)} P e^{(-A\tau)} e, \quad (9)$$

and observe that there exist positive scalars  $c_1$  and  $c_2$  satisfying:

$$c_1 |e|^2 \leq V(e, \tau) \leq c_2 |e|^2, \quad \forall e \in \mathbb{R}^n, \tau \in [0, T_M], \quad (10)$$

where, denoting by  $\lambda_m(S)$  and  $\lambda_M(S)$  the minimum and the maximum eigenvalues of symmetric matrix  $S$ , respectively, we selected:

$$c_1 := \min_{\tau \in [0, T_M]} \lambda_m(e^{(-A^\top \tau)} P e^{(-A\tau)}), \quad (11a)$$

$$c_2 := \max_{\tau \in [0, T_M]} \lambda_M(e^{(-A^\top \tau)} P e^{(-A\tau)}), \quad (11b)$$

which are well defined and positive, from positive definiteness of  $P$  and invertibility of  $e^{(-A\tau)}$ .

The variation of  $V$  along flowing solutions of (6) is:

$$\begin{aligned} \dot{V}(e, \tau) &:= \\ &2e^\top e^{(-A^\top \tau)} P e^{(-A\tau)} \dot{e} + e^\top (-A^\top) e^{(-A^\top \tau)} P e^{(-A\tau)} e \\ &+ e^\top e^{(-A^\top \tau)} P e^{(-A\tau)} (-A) e = \\ &e^\top e^{(-A^\top \tau)} P e^{(-A\tau)} (2Ae - Ae - Ae) = 0, \\ &\forall (e, \tau) \in \mathbb{R}^n \times [0, T_M]. \end{aligned} \quad (12)$$

The variation of  $V$  across jumping solutions of (6) is:

$$\begin{aligned} \Delta V(e, \tau) &:= V(e^+, \tau^+) - V(e, \tau) = \\ &e^\top (I - K(\tau)C)^\top P (I - K(\tau)C) e - e^\top e^{(-A^\top \tau)} P e^{(-A\tau)} e = \\ &- e^\top \underbrace{(e^{(-A^\top \tau)} P e^{(-A\tau)} - (I - K(\tau)C)^\top P (I - K(\tau)C))}_{M(\tau)} e, \\ &\forall (e, \tau) \in \mathbb{R}^n \times [T_m, T_M], \end{aligned} \quad (13)$$

where we recall that  $\tau \in [T_m, T_M]$  for all  $(e, \tau) \in \mathcal{D}$ . Consider now (8), which implies:

$$\begin{bmatrix} e^{(-A^\top \tau)} P e^{(-A\tau)} & \star \\ P(I - K(\tau)C) & P \end{bmatrix} > 0 \quad \forall \tau \in [T_m, T_M], \quad (14)$$

and after a Schur-complement:

$$e^{(-A^\top \tau)} P e^{(-A\tau)} - (I - K(\tau)C)^\top P (I - K(\tau)C) > 0,$$

namely  $M(\tau) > 0$ ,  $\forall \tau \in [T_m, T_M]$ . Define the positive scalar  $c_3$  as follows:

$$c_3 := \min_{\tau \in [T_m, T_M]} \lambda_m(M(\tau)). \quad (15)$$

Then one gets from (13):

$$\Delta V(e, \tau) \leq -c_3 |e|^2, \quad \forall (e, \tau) \in \mathcal{D}. \quad (16)$$

The proof is completed by first noting that, for attractor  $\mathcal{A}$  in (7),  $|(e, \tau)|_{\mathcal{A}} := \inf_{y \in \mathcal{A}} |(e, \tau) - y| = |e|$ , and then exploiting the fact that solutions to (6) are persistently jumping at least every  $T_M$  ordinary time. Indeed, for each solution  $\phi$  and for each  $(t, j) \in \text{dom } \phi$ , it is immediate to check that  $j \geq \frac{t}{T_M} - 1$ .

Then uniform global asymptotic stability of  $\mathcal{A}$  follows from (10), (12), (16) and [20, Proposition 3.24] with  $N_r = 1$  and  $\gamma_r(t) = \frac{t}{T_M}$ .  $\square$

Based on the analysis result of Lemma 1, we can now prove a few relevant constructions for the gain  $K(\cdot)$ , corresponding to a few special cases. The first case is relatively straightforward and corresponds to the case where  $C$  is invertible (namely the state is completely accessible at the sampling instants). This case is somewhat interesting because it corresponds to the source of inspiration of the subsequent construction, and has been used in a dedicated application by the first author in [27]. It is reported below.

*Theorem 1:* If  $C$  is invertible, then for any  $P = P^\top > 0$  and any  $\lambda \in [0, 1)$ , inequality (8) is satisfied with:

$$K(\tau) = (I - \lambda e^{(-A\tau)}) C^{-1}, \quad (17)$$

which then guarantees UGAS of  $\mathcal{A}$  for system (6).

*Proof:* By virtue of selection (17), we have

$$P(\tau)K(\tau)C = P(I - \lambda e^{(-A\tau)}). \quad (18)$$

Then condition (8) in Lemma 1, becomes:

$$\begin{bmatrix} e^{(-A^\top \tau)} P e^{(-A\tau)} & \lambda e^{(-A^\top \tau)} P \\ \lambda P e^{(-A\tau)} & P \end{bmatrix} > 0 \iff \begin{bmatrix} P & \lambda P \\ \lambda P & P \end{bmatrix} > 0. \quad (19)$$

The last one is always verified for  $0 \leq \lambda < 1$ , and for any positive definite matrix  $P$ .  $\square$

*Remark 1:* Replacing the gain  $K(\tau)$  of (17) in (6b), we obtain:

$$e^+ = \lambda e^{(-A\tau)} e, \quad (20)$$

that clearly reveals that the choice  $\lambda = 0$  leads to a dead-beat controller, while the choice  $\lambda = 1$  leads to a nontrivial reset that resets back the estimation error to the value that it had immediately after the previous sample (this fact is evident by keeping in mind the explicit expression of the error  $e(t, k) = e^{(A\tau)} e(t_k, k)$  for all  $t \in [t_k, t_{k+1}]$ ). Clearly, the choice  $\lambda = 1$  is not allowed in our result because it leads to a bounded, but non converging, response.  $\dashv$

The solution of Theorem 1 is only viable under demanding conditions on the available measurements, that are only seldom verified. Due to this reason, one of the main contributions of this paper resides in a construction for the gain  $K(\cdot)$  as long as one can find a constant matrix  $P$  satisfying the following infinite set of matrix inequalities:

$$\Xi_P(\tau) := \begin{bmatrix} (C^\perp)^\top e^{(-A^\top \tau)} P e^{(-A\tau)} C^\perp & \star \\ P C^\perp & P \end{bmatrix} > 0, \quad \forall \tau \in [T_m, T_M], \quad (21)$$

where  $C^\perp$  denotes the orthonormal complement of  $C^\top$ . Despite the non-uniqueness of  $C^\perp$ , feasibility of (21) is independent of the specific selection. Indeed replacing  $C^\perp$  by any of the alternative selections  $C^\perp S$  (with  $S$  being any unitary matrix) does not affect feasibility of (21) because one can factor out matrix  $\text{diag}(S, I)$  without affecting feasibility. Matrix inequality (21) is not easy to solve, but we provide in Section IV a numerical algorithm that is guaranteed to converge to a solution, whenever it exists, in a finite number of steps. Insight about the implication of (21), at least for the periodic case  $T_m = T_M$ , can be given by the following proposition, whose proof is postponed to the end of this Section.

*Proposition 2:* Consider any positive value of  $T = T_m = T_M$ . LMI (21) is feasible if and only if pair  $(C, e^{AT})$  is detectable.

While Proposition 2 characterizes feasibility (and non-conservativeness) of (21) for the periodic case, in the general case  $T_m < T_M$  some level of conservativeness may arise from the use of a common  $P$  for all  $\tau \in [T_m, T_M]$ . Nevertheless condition (21) is relatively mild and for the following planar example it is shown to be never conservative.

*Example 1:* Consider system (1)-(2) where  $A = \begin{bmatrix} \alpha & 1 \\ -1 & \alpha \end{bmatrix}$ ,  $C = [1 \ 0]$ ,  $C^\perp = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ , with  $\alpha \geq 0$  to avoid trivialities. Due to the oscillatory response with period  $2\pi$ , the state is not

detectable, regardless of  $\alpha \geq 0$ , if  $k\pi \in [T_m, T_M]$  for some  $k \in \mathbb{Z}_{\geq 0}$ . Ruling out those infeasible cases corresponds to requiring:

$$k\pi < T_m \leq T_M < (k+1)\pi \quad (22)$$

for some  $k \in \mathbb{Z}_{\geq 0}$ , which is a necessary condition for detectability. We construct below a solution  $P$  to (21) under assumption (22). By replacing  $e^{-A\tau} = e^{-\alpha\tau} \begin{bmatrix} \cos(\tau) & -\sin(\tau) \\ \sin(\tau) & \cos(\tau) \end{bmatrix}$  in (21), choosing  $P = \begin{bmatrix} p_{11} & 0 \\ 0 & p_{22} \end{bmatrix}$ , and applying a Schur complement, inequality (21) is satisfied if and only if:

$$e^{-2\alpha\tau} (p_{11} \sin^2(\tau) + p_{22} \cos^2(\tau)) - p_{22} > 0. \quad (23)$$

Consider now any selection of  $p_{11}, p_{22}$  satisfying

$$p_{11} > \frac{e^{2\alpha T_M} p_{22}}{\min\{\sin^2(T_m), \sin^2(T_M)\}}. \quad (24)$$

which is well defined from (22). Then inequality (23) holds because  $0 < p_{11} (\min\{\sin^2(T_m), \sin^2(T_M)\}) - e^{2\alpha T_M} p_{22} \leq p_{11} \sin^2(\tau) - e^{2\alpha\tau} p_{22} \leq p_{11} \sin^2(\tau) + p_{22} \cos^2(\tau) - e^{2\alpha\tau} p_{22}$ . Note that this selection applies for any (destabilizing) choice of  $\alpha \geq 0$  even though, through  $p_{11}$  in (24), the Lyapunov function is stretched for larger values of  $\alpha$  and  $T_M$ .

We report below the explicit expression of  $K(\cdot)$ , which induces UGAS of attractor  $\mathcal{A}$  for the observation error dynamics, as long as (21) is satisfied.

*Theorem 2:* Assume that  $C$  is full row rank and denote by  $C^\perp$  a basis of the orthogonal complement of  $C^\top$ . If there exists  $P = P^\top > 0$  satisfying (21), then selection:

$$K(\tau) := \left( C^\top - C^\perp \left( (C^\perp)^\top e^{(-A^\top \tau)} P e^{(-A\tau)} C^\perp \right)^{-1} (C^\perp)^\top \right) \left( e^{(-A^\top \tau)} P e^{(-A\tau)} \right)^\top C^\top (C C^\top)^{-1}, \quad (25)$$

guarantees UGAS of  $\mathcal{A}$  for system (6).

Note that  $K(\cdot)$  in (25) does not depend on the selection of  $C^\perp$ . Indeed all such selections are parametrized by  $C^\perp S$ , with any unitary  $S$ , and  $S$  does not affect the value of  $K(\cdot)$  in (25) because  $C^\perp S \left( S^\top (C^\perp)^\top e^{(-A^\top \tau)} P e^{(-A\tau)} C^\perp S \right)^{-1} S^\top (C^\perp)^\top = C^\perp \left( (C^\perp)^\top e^{(-A^\top \tau)} P e^{(-A\tau)} C^\perp \right)^{-1} (C^\perp)^\top$ . Coming back to Example 1, we can compute the observer gain using (25):

$$K(\tau) = \begin{bmatrix} 1 & \frac{(p_{11} - p_{22}) \sin(\tau) \cos(\tau)}{p_{11} \sin^2(\tau) + p_{22} \cos^2(\tau)} \end{bmatrix}^\top, \quad (26)$$

where  $p_{11}$  and  $p_{22}$  are any positive constant satisfying (24).

*Proof of Theorem 2:* The proof is divided into two parts. In the first part we show that condition (21) is enough to ensure the existence of a gain  $K(\cdot)$  such that (8) is satisfied. In the second part we show that given a matrix  $P$  satisfying condition (21), then (8) is satisfied for the gain  $K(\cdot)$  selected as in (25). Then the result follows from Lemma 1.

*Part 1 (Proof of the existence):* In this first part we have to show that if (21) holds, then there exists  $K(\cdot)$  (equivalently  $Y(\cdot)$ ) such that the following inequality holds:

$$\begin{bmatrix} \Psi(\tau) & \star \\ P + Y(\tau)C & P \end{bmatrix} := \begin{bmatrix} e^{(-A^\top \tau)} P e^{(-A\tau)} & \star \\ P - PK(\tau)C & P \end{bmatrix} > 0, \quad (27)$$

where we introduced  $\Psi(\tau) := e^{(-A^\top \tau)} P e^{(-A\tau)}$  and  $Y(\tau) := -PK(\tau)$ . Equation (27) can be written as:

$$\underbrace{\begin{bmatrix} \Psi(\tau) & \star \\ P & P \end{bmatrix}}_{Q(\tau)} + \underbrace{\begin{bmatrix} 0 \\ I \end{bmatrix}}_H Y(\tau) \underbrace{\begin{bmatrix} C & 0 \end{bmatrix}}_{G^\top} + \begin{bmatrix} C^\top \\ 0 \end{bmatrix} Y(\tau) \begin{bmatrix} 0 & I \end{bmatrix} > 0. \quad (28)$$

Applying the elimination lemma (see, e.g. [28, Equations (2.27)-(2.28)]) for each  $\tau$  there exists a matrix  $Y(\tau)$  such that (28) is satisfied if and only if the following relations hold:

$$(H^\perp)^\top Q(\tau) (H^\perp) > 0, \quad (G^\perp)^\top Q(\tau) (G^\perp) > 0, \quad (29)$$

where  $H^\perp = \begin{bmatrix} I \\ 0 \end{bmatrix}$  is a basis of the Kernel of  $H^\top$ , and  $G^\perp = \begin{bmatrix} C^\perp & 0 \\ 0 & I \end{bmatrix}$  is a basis of the Kernel of  $G^\top$ . Note that all possible selections of  $H^\perp$  and  $G^\perp$  are parametrized by  $H^\perp S_H$  and  $G^\perp S_G$ , with any unitary matrices  $S_H, S_G$ , which can be factored out and do not affect the feasibility of (29).

Using the above relations, the left equation in (29) becomes:

$$\begin{bmatrix} I & 0 \end{bmatrix} \begin{bmatrix} \Psi(\tau) & \star \\ P & P \end{bmatrix} \begin{bmatrix} I \\ 0 \end{bmatrix} = \Psi(\tau) = e^{(-A^\top \tau)} P e^{(-A\tau)} > 0, \quad (30)$$

which is always satisfied because  $P > 0$ . Regarding the right equation in (29) we have:

$$\begin{bmatrix} (C^\perp)^\top & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \Psi(\tau) & \star \\ P & P \end{bmatrix} \begin{bmatrix} C^\perp & 0 \\ 0 & I \end{bmatrix} = \begin{bmatrix} (C^\perp)^\top \Psi(\tau) C^\perp & \star \\ PC^\perp & P \end{bmatrix} > 0, \quad (31)$$

which is satisfied by hypothesis (21). This means that if condition (21) is satisfied, then there exists  $Y(\tau)$  (and consequently a matrix gain  $K(\tau) = -P^{-1}Y(\tau)$ ) such that (28) (therefore (8)) is satisfied.

*Part 2 (Selection of  $K(\cdot)$ ):* We show next that for a given matrix  $P$  satisfying condition (21), inequality (8) is satisfied for the gain  $K(\cdot)$  proposed in (25).

Since  $C$  is full row rank, then  $R := [C^\perp \ C^\top] \in \mathbb{R}^{n \times n}$  is nonsingular, and inequality (27) holds if and only if:

$$\begin{bmatrix} R & 0 \\ 0 & I \end{bmatrix}^\top \begin{bmatrix} \Psi(\tau) & \star \\ P + Y(\tau)C & P \end{bmatrix} \begin{bmatrix} R & 0 \\ 0 & I \end{bmatrix} = \begin{bmatrix} (C^\perp)^\top \Psi(\tau) C^\perp & \star \\ C\Psi(\tau)C^\perp & C\Psi(\tau)C^\top \\ PC^\perp & PC^\top + Y(\tau)CC^\top \end{bmatrix} \begin{bmatrix} \star \\ \star \\ P \end{bmatrix} > 0, \quad \forall \tau \in [T_m, T_M], \quad (32)$$

where we used  $CC^\perp = 0$ . By applying a Schur-complement, and using the property  $(C^\perp)^\top \Psi(\tau) C^\perp > 0, \forall \tau \in [T_m, T_M]$  (ensured by the upper left entry of (21)), we obtain that inequality (32) is equivalent to the following constraint:

$$\begin{bmatrix} M_{11}(\tau) & \star \\ M_{21}(\tau) & M_{22}(\tau) \end{bmatrix} = \begin{bmatrix} C\Psi(\tau)C^\top & \star \\ PC^\top + Y(\tau)CC^\top & P \end{bmatrix} - \underbrace{\begin{bmatrix} C\Psi(\tau)C^\perp \\ PC^\perp \end{bmatrix}}_{\Sigma^\top} \left[ (C^\perp)^\top \Psi(\tau) C^\perp \right]^{-1} \Sigma > 0 \quad (33)$$

Since we already established (in Part 1 of this proof) the existence of a solution  $Y(\tau)$  to (33), the diagonal terms  $M_{11}(\tau)$  and  $M_{22}(\tau)$  in (33), which are independent of  $Y(\tau)$ , must necessarily be positive definite. So in order to ensure that inequality (33) is satisfied, it is enough to show that  $M_{21}(\tau) = 0, \forall \tau \in [T_m, T_M]$ , which is done below. In particular, noting that expression (25) corresponds to:

$$Y(\tau) = -PK(\tau) = P \left( C^\perp ((C^\perp)^\top \Psi(\tau) C^\perp)^{-1} (C\Psi(\tau)C^\perp)^\top - C^\top \right) (CC^\top)^{-1}. \quad (34)$$

Then simple manipulations are sufficient to show that:

$$M_{21}(\tau) = PC^\top + Y(\tau)CC^\top - PC^\perp ((C^\perp)^\top \Psi(\tau) C^\perp)^{-1} (C\Psi(\tau)C^\perp)^\top = 0, \quad (35)$$

Thus completing the proof.  $\square$

*Remark 2:* Selection (25) for gain  $K(\cdot)$  can well be understood as follows:

$$K(\tau) := \left( PC^\perp ((C^\perp)^\top \Psi(\tau) C^\perp)^{-1} (C\Psi(\tau)C^\perp)^\top M_{21}(\tau) - PC^\top \right) (CC^\top)^{-1}, \quad (36)$$

where  $\Psi(\tau) := e^{(-A^\top \tau)} P e^{(-A\tau)}$ , and where any matrix  $M_{12}(\tau)$  can be selected, as long as it guarantees inequality (27). Part 1 of the proof of the theorem guarantees that both  $M_{11}(\tau)$  and  $M_{22}(\tau)$  are uniformly positive definite in  $[T_m, T_M]$ . We may then be inspired by the parametric selection (17) of Theorem 1 and pick:

$$M_{21}(\tau) = \lambda \left( M_{22}^{\frac{1}{2}}(\tau) \right)^\top M_{11}^{\frac{1}{2}}(\tau), \quad \lambda \in [0, 1], \quad (37)$$

where  $M_{11}^{\frac{1}{2}}(\tau)$  and  $M_{22}^{\frac{1}{2}}(\tau)$  are the Cholesky's factorizations of the positive definite matrices  $M_{11}(\tau)$  and  $M_{22}(\tau)$ , respectively. With selection (37), we always satisfy (33) because a Schur-complement ensures (33) as long as:

$$M_{11}(\tau) - M_{21}^\top(\tau) M_{22}^{-1}(\tau) M_{21}(\tau) = (1 - \lambda^2) M_{11}(\tau) > 0, \quad (38)$$

which holds true as long as  $\lambda \in [0, 1]$ . For any  $\lambda \neq 0$ , selection (37) makes the observer heavy from the computational point of view, because  $M_{11}(\tau)$  and  $M_{22}(\tau)$  are time-varying matrices, so the Cholesky's factorization must be performed at each sampling time in order to determine  $M_{21}(\tau)$  in (37), and consequently the gain  $K(\tau)$  in (36).

This computational aspect motivated us to present an efficient solution corresponding to  $\lambda = 0$  in Theorem 2. However, based on similar arguments to those presented in Remark 1, it might be desirable to pick larger values of  $\lambda$  to reduce the aggressiveness of the output error injection and increasing the filtering action of the sampled-data observer.  $\lrcorner$

Based on Theorem 2 we can now prove Proposition 2.

*Proof of Proposition 2:* The first implication, i.e. if LMI (21) is feasible, then the pair  $(C, e^{AT})$  is detectable, is trivial, because in the case  $T = T_m = T_M$  only periodic sampling (with period  $T$ ) is allowed by the observer dynamics. Then the definition of detectability implies that there does not exist

an asymptotic state observer and condition (21) cannot be feasible.

Let us prove the converse implication. If the pair  $(C, e^{AT})$  is detectable, then there exist matrices  $Q > 0$  and  $L$  such that:

$$Q - (e^{AT} - LC)^\top Q (e^{AT} - LC) > 0. \quad (39)$$

Condition (39) is equivalent to the following condition, after a Schur-complement:

$$\begin{bmatrix} Q & \star \\ (e^{AT} - LC) & Q^{-1} \end{bmatrix} > 0. \quad (40)$$

Since  $e^{-AT} \in \mathbb{R}^{n \times n}$  is nonsingular, inequality (40) holds if and only if:

$$\begin{bmatrix} e^{-AT} & 0 \\ 0 & I \end{bmatrix}^\top \begin{bmatrix} Q & \star \\ (e^{AT} - LC) & Q^{-1} \end{bmatrix} \begin{bmatrix} e^{-AT} & 0 \\ 0 & I \end{bmatrix} = \begin{bmatrix} e^{-A^\top T} Q e^{-AT} & \star \\ (I - LC e^{-AT}) & Q^{-1} \end{bmatrix} > 0. \quad (41)$$

Equation (41) can be written as:

$$\underbrace{\begin{bmatrix} e^{-A^\top T} Q e^{-AT} & \star \\ I & Q^{-1} \end{bmatrix}}_{N:=} + \text{He} \left( \underbrace{\begin{bmatrix} 0 \\ -I \end{bmatrix}}_{E:=} L \underbrace{\begin{bmatrix} C e^{-AT} & 0 \end{bmatrix}}_{F^\top:=} \right) > 0. \quad (42)$$

Applying the elimination lemma, as in Theorem 2, there exists a matrix  $L$  such that (42) is satisfied if and only if the following relations hold:

$$(E^\perp)^\top N (E^\perp) > 0, \quad (F^\perp)^\top N (F^\perp) > 0, \quad (43)$$

where  $E^\perp = \begin{bmatrix} I & 0 \end{bmatrix}^\top$  is a basis of the Kernel of  $E^\top$ , and  $F^\perp = \begin{bmatrix} e^{AT} C^\perp & 0 \\ 0 & I \end{bmatrix}$  is a basis of the Kernel of  $F^\top$ .

Using the above relations, the left equation in (43) becomes:

$$\begin{bmatrix} I & 0 \end{bmatrix} \begin{bmatrix} e^{-A^\top T} Q e^{-AT} & \star \\ I & Q^{-1} \end{bmatrix} \begin{bmatrix} I \\ 0 \end{bmatrix} = e^{(-A^\top T)} Q e^{(-AT)} > 0, \quad (44)$$

which is always satisfied because  $Q > 0$ . Regarding the right equation in (43) we have:

$$\begin{bmatrix} (C^\perp)^\top e^{A^\top T} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} e^{-A^\top T} Q e^{-AT} & \star \\ I & Q^{-1} \end{bmatrix} \begin{bmatrix} e^{AT} C^\perp & 0 \\ 0 & I \end{bmatrix} = \begin{bmatrix} (C^\perp)^\top Q C^\perp & \star \\ e^{AT} C^\perp & Q^{-1} \end{bmatrix} > 0, \quad (45)$$

which is equivalent to:

$$(C^\perp)^\top Q C^\perp - (C^\perp)^\top e^{A^\top T} Q e^{AT} C^\perp > 0, \quad (46)$$

after a Schur-complement. Let us now select  $P := e^{A^\top T} Q e^{AT}$ , which is positive definite because  $Q > 0$  and  $e^{AT}$  is nonsingular. Then using  $P = P P^{-1} P$ , we obtain:

$$(C^\perp)^\top e^{-A^\top T} P e^{-AT} C^\perp - (C^\perp)^\top P P^{-1} P C^\perp > 0, \quad (47)$$

which implies (21) after a Schur-complement.  $\square$

#### IV. DESIGN ALGORITHM

We propose here an algorithm to solve the infinite-dimensional problem (21) in a finite number of steps. To this end let us introduce the following optimization problem:

$$(P^*, p^*) = \arg \min_{P=P^\top, p_M} p_M, \quad \text{subject to:} \quad (48)$$

$$\begin{aligned} \Xi_P(\tau) &> 2\mu I, \quad \forall \tau \in [T_m, T_M], \\ I &\leq P \leq p_M I, \end{aligned}$$

where  $\Xi_P(\tau)$  is defined in (21), and  $\mu > 0$  is a positive scalar constant. Problem (48) is again infinite dimensional, but it avoids numerical problems and solutions that lead to large values of  $P$  because its upper bound is minimized. The feasibility of problem (48) is equivalent to the feasibility of problem (21) as established in the following result.

*Lemma 2:* The optimization problem (48) is feasible if and only if relation (21) is feasible, moreover any matrix  $P$  solution to (48) is also a solution to (21).

*Proof:* Any solution to (48) is also a solution of (21), because (21) has relaxed constraint. Vice versa, consider any  $\bar{P}$  satisfying (21) and denote by  $\bar{p}_m, \bar{p}_M$  its minimum and maximum eigenvalues. Also denote by:

$$\xi_m := \min_{\tau \in [T_m, T_M]} \lambda_m(\Xi_{\bar{P}}(\tau)),$$

where  $\lambda_m(\cdot)$  denotes the smallest (real) eigenvalue of the symmetric matrix at argument. Then it is straightforward to verify that  $P = \max \left\{ \frac{1}{\bar{p}_m}, 2\mu\xi_m \right\} \bar{P}$  satisfies (48) with  $\bar{p}_M = \max \left\{ \frac{\bar{p}_M}{\bar{p}_m}, 2\mu\xi_m \bar{p}_M \right\} \bar{P}$ .  $\square$

Focusing on (48) we introduce now a numerical algorithm, aiming at finding a matrix  $P$  solution to condition (21)  $\forall \tau \in [T_m, T_M]$ . The scheme of the algorithm is shown at the top of next page. The algorithm can be roughly divided into three parts: the initialization, the synthesis, and the analysis phase. During the initialization we establish an exponential bound on  $e^{-AT}$  by finding a solution  $\Pi = \Pi^\top > 0$  and  $\beta \geq 0$  to the generalized eigenvalue problem:

$$(A + \beta I)^\top \Pi + \Pi (A + \beta I) > 0, \quad (49)$$

which is a quasi-convex problem easily solved by bisection algorithms (e.g. with the Matlab command `gevp`), and then selecting  $\gamma := \sqrt{\frac{\lambda_M(\Pi)}{\lambda_m(\Pi)}}$ , as established in Lemma 3 below. Then, during the synthesis phase, we solve the finite dimensional optimization:

$$(P_\tau^*, p_\tau^*) = \arg \min_{P=P^\top, p_M} p_M, \quad \text{subject to:} \quad (50)$$

$$\begin{aligned} \Xi_P(\tau) &> 2\mu I, \quad \forall \tau \in \mathcal{T}, \\ I &\leq P \leq p_M I, \end{aligned}$$

where  $\tau$  ranges over a finite number of points collected in the discrete set  $\mathcal{T}$  (in the first step  $\mathcal{T} = \{T_m, T_M\}$ ). Given an optimal solution  $(P_\tau^*, p_\tau^*)$  to (50), during the analysis phase we check the following eigenvalue conditions, relaxing the constraints in (50) to half of their values:

$$\Xi_{P_\tau^*}(\tau) > \mu I, \quad \forall \tau \in \mathcal{T}_d, \quad (51)$$

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**Algorithm 1** NUMERICAL PROCEDURE TO SOLVE (50)-(51) (NOTATION `set` AND `solvesdp` ARE CONSISTENT WITH *Yalmip* [29])

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1: Initialize the parameters:  $\beta, \Pi$  from (49) and  $\gamma = \sqrt{\frac{\lambda_M(\Pi)}{\lambda_m(\Pi)}}$ ; ▷ Initialize parameters.
2: Initialize the internal variables:  $\mathcal{T} = \{T_m, T_M\}; \mu = 1$ ; ▷ Initialize variables.
3:  $constr = \text{set}(P \geq I_n) + \text{set}(P \leq p_M I_n)$ ; ▷ Define the constraints (here called 'constr'): positivity of  $P$  and  $P$  bounded.
4: for  $i$  from 1 to  $\text{length}(\mathcal{T})$  do
5:    $constr = constr + \text{set}(\Xi_P(\mathcal{T}(i)) > 2\mu I_{2n-m})$ ; ▷ The constraints of problem (50) are included.  $\Xi_P(\cdot)$  is defined in (21).
6: end for
   Until END do
7:  $(P_{\mathcal{T}}^*, P_{\mathcal{T}}^*) = \text{solvesdp}(constr, p_M)$ ; ▷ Find a pair  $(P_{\mathcal{T}}^*, P_{\mathcal{T}}^*)$  solution to the LMI optimization (50).
8: if The problem is not feasible then
9:   END: (48) and (21) are not feasible.
10: end if
11: Define  $\delta_{\mathcal{T}}^* = \mu (p_{\mathcal{T}}^* \|A\| \gamma e^{\beta T_M})^{-1}$ ; ▷ Define  $\delta_{\mathcal{T}}^*$  as in (52).
12: Define  $\mathcal{T}_d = [T_m : 2\delta_{\mathcal{T}}^* : T_M]$ ; ▷ Generate  $\mathcal{T}_d$  in (51) as a set equally spaced values with step  $2\delta_{\mathcal{T}}^*$ .
13: for  $j$  from 1 to  $\text{length}(\mathcal{T}_d)$  do
14:    $\text{mineigs}(j, 1) = \lambda_m(\Xi_{P_{\mathcal{T}}^*}(\mathcal{T}_d(j)))$ ;  $\text{mineigs}(j, 2) = \mathcal{T}_d(j)$ ; ▷ For each  $\tau \in \mathcal{T}_d$  store  $(\lambda_m(\Xi_{P_{\mathcal{T}}^*}(\tau)), \tau)$ .  $\Xi_{P_{\mathcal{T}}^*}(\cdot)$  is defined in (21) with  $P = P_{\mathcal{T}}^*$ .
15: end for
16: if  $\text{mineigs}(j, 1) > \mu \forall j$  then
17:   END:  $P_{\mathcal{T}}^*$  solves (21); ▷ If all the minimum eigenvalues are larger than  $\mu$ , then  $P_{\mathcal{T}}^*$  is a solution to (21).
18: else
19:    $\bar{k} \in \arg \min_j (\text{mineigs}(j, 1))$ ;  $\bar{\tau} = \text{mineigs}(\bar{k}, 2)$ ; ▷ Locate a worst-case value of  $\tau \in \mathcal{T}_d$ .
20:    $constr = constr + \text{set}(\Xi_P(\bar{\tau}) > 2\mu I_{2n-m})$  ▷ A new constraint is included in (50) by adding  $\bar{\tau}$  to set  $\mathcal{T}$ .
21: end if

```

---

where  $\mathcal{T}_d \subset [T_m, T_M]$  contains an ordered set of scalars  $T_m = \tau_1 < \tau_2 < \dots < \tau_{\nu^*} = T_M$  satisfying:

$$\tau_{k+1} - \tau_k \leq 2\delta_{\mathcal{T}}^* := \frac{2\mu}{p_{\mathcal{T}}^* \|A\| \gamma e^{\beta T_M}}, \quad \forall k = 1, \dots, \nu^* - 1. \quad (52)$$

Finally, if this analysis phase is successful, then the algorithm stops and returns  $P_{\mathcal{T}}^*$  as a solution to (21). Otherwise, a value:

$$\bar{\tau} \in \arg \min_{\tau \in \mathcal{T}_d} (\lambda_m(\Xi_{P_{\mathcal{T}}^*}(\tau))) \quad (53)$$

is added to the set  $\mathcal{T}$ , and the algorithm restarts from the synthesis phase.

A useful property of this algorithm is reported below in Theorem 3, ensuring that if there exists a pair  $(P^*, p^*)$ , solution to (48), then the algorithm terminates successfully in a finite number of steps, thus providing a solution to (21).

For stating Theorem 3, the following useful results are presented. The first result is a straightforward consequence of standard Lyapunov theory applied to the linear time-invariant systems  $\dot{x} = -Ax$ . The second result is proven after Theorem 3 to avoid breaking the flow of the exposition.

**Lemma 3:** For any square matrix  $A$ , there exist a symmetric positive-definite matrix  $\Pi > 0$  and a scalar  $\beta \geq 0$  satisfying (49). Moreover for any values of  $\Pi$  and  $\beta$ , the following holds:

$$\|e^{(-A\tau)}\| \leq \gamma e^{\beta\tau} := \sqrt{\frac{\lambda_M(\Pi)}{\lambda_m(\Pi)}} e^{\beta\tau}, \quad \forall \tau \geq 0. \quad (54)$$

**Lemma 4:** Consider a symmetric matrix  $0 < P \leq pI$ , a constant  $\mu > 0$  and a value  $\bar{\tau} \in [T_m, T_M]$  such that:

$$\Xi_P(\bar{\tau}) > 2\mu I. \quad (55)$$

If  $\gamma$  and  $\beta$  are chosen as in Lemma 3, the following holds:

$$\Xi_P(\tau) > \mu I, \quad \forall \tau \in [\bar{\tau} - \delta, \bar{\tau} + \delta], \quad (56)$$

as long as  $\delta$  satisfies:

$$\delta \leq \mu (p \|A\| \gamma e^{\beta T_M})^{-1}. \quad (57)$$

Now the main theorem can be given. Note that by Lemma 2 assuming the existence of a solution to (48) is equivalent to assuming the existence of a solution to (21).

**Theorem 3:** If there exists a solution  $P^* \leq p^* I$  to problem (48), then the proposed algorithm terminates successfully at line 17, providing an output  $P_{\mathcal{T}}^*$ , after a finite number of iterations  $N$  satisfying:

$$N \leq (T_M - T_m)/\delta^*, \quad \delta^* = \frac{\mu}{p^* \|A\| \gamma e^{\beta T_M}}. \quad (58)$$

where  $(\gamma, \beta)$  are any solution to (49). Moreover such an output  $P_{\mathcal{T}}^*$  is a solution to the infinite-dimensional problem (21).

*Proof:* Consider the solution  $(P^*, p^*)$  to (48). For any finite set of points  $\mathcal{T}$  in (50), we have  $\mathcal{T} \subset [T_m, T_M]$ , therefore  $(P^*, p^*)$  is also solution to (50) for any selection of  $\mathcal{T}$ . As a consequence, for each  $\mathcal{T}$  we have that (50) is feasible and its solution  $(P_{\mathcal{T}}^*, p_{\mathcal{T}}^*)$  satisfies  $p_{\mathcal{T}}^* \leq p^*$ . Then, line 7 of the algorithm always gives a solution.

Consider now a solution  $(P_{\mathcal{T}}^*, p_{\mathcal{T}}^*)$  at some step of the algorithm iteration and note that for each  $\tau \in \mathcal{T}$  we have  $\Xi_{P_{\mathcal{T}}^*}(\tau) > 2\mu I$ . Then from Lemma 4 we have that inequality:

$$\Xi_{P_{\mathcal{T}}^*}(\tau) > \mu I, \quad \forall \tau \in \mathcal{T} + \delta^* \mathbb{B} \subset \mathcal{T} + \delta_{\mathcal{T}}^* \mathbb{B}, \quad (59)$$



where  $\delta^*$  given in (58), and  $\delta_{\mathcal{T}}^*$  given in (52), satisfy  $\delta^* \leq \delta_{\mathcal{T}}^*$  because  $p^* \geq p_{\mathcal{T}}^*$ .

Consider now the case  $T_M - T_m \leq \delta^*$ . Then  $\{T_M, T_m\} + \delta^*\mathbb{B}$  contains  $[T_M, T_m]$  and the theorem is proven with  $N = 1$  iterations. In the less trivial case when  $T_M - T_m > \delta^*$ , either the analysis step verifying (51) (see line 17 of the algorithm) is successful, or it identifies a new value  $\bar{\tau} \notin \mathcal{T} + \delta^*\mathbb{B}$  that is added to  $\mathcal{T}^+$  (the value of  $\mathcal{T}$  at the next synthesis step). The previous reasoning (together with  $T_M - T_m > \delta^*$ ) implies that  $\mathcal{T}$  only contains elements whose mutual distance is larger than  $\delta^*$ . Since  $\mathcal{T}$  increases by one element at each iteration, the algorithm must terminate successfully when  $\mathcal{T}$  has at most  $\frac{T_M - T_m}{\delta^*} + 1$  elements. Since  $\mathcal{T}$  has two elements at the first iteration, an upper bound on the number of iterations before termination is given by:

$$N = \left\lceil \frac{T_M - T_m}{\delta^*} + 1 - 2 \right\rceil < \frac{T_M - T_m}{\delta^*}, \quad (60)$$

where  $\lceil \cdot \rceil$  denotes the smallest integer upper bound of its argument.

When the algorithm stops, it provides a matrix  $P_{\mathcal{T}}^*$  satisfying (51), (52). Then Lemma 4 with  $\delta = \delta_{\mathcal{T}}^*$  and (51), (52) imply that  $\Xi_{P_{\mathcal{T}}^*}(\tau) > 0 \ \forall \tau \in \mathcal{T} + \delta_{\mathcal{T}}^*\mathbb{B} \subset [T_m, T_M]$ , where the last inclusion follows from comparing (52) and (57). As a consequence inequality  $\Xi_{P_{\mathcal{T}}^*}(\tau) > 0$  is satisfied for all  $\tau$  in  $[T_m, T_M]$ , which implies (21) with  $P = P_{\mathcal{T}}^*$ .  $\square$

*Remark 3:* If no solution  $P$  exists to (21), then either the algorithm terminates at step 10 with a certified infeasibility (because infeasibility with  $\tau \in [T_m, T_M]$  implies infeasibility with  $[T_m, T_M]$ ), or it runs indefinitely, eventually meeting numerical problems. A stopping condition could be imposed by adding an extra well-conditioning constraint  $P \leq \bar{p}I$  to (21), (48) and (50), for some reasonably large  $\bar{p} \in \mathbb{R}_{\geq 0}$ . Then the algorithm would be guaranteed to terminate successfully whenever a solution to (21) with  $P \leq \bar{p}I$  exists and to terminate negatively when such a solution does not exist.  $\dashv$

*Remark 4:* In our preliminary work [25] we proposed a simple discretization algorithm to get an appropriate solution to (21). Instead Theorem 3 certifies that whenever (21) is feasible, Algorithm 1 provides an exact solution to (21).  $\dashv$

*Proof of Lemma 4.* Since matrix  $\Xi_P(\bar{\tau})$  is symmetric positive definite, from [30, Corollary 2.5.11] it is possible to decompose it as:

$$\Xi_P(\bar{\tau}) = \bar{N}\bar{\Lambda}\bar{N}^\top, \quad \bar{N}^\top\bar{N} = I, \quad (61)$$

where  $\bar{\Lambda} = \text{diag}(\bar{\lambda}_1, \dots, \bar{\lambda}_n)$ ,  $\bar{N} = [\bar{v}_1, \dots, \bar{v}_n]$  contain, respectively, the eigenvalues and an orthonormal set of eigenvectors of  $\Xi_P(\bar{\tau})$ . Under these conditions, and based on the fact that the eigenvalues  $\lambda_i(\tau)$  of  $\Xi_P(\tau)$  are continuous functions of  $\tau$ , in [31, Eq. 1.3] it is shown that the first order derivatives of the eigenvalues  $\lambda_i$  are:

$$\frac{\partial \lambda_i(\bar{\tau})}{\partial \tau} = \bar{v}_i^\top \frac{\partial \Xi_P(\bar{\tau})}{\partial \tau} \bar{v}_i, \quad (62)$$

where, in our case,

$$\frac{\partial \Xi_P(\bar{\tau})}{\partial \tau} = \begin{bmatrix} (C^\perp)^\top e^{(-\bar{A}^\top \bar{\tau})} (-\bar{A}^\top P - P \bar{A}) e^{(-\bar{A} \bar{\tau})} C^\perp & 0 \\ 0 & 0 \end{bmatrix}. \quad (63)$$

Therefore, taking into consideration the fact that  $\{\bar{v}_i\}$  is an orthonormal set, we obtain from (62):

$$\left| \frac{\partial \lambda_i(\bar{\tau})}{\partial \tau} \right| = \left| \bar{v}_i^\top \frac{\partial \Xi_P(\bar{\tau})}{\partial \tau} \bar{v}_i \right| \leq 2 \|P\| \|A\| \|e^{(-A\bar{\tau})}\|, \quad (64)$$

where we used the sub-multiplicativity of the norm, and the fact that  $\|C^\perp\| = 1$ . Based on bounds (49) and (54), and on the assumption that  $P \leq pI$ , inequality (64) implies:

$$\left| \frac{\partial \lambda_i(\bar{\tau})}{\partial \tau} \right| \leq 2p \|A\| \gamma e^{\beta \bar{\tau}} \leq 2p \|A\| \gamma e^{\beta T_M}. \quad (65)$$

where we used  $\bar{\tau} \in [T_m, T_M]$ .

If inequality (55) is satisfied, then  $\lambda_m(\Xi_P(\bar{\tau})) \geq 2\mu$ . Therefore, from (65), the minimum eigenvalue of  $\Xi_P(\tau)$  cannot be less than  $\mu$  as long as:

$$\tau \in [\bar{\tau} - \mu(p \|A\| \gamma e^{\beta T_M})^{-1}, \bar{\tau} + \mu(p \|A\| \gamma e^{\beta T_M})^{-1}].$$

## V. NUMERICAL EXAMPLE

Consider system (1)-(2) with the following data:

$$A = \begin{bmatrix} -0.02 & -1.4 & 9.8 \\ -0.01 & -0.4 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 9.8 \\ 6.3 \\ 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}^\top, \quad (66)$$

where  $\text{eig}(A) = \{-0.656, 0.118 + 0.368i, 0.118 - 0.368i\}$ .

For this selection we fix  $C^\perp = \begin{bmatrix} 0 & 1 & 0 \\ 0.7071 & 0 & -0.7071 \end{bmatrix}^\top$ .

Algorithm 1 is applied for three different choices of  $[T_m, T_M]$ . In particular for  $\tau \in [1, 3]$ ,  $\tau \in [3, 4]$  and  $\tau \in [4, 8]$ . In the first case,  $\tau \in [T_m, T_M] = [1, 3]$ , the algorithm finds a solution with only one iteration, as shown in Figure 1(a). The value of  $P$ , solution to the problem is:

$$P = \begin{bmatrix} 0.8334 & -0.0041 & 0.8333 \\ -0.0041 & 2.0116 & -0.0359 \\ 0.8333 & -0.0359 & 0.8341 \end{bmatrix}. \quad (67)$$

In a second case we select  $\tau \in [T_m, T_M] = [3, 4]$  and the algorithm does not find a solution because the periodically sampled plant is not observable for  $\tau^* = 3.425 \in [3, 4]$ . This fact is clear looking at Figure 2 where the minimum singular values of the observability matrix

$$\mathcal{O}(\tau) = \begin{bmatrix} C & Ce^{A\tau} & \dots & C(e^{A\tau})^{n-1} \end{bmatrix}^\top$$

are shown. Moreover it is confirmed by Figure 1(b), where it is shown that, after four iterations, the minimum eigenvalue of matrix (21) is always negative in a neighborhood of  $\tau = 3.425$ . Finally, in the last case, we select  $\tau \in [T_m, T_M] = [4, 8]$ , and the algorithm finds a solution with two iterations, as shown in Figure 1(c). The value of  $P$ , solution to the problem is:

$$P = \begin{bmatrix} 0.9573 & -0.0031 & 0.9571 \\ -0.0031 & 2.2754 & -0.0122 \\ 0.9571 & -0.0122 & 0.9573 \end{bmatrix}. \quad (68)$$

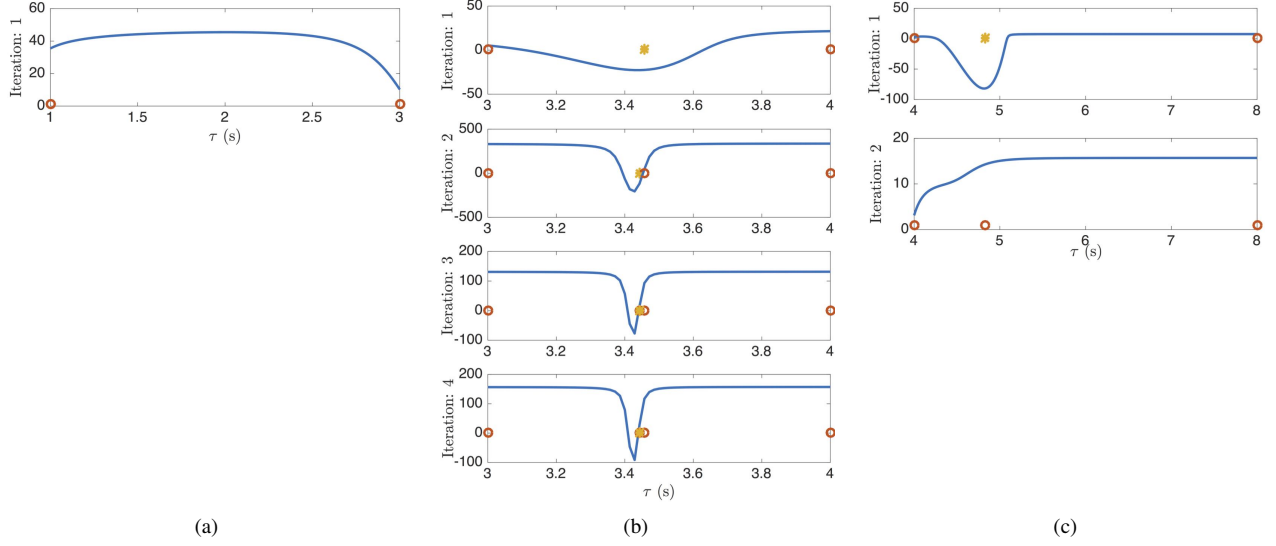


Figure 1. Minimum eigenvalues of matrix (21), computed at line 14 of Algorithm 1 for:  $\tau \in [1, 3]$  (a),  $\tau \in [3, 4]$  (b) and  $\tau \in [4, 8]$  (c).

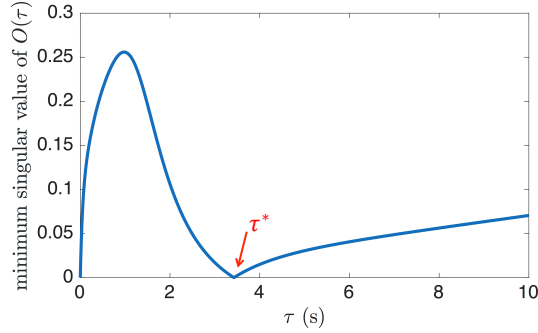


Figure 2. Minimum singular values of the observability matrix  $\mathcal{O}(\tau)$  for the periodically sampled plant.

### A. Simulation results

Initially, the unstable plant (66) has been stabilized by means of a state feedback using a low gain:

$$K_u = 10^{-2} [0.16 \quad 5.47 \quad -0.01],$$

such that:

$$\text{eig}(A + BK_u) = \{-0.01, -0.02, -0.03\}.$$

The corresponding slow transient ensures that the signals do not blow up during the simulation, but they are associated to a sufficiently rich behavior.

The dynamics expressed in (4) has been implemented together with the observer (5) in the MATLAB®-Simulink environment. The gain  $K(\tau)$  is computed on-line according to (25) by using matrix  $P$  in (67) for  $\tau \in [1, 3]$  and  $P$  in (68) for  $\tau \in [4, 8]$ . Moreover, in order to implement a random value of the time-instant of the measurements, we implement the following modified error dynamics, corresponding to (6) with random selection of the inter-measurement intervals:

$$\begin{cases} \dot{e} = Ae, \\ \dot{\tau} = 1, \\ \dot{\tau}_r = -1, \end{cases} \quad \tau_r \in [T_M, 0], \quad (69a)$$

$$\begin{cases} e^+ = (I - K(\tau)C)e, \\ \tau^+ = 0, \\ \tau_r^+ = T_m + (T_M - T_m)\nu^+, \end{cases} \quad \tau_r = 0. \quad (69b)$$

where  $\nu^+$  is a random variable uniformly distributed in the interval  $[0, 1]$ . This modified dynamics is inspired, with the same notation, by [32].

In Figure 3 the real and estimated state vector components  $x_i$ ,  $\hat{x}_i$ ,  $i = 1, 2, 3$ , as well as estimation errors  $e_i = x_i - \hat{x}_i$ ,  $i = 1, 2, 3$ , are shown, during the first test with  $\tau \in [1, 3]$ . Moreover, for the same test, the waveforms of the Lyapunov function  $V$ , of the variables  $\tau$  and  $\tau_r$  and of the output error  $y - \hat{y}$  are shown in Figure 4.

From Figure 3 it is evident that the estimated variables track very well the corresponding state variables and all the errors go to zero asymptotically. Moreover, it is possible to note the impulsive behavior of the estimate especially during the initial transient. From Figure 4 we note that the Lyapunov function is constant during flow, and decreases across jumps, as expected from the theoretical results (Lemma 1). Finally, from the waveforms of  $\tau$  and  $\tau_r$  we see that the jumps occur randomly in the interval  $[1, 3]$  according to the described dynamics.

Figures 5-6 show the results for the same test described above, but when the measurements are provided more sporadically,  $\tau \in [4, 8]$ . In this case the same comments given for the first test can be provided, confirming the effectiveness of the proposed approach. Obviously, the convergence rate in this case is slower because the measurements are accessible less frequently.

## VI. CONCLUSION

In this work an observer with a time-varying output error injection has been proposed for a linear continuous-time plant with asynchronous sampled measurements. In particular some theoretical tools have been provided, in terms of LMIs, certifying asymptotic stability of a certain compact set where the estimation error is zero. Two solutions have been proposed,

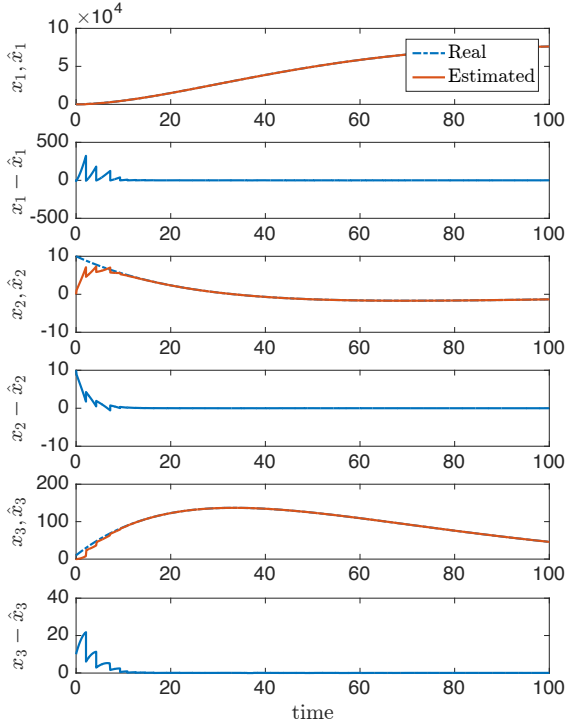


Figure 3. Real and estimated state vector components  $x_i, \hat{x}_i, i = 1, 2, 3$ , as well as estimation errors  $x_i - \hat{x}_i, i = 1, 2, 3$ , during a test with  $\tau \in [1, 3]$ .

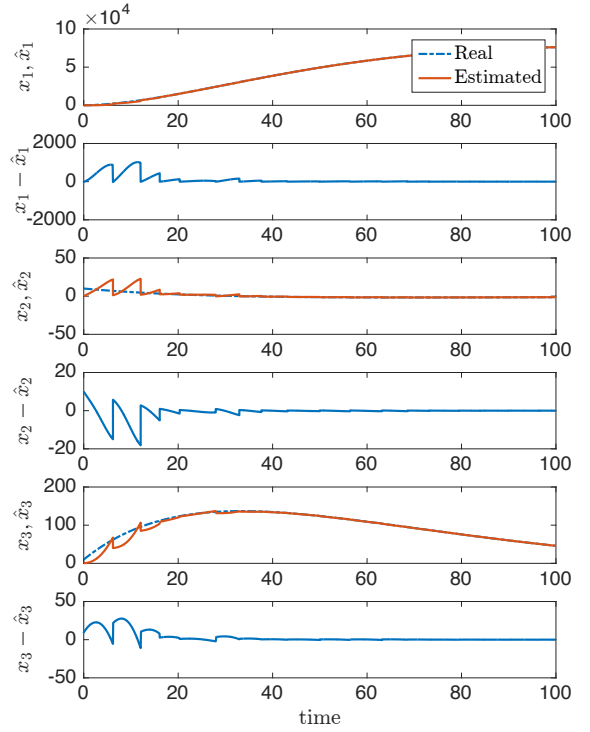


Figure 5. Real and estimated state vector components  $x_i, \hat{x}_i, i = 1, 2, 3$ , as well as estimation errors  $x_i - \hat{x}_i, i = 1, 2, 3$ , during a test with  $\tau \in [4, 8]$ .

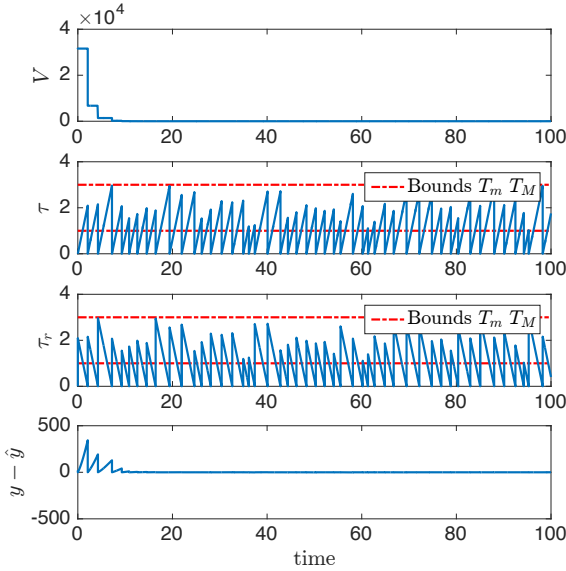


Figure 4. Waveforms of the Lyapunov function  $V$ , of the variables  $\tau$  and  $\tau_r$  and of the output error  $y - \hat{y}$ , during a test with  $\tau \in [1, 3]$ .

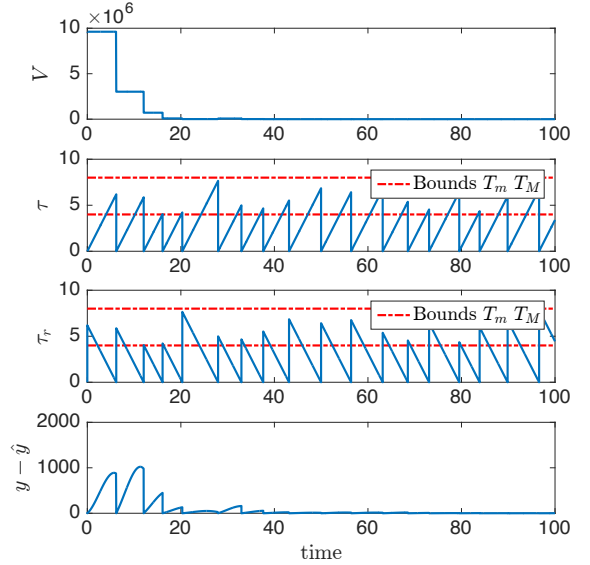


Figure 6. Waveforms of the Lyapunov function  $V$ , of the variables  $\tau$  and  $\tau_r$  and of the output error  $y - \hat{y}$ , during a test with  $\tau \in [4, 8]$ .

one under the restrictive assumption that the output matrix is invertible, and one for the more general case of a detectable pair, under the assumption that some LMI conditions hold. Moreover, necessary conditions for the feasibility of those LMI have been established. Since the proposed time-varying observer is based on a solution to an infinite-dimensional LMI, a numerical algorithm has been introduced which is guaranteed to converge after a finite number of iterations to

a solution to the infinite dimensional problem whenever one exists. The results provided by a numerical example show the effectiveness of the proposed approach, confirming the theoretical results and the feasibility of the proposed numerical solution.

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