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► **To cite this version:**

Antonino Sferlazza, Sophie Tarbouriech, Luca Zaccarian. Time-varying Sampled-data Observer with Asynchronous Measurements. [Research Report] Rapport LAAS n° 17058, Department of Energy Information engineering and Mathematical models, University of Palermo, Viale delle Scienze, 90128 Palermo, Italy; CNRS, LAAS, 7 av. du Colonel Roche, F-31400, Toulouse, France. 2017. hal-01488636v1

HAL Id: hal-01488636

<https://hal.science/hal-01488636v1>

Submitted on 13 Mar 2017 (v1), last revised 6 Dec 2017 (v2)

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Time-varying Sampled-data Observer with Asynchronous Measurements (Extended Version)

Antonino Sferlazza, *Member IEEE*, Sophie Tarbouriech, *Member IEEE*, Luca Zaccarian, *Fellow Member IEEE*.

Abstract—In this paper a time-varying observer for a linear continuous-time plant with asynchronous sampled measurements is proposed. The observer is contextualized in the hybrid systems framework providing an elegant setting for the proposed solution. In particular some theoretical tools are provided, in terms of LMIs, certifying asymptotic stability of a certain compact set where the estimation error is zero. We consider sampled asynchronous measurements that occur at arbitrary times in a certain window with an upper and lower bound. The design procedure, that we propose for the selection of the time-varying gain, is based on a constructive algorithm that is guaranteed to find a solution to an infinite-dimensional LMI whenever a feasible solution exists. Finally a numerical example shows the effectiveness of the proposed approach.

Index Terms—Sample data observer, discrete asynchronous measurements, hybrid systems, linear systems, linear matrix inequalities.

I. INTRODUCTION

In the last years, the design of observers for systems with sampled measurements has received great attention. This interest is motivated by many engineering applications, such as sampled-data systems, quantized systems, networked systems, localization of mobile vehicles, etc. [1], [2], [3]. In these cases, the output is available only at sampling instants, and, for this reason, classical observer structures cannot be used.

This problem is not new in control engineering and there are many works in the literature dealing with these issues, providing several solutions. In particular this problem has been considered in a stochastic framework, and particular Kalman filters have been developed for these purposes. For example in [4], a Kalman filter with intermittent observations is developed starting from the discrete Kalman filtering formulation, and modeling the input of the observation as a random process. A similar approach is followed in [5] where the observations are available according to a Bernoulli process. Further convergence analysis and boundedness analysis on the estimation error have been recently analyzed in [6] and [7]. Other examples are given in [8] where the intermittent observations are considered in the development of an unscented Kalman

filter. Another stochastic approach, is proposed in [9], where an intrinsic filtering on the special orthogonal group $SO(3)$ is shown. Here the problem of the continuous-time dynamics with discrete measurements is cast into a rigorous stochastic and geometric framework.

A deterministic approach has been followed in other works. For example an interesting approach is proposed in [10] where the finite time convergence of an observer is proven for linear systems with sampled measurements, and subsequently in [11] and [12], where a similar method has been proposed for nonlinear Lipschitz systems with sampled measurements. This approach has been extended in [13] by means of new conditions in terms of linear matrix inequalities (LMIs). Moreover similar types of observers has been developed, such as in [14], where the same category of systems have been treated, but obtaining conditions in different form, or in [15] and [16], where nonlinear uniformly observable single output systems [17] are addressed. Finally, by using Lyapunov tools adapted to impulsive systems, some classes of systems with both sampled and delayed outputs are addressed in [18] and [19].

Recently, different approaches have been proposed using the hybrid system formalism of [20]. The use of the hybrid formalism provides a natural setting for the modeling of this type of observers, where both continuous-time and discrete-time dynamics coexist. Indeed a sampled-data observer can be modeled by a "flow map", which describes the continuous-time dynamics when the measurement is not available, while the measurement can be considered as a discrete event and can be modeled by a suitable "jump map". This kind of formalism is applied in [21], [22], where the estimation of the state of a linear time invariant systems is proposed, with asynchronous measurements and a constant output error injection gain. In the same context [23] proposes a hybrid observer for linear systems producing an estimate that converges to the plant state in finite time. These concepts have also been applied to distributed systems such as in [24], where the problem of estimating the state of a linear time-invariant plant is addressed in a distributed fashion over networks allowing only intermittent transmission of information.

In this paper we address the presence of asynchronous sampled measurements for continuous-time plants using a hybrid formalism. Differently from the other papers presented in the literature, a design procedure based on a constructive solution to an infinite-dimensional LMI is given, leading to a time-varying observer gain. In particular some theoretical tools are provided, in terms of LMIs, certifying asymptotic

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stability of a certain compact set where the estimation error is zero. Moreover, necessary conditions for the existence of the proposed observer are also given. Since the proposed hybrid time-varying observer is based on a solution to an infinite-dimensional LMI, a solution algorithm is proposed, which is proven to find a solution to the infinite-dimensional problem in a finite number of iterations, whenever the problem admits one.

Preliminary results in the directions of this paper have been presented in [25]. Here, as compared to [25], we provide proofs, prove an insightful necessary condition, which is also sufficient in some special cases. We also reformulate completely the numerical algorithm, which is here proven to always lead to a solution, whenever the (infinite-dimensional) observer design conditions are feasible.

The paper is organized as follows. In Section II the problem statement is formalized by fitting our problem in the hybrid systems framework. Then in Section III an analysis result is presented certifying asymptotic stability of a certain compact set where the estimation error is zero. Subsequently two possible time-varying gain selections are proposed, one under the restrictive assumption that the output matrix is invertible, and one for the more general case of a detectable pair, under the assumption that some LMI conditions hold. Moreover, necessary conditions for the existence of the proposed observer are given establishing an interesting link between the feasibility of the proposed construction and the detectability of the sampled plant. In Section IV a numerical algorithm is proposed in order to solve the infinite-dimensional LMIs given in Section III in a finite number of steps. In Section V a numerical example is given in order to show the effectiveness of the proposed approach. Finally the conclusion are given in Section VI.

II. PROBLEM STATEMENT

In this work we consider a class of systems described by the following equation:

$$\dot{x} = Ax + Bu, \quad (1)$$

where $x \in \mathbb{R}^n$ is the state of the system, $u : [0, \infty) \rightarrow \mathbb{R}^q$ is a known input that belongs to the class of locally bounded measurable functions, $A \in \mathbb{R}^{n \times n}$, and $B \in \mathbb{R}^{n \times q}$. Let us assume that an output of system (1) is accessible at discrete instants of time, resulting in a sequence of m dimensional vectors y_k , $k \in \mathbb{Z}_{\geq 1}$ defined as:

$$y_k := Cx(t_k), \quad (2)$$

where $C \in \mathbb{R}^{m \times n}$ is full row rank and t_k , $k \in \mathbb{Z}_{\geq 1}$, is a sequence of increasing non-negative real numbers that satisfies the following assumption:

Assumption 1: There exist scalars T_m and T_M , with $0 < T_m \leq T_M$, such that:

$$T_m \leq |t_{k+1} - t_k| \leq T_M, \quad \forall k \in \mathbb{Z}_{\geq 1}. \quad (3)$$

Assumption 1 requires that we are considering the case of asynchronous discrete-time measurements with a sampling interval lower and upper bounded by two known positive constants T_m and T_M . Note that T_m must be strictly greater than

zero to avoid Zeno behaviors in the hybrid model developed below.

Taking inspiration from the hybrid systems formalism of [20], it is possible to represent the sampled-data system associated with this setting as follows:

$$\begin{cases} \dot{x} = Ax + Bu, \\ \dot{\tau} = 1, \end{cases} \quad (x, \tau) \in \mathcal{C}_x := \mathbb{R}^n \times [0, T_M], \quad (4a)$$

$$\begin{cases} x^+ = x, \\ \tau^+ = 0, \end{cases} \quad (\hat{x}, \tau) \in \mathcal{D}_x := \mathbb{R}^n \times [T_m, T_M], \quad (4b)$$

$$y = Cx, \quad (4c)$$

where the variable τ is a timer keeping track of the elapsed time since the last sample, and the impulsive nature of the available measurement is represented by the extra property that output y is only available at jump times. With model (4), one readily sees that for any sequence y_k in (2), satisfying (3), there exists a solution to (4) such that $y_k = y(t_k, k)$, $k \in \mathbb{Z}_{\geq 1}$.

Constraining the jump set to be included in the set where $\tau \in [T_m, T_M]$ ensures that Assumption 1 is verified as clarified in the next statement, which is a corollary of [26, Props 1.1 & 1.2, page 747].

Proposition 1: Consider any solution to (4) and denote by t_k , $k \in \mathbb{Z}_{\geq 1}$, its jump times. Then the sequence t_k satisfies Assumption 1. Moreover, consider any sequence t_k , $k \in \mathbb{Z}_{\geq 1}$, satisfying Assumption 1. For each $x_0 \in \mathbb{R}^n$ there exists $\tau_0 \in [0, T_M]$ such that a solution ϕ to (4) with $\phi(0, 0) = (x_0, \tau_0)$ has jump times coinciding with t_k , $k \in \mathbb{Z}_{\geq 1}$.

In this paper we propose an observer whose structure implicitly complies with the restriction specified in Assumption 1 on the available output. Our observer is capable of providing an asymptotic estimate of the plant state, regardless of the sequence of times t_k at which the sampled output is available. The hybrid structure of the proposed observer is the following:

$$\begin{cases} \dot{\hat{x}} = A\hat{x} + Bu, \\ \hat{x}^+ = \hat{x} + K(\tau)(y - C\hat{x}), \end{cases} \quad (\hat{x}, x, \tau) \in \mathbb{R}^n \times \mathcal{C}_x, \quad (5)$$

where the matrix function $K : [T_m, T_M] \rightarrow \mathbb{R}^{n \times m}$ corresponds to the time-varying gain of the observer responsible for the *discrete* output injection term. It is clear that with dynamics (5) and due to Proposition 1 output y is only used at the sampling instants t_k compliant with Assumption 1.

The design of the time-varying gain $K(\cdot)$ will be performed in the next section. Note that as compared to a standard LTI Luenberger architecture (such as the one used in [21]), observer (5) is based on an injection term that depends on the elapsed time since the last measurement. Such an elapsed time is known to the observer by way of state τ in (4).

III. STABILITY CONDITIONS GAIN SELECTION

One of the main goals of this work is to give design rules to select the gain function $K(\cdot)$ in (5) such that the estimation error $e := x - \hat{x}$ converges asymptotically to zero. Such a property is well characterized in terms of the stability of the following error dynamics, issued from (4)-(5):

$$\begin{cases} \dot{e} = Ae, \\ \dot{\tau} = 1, \end{cases} \quad (e, \tau) \in \mathcal{C} := \mathbb{R}^n \times [0, T_M], \quad (6a)$$

$$\begin{cases} e^+ = (I - K(\tau)C)e, \\ \tau^+ = 0, \end{cases} \quad (e, \tau) \in \mathcal{D} := \mathbb{R}^n \times [T_m, T_M]. \quad (6b)$$

We first present an analysis result certifying asymptotic stability of the compact set:

$$\mathcal{A} := \left\{ (e, \tau) : e = 0, \tau \in [0, T_M] \right\}, \quad (7)$$

corresponding to the set where the estimation error is zero.

Note that Lemma 1 below is an extension of Theorem 1 in [21] to the case of time-varying injection gain $K(\cdot)$.

Lemma 1: Assume that there exists a matrix $P = P^\top > 0$, and a continuous matrix function $\tau \rightarrow K(\tau)$ such that:

$$\begin{bmatrix} e^{(-A^\top \tau)} P e^{(-A\tau)} & P \\ P & P \end{bmatrix} > \begin{bmatrix} 0 & \star \\ PK(\tau)C & 0 \end{bmatrix}, \quad \forall \tau \in [T_m, T_M]. \quad (8)$$

Then set \mathcal{A} in (7) is uniformly globally asymptotically stable¹ (UGAS) for the error dynamics in (6).

Proof: Consider the Lyapunov function:

$$V(e, \tau) = e^\top e^{(-A^\top \tau)} P e^{(-A\tau)} e, \quad (9)$$

and observe that there exist positive scalars c_1 and c_2 satisfying:

$$c_1 |e|^2 \leq V(e, \tau) \leq c_2 |e|^2, \quad \forall e \in \mathbb{R}^n, \tau \in [0, T_M], \quad (10)$$

where, denoting by $\lambda_m(S)$ and $\lambda_M(S)$ the minimum and the maximum eigenvalues of symmetric matrix S , respectively, we selected:

$$c_1 := \min_{\tau \in [0, T_M]} \lambda_m \left(e^{(-A^\top \tau)} P e^{(-A\tau)} \right), \quad (11a)$$

$$c_2 := \max_{\tau \in [0, T_M]} \lambda_M \left(e^{(-A^\top \tau)} P e^{(-A\tau)} \right), \quad (11b)$$

which are well defined and positive, from positive definitiveness of P and invertibility of $e^{(-A\tau)}$.

The variation of V along flowing solutions of (6) is:

$$\begin{aligned} \dot{V}(e, \tau) &:= \\ & 2e^\top e^{(-A^\top \tau)} P e^{(-A\tau)} \dot{e} + e^\top (-A^\top) e^{(-A^\top \tau)} P e^{(-A\tau)} e \\ & + e^\top e^{(-A^\top \tau)} P e^{(-A\tau)} (-A) e = \\ & e^\top e^{(-A^\top \tau)} P e^{(-A\tau)} (2Ae - Ae - Ae) = 0, \\ & \quad \forall (e, \tau) \in \mathbb{R}^n \times [0, T_M]. \end{aligned} \quad (12)$$

The variation of V across jumping solutions of (6) is:

$$\begin{aligned} \Delta V(e, \tau) &:= V(e^+, \tau^+) - V(e, \tau) = \\ & e^\top (I - K(\tau)C)^\top P (I - K(\tau)C) e - e^\top e^{(-A^\top \tau)} P e^{(-A\tau)} e = \\ & - e^\top \underbrace{\left(e^{(-A^\top \tau)} P e^{(-A\tau)} - (I - K(\tau)C)^\top P (I - K(\tau)C) \right)}_{M(\tau)} e, \\ & \quad \forall (e, \tau) \in \mathbb{R}^n \times [T_m, T_M], \end{aligned} \quad (13)$$

where we recall that $\tau \in [T_m, T_M]$ for all $(e, \tau) \in \mathcal{D}$. Consider now (8), which implies:

$$\begin{bmatrix} e^{(-A^\top \tau)} P e^{(-A\tau)} & \star \\ P(I - K(\tau)C) & P \end{bmatrix} > 0 \quad \forall \tau \in [T_m, T_M], \quad (14)$$

and after a Schur-complement:

$$e^{(-A^\top \tau)} P e^{(-A\tau)} - (I - K(\tau)C)^\top P (I - K(\tau)C) > 0,$$

namely $M(\tau) > 0$, $\forall \tau \in [T_m, T_M]$. Define the positive scalar c_3 as follows:

$$c_3 := \min_{\tau \in [T_m, T_M]} \lambda_m \left(M(\tau) \right). \quad (15)$$

Then one gets from (13):

$$\Delta V(e, \tau) \leq -c_3 |e|^2, \quad \forall (e, \tau) \in \mathcal{D}. \quad (16)$$

The proof is completed by first noting that, for attractor \mathcal{A} in (7), $|(e, \tau)|_{\mathcal{A}} = |e|$, and then exploiting the fact that solutions to (6) are persistently jumping at least every T_M ordinary time. Indeed, for each solution ϕ and for each $(t, j) \in \text{dom } \phi$, it is immediate to check that $j \geq \frac{t}{T_M} - 1$.

Then uniform global asymptotic stability of \mathcal{A} follows from (10), (12), (16) and [20, Proposition 3.24] with $N_r = 1$ and $\gamma_r(t) = \frac{t}{T_M}$. \square

Based on the analysis result of Lemma 1, we can now prove a few relevant constructions for the gain $K(\cdot)$, corresponding to a few special cases. The first case is relatively straightforward and corresponds to the case where C is invertible (namely the state is completely accessible at the sampling instants). This case is somewhat interesting because it corresponds to the source of inspiration of the subsequent construction, and has been used in a dedicated application by the first author in [27]. It is reported below.

Theorem 1: If C is invertible, then for any $P = P^\top > 0$ and any $\lambda \in [0, 1)$, inequality (8) is satisfied with:

$$K(\tau) = \left(I - \lambda e^{(-A\tau)} \right) C^{-1}, \quad (17)$$

which then guarantees UGAS of \mathcal{A} for system (6).

Proof: By virtue of selection (17), we have

$$P(\tau)K(\tau)C = P(I - \lambda e^{(-A\tau)}). \quad (18)$$

Then condition (8) in Lemma 1, becomes:

$$\begin{bmatrix} e^{(-A^\top \tau)} P e^{(-A\tau)} & \lambda e^{(-A^\top \tau)} P \\ \lambda P e^{(-A\tau)} & P \end{bmatrix} > 0 \iff \begin{bmatrix} P & \lambda P \\ \lambda P & P \end{bmatrix} > 0. \quad (19)$$

The last one is always verified for $0 \leq \lambda < 1$, and for any positive definite matrix P . \square

Remark 1: The effect of selection (17) in Theorem 1 on the jump equation (6b) of the error dynamics is insightful in terms of the selection of constant λ in (17). In particular, using (17) in (6b), we obtain:

$$e^+ = \lambda e^{(-A\tau)} e, \quad (20)$$

which clearly reveals that the choice $\lambda = 0$ leads to a dead-beat controller, while the choice $\lambda = 1$ leads to a nontrivial reset that resets back the estimation error to the value that it had

¹For a definition of uniform global asymptotic stability of a compact attractor for a hybrid system, see [20, Ch 3].

immediately after the previous sample (this fact is evident by keeping in mind the explicit expression of the error $e(t, k) = e^{(A\tau)}e(t_k, k)$ for all $t \in [t_k, t_{k+1}]$). Clearly, the choice $\lambda = 1$ is not allowed in our result because it leads to a bounded, but non converging, response. \lrcorner

The solution of Theorem 1 is only viable under demanding conditions on the available measurements, that are only seldom verified. Due to this reason, one of the main contributions of this paper is the result given next, which provides a construction for the gain $K(\cdot)$ as long as one can find a constant matrix P satisfying the following infinite set of matrix inequalities:

$$\Xi_P(\tau) := \begin{bmatrix} (C^\perp)^\top e^{(-A^\top \tau)} P e^{(-A\tau)} C^\perp & \star \\ PC^\perp & P \end{bmatrix} > 0, \quad \forall \tau \in [T_m, T_M], \quad (21)$$

where C^\perp denotes the orthogonal complement of C^\top , namely a matrix such that $[C^\perp \ C^\top]$ is square and nonsingular, and such that $CC^\perp = 0$.

Matrix inequality (21) is not easy to solve, but we discuss in Section IV a numerical algorithm that is able to perform this task. We report below the explicit expression of $K(\cdot)$, which induces UGAS of attractor \mathcal{A} for the observation error dynamics, as long as (21) is satisfied.

Theorem 2: Assume C is full row rank and denote by C^\perp a basis of the orthogonal complement of C^\top . If there exists $P = P^\top > 0$ satisfying (21), then selection:

$$K(\tau) := \left(C^\top - C^\perp \left((C^\perp)^\top e^{(-A^\top \tau)} P e^{(-A\tau)} C^\perp \right)^{-1} \left(C e^{(-A^\top \tau)} P e^{(-A\tau)} C^\perp \right)^\top \right) (CC^\top)^{-1}, \quad (22)$$

guarantees UGAS of \mathcal{A} for system (6).

Proof: The proof is divided into two parts. In the first part we show that condition (21) is enough to ensure the existence of a gain $K(\cdot)$ such that (8) is satisfied. In the second part we show that given a matrix P satisfying condition (21), then (8) is satisfied for the gain $K(\cdot)$ selected as in (22). Then the result follows from Lemma 1.

Part 1 (Proof of the existence): In this first part we have to show that if (21) holds, then there exists $K(\cdot)$ (equivalently $Y(\cdot)$) such that following inequality holds:

$$\begin{bmatrix} \Psi(\tau) & \star \\ P + Y(\tau)C & P \end{bmatrix} := \begin{bmatrix} e^{(-A^\top \tau)} P e^{(-A\tau)} & \star \\ P - PK(\tau)C & P \end{bmatrix} > 0, \quad (23)$$

where we introduced $\Psi(\tau) := e^{(-A^\top \tau)} P e^{(-A\tau)}$ and $Y(\tau) := -PK(\tau)$. Equation (23) can be written as:

$$\underbrace{\begin{bmatrix} \Psi(\tau) & \star \\ P & P \end{bmatrix}}_{Q(\tau)} + \underbrace{\begin{bmatrix} 0 \\ I \end{bmatrix}}_H Y(\tau) \underbrace{\begin{bmatrix} C & 0 \end{bmatrix}}_{G^\top} + \begin{bmatrix} C^\top \\ 0 \end{bmatrix} Y(\tau) \begin{bmatrix} 0 & I \end{bmatrix} > 0. \quad (24)$$

Applying the elimination lemma (see, e.g. [28, Equations (2.27)-(2.28)]) for each τ there exists a matrix $Y(\tau)$ such that (24) is satisfied if and only if the following relations hold:

$$(H^\perp)^\top Q(\tau) (H^\perp) > 0, \quad (G^\perp)^\top Q(\tau) (G^\perp) > 0, \quad (25)$$

where $H^\perp = \begin{bmatrix} I \\ 0 \end{bmatrix}$ is a basis of the Kernel of H^\top , and $G^\perp = \begin{bmatrix} C^\perp & 0 \\ 0 & I \end{bmatrix}$ is a basis of the Kernel of G^\top .

Using the above relations, the left equation in (25) becomes:

$$\begin{bmatrix} I & 0 \end{bmatrix} \begin{bmatrix} \Psi(\tau) & \star \\ P & P \end{bmatrix} \begin{bmatrix} I \\ 0 \end{bmatrix} = \Psi(\tau) = e^{(-A^\top \tau)} P e^{(-A\tau)} > 0, \quad (26)$$

which is always satisfied because $P > 0$. Regarding the right equation in (25) we have:

$$\begin{bmatrix} (C^\perp)^\top & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \Psi(\tau) & \star \\ P & P \end{bmatrix} \begin{bmatrix} C^\perp & 0 \\ 0 & I \end{bmatrix} = \begin{bmatrix} (C^\perp)^\top \Psi(\tau) C^\perp & \star \\ PC^\perp & P \end{bmatrix} > 0, \quad (27)$$

which is satisfied by hypothesis (21). This means that if condition (21) is satisfied, then there exists $Y(\tau)$ (and consequently a matrix gain $K(\tau) = -P^{-1}Y(\tau)$) such that (24) (therefore (8)) is satisfied.

Part 2 (Selection of $K(\cdot)$): We show next that for a given matrix P satisfying condition (21), inequality (8) is satisfied for the gain $K(\cdot)$ proposed in (22).

Since C is full row rank, then $R := [C^\perp \ C^\top] \in \mathbb{R}^{n \times n}$ is nonsingular, and inequality (23) holds if and only if:

$$\begin{bmatrix} R & 0 \\ 0 & I \end{bmatrix}^\top \begin{bmatrix} \Psi(\tau) & \star \\ P + Y(\tau)C & P \end{bmatrix} \begin{bmatrix} R & 0 \\ 0 & I \end{bmatrix} = \begin{bmatrix} (C^\perp)^\top \Psi(\tau) C^\perp & \star \\ \frac{C\Psi(\tau)C^\perp}{PC^\perp} & \frac{C\Psi(\tau)C^\top}{PC^\top + Y(\tau)CC^\top} \Big| \frac{\star}{P} \end{bmatrix} > 0, \quad \forall \tau \in [T_m, T_M], \quad (28)$$

where we used $CC^\perp = 0$. By applying a Schur-complement, and using the property $(C^\perp)^\top \Psi(\tau) C^\perp > 0, \forall \tau \in [T_m, T_M]$ (ensured by the upper left entry of (21)), we obtain that inequality (28) is equivalent to the following constraint:

$$\begin{bmatrix} M_{11}(\tau) & \star \\ M_{21}(\tau) & M_{22}(\tau) \end{bmatrix} = \begin{bmatrix} C\Psi(\tau)C^\top & \star \\ PC^\top + Y(\tau)CC^\top & P \end{bmatrix} - \underbrace{\begin{bmatrix} C\Psi(\tau)C^\perp \\ PC^\perp \end{bmatrix}}_{\Sigma^\top :=} \left[(C^\perp)^\top \Psi(\tau) C^\perp \right]^{-1} \Sigma > 0 \quad (29)$$

Since we already established (in Part 1 of this proof) the existence of a solution $Y(\tau)$ to (29), the diagonal terms $M_{11}(\tau)$ and $M_{22}(\tau)$ in (29), which are independent of $Y(\tau)$, must necessarily be positive definite. So in order to ensure that inequality (29) is satisfied, it is enough to show that $M_{21}(\tau) = 0, \forall \tau \in [T_m, T_M]$, which is done below. In particular, noting that expression (22) corresponds to:

$$Y(\tau) = PK(\tau) = P \left(C^\perp \left((C^\perp)^\top \Psi(\tau) C^\perp \right)^{-1} \left(C\Psi(\tau)C^\perp \right)^\top - C^\top \right) (CC^\top)^{-1}. \quad (30)$$

Then simple manipulations are sufficient to show that:

$$M_{21}(\tau) = PC^\top + Y(\tau)CC^\top - PC^\perp ((C^\perp)^\top \Psi(\tau)C^\perp)^{-1} (C\Psi(\tau)C^\perp)^\top = 0, \quad (31)$$

Thus completing the proof. \square

Remark 2: Selection (22) for gain $K(\cdot)$ can well be understood as follows:

$$K(\tau) := \left(PC^\perp ((C^\perp)^\top \Psi(\tau)C^\perp)^{-1} (C\Psi(\tau)C^\perp)^\top M_{21}(\tau) - PC^\top \right) (CC^\top)^{-1}, \quad (32)$$

where $\Psi(\tau) := e^{(-A^\top \tau)} P e^{(-A\tau)}$, and where any matrix $M_{12}(\tau)$ can be selected, as long as it guarantees inequality (23). Part 1 of the proof of the theorem guarantees that both $M_{11}(\tau)$ and $M_{22}(\tau)$ are uniformly positive definite in $[T_m, T_M]$. We may then get inspired by the parametric selection (17) of Theorem 1 and pick:

$$M_{21}(\tau) = \lambda \left(M_{22}^{\frac{1}{2}}(\tau) \right)^\top M_{11}^{\frac{1}{2}}(\tau), \quad \lambda \in [0, 1), \quad (33)$$

where $M_{11}^{\frac{1}{2}}(\tau)$ and $M_{22}^{\frac{1}{2}}(\tau)$ are the Cholesky's factorizations of the positive definite matrices $M_{11}(\tau)$ and $M_{22}(\tau)$ respectively. With selection (33), we always satisfy (29) because a Schur-complement ensures (29) as long as:

$$M_{11}(\tau) - M_{21}^\top(\tau)M_{22}^{-1}(\tau)M_{21}(\tau) = (1 - \lambda^2)M_{11}(\tau) > 0, \quad (34)$$

which holds true as long as $\lambda \in [0, 1)$. For any $\lambda \neq 0$, selection (33) makes the observer heavy from the computational point of view, because $M_{11}(\tau)$ and $M_{22}(\tau)$ are time-varying matrices, so the Cholesky's factorization must be performed at each sampling time in order to determine $M_{21}(\tau)$ in (33), and consequently the gain $K(\tau)$ in (32).

This computational aspect motivated us to present an efficient solution corresponding to $\lambda = 0$ in Theorem 2. However, based on similar arguments to those presented in Remark 1, it might be desirable to pick larger values of λ to reduce the aggressiveness of the output error injection and increasing the filtering action of the sampled data observer. \lrcorner

The next result shows an interesting link between the feasibility of construction in Theorem 2 and the detectability of the sampled plant when $T_m = T_M$, thereby giving some insight about the conservativeness of the design conditions.

Proposition 2: Consider any positive value of $T = T_m = T_M$. LMI (21) is feasible if and only if pair (C, e^{AT}) is detectable, i.e. if the minimum singular value of the matrix $\mathcal{O}(T)$ defined as:

$$\mathcal{O}(T) = \begin{bmatrix} C & Ce^{(-A^\top T)} & \dots & C \left(e^{(-A^\top T)} \right)^{n-1} \end{bmatrix}^\top \quad (35)$$

is greater than zero.

Proof: The first implication, i.e. if LMI (21) is feasible, then the pair (C, e^{AT}) is detectable, is trivial, because in the case $T = T_m = T_M$ only periodic sampling (with period T) is allowed by the observer dynamics. Then the definition of detectability implies that there does not exist an asymptotic state observer and condition (21) cannot be feasible.

Let us prove the converse implication. If the pair (C, e^{AT}) is detectable, then there exist matrices $Q > 0$ and $L > 0$ such that:

$$Q - (e^{AT} - LC)^\top Q (e^{AT} - LC) > 0. \quad (36)$$

Condition (36) is equivalent to the following condition, after a Schur-complement:

$$\begin{bmatrix} Q & \star \\ (e^{AT} - LC) & Q^{-1} \end{bmatrix} > 0. \quad (37)$$

Since $e^{-AT} \in \mathbb{R}^{n \times n}$ is nonsingular, inequality (37) holds if and only if:

$$\begin{bmatrix} e^{-AT} & 0 \\ 0 & I \end{bmatrix}^\top \begin{bmatrix} Q & \star \\ (e^{AT} - LC) & Q^{-1} \end{bmatrix} \begin{bmatrix} e^{-AT} & 0 \\ 0 & I \end{bmatrix} = \begin{bmatrix} e^{-A^\top T} Q e^{-AT} & \star \\ (I - LC e^{-AT}) & Q^{-1} \end{bmatrix} > 0. \quad (38)$$

Equation (38) can be written as:

$$\underbrace{\begin{bmatrix} e^{-A^\top T} Q e^{-AT} & \star \\ I & Q^{-1} \end{bmatrix}}_{N:=} + \text{He} \left(\underbrace{\begin{bmatrix} 0 \\ -I \end{bmatrix}}_{E:=} L \underbrace{\begin{bmatrix} C e^{-AT} & 0 \end{bmatrix}}_{F^\top:=} \right) > 0. \quad (39)$$

Applying the elimination lemma, as in Theorem 2, there exists a matrix L such that (39) is satisfied if and only if the following relations hold:

$$(E^\perp)^\top N (E^\perp) > 0, \quad (F^\perp)^\top N (F^\perp) > 0, \quad (40)$$

where $E^\perp = \begin{bmatrix} I \\ 0 \end{bmatrix}$ is a basis of the Kernel of E^\top , and $F^\perp = \begin{bmatrix} e^{AT} C^\perp & 0 \\ 0 & I \end{bmatrix}$ is a basis of the Kernel of F^\top .

Using the above relations, the left equation in (40) becomes:

$$\begin{bmatrix} I & 0 \end{bmatrix} \begin{bmatrix} e^{-A^\top T} Q e^{-AT} & \star \\ I & Q^{-1} \end{bmatrix} \begin{bmatrix} I \\ 0 \end{bmatrix} = e^{(-A^\top T)} Q e^{(-AT)} > 0, \quad (41)$$

which is always satisfied because $Q > 0$. Regarding the right equation in (40) we have:

$$\begin{bmatrix} (C^\perp)^\top e^{A^\top T} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} e^{-A^\top T} Q e^{-AT} & \star \\ I & Q^{-1} \end{bmatrix} \begin{bmatrix} e^{AT} C^\perp & 0 \\ 0 & I \end{bmatrix} = \begin{bmatrix} (C^\perp)^\top Q C^\perp & \star \\ e^{AT} C^\perp & Q^{-1} \end{bmatrix} > 0, \quad (42)$$

which is equivalent to:

$$(C^\perp)^\top Q C^\perp - (C^\perp)^\top e^{A^\top T} Q e^{AT} C^\perp > 0, \quad (43)$$

after a Schur-complement. Let us now select $P := e^{A^\top T} Q e^{AT}$, which is positive definite because $Q > 0$ and $e^{A^\top T}$ is nonsingular. Then using $P = PP^{-1}P$, we obtain:

$$(C^\perp)^\top e^{-A^\top T} P e^{-AT} C^\perp - (C^\perp)^\top P P^{-1} P C^\perp > 0, \quad (44)$$

which implies (21) after a Schur-complement. \square

Algorithm 1 A NUMERICAL PROCEDURE TO SOLVE (47)-(48)

```

1:  $\beta \leftarrow$  such that (46) is satisfied;  $\gamma \leftarrow$  root square condition number of  $\Pi$  in (46)
2:  $\mathcal{T} \leftarrow \{T_m, T_M\}$ ;  $\mu \leftarrow 1$ ;  $P \leftarrow \text{sdpvar}(n, n)$ ;  $p_M \leftarrow \text{sdpvar}(1, 1)$  ▷ Initialize variables.
3:  $\text{constr} = \text{set}(P \geq I_n) + \text{set}(P \leq p_M I_n)$  ▷ Define the constraints: positivity of  $P$  and  $P$  bounded.
4: for  $i \leftarrow 1, \text{length}(\mathcal{T})$  do
5:    $\text{constr} = \text{constr} + \text{set}(\Xi_P(\mathcal{T}(i)) > 2\mu I_{2n-m})$  ▷ The other constraints of problem (47) are included.
6: end for
7: while 1 do
8:    $P_{\mathcal{T}}^* = P \leftarrow \text{solvesdp}(\text{constr}, p_M)$  ▷ Find a matrix  $P_{\mathcal{T}}^*$  solution to the optimization (47).
9:   if The problem is not feasible then
10:    (45) and (21) not feasible.
11:   end if
12:    $\mathcal{T}_d = T_m : 2\delta_{\mathcal{T}}^* : T_M$  ▷ Generate the set  $\mathcal{T}_d$  used in (48).
13:   for  $j \leftarrow 1, \text{length}(\mathcal{T}_d)$  do
14:     $\text{mineigs}(j) \leftarrow \min \text{eig}(\Xi_{P_{\mathcal{T}}^*}(\mathcal{T}_d(j)))$  ▷ Compute the minimum eigenvalues of  $\Xi_{P_{\mathcal{T}}^*}(\cdot) \forall \tau \in \mathcal{T}_d$ .
15:   end for
16:   if  $\text{mineigs}(j) > \mu \forall j$  then
17:    break return( $P_{\mathcal{T}}^*$ ) ▷ If all the minimum eigenvalues are larger than  $\mu$ , then  $P_{\mathcal{T}}^*$  is a solution.
18:   else
19:     $\bar{\tau} \leftarrow \arg \min_{\tau \in \mathcal{T}_d}(\text{mineigs})$ 
20:     $\text{constr} = \text{constr} + \text{set}(\Xi_P(\bar{\tau}) > 2\mu I_{2n-m})$  ▷ A new constraint is included in (47) by adding  $\bar{\tau}$  to set  $\mathcal{T}$ .
21:   end if
22: end while

```

IV. DESIGN ALGORITHM

We propose here an algorithm to solve the infinite-dimensional problem (21) in a finite number of steps. To this end let us introduce the following optimization problem:

$$(P^*, p^*) = \arg \min_{P=P^\top, p_M} p_M, \quad \text{subject to:} \quad (45)$$

$$\begin{aligned} \Xi_P(\tau) &> 2\mu I, & \forall \tau \in [T_m, T_M], \\ I &\leq P \leq p_M I, \end{aligned}$$

where $\Xi_P(\tau)$ is defined in (21), and $\mu > 0$ is a positive scalar constant. Problem (45) is again infinite dimensional, but it avoids numerical problems and solutions that lead to large values of P because its upper bound is minimized. The feasibility of problem (45) is equivalent to the feasibility of problem (21) as established in the following result.

Lemma 2: The optimization problem (45) is feasible if and only if relation (21) is feasible, moreover any matrix P solution to (45) is also a solution to (21).

Proof: Any solution to (45) is also a solution of (21), because (21) has relaxed constraint. Vice versa, consider any \bar{P} satisfying (21) and denote by \bar{p}_m, \bar{p}_M its minimum and maximum eigenvalues. Also denote by:

$$\xi_m := \min_{\tau \in [T_m, T_M]} \lambda_m(\Xi_{\bar{P}}(\tau)),$$

where $\lambda_m(\cdot)$ denotes the smallest (real) eigenvalue of the symmetric matrix at argument. Then it is straightforward to verify that $P = \max\left\{\frac{1}{\bar{p}_m}, 2\mu\xi_m\right\}\bar{P}$ satisfies (45) with $\bar{p}_M = \max\left\{\frac{\bar{p}_M}{\bar{p}_m}, 2\mu\xi_m\bar{p}_M\right\}\bar{P}$. \square

Focusing on (45) we introduce now a numerical algorithm, aiming at finding a matrix P solution of problem (21) $\forall \tau \in$

$[T_m, T_M]$. The scheme of the algorithm is shown at the top of this page. The algorithm can be roughly divided into three parts: the initialization, the synthesis, and the analysis phase. During the initialization we establish an exponential bound on e^{-AT} by finding a solution $\Pi = \Pi^\top > 0$ and $\beta \geq 0$ such that:

$$(A + \beta I)^\top \Pi + \Pi (A + \beta I) > 0, \quad (46)$$

and then selecting $\gamma := \sqrt{\frac{\lambda_M(\Pi)}{\lambda_m(\Pi)}}$, as established in Lemma 3 below. Then, during the synthesis phase, we solve the finite dimensional optimization:

$$(P_{\mathcal{T}}^*, p_{\mathcal{T}}^*) = \arg \min_{P=P^\top, p_M} p_M, \quad \text{subject to:} \quad (47)$$

$$\begin{aligned} \Xi_P(\tau) &> 2\mu I, & \forall \tau \in \mathcal{T}, \\ I &\leq P \leq p_M I, \end{aligned}$$

where τ ranges over a finite number of points collected in the discrete set \mathcal{T} (in the first step $\mathcal{T} = \{T_m, T_M\}$). Given an optimal solution $(P_{\mathcal{T}}^*, p_{\mathcal{T}}^*)$ of (47), during the analysis phase we check the following eigenvalue conditions, relaxing the constraints in (47) to half of their values:

$$\Xi_{P_{\mathcal{T}}^*}(\tau) > \mu I, \quad \forall \tau \in \mathcal{T}_d, \quad (48)$$

where $\mathcal{T}_d \subset [T_m, T_M]$ contains an ordered set of scalars $T_m = \tau_1 < \tau_2 < \dots < \tau_{\nu^*} = T_M$ satisfying:

$$\tau_{k+1} - \tau_k \leq 2\delta_{\mathcal{T}}^* := \frac{2\mu}{p_{\mathcal{T}}^* \|A\| \gamma e^{\beta T_M}}, \quad \forall k = 1, \dots, \nu^* - 1. \quad (49)$$

Finally, if this analysis phase is successful, then the algorithm stops and returns $P_{\mathcal{T}}^*$ as a solution to (21). Otherwise, a value:

$$\bar{\tau} \in \arg \min_{\tau \in \mathcal{T}_d} (\lambda_m(\Xi_{P_{\mathcal{T}}^*}(\tau))) \quad (50)$$

is added to the set \mathcal{T} , and the algorithm restarts from the synthesis phase.

A useful property of this algorithm is reported below in Theorem 3, ensuring that if there exists a pair (P^*, p^*) , solution to (45), then the algorithm terminates successfully in a finite number of steps, thus providing a solution to (21).

For stating Theorem 3, the following useful results are presented. The first result is a straightforward consequence of standard Lyapunov theory applied to the linear time-invariant systems $\dot{x} = -Ax$. The second result is proven after Theorem 3 to avoid breaking the flow of the exposition.

Lemma 3: For any square matrix A , there exist a symmetric positive-definite matrix $\Pi > 0$ and a scalar $\beta \geq 0$ satisfying (46). Moreover for any such values of Π and β , the following holds:

$$\|e^{(-A\tau)}\| \leq \gamma e^{\beta\tau} := \sqrt{\frac{\lambda_M(\Pi)}{\lambda_m(\Pi)}} e^{\beta\tau}, \quad \forall \tau \geq 0. \quad (51)$$

Lemma 4: Consider a symmetric positive definite matrix $P \leq pI$, a positive constant μ and a value $\bar{\tau} \in [T_m, T_M]$ such that

$$\Xi_P(\bar{\tau}) > 2\mu I. \quad (52)$$

If γ and β are chosen as in Lemma 3, then the following holds:

$$\Xi_P(\tau) > \mu I, \quad \forall \tau \in [\bar{\tau} - \delta, \bar{\tau} + \delta], \quad (53)$$

as long as δ satisfies:

$$\delta \leq \mu \left(p \|A\| \gamma e^{\beta T_M} \right)^{-1}. \quad (54)$$

Now the main theorem can be given. Note that by Lemma 2 assuming the existence of a solution to (45) is equivalent to assuming the existence of a solution to (21).

Theorem 3: If there exists a solution $P^* \leq p^*I$ to problem (45), then the proposed algorithm terminates successfully at line 17, providing an output $P_{\mathcal{T}}^*$, after a finite number of iterations N satisfying:

$$N \leq (T_M - T_m)/\delta^*, \quad \delta^* = \frac{\mu}{p^* \|A\| \gamma e^{\beta T_M}}. \quad (55)$$

where (γ, β) come from any solution to (46). Moreover such an output $P_{\mathcal{T}}^*$ is a solution to the infinite-dimensional problem (21).

Proof:

Consider the solution (P^*, p^*) to (45). For any finite set of points \mathcal{T} in (47), we have $\mathcal{T} \subset [T_m, T_M]$, therefore (P^*, p^*) is also solution to (47) for any selection of \mathcal{T} . As a consequence, for each \mathcal{T} we have that (47) is feasible and its solution $(P_{\mathcal{T}}^*, p_{\mathcal{T}}^*)$ satisfies $p_{\mathcal{T}}^* \leq p^*$. Then, line 8 of the algorithm always gives a solution.

Consider now a solution $(P_{\mathcal{T}}^*, p_{\mathcal{T}}^*)$ at some step of the algorithm iteration and note that for each $\tau \in \mathcal{T}$ we have $\Xi_{P_{\mathcal{T}}^*}(\tau) > 2\mu I$. Then from Lemma 4 we have that inequality:

$$\Xi_{P_{\mathcal{T}}^*}(\tau) > \mu I, \quad \forall \tau \in \mathcal{T} + \delta^* \mathbb{B} \subset \mathcal{T} + \delta_{\mathcal{T}}^* \mathbb{B}, \quad (56)$$

where δ^* given in (55), and $\delta_{\mathcal{T}}^*$ given in (49), satisfy $\delta^* \leq \delta_{\mathcal{T}}^*$ because $p^* \geq p_{\mathcal{T}}^*$.

Consider now the case $T_M - T_m \leq \delta^*$. Then $\{T_M, T_m\} + \delta^* \mathbb{B}$ contains $[T_M, T_m]$ and the theorem is proven with $N = 1$ iterations. In the less trivial case when $T_M - T_m > \delta^*$, either the analysis step verifying (48) (see line 17 of the algorithm) is successful, or it identifies a new value $\bar{\tau} \notin \mathcal{T} + \delta^* \mathbb{B}$ that is added to \mathcal{T}^+ (the value of \mathcal{T} at the next synthesis step). The previous reasoning (together with $T_M - T_m > \delta^*$) implies that \mathcal{T} only contains elements whose mutual distance is larger than δ^* . Since \mathcal{T} increases by one element at each iteration, the algorithm must terminate successfully when \mathcal{T} has at most $\frac{T_M - T_m}{\delta^*} + 1$ elements. Since \mathcal{T} has two elements at the first iteration, an upper bound on the number of iterations before termination is given by:

$$N = \left\lceil \frac{T_M - T_m}{\delta^*} + 1 - 2 \right\rceil < \frac{T_M - T_m}{\delta^*}, \quad (57)$$

where $\lceil \cdot \rceil$ denotes the smallest integer upper bound of its argument.

When the algorithm stops, it provides a matrix $P_{\mathcal{T}}^*$ satisfying (48), (49). Then Lemma 4 with $\delta = \delta_{\mathcal{T}}^*$ and (48), (49) imply that $\Xi_{P_{\mathcal{T}}^*}(\tau) > 0 \quad \forall \tau \in \mathcal{T} + \delta_{\mathcal{T}}^* \mathbb{B} \subset [T_m, T_M]$, where the last inclusion follows from comparing (49) and (54). As a consequence inequality $\Xi_{P_{\mathcal{T}}^*}(\tau) > 0$ is satisfied for all τ in $[T_m, T_M]$, which implies (21) with $P = P_{\mathcal{T}}^*$. \square

Remark 3: If no solution P exists to (21), then either the algorithm terminates at step 10 with a certified infeasibility (because infeasibility with $\tau \in [T_m, T_M]$ implies infeasibility with $[T_m, T_M]$), or it runs indefinitely, eventually meeting numerical problems. A stopping condition could be imposed by adding an extra well-conditioning constraint $P \leq \bar{p}I$ to (21), (45) and (47), for some reasonably large $\bar{p} \in \mathbb{R}$. Then the algorithm would be guaranteed to terminate successfully whenever a solution to (21) with $P \leq \bar{p}I$ exists and to terminate negatively when such a solution does not exist. \lrcorner

A. Proof of Lemma 4

Since matrix $\Xi_P(\bar{\tau})$ is symmetric positive definite, from [29, Corollary 2.5.11] it is possible to decompose it as:

$$\Xi_P(\bar{\tau}) = \bar{N} \bar{\Lambda} \bar{N}^T, \quad \bar{N}^T \bar{N} = I, \quad (58)$$

where:

$$\bar{\Lambda} = \text{diag}(\bar{\lambda}_1, \dots, \bar{\lambda}_n), \quad \bar{N} = [\bar{v}_1, \dots, \bar{v}_n]. \quad (59)$$

Thus $\{\bar{\lambda}_i\}$ and $\{\bar{v}_i\}$ are, respectively, the eigenvalues and an orthonormal set of eigenvectors of $\Xi_P(\bar{\tau})$. Under these conditions, and based on the fact that the eigenvalues $\lambda_i(\tau)$ of $\Xi_P(\tau)$ are continuous functions of τ , in [30, Eq. 1.3] it is shown that the first order derivatives of the eigenvalues $\bar{\lambda}_i$ are:

$$\frac{\partial \lambda_i(\bar{\tau})}{\partial \tau} = \bar{v}_i^T \frac{\partial \Xi_P(\bar{\tau})}{\partial \tau} \bar{v}_i, \quad (60)$$

where, in our case,

$$\frac{\partial \Xi_P(\bar{\tau})}{\partial \tau} = \left[\begin{array}{c} (C^\perp)^T e^{(-A^T \bar{\tau})} (-\bar{A}^T P - P \bar{A}) e^{(-A \bar{\tau})} C^\perp \quad 0 \\ 0 \end{array} \right]. \quad (61)$$

Therefore, taking into consideration the fact that $\{\bar{v}_i\}$ is an orthonormal set, we obtain from (60):

$$\begin{aligned} \left| \frac{\partial \lambda_i(\bar{\tau})}{\partial \tau} \right| &= \left| \bar{v}_i^\top \frac{\partial \Xi_P(\bar{\tau})}{\partial \tau} \bar{v}_i \right| \\ &\leq \left\| e^{(-A^\top \bar{\tau})} \right\| \left\| A^\top P \right\| + \left\| PA \right\| \left\| e^{(-A \bar{\tau})} \right\| \\ &\leq 2 \|P\| \|A\| \left\| e^{(-A \bar{\tau})} \right\|, \end{aligned} \quad (62)$$

where we used the sub-multiplicativity of the norm, and the fact that $\|C^\perp\| = 1$. Based on bounds (46) and (51), and on the assumption that $P \leq pI$, inequality (62) implies:

$$\left| \frac{\partial \lambda_i(\bar{\tau})}{\partial \tau} \right| \leq 2p \|A\| \gamma e^{\beta \bar{\tau}} \leq 2p \|A\| \gamma e^{\beta T_M}. \quad (63)$$

where we used $\bar{\tau} \in [T_m, T_M]$.

If inequality (52) is satisfied, then $\lambda_m(\Xi_P(\bar{\tau})) \geq 2\mu$. Therefore, from (63), the minimum eigenvalue of $\Xi_P(\tau)$ cannot be less than μ as long as:

$$\tau \in \left[\bar{\tau} - \mu \left(p \|A\| \gamma e^{\beta T_M} \right)^{-1}, \bar{\tau} + \mu \left(p \|A\| \gamma e^{\beta T_M} \right)^{-1} \right].$$

V. NUMERICAL EXAMPLE

Let us consider system (1)-(2) with the following unstable plant:

$$A = \begin{bmatrix} -0.02 & -1.4 & 9.8 \\ -0.01 & -0.4 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 9.8 \\ 6.3 \\ 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}^\top, \quad (64)$$

for which $\text{eig}(A) = \{-0.656, 0.118+0.368i, 0.118-0.368i\}$. For the matrix C in (64) we have $C^\perp = \begin{bmatrix} 0 & 0.7071 & 1 \\ 0 & 0 & -0.7071 \end{bmatrix}^\top$.

In this case the proposed algorithm is applied for three different choices of $[T_m, T_M]$. In particular for $\tau \in [1, 3]$, $\tau \in [3, 4]$ and $\tau \in [4, 8]$.

In the first case, $\tau \in [T_m, T_M] = [1, 3]$, the algorithm finds a solution with only one iteration. Indeed, at the first iteration, the algorithm considers only $\mathcal{T} = \{T_m, T_M\}$, and, during the analysis phase, conditions (48) is satisfied for all $\tau \in T_d$, as shown in Figure 1. The obtained matrix P is:

$$P = 10^4 \begin{bmatrix} 0.8334 & -0.0041 & 0.8333 \\ -0.0041 & 2.0116 & -0.0359 \\ 0.8333 & -0.0359 & 0.8341 \end{bmatrix}, \quad \tau \in [1, 3]. \quad (65)$$

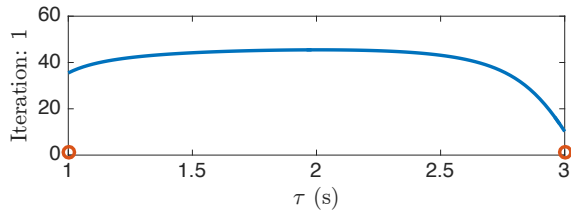


Figure 1. Minimum eigenvalues of matrix $\Xi_P(\tau)$ for $\tau \in [1, 3]$.

In the second case $\tau \in [T_m, T_M] = [3, 4]$ the algorithm does not find a solution because the plant is not observable

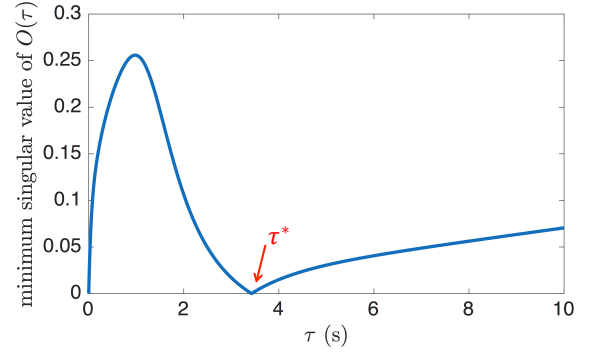


Figure 2. Minimum singular values of the observability matrix $\mathcal{O}(\tau)$ for the sampled plant.

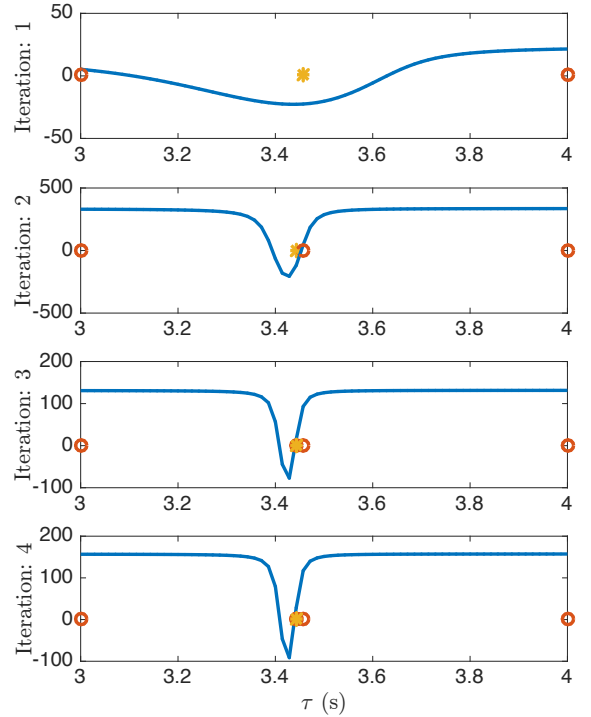


Figure 3. Minimum eigenvalues of matrix $\Xi_P(\tau)$ for $\tau \in [3, 4]$.

for $\tau = 3.425 \in [3, 4]$. This fact is clear looking at Figure 2 where the minimum singular values of the observability matrix $\mathcal{O}(\tau)$ are shown, and for $\tau = 3.425$ the minimum singular value is zero, revealing a rank loss for the observability matrix. This is confirmed by Figure 3, where it is shown that, after four iterations, the analysis phase fails in a neighborhood of $\tau = 3.425$.

Finally, in the last case, $\tau \in [T_m, T_M] = [4, 8]$, the algorithm finds a solution with two iterations as shown in Figure 4. Indeed, during the first iteration the algorithm considers $\mathcal{T} = \{T_m, T_M\}$, and, during the analysis phase, conditions (48) is not satisfied for all $\tau \in T_d$ (see the upper subplot of Figure 4). In particular it results that the minimum negative eigenvalue is obtained at $\tau = 4.84$, so an additional constraint is considered evaluating (47) also for $\bar{\tau} = 4.84$. Finally, in the second iteration, the algorithm finds a solution

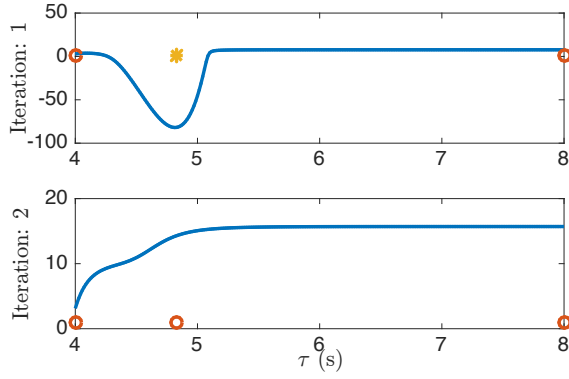


Figure 4. Minimum eigenvalues of matrix $\Xi_P(\tau)$ for $\tau \in [4, 8]$.

as it is depicted in the lower subplot of Figure 4. The obtained matrix P is:

$$P = 10^4 \begin{bmatrix} 0.9573 & -0.0031 & 0.9571 \\ -0.0031 & 2.2754 & -0.0122 \\ 0.9571 & -0.0122 & 0.9573 \end{bmatrix} \quad \tau \in [4, 8]. \quad (66)$$

Initially, the unstable plant (64) has been stabilized by means of a state feedback using a low gain:

$$K_u = 10^{-2}[0.16 \quad 5.47 \quad -0.01],$$

such that:

$$\text{eig}(A + BK_u) = \{-0.01, -0.02, -0.03\}.$$

The corresponding slow transient ensures that the signals do not blow up during the simulation, but they are associated to a sufficiently rich behavior.

The dynamics expressed in (4) has been implemented together with the observer (5) in the MATLAB[®]-Simulink environment. The gain $K(\tau)$ is computed on-line according to (22) by using matrix P in (65) for $\tau \in [1, 3]$ and P in (66) for $\tau \in [4, 8]$. Moreover, in order to implement a random value of the time-instant of the measurements, we implement the following modified error dynamics, corresponding to (6) with random selection of the inter-measurement intervals:

$$\begin{cases} \dot{e} = Ae, \\ \dot{\tau} = 1, \\ \dot{\tau}_r = -1, \end{cases} \quad \tau_r \in [T_M, 0], \quad (67a)$$

$$\begin{cases} e^+ = (I - K(\tau)C)e, \\ \tau^+ = 0, \\ \tau_r^+ = T_m + (T_M - T_m)\nu^+, \end{cases} \quad \tau_r = 0. \quad (67b)$$

where ν^+ is a random variable uniformly distributed in the interval $[0, 1]$. This modified dynamics is inspired, with the same notation, by [31].

In Figure 5 the real and estimated state vector components $x_i, \hat{x}_i, i = 1, 2, 3$, as well as estimation errors $e_i = x_i - \hat{x}_i, i = 1, 2, 3$, are shown, during the first test with $\tau \in [1, 3]$. Moreover, for the same test, the waveforms of the Lyapunov function V , of the variables τ and τ_r and of the output error $y - \hat{y}$ are shown in Figure 6.

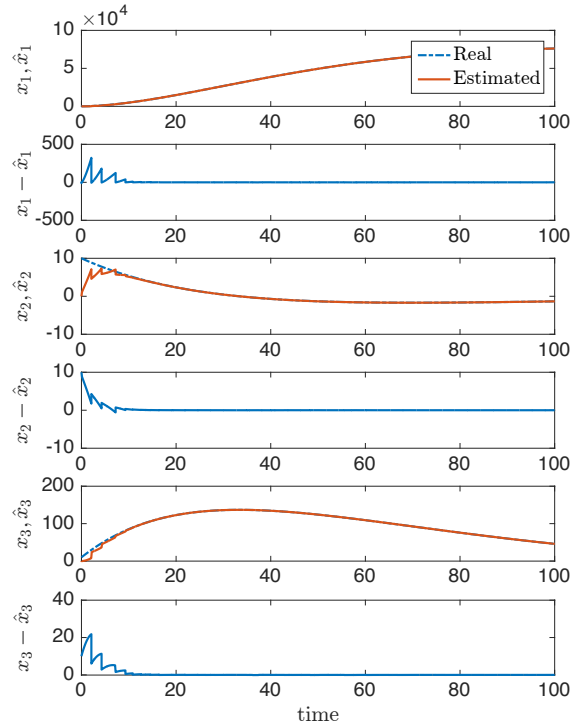


Figure 5. Real and estimated state vector components $x_i, \hat{x}_i, i = 1, 2, 3$, as well as estimation errors $x_i - \hat{x}_i, i = 1, 2, 3$, during a test with $\tau \in [1, 3]$.

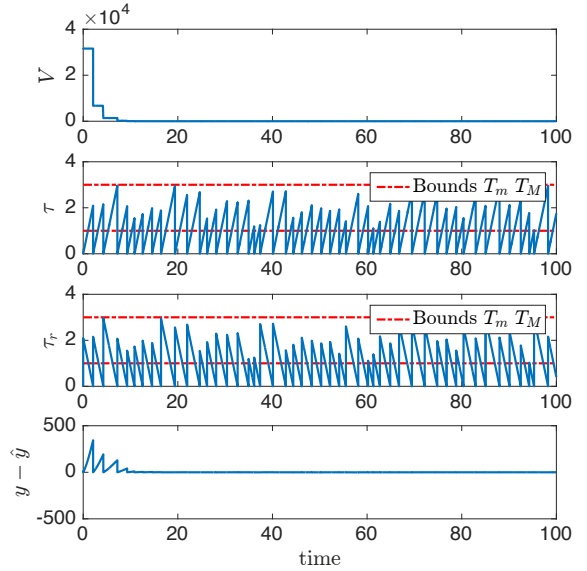


Figure 6. Waveforms of the Lyapunov function V , of the variables τ and τ_r and of the output error $y - \hat{y}$, during a test with $\tau \in [1, 3]$.

From Figure 5 it is evident that the estimated variables track very well the corresponding state variables and all the errors go to zero asymptotically. Moreover, it is possible to note the impulsive behavior of the estimate especially during the initial transient. From Figure 6 we note that the Lyapunov function is constant during flow, and decreases across jumps, as expected from the theoretical results (Lemma 1). Finally, from the waveforms of τ and τ_r we see that the jumps occur

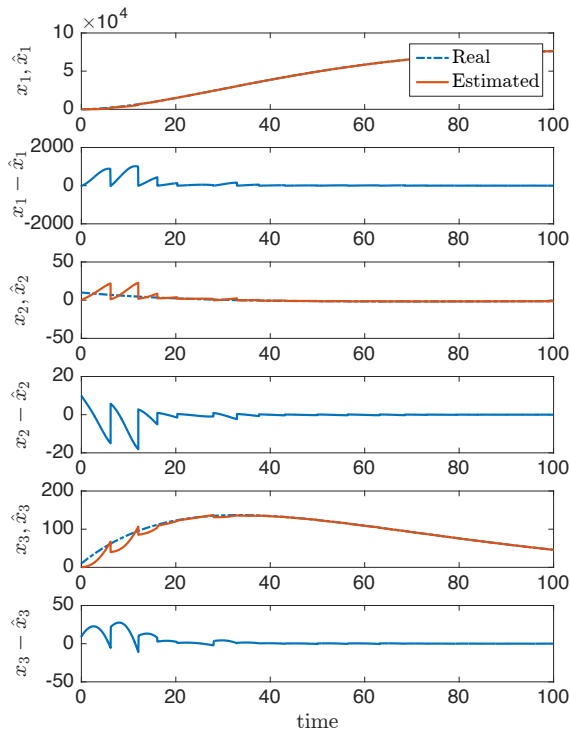


Figure 7. Real and estimated state vector components x_i , \hat{x}_i , $i = 1, 2, 3$, as well as estimation errors $x_i - \hat{x}_i$, $i = 1, 2, 3$, during a test with $\tau \in [4, 8]$.

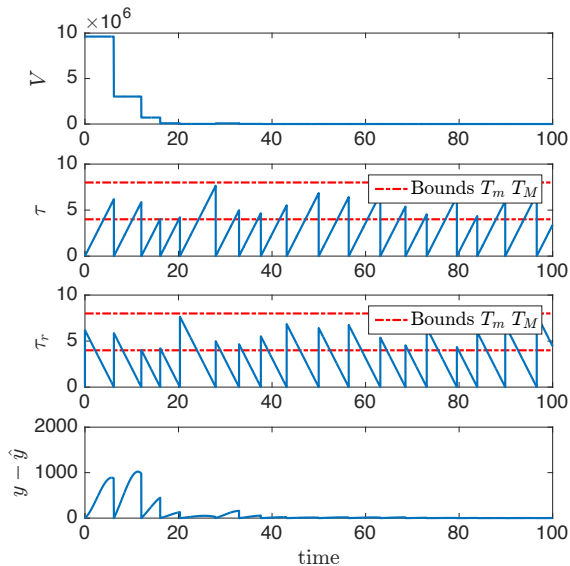


Figure 8. Waveforms of the Lyapunov function V , of the variables τ and τ_r and of the output error $y - \hat{y}$, during a test with $\tau \in [4, 8]$.

randomly in the interval $[1, 3]$ according to the described dynamics.

Figures 7-8 show the results for the same test described above, but when the measurements are provided more sporadically, $\tau \in [4, 8]$. In this case the same comments given for the first test can be provided, confirming the effectiveness of the proposed approach. Obviously, the convergence rate in this case is slower because the measurements are accessible less

frequently.

VI. CONCLUSION

In this work an observer with a time-varying output error injection has been proposed for a linear continuous-time plant with asynchronous sampled measurements. In particular some theoretical tools have been provided, in terms of LMIs, certifying asymptotic stability of a certain compact set where the estimation error is zero. Two solutions have been proposed, one under the restrictive assumption that the output matrix is invertible, and one for the more general case of a detectable pair, under the assumption that some LMI conditions hold. Moreover, necessary conditions for the feasibility of those LMI have been established. Since the proposed time-varying observer is based on a solution to an infinite-dimensional LMI, a numerical algorithm has been introduced which is guaranteed to converge after a finite number of iterations to a solution to the infinite dimensional problem whenever one exists. The results provided by a numerical example show the effectiveness of the proposed approach, confirming the theoretical results and the feasibility of the proposed numerical solution.

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