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OPTIMAL CONTROL OF TRAFFIC FLOW ON PERIURBAN RINGWAYS WITH APPLICATION TO THE BOULEVARD PERIPHERIQUE IN PARIS

Nella BHDURI⁽¹⁾, Markos PAPAGEORGIOU⁽²⁾, Jean-Marc BLOSSEVILLE⁽¹⁾

(1) Institut National de Recherche sur les Transports et leur Sécurité (INRETS)
Département Analyse et Régulation du Trafic (DART)
2, Avenue du Général Malleret-Joinville - 94110 Arcueil - FRANCE

(2) Lehrstuhl für Steuerungs- und Regelungstechnik
Technische Universität München
P.O. Box 202420 D-8000 München 2 - R.F.A.

Abstract. An optimal control algorithm based on the discrete maximum principle is applied to regulate traffic flow on ringways by use of ramp metering.

The optimal on-ramp metering strategy is applied to the Boulevard Périphérique of Paris which has a total length of 35 km and includes 34 on-ramps and 36 off-ramps. In view of the high problem dimension, the non-linear model used is a simplified version of an accurate macroscopic model of traffic flow on motorways which is called META. Space/time discretization is effectuated on the basis of 3 km/60 s intervals. Simulation tests have been performed in order to test the accuracy of the model. The optimal control problem is formulated subject to several technical and physical restrictions which are expressed in form of inequality constraints. The optimization objective is mainly the minimization of the total time spent in the ringway system.

The numerical solution of the formulated non-linear optimal control problem is performed by powerful feasible direction algorithms using conjugate gradient techniques.

1. INTRODUCTION

Traffic flow on uncontrolled periurban ringways which are fed by ramps and motorways, is characterized by a natural (rather than optimal) space and time distribution of traffic demand. This causes severe congestions which reduce traffic throughput at several locations of the ringway during daily rush hours, at the begin resp. end of the vacation periods, week-ends, etc. This paper investigates the possibility of improving the traffic situation on periurban ringways by on-ramp metering strategies based on optimal control theory.

Application of optimal control theory to regulate traffic flow on motorways has attained considerable attention in the last two decades. Because of the high computational effort required for the solution of non-linear optimal control problems, several proposed strategies have been based on Linear Quadratic (LQ) optimization theory. However, due to the high order of non-linearities of congested traffic flow, strategies based on LQ-problems may not be able to dissolve severe congestions. Furthermore, any restrictions and system constraints are not considered in the formulation of LQ-optimal control problems.

Non-linear optimal control problems for coordinated ramp metering have been formulated by Kaya (1972) and Tabac (1972). These papers considered the minimization of the total time spent as an optimization objective and used a relatively simple non-linear traffic flow model. Greenlee and Payne (1977) proposed a different optimal control approach for responding to incidents. Unfortunately, none of these papers includes a numerical algorithm for solution of the optimization problem. On the other hand, Blinkin (1976), Cremer (1976) and Papageorgiou & Mayr (1982) have resolved the optimal control problem

on the basis of more accurate non-linear models. Blinkin (1976), Papageorgiou & Mayr (1982) used algorithms based on the maximum principle. Cremer (1976) solved the problem using a non-linear programming algorithm. In these three cases the optimal ramp-metering problem is solved subject to artificial optimization objectives, for a relatively low number of on-ramps.

The optimal control methodology is applied in this paper to the Boulevard Périphérique de Paris, which has a total length of 35 km and includes 34 on-ramps and 36 off-ramps. In view of the high problem dimension, the non-linear mathematical model used is a simplified version of an accurate non-linear macroscopic model of traffic flow on motorways which is called META (Modèle d'Écoulement de Trafic sur Autoroute). (Papageorgiou, 1988). Space/time discretization of the simplified model is effectuated on the basis of 3 km/60 s intervals. In order to test the efficiency of the chosen model with such a rude discretization, simulation tests have been performed.

The optimization objective considered is essentially minimization of the total time spent by all drivers on the ringway. This includes the total travel time on the ringway itself and the total waiting time at the on-ramps. Several technical and physical restrictions which are present for the ringway system, are expressed in form of inequality constraints.

The numerical solution of the formulated non-linear optimal control problem is performed by a powerful algorithm based on the discrete maximum principle and using conjugate gradients techniques. The main advantage of the algorithm used is the moderate computational effort as well as the fact that both storage space and CPU time increase roughly linearly with the problem dimension.

The presented optimal control algorithm for ramp metering is applicable either as a design tool (investigation of different on-ramp metering options, closure of on-ramps, potential improvements, etc...) or as a real-time control tool proposing optimal, traffic responsive on-ramp metering strategies to the traffic operator or to local control loops.

2. MODELLING

The efficiency of an optimal control strategy depends up to a certain degree on the properties of the model used. An available macroscopic non-linear traffic flow model called META (Modèle d'Écoulement de Trafic sur Autoroute) has been tested on the basis of real traffic data and results were very satisfactory. As shown in Papageorgiou (1988), META reproduces in a natural way complicated real traffic phenomena such as discontinuities, hysteresis phenomena and breakdown of traffic flow due to overload. However, application of META implies subdivision of the ringway into sections of some 500 m in length. Moreover, META uses two difference equations to describe the evolution of traffic state in each section. Hence a set of 140 state variables is needed for a 35 km long ringway like the Boulevard Périphérique in Paris. The required sample time interval is in the order of 10-20 seconds. This leads to a considerably high computational effort for the solution of an optimal control problem based on META. For this reason, a macroscopic non-linear simplified model is used in this paper as a basis for derivation of optimal control trajectories. The simplifications introduced in the model equations permit a space/time discretization in the order of 3 km/min. Furthermore, the simplified model uses only one state variable to describe the dynamic evolution of the ringway traffic flow. In this way, only 12 state variables are needed for the 35 km long ringway. Moreover, the computation time is reduced by using 1 min time interval discretization instead of 10-20 s required by META.

One of the problems faced due to the simplification is that a 3 km long ringway section includes more than one on-ramp and one off-ramp. Thus, the on-ramps and off-ramps of a section must be aggregated into one single on-ramp and one off-ramp.

In the first part of this section there is a brief presentation of the traffic variables followed by the presentation of META equations and the simplifications introduced. The second part of this section is concerned with on-ramp and off-ramp aggregations. The final part presents the simplified model validation.

2.1. Traffic variables

Macroscopic description of traffic flow implies the definition of adequate flow variables expressing the average behaviour of the vehicles in a given motorway section i . For a space/time-discretized presentation, we define traffic density $\rho_i(k)$ as the number of cars in the section i at time $t = kT$ divided by the section length Δ_i where $k = 0, 1, \dots$ is the discrete time index and T denotes the sample time interval. Similarly, we define mean speed $v_i(k)$ as the mean speed of the vehicles included in the section at time $t = kT$. Finally, $q_i(k)$ is the number of vehicles leaving the section during $kT < t < (k+1)T$, divided by T . On-ramp resp. off-ramp volumes $u_i(k)$ resp. $s_i(k)$ are defined in an analogous way.

2.2. The macroscopic traffic flow model META

The macroscopic traffic flow model META, as presented in Papageorgiou et al. (1988) consists of the following equations applying to each section i .

(1) Conservation of vehicles equation

$$\rho_i(k+1) = \rho_i(k) + T/\Delta_i (q_{i-1}(k) - q_i(k) + u_i(k) - s_i(k)) \quad (1)$$

(2) A volume-density-mean speed relation which is a consequence of the variables definitions and which has a direct analogon in hydrodynamic flow

$$q_i(k) = \rho_i(k) v_i(k) \quad (2)$$

(3) A dynamic mean speed-density relationship

$$v_i(k+1) = v_i(k) + \frac{T}{\tau \Delta_i} \{V(\rho_i(k)) - v_i(k)\} + \frac{T}{\Delta_i} v_i(k) [v_{i-1}(k) - v_i(k)] - \frac{\nu T}{\tau \Delta_i} \frac{\rho_{i+1}(k)/\lambda_{i+1} - \rho_i(k)/\lambda_i}{\rho_i(k)/\lambda_i + \kappa} - \frac{\delta T}{\Delta_i} \frac{v_i(k) v_i(k)}{\rho_i(k) + \kappa \lambda_i} \quad (3)$$

Where $V(\rho_i(k))$ is a steady-state homogeneous speed-density characteristic which is a monotonically decreasing function of the density ρ_i :

$$V(\rho_i) = v_f \exp [-0.5 \left(\frac{\rho_i}{\lambda_i \rho_{cr}} \right)^2] \quad (4)$$

The constant parameters are v_f (free speed), ρ_{cr} (critical density per lane), τ (time constant), ν (anticipation constant), δ (on-ramp constant) and κ , and λ_i denotes the number of lanes in section i .

Combining (2) and (4), one obtains a function $Q(\rho_i) = \rho_i V(\rho_i)$ which is broadly known as the fundamental diagram of traffic engineering. One can easily show that $Q_{max} = Q(\lambda_i \rho_{cr})$, i.e. traffic flow obtains its maximum value for $\rho = \lambda_i \rho_{cr}$. Mean speed resulting from (3) is limited by v_{min} , e.g., $v_{min} = 5$ km/h. The presented macroscopic traffic flow model is based on a model originally proposed by Payne (1971). The last term of equation (3) was added in order to account for speed decrease caused by merging phenomena in on-ramp sections, see Papageorgiou et al., 1988. A further cause of speed reduction due to weaving phenomena results from lane drop on the mainstream. For this case, Papageorgiou (1988) proposed the following term:

$$-\left(\phi T/\Delta_i\right) \left[\left(\lambda_{i+1} - \lambda_i\right)/\lambda_i\right] \left[\rho_i(k)\right] / \left[\left(\rho_{cr} \lambda_i\right)\right] v_i(k)^2$$

Where ϕ is a new constant parameter. This term is added to the right-hand side of (3) whenever $\lambda_{i+1} < \lambda_i$.

The conservation of vehicles equation (1) is slightly modified, by using a mainstream volume rate instead of the direct off-ramp volume $s_i(k)$.

Hence, eq(1) becomes

$$\rho_1(k+1) = \rho_1(k) + \frac{T}{\Delta_1} [(1-\gamma_1)q_{1-1}(k) - q_1(k) + u_1(k)] \quad (1.a)$$

where γ_1 is the off-ramp volume rate which is supposed to be known. In fact real measurements indicate that exit rates γ_1 are roughly constant during rush hours.

Eqs (1.a), (2) and (3) permit to describe the dynamic evolution of traffic flow on the mainstream. In order to reproduce the traffic behaviour on on-ramps, another equation is needed which expresses the conservation of vehicles for each ramp queue :

$$l_1(k+1) = l_1(k) + T \cdot [d_1(k) - u_1(k)] \quad (5)$$

where $l_1(k)$ is the queue length (veh) on on-ramp 1 at time $t = kT$, and $d_1(k)$ is the corresponding on-ramp demand which is assumed to be known.

As we have noted earlier, an optimal control problem based on META requires a high computational effort. For this reason a simplified model will be developed next.

2.3. Simplified model

The main simplification is introduced in the dynamic speed-density relationship of META. It consists in neglecting the time constant τ in eq(3). The parameter expresses the fact that drivers adjust their speed with a small time delay when density changes occur. The physical meaning of this simplification consists, thus, in assuming that drivers adjust instantaneously their speed when any density changes occur. By setting $\tau \rightarrow 0$ in eq(3), one obtains.

$$v_1(k) = v[d_1(k)] = \frac{v \rho_{1+1}(k)/\lambda_{1+1} - \rho_1(k)/\lambda_1}{\Delta_1 \rho_1(k)/\lambda_1 + \kappa} \quad (6)$$

see Papageorgiou et al. (1989) for more details.

We note that eq(6) does not include the convection term. Furthermore, deceleration due to the existence of an on-ramp in the section or due to a lane drop is neglected.

Combining (2) and (6), one obtains a new simplified traffic volume-density relationship

$$q_1(k) = Q[\rho_1(k)] = v/\Delta_1 \cdot [\rho_{1+1}(k)\lambda_{1+1} - \rho_1(k)\lambda_1] \quad (7)$$

The constant parameter κ added in eqs (3) and (6) to limit the effect of the gradient density term in case of very low density values, is dropped in eq (7) because it become useless.

Eqs(1.a), (5) and (7) constitute the simplified model. It is interesting to note that numerical instability would appear if the second term of eq(7) would be dropped, see Papageorgiou et al. 1989. Furthermore, in view of (7), it becomes apparent that even under steady-state conditions, maximum traffic volume is not constant but depends on the density gradient.

This paper considers application of the optimal control methodology to ramp metering on the Boulevard Périphérique de Paris (BPP). The optimal control problem is formulated on the basis of the simplified model with 3 km long space

discretization samples. In what follows, we will therefore describe the particular characteristics of BPP and will then proceed to the problem of aggregation of on-ramps and off-ramps included in the 3 km long sections.

2.4. Description of the Boulevard Périphérique de Paris (BPP)

The BPP is a 35.17 km long closed ringway. It is a dual carriageway but in this study we only consider the inner part of it. The inner part includes 34 on-ramps and 36 off-ramps, of which 5 on-ramps and 6 off-ramps are motorways. The greatest part of the BPP has 4 lanes except for the southern stretch (7 km) which reduces to 3 and even to 2 lanes.

Although it is similar to normal motorways, the BP has particular features like the high density of on-ramps and off-ramps, the priority of vehicles entering from on-ramps over mainstream vehicles, and the speed limitation of 80 km/h.

2.5. Subdivision into sections and on-ramp / off-ramp aggregation

In order to apply the simplified model, the BPP is subdivided into 12 sections of some 3 km in length each. Unfortunately, there are more than one on-ramp and one off-ramp included in each 3 km interval of the BPP, as shown in Fig.1. In the aim of getting only one control variable for each section, on-ramps of the BPP are aggregated into one per 3 km long section. At the same time, density in each section is assumed to be homogeneously distributed, i.e., we have one state variable per section. Finally off-ramps are also, aggregated into one off-ramp per section.

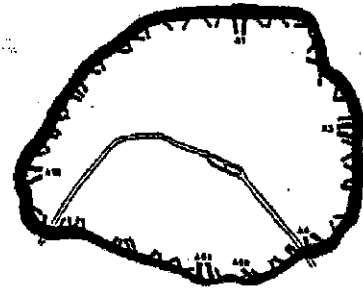


Fig.1 : sketch of the BPP

In order to preserve the main characteristics of traffic flow, the importance of each on-ramp is taken into account in the aggregation procedure. More precisely, important ramps (which have the highest traffic volume) have been conserved in their location whereas less important ramps are grouped with the important ones. Moreover, a ramp located at the entry (resp. exit) of a section can be grouped with the ones located at the exit of the previous (resp. entry of the next) one. For example, Fig.2 shows a 3.2 km long section including 3 on-ramps and 3 off-ramps.

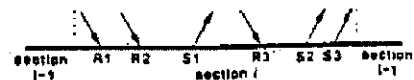


Fig.2 : 3.2 km long section of the BPP

By inspection of the usual traffic volumes entering by these ramps, we found out that R3 is the most important one. Thus, the traffic state of the section in the simplified model corresponds to the real traffic state downstream of R3. On the other hand, the on-ramps R1 and R2 provide a much lower traffic volume and moreover their effect on mainstream is roughly cancelled by the next exit S1. In summary, the stretch between R3 and S2 is the main bottleneck of this section and therefore a reasonable simplification is the one given by Fig.3.

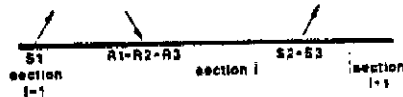


Fig.3 : Example of on-ramp/off-ramp aggregation

Furthermore, since there are off-ramps in the exit of the previous section, S1 is grouped with them. Similar arguments apply to aggregation of each of the twelve sections in the simplified model.

2.6. Simulation tests

The time interval T should be chosen $T < \Delta/V_f$ (8), in order to guarantee numerical model stability. In our case, the smallest section is 2.5 km long and free speed equals 90 km/h, which gives $T < 1,7$ mn. Thus, T is chosen to be 1 mn.

In order to make sure that the model preserves a sufficient accuracy even with such a rude space/time discretization, we have performed a simulation test. Since real traffic data for the whole of BPP were not available for comparison, another simulation has been effectuated with the macroscopic traffic flow model META.

2.7. Traffic simulation using META

To apply META, the BPP is subdivided into 70 sections of some 500 m in length, each including at most one on-ramp and one off-ramp. Since the shortest section is 330 m long and according to eq(8), the time interval should be $T < 13,5$ s. A 10 s sample is chosen for time discretization.

The model parameters are v_f , ρ_{CR} , ν , τ , κ , ϕ , and δ . The same parameter values are used for all road sections. All parameter values are taken from Papageorgiou, 1988, where parameter estimation was based on real traffic data collected from 6 km of the BPP. These values are :

$$\begin{aligned} v_f &= 90 \text{ veh/km} ; \quad \rho_{CR} = 37.3 \text{ veh/km/lane} ; \\ \nu &= 35 \text{ km}^2/\text{h} ; \quad \tau = 0.01 ; \quad \phi = 2 ; \quad \delta = 0.7 \\ \kappa &= 40 \text{ veh/km} ; \end{aligned}$$

The simulation is performed for a 10 hours period (from 6 p.m. to 4 a.m.) which includes congestion creation and disappearance for two rush hours periods (the morning and the afternoon peaks).

2.8. Simplified model simulation test

The simplified model includes three parameters, namely two parameters of the fundamental diagram (v_f and ρ_{CR}) and the anticipation parameter ν .

In this paper, the fundamental diagram used for all sections is the same as for application of META, i.e. $v_f = 90$ km/h and $\rho_{CR} = 37.3$ veh/km/lane

is used. Parameter ν is adjusted so as to reach best agreement with the results provided by META. This leads to a value $\nu = 48$ km/h.

Comparison with simulation results of META showed that the simplified model reproduces congestion creation in the same locations. However there is a 35 mn time delay and moreover congestion propagation is not everywhere equivalent to the results obtained by META.

Although the introduced simplifications may appear fairly rude, we would like to make the following remarks :

(i) The simplified model is not needed for simulation but for control purposes. Accuracy requirements are therefore less pronounced. Since, the simplified model does reproduce the main congestions, we expect that the resulting optimal control will seek to avoid them by adequate ramp-metering measures.

(ii) Robustness of the resulting control will be considerably enhanced if the optimal control results are used as desired values (set values) for inferior local control loops.

(iii) The efficacy of the resulting control may be tested by simulation studies on the basis of META. However these tests were not available at the time of the writing of this paper.

(iv) Simplifications aim at reducing the computational effort required for the solution of the optimal control problem. There is an apparent trade-off of accuracy versus computational effort (e.g. sections may be chosen to be 2 km or 1 km or 500 m long) which will be considered in future investigations.

3. SYSTEM CONSTRAINTS

Optimal specification of on-ramp volumes must be performed subject to some constraints. Due to the geometric characteristics of the entrance ramps, there is a maximum on-ramp volume value which cannot be exceeded

$$u_i(k) \leq u_{i,\max} \quad (9)$$

On the other hand, too low rates of on-ramp volumes lead waiting drivers to judge the metering signal to be malfunctioning. Hence a reasonable lower limit should be posed

$$u_i(k) \geq u_{i,\min} > 0 \quad (10)$$

Besides these input volume constraints which are constant for each on-ramp, there are some other constraints which depend upon the traffic state either on the mainstream or on the entrance ramp. First, the queue length cannot be negative : $l_i(k) \geq 0$. Substituting this in eq. (5), we obtain

$$\begin{aligned} u_i(k+1) &\leq d_i(k) + l_i(k)/T \\ &= u_{\max}(l_i(k)) \end{aligned} \quad (11)$$

Second, the density at the main street section also cannot be negative : $\rho_i(k) \geq 0$. By substitution in (1.a) one obtains:

$$\begin{aligned} u_i(k) &\leq -(\Delta_i/T) \cdot \rho_i(k) - (1-\tau_i) q_{i-1}(k) + q_i(k) \\ &= u_{\min}(\rho_i(k)) \end{aligned} \quad (12)$$

Summarizing equations (9), (10), (11) and (12), the admissible control region for on-ramp volumes is given by :

$$\max [u_{i,\min}, u_{\min}(\rho_i(k))] \leq u_i(k) \leq \min [u_{i,\max}, u_{\max}(\rho_i(k))] \quad (13)$$

Moreover a maximum queue length should not be exceeded, else interference with traffic in surface streets could occur

$$l_i(k) \leq l_{i,\max} \quad (14)$$

Finally, a maximum traffic density value is imposed by the geometric characteristics of the mainstream :

$$\rho_i(k) \leq \rho_{i,\max} \quad (15)$$

Constraints (14), (15) will be considered in a smooth way in order to guarantee that the admissible region of the optimization problem can not be equal to the zero set. This could arise e.g. due to constraints (9), (14) and due to eq.(15) for a sufficiently high demand.

4. CONTROL OBJECTIVE

Several control objectives for freeway traffic have been proposed by various researchers. Minimization of the total time spent by all drivers on the ringway system (T_s) seems to be the most suitable one (see Papageorgiou, 1983 for details). Total time spent includes the total travel time on the ringway itself and the total waiting time at the on-ramps. Minimization of total time spent on the ringway implies minimization of delays caused by congestions and is thus a suitable requirement in order to prevent or eliminate congestions. T_s is given by

$$T_s = T \sum_{k=0}^K \sum_{i=1}^N [\rho_i(k) \cdot \Delta_i + l_i(k)] \quad (16)$$

where K is the fixed optimization horizon. The first term of eq.(16) represents the total travel time for a given time horizon KT , the second term corresponds to the total waiting time for the same time horizon KT .

Using only the total time spent as a cost function may lead to discontinuous (bang bang) optimal control trajectories which cannot be accepted for on-ramp traffic volumes. Therefore, a term which penalize the temporal variations of the input variables is added to the cost function

$$J = T_s + \beta_1 \sum_{k=0}^K [(u(k) - u(k-1))]^2 \quad (17)$$

where J is the cost function to be minimized, $\beta_1 < 1$, is a weighting parameter.

In order to consider state variable constraints (14) and (15), penalty functions are added to the basic cost function (17):

$$J' = J + \sum_{k=0}^K \sum_{i=1}^N \{ \beta_2 \cdot [\psi(l_{i,\max} - l_i(k))]^2 + \beta_3 \cdot [\psi(\rho_{i,\max} - \rho_i(k))]^2 \} \quad (18)$$

where $\psi(\eta)$ is a vector penalty function with components:

$$\psi_i(\eta) = \min(0, \eta_i)$$

and $\beta_2, \beta_3 \ll 1$ are weighting parameters. It is well known that higher values of the weighting parameters lead to shorter exceeding of constraints (14) and (15) and lower convergence rate.

5. CONTROL PROBLEM FORMULATION

The non-linear optimization problem (P) solved in this paper is formulated as following :

(P): Given an initial condition of state variables $\rho(0)$, $l(0)$ and predicted trajectories $d(k)$, $\gamma(k)$, $k = 0, \dots, K-1$. Find the optimal on-ramp trajectories $u(k)$, $k = 0, \dots, K-1$ which minimize the augmented performance index (18), subject to the simplified model equations (1a), (5) & (7) and to the inequality constraints (13).

In order to solve this problem we use the discrete maximum principle.

6. SOLUTION OF THE OPTIMIZATION PROBLEM

6.1. General problem formulation

Optimal control problem (P) formulated above belongs to the following general class of optimal control problems : Minimize

$$J = \phi(x(K)) + \sum_{k=0}^{K-1} \theta[x(k), u(k), k] \quad (19)$$

subject to the state equation

$$x(k+1) = f(x(k), u(k), k) \quad k=0, \dots, K-1 \quad (20)$$

$$x(0) = x_0 \quad (21)$$

and the control constraints

$$u_{\min}(x(k)) \leq u(k) \leq u_{\max}(x(k)) \quad (22)$$

where $x \in R^n$ and $u \in R^m$ are the state and control vectors respectively, ϕ , θ and f are non-linear differentiable functions, and K is the fixed optimization horizon.

In order for a control trajectory to be admissible, it must satisfy the equality constraints (20) and (21) and the inequalities (22). For the inequality constraints (22), we introduce

$$h^1(x(k), u(k), k) = u(k) - u_{\min}(x(k)) \quad (23)$$

$$h^2(x(k), u(k), k) = u_{\max}(x(k)) - u(k) \quad (24)$$

h^1 and $h^2 \in R^m$ are assumed to be continuously differentiable functions. It is clear that eqn(22) is satisfied when

$$h^1(x(k), u(k), k) \geq 0 \quad (25)$$

$$h^2(x(k), u(k), k) \geq 0 \quad (26)$$

Consequently, an admissible trajectory, must now satisfy eqn(20), (21), (25) and (26).

6.2. Solution algorithm

We introduce the Hamiltonian:

$$H = \phi + \lambda(k+1)^T f + \mu^1(k)^T h^1 + \mu^2(k)^T h^2 \quad (27)$$

where the arguments $(x(k), u(k), k)$ are suppressed in functions H, ϕ, f, h^1, h^2 for convenience; λ denotes transpose, λ is the costate vector and μ^1 and μ^2 are Lagrange multipliers.

The adjoint variables $\lambda \in \mathbb{R}^n$ satisfy:

$$\lambda(k) = \delta\phi/\delta x(k) + \delta f^T/\delta x(k) \cdot \lambda(k+1) + \delta h^{1T}/\delta x(k) \cdot \mu^1(k) + \delta h^{2T}/\delta x(k) \cdot \mu^2(k) \quad (28)$$

$$k = 0, \dots, K-1$$

$$\lambda(K) = \delta\phi/\delta x(K) \quad (29)$$

Application of the discrete maximum principle requires that, for an optimal control,

$$\delta H/\delta u = 0$$

From this condition and due to the fact that the Lagrange multipliers are equal to zero if the control variables do not lie on the corresponding constraints, we obtain

$$\mu^1(k) = \min \{ 0, -\delta\phi/\delta u(k) + \delta f^T/\delta u(k) \cdot \lambda(k+1) \} \quad (30)$$

and

$$\mu^2(k) = \min \{ 0, \delta\phi/\delta u(k) + \delta f^T/\delta u(k) \cdot \lambda(k+1) \} \quad (31)$$

Furthermore we define the gradient of J with respect to $u(k)$ on the equality constraints surface

$$g(k) = \delta\phi/\delta u(k) + \delta f^T/\delta u(k) \cdot \lambda(k+1) \quad (32)$$

The component N^i of the projected gradient with respect to the admissible control region (22) is given by:

$$\gamma_i(k) = \begin{cases} g_i(k) & \text{if } \mu_1^1(k) = 0 \text{ and } \mu_2^2(k) = 0 \\ 0 & \text{otherwise} \end{cases} \quad (33)$$

In analogy to the L_2 space of continuous-time systems, we define the scalar product of two vector trajectories $\eta(k)$, $\xi(k)$, $k = 0, \dots, K-1$, as follows:

$$[\eta(k), \xi(k)] = \sum_{k=0}^{K-1} \eta^T(k) \xi(k) \quad (34)$$

Furthermore, we define a saturation vector function $\text{sat}(\eta)$ with components:

$$\begin{aligned} \text{sat}_1(\eta) &= \eta_{1,\max} & \eta_1 > \eta_{1,\max} \\ \text{sat}_1(\eta) &= \eta_{1,\min} & \eta_1 < \eta_{1,\min} \\ \text{sat}_1(\eta) &= \eta_1 & \text{otherwise} \end{aligned}$$

With above definitions, the necessary conditions of optimality are given by (20), (21), (28), (29), (30), (31), (32), (33), $\mu_1 \leq 0$, $\mu_2 \leq 0$ and

$$\gamma_i(k) = 0 \quad \forall i, \forall k \quad (35)$$

Hence if these equations are satisfied, a stationary point of the optimal control problem has been found. If the state functions f , h are linear and the objective functional J is convex, these conditions become necessary and sufficient [Schoeffler, 1971].

The necessary conditions for optimality constitute a TPBVP. A well-known algorithmical structure for direct solution of the TPBVP is given in Sage, (1988), see also Papageorgiou & Mayr (1988). Some modifications are introduced to this algorithm in order to take into account inequality constraints. The algorithm used can be described as follows.

Step 1: Select an admissible initial control trajectory $u^0(k)$, $k = 0, \dots, K-1$; set $i = 0$.
Step 2: Using $u^i(k)$, $x^i(k)$, $k = 0, \dots, K-1$, calculate $(h^1)^i(k)$ and $(h^2)^i(k)$ from (23) and (24); solve (20) from initial condition (21) to obtain $X^i(k+1)$; beginning from the terminal condition (28), calculate simultaneously $(\mu^1)^i(k)$ and $(\mu^2)^i(k)$ and $\lambda_i(k+1)$ from (30), (31) and (28), respectively; $k = K-1, \dots, 0$.

Step 3: Using $u^i(k)$, $x^i(k)$, $\lambda^i(k+1)$, $(\mu^1)^i(k)$ and $(\mu^2)^i(k)$, $k = 0, \dots, K-1$, calculate the gradients $g^i(k)$ and $\gamma^i(k)$ from (32) and (33), respectively.

Step 4: Specify a search direction $P^i(k)$, $k = 0, \dots, K-1$. Step 5: apply a one-dimensional search routine along the P^i -direction to obtain u^{i+1} ; i.e. $J(u^{i+1}) = \min_{\alpha} J(\text{sat}(u^i + \alpha P^i))$ where α^i is the scalar step length. Step 6: if for a given scalar $\epsilon > 0$, $|\gamma_i(k), \gamma_i(k)| < \epsilon$ holds, stop; otherwise, set $i=i+1$ and go back to step 2.

Several techniques for specifying a search direction P^i in step 4 can be applied. As a first approach, a conjugate gradient techniques is used; comparisons with other methods for the P^i -specification may be effectuated in the future. Appendix A provides the equations for calculating the search direction P^i on the basis of the Fletcher-Reeves formula. Strategies for restarting the algorithm whenever a sufficient negativity condition is not fulfilled are also described in Appendix A. The one-dimensional search routine applied to step 5 is based on parabolic and cubic interpolations and is described in Appendix B.

The described algorithm is known to be insensitive with respect to poor initial guess trajectories $u^0(k)$. Alternative methods, such as quasi-linearization, are known to suffer by divergence problems if the initial guess trajectory is far from the optimal solution trajectory [Sage, 1988].

7. APPLICATION TO THE BOULEVARD PERIPHERIQUE OF PARIS

A test of the optimal control strategy is performed on the basis of a 90 min demand scenario collected from the BPP. Due to incomplete information provided by the detector loops installed on the BPP, the utilised demand values are approximations of the real ones.

7.1 No-control case

In order to compare the optimal control solution with the no-control case, a simulation of traffic flow on the BPP with a natural distribution of traffic demand is performed. When no on-ramp metering strategy is applied to the ringway, the entering procedure may be described by:

$$u_i(k) = \min \{ u_{\max}(\rho_i(k)), d_i(k) + l_i(k)/T \} \quad (36)$$

where $u_{\max}(\rho_i)$ is equal to the maximum on-ramp volume value given by eq.(8) if the traffic is fluid in the ringway section; if a congestion is present on the mainstream, $u_{\max}(\rho_i)$ decreases according to the relation:

$$u_{\max}(\rho_i(k)) = u_{1,\max} \sin \left\{ 1, \frac{(\rho_i(k) - \lambda_{1,\text{cr}}) \rho_{1,\text{cr}}}{\lambda_{1,\text{cr}} \rho_{1,\text{cr}} - \lambda_{1,\text{cr}} \rho_i(k)} \right\} \quad (37)$$

The maximum on-ramp volume fixed by the geometric characteristics of the entrance ramp is assumed to be $u_{1,\max} = 1600$ veh/h if the entrance is a normal

ramp; it is set equal to 3200 veh/h if the entrance is a motorway. The maximum density ρ_{max} is set equal to 100 veh/km/lane.

Equations (35) and (36) make clear that an on-ramp queue may be created even without ramp metering measures, namely if the demand d_i becomes higher than the on-ramp capacity $u_{max}(\rho_i)$. For the BPP, this is particularly the case on the motorway ramps which experience queues of several kilometers during rush hours due to the congestion on the BPP.

The initial density and queue length values are assumed to be the same for the different ringway sections resp. on-ramps:

$$\rho_i(0) = 100 \text{ veh/km} ; l_i(0) = 50 \text{ veh}$$

Figures 4 and 5 depict simulation results for traffic densities and queue lengths on the 12 ringway sections and the 12 on-ramps for the no-control case.

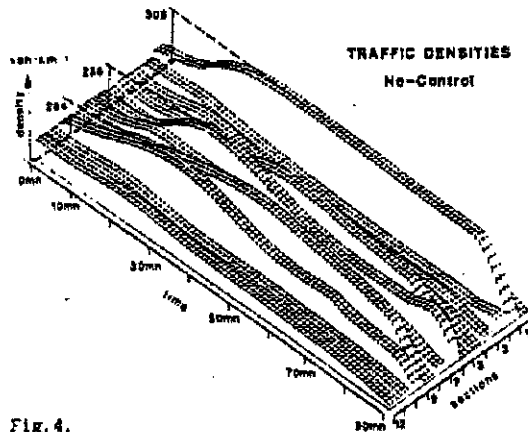


Fig. 4.

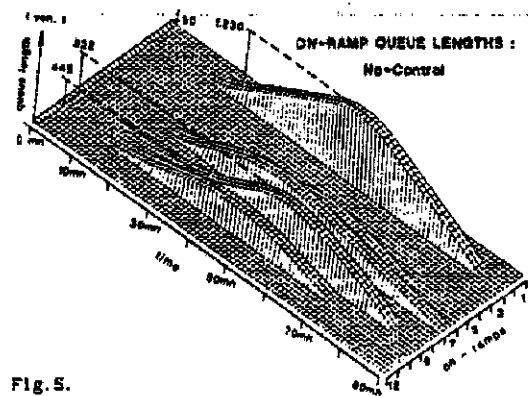


Fig. 5.

7.2 Optimal control case

The resulting on-ramp volume trajectories of the no-control case are used as an initial guess for the optimal control algorithm. The maximum queue length is set equal to 400 veh for a normal ramp and 800 veh for a motorway entrance. The maximum queue length of an aggregated ramp is given by the sum of the maximum queue lengths of the corresponding real ramps.

The sample time interval is chosen $T=1$ min, the weighting factors of the optimization objective function (18) are taken:

$$\beta_1 = 10^{-4}, \beta_2 = 10^{-2} \text{ and } \beta_3 = 5 \cdot 10^{-3}$$

The value of the total time spent for the initial guess (no-control) is 8870 veh.min. The whole objective function has nearly the same value since the three other terms are very small. The total time spent is reduced by the optimization to 6944 veh.min after 25 iterations which took 4 min 46 sec CPU time on a slow DPS8/52 biprocessor under a multics system. A further reduced value $T=6888$ veh.h is obtained after 118 iterations which corresponds to 20 min 58 sec CPU time on the same digital computer.

Simulation results for the optimal densities and queue lengths for the 90 min traffic demand scenario are depicted on figures 6 and 7.

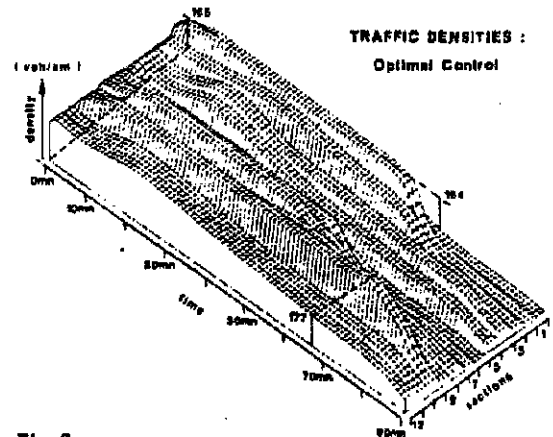


Fig. 6.

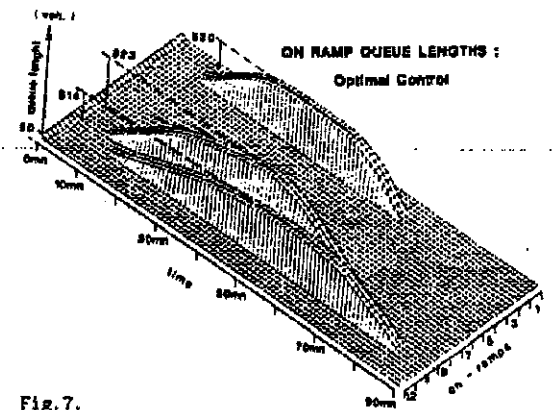


Fig. 7.

A comparison of the optimal control results with the no-control case shows a reduction of 30.9% of the total time spent by all drivers in the ringway system. An examination of fig. 6 shows that the optimal control strategy chooses automatically among the high number of possible on-ramp volume combinations the optimal on-ramp distribution in space and time so as to maintain the different sections density near the critical one. This leads to an optimal utilization of the ringway infrastructure capacity.

Furthermore, comparison between fig. 6 and 7 shows that application of the optimal control strategy reduces the on-ramp queue lengths and hence the total waiting time. Reduction of waiting queue lengths by application of ramp metering may appear to be paradoxical. In fact, this reduction is a consequence of the increased throughput on the mainstream due to congestion avoidance.

8. CONCLUSION

An on-ramp metering strategy based on optimal control theory is developed and applied to the whole of the Boulevard Périphérique of Paris.

The optimal control algorithm used is based on the discrete maximum principle; a simplified macroscopic non linear traffic flow model with a rather rude space/time discretization (3 km/1 mn) is used as a basis for the optimal control approach. In order to test the accuracy of the simplified model a simulation test has been performed with a more accurate traffic flow model for motorways which is called META. Comparison of the simplified model with META indicates that the main characteristics of traffic flow are reproduced with sufficient accuracy for optimization purposes.

The performance index considered is mainly to minimize the total time spent in the ringway system (including the on-ramp queues). Several technical and physical restrictions which are present are expressed in form of inequality constraints.

Comparison between the optimal control solution and the no-control case showed a gain of 30.8 % for the total time spent by all drivers. It is worth noticing that application of the optimal control strategy reduce the on-ramp queue lengths and eventually the on-ramp waiting time.

Results of the optimal control strategy can be used for a real time application or as desired values (set values) for inferior local control loops. A trade-off of accuracy versus computational effort (e.g. space discretization may be chosen to be 2 km or 1 km or 500 m long) will be considered in future investigations. Also the robustness of the resulting control will be tested by simulation studies on the basis of META.

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APPENDIX A: SPECIFICATION OF A SEARCH DIRECTION

The search direction P^i in step 4 of the described algorithm can be specified by use of the Fletcher-Reeves conjugate gradient method generalized to apply to optimal control problems with inequality control constraints. The according equations are given below:

$$P^i = -g^i + \beta^i P^{i-1}$$

with $\beta^0 = 0$ and

$$\beta^i = (r^i, r^i) / (r^{i-1}, r^{i-1}), \quad i \geq 1$$

As suggested by Fagurek & Woodside, (1968), the reduced gradient vector r_i is used in order for the search direction component P_i not to stick on the input constraints.

Restarting with $\beta^i = 0$ any time a sufficient negativity condition

$$(P^i, r^i) \leq -B [(P^i, P^i)(r^i, r^i)]^{1/2}$$

does not hold, is known to better results. B is a positive constant parameter. Periodical restarting every N iterations has been proposed by Bersekas, (1974). Both restarting procedures have been used in the algorithm presented in this paper.

APPENDIX B: ONE DIMENSION SEARCH ROUTINE

The following one-dimension search routine is applied in step 5 of the described algorithm.

(1) Assuming constant cost functional decrease at each iteration 1 and applying parabolic interpolation, we get an estimate for the step α^i

$$\alpha^i = -2(j^{i-1} - j^i) / (dj/d\alpha)_\alpha$$

(2) If for a fixed parameter A

$$(dj/d\alpha)_\alpha / (dj/d\alpha)_0 \leq A < 1$$

holds, the estimation is considered satisfactory. Otherwise, another value for α_i is calculated using cubic interpolation as described by Fletcher and Reeves, (1964).